A DIRECT APPROACH TO PLATEAU'S PROBLEM IN ANY CODIMENSION

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Abstract. This paper proposes a direct approach to solve the Plateau's problem in codimension higher than one. The problem is formulated as the minimization of the Hausdorff measure among a family of d-rectifiable closed subsets of $\mathbb{R}^n$: following the previous work [DGM14], the existence result is obtained by a compactness principle valid under fairly general assumptions on the class of competitors. Such class is then specified to give meaning to boundary conditions. We also show that the obtained minimizers are regular up to a set of dimension less than $(d-1)$.

1. Introduction

Plateau's problem consists in looking for a surface of minimal area among those surfaces spanning a given boundary. A considerable amount of effort in Geometric Measure Theory during the last fifty years has been devoted to provide generalized concepts of surface, area and of "spanning a given boundary", in order to apply the direct methods of the calculus of variations to the Plateau's problem. In particular we recall the notions of sets of finite perimeter [De54, De55], of currents [FF60] and of varifolds [All72, All75, Alm68], introduced respectively by De Giorgi, Federer, Fleming, Almgren and Allard. A more "geometric" approach was proposed by Reifenberg in [Rei60], where Plateau's problem was set as the minimization of Hausdorff d-dimensional measure among compact sets and the notion of spanning a given boundary was given in term of inclusions of homology groups.

Any of these approach has some drawbacks: in particular, not all the "reasonable" boundaries can be obtained by the above notions and not always the solutions are allowed to have the type of singularities observed in soap bubbles (the so called Plateau's laws). Recently in [HP13] Harrison and Pugh, see also [Har14], proposed a new notion of spanning a boundary, which seems to include reasonable physical boundaries and they have been able to show, in the codimension one case, existence of least area surfaces spanning a given boundary.

In the recent paper [DGM14], De Lellis, Maggi and the third author have proposed a direct approach to the Plateau's problem, based on the "elementary" theory of Radon measures and on a deep result of Preiss concerning rectifiable measures. Roughly speaking they showed, in the codimension one case, that every time one has a class which contains "enough" competitors (namely the cone and the cup competitors, see [DGM14, Definition 1]) it is always possible to prove that the infimum of the Plateau's problem is achieved by the area of a rectifiable set. They then applied this result to provide a new proof of Harrison and Pugh theorem as well as to show the existence of sliding minimizers, a new notion of minimal sets proposed by David in [Dav14, Dav13] and inspired by Almgren's $(\mathbf{M}, 0, \infty)$, [Alm76].

In this note, we extend the result [DGM14] to any codimension. More precisely, we prove that every time the class of competitors for the Plateau's Problem consists of rectifiable sets and it is closed by Lipschitz deformations, it is possible to show that the infimum is achieved by a compact set K which is, away from the "boundary", an analytic manifold outside a closed set of Hausdorff dimension at most $(d-1)$, see Theorem 1.3 below for the precise statement. We then apply this result to provide existence of sets spanning a given boundary according to the natural generalization of the notion introduced by Harrison and Pugh, Theorem 1.3, and to show the existence of sliding minimizers in any codimension, Theorem 1.8.

Although the general strategy of the proof is the same of [DGM14], some non-trivial modifications have to be done in order to deal with sets of any co-dimension. In particular, with respect to [DGM14], we use a different notion of "good class", the main reason being the following: one of the key steps of the proof of our main result consists in showing a precise density lower bound for the measure obtained as limit of the sequence of Radon measures naturally associated to a minimizing sequence (K_j), see Steps 1 and 4 in the proof of Theorem 1.3. In order to obtain such a lower bound, instead of relying on relative isopermetric inequalities on the sphere as in [DGM14] (which are peculiar of the co-dimension one case), we use the deformation theorem of David and Semmes in [DS00] to obtain suitable competitors, following a strategy already introduced by Federer and Fleming for rectifiable currents, see [FF60] and [Alm76]. Moreover, since our class is essentially closed by Lipschitz deformations, we are actually able to prove that any set achieving the infimum is a stationary varifold and that, in addition, it is smooth outside a closed set of relative co-dimension one (this does not directly follows by Allard's regularity theorem, see Step 7 in the proof of Theorem 1.3). Simple examples show that this regularity is actually optimal.

In order to precisely state our main results, let us introduce some notations and definitions. We will always work in $\mathbb{R}^n$ and $1 \leq d \leq n$ will always be an integer number, we recall that a set K is said to be d-rectifiable if it can be covered, up to an $\mathcal{H}^d$ negligible set, by countably many C^1 manifolds, see [Sim83, Chapter 3], where $\mathcal{H}^d$ is the d-dimensional Hausdorff measure. We also let $\mathsf{Lip}(\mathbb{R}^n)$ be the space of Lipschitz maps in $\mathbb{R}^n$.

Definition 1.1 (Lipschitz deformations)**.** Given a ball $B_{x,r}$, we let $\mathfrak{D}(x,r)$ be the set of functions $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\varphi(z) = z$ in $\mathbb{R}^n \setminus B_{x,r}$ and which are smoothly isotopic to the identity inside $B_{x,r}$, namely those for which there exists an isotopy $\lambda \in C^\infty([0,1] \times \mathbb{R}^n; \mathbb{R}^n)$ such that

$$\lambda(0,\cdot) = \mathrm{Id}, \quad \lambda(1,\cdot) = \varphi, \quad \lambda(t,h) = h \quad \forall\,(t,h) \in [0,1] \times (\mathbb{R}^n \setminus B_{x,r}) \quad \text{and}$$

$$\lambda(t,\cdot) \text{ is a diffeomorphism of } \mathbb{R}^n \ \forall t \in [0,1].$$

We finally set $\mathsf{D}(x,r) := \overline{\mathfrak{D}(x,r)}^{C^0} \cap \mathsf{Lip}(\mathbb{R}^n)$, the intersection of the Lipschitz maps with the closure of $\mathfrak{D}(x,r)$ with respect to the uniform topology.

Observe that in the definition of $\mathsf{D}(x,r)$ it is equivalent to require any C^k regularity on the isotopy λ, for $k \geq 1$, as C^k isotopies supported in $B(x,r)$ can be approximated in C^k by smooth ones also supported in the same set.

The following definition describes the properties required on comparison sets: the key property for K' to be a competitor of K is that K' is close in energy to sets obtained from K via deformation maps as in Definition 1.1. This allows a larger flexibility on the choice of the admissible sets, since a priori K' might not belong to the competition class.

Definition 1.2 (Deformed competitors and good class)**.** Let $H \subset \mathbb{R}^n$ be closed, $K \subset \mathbb{R}^n \setminus H$ be a relatively closed countably $\mathcal{H}^d$-rectifiable and $B_{x,r} \subset \mathbb{R}^n \setminus H$. A *deformed competitor* for K in $B_{x,r}$ is any set of the form

$$\varphi\,(K) \quad \text{ where } \quad \varphi \in \mathsf{D}(x,r).$$

Given a family $\mathcal{P}(H)$ of relatively closed d-rectifiable subsets $K \subset \mathbb{R}^n \setminus H$, we say that $\mathcal{P}(H)$ is a *good class* if for every $K \in \mathcal{P}(H)$, for every $x \in K$ and for a.e. $r \in (0, \mathrm{dist}(x,H))$

$$\inf\left\{\mathcal{H}^d(J) : J \in \mathcal{P}(H)\,, J \setminus \overline{B_{x,r}} = K \setminus \overline{B_{x,r}}\right\} \leq \mathcal{H}^d(L) \tag{1.1}$$

whenever L is any deformed competitor for K in $B_{x,r}$.

Once we fix a closed set H, we can formulate Plateau's problem in the class $\mathcal{P}(H)$:

$$m_0 := \inf\left\{\mathcal{H}^d(K) : K \in \mathcal{P}(H)\right\}. \tag{1.2}$$

We will say that a sequence $(K_j) \subset \mathcal{P}(H)$ is a *minimizing sequence* if $\mathcal{H}^d(K_j) \downarrow m_0$. The following theorem is our main result and establishes the behavior of minimizing sequences.

Theorem 1.3. *Let $H \subset \mathbb{R}^n$ be closed and $\mathcal{P}(H)$ be a good class. Assume the infimum in Plateau's problem* (1.2) *is finite and let $(K_j) \subset \mathcal{P}(H)$ be a minimizing sequence. Then, up to subsequences, the measures $\mu_j := \mathcal{H}^d \llcorner K_j$ converge weakly* in $\mathbb{R}^n \setminus H$ to a measure $\mu = \mathcal{H}^d \llcorner K$, where $K = \operatorname{spt}\mu \setminus H$ is a countably $\mathcal{H}^d$-rectifiable set. Furthermore:*

(a) *the integral varifold naturally associated to μ is stationary in $\mathbb{R}^n \setminus H$;*
(b) *K is a real analytic submanifold outside a relatively closed set $\Sigma \subset K$ with $\dim_{\mathcal{H}}(\Sigma) \le d-1$.*

In particular, $\liminf_j \mathcal{H}^d(K_j) \ge \mathcal{H}^d(K)$ and if $K \in \mathcal{P}(H)$, then K is a minimum for (1.2).

We wish to apply Theorem 1.3 to two definitions of boundary conditions. The first one is the natural generalization of the one considered in [HP13]:

Definition 1.4. Let H be a closed set in $\mathbb{R}^n$.

Let us consider the family

$$\mathcal{C}_H = \left\{\gamma : S^{n-d} \to \mathbb{R}^n \setminus H : \gamma \text{ is a smooth embedding of } S^{n-d} \text{ into } \mathbb{R}^n\right\}.$$

We say that $\mathcal{C} \subset \mathcal{C}_H$ is closed by isotopy (with respect to H) if $\mathcal{C}$ contains all elements $\gamma' \in \mathcal{C}_H$ belonging to the same smooth isotopy class $[\gamma]$ in $\mathbb{R}^n \setminus H$ of any $\gamma \in \mathcal{C}$, see [Hir94, Ch. 8]. Given $\mathcal{C} \subset \mathcal{C}_H$ closed by isotopy, we say that a relatively closed subset K of $\mathbb{R}^n \setminus H$ is a $\mathcal{C}$-spanning set of H if

$$K \cap \gamma \neq \emptyset \text{ for every } \gamma \in \mathcal{C}.$$

We denote by $\mathcal{F}(H, \mathcal{C})$ the family of countably $\mathcal{H}^d$-rectifiable sets which are $\mathcal{C}$-spanning sets of H.

We can prove the following closure property for the class $\mathcal{F}(H, \mathcal{C})$:

Theorem 1.5. *Let H be closed in $\mathbb{R}^n$ and $\mathcal{C}$ be closed by isotopy with respect to H, then:*

(a) *$\mathcal{F}(H, \mathcal{C})$ is a good class in the sense of Definition 1.2.*
(b) *If the infimum* (1.2) *corresponding to $\mathcal{P}(H) = \mathcal{F}(H, \mathcal{C})$ is finite, then the set K provided by Theorem 1.3 belongs to $\mathcal{F}(H, \mathcal{C})$. In particular the Plateau's problem in the class $\mathcal{F}(H, \mathcal{C})$ has a solution.*

The second type of boundary condition we want to consider is the one related to the notion of "sliding minimizers" introduced by David in [Dav14, Dav13].

Definition 1.6 (Sliding minimizers)**.** Let $H \subset \mathbb{R}^n$ be closed and $K_0 \subset \mathbb{R}^n \setminus H$ be relatively closed. We denote by $\Sigma(H)$ the family of Lipschitz maps $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that there exists a continuous map $\Phi : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ with $\Phi(1, \cdot) = \varphi$, $\Phi(0, \cdot) = \mathrm{Id}$ and $\Phi(t, H) \subset H$ for every $t \in [0,1]$. We then define

$$\mathcal{A}(H, K_0) = \left\{K : K = \varphi(K_0) \text{ for some } \varphi \in \Sigma(H)\right\}$$

and say that K_0 is a sliding minimizer if $\mathcal{H}^d(K_0) = \inf\{\mathcal{H}^d(J) : J \in \mathcal{A}(H, K_0)\}$.

Remark 1.7. For every $K_0 \subset \mathbb{R}^n \setminus H$ relatively closed and d-rectifiable, $\mathcal{A}(H, K_0)$ is a good class in the sense of Definition 1.2, since $\mathsf{D}(x, r) \subset \Sigma(H)$ for every $B_{x,r} \subset \mathbb{R}^n \setminus H$.

Applying Theorem 1.3 to the framework of sliding minimizers we obtain the following result which is the analogous of [DGM14, Theorem 7] in any codimension. Here and in the following $U_\delta(E)$ denotes the δ-neighborhood of a set $E \subset \mathbb{R}^n$.

Theorem 1.8. *Assume that*

(i) K_0 *is bounded d-rectifiable with* $\mathcal{H}^d(K_0) < \infty$*;*

(ii) $\mathcal{H}^d(H) = 0$ *and for every* $\eta > 0$ *there exist* $\delta > 0$ *and* $\Pi \in \Sigma(H)$ *such that*

$$\operatorname{Lip}\Pi \le 1 + \eta\,, \qquad \Pi(U_\delta(H)) \subset H\,.$$

Then, given any minimizing sequence (K_j) *in the Plateau's problem corresponding to* $\mathcal{P}(H) = \mathcal{A}(H, K_0)$ *and any set* K *as in Theorem 1.3, we have*

$$\inf\left\{\mathcal{H}^d(J) : J \in \mathcal{A}(H, K_0)\right\} = \mathcal{H}^d(K) = \inf\left\{\mathcal{H}^d(J) : J \in \mathcal{A}(H, K)\right\}.$$

In particular K *is a sliding minimizer.*

Remark 1.9. It is far from obvious to prove the existence of a minimizer in the class $\mathcal{A}(H, K_0)$. It is indeed false in general that the sliding minimizer K in Theorem 1.8 belongs to $\mathcal{A}(H, K_0)$ (see the discussion in [DGM14, Remark 8]).

The paper is structured as follows, in Section 2 we will recall some basic definitions and recall some known theorems we are going to use, in particular Preiss rectifiability criterion and a version of the deformation theorem due to David and Semmes. In Section 3 we prove Theorem 1.3 and in Section 4 we prove Theorems 1.5 and 1.8.

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2. Notation and preliminaries

We are going to use the following notations: $Q_{x,l}$ denotes the closed cube centered in x, with edge length l; moreover we set

$$R_{x,a,b} := x + \Big[-\frac{a}{2}, \frac{a}{2}\Big]^d \times \Big[-\frac{b}{2}, \frac{b}{2}\Big]^{n-d} \quad \text{and} \quad B_{x,r} := \{y \in \mathbb{R}^n \ : \ |y - x| < r\}. \tag{2.1}$$

When cubes, rectangles and balls are centered in the origin, we will simply write Q_l, $R_{a,b}$ and B_r. Cubes and balls in the subspace $\mathbb{R}^d \times \{0\}^{n-d}$ are denoted with $Q^d_{x,l}$ and $B^d_{x,r}$ respectively. We also let ω_d be the Lebesgue measure of the unit ball in $\mathbb{R}^d$.

Let us recall the following deep structure result for Radon measures due to Preiss [Pre87, DeL08] which will play a key role in the proof of Theorem 1.3.

Theorem 2.1. *Let d be an integer and μ a locally finite measure on $\mathbb{R}^n$ such that the d-density*

$$\theta(x) := \lim_{r\to 0} \frac{\mu(B_{x,r})}{\omega_d r^d}$$

exists and satisfies $0 < \theta(x) < +\infty$ *for μ-a.e. x. Then* $\mu = \theta\mathcal{H}^d \llcorner K$*, where K is a countably $\mathcal{H}^d$-rectifiable set.*

In order to apply Preiss' Theorem, we will rely on the monotonicity formula for minimal surfaces, which roughly speaking can be obtained by comparing the given minimizer with a cone. To this aim let us introduce the following definition:

Definition 2.2 (Cone competitors)**.** In the setting of Definition 1.2, the cone competitor for K in $B_{x,r}$ is the following set

$$\mathbf{C}_{x,r}(K) = \big(K \setminus B_{x,r}\big) \cup \left\{\lambda x + (1-\lambda) z : z \in K \cap \partial B_{x,r}\,, \lambda \in [0,1]\right\}. \tag{2.2}$$

Let us note that in general a cone competitor in $B_{x,r}$ is not a deformed competitor in $B_{x,r}$. On the other hand as in [DGM14] we can show that:

Lemma 2.3. *Given a good class $\mathcal{P}(H)$ in the sense of Definition 1.2, for any $K \in \mathcal{P}(H)$ countably $\mathcal{H}^d$-rectifiable and for every $x \in K$, the set K verifies the following inequality for a.e. $r \in (0, \operatorname{dist}(x, H))$:*

$$\inf\left\{\mathcal{H}^d(J) : J \in \mathcal{P}(H)\,, J \setminus \overline{B_{x,r}} = K \setminus \overline{B_{x,r}}\right\} \le \mathcal{H}^d(\mathbf{C}_{x,r}(K)).$$

Proof. Without loss of generality, let us consider balls B_r centered at 0 with $B_r \subset\subset \mathbb{R}^n \setminus H$. We assume in addition that $K \cap \partial B_r$ is $\mathcal{H}^{d-1}$-rectifiable with $\mathcal{H}^{d-1}(K \cap \partial B_r) < \infty$ and that r is a Lebesgue point of $t \in (0, \infty) \mapsto \mathcal{H}^{d-1}(K \cap \partial B_t)$. All these conditions are fulfilled for a.e. r and, again by scaling, we can assume that $r = 1$ and use B instead of B_1. For $s \in (0, 1)$ let us set

$$\varphi_s(r) = \begin{cases} 0\,, & r \in [0, 1-s)\,, \\ \frac{r-(1-s)}{s}\,, & r \in [1-s, 1]\,, \\ r\,, & r \ge 1\,, \end{cases}$$

and $\phi_s(x) = \varphi_s(|x|)\frac{x}{|x|}$ for $x \in \mathbb{R}^n$. In this way, one easily checks that $\phi_s : \mathbb{R}^n \to \mathbb{R}^n \in \mathsf{D}(0, 1)$.

Since $\phi_s(K \cap \overline{B_{1-s}}) = \{0\}$, we need to show that

$$\limsup_{s\to 0^+} \mathcal{H}^d\big(\phi_s(K \cap (B \setminus B_{1-s}))\big) \le \frac{\mathcal{H}^{d-1}(K \cap \partial B)}{d} = \mathcal{H}^d(\mathbf{C}_{x,r}(K))\,.$$

Let $x_0 \in K \cap \partial B_t$ and let us fix an orthonormal base $\nu_1, \dots, \nu_d$ of the approximate tangent space $T_{x_0}K$ such that $\nu_i \in T_{x_0}K \cap T_{x_0}\partial B_t$ for $i \le d-1$. Let

$$J_d^K \phi_s = \Big|(\bigwedge^d D\phi_s)(T_{x_0}K)\Big| = |D\phi_s(\nu_1) \wedge \cdots \wedge D\phi_s(\nu_d)|$$

be the d-dimensional tangential Jacobian of ϕ_s with respect to K. A simple computation shows that

$$\begin{aligned} J_d^K \phi_s(x) &\le \Big(\frac{\varphi_s(|x|)}{|x|}\Big)^d + |\nu_d \cdot \hat{x}|\, \varphi_s'(|x|)\,\Big(\frac{\varphi_s(|x|)}{|x|}\Big)^{d-1} \\ &\le 1 + |\nu_d \cdot \hat{x}|\, \varphi_s'(|x|)\Big(\frac{\varphi_s(|x|)}{|x|}\Big)^{d-1}, \qquad \text{for } \mathcal{H}^d\text{-a.e. } x \in K\,. \end{aligned} \tag{2.3}$$

Here $\hat{x} = x/|x|$ and in the last inequality we have exploited that $\varphi(r) \le r$ for $r \in [1-s, 1]$. Using that $|\nu_d \cdot \hat{x}|$ is the tangential co-area factor of the map $f(x) = |x|$, we find with the aid of the area and co-area formulas,

$$\begin{aligned} \mathcal{H}^d\big(\phi_s(K \cap (B \setminus B_{1-s}))\big) &= \int_{K\cap(B\setminus B_{1-s})} J_d^K \phi_s \, d\mathcal{H}^d \\ &= \int_{K\cap(B\setminus B_{1-s})\cap\{|\nu_d\cdot\hat{x}|\neq 0\}} J_d^K \phi_s \, d\mathcal{H}^d + \int_{K\cap(B\setminus B_{1-s})\cap\{|\nu_d\cdot\hat{x}|=0\}} J_d^K \phi_s \, d\mathcal{H}^d \\ &\le \int_{1-s}^1 dt \int_{K\cap\partial B_t} \frac{J_d^K \phi_s}{|\nu_d \cdot \hat{x}|}\, d\mathcal{H}^{d-1} + \mathcal{H}^d\big(K \cap (B \setminus B_{1-s}) \cap \{|\nu_d \cdot \hat{x}| = 0\}\big), \end{aligned} \tag{2.4}$$

since $|J_d^K \phi_s| \le 1$ where $|\nu_d \cdot \hat{x}| = 0$. Using

$$\lim_{s\to 0} \mathcal{H}^d\big(K \cap (B \setminus B_{1-s})\big) = 0\,,$$

the second term in (2.4) can be ignored. Moreover, being $t = 1$ a Lebesgue point of $t \in (0, \infty) \mapsto \mathcal{H}^{d-1}(K \cap \partial B_t)$, we have

$$\lim_{s\to 0} \frac{1}{s} \int_{1-s}^1 |\mathcal{H}^{d-1}(K \cap \partial B_t) - \mathcal{H}^{d-1}(K \cap \partial B)|\, dt = 0\,.$$

Thanks to this and to the estimate (2.3), we infer from (2.4) that

$$\limsup_{s\to 0^+} \mathcal{H}^d(\varphi_s(K\cap B)) \le \mathcal{H}^{d-1}(K\cap\partial B) \limsup_{s\to 0^+} \frac{1}{s}\int_{1-s}^{1}\Big(\frac{\varphi_s(t)}{t}\Big)^{d-1} dt = \frac{\mathcal{H}^{d-1}(K\cap\partial B)}{d},$$

as required.

□

The second key result we are going to use is a deformation theorem for closed sets due to David and Semmes [DS00], analogous to the one for rectifiable currents [Sim83, Fed69]. We provide a slightly extended statement for the sake of forthcoming proofs.

Before stating the theorem, let us introduce some further notation. Given a closed cube $Q = Q_{x,l}$ and $\varepsilon > 0$, we cover Q with a grid of closed smaller cubes with edge length $\varepsilon \ll l$, with non empty intersection with $\mathrm{Int}(Q)$ and such that the decomposition is centered in x (i.e. one of the subcubes is centered in x). The family of this smaller cubes is denoted $\Lambda_\varepsilon(Q)$. We set

$$\begin{aligned} C_1 &:= \bigcup \{T\cap Q : T\in\Lambda_\varepsilon(Q), T\cap\partial Q \neq \emptyset\}, \\ C_2 &:= \bigcup \{T\in\Lambda_\varepsilon(Q) : (T\cap Q)\not\subset C_1, T\cap \partial C_1 \neq\emptyset\}, \\ Q^1 &:= \overline{Q\setminus(C_1\cup C_2)} \end{aligned} \tag{2.5}$$

and consequently

$$\Lambda_\varepsilon(Q^1\cup C_2) := \big\{T\in\Lambda_\varepsilon(Q) : T\subset (Q^1\cup C_2)\big\}.$$

For each nonnegative integer $m\le n$, let $\Lambda_{\varepsilon,m}(Q^1\cup C_2)$ denote the collection of all m-dimensional faces of cubes in $\Lambda_\varepsilon(Q^1\cup C_2)$ and $\Lambda^*_{\varepsilon,m}(Q^1\cup C_2)$ will be the set of the elements of $\Lambda_{\varepsilon,m}(Q^1\cup C_2)$ which are not contained in $\partial(Q^1\cup C_2)$. We also let $S_{\varepsilon,m}(Q^1\cup C_2) := \bigcup \Lambda_{\varepsilon,m}(Q^1\cup C_2)$ be the m-skeleton of order ε in $Q^1\cup C_2$.

Theorem 2.4. *Let $r>0$ and E be a compact subset of Q such that $\mathcal{H}^d(E)<+\infty$ and $Q\subset B_{x_0,r}$. There exists a map $\Phi_{\varepsilon,E}\in \mathsf{D}(x_0,r)$ satisfying the following properties:*

1. *$\Phi_{\varepsilon,E}(x) = x$ for $x\in\mathbb{R}^n\setminus(Q^1\cup C_2)$;*
2. *$\Phi_{\varepsilon,E}(x) = x$ for $x\in S_{\varepsilon,d-1}(Q^1\cup C_2)$;*
3. *$\Phi_{\varepsilon,E}(E\cap(Q^1\cup C_2))\subset S_{\varepsilon,d}(Q^1\cup C_2)\cup\partial(Q^1\cup C_2)$;*
4. *$\Phi_{\varepsilon,E}(T)\subset T$ for every $T\in\Lambda_{\varepsilon,m}(Q^1\cup C_2)$, with $m=d,...,n$;*
5. *either $\mathcal{H}^d(\Phi_{\varepsilon,E}(E)\cap T)=0$ or $\mathcal{H}^d(\Phi_{\varepsilon,E}(E)\cap T)=\mathcal{H}^d(T)$, for every $T\in\Lambda^*_{\varepsilon,d}(Q^1)$;*
6. *$\mathcal{H}^d(\Phi_{\varepsilon,E}(E\cap T))\le k_1\mathcal{H}^d(E\cap T)$ for every $T\in\Lambda_\varepsilon(Q^1\cup C_2)$;*

where k_1 depends only on n and d (but not on ε).

Proof. Proposition 3.1 in [DS00] provides a map $\widetilde{\Phi_{\varepsilon,E}}\in\mathsf{D}(x_0,r)$ satisfying properties (1)-(4) and (6). We want to set

$$\Phi_{\varepsilon,E} := \Psi\circ\widetilde{\Phi_{\varepsilon,E}},$$

where Ψ will be defined below. We first define Ψ on every $T\in\Lambda_{\varepsilon,d}(Q^1\cup C_2)$ distinguishing two cases

(a) if either $\mathcal{H}^d(\widetilde{\Phi_{\varepsilon,E}}(E)\cap T)=0$ or $\mathcal{H}^d(\widetilde{\Phi_{\varepsilon,E}}(E)\cap T)=\mathcal{H}^d(T)$ or $T\notin\Lambda^*_{\varepsilon,d}(Q^1)$, then we set $\Psi_{|T}=\mathrm{Id}$;

(b) otherwise, since $\widetilde{\Phi_{\varepsilon,E}}(E)$ is compact, there exists $y_T\in T$ and $\delta_T>0$ such that $B_{\delta_T}(y_T)\cap\widetilde{\Phi_{\varepsilon,E}}(E)=\emptyset$; we define

$$\Psi_{|T}(x) = x+\alpha(x-y_T)\min\left\{1,\frac{|x-y_T|}{\delta_T}\right\},$$

where $\alpha>0$ such that the point $x+\alpha(x-y_T)\in(\partial T)\times\{0\}^{n-d}$.

The second step is to define Ψ on every $T' \in \Lambda_{\varepsilon,d+1}(Q^1 \cup C_2)$. Without loss of generality, we can assume T' centered in 0. We divide T' in pyramids $P_{T,T'}$ with base $T \in \Lambda_{\varepsilon,d}(Q^1 \cup C_2)$ and vertex 0. Assuming $T \subset \{x_{d+1} = -\frac{\varepsilon}{2}, x_{d+2}, ..., x_n = 0\}$ and $T' \subset \{x_{d+2}, ..., x_n = 0\}$, we set

$$\Psi_{|P_{T,T'}}(x) = -\frac{2x_{d+1}}{\varepsilon}\Psi_{|T}\left(-\frac{x}{x_{d+1}}\frac{\varepsilon}{2}\right).$$

We iterate this procedure on all the dimensions till to n, defining it well in $Q^1 \cup C_2$. Since $\Psi_{|\partial(Q^1\cup C_2)} = \mathrm{Id}$, we can extend the map as the identity outside $Q^1 \cup C_2$. In addition, one can easily check that $\Psi \in \mathsf{D}(x_0, r)$ and thus, since $\widetilde{\Phi_{\varepsilon,E}} \in \mathsf{D}(x_0, r)$ and the class $\mathsf{D}(x_0, r)$ is closed by composition, this concludes the proof. $\square$

Later we will need to implement the above deformation of a set E on a rectangle rather than a cube. The deformation theorem can be proved for very general cubical complexes, [Alm86]; however, for the sake of exposition, we limit ourselves to the simple case of a rectangular complex, which can be deduced by Theorem 2.4 through a bi-Lipschitz (linear) transformation of $\mathbb{R}^n$. More precisely, let us consider a closed rectangle

$$R := [0, \ell_1] \times \cdots \times [0, \ell_n] \qquad \ell_1 \le \cdots \le \ell_n$$

and a tiling of $\mathbb{R}^n$ made of rectangle ε-homothetic to R. Let $\Lambda^R_\varepsilon(R)$ denote the family of the translated and ε scaled copies of R and let us set

$$C^R_1 := \bigcup \left\{T \cap R : T \in \Lambda^R_\varepsilon(R), T \cap \partial R \neq \emptyset\right\},$$
$$C^R_2 := \bigcup \left\{T \in \Lambda^R_\varepsilon(R) : (T \cap R) \not\subset C^R_1, T \cap \partial C^R_1 \neq \emptyset\right\},$$
$$R^1 := \overline{R \setminus (C^R_1 \cup C^R_2)}.$$

As before, for each nonnegative integer $m \le n$, we let $\Lambda^R_{\varepsilon,m}(R^1 \cup C^R_2)$ denote the collection of all m-dimensional faces of rectangles in $\Lambda^R_\varepsilon(R^1 \cup C^R_2)$ and $\Lambda^{R*}_{\varepsilon,m}(R^1 \cup C^R_2)$ will be the set of the elements of $\Lambda^R_{\varepsilon,m}(R^1 \cup C^R_2)$ which are not contained in $\partial(R^1 \cup C^R_2)$. We also let $S^R_{\varepsilon,m}(R^1 \cup C^R_2) := \bigcup \Lambda^R_{\varepsilon,m}(R^1 \cup C^R_2)$ be the m-skeleton of order ε in $R^1 \cup C^R_2$. Then the following theorem is an immediate consequence of Theorem 2.4:

Theorem 2.5. *Let $r > 0$ and E be a compact subset of R such that $\mathcal{H}^d(E) < +\infty$ and $R \subset B_{x_0,r}$. There exists a map $\Phi_{\varepsilon,E} \in \mathsf{D}(x_0, r)$ satisfying the following properties:*

1. *$\Phi_{\varepsilon,E}(x) = x$ for $x \in \mathbb{R}^n \setminus (R^1 \cup C^R_2)$;*
2. *$\Phi_{\varepsilon,E}(x) = x$ for $x \in S^R_{\varepsilon,d-1}(R^1 \cup C^R_2)$;*
3. *$\Phi_{\varepsilon,E}(E \cap (R^1 \cup C^R_2)) \subset S^R_{\varepsilon,d}(R^1 \cup C^R_2) \cup \partial(R^1 \cup C^R_2)$;*
4. *$\Phi_{\varepsilon,E}(T) \subset T$ for every $T \in \Lambda^R_{\varepsilon,m}(R^1 \cup C^R_2)$, with $m = d, ..., n$;*
5. *either $\mathcal{H}^d(\Phi_{\varepsilon,E}(E) \cap T) = 0$ or $\mathcal{H}^d(\Phi_{\varepsilon,E}(E) \cap T) = \mathcal{H}^d(T)$, for every $T \in \Lambda^{R*}_{\varepsilon,d}(R^1)$;*
6. *$\mathcal{H}^d(\Phi_{\varepsilon,E}(E \cap T)) \le k_1 \mathcal{H}^d(E \cap T)$ for every $T \in \Lambda^R_\varepsilon(R^1 \cup C^R_2)$;*

where k_1 depends only on n, d and ℓ_n/ℓ_1 (but not on ε).

Note that this time the constant k_1 depends also from the ratio ℓ_n/ℓ_1. In the sequel we will apply this construction only to rectangles where this ratio is between 1 and 4: $1 \le \ell_n/\ell_1 \le 4$, thus obtaining a constant k_1 actually depending just on n and d.

3. Proof of Theorem 1.3

Proof of Theorem 1.3. Up to extracting subsequences we can assume the existence of a Radon measure μ on $\mathbb{R}^n \setminus H$ such that

$$\mu_j \overset{*}{\rightharpoonup} \mu, \qquad \text{as Radon measures on } \mathbb{R}^n \setminus H, \tag{3.1}$$

where $\mu_j = \mathcal{H}^d \llcorner K_j$. We set $K = \operatorname{spt}\mu \setminus H$ and divide the argument in several steps.

Step one: We show the existence of $\theta_0 = \theta_0(n,d) > 0$ such that

$$\mu(B_{x,r}) \geq \theta_0\, \omega_d r^d\,, \qquad x \in \operatorname{spt}\mu \text{ and } r < d_x := \operatorname{dist}(x,H). \tag{3.2}$$

To this end, it is sufficient to prove the existence of $\beta = \beta(n,d) > 0$ such that

$$\mu(Q_{x,l}) \geq \beta\, l^d\,, \qquad x \in \operatorname{spt}\mu \text{ and } l < 2d_x/\sqrt{n}\,.$$

Let us assume by contradiction that there exist $x \in \operatorname{spt}\mu$ and $l < 2d_x/\sqrt{n}$ such that

$$\frac{\mu(Q_{x,l})^{\frac{1}{d}}}{l} < \beta.$$

We claim that this assumption, for β chosen sufficiently small depending only on d and n, implies that for some $l_\infty \in (0,l)$

$$\mu(Q_{x,l_\infty}) = 0, \tag{3.3}$$

which is a contradiction with the property of x to be a point of $\operatorname{spt}\mu$. In order to prove (3.3), we assume that $\mu(\partial Q_{x,l}) = 0$, which is true for a.e. l.

To prove (3.3), we construct a sequence of nested cubes $Q_i = Q_{x,l_i}$ such that, if β is sufficiently small, the following holds:

(i) $Q_0 = Q_{x,l}$;
(ii) $\mu(\partial Q_{x,l_i}) = 0$;
(iii) setting $m_i := \mu(Q_i)$ then:

$$\frac{m_i^{\frac{1}{d}}}{l_i} < \beta;$$

(iv) $m_{i+1} \leq (1 - \frac{1}{k_1})m_i$, where k_1 is the constant in Theorem 2.4 (6);
(v) $(1-4\varepsilon_i)l_i \geq l_{i+1} \geq (1-6\varepsilon_i)l_i$, where

$$\varepsilon_i := \frac{1}{k\beta}\frac{m_i^{\frac{1}{d}}}{l_i} \tag{3.4}$$

and $k = \max\{6, 6/(1-(\frac{k_1-1}{k_1})^{\frac{1}{d}})\}$ is a universal constant.
(vi) $\lim_i m_i = 0$ and $\lim_i l_i > 0$.

Following [DS00], we are going to construct the sequence of cubes by induction: the cube Q_0 satisfies by construction hypotheses (i)-(iii). Suppose that cubes until step i are already defined.

Setting $m_i^j := \mathcal{H}^d(K_j \cap Q_i)$, we cover Q_i with the family $\Lambda_{\varepsilon_i l_i}(Q_i)$ of closed cubes with edge length $\varepsilon_i l_i$ as described in Section 2 and we set C_1^i and C_2^i for the corresponding sets defined in (2.5). We define Q_{i+1} to be the internal cube given by the construction, and we note that C_2^i and Q_{i+1} are non-empty if, for instance,

$$\varepsilon_i = \frac{1}{k\beta}\frac{m_i^{\frac{1}{d}}}{l_i} < \frac{1}{k} \leq \frac{1}{6},$$

which is guaranteed by our choice of k. Observe moreover that $C_1^i \cup C_2^i$ is a strip of width at most $2\varepsilon_i l_i$ around ∂Q_i, hence the side l_{i+1} of Q_{i+1} satisfies $(1-4\varepsilon_i)l_i \leq l_{i+1} < (1-2\varepsilon_i)l_i$.

Now we apply Theorem 2.4 to Q_i with $E = K_j$ and $\varepsilon = \varepsilon_i l_i$, obtaining the map $\Phi_{i,j} = \Phi_{\varepsilon_i l_i, K_j}$. We claim that, for every j sufficiently large,

$$m_i^j \leq k_1(m_i^j - m_{i+1}^j) + o_j(1). \tag{3.5}$$

Indeed, since (K_j) is a minimizing sequence, by the definition of good class we have that

$$\begin{aligned} m_i^j &\le m_i + o_j(1) \le \mathcal{H}^d\left(\Phi_{i,j}\left(K_j \cap Q_i\right)\right) + o_j(1) \\ &= \mathcal{H}^d\left(\Phi_{i,j}\left(K_j \cap Q_{i+1}\right)\right) + \mathcal{H}^d\left(\Phi_{i,j}\left(K_j \cap (C_1^i \cup C_2^i)\right)\right) + o_j(1) \\ &\le k_1 \mathcal{H}^d\left(K_j \cap (C_1^i \cup C_2^i)\right) + o_j(1) = k_1(m_i^j - m_{i+1}^j) + o_j(1). \end{aligned}$$

The last inequality holds because $\mathcal{H}^d\left(\Phi_{i,j}\left(K_j \cap Q_{i+1}\right)\right) = 0$ for j large enough: otherwise, by property (5) of Theorem 2.4, there would exist $T \in \Lambda^*_{\varepsilon_i l_i, d}(Q_{i+1})$ such that $\mathcal{H}^d(\Phi_{i,j}(K_j \cap T)) = \mathcal{H}^d(T)$. Together with property (ii), this would imply

$$l_i^d \varepsilon_i^d = \mathcal{H}^d(T) \le \mathcal{H}^d\left(\Phi_{i,j}\left(K_j\right) \cap Q_i\right) \le k_1 \mathcal{H}^d\left(K_j \cap Q_i\right) \le k_1 m_i^j \to k_1 m_i$$

and therefore, substituting (3.4),

$$\frac{m_i}{k^d \beta^d} \le k_1 m_i,$$

which is false if β is sufficiently small ($m_i > 0$ because $x \in \operatorname{spt}(\mu)$). Passing to the limit in j in (3.5) we obtain (iv):

$$m_{i+1} \le \frac{k_1 - 1}{k_1} m_i. \tag{3.6}$$

Since $l_{i+1} \ge (1 - 4\varepsilon_i) l_i$, we can slightly shrink the cube Q_{i+1} to a concentric cube Q'_{i+1} with $l'_{i+1} \ge (1 - 6\varepsilon_i) l_i > 0$, $\mu(\partial Q'_{i+1}) = 0$ and for which (iv) still holds, just getting a lower value for m_{i+1}. With a slight abuse of notation, we rename this last cube Q'_{i+1} as Q_{i+1}.

We now show (iii). Using (3.6) and condition (iii) for Q_i, we obtain

$$\frac{m_{i+1}^{\frac{1}{d}}}{l_{i+1}} \le \left(\frac{k_1 - 1}{k_1}\right)^{\frac{1}{d}} \frac{m_i^{\frac{1}{d}}}{(1 - 6\varepsilon_i) l_i} < \left(\frac{k_1 - 1}{k_1}\right)^{\frac{1}{d}} \frac{\beta}{1 - 6\varepsilon_i}.$$

The last quantity will be less than β if

$$\left(\frac{k_1 - 1}{k_1}\right)^{\frac{1}{d}} \le 1 - 6\varepsilon_i = 1 - \frac{6}{k\beta} \frac{m_i^{\frac{1}{d}}}{l_i}. \tag{3.7}$$

In turn, inequality (3.7) is true because (iii) holds for Q_i, provided we choose $k \ge 6/\big(1 - (1 - 1/k_1)^{\frac{1}{d}}\big)$. Furthermore, estimating $\varepsilon_0 < 1/k$ by (iii) and (v), we also have $\varepsilon_{i+1} \le \varepsilon_i$.

We are left to prove (vi): $\lim_i m_i = 0$ follows directly from (iv); regarding the non degeneracy of the cubes, note that

$$\begin{aligned} \frac{l_\infty}{l_0} := \liminf_i \frac{l_i}{l_0} &\ge \prod_{i=0}^{\infty} (1 - 6\varepsilon_i) = \prod_{i=0}^{\infty} \left(1 - \frac{6}{k\beta} \frac{m_i^{\frac{1}{d}}}{l_i}\right) \\ &\ge \prod_{i=0}^{\infty} \left(1 - \frac{6 m_0^{\frac{1}{d}}}{k\beta l_0 \prod_{h=0}^{i-1} (1 - 6\varepsilon_h)} \left(\frac{k_1 - 1}{k_1}\right)^{\frac{i}{d}}\right) \\ &\ge \prod_{i=0}^{\infty} \left(1 - \frac{6}{k(1 - 6\varepsilon_0)^i} \left(\frac{k_1 - 1}{k_1}\right)^{\frac{i}{d}}\right), \end{aligned}$$

where we used $\varepsilon_h \le \varepsilon_0$ in the last inequality. Since $\varepsilon_0 < 1/k$, the last product is strictly positive, provided

$$k > \frac{6}{1 - \left(\frac{k_1 - 1}{k_1}\right)^{\frac{1}{d}}},$$

which is guaranteed by our choice of k. We conclude that $l_\infty > 0$, which ensures claim (3.3).

Step two: We fix $x \in \operatorname{spt}\mu \setminus H$, and prove that

$$r \mapsto \frac{\mu(B_{x,r})}{r^d} \quad \text{is increasing on } (0, d_x). \tag{3.8}$$

The proof is a straightforward adaptation of the corresponding one in [DGM14, Theorem 2], and amounts to prove a differential inequality for the function $f(r) := \mu(B_{x,r})$. In turn, this inequality is obtained in a two step approximation: first one exploits the rectifiability of the minimizing sequence (K_j) and property (1.1) to compare K_j with the cone competitor $\mathbf{C}_{x,r}(K_j)$, see (2.2). The comparison, a priori, is only allowed with elements of $\mathcal{P}(H)$, so for almost every $r < d_x$ the following holds:

$$\begin{aligned} f_j(r) &= \mathcal{H}^d(K_j) - \mathcal{H}^d(K_j \setminus \overline{B_{x,r}}) \le m_0 + o_j(1) - \mathcal{H}^d(K_j \setminus \overline{B_{x,r}}) \\ &\le o_j(1) + \inf_{K' \in \mathcal{P}(H)} \mathcal{H}^d(K') - \mathcal{H}^d(K_j \setminus \overline{B_{x,r}}) \le o_j(1) + \inf_{\substack{K' \in \mathcal{P}(H) \\ K' \setminus \overline{B_{x,r}} = K_j \setminus \overline{B_{x,r}}}} \mathcal{H}^d(K' \cap \overline{B_{x,r}}), \end{aligned}$$

where $f_j(r) := \mathcal{H}^d(K_j \cap B_{x,r})$. Nevertheless, K_j can be compared with its cone competitor, up to an error infinitesimal in j, thanks to Lemma 2.3. We recover

$$\begin{aligned} \inf_{\substack{K' \in \mathcal{P}(H) \\ K' \setminus \overline{B_{x,r}} = K_j \setminus \overline{B_{x,r}}}} \mathcal{H}^d(K' \cap \overline{B_{x,r}}) &\le o_j(1) + \mathcal{H}^d(\mathbf{C}_{x,r}(K_j) \cap \overline{B_{x,r}}) \\ &\le o_j(1) + \frac{r}{d}\mathcal{H}^{d-1}(K_j \cap \partial B_{x,r}) = o_j(1) + \frac{r}{d} f_j'(r)\,. \end{aligned}$$

One then passes to the limit in j and obtains the desired monotonicity formula. We refer to [DGM14, Theorem 2] for the conclusion of the proof of (3.8).

Step three: By (3.2) and (3.8), the d-dimensional density of the measure μ, namely:

$$\theta(x) = \lim_{r \to 0^+} \frac{f(r)}{\omega_d r^d} \ge \theta_0\,,$$

exists, is finite and positive μ-almost everywhere. Preiss' Theorem 2.1 implies that $\mu = \theta \mathcal{H}^d \llcorner \tilde{K}$ for some countably $\mathcal{H}^d$-rectifiable set $\tilde{K}$ and some positive Borel function θ. Since K is the support of μ, then $\mathcal{H}^d(\tilde{K} \setminus K) = 0$. On the other hand, by differentiation of Hausdorff measures, (3.2) yields $\mathcal{H}^d(K \setminus \tilde{K}) = 0$. Hence K is d-rectifiable and $\mu = \theta \mathcal{H}^d \llcorner K$.

Step four: We prove that $\theta(x) \ge 1$ for every $x \in K$ such that the approximate tangent space to K exists (thus, $\mathcal{H}^d$-a.e. on K). For further use (see step 7 below) we actually prove a slightly more general results: $\theta(x) \ge 1$ for every $x \in K \setminus H$ such that there exists a sequence $r_k \downarrow 0$ for which

$$\frac{\mu_{x,r_k}}{r_k^d} \overset{*}{\rightharpoonup} \theta(x) \mathcal{H}^d \llcorner \pi\,, \qquad \text{as } k \to +\infty \tag{3.9}$$

where π is a d-dimensional plane. Here the measures $\mu_{x,r}$ are defined as $\mu_{x,r}(A) = \mu(x + rA)$ for every Borel set A.

Let us assume without loss of generality that $x = 0$ and $\pi = \{x_{d+1} = ... = x_n = 0\}$. Note that $\mu_{x,r}$ are supported on $(K - x)/r$ and that (3.9) and the lower density estimates (3.2) imply that the support of μ_{x,r_k} has to converge in the Kuratowski sense to the support of $\mathcal{H}^d \llcorner \pi$. In particular, for every $\varepsilon > 0$, there are infinitely many small $\rho > 0$ such that

$$K \cap B_\rho \subset \left\{ y \in \mathbb{R}^n : |y_{d+1}|, ..., |y_n| < \frac{\varepsilon}{100}\rho \right\}. \tag{3.10}$$

Let us now assume, by contradiction, that $\theta(0) < 1$. Thanks to (3.8) and (3.10) we can slightly tilt ρ to find $r > 0$ and $\alpha < 1$ such that $\mu(\partial Q_r) = 0$ and

$$\frac{\mu(Q_r)}{r^d} \le \alpha < 1, \quad K \cap (Q_r \setminus R_{r,\varepsilon r}) = \emptyset\,, \tag{3.11}$$

where $R_{r,\varepsilon r}$ is defined as in (2.1). In particular, since μ_j are weakly converging to μ, we get that for $j \geq j(r)$

$$\frac{\mu_j(Q_r)}{r^d} \leq \alpha < 1 \quad \text{and} \quad \mu_j(Q_r \setminus R_{r,\varepsilon r}) = o_j(1), \tag{3.12}$$

We now wish to clear the small amount of mass appearing in the complement of $R_{r,\varepsilon r}$: we achieve this by repeatedly applying Theorem 2.5. We set $Q_r \cap \{x_{d+1} \geq \frac{\varepsilon}{2} r\} =: R$, and we apply Theorem 2.5 to this rectangle with $E = K_j^0 := K_j$, obtaining the map $\varphi_{1,j}$. We recall that the obtained constant k_1 for the area bound is universal, since it depends on the side ratio of R, which is bounded from below by 1 and from above by 4, provided ε small enough. We set $K_j^1 := \varphi_{1,j}(K_j^0)$ and repeat the argument with $Q_r \cap \{x_{d+1} \leq -\frac{\varepsilon}{2} r\} =: R$ and $E := K_j^1$, obtaining the map $\varphi_{2,j}$. We again set $K_j^2 := \varphi_{2,j}(K_j^1)$ and iterate this procedure to the rectangles $Q_r \cap \{x_{d+2} \geq \frac{\varepsilon}{2} r\}, ..., Q_r \cap \{x_n \leq -\frac{\varepsilon}{2} r\}$. After $2(n-d)$ iteration, we set

$$K_j^{2(n-d)} := \varphi_{2(n-d),j} \circ ... \circ \varphi_{1,j}(K_j).$$

We are going to use the cube $Q_{r(1-\sqrt{\varepsilon})}$ because, taking ε small enough, then $\sqrt{\varepsilon} > 4\overline{C}\varepsilon$, where $\overline{C} > 1$ is the side ratio considered before. This allows us to claim that

$$\mathcal{H}^d(K_j^{2(n-d)} \cap (Q_{r(1-\sqrt{\varepsilon})} \setminus R_{r(1-\sqrt{\varepsilon}),6\varepsilon r})) = 0. \tag{3.13}$$

Otherwise there would exist a d-face of a smaller rectangle $T \subset (Q_r \setminus R_{r,\varepsilon r})$ such that

$$\mathcal{H}^d(K_j^{2(n-d)} \cap T) = \mathcal{H}^d(T) \geq \varepsilon^d r^d\,,$$

which would lead to the following contradiction for j large:

$$\varepsilon^d r^d \leq \mathcal{H}^d(T) \leq \mathcal{H}^d\left(K_j^{2(n-d)} \cap (Q_r \setminus R_{r,\varepsilon r})\right) \leq k_1^{2(n-d)} \mathcal{H}^d\left(K_j \cap (Q_r \setminus R_{r,\varepsilon r})\right) = o_j(1).$$

In particular, we cleared any measure on every slab

$$\bigcup_{i=d+1}^{n} \left\{3\varepsilon r < |x_i| < (1-\sqrt{\varepsilon})\frac{r}{2}\right\} \cap Q_{r(1-\sqrt{\varepsilon})}.$$

We want now to construct a map $P \in \mathsf{D}(0,r)$, collapsing $R_{r(1-\sqrt{\varepsilon}),6\varepsilon r}$ onto the tangent plane. To this end, for $x \in \mathbb{R}^n$, $x = (x', x'')$ with $x' \in \mathbb{R}^d$ and $x'' \in \mathbb{R}^{n-d}$, we set

$$\|x'\| := \max\{|x_i| : i = 1, \dots, d\} \qquad \|x''\| := \max\{|x_i| : i = d+1, \dots, n\} \tag{3.14}$$

and we define P as follows:

$$P(x) = \begin{cases} \left(x', g(\|x'\|)\frac{(\|x''\|-3\varepsilon r)_+}{1-6\varepsilon}\frac{x''}{\|x''\|} + (1-g(\|x'\|))x''\right) & \text{if } \max\{\|x'\|, \|x''\|\} \leq r/2 \\ \mathrm{Id} & \text{otherwise,} \end{cases} \tag{3.15}$$

where $g : [0, r/2] \to [0, 1]$ is a compactly supported cut off function such that

$$g \equiv 1 \quad \text{on } [0, r(1-\sqrt{\varepsilon})/2] \qquad \text{and} \qquad |g'| \leq 10/r\sqrt{\varepsilon}\,.$$

It is not difficult to check that $P \in \mathsf{D}(0,r)$ and that $\operatorname{Lip} P \leq 1 + C\sqrt{\varepsilon}$, for some dimensional constant C.

We now set $\widetilde{K_j} := P(K_j^{2(n-d)})$, which verifies, thanks to (3.13),

$$\mathcal{H}^d\Big(\widetilde{K_j} \cap \big(Q_{(1-\sqrt{\varepsilon})r} \setminus Q^d_{(1-\sqrt{\varepsilon})r}\big)\Big) = 0 \tag{3.16}$$

and

$$
\begin{aligned}
&\mathcal{H}^d\big(\widetilde{K_j}\cap\big(Q_r\setminus Q_{r(1-\sqrt{\varepsilon})}\big)\big)\le(1+C\sqrt{\varepsilon})^d\mathcal{H}^d\big(K_j^{2(n-d)}\cap\big(Q_r\setminus Q_{r(1-\sqrt{\varepsilon})}\big)\big)\\
&\le(1+C\sqrt{\varepsilon})k_1^{2(n-d)}\mathcal{H}^d\big(K_j\cap\big(Q_r\setminus(Q_{r(1-\sqrt{\varepsilon})}\cup R_{r,\varepsilon r})\big)\big)\\
&\quad+(1+C\sqrt{\varepsilon})\mathcal{H}^d\big(K_j\cap\big(R_{r,\varepsilon r}\setminus Q_{r(1-\sqrt{\varepsilon})}\big)\big)\\
&\le o_j(1)+(1+C\sqrt{\varepsilon})\mathcal{H}^d\big(K_j\cap\big(R_{r,\varepsilon r}\setminus Q_{r(1-\sqrt{\varepsilon})}\big)\big)\,,
\end{aligned}
\tag{3.17}
$$

where in the last inequality we have used (3.12). Moreover, by using (3.11), (3.12) and (3.16), we also have that, for ε small and j large:

$$
\begin{aligned}
\frac{\mathcal{H}^d(\widetilde{K_j}\cap Q^d_{r(1-\sqrt{\varepsilon})})}{r^d(1-\sqrt{\varepsilon})^d}=\frac{\mathcal{H}^d(\widetilde{K_j}\cap Q_{r(1-\sqrt{\varepsilon})})}{r^d(1-\sqrt{\varepsilon})^d}&\le(1+C\sqrt{\varepsilon})\frac{\mathcal{H}^d(K_j^{2(n-d)}\cap Q_r)}{r^d}\\
&\le(1+C\sqrt{\varepsilon})\frac{\mathcal{H}^d(K_j\cap Q_r)+o_j(1)}{r^d}\\
&\le\alpha+o_j(1)<1.
\end{aligned}
\tag{3.18}
$$

As a consequence of (3.18) and the compactness of $\widetilde{K_j}$, there exist $y_j'\in Q^d_{(1-\sqrt{\varepsilon})r}$ and $\delta_j>0$ such that, if we set $y_j:=(y_j',0)$, then

$$
\widetilde{K_j}\cap B^d_{y_j,\delta_j}=\emptyset\quad\text{and}\quad B^d_{y_j,\delta_j}\subset Q^d_{(1-\sqrt{\varepsilon})r}.
\tag{3.19}
$$

After the last deformation, our set $\widetilde{K_j}\cap Q_{r(1-\sqrt{\varepsilon})}$ is contained in the tangent plane and we want to use the property (3.19) to collapse $\widetilde{K_j}\cap Q_{r(1-\sqrt{\varepsilon})}$ into $\left(\partial Q^d_{(1-\sqrt{\varepsilon})r}\right)\times\{0\}^{n-d}$. To this end, for every $j\in\mathbb{N}$ let us define the following Lipschitz map:

$$
\varphi_j(x)=\begin{cases}\big(x'+z'_{j,x},\,x''\big) & \text{if } x\in R_{r(1-\sqrt{\varepsilon}),r}\\ x & \text{otherwise,}\end{cases}
$$

with

$$
z'_{j,x}:=\min\left\{1,\frac{\left|x'-y_j'\right|}{\delta_j}\right\}\frac{(r-4\|x''\|)_+}{r}\gamma_{j,x}(x'-y_j'),
$$

where $\gamma_{j,x}>0$ is such that $x'+\gamma_{j,x}(x'-y_j')\in\partial Q^d_{(1-\sqrt{\varepsilon})r}\times\{0\}^{n-d}$ and $\|x''\|$ is defined in (3.14). One can easily check that $\varphi_j\in\mathsf{D}(0,r)$. Moreover, setting $\varphi_j(\widetilde{K_j})=:K_j'$, we have that

$$
K_j'\setminus Q_r=K_j\setminus Q_r
$$

and

$$
\mathcal{H}^d(K_j'\cap Q_{r(1-\sqrt{\varepsilon})})=0,
\tag{3.20}
$$

thanks to (3.16), since

$$
\mathcal{H}^d\left(\partial Q^d_{(1-\sqrt{\varepsilon})r}\times\{0\}^{n-d}\right)=0.
$$

Since $\mathcal{P}(H)$ is a good class, by (1.1) there exists a sequence of competitors $(J_j)_{j\in\mathbb{N}}\subset\mathcal{P}(H)$ such that $J_j\setminus\overline{B}_{0,r}=K_j\setminus\overline{B}_{0,r}$ and $\mathcal{H}^d(J_j)=\mathcal{H}^d(K_j')+o_j(1)$. Hence, thanks to (3.17) and (3.20), we get

$$
\begin{aligned}
\mathcal{H}^d(K_j)-\mathcal{H}^d(J_j) &\ge \mathcal{H}^d(K_j)-\mathcal{H}^d(K_j')-o_j(1)=\mathcal{H}^d(K_j\cap Q_r)-\mathcal{H}^d(K_j'\cap Q_r)-o_j(1)\\
&\ge \mathcal{H}^d\left(K_j\cap Q_{r(1-\sqrt{\varepsilon})}\right)+\mathcal{H}^d\left(K_j\cap(R_{r,\varepsilon r}\setminus Q_{r(1-\sqrt{\varepsilon})})\right)+\\
&\quad -o_j(1)-(1+C\sqrt{\varepsilon})\mathcal{H}^d\left(K_j\cap(R_{r,\varepsilon r}\setminus Q_{r(1-\sqrt{\varepsilon})})\right)\\
&\ge \mathcal{H}^d\left(K_j\cap Q_{r(1-\sqrt{\varepsilon})}\right)-C\sqrt{\varepsilon}\mathcal{H}^d\left(K_j\cap(R_{r,\varepsilon r}\setminus Q_{r(1-\sqrt{\varepsilon})})\right)-o_j(1).
\end{aligned}
$$

Passing to the limit as $j\to\infty$ and using (3.1), (3.2) and (3.11), we get

$$
\begin{aligned}
\liminf_j \mathcal{H}^d(K_j) &\ge \liminf_j \mathcal{H}^d(J_j)+\mu(Q_{r(1-\sqrt{\varepsilon})})-C\sqrt{\varepsilon}r^d\\
&\ge \liminf_j \mathcal{H}^d(J_j)+(\theta_0(1-\sqrt{\varepsilon})^d-C\sqrt{\varepsilon})r^d.
\end{aligned}
$$

Since, for ε small, this is in contradiction with K_j be a minimizing sequence, we finally conclude that $\theta(0)\ge 1$.

Step five: We now show that $\theta(x)\le 1$ for every $x\in K$ such that the approximate tangent space to K exists. Again, for further purposes, we will actually show that $\theta(x)\le 1$ for every $x\in K\setminus H$ such that (3.9) holds. Arguing by contradiction, we assume that $\theta(x)=1+\sigma>1$. As usually, we assume that $x=0$ and $\pi=\{y: y_{d+1},\dots,y_n=0\}$. By the monotonicity of the density established in Step 2, for every $\varepsilon>0$ we can find $r>0$ such that

$$
K\cap Q_r\subset R_{r,\varepsilon r}\,,\qquad 1+\sigma\le\frac{\mu(Q_r)}{r^d}\le 1+\sigma+\varepsilon\,\sigma. \tag{3.21}
$$

Since $\mathcal{H}^d\llcorner K_j$ converges to μ we have

$$
\mathcal{H}^d(K_j\cap Q_r)>\Big(1+\frac{\sigma}{2}\Big)\,r^d\,,\qquad \mathcal{H}^d((K_j\cap Q_r)\setminus R_{r,\varepsilon r})<\frac{\sigma}{4}\,r^d.\qquad \forall j\ge j_0(r)\,, \tag{3.22}
$$

Consider the map $P:\mathbb{R}^n\to\mathbb{R}^n\in\mathsf{D}(0,r)$ with $\operatorname{Lip}P\le 1+C\sqrt{\varepsilon}$ defined in (3.15), which collapses $R_{r(1-\sqrt{\varepsilon}),\varepsilon r}$ onto the tangent plane. By exploiting the fact that $\mathcal{P}(H)$ is a good class, we find that

$$
\begin{aligned}
\mathcal{H}^d(K_j\cap Q_r)-o_j(1)\le{}& \underbrace{\mathcal{H}^d(P(K_j\cap R_{(1-\sqrt{\varepsilon})r,\varepsilon r}))}_{I_1}+\underbrace{\mathcal{H}^d(P(K_j\cap(R_{r,\varepsilon r}\setminus R_{(1-\sqrt{\varepsilon})r,\varepsilon r})))}_{I_2}\\
&+\underbrace{\mathcal{H}^d(P(K_j\cap(Q_r\setminus R_{r,\varepsilon r})))}_{I_3}\,.
\end{aligned}
$$

By construction, $I_1\le r^d$, while, by (3.22),

$$
I_3\le(\operatorname{Lip}P)^d\,\mathcal{H}^d(K_j\cap(Q_r\setminus R_{r,\varepsilon r}))<(1+C\sqrt{\varepsilon})^d\,\frac{\sigma}{4}\,r^d\,.
$$

Hence, as $j\to\infty$,

$$
\Big(1+\frac{\sigma}{2}\Big)r^d\le r^d+\liminf_{j\to\infty}I_2+(1+C\sqrt{\varepsilon})^d\,\frac{\sigma}{4}\,r^d\,,
$$

that is,

$$
\Big(\frac{1}{2}-\frac{(1+C\sqrt{\varepsilon})^d}{4}\Big)\,\sigma\le\liminf_{j\to\infty}\frac{I_2}{r^d}\,. \tag{3.23}
$$

By (3.21), we finally estimate that

$$
\begin{aligned}
\limsup_{j\to\infty} I_2 &\le (1+C\sqrt{\varepsilon})^d\,\mu(Q_r\setminus Q_{(1-\sqrt{\varepsilon})r})\\
&\le (1+C\sqrt{\varepsilon})^d\Big((1+\sigma+\varepsilon\sigma)-(1+\sigma)(1-\sqrt{\varepsilon})^d\Big)\,r^d.
\end{aligned} \tag{3.24}
$$

By choosing ε sufficiently small, (3.23) and (3.24) provide the desired contradiction. In particular, by combining this with the previous step we deduce that $\theta = 1$ for every x such that K admits an approximate tangent space at x, that is for $\mathcal{H}^d$ almost every x. Classical argument in measure theory then implies that $\mu = \mathcal{H}^d \llcorner K$.

Step six: We now show that the canonical density one rectifiable varifold associated to K is stationary in $\mathbb{R}^n \setminus H$. In particular, applying Allard's regularity theorem, see [Sim83, Chapter 5], we will deduce that there exists an $\mathcal{H}^d$-negligible closed set $\Sigma \subset K$ such that $\Gamma = K \setminus \Sigma$ is a real analytic manifold. Since being a stationary varifold is a local property, to prove our claim it is enough to show that for every ball $B \subset\subset \mathbb{R}^n \setminus H$ we have

$$\mathcal{H}^d(K) \leq \mathcal{H}^d(\phi(K)) \tag{3.25}$$

whenever ϕ is a diffeomorphism such that $\operatorname{spt}\{\phi - \mathrm{Id}\} \subset B$. Indeed, by exploiting (3.25) with $\phi_t = \mathrm{Id} + tX$, $X \in C_c^1(B)$ we deduce the desired stationarity property.

To prove (3.25) we argue as in [DGM14, Theorem 7]. Given $\varepsilon > 0$ we can find $\delta > 0$ and a compact set $\hat{K} \subset K \cap B$ with $\mathcal{H}^d((K \setminus \hat{K}) \cap B) < \varepsilon$ such that K admits an approximate tangent plane $\pi(x)$ at every $x \in \hat{K}$,

$$\sup_{x \in \hat{K}} \sup_{y \in B_{x,\delta}} |\nabla\phi(x) - \nabla\phi(y)| \leq \varepsilon\,, \qquad \sup_{x \in \hat{K}} \sup_{y \in \hat{K} \cap B_{x,\delta}} \mathrm{d}(\pi(x), \pi(y)) < \varepsilon\,, \tag{3.26}$$

where d is a distance on $G(d)$, the d-dimensional Grassmanian. Moreover, denoting by $S_{x,r}$ the set of points in $B_{x,r}$ at distance at most $\varepsilon\, r$ from $x + \pi(x)$, then $K \cap B_{x,r} \subset S_{x,r}$ for every $r < \delta$ and $x \in \hat{K}$. By Besicovitch covering theorem we can find a finite disjoint family of closed balls $\{\overline{B}_i\}$ with $B_i = B_{x_i,r_i} \subset B \subset\subset \mathbb{R}^n \setminus H$, $x_i \in \hat{K}$, and $r_i < \delta$, such that $\mathcal{H}^d(\hat{K} \setminus \bigcup_i B_i) < \varepsilon$. By exploiting the construction of Step four, we can find $j(\varepsilon) \in \mathbb{N}$ and maps $P_i : \mathbb{R}^n \to \mathbb{R}^n$ with $\mathrm{Lip}\,(P_i) \leq 1 + C\sqrt{\varepsilon}$ and $P_i = \mathrm{Id}$ on B_i^c, such that, for a certain $X_i \subset S_i = S_{x_i, \varepsilon\, r_i}$,

$$\begin{aligned} &P_i(X_i) \subset B_i \cap (x_i + \pi(x_i))\,, \\ &\mathcal{H}^d\Big(P_i\big((K_j \cap B_i) \setminus X_i\big)\Big) \leq C\sqrt{\varepsilon}\,\omega_d\, r_i^d\,, \qquad \forall j \geq j(\varepsilon)\,. \end{aligned} \tag{3.27}$$

Denoting with J_d^π the d-dimensional tangential jacobian with respect to the plane π and by J_d^K the one with respect to K and exploiting (3.26), (3.27), the area formula and that $\omega_d\, r_i^d \leq \mathcal{H}^d(K \cap B_i)$ (by the monotonicity formula), and setting $\alpha_i = \mathcal{H}^d((K \setminus \hat{K}) \cap B_i)$, we get

$$\begin{aligned} \mathcal{H}^d(\phi(P_i(K_j \cap X_i))) &= \int_{P_i(K_j \cap X_i)} J_d^{\pi(x_i)}\phi(x)\, d\mathcal{H}^d(x) \leq (J_d^{\pi(x_i)}\phi(x_i) + \varepsilon)\,\omega_d\, r_i^d \\ &\leq (J_d^{\pi(x_i)}\phi(x_i) + \varepsilon)\,\mathcal{H}^d(K \cap B_i) \leq (J_d^{\pi(x_i)}\phi(x_i) + \varepsilon)\,(\mathcal{H}^d(\hat{K} \cap B_i) + \alpha_i) \\ &\leq \int_{\hat{K} \cap B_i} (J_d^K\phi(x) + 2\varepsilon)\; d\mathcal{H}^d(x) + ((\mathrm{Lip}\,\phi)^d + \varepsilon)\,\alpha_i \\ &= \mathcal{H}^d(\phi(\hat{K} \cap B_i)) + 2\varepsilon\,\mathcal{H}^d(\hat{K} \cap B_i) + ((\mathrm{Lip}\,\phi)^d + \varepsilon)\,\alpha_i\,, \end{aligned} \tag{3.28}$$

where in the last identity we have used the area formula and the injectivity of ϕ. Since $P_i = \mathrm{Id}$ on B_i^c, $\phi = \mathrm{Id}$ on B^c, $B_i \subset B$ and the balls B_i are disjoint, the map $\tilde{\phi}$ which is equal to ϕ on $B \setminus \cup_i B_i$, equal to the identity on B^c and equal to $\phi \circ P_i$ on B_i is well defined. Moreover, by (3.28), we get

$$\mathcal{H}^d(\tilde{\phi}(K_j)) \leq \mathcal{H}^d(\phi(K)) + C\varepsilon$$

where C depends only on K. By exploiting the definition of good class, we get that

$$\mathcal{H}^d(K) \leq \mathcal{H}^d(\tilde{\phi}(K_j)) + o_j(1) \leq \mathcal{H}^d(\phi(K)) + C\varepsilon + o_j(1).$$

Letting $j \to \infty$ and $\varepsilon \to 0$ we obtain (3.25).

Step seven: We finally address the dimension of the singular set. Recall that, by monotonicity, the density function

$$\Theta^d(K,x) = \lim_{r\to 0} \frac{\mathcal{H}^d(K\cap B_{x,r})}{\omega_d r^d}$$

is everywhere defined in $\mathbb{R}^n \setminus H$ and equals 1 $\mathcal{H}^d$-almost everywhere in K. Fixing $x \in K$ and a sequence $r_k \downarrow 0$, the monotonicity formula, the stationarity of $\mathcal{H}^d \llcorner K$ and the compactness theorem for integral varifolds [All72, Theorem 6.4] imply that (up to subsequences)

$$\mathcal{H}^d \llcorner \left(\frac{K-x}{r_k}\right) \rightharpoonup V \quad \text{locally in the sense of varifolds,} \tag{3.29}$$

where

(a) V is a stationary integral varifold: in particular $\Theta^d(\|V\|,y) \geq 1$ for $y \in \mathrm{spt}(V)$;
(b) V is a cone, namely $(\delta_\lambda)_\# V = V$, where $\delta_\lambda(x) = \lambda x$, $\lambda > 0$;
(c) $\Theta^d(\|V\|,0) = \Theta^d(K,x) \geq \Theta^d(\|V\|,y)$ for every $y \in \mathbb{R}^n$.

Recall that the tangent varifold V depends (in principle) on the sequence (r_k). We denote by $\mathrm{TanVar}(K,x)$ the (nonempty) set of all possible limits V as in (3.29), varying among all sequences along which (3.29) holds. Given a cone W we set

$$\mathrm{Spine}(W) := \{y \in \mathbb{R}^n : \Theta^d(\|W\|,y) = \Theta^d(\|W\|,0)\}. \tag{3.30}$$

By [Alm00, 2.26], $\mathrm{Spine}(W)$ is a vector subspace of $\mathbb{R}^n$, see also [Whi97, Theorem 3.1]. We can stratify K in the following way: for every $k = 0,\dots,n$ we let

$$A_k := \{x \in K : \text{ for all } V \in \mathrm{TanVar}(K,x),\ \dim \mathrm{Spine}(V) \leq k\}.$$

Clearly $A_0 \subset \cdots \subset A_d = \cdots = A_n$; moreover the following holds: $\dim_{\mathcal{H}} A_k \leq k$, see [Alm00, 2.28] and [Whi97, Theorem 2.2]. In order to prove our claim, we need to show that $A_d \setminus A_{d-1} \subset K \setminus \Sigma$, where Σ, as in Step six. To this end we note that the monotonicity formula for stationary varifolds implies that if W is a d-dimensional stationary cone with $\dim \mathrm{Spine}(W) = d$, then $\|W\| = \Theta^d(\|W\|,0)\mathcal{H}^d \llcorner \pi$ for some d-dimensional plane [1] π. In particular since every $x \in A_d \setminus A_{d-1}$ admits at least one flat tangent varifold, for every such x there exists a sequence r_k satisfying

$$\mathcal{H}^d \llcorner \frac{K-x}{r_k} \to m\mathcal{H}^d \llcorner \pi\,;$$

moreover $m = \Theta^d(K,x)$ by (c). But then, the very same proof of Step five above implies that $\Theta^d(K,x) = 1$. Thus every $x \in A_d \setminus A_{d-1}$ satisfies the hypotheses of Allard's regularity Theorem [All72, Regularity Theorem, Section 8], implying that $K \cap Q_{x,\frac{r}{2}}$ is a real analytic submanifold. Equivalently $x \notin \Sigma$ and this concludes the proof.

□

4. Proof of Theorems 1.5 and 1.8

In this Section we prove Theorem 1.5 and 1.8. With Theorem 1.3 at hand, the proofs are quite similar to the corresponding ones in [DGM14] (see Theorems 4 and 7 there), hence we limit ourselves to provide a short sketch underlying only the main differences.

Proof of Theorem 1.5. We start by proving that $\mathcal{F}(H,\mathcal{C})$ is a good class in the sense of Definition 1.2: let $\widetilde{K} \in \mathcal{F}(H,\mathcal{C})$, $x \in \widetilde{K}$, $r \in (0,\mathrm{dist}(x,H))$ and $\varphi \in \mathsf{D}(x,r)$. We show that $\varphi(\widetilde{K}) \in \mathcal{F}(H,\mathcal{C})$

[1] Indeed up to a rotation $\mathrm{spt}(W) = \mathrm{Spine}(W) \times \Gamma$, where Γ is a cone in $\mathbb{R}^{n-d}$. If $\Gamma \neq \{0\}$ then $\Theta^d(\|W\|,0) > \Theta^d(\|W\|,y)$ for any $y \in \mathrm{Spine}(W) \setminus \{0\}$, which contradicts (3.30).

arguing by contradiction: assume that $\gamma(S^{n-d}) \cap \varphi(\widetilde{K}) = \emptyset$ for some $\gamma \in \mathcal{C}$ and, without loss of generality, suppose also that $\gamma(S^{n-d}) \cap (\widetilde{K} \setminus B_{x,r}) = \emptyset$. By Definition 1.1 there exists a sequence

$$(\varphi_j) \subset \mathfrak{D}(x,r) \qquad \text{such that} \qquad \lim_j \|\varphi_j - \varphi\|_{C^0} = 0.$$

Since $\gamma(S^{n-d})$ is compact and $\varphi_j = \mathrm{Id}$ outside $B_{x,r}$, for j sufficiently large $\gamma(S^{n-d}) \cap \varphi_j(\widetilde{K}) = \emptyset$; moreover φ_j is invertible, hence $\varphi_j^{-1}\left(\gamma(S^{n-d})\right) \cap \widetilde{K} = \emptyset$. But the property for φ_j of being isotopic to the identity implies $\varphi_j^{-1} \circ \gamma \in \mathcal{C}$, which contradicts $\widetilde{K} \in \mathcal{F}(H, \mathcal{C})$. This proves (a).

Given a minimizing sequence $(K_j) \subset \mathcal{F}(H, \mathcal{C})$ which consists of rectifiable sets, we can therefore find a set K with the properties stated in Theorem 1.3. In order to conclude (b), namely that $K \in \mathcal{F}(H, \mathcal{C})$, we refer to [DGM14, Theorem 4(b)]: the proof is the same.

□

Proof of Theorem 1.8. As already observed in Remark 1.7, $\mathcal{A}(H, K_0)$ is a good class and we can therefore apply Theorem 1.3. We thus know that $\mathcal{H}^d \llcorner K_j \overset{*}{\rightharpoonup} \mu = \mathcal{H}^d \llcorner K$ and that K is a smooth set away from H and from a relatively closed set Σ of dimension less or equal than $(d-1)$. The conclusion of the proof can now be obtained by repeating verbatim Steps 4 and 6 in the proof of Theorem 7 in [DGM14].

□

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