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Some optimization problems in mass transport theory

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Introduction

In the last years the study of optimal transport problems has received a lot of attention and the underlying theory has found applications in many fields like non-linear partial differential equations, calculus of variations, probability, economics, fluidodynamics and many others fields. In particular we can approach problems as optimization of transport networks, urban planning, location and irrigation, traffic models with congestion or concentration effects and many others with the Monge-Kantorovich mass transport theory.

This thesis is dedicated to the study of some new models in the framework of location problems and in traffic problems with congestion, based on the works [23] and [22], in collaboration with Giuseppe Buttazzo, Guillaume Carlier and Fabrizio Oliviero.

Mass transport theory starts in 1781 with the work of Gaspard Monge (see his famous work [51]), that proposes a mathematical model to describe the way to transport a given mass density to a final configuration with the minimum total cost. Given two densities ρ_0 and ρ_1, the mass in each point x, that is $\rho_0(x)dx$, has to be moved to the destination $T(x)$. Supposing that all the mass located in some point must go to the same destination, the infinitesimal work to transport the mass in x to $T(x)$ is $|x - T(x)|\rho_0 dx$. Summing all the contributions, the total cost is given by

$$\int_{\mathbb{R}^n} |x - T(x)|\rho_0(x)\, dx. \tag{1}$$

The above quantity has to be minimized among all the so called *transport maps* T, i.e. measurable functions such that for every Borel set A the following mass balance condition

$$\int_{T^{-1}(A)} \rho_0(x)dx = \int_A \rho_1(y)dy$$

holds. Clearly this means that the initial and final densities have the same total mass.

Many generalizations can be taken into account. The unit transport $|x-T(x)|$ cost can be replaced by a more general function, for examples of the type $|x - T(x)|^p$ and ρ_0 and ρ_1 can be replaced by two positive measures μ and ν, that we may take as probabilities if we normalize to 1 the initial and final total masses.

The problem of the minimization of the transport cost (1), among all transport maps T sending ρ_0 into ρ_1 contains some substantial intrinsic difficulties mainly due to the non-linearity of the unknown and to the strong requirement that all the mass located in x must go to the same destination $T(x)$. There has been no significant progress until 1940 when Leonid Kantorovich proposed (see his famous papers [43], [44]) an alternative approach to the mass transportation problem. In the Kantorovich point of view the transport of ρ_0 into ρ_1 is represented by a probability measure $\pi \in \mathcal{P}(\mathbb{R}^n \times \mathbb{R}^n)$, called *transport plan*, such that $d\pi(x, y)$ is the quantity of mass in x which is sent into y. Of course each transport plan has to transport ρ_0 into ρ_1: this condition is expressed by requiring that $\pi(A \times \mathbb{R}^n) = \rho_0(A)$ and $\pi(\mathbb{R}^n \times B) = \rho_1(B)$, i.e. π has ρ_0 and ρ_1 as marginals. In this framework the total cost of transporting ρ_0 into ρ_1 with respect to Monge's criterium is given by

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y| \, d\pi(x, y). \tag{2}$$

The cost functional is linear with respect to the transport plans π so the existence of an optimal transport plan minimizing the above cost functional, follows by a standard argument. Kantorovich's approach allows to easily generalize the transport problem to the case of general cost function $c(x, y)$.

Kanotorovich's formulation is a relaxed version of Monge's one. The advantages of this approach can be found in the linearity of the functional with respect to the unknown, in the good properties of the set of admissible plans (that is non-empty convex and weakly compact in the space of probability measures $\mathcal{P}(\mathbb{R}^n \times \mathbb{R}^n)$) and in the admissibility of mass splitting. The existence of an optimal transport map is, on the contrary a very difficult issues; after the work of Sudakov on the existence of a solution (see [55]) many other authors develop the subject with a lot of generalizations.

In many optimal transport problems the initial configuration of the mass is known, while the final measure is not prescribed but only has to satisfies some suitable requirements. A typical example are the location problems, in which the final measure is concentrated in a finite number of points.

Let us now focus on the location problem, that can be described as follows: a given bounded and closed region $\Omega \subset \mathbb{R}^n$ is considered, together with a given nonnegative function $\rho : \Omega \to \mathbb{R}^+$ which represents the distribution density of resources in Ω. The goal is to concentrate the resources into a given number N of points $x_1, \dots, x_N$ in an optimal way. The simplest way to find the optimal configuration is to consider the average distance between the initial and final location of a unit of resources and minimize it. Let us denote by m_i the quantity of resources that will be concentrated at the point x_i and by Ω_i the so-called *Voronoi cell* corresponding to x_i, that is, the subregion of Ω that sends its resources to the point x_i. In other words, we have

$$m_i = \int_{\Omega_i} \rho(x) \, dx.$$

Then we have that the total cost to concentrate the resources spread on Ω_i into every x_i is given by

$$A \int_{\Omega_i} |x - x_i|^p \rho(x) \, dx$$

where A is a proportionality constant and p is a given number. We are assuming that the cost to move a unit mass along a distance l is Al^p. Summing up over all the N, we have that the total cost is given by

$$A\sum_{i=1}^{N}\int_{\Omega_i}|x-x_i|^p\rho(x)\,dx$$

that can be written also in the form

$$A\int_{\Omega}\big(\mathrm{dist}(x,\Sigma)\big)^p\rho(x)\,dx,$$

where Σ is the unknown set of N points to be determined.

The most efficient choice of the positions is then obtained by solving the minimization problem

$$\min\Big\{F(\Sigma)=\int_{\Omega}\big(\,\mathrm{dist}(x,\Sigma)\big)^p\rho(x)\,dx\ :\ \Sigma\subset\Omega,\ \#\Sigma=N\Big\}. \tag{3}$$

Here $\#\Sigma$ is the cardinality of Σ, and $\mathrm{dist}(x,\Sigma)$ is the distance function

$$\mathrm{dist}(x,\Sigma)=\min\big\{|x-y|\ :\ y\in\Sigma\big\}.$$

The existence of a solution Σ_N for a fixed N follows immediately by applying the direct methods of the calculus of variations but the numerical approximation of the solution is a hard problem, when N is large, because of the high number of local minima of the functional $F(\Sigma)$ in the class of admissible sets. This prevents the use of fast gradient methods and require the slow global optimization methods, as is often the case in this kind of problems.

For this reason, it is interesting to study the asymptotic behavior of the optimal sets Σ_N as $N\to+\infty$. In order to do it, we identify each set $\Sigma\subset\Omega$ of N points with the measure

$$\mu_N=\frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}.$$

Thanks to the theory of Γ-convergence introduced by De Giorgi in the seventies (see [38], and the book [37]), in [11] it was shown that the location cost for large N is asymptotically equivalent to the limit cost:

$$AC_{p,d}\,N^{-p/d}\int_{\Omega}\frac{\rho(x)}{\big(\mu(x)\big)^{p/d}}\,dx,$$

where μ is the absolutely continuous part of the limit of the μ_N and $C_{p,d}$ is a constant depending on the dimension d and the exponent p. The value of the constant $C_{p,d}$ is known only for $d=2$:

$$C_{p,2}=\int_E|x|^p dx,$$

where E is the regular hexagon of unitary area centered at the origin. In Figure 1 is depicted a plot of the value of $C_{p,2}$ for $p \in [0, 2)$.

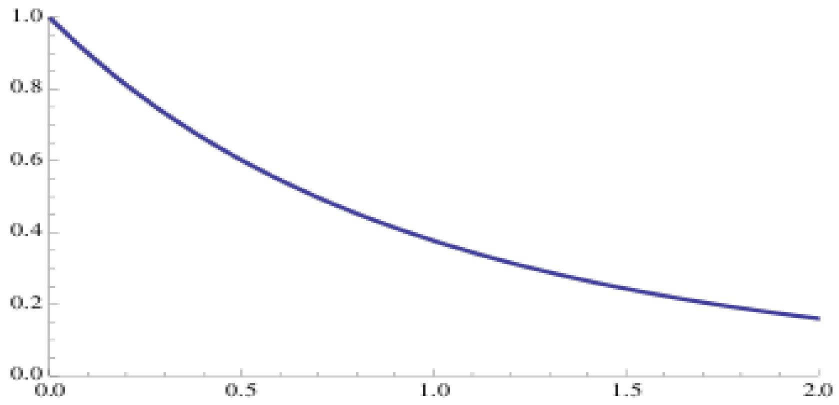


Figure 1: Plot of the value of $C_{p,2}$ for $p \in [0, 2)$.

Determining $C_{p,d}$ for $d \geq 3$ is still an open problem but upper and lower estimates are possible. Observing that $C_{p,d} \geq \min\{\int_K |x|^p \, dx \; : \; |K| = 1\}$, the minimum is realized by the ball so that we have the lower bound $C_{p,d} \geq \omega_d^{-p/d} \frac{d}{p+d}$. The upper bound is obtained considering a random distribution of points with a certain law as in [35] where is proved the estimate $C_{p,d} \leq \omega_d^{-p/d} \Gamma(1 + \frac{p}{d})$. Then we have

$$\omega_d^{-p/d} \frac{d}{p+d} \leq C_{p,d} \leq \omega_d^{-p/d} \Gamma(1 + \frac{p}{d}).$$

In Figure 2 we plot the value of the estimates of $C_{2,d}\omega_d^{2/d}$.

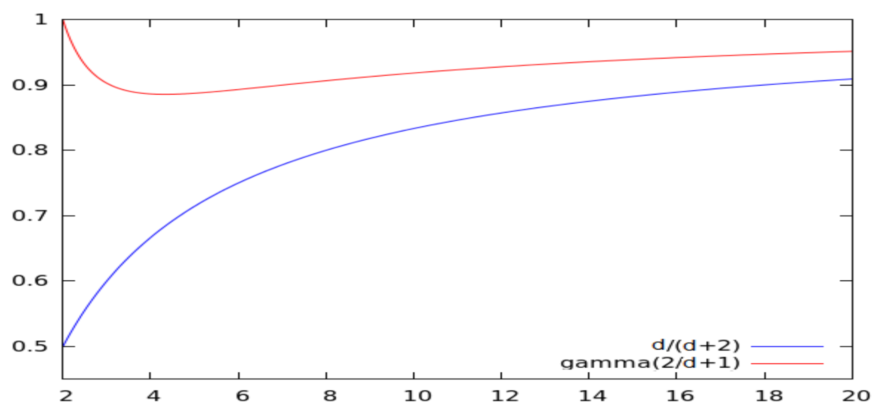


Figure 2: Plot of the value of $\frac{d}{d+2}$ and $\Gamma(1 + \frac{2}{d})$

One of the main objects of our study is a location problem which models an airfreigth system with an extra term in the cost in functional $F(\Sigma)$, which appears due to the transport of mass between the various airports of the system. We start thinking at $\Omega \subset \mathbb{R}^2$ as a region or a state with a distribution of resources ρ in which we want to locate N airports, say $x_1, x_2, \cdots, x_N$, in the optimal way as in the location problem. The airports collect the resources, distributed in Ω, which are then transported between the airports along point to point trajectories. This possibility is modeled by an extra cost in the optimization problem that we will call *routing cost*. If we suppose that the cost of connection between two airports x_i, x_j does not depend on the transported mass but only on a q-power of the distance, multiplied by a suitable constant K, the total routing cost is given by:

$$K \sum_{i,j} |x_i - x_j|^q = KN^2 \int_{\Omega\times\Omega} |x-y|^q \, d(\mu_N \otimes \mu_N).$$

Again, we are interested in the asymptotical problem: taking into account location and routing contributions and setting $\varepsilon = A\, C_{p,d}\, N^{-2-p/d}/K$, we have to solve the minimum problem

$$\min \left\{ \varepsilon \int_\Omega \frac{\rho(x)}{(\mu(x))^{p/d}}\, dx + \int_{\Omega\times\Omega} |x-y|^q \, d(\mu\otimes\mu) \right\}.$$

When $\varepsilon \to 0$ the optimal densities μ_ε of this problem tend to a Dirac mass δ_{x_0} for a suitable point x_0. In order to identify the limit problem as $\varepsilon \to 0$, and so to identify the point x_0 around which the optimal densities μ_ε concentrate, it is convenient to rescale the cost above dividing it by its minimum value; this minimum value is shown to be asymptotical to $\varepsilon^{1/(1+p/d)}$ so that the quantity to be minimized is

$$G_\varepsilon(\mu) = \varepsilon^{(p/d)/(1+p/d)} \int_\Omega \frac{\rho(x)}{\big(\mu(x)\big)^{p/d}}\, dx + \varepsilon^{-1/(1+p/d)} \int_{\Omega\times\Omega} V(x-y)\, d(\mu\otimes\mu).$$

We prove the following

Theorem. *The Γ-limit of the sequence of functionals G_ε, computed on the Dirac mass δ_{x_0} and with respect to the weak* convergence of measures, coincides with the functional*

$$H(\delta_{x_0}) = C \int_\Omega \big(\rho(x)\big)^\beta |x - x_0|^{\alpha q}\, dx,$$

where $\alpha = \frac{p/d}{1+p/d}$, $\beta = \frac{1}{1+p/d}$ *and* $C = \left(1 + \frac{p}{d}\right)\left(\frac{2d}{p}\right)^\alpha$.

One of our main motivations is the application of the location-routing model to the real case, so the choice of the exponents p and q will be done according to the specific case considered. In particular, p is related to the ground transportation cost and q to the air transportation cost. For the location term we may suppose a linear dependence with respect to the distance but we cannot assume the same for the routing term.

Specifically, the air cost is mainly related to the fuel consumption during the flight: a suitable value for the exponent q (see Paragraph 5.2 in [23]) is:

$$q = 0.7.$$

Notice that a value of q lower than 1 is reasonable for our case because in real aircraft for a certain travel the costs of taking off and landing are fixed and not negligible, so the air cost transport is well modeled by a concave function of the distance.

Starting from the theoretical results, we perform some numerical simulations using the parameters described above. We use an iterative scheme based on an optimality condition for the minimizer of the total cost functional. Starting from the uniform distribution $\mathcal{U}(\Omega)$ with total mass 1, the scheme is given by

$$\begin{cases} \mu_0 = \mathcal{U}(\Omega) \\ \mu_{n+1} = \left(\frac{\varepsilon\rho}{c+V*\mu_n}\right)^{p/d+1}, \end{cases}$$

where $V(x) = x^q$ and c is a Lagrange multiplier.

The applications to some real cases require also a good model for the distribution of the population ρ; such models are available for some occidental areas: we will focus for numerical simulations on the USA airfreight system.

This iterative scheme allows us to identify the main hub for the interesting example of the USA.

Of course the presented model is obtained through many simplifications but a correct choice of the involved parameters, first of all the exponents p and q, makes it quite appropriate to describe reality and the numerical simulations confirm the theoretical results.

The second part of the thesis is devoted to a model of traffic congestion: in some application in fact it is necessary to take into account how much the transportation appears to be concentrated. The effects of traffic congestion in the modeling of a road network has to reflect the very natural situation in which the transport speed decreases as soon as the traffic density increases. So it seems reasonable to find a way to measure how much the transportation is concentrated. From this point of view, Monge's problem can be seen as a concentration-neutral one and all the variants depart from this one.

The first formalization of congestion effects is due to Wardrop in the 50's (see [58]) and is based on two principles: all paths connecting two points which are actually followed by some vehicles must provide the same traveling time (which depends on their length as well as congestion) and all other paths provide much time. This means that we use only the geodesic paths for a metric that is induced by the use of the paths themselves. This gives an equilibrium problem, that can be seen as a fixed point and that has a variational characterization discovered by Beckmann (see [8]).

In the continuous transportation model of Beckmann, an open bounded connected region $\Omega \subset \mathbb{R}^d$ is given, with two probability measures ρ_0 and ρ_1 representing says, the distribution of residents and working places in Ω. We assume that the traffic is a vector field $\sigma : \Omega \to \mathbb{R}^d$ whose direction is the travel direction and whose modulus $|\sigma|$ is the intensity of traffic. Following Beckmann's model the relationship between the excess of demand $\rho = \rho_1 - \rho_0$ and the traffic flow σ is given by the equilibrium of the outflow of consumers and the excess of demand in very subregion $K \subset \Omega$:

$$\int_{\partial K} \sigma \cdot n \, d\mathcal{H}^{d-1} = (\rho_1 - \rho_0)(K),$$

where n is the outer unit vector. The previous relationship has to hold for arbitrary K, so formally we have

$$\operatorname{div} \sigma = \rho_0 - \rho_1.$$

We suppose also that the region is isolated, i.e. no traffic flow should cross the boundary:

$$\sigma \cdot n = 0 \text{ on } \partial\Omega.$$

In order to take into account congestion effects, we assume that the transportation cost per consumer at a point x depends on the intensity of traffic at x itself. Let $H : \mathbb{R}^+ \to \mathbb{R}^+$ be a convex nonneagtive function and assume that the transportation cost at x is $H(|\sigma(x)|)$. Then we may define the total transportation cost between ρ_0 and ρ_1 as the value of the minimal flow problem

$$\inf \left\{ \int_\Omega H(|\sigma(x)|)dx \;:\; -\operatorname{div} = \rho_1 - \rho_0,\; \sigma \cdot n = 0 \text{ on } \partial\Omega \right\}.$$

This problem is tightly connected with the Monge-Kantorovich optimal transportation: when $H(|\sigma|) = |\sigma|$ no congestion effect occurs and the transport problem reduces to Monge's transport where particles travels along geodesics. On the contrary, when H is superlinear congestion effect may occur and the mass particles trajectories follow more complicated paths.

The question we address concerns the design of low congested regions; more precisely, two congestion functions H_1 and H_2 are given, with $H_1 \leq H_2$, and the goal is to find a region $C \subset \Omega$ where the low congested traffic may travel. Since reducing the congestion in a region C is a costly issue, a term $m(C)$ is added, to describe the cost of improving the region C, thus penalizing too large low congested regions. On the region $\Omega \setminus C$ we then have the normally congested traffic governed by the function H_2, while on the low congested region C the traffic is governed by the function H_1.

Putting the problem into a mathematical formulation, for every region C we consider the cost function

$$F(C) = \min \left\{ \int_{\Omega \setminus C} H_2(\sigma)\, dx + \int_C H_1(\sigma)\, dx \;:\; -\operatorname{div} \sigma = \rho \text{ in } \Omega,\; u \cdot n = 0 \text{ on } \partial\Omega \right\}$$

so that we deal with the minimization problem

$$\min \big\{ F(C) + m(C) \;:\; C \subset \Omega \big\}. \tag{4}$$

We consider several choices of the congestion region C and the corresponding cost $m(C)$, related to different traffic congestion models. Our first result concern the case when C is a d-dimensional subdomain of Ω and the penalization $m(C)$ involves the perimeter of C: in this situation an optimal region C is shown to exist.

Theorem. *Assume that the cost $F(C)$ is finite for at least a subset C of $\overline{\Omega}$ with finite perimeter and that $m(C) = k\,\mathrm{Per}(C)$ with $k > 0$. Then there exists at least an optimal set C_{opt} for problem* (4).

When $m(C)$ is simply proportional to the Lebesgue measure of C, the optimal choice for the planner is to have a low congested area C_0, a normally congested area C_1, together with an area $\Omega \setminus (C_0 \cup C_1)$ with intermediate congestion.

In the case with volume constraint, passing to a relaxed formulation in which the set C is replaced by a density function $\theta(x)$ with $0 \le \theta(x) \le 1$, we end up with the minimization problem

$$\min\left\{\int_\Omega \Big(H_2(\sigma) \wedge \big(H_1(\sigma) + k\big)\Big)^{**} dx \;:\; \sigma \in \Gamma_\rho\right\}$$

that is of type (13) where $H(\sigma) = (H_2(\sigma) \wedge (H_1(\sigma) + k))^{**}$ and

$$\Gamma_\rho = \big\{\sigma \in L^1(\Omega;\mathbb{R}^d) \;:\; -\operatorname{div}\sigma = \rho \text{ in } \Omega,\ \sigma\cdot n = 0 \text{ on } \partial\Omega\big\}.$$

Using a convex analysis formula, we can write the dual formulation

$$\min\left\{\int_\Omega H(\sigma)\,dx \;:\; \sigma \in \Gamma_\rho\right\} = \sup\left\{\int_\Omega u\,d\rho - \int_\Omega H^*(\nabla u)\,dx\right\} \tag{5}$$

where the flux σ and the dual variable u are linked by $\sigma = \nabla H^*(\nabla u)$ and the Euler-Lagrange equation of problem (5) is formally written as

$$\begin{cases} -\operatorname{div}\nabla H^*(\nabla u) = \rho & \text{in } \Omega \\ \nabla H^*(\nabla u)\cdot\nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{6}$$

When the admissible sets C are networks, that is closed connected one-dimensional sets, and the penalization cost $m(C)$ is proportional to the total length of C (the 1-dimensional Hausdorff measure $\mathcal{H}^1(C)$), we obtain the existence of optimal configurations.

We observe that the analysis above is similar to a two-phase optimization problem (a complete analysis and numerical methods to treat them are available in [1]). This consists in finding an optimal design for a domain that is occupied by two constituent media with constant conductivities α and β with $0 < \alpha < \beta < +\infty$, under an objective function and a state equation that have a form similar to (5). Some numerical calculations and examples have been done using finite element method.

Plan of the work

This thesis consists of three chapters. We give here a brief summary of each Chapter.

In Chapter 1 we give a non exhaustive presentation of the theory of optimal transportation starting from the classical Monge-Kantorovich formulation up to the recent developments performed by Benamou and Brenier. We begin with the description of Monge and Kantorovich problems in a quite general setting.

Problem (Monge)**.** *Let X and Y be two Polish spaces and denote by $\mathcal{M}_1(X)$ (respectively $\mathcal{M}_1(Y)$), the set of probability measures on X, (respectively Y). Given $\rho_0 \in \mathcal{M}_1(X)$, $\rho_1 \in \mathcal{M}_1(Y)$ and a cost function $c : X \times Y \to [0, +\infty)$, the Monge problem is:*

$$\inf_{T_\sharp \rho_0 = \rho_1} \int_X c(x, T(x))\, d\rho_0(x). \tag{7}$$

Problem (Kantorovich)**.** *Let X and Y be two Polish spaces. Given a cost function $c(x, y)$ and two densities $\rho_0 \in \mathcal{M}_1(X)$ and $\rho_1 \in \mathcal{M}_1(Y)$, minimize the total cost, among all possible transport plans π with marginals ρ_0 and ρ_1:*

$$\min \int_{X \times Y} c(x, y)\, d\pi(x, y). \tag{8}$$

It is discussed the existence of an optimal transport map or plan and the connection between the two problems. Then a dual formulation of Kantorovich problem is presented, the so called *Kantorovich duality* and the main tools necessary to its proof.

Theorem. *Let X and Y be two Polish spaces and let $\rho_0 \in \mathcal{M}_1(X)$ and $\rho_1 \in \mathcal{M}_1(Y)$ be probability measures. Define $\Pi(\rho_0, \rho_1)$ to be the set of all Borel probability measures π on $X \times Y$ with marginals ρ_0 and ρ_1. Given a lower semicontinuous function $c : X \times Y \to \mathbb{R}^+ \cup \{+\infty\}$, then:*

$$\min_{\pi \in \Pi(\rho_0, \rho_1)} \int_{X \times Y} c(x, y)\, d\pi(x, y) = \sup_{(\varphi, \psi)} \left\{ \int_X \varphi\, d\rho_0 + \int_Y \psi\, d\rho_1 \right\} \tag{9}$$

with the pair $(\varphi, \psi) \in L^1(\rho_0) \times L^1(\rho_1)$ and satisfying $\varphi + \psi \le c$.

A different, dynamical approach to mass transportation theory was proposed by Benamou and Brenier in [9]. Although Monge mentioned the notion of trajectories between given data in his work (see [51]), the frst time-dependent optimal transport problem was not formulated until 1999 by J.D. Benamou and Y. Brenier. Looking at the trajectory of each particle T_t, it is studied the evolution of the measure $\rho_t = (T_t)_\sharp \rho_0$ at intermediate time t. It is convenient to switch to the Eulerian point of view and to introduce the variables ρ, v representing respectively the density and velocity field, and the flux vector $q = \rho v$. The dynamical formulation of mass transport problem is then

$$\min\{\mathcal{A}(\rho, v) \,:\, \partial_t \rho + \nabla \cdot q = 0\,,\ \rho(0, \cdot) = \rho_0\,,\ \rho(1, \cdot) = \rho_1\} \tag{10}$$

where $\mathcal{A}(\rho, v)$ is the time integral of energy

$$\mathcal{A}(\rho, v) = \int_0^T \Big(\int_{\mathbb{R}^d} \rho(t,x)|v(t,x)|^2 dx \Big) dt.$$

Denote by $Q = [0,1] \times \Omega \subset \mathbb{R}^{d+1}$ the time-space domain, by n its outer normal versor and by $\sigma = (\rho, v)$ the measure with value in $\mathbb{R}^{d+1}$ belonging to the space $\mathcal{M}_b(\bar{Q}, \mathbb{R}^{d+1})$. Taking the scalar measure $f = \delta_1(t) \otimes \rho_1(x) - \delta_0(t) \otimes \rho_0(x)$, we can rewrite (10) in the more common form:

$$\min\{\mathcal{A}(\sigma) \,:\, -\operatorname{div} \sigma = f \text{ in } \bar{Q},\ \sigma \cdot n = 0 \text{ on } [0,1] \times \partial\Omega\} \tag{11}$$

Afterwards, we discuss the main properties of the p-Wasserstein distance, defined as follows.

Definition. *The p-Wasserstein distance between $\mu, \nu \in \mathcal{P}_p(X)$ is:*

$$W_p(\mu, \nu) = \Big(\inf_{\pi \in \Pi(\mu,\nu)} \int_{X \times X} d(x,y)^p d\pi(x,y) \Big)^{1/p}. \tag{12}$$

Another important preliminary notion is the theory of Γ-convergence, introduced by De Giorgi, that permits to characterize the asymptotic behavior of families of infimum problems.

Definition. *Let $F_h : X \to \mathbb{R} \cup \{+\infty\}$ be a sequence of functionals on a metric space X. Then we define the $\Gamma - \liminf : X \to \mathbb{R} \cup \{+\infty\}$ and $\Gamma - \limsup : X \to \mathbb{R} \cup \{+\infty\}$ as:*

- $(\Gamma - \liminf F_h)(x) = \inf_{x_h \to x} \liminf_{h \to \infty} F_h(x_h)$,
- $(\Gamma - \limsup F_h)(x) = \inf_{x_h \to x} \limsup_{h \to \infty} F_h(x_h)$.

If $F = (\Gamma - \limsup F_h) = (\Gamma - \liminf F_h)$ we say that the sequence Γ-converges to the Γ-limit F, and we write, $F_h \xrightarrow{\Gamma} F$

In the further Paragraphs of the first Chapter, we describe some models present in the literature which are used in the successive Chapters of this thesis. In particular we describe the location problem and one of its generalization, the so called irrigation problem. In the last Paragraph we describe an extension of the theory introduced by Benamou-Brenier modelling the situation in which congestion effects are present, for instance traffic on a highway, crowds moving in a domain with obstacles and in general when the transportation does not behave in the classical Monge setting.

In Chapter 2, we present a generalization of the classical location model. Precisely, we consider a region $\Omega \subset \mathbb{R}^2$ with a distribution of resources $\rho : \Omega \to \mathbb{R}^+$ and we want to locate N points, say $x_1, x_2, \cdots, x_N$, representing N airports, according to some optimization criteria. The idea is that the airports collect the resources distributed in Ω and the goods can travel on point-to-point basis.

Essentially, we present two possibilities for the choice of the cost to transport a unit of mass from two points regarding the mass dependence. We can suppose that the unitary cost depends on the distance between two airports or that it also depends on the transported mass. In both cases, we suppose that the ground transportation cost and the routing cost are proportional respectively to the p-power and the q-power of the distance.

In the independent-mass case, we study the following minimization problem, written in terms of the measures $\mu_N = \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}$:

$$\min\left\{C_{p,d}N^{-p/d}\int_\Omega \frac{\rho(x)}{(\mu(x))^{p/d}}\,dx + KN^2\int_{\Omega\times\Omega}|x-y|^q\,d(\mu\otimes\mu)\right\},$$

where A and K are the proportionality constants.

On the other hand, we can assume that the amount of mass m_i located in x_i is dispatched in the other points proportionally to the masses m_j so, using the measures $\nu_N = \sum_{i=1}^{N} m_i\delta_{x_i}$, the model is:

$$\min\left\{AW_p^p(\rho,\nu) + \frac{B}{m}\int_{\Omega\times\Omega} V(x-y)\,d(\nu\otimes\nu)\ :\ \#(\mathrm{spt}\nu) = N\right\}.$$

In both cases, we study the asymptotic characterization of the limit problem as $N\to\infty$.

In Chapter 3, we introduce the a model of traffic congestion. We suppose to be in the framework of continuous traffic congestion even if the original model was introduced for the discrete case of networks. Let us consider a region $\Omega\subset\mathbb{R}^2$ in which there is a so-called traffic intensity, i.e. a density of traffic congestion. To correctly describe the model, we use the formalism of measures on the set of paths, which is a classical tool in transport theory, in connection with optimal transport. So, denoting by $C^{x,y}$ the subset of $C = W^{1,\infty}([0,1],\overline{\Omega})$ of continuous paths from x to y, we consider a Monge transport problem in which the transportation cost depends also on the possible paths $\tau\in C^{x,y}$ followed by the mass transported. Then we define a probability measure Q on C concentrated on absolutely continuous curves compatible with mass conservation and we define a measure representing the intensity of traffic i_Q associated to Q. The congestion effects are then captured by the metric associated to Q:

$$\xi_Q(x) = g(x, i_Q(x)),$$

for a given increasing function $g:\mathbb{R}^+\to\mathbb{R}^+$. Considering a region Ω in which the densities of residents and of working places are known represented by two probability measures ρ_0 and ρ_1, we consider the following Kantorovich type problem:

$$\inf_{\pi\in\Pi(\rho_0,\rho_1)}\int_{\overline{\Omega}\times\overline{\Omega}} c_{\xi_Q}(x,y)\,d\pi(x,y).$$

Using a Moser type approach, and denoting by ρ the difference $\rho=\rho_0-\rho_1$ and by σ the traffic flux, the model with congestion effects, in the stationary regime, reduces to the minimal flow problem (see [18])

$$\min\left\{\int_\Omega H(\sigma)\,dx \ :\ -\operatorname{div}\sigma=\rho \text{ in } \Omega,\ \sigma\cdot n=0 \text{ on } \partial\Omega\right\}. \tag{13}$$

We study a very simplified model of a given region where the traffic flows according to two regimes: in a region C we have a low congestion, where in the remaining part $\Omega\setminus C$ the congestion is higher. The two congestion functions H_1 and H_2 are given, but the region C has to be determined in an optimal way in order to minimize the total transportation cost. So the problem is of the following form:

$$\min\left\{F(C)+m(C)\ :\ C\subset\Omega\right\},$$

where

$$F(C)=\min\left\{\int_{\Omega\setminus C} H_2(\sigma)\,dx+\int_C H_1(\sigma)\,dx\ :\ \sigma\in\Gamma_f\right\}$$

and $m(C)$ is a penalization term. In case of perimeter constraint, $m(C)=kPer(C)$, the existence of an optimal C is proved and an optimality condition is computed. The optimal set is convex, but higher regularity for ∂C is an open problem.

When $m(C)=|C|$ is the Lebesgue measure, a relaxed formulation of the problem is considered:

$$\min\left\{\int_\Omega \Big(H_2(\sigma)\wedge\big(H_1(\sigma)+k\big)\Big)^{**}dx\ :\ \sigma\in\Gamma_f\right\}.$$

It is proved that the optimal set is composed by a low congested region, a normal congested region and an area with intermediate congestion.

Finally, we consider the optimization problem in the class of one dimensional sets. The existence is proved for the enlarged sets $\Sigma_r=\Sigma+B_r(0)$, where $B_r(0)$ is the ball of radius r centered in the origin but in general the optimization problem with networks does not admit a solution, because the limits of minimizing sequences may develop multiplicities.

Notations

$\mathcal{M}(X)$	set of measures on X;
$\mathcal{M}^+(X)$	set of positive measures on X;
$\mathcal{M}_1(X)$ or $\mathcal{P}(X)$	set of probability measures on X;
$\mathcal{P}_p(X)$	set of probability measures on X with finite p-moment;
$\mathcal{M}_{ac}(X)$ or $\mathcal{P}_{ac}(X)$	set of absolutely continuous (w.r.t. Lebesgue measure) probability measures on X;
$C(X)$	set of continuous functions on X;
$C_b(X)$	set of continuous bounded functions on X;
$C_0(X)$	set of continuous functions on X going to 0 at infinity;
$\mathcal{D}(X)$	set of $C^\infty(X)$ functions with compact support on X;
$\mathcal{D}^{'}(X)$	set of distributions on X;
$\mathrm{Lip}(X)$	set of Liptchitz functions on X;
$\pi_i : \mathbb{R}^n \to \mathbb{R}$	is the project function on the i-th component;
$\mathcal{B}(X)$	the Borel subsets of X;
$\mathcal{L}^d$	d-dimensional Lebesgue measure;
$\mathcal{H}^d$	d-dimensional Hausdorff mesure;
$\lvert A \rvert$	the Lebesgue measure of a set A;
$(f)_\sharp \mu$	the push forward of the measure μ through the map f;
$\frac{d\mu}{dm}$	Radon-Nikodym derivative of μ with respect to m;
$\mu \ll m$	the measure μ is absolutely continuous with respect to m;
f^*	the Legendre-Fenchel conjugate function of f;
$\operatorname{div} v$	the divergence operator $\sum_{i=1}^n \frac{\partial v_i}{\partial x_i}$,
∇v	the gradient operator $\left(\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}, \cdots, \frac{\partial v}{\partial x_d}\right)$;
Δv	the Laplace operator $\sum_{i=1}^n \frac{\partial^2 v_i}{\partial x_i^2}$.

Chapter 1

Preliminaries

In this chapter we will describe some preliminaries useful for chapters two and three. First we will describe some generalities in theory of optimal transportation and on Γ-convergence theory. Then we will illustrate the so-called location and irrigation problems. In the last paragraph, we describe the model of optimal transport with congestion effect.

1.1 Monge-Kantorovich's problem

Monge's transportation problem, in a simple form is the following: consider an open subset $\Omega \subset \mathbb{R}^d$ and two mass distributions, identified by two measures ρ_0 and ρ_1. Suppose that one wants to transport the amount of mass ρ_0 on the place identified by ρ_1. The transportation is put through by a *transport map* T, that is an application $T : \Omega \to \Omega$ such that:

$$\int_A d\rho_0(x) = \int_{T^{-1}(A)} d\rho_1(y)$$

for every $A \subset \Omega$. This condition explains the fact that the image of the measure ρ_0 through the map T has to be ρ_1. Usually, this condition is written using the notion of *push-forward* denoted by $\rho_1 = T_\sharp \rho_0$. We suppose that the distributions have the same mass (that often in the following will be assumed equal to 1):

$$\int_\Omega \rho_0(x) = \int_\Omega \rho_1(x).$$

If we suppose that the *cost* of moving a unit of mass is proportional to the distance $|x - T(x)|$, the total cost to transport all ρ_0 on ρ_1 using the map T is given by:

$$\int_\Omega |x - T(x)| \rho_0(x)\, dx.$$

Now, a very basic question could be: *Is it possible to find an optimal map (possibly unique), such that the total cost is minimum?*. Giving an answer to this question is equivalent to solve a minimization problem, precisely:

$$\min\left\{\int_\Omega |x-T(x)|\,d\rho_0(x)\,:\,T_\sharp\rho_0=\rho_1\right\}. \tag{1.1}$$

This problem, written in *Monge's formulation*, has some intrinsic difficulties as the following examples explain.

A first observation is: it is not so obvious that the set of the admissible transport maps is not empty and also if it exists a transport map, the minimun cannot be always attained. The following two examples show that these situations may occur:

Example 1.1 (Non-existence of transport maps)**.** Let $\Omega=\mathbb{R}$, $\rho_0=\delta_0$ and $\rho_1=\delta_1+\delta_{-1}$. There is no map T such that $T_\sharp\rho_1=\rho_0$, so the Monge's problem has no solution.

Example 1.2 (Non-existence of minimizer)**.** Let X_0, X_1, $X_2\subset\mathbb{R}^2$ be the sets given by:

$$X_0=\{(x,0)\,:\,0\le x\le 1\}$$

$$X_1=\{(x,d)\,:\,0\le x\le 1\}$$

$$X_2=\{(x,-d)\,:\,0\le x\le 1\}$$

where $d>0$, and let $\rho_0=\mathcal{H}^1\llcorner X_0$, $\rho_1=\frac{1}{2}\mathcal{H}^1\llcorner X_1+\frac{1}{2}\mathcal{H}^1\llcorner X_2$. The set of admissible transport maps is nonempty, but the minimal value of transport cost maps cannot be achieved.

Another issue is concerned with the uniqueness of the optimal transport map: the next example shows how it is possible that every transport map is optimal:

Example 1.3 (Non-uniqueness of minimizer)**.** Let $\Omega=\mathbb{R}^2$, $\rho_1=\delta_A+\delta_B$ $(A,B\in\mathbb{R}^2)$ and ρ_0 is supported on the middle axis between A and B. Then

$$\int_\Omega |x-T(x)|\,d\rho_0(x)=\int_\Omega |x-A|\,d\rho_0(x)$$

whenever $T(x)\in\{A,B\}$. Hence any admissible transport map is optimal.

So far, we suppose the elementary cost to be the Euclidean distance and the ambient space to be $\mathbb{R}^d$, but the problem can be reformulated for more general costs $c(x,y)$ and spaces. An abstract version of Monge's problem is:

Problem 1.4 (Monge)**.** *Let X and Y be two Polish spaces and denote by $\mathcal{M}_1(X)$ (respectively $\mathcal{M}_1(Y)$), the set of probability measures on X, (respectively Y). Given $\rho_0\in\mathcal{M}_1(X)$, $\rho_1\in\mathcal{M}_1(Y)$ and a cost function $c:X\times Y\to[0,+\infty]$, the Monge problem is:*

$$\inf_{T_\sharp\rho_0=\rho_1}\int_X c(x,T(x))\,d\rho_0(x). \tag{1.2}$$

As we have seen in the previous examples, the problem, at least in Monge's formulation, is not well-posed and it is of difficult direct approach. In order to apply the direct methods of the calculus of variations, it would be necessary to find a topology that makes $c(\cdot,\cdot)$ lower semicontinuous and the set of admissible maps compact, but in general there is no such a topology. Only one hundred and fifty years after Monge, with the work of Kantorovich, a more general viewpoint permits to deduce the existence of optimal transport maps in several situations.

A big problem in Monge's version is that the mass can be put together, but cannot be split. In the 1940's, Kantorovich proposed a relaxed formulation (see [43], [44]) that allows mass splitting. We discuss here Kantorovich's approach to mass transportation problem: instead of looking for an optimal transportation map, his idea was to consider *transport plans.*

Definition 1.5. Let (X, ρ_0) and (Y, ρ_1) be two probability spaces. We define the set of "transport plans", $\Pi(\rho_0, \rho_1)$ as the set of all probability measures on $X \times Y$ with marginals ρ_0 and ρ_1 respectively on X and Y. Precisely, $\pi \in \Pi(\rho_0, \rho_1)$ if and only if, for all measurable subsets $A \subseteq X$ and $B \subseteq Y$:

$$\pi(A \times Y) = \rho_0(A) \quad \text{and} \quad \pi(X \times B) = \rho_1(B).$$

We can reformulate Monge's problem using transport plans:

Problem 1.6 (Kantorovich)**.** *Let X and Y be two Polish spaces. Given a cost function $c(x, y)$ and two densities $\rho_0 \in \mathcal{M}(X)$ and $\rho_1 \in \mathcal{M}(Y)$, minimize the total cost, among all possible transport plans:*

$$\min \int_{X\times Y} c(x, y)\, d\pi(x, y) \;:\; \pi \in \Pi(\rho_0, \rho_1). \tag{1.3}$$

Of course, since not all transport plans are induced by a suitable transport map T, it may happen that the optimal value of Problem 1.6 is strictly less than the one of Problem 1.4.

Clearly, the set of admissible plans $\Pi(\rho_0, \rho_1)$ is nonempty, as the product measure $\mu = \rho_0 \otimes \rho_1$ is always admissible, so this formulation avoids "non-existence" of admissible transports. Thanks to the weak*-compactness of probability measures (using Prokhorov's theorem 1.8) and the direct methods of the calculus of variations, it is possible to prove the existence of an optimal transport plan in Problem 1.6. This will be proven in Theorem 1.9. For its proof we need some classical results, see [57].

Lemma 1.7. *Let ρ_0 and ρ_1 be Borel probability measures on a Polish space X. The set $\Pi(\rho_0, \rho_1)$ of transport plans is tight.*

Theorem 1.8 (Prokhorov)**.** *Let S be a tight set of Borel probability measures on a Polish space X. Then S is relatively sequentially compact with respect to the weak convergence. That is, given a sequence of $\{\rho_n\}$ in S there exists a Borel probability measure ρ such that for a suitable subsequence ρ_{n_k} we have*

$$\lim_{k\to+\infty} \int_X \varphi\, d\rho_{n_k} = \int_X \varphi\, d\rho$$

for every $\varphi \in C_b(X)$.

Theorem 1.9 (Existence of optimal transport plan)**.** *Let ρ_0 and ρ_1 be Borel probability measures on a Polish space X. Let $c : X \times X \to \mathbb{R}^+ \cup \{+\infty\}$. Then there exists a solution of Problem 1.6.*

Proof. If the cost function c is continuous and bounded, thanks to tightness of the set of transport plans (Lemma 1.7) and Prokhorov's theorem, the functional (1.3) is continuous. Also in the case of lower semicountinuous cost function c, Kantorovich functional is lower semicontinuous (as a consequence of motonotone convergence theorem). In order to apply the direct methods, we need to prove that the set of transport plans is closed under the weak topology. Let $\{\pi_n\}_{n\in\mathbb{N}}$ be a weakly convergent sequence in $\Pi(\rho_0, \rho_1)$ and let π such that $\pi_n \rightharpoonup \pi$. For every couple of functions $f,\ g \in C_b(X)$ by passing to the limit as $\pi_n \rightharpoonup \pi$, we have:

$$\int_{X\times X} (f, g)\, d\pi_n = \int_{X\times X} (f, g)\, d\pi,$$

that is, $(\pi_1)_\sharp \pi = \rho_0$ and $(\pi_2)_\sharp \pi = \rho_1$.

Let now $\{\pi_n\}_{n\in\mathbb{N}}$ be a minimizing sequence. By Theorem 1.8, we can extract a convergent subsequence, $\pi_{n_k} \rightharpoonup \pi^*$ for some Borel probability measure π^*. Thanks to the closedness of $\Pi(\rho_0, \rho_1)$ with respect to weak topology, $\pi^* \in \Pi(\rho_0, \rho_1)$ and by the lower semicontinuity of (1.3), π^* is a minimizer. □

Remark 1.10. This existence theorem does not imply that the optimal cost is finite. It might be that all transport plans lead to an infinite total cost, i.e. $\int c\, d\pi = +\infty$ for all $\pi \in \Pi(\rho_0, \rho_1)$. A simple condition to rule out this annoying possibility is

$$\int c(x, y)\, d\rho_0(x)\, d\rho_1(y) < +\infty.$$

Hereafter, we indicate with (MK) the Monge-Kantorovich's problem, referring to Problem 1.4 or 1.6. Of course the infimum in (1.3) could be strictly less than the infimum in (1.2). The following proposition clarifies the "connection"between maps and plans.

Proposition 1.11. *Any transport map $T : X \to Y$ between ρ_0 and ρ_1 induces a transport plan $\gamma_T \in \Pi(\rho_0, \rho_1)$ given by*

$$\gamma_T = (Id \times T)_\sharp \rho_0.$$

Vice versa, a transport plan γ is induced by a transport map if γ is concentrated on a γ-measurable graph Γ.

Proof. If T is a map that transports ρ_0 in ρ_1, we have $\pi_1(\gamma_T)_\sharp = \rho_0$ and $\pi_2(\gamma_T)_\sharp = \rho_1$, where π_i is the projection on the i-th component.

On the other side, say that γ is concentrated on a γ-mesurable graph Γ and consider $\Gamma_0 \in \mathcal{B}(X \times Y)$, $\Gamma_0 \subset \Gamma$. Let $\{K_n\}_{n\in\mathbb{N}}$ be an increasing sequence of compact subsets of Γ_0 such that $\gamma(\Gamma_0 \setminus K_n) \to 0$ and denote by $C_n = \pi_1(K_n)$, that is compact in X. By the disintegration theorem (see [5]), for every $x \in C_n$ there exists unique $y = f_n(x)$ such that $(x, y) \in K_n$. By compactness, functions $f_n : C_n \to Y$ are continuous and $f_n = f_m$ on C_n if $m \geq n$. Let us define a Borel map $T : \bigcup C_n \to Y$ such that $T_{|C_n} = f_n$ and $T(x) = y_0$ for $x \in X \setminus \bigcup C_n$. Then $T(x) = y$ γ-a.e. in $X \times Y$ and we have:

$$\begin{aligned}\int_{X\times Y} \varphi(x,y)\, d\gamma(x,y) &= \int_{X\times T(X)} \varphi(x,T(x))\, d\gamma(x,y) \\ = \int_X \varphi(x,T(x))\, d\rho_0(x) &= \int_{X\times Y} \varphi\, d(Id\times T)_\sharp \rho_0\end{aligned}$$

□

Two basic concepts in the theory of optimal transport are the geometric property of cyclical monotonicity and the so-called *Kantorovich duality*, actually the dual formulation of the mass transport problem that permits us to see a minimization problem as a maximization one.

Definition 1.12. A subset $S \subseteq X \times Y$ is said to be c-cyclically monotone if for any $n \in \mathbb{N}$ and for any couple $(x_i, y_i) \in S$, and for any permutation σ of n elements,

$$\sum_{i=1}^n c(x_i, y_i) \leq \sum_{i=1}^n c(x_{\sigma(i)}, y_i).$$

We go on with the dual Kantorovich problem.

Theorem 1.13. *Let X and Y be two Polish spaces and let $\rho_0 \in \mathcal{M}_1(X)$ and $\rho_1 \in \mathcal{M}_1(Y)$ be probability measures. Given a lower semicontinuous function $c : X \times Y \to \mathbb{R}^+ \cup \{+\infty\}$, then:*

$$\min_{\pi\in\Pi(\rho_0,\rho_1)} \int_{X\times Y} c(x,y)\, d\pi(x,y) = \sup_{(\varphi,\psi)} \left\{ \int_X \varphi\, d\rho_0 + \int_Y \psi\, d\rho_1 \right\} \tag{1.4}$$

with the pair $(\varphi, \psi) \in L^1(\rho_0) \times L^1(\rho_1)$ and satisfying $\varphi + \psi \leq c$.

If in the original (MK) problem the central notion is cost, in the dual problem is price. There is an interesting interpretation of the duality in terms of "minimum cost - maximum profit" (that is a Caffarelli example in Villani's book [57]): suppose to be an industrialist and to have to transport a certain quantity of your product from centers of production to distribution centers. You can do this work to a transportation company that applies a cost of $c(x, y)$ for each unit of product from x to y. You want to solve (MK) in order to pay as less than possible. Now, another company says that will do the same work fixing a loading price $\varphi(x)$ at place x and an unloading price $\psi(y)$ at place y in such a way that will be convenient for you. Of course you think that necessarily it will be $\varphi(x) + \psi(y) \leq c(x, y)$. Kantorovich's duality tells you that the second company can arrange the prices in such a way that you will pay at least how much it is requested by the first company.

As observed, a competitive pair (φ, ψ) satisfies $\varphi(x) + \psi(y) \leq c(x, y)$, so for fixed φ (and similarly for ψ), the best ψ compatible with the constraint is:

$$\psi(y) = \inf_x \{c(x, y) - \varphi(x)\}. \tag{1.5}$$

From this observation we can reconstruct ψ in terms of φ, so we can just take φ as the only unknown in our problem. The unknown cannot be any function, it has to satisfy (1.5): the next definitions will be useful.

Definition 1.14. A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be c-concave if it is of kind

$$f(x) = \inf_{(y,k)\in\mathcal{A}} c(x, y) + k$$

for some subset $\mathcal{A} \subseteq X \times \mathbb{R}$

Definition 1.15. Given a function $f : X \to \mathbb{R}$, the c-subdifferential $\partial_c f$ of f is defined by:

$$\partial_c f = \{(x, y) \in X \times Y \,:\, f(z) \geq f(x) + c(z, y) - c(x, y),\, \forall z \in X\}$$

Definition 1.16. Let X and Y be non-empty sets and $c : X \times Y \to \mathbb{R}$. Given $\varphi : X \to \mathbb{R}$, the c-transform of φ is defined by

$$\varphi^c(y) = \inf_x \{c(x, y) - \varphi(x)\}$$

Theorem 1.17 (Rockafellar)**.** *Let X and Y be two Polish spaces. A non empty subset $\Gamma \subseteq X \times Y$ is c-cyclically monotone if and only if is included in the subdifferential of a lower semicontinuous c-convex function f, that is $\Gamma \subseteq \partial_c f$.*

As a particular case, we note that if $c(x, y) = -x \cdot y$ on $\mathbb{R}^n \times \mathbb{R}^n$, then the c-transform coincides with the usual Legendre transform, and c-convexity is just plain convexity on $\mathbb{R}^n$.

Definition 1.18. Let X be a normed vector space and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a convex function. The function $f^* : X^* \to \mathbb{R} \cup \{+\infty\}$ defined on the dual of X by:

$$f^*(x^*) = \sup_{x\in X} [\langle x^*, x\rangle - f(x)]$$

is called *Legendre-Fenchel transform* of f.

Theorem 1.19 (Fenchel-Rockafellar duality)**.** *Let X be a normed vector space and let $F, G : X \to \mathbb{R} \cup \{+\infty\}$ be convex functionals. Suppose that there exists $x_0 \in X$ such that:*

- *$F(x_0) < +\infty$ and $G(x_0) < +\infty$;*
- *F is continuous at x_0.*

Then

$$\inf_{x \in X} [F(x) + G(x)] = \max_{x^* \in X^*} [-F^*(x^*) - G^*(x^*)]$$

We can restrict to one variable in the right-hand side of (1.4), rewriting Problem 1.13 as

$$\sup \left\{ \int_X \varphi \, d\rho_0 + \int_Y \varphi^c \, d\rho_1 \right\}.$$

Remark 1.20. We observe that the requirement $c(x, y) \leq \varphi(x) + \varphi^c(y)$ is necessary. In fact, if we remove this hypothesis, there is a counterexample by A. Pratelli that contradicts the thesis. Let $X = Y = [0, 1]$, consider $c(x, y) = \psi(x, y - 1)$, where

$$\psi(x, y) = \begin{cases} 1 & \text{if } y = x \\ 2 & \text{if } y = x + \alpha \\ \infty & \text{otherwise} \end{cases},$$

for a fixed $\alpha \in \mathbb{R}$. Taking $T(x) = x + \alpha$, it can be observed that it is minimal but not concentrated on a c-monotone set.

For a complete proof of the Theorem 1.13 we refer to [57] and we give here an informal idea of the proof.

Idea of the proof of Theorem 1.13. Let π be an optimal transport plan (from existence theorem), and let (φ, ψ) a couple satisfies the condition $\varphi(x) + \psi(y) \leq c(x, y)$. We have

$$\int c(x, y) \, d\pi(x, y) \geq \int \varphi(x) \, d\rho_0(x) + \int \psi(y) \, d\rho_1(y) = \int (\varphi(x) + \psi(y)) \, d\pi(x, y).$$

So if both quantities are equal, then $\int (c - \varphi - \psi) \, d\pi = 0$, and since the integrand is nonnegative, necessarily

$$c(x, y) = \varphi(x) + \psi(y).$$

Now let $(x_i, y_i)_{0 \leq i \leq n} \in \operatorname{spt} \pi$, so there is indeed some transfer from x_i to y_i. Then we hope that

$$\begin{cases} \varphi(x_0) + \psi(y_0) = c(x_0, y_0) \\ \varphi(x_1) + \psi(y_1) = c(x_1, y_1) \\ \dots \\ \varphi(x_n) + \psi(y_n) = c(x_n, y_n). \end{cases}$$

On the other hand, if x is an arbitrary point,

$$\begin{cases} \varphi(x_1) + \psi(y_0) \leq c(x_1, y_0) \\ \varphi(x_2) + \psi(y_1) \leq c(x_2, y_1) \\ \dots \\ \varphi(x) + \psi(y_n) = c(x, y_n). \end{cases}$$

By subtracting these inequalities from the previous inequalities and adding up everything, (we can arbitrarily choose $\varphi(x_0) = 0$), one obtains

$$\varphi(x) \geq (c(x_0, y_0) - c(x_1, y_0)) + \cdots + (c(x_n, y_n) - c(x, y_n))$$

and this should be true for all choice of (x_i, y_i) in the support of π. So it becomes natural to define ψ as the supremum of all the functions appearing in the right-hand side. It means that ψ satisfies the equation

$$\psi^c(y) = c(x, y) - \psi(y).$$

Then, if ψ^c and ψ are integrable, we can write

$$\int c\, d\pi = \int \psi^c\, d\pi + \int \psi(y)\, d\pi = \int \psi^c\, d\rho_0 + \int \psi(y)\, d\rho_1.$$

This shows at the same time that π is optimal in the Kantorovich problem and that the pair (ϕ^c, ϕ) is optimal in the dual Kantorovich problem. □

We discuss now some remarkable cases of problem with different cost function $c(x, y)$.

Since the case of quadratic cost has a prominent role because of the simplicity of results and the importance of applications, we analyze here a characterization of *classical* solutions in the setting $X = Y = \mathbb{R}^d$ and $c(x, y) = |x - y|^2$. The following theorem was proved first by Y.Brenier in [19].

Theorem 1.21 (Brenier)**.** *Let $X = Y = \mathbb{R}^d$ and $\rho_0, \rho_1 \in \mathcal{M}_1(\mathbb{R}^d)$ with second finite moment (i.e., $\int_X |x|^2\, d\rho_i(x) < +\infty$), $\rho_0 << \mathcal{L}^d$. If $c(x, y) = \frac{|x-y|^2}{2}$, then:*

- *there exists a unique $\pi \in \Pi(\rho_0, \rho_1)$, optimal for Problem 1.6 induced by a transport map T;*
- *$T = \nabla\varphi$ for some convex l.s.c. φ (moreover, ρ_0 is concentrated on $Dom(\varphi)$).*

Conversely, if $T = \nabla\varphi \in L^2(\rho_0, \mathbb{R}^d)$ for some convex φ, then T is optimal between ρ_0 and $T_\sharp \rho_0$

We give here a proof based on Theorem 1.17 but in [57] it can be found a duality-based proof.

Proof. Let $\psi : X \to \mathbb{R}$ be a c-convex function such that the graph Γ of its subdifferential contains the support of any optimal transport plan π. Setting $\varphi(x) = \psi(x) - |x|^2/2$, it can be seen that

$$(x_0, y_0) \in \Gamma \Leftrightarrow y_0 \in \partial_-\varphi(x_0).$$

Since a convex function is almost everywhere differentiable with respect to the Lebesgue measure and hence with respect to ρ_0, for ρ_0-a.e $x_0 \in X$ there exists a unique point y_0 such that (x_0, y_0), that is $y_0 = \nabla\varphi(x_0)$. Since $\operatorname{spt}\pi \subset \Gamma$, we have $\pi = (Id \times \varphi)_\sharp \rho_0$.

Finally, assuming the existence of an optimal plan π induced by a transport map $T = \nabla\varphi$, the uniqueness follows by observing that the combination of two optimal plans $\pi^{''} = \frac{1}{2}(\pi + \pi^{'})$ is still optimal and induced by a transport if and only if $T = T^{'}$, ρ_0-a.e.

□

Another result was obtained by Gangbo and McCann for a strictly convex cost in [42].

Theorem 1.22. *Let $c(x, y)$ be a strictly convex, superlinear cost on $\mathbb{R}^d$, and let ρ_0, ρ_1 be probability measures on $\mathbb{R}^d$ such that ρ_0 is absolutely continuous with respect to Lebesgue measure. Then there exist a unique optimal plan π for (MK) problem, and it has the form*

$$\pi = (Id \times T)_\sharp \rho_0,$$

where T is uniquely determined by

$$T(x) = x - \nabla c^*(\nabla \varphi(x))$$

for some c-concave function φ.

Another possible generalization is given by considering a Riemannian setting, taking $X = Y = M$ a smooth, complete, Riemannian manifold and as cost $c(x, y) = d^2(x, y)$ the square of geodesic distance on M. At the end of nineties McCann in [50] generalized most of Theorem 1.21 in this sense:

Theorem 1.23 (McCann)**.** *Let M be a connected, complete smooth Riemannian manifold equipped with standard volume measure. Let ρ_0, ρ_1 be two probability measures on M with compact support, absolutely continuous with respect to volume measure. Let $c(x, y)$ be equal to the geodesic distance on M, denoted by $d(x, y)^2$. Then, (MK) problem between ρ_0 and ρ_1 admits a unique optimal transport plan $\pi = (Id \times T)_\sharp \rho_0$ for T satisfying $T_\sharp \rho_0 = \rho_1$. Moreover, T is uniquely determined by:*

$$T(x) = \exp_x(-\nabla \varphi(x))$$

for some d^2-concave function φ.

Until now we focused on the basic question about existence and characterization of Monge-Kantorovich problem. Another interesting question is what informations on ρ_0 and ρ_1 give us the knowledge of optimal cost? We want to look at optimality from the metric point of view so we shall work under quite general assumptions on space X.

Let (X, d) be a Polish metric space and denote with $\mathcal{P}(X)$ the set of Borel probability measures on X. Let $p > 0$ be a positive real number and denote with:

$$\mathcal{P}_p(X) = \left\{ \mu \in \mathcal{P}(X) \ : \ \int_X d^p(x, x_0)\, d\mu(x) < +\infty \right\}$$

the set of probability measures with finite p-th moment (the finiteness of the integral formula is independent on the choice of the point x_0).

Definition 1.24. The p-Wasserstein distance between $\mu, \nu \in \mathcal{P}_p(X)$ is:

$$W_p(\mu, \nu) = \left(\inf_{\pi \in \Pi(\mu,\nu)} \int_{X \times X} d^p(x, y)\, d\pi(x, y) \right)^{1/p} \tag{1.6}$$

provided $\mu, \nu \in \mathcal{P}_p(X)$.

The Wasserstein distance is really a metric since it satisfies the conditions:

1. $W_p(\mu,\nu) = 0$ iff $\mu = \nu$;
2. $W_p(\mu,\nu) = W_p(\nu,\mu)$;
3. $W_p(\mu^1,\mu^3) \le W_p(\mu^1,\mu^2) + W_p(\mu^2,\mu^3)$.

Observe that, by Jensen's inequality, it follows that $W_p \le W_{p'}$ whenever $p \le p^{'}$.

Some topological properties of Wasserstein distance are known. The following proposition is a result on the stability of optimality and narrow lower semicontinuity of the Wasserstein distance (see [57] for a proof):

Proposition 1.25. *Let (X,d) be a metric space. Let $\{\mu_n^1\}$, $\{\mu_n^2\} \subset \mathcal{P}_p(X)$ be two sequences narrowly converging to μ^1,μ^2 respectively, and let $\pi_n \in \Pi(\mu_n^1,\mu_n^2)$ be the sequence of corresponding optimal transport plans. Then π_n is narrowly relatively compact in $\mathcal{P}_p(X \times X)$ and any narrow limit point $\pi \in \Pi(\mu^1,\mu^2)$ is an optimal transport plan for (μ^1,μ^2) with:*

$$W_p(\mu^1,\mu^2) \le \liminf_{n\to+\infty} W_p(\mu_n^1,\mu_n^2)$$

There are nice properties of the metric space $(\mathcal{P}_p(X),W_p)$ that can be recovered by analogous properties of (X,d), like compactness, completeness, the property of being a geodesic space or non-branching. We resume most of them in next theorems.

Theorem 1.26. *If (X,d) is a complete metric space, then so is $(\mathcal{P}_p(X),W_p)$.*

Theorem 1.27. *Let (X,d) be a complete and separable metric space. Let $\{\mu_n\}$ be a sequence in $\mathcal{P}_p(X)$. Then the following are equivalent:*

1. $W_p(\mu_n,\mu) \to 0$;
2. *μ_n narrowly converges to μ for $n \to \infty$ and*

$$\int_X d^p(x,x_0)\,d\mu_n \to \int_X d^p(x,x_0)\,d\mu$$

for all $x_0 \in X$.

1.2 Dynamic Benamou-Brenier formulation

In this section we will see a dynamical formulation of the mass transport problem described in the previous section. A reformulation of Monge-Kantorovich's problem is possible by introducing a continuous time variable $t \in [0,T]$, where a time interval $[0,T]$ is arbitrarily chosen and fixed. The main motivations of this version of optimal transport lie in the fact that a time-dependent model gives a more complete description of the transport and in the rich mathematical structure intrinsically useful.

This different, dynamical approach to mass transportation problem was proposed by Benamou and Brenier in [9] (see also [19]). Such a time-continuous reformulation was implicitly contained in the original problem addressed by Monge. The elimination of time variable was just a clever way for reducing the dimension of the problem but we will see in the following some reasons to keep to the time continuous formulation. Roughly speaking, if the mass transportation problem is viewed as a distance problem, then time-dependent minimization problem can be viewed as a geodesic problem.

Time-dependence approach is used in a wide range of applications. First examples could be in fluid dynamics framework, data assimilation in weather forecasting or imaging processing etc..

So far, we have only considered a time-independent minimization problem in which the cost function for transporting one unit of mass from one location to another does not depend on the actual history of the transportation. Let us recall briefly, just to fix the notation, (MK) problem: two bounded nonnegative measurable functions ρ_0 and ρ_T, with compact support in a Polish space X and same mass, are given. Monge's problem is: select a map T transporting $\rho_0 dx$ on $\rho_T dx$ and minimize the total transportation cost:

$$\int_{\mathbb{R}^d} c(x, T(x))\rho_0(x)\, dx. \tag{1.7}$$

We saw in Section 1.1 a brief overview on the subject and the main results and tools. In particular, we saw that in the quadratic case, there is a unique optimal map T_{opt} defined on the support of ρ_0 in the class of admissible transport maps. Moreover, T_{opt} is characterized as the unique map in this class which can be written as gradient of a convex potential:

$$T_{opt}(x) = \nabla\Phi(x).$$

We describe here the time-dependent formulation and its connection with the classical Monge-Kantorovich problem. The introduction of a time variable will allow us to look also at the *transport history* of the entire process.

In the new model we shall study a transportation process via the family of the trajectories of all points. Fix here, for convenience, a time interval $[0, 1]$; to each x associate a trajectory $(T_t(x))_{0\le t\le 1}$, and denote by $C[(T_t x)_{0\le t\le 1}]$ the corresponding displacement cost that is the cost of transporting x along the trajectory $(T_t x)_{0\le t\le 1}$. Requiring $t \to T_t x$ to be continuous, the time-dependent minimization problem reads as:

$$\inf\Big\{ \int_X C[(T_t(x))_{0\le t\le 1}]\, d\rho_0(x) \,:\, T_0 = \mathrm{Id},\, (T_1)_\sharp\rho_0 = \rho_1 \Big\}, \tag{1.8}$$

where the infimum is taken over all trajectories $(T_t(x))$.

A sufficient condition that ensures the same total cost and the same displacement map in Problems 1.7 and 1.8 is:

$$c(x, y) = \inf\{C(z_t)\ ,\ z_0 = x\, ,\, z_1 = y\},$$

where z_t denotes a continuous curve from x to y. We can enforce the requirement asking the trajectory to be optimal, i.e. for ρ_0-almost x:

$$c(x, T(x)) = C(T_t(x)).$$

The existence of an optimal solution for the time-dependent problem with the above condition, using known result for time-independent problem, is guaranteed by the following theorem, consequence of the analogous in time-independent (see [57] for a proof):

Theorem 1.28. *Let $c(x,y) = c(x-y)$ be a strictly convex function with $c(0) = 0$, $C(z_t) = \int_0^1 c(\dot{z_t})\,dt$. Let ρ_0, ρ_1 be two probability functions absolutely continuous with respect to Lebesgue measure and let $\nabla\Psi$ be the unique gradient of a c-concave function such that $(Id - \nabla c^*(\nabla\Psi))_\sharp \rho_0 = \rho_1$. Then the solution of (1.8), for $0 \le t \le 1$, is given by:*

$$T_t(x) = x - t\nabla c^*(\nabla\Psi). \tag{1.9}$$

Again, an important particular case occurs when $c(x,y) = |x-y|^2$ and $X = \mathbb{R}^n$. In this setting the solution of time-dependent problem coincides with the displacement interpolation.

Now, we think to the initial and final probabilities ρ_0 and ρ_1 of Monge-Kantorovich's problem as densities of some set of particles at time $t = 0$, $t = 1$ respectively. If $(T_t)_{0\le t\le 1}$ is the solution of (1.8), consider the probability measure at intermediate times:

$$\rho_t = (T_t)_\sharp \rho_0. \tag{1.10}$$

The question at this level is: "what is the natural evolution equation for ρ_t"? To answer, and using the time dependence, it is convenient to pass from Lagrangian to Eulerian point of view. If in the Lagrangian viewpoint it is taken into account the collection of all trajectories, in the Eulerian scheme it is studied the velocity field of the particles in the interpolation process.

To switch between these formulations, for every time $t \in [0,1]$, we introduce the velocity field $v(t,x)$ that represents the velocity of some given particle x at time t and we set:

$$\begin{cases} v(t, T_t(x)) = \frac{\partial T_t(x)}{\partial t} \\ T_0(x) = x. \end{cases} \tag{1.11}$$

Of course it is necessary to assume sufficient regularity, for instance, let v be uniformly Lipschitz, then Cauchy-Lipschitz theory ensures the existence of a well-defined flow in time interval $0 \le t \le 1$. Denote by $\rho(t,x) \ge 0$ the density field and look at pairs (ρ, v). We have:

Theorem 1.29. *Let X be $\mathbb{R}^d$. If (T_t) is the solution of (1.8), $v = v(t,x)$ is the associated velocity field and $\rho(t,x) = \rho_t = (T_t)_\sharp\rho_0$, then ρ_t is the unique solution of:*

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho v) = 0 \qquad 0 \le t \le 1 \tag{1.12}$$

in $C([0,1], \mathcal{P}(X))$.

Note that equation (1.12) is known in physics as the identity of conservation of mass and in fact the notation ρ reminds the density of a fluid.

Proof. The idea is to show that for all $\varphi \in C_c^\infty$ the map $t \mapsto \int_{\mathbb{R}^d} \varphi \, d\rho_t$ is Lipschitz on $(0,1)$ and has derivative:

$$\frac{d}{dt}\int_{\mathbb{R}^d} \varphi \, d\rho_t = \int_{\mathbb{R}^d} \nabla\varphi v_t \, d\rho_t$$

We have $\rho_t = (T_t)_\sharp \rho_0$, which means:

$$\int_{\mathbb{R}^d} \varphi \, d\rho_t = \int_{\mathbb{R}^d} (\varphi \circ T_t) \, d\rho_0$$

The function φ is compactly supported and T_t^{-1} is continuous, then the function $(\varphi \circ T_t)$ is supported on a compact set and it is Lipschitz with derivative:

$$\frac{\partial}{\partial t}(\varphi \circ T_t) = (\nabla\varphi \circ T_t) \cdot \frac{\partial T_t}{\partial t} = (\nabla\varphi \circ T_t) \cdot (v_t \circ T_t).$$

Fixing $h > 0$, we have:

$$\frac{1}{h}\left(\int_{\mathbb{R}^d} \varphi \, d\rho_{t+h} - \int_{\mathbb{R}^d} \varphi \, d\rho_t\right) = \int_{\mathbb{R}^d}\left(\frac{\rho \circ T_{t+h} - \rho \circ T_t}{h} \, d\rho_0\right).$$

The term on the right is uniformly bounded on $[0, 1-h] \times \mathbb{R}^d$ and it converges to $(\nabla\varphi \circ T_t) v_t$ as $h \to 0$. We deduce, from Lebesgue dominated convergence theorem, that the map $t \mapsto \int_{\mathbb{R}^d} \varphi \, d\rho_t$ is differentiable for almost all t and:

$$\frac{d}{dt}\int_{\mathbb{R}^d} \varphi \, d\rho_t = \int_{\mathbb{R}^d} (\nabla\varphi \circ T_t) \cdot (v_t \circ T_t) \, d\rho_0 = \int_{\mathbb{R}^d} \nabla\varphi \cdot v_t \, d\rho_t.$$

So ρ_t solves the continuity equation. Regarding the uniqueness, by linearity, it is sufficient to prove that if a time-dependent measure ρ_t solves (1.12), then for all time $s < 1$,

$$\rho_0 = 0 \Rightarrow \rho_s = 0.$$

Assume we can construct a Lipschitz function $\varphi(t,x)$, defined on the time-interval $[0,1]$, compactly supported, and solving

$$\begin{cases} \frac{\partial\varphi}{\partial t} = -v \cdot \nabla\varphi, \\ \varphi_{|t=s} = \varphi_s \end{cases} \tag{1.13}$$

where φ_s is arbitrarily chosen in $\mathcal{D}(X)$. Then, we deduce that $t \mapsto \int \varphi_t \, d\rho_t$ is Lipschitz and satisfies

$$\frac{d}{dt}\int \varphi_t \rho_t = \int \frac{\partial \varphi_t}{\partial t} d\rho_t + \int \varphi_t d\left(\frac{\partial \rho_t}{\partial t}\right) = -\int v_t \cdot \nabla\varphi_t \, d\rho_t + \int \varphi_t \, d(\nabla \cdot (v_t \rho_t)) = 0$$

for almost all t. So

$$\int \varphi_s \, d\rho_s = \int \varphi_0 \, d\rho_0 = 0.$$

By arbitrariness of φ_s, it follows that $\rho_s = 0$. It remains to construct a solution of (1.13) with time condition at $t = s$. Since (T_t) is a locally Lipschitz function with compact support satisfying (1.13) almost everywhere, the proof is completed. $\square$

If we define the kinetic energy of particles at each time t as:

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^d} \rho(t,x)|v(t,x)|^2\,dx,$$

one can think to the total effort to be spent for moving particles at speed v: at each velocity field, it is associated a time integral of energy (up to factor $1/2$):

$$\mathcal{A}(\rho, v) = \int_0^T \Big(\int_{\mathbb{R}^d} \rho(t,x)|v(t,x)|^2\,dx\Big)dt. \tag{1.14}$$

From now, the discussion is made for the case $c(x,y) = |x-y|^2$ but as underlined in Remark 1.31, similar arguments hold in more general cases.

Benamou and Brenier stated the following equivalence:

Theorem 1.30. *Let $\rho_0, \rho_1 \in \mathcal{M}_{ac}(\mathbb{R}^d)$ (endowed with weak* topology) be compactly supported. Then we have the equivalence:*

$$W_2^2(\rho_0, \rho_1) = \inf\{\mathcal{A}(\rho, v)\ :\ (\rho, v) \in \mathcal{V}(\rho_0, \rho_1)\} \tag{1.15}$$

where $\mathcal{V}(\rho_0, \rho_1)$ denotes the set of all pairs $(\rho, v)_{0\le t\le 1}$ satisfying the continuity equation (1.12) with the boundary conditions:

$$\rho(0,\cdot) = \rho_0, \qquad \rho(1,\cdot) = \rho_1. \tag{1.16}$$

Moreover, the infimum is achieved by the unique pair (ρ, v) defined from Ψ in (1.9) by:

$$\int f(t,x)\rho(t,x)\,dt\,dx = \int f\left(t, x + t\frac{\nabla\Psi(x) - x}{T}\right)\rho_0(x)\,dt\,dx. \tag{1.17}$$

$$\int f(t,x)\rho(t,x)v(t,x)\,dt\,dx = \int \frac{\nabla\Psi(x) - x}{T} f\left(t, x + t\frac{\nabla\Psi(x) - x}{T}\right)\rho_0(x)\,dt\,dx \tag{1.18}$$

for all continuous functions f.

Proof. Here we sketch the main steps. Fixing ρ and v smooth, bounded, of class C^1, the steps of the proof will be:

i. the lower bound $W_2^2(\rho_0, \rho_1) \le \inf\{\mathcal{A}(\rho, v)\ :\ (\rho, v) \in \mathcal{V}(\rho_0, \rho_1)\}$ holds;

ii. it exists $(\rho, v) \in \mathcal{V}(\rho_0, \rho_1)$ attaining the minimum in (1.15).

Step (i). To achieve step (i), it is sufficient to consider an admissible pair (ρ, v) and the flow map of the vector field v, as in (1.11) and the curve ρ_t as in (1.10). In particular,

$$\int \rho_t(x)|v_t(x)|^2\,dx = \int \rho_0(x)|v_t(T_t x)|^2\,dx = \int \rho_0(x)\left|\frac{d}{dt}T_t x\right|^2\,dx.$$

Integrating in t:

$$\begin{aligned}\mathcal{A}(\rho, v) \geq \int \rho_0(x) \left(\int_0^1 \left| \frac{d}{dt} T_t x \right|^2 dt \right) dx \geq \\ \int \rho_0(x) |T_1 x - T_0 x|^2 \, dx = \int \rho_0(x) |T_1 x - x|^2 \, dx.\end{aligned} \tag{1.19}$$

Now, an approximation argument is necessary to reduce the minimization to the case of smooth velocity field and to conclude

$$\inf\{\mathcal{A}(\rho, v) \ : \ (\rho, v) \in \mathcal{V}(\rho_0, \rho_1)\} \geq W_2^2(\rho_0, \rho_1).$$

A change of variable is necessary, passing form (ρ, v) to $(\rho, \rho v)$, in order to convexify the functional. We do not detail more and we refer to [57] for the precise argument. *Step (ii).* We show a couple $(\rho, v) \in \mathcal{V}(\rho_0, \rho_1)$ such that the equality holds in (1.15). Consider $T = \nabla\Psi$ the optimum in (MK) problem and set:

$$v_t = \left(\frac{d}{dt} T_t \right) \circ T_t^{-1} = (T - Id) \circ T_t^{-1}.$$

By an argument similar to Theorem 1.29, it is not difficult to see that (ρ_t, v_t) solves (1.12) in the weak sense:

$$\int \rho_t \Phi(v_t) \, dx = \int \rho_0(x) \Phi(T(x) - x) \, dx.$$

Choosing $\Phi(v) = |v|^2$, we conclude. □

Remark 1.31. We observe that this equivalence between optimal transportation and fluid mechanics holds in a more general contex. In particular, it holds for $c(x, y) = |x - y|$. Anyway, Theorem 1.30 can be generalized in the setting of a smooth manifold M: the first part of Step (i) and the Step (ii) work without problems in this situation; the delicate point is the approximation by smooth velocities, but using an embedding argument it also holds.

Thinking to the fluid dynamics framework, as in the previous discussion, it is convenient to introduce the variable $q = \rho v$ that represents the flux of the transported mass; the continuity equation, in the new variable (ρ, q)

$$\partial_t \rho + \nabla \cdot q = 0$$

has to be given in the sense of distribution, that is:

$$\int_0^1 \int_\Omega \partial_t \varphi(t, x) \, d\rho(x) + \int_\Omega \nabla \cdot \varphi(x, t) \, dq(x) = 0$$

for every smooth function φ with $\varphi(0, x) = \varphi(1, x) = 0$. Note that, with this change of variable, the continuity equation is linear with respect to density ρ and momentum q.

The general dynamical formulation of mass transportation problems, following this Eulerian point of view, then becomes:

$$\min\{\mathcal{A}(\rho, v)\ :\ \partial_t \rho + \nabla \cdot q = 0\,,\ \rho(0,\cdot) = \rho_0\,,\ \rho(1,\cdot) = \rho_1\}. \tag{1.20}$$

Existence of minimizers for the problem above, with the linear constraint of continuity equation, follows also by direct methods of the calculus of variation. Denote by $Q = [0,1] \times \Omega \subset \mathbb{R}^{d+1}$ the time-space domain and by n its outer normal versor. Let $\sigma = (\rho, v)$ be the measure with value in $\mathbb{R}^{d+1}$ belonging to the space $\mathcal{M}_b(\bar{Q}, \mathbb{R}^{d+1})$. Taking the scalar measure $f = \delta_1(t) \otimes \rho_1(x) - \delta_0(t) \otimes \rho_0(x)$, (1.20) can be rewritten in the more common form:

$$\min\{\mathcal{F}(\sigma)\ :\ -\operatorname{div}\sigma = f \,\text{in}\, \bar{Q},\ \ \sigma \cdot n = 0 \,\text{on}\, \partial Q\} \tag{1.21}$$

where we rename $\mathcal{A}$ with $\mathcal{F}$ to emphasize the change of variable. It follows:

Theorem 1.32. *Let $\mathcal{F} : \mathcal{M}_b(\bar{Q}, \mathbb{R}^{d+1}) \to [0, +\infty]$ be lower semicontinuous for the weak* convergence, and assume the coercivity condition for a suitable constant C:*

$$\mathcal{F}(\sigma) \geq C|\sigma| - \frac{1}{C} \qquad \forall \sigma \in \mathcal{M}_b(\bar{Q}, \mathbb{R}^{d+1}).$$

Assume that $\mathcal{F}(\sigma_0) < +\infty$ for at least one σ_0 satisfying continuity equation with boundary condition as in (1.21). Then the minimum Problem 1.21 admits a solution. If $\mathcal{F}$ is strictly convex, the solution is unique.

Proof. Let $(\sigma_n)_{n\in\mathbb{N}}$ be a minimizing sequence for (1.21). The coercivity assumption ensures that (σ_n) is bounded and then it exists a subsequence (σ_{n_k}) weakly* convergent to $\sigma \in \mathcal{M}_b(\bar{Q}, \mathbb{R}^{d+1})$. Passing to the limit as $k \to +\infty$ in the constraint of the problem, we get an admissible σ. By lower semicontinuity, we have:

$$\inf_{\sigma \text{ admis.}} \mathcal{F}(\sigma) = \lim_{k\to+\infty} \mathcal{F}(\sigma_{n_k}) \geq \mathcal{F}(\sigma)$$

that proves the optimality of minimizer σ. □

1.3 Γ-convergence

In this section we make a summary on an important tool of calculus of variations: Γ-convergence. Introduced by De Giorgi in the seventies, it allows to characterize the asymptotic behavior of families of infimum problems. It establishes a link between minima (minimizers) of a sequence of functionals and the minimum (minimizer) of the limit functional. We consider here the framework of metric spaces and we refer to the books of Dal Maso [37] and Braides [13].

Let X be a metric space. Since a basic notion in many of our problem is that of lower semicontinuous function, we start with:

Definition 1.33. A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be (sequentially) lower semicontinuous at $x \in X$ if, for every sequence $\{x_j\}$ converging to x, we have

$$f(x) \leq \liminf_{j \to +\infty} f(x_j)$$

We say that f is lower semicontinuous if it is l.s.c. at x for all $x \in X$.

We remark the following condition, equivalent to lower semicontinuity: f has the sublevel closed for all $t \in \mathbb{R}$, or $f(x) = \liminf_{y \to x} f(y)$. Moreover, the following properties hold: the sum of l.s.c. functions is l.s.c., the supremum of family of continuous functions is l.s.c. and the characteristic function of a set E is l.s.c. if and only if the set is open. Similar can be said for upper semicontinuous functions (f is upper semicontinuous if and only if $-f$ is l.s.c.).

Definition 1.34. Let $F_h : X \to \mathbb{R} \cup \{+\infty\}$ be a sequence of functionals. Then we define the $\Gamma - \liminf : X \to \mathbb{R} \cup \{+\infty\}$ and $\Gamma - \limsup : X \to \mathbb{R} \cup \{+\infty\}$ as:

- $(\Gamma - \liminf F_h)(x) = \inf_{x_h \to x} \liminf_{h \to \infty} F_h(x_h)$,
- $(\Gamma - \limsup F_h)(x) = \inf_{x_h \to x} \limsup_{h \to \infty} F_h(x_h)$.

If $F = (\Gamma - \limsup F_h) = (\Gamma - \liminf F_h)$ we say that the sequence Γ-converges to the Γ-limit F, and we write, $F_h \xrightarrow{\Gamma} F$

A more useful characterization of Γ-convergence is shown in the next theorem.

Theorem 1.35. *The sequence F_h Γ-converges to F if and only if:*

i. for every $x \in X$ and every sequence $x_h \in X$ converging to x, there holds

$$F(x) \leq \liminf_h F_h(x_h);$$

ii. for every $x \in X$, there exists a sequence $x_h \in X$ converging to x such that:

$$F(x) \geq \limsup F_h(x_h).$$

In Theorem 1.35, condition (ii) can be replaced by one of the following equivalent conditions:

$(ii)'$ for every $x \in X$, there exist a sequence $x_h \in X$ converging to x such that:

$$F(x) = \lim_h F_h(x_h)$$

$(ii)''$ for every $x \in X$ and for every $\varepsilon > 0$ there exist a sequence $x_h \in X$ converging to x such that:

$$F(x) \geq \limsup_h F_h(x_h) - \varepsilon$$

In the next proposition, a characterization of Γ-limit is given, when it exists.

Proposition 1.36. *If* $F_h \xrightarrow{\Gamma} F$*, then*

$$F(x) = \inf\{\liminf_h F_h(x_h) \,:\, x_h \to x\}$$

There are many examples showing Γ-convergence is independent from pointwise convergence. We summarize in the next remarks some properties of Γ-convergence and some no-properties for which counterexamples can be found in [37], [13].

Remark 1.37. An important property of Γ-convergence is its stability under continuity perturbations. Given $F_h \xrightarrow{\Gamma} F$, $G_h \xrightarrow{\Gamma} G$, in general it is not true that the sum $F_h + G_h$ Γ-converges to $F + G$, but if we assume G continuous, then it holds:

$$(F_h + G_h) \xrightarrow{\Gamma} (F + G)$$

Remark 1.38. The uniform convergence of F_h to F, in general does not imply Γ-convergence, but if we assume F l.s.c., then $F_h \xrightarrow{\Gamma} F$.

Remark 1.39. The $\Gamma - \liminf$ and $\Gamma - \limsup$ (and, if it exists, the $\Gamma - \lim$) are lower semicontinuous functions.

If F is not semicontinuous, it can be useful to do an operation of *relaxation*, that is, to compute the lower semicontinuous envelope of F.

Definition 1.40. Let $F : X \to \mathbb{R} \cup \{+\infty\}$. The lower semicontinuous envelope of F is the greatest lower semicontinuous function non greater than F:

$$\bar{F}(x) = \sup\{G(x) \,:\, G\, l.s.c\ , G \leq F\}$$

Since the supremum of l.s.c. function is still l.s.c., $\bar{F}$ is l.s.c..

Proposition 1.41. *We have*

$$\Gamma - \liminf F_h = \Gamma - \liminf \bar{F}_h,$$

$$\Gamma - \limsup F_h = \Gamma - \limsup \bar{F}_h.$$

We consider now three classes of Γ-convergent sequences for which the Γ-limit is determined. They are the monotone sequences: constant, increasing, and decreasing.

- Consider the simplest case $F_h = F$ for all $h \in \mathbb{N}$. It is easy to show that F_h Γ-converges but in general $\Gamma - \liminf F_h \neq F$. The equality holds if F is lower semicontinuous and in such a case we have $\Gamma - \lim F_h = \bar{F} = F$.

- Consider a sequence F_h such that $F_h \leq F_{h+1}$ for all $h \in \mathbb{N}$, then

$$F_h \xrightarrow{\Gamma} \lim_h \bar{F}_h = \sup_h \bar{F}_h.$$

- Consider a sequence F_h such that $F_h \geq F_{h+1}$ for all $h \in \mathbb{N}$. Then

$$F_h \xrightarrow{\Gamma} \overline{\inf_h F_h}.$$

We see now as, under some equi-coercivity assumptions, the Γ-convergence of a sequence F_h to a function F implies the convergence of the minimum values of F_h to the minimum value of F. If we assume also that F_h and F have a unique minimum point, the sequence of minimizers of F_h converges to the minimizer of F.

We recall here some definitions and the main results about convergence of minima and minimizers of Γ-convergent sequences.

Definition 1.42. A function $F : X \to \mathbb{R} \cup \{+\infty\}$ is coercive if for every $t \in \mathbb{R}$ the set $\{F \leq t\}$ is precompact (its closure is compact in X). We say that a sequence F_h is equi-coercive if there exists a non empty compact set $K \subset X$ such that $\inf_X F_h = \inf_K F_h$ for all $h \in \mathbb{N}$.

Theorem 1.43. *Let F_h be an equi-coercive sequence that Γ-converges to a function F. Then F is coercive and*

$$\min_{x \in X} F = \lim_h \inf_{x \in X} F_h$$

Moreover, considering $x_h \in \operatorname{argmin} F_h$ converging to $x \in X$, then x is a minimizer for F and

$$F(x) = \lim_h F_h(x_h).$$

We conclude the overview on Γ-convergence with the property of compactness for functions from X to $\mathbb{R} \cup \{+\infty\}$.

Theorem 1.44. *Let (X, d) be a separable metric space and let $\{F_h : X \to \mathbb{R} \cup \{+\infty\}\}$ be a sequence. Then there is a Γ-convergent subsequence $\{F_{h_k}\}$.*

1.4 Location problem

In this paragraph, we will present in detail the so-called *location problem* and the known results about it. Moreover it is presented a brief overview of the various models that have been studied in the literature in the last years, stressing on the main differences between them.

In everybody experience, we think that a well-organized city makes life happier for the citizens that work and live and have to travel every day from home to office. In general, planning economic activities is an extremely complex problem because of a high number of parameters that often intervene. Depending on what is the objective of the modelization, one can do different choices and reach different models.

The common setting of all models is an open connected bounded domain $\Omega \subset \mathbb{R}^d$ that could represent a region, city or state; in case of explicit examples referring to real cases, the dimension of the ambient space will be $d = 2$ or $d = 1$. One can think that in

Ω there is a distribution of citizens or resources represented by a nonnegative function $\rho(x) : \Omega \to \mathbb{R}$ and activity planner consists in determining a configuration of facilities (for examples, shops, cinemas, offices, etc.) in such a way to minimize the total cost of travel for moving or concentrating population or resources. The facilities are represented by a given number N of points $x_1, \dots, x_N$ in Ω and location problem consists in determining the N points in order to minimize the work necessary to concentrate the resources in x_i. Assuming that the work to move a unit mass from x to y is $|x-y|^p$, then the total concentration cost in a region Ω_i, corresponding to the point x_i inside Ω_i, is given by

$$\int_{\Omega_i} |x - x_i|^p \rho(x)\, dx.$$

Then, the total cost is given by

$$\sum_{i=1}^{N} \int_{\Omega_i} |x - x_i|^p \rho(x)\, dx,$$

that can be also written in the form

$$\int_{\Omega} \big(\mathrm{dist}(x, \Sigma)\big)^p \rho(x)\, dx,$$

where $\Sigma = \{x_1, \dots, x_N\}$ is the set of N points.

Precisely, the problem we faced with is the following:

Problem 1.45 (Location)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open bounded set with Lipschitz boundary and let $\rho(x) : \Omega \to \mathbb{R}$ be a nonnegative function. Given $N \in \mathbb{N}$, the location problem consists in minimizing the functional:*

$$\mathcal{F}_N(\rho, \Omega) = \min \Big\{ \int_{\Omega} \big(\mathrm{dist(x, \Sigma)}\big)^{\mathrm{p}} \rho(\mathrm{x})\, \mathrm{dx} \ : \ \Sigma \subset \Omega,\ \#\Sigma = \mathrm{N} \Big\}. \tag{1.22}$$

Here Σ is the unknown set of N points to be determined, $\#\Sigma$ is the cardinality of Σ, and $\mathrm{dist}(x, \Sigma)$ is the distance function:

$$\mathrm{dist}(x, \Sigma) = \min \big\{ |x - y| \ : \ y \in \Sigma \big\}.$$

The existence of an optimal configuration Σ_N for N points is straightforward and has been studied by several authors (see for instance [25], [52] and the references therein).

In particular, for a proof see Theorem 1.55 of the next section.

Remark 1.46. Theorem 1.55 can be easily generalized to the case of sets Σ with finite numbers of connected components, that, in case $l = 0$ is just the location problem.

Moreover, in [52], Morgan and Bolton proved that regular hexagons beat any other collections of congruent or noncongruent shapes of equal or nonequal areas, in finite or infinite domains. We cite the main theorems proved in [52] that solve location problem.

Theorem 1.47. *Consider a partitioning of a square area A into N regions P_i with associated points (the "centers of production") p_i in $\mathbb{R}^2$. Then the average distance ρ, defined as*

$$\rho = \frac{1}{A}\sum_{i=1}^{N}\int_{P_i}|x - p_i|,$$

is greater than the corresponding average distance for a regular hexagon of area A/N centered at the origin:

$$\rho > (A/N)^{1/2}\rho_0(6),$$

where $\rho_0(N)$ is the average distance from the center in a regular hexagon of unit area.

Observe that approximating an ideal hexagon with a lots of hexagon, it is possible to generalize the result of Theorem 1.47 to domains that are not squares.

Also we have an analogous result for unbounded domains:

Theorem 1.48. *Consider a tilling of the plane by convex polygons P_i with area A_i and associated points p_i in $\mathbb{R}^2$. Let $Q(r)$ denote the square $[-r/2, r/2]^2$. Suppose that $\{P_i \subset Q(r)\}$ is finite and that the average area of every P_i tilling the whole palne is at least 1, that is*

$$\liminf_{r\to+\infty} ave\{A_i \,:\, P_i \subset Q(r)\} \geq 1.$$

Let

$$\rho(r) = \frac{1}{r^2}\int_{Q(r)}|x - p_i(x)|,$$

where $p_i(x)$ denotes the point associated to the polygon containing x. Then

$$\liminf_{r\to+\infty}\rho(r) \geq \rho_0(6).$$

In [25] and [52] there are some nice figures that show the optimal position of few points. If we consider the case when Ω is the unit disc in $\mathbb{R}^2$, some considerations can be done. In case $N = 1$, it is not difficult to prove that the only minimizer is the center of the disc. When $N \geq 2$ is not too large, some numerical approximations, using the classical finite difference method, done in [25], show that the optimal set Σ_N is given by the vertices of a centered regular polygon, as in Figure 1.1.

On the other hand, the numerical computation of an optimal set when the number N is large, presents big difficulties, essentially due to the fact that the cost in (1.22) admits a huge number of local minima, which prevent the use of fast gradient methods and make necessary the implementation of global optimization methods that are in general much slower.

So when N is large it is better to perform an asymptotical analysis that gives us important informations about the limit density of optimal points (see [11], [28] and references therein).

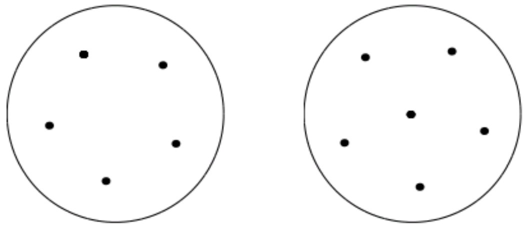


Figure 1.1: Optimal location of 5 and 6 points in the unit disc

As $N \to +\infty$, instead of looking at the precise position of the x_i, we are interested in the limit density. Observe that, if we identify each set $\Sigma_N \subset \Omega$ of N points with the measure

$$\mu_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i},$$

it is possible to write the optimal location problem in terms of the Wasserstein distance W_p; for all $\Sigma \subset \mathbb{R}^d$ we have:

$$\int_\Omega \big(\mathrm{dist}(x,\Sigma)\big)^p \rho(x)\,dx = \inf\{W_p^p(\rho,\nu) : \nu \in \mathcal{M}_+(\bar{\Omega})\,,\ \mathrm{spt}(\nu) \subset \Sigma\}. \tag{1.23}$$

So, when N is large, we want to study the asymptotic behaviour of Σ_N as $N \to +\infty$. In order to do this, we define, for all $\varepsilon > 0$, the functional $F_\varepsilon : (\mathcal{M}_+(\bar{\Omega}))^2 \to \mathbb{R}$ as:

$$F_\varepsilon(\nu,\mu) = \begin{cases} \frac{1}{\varepsilon} W_p(\rho,\nu) & \text{if} \mu(\cdot) = \varepsilon^d G(\nu,\cdot) \,\text{and}\, \sharp(\mathrm{spt}\,\nu) < +\infty \\ +\infty & \text{otherwise} \end{cases} \tag{1.24}$$

where $G(\nu,\cdot)$ is the function defined for all Borel sets B by:

$$G(\nu,B) = \sharp(\mathrm{spt}(\nu) \cap B).$$

Setting $\varepsilon = N^{-p/d}$ and using (1.23), we obtain:

$$N^{p/d} F_N(\rho,\Omega) = \inf_\nu \left\{\frac{1}{\varepsilon} W_p(\rho,\nu) : \sharp(\mathrm{spt}\,\nu) \le \frac{1}{\varepsilon^d}\right\} = \inf_{(\nu,\mu)} \{F_\varepsilon(\nu,\mu) : \mu(\bar{\Omega}) \le 1\} \tag{1.25}$$

From the equality above, it is clear that the convergence of $N^{p/d}\mathcal{F}_N(\rho,\Omega)$ will be connected to Γ-convergence of F_ε with respect to weak* topology of $(\mathcal{M}_+(\bar{\Omega}))^2$. The Γ-convergence result is the following:

Theorem 1.49. *If ρ is l.s.c. and positive, then the sequence F_ε as above* (1.24)*, Γ-converge to functional:*

$$F(\nu,\mu) = \begin{cases} C_{p,d} \int_\Omega \frac{\rho(x)}{\mu_a(x)^{p/d}}\,dx & \textit{if} \nu = \rho \\ +\infty & \textit{otherwise} \end{cases} \tag{1.26}$$

where $\mu_a = \frac{d\mu}{dx}$ *is the absolutely continuous part of* μ *with respect to the Lebesgue measure on* Ω *and* $C_{p,d}$ *is a constant depending on d, p.*

From (1.25) and Theorem 1.49 we can deduce:

Corollary 1.50. *The sequence* $N^{p/d}\mathcal{F}_N(\rho,\Omega)$ *converges for* $N \to +\infty$ *to a finite positive limit given by:*

$$\begin{aligned}\mathcal{F}_\infty(\rho,\Omega) &= \min\Big\{C_{p,d}\int_\Omega \frac{\rho(x)}{\mu_a(x)^{p/d}}\,dx \,:\, \mu \in \mathcal{M}_+\,,\ \int_\Omega \mu \le 1\Big\}\\ &= C_{p,d}\Big(\int_\Omega \rho(x)^{pd/(d+1)}\Big)^{(d+1)/d}.\end{aligned} \tag{1.27}$$

For the proof of Theorem 1.49, we need a lemma that identifies also the constant $C_{p,d}$ in Γ-limit (1.26).

Lemma 1.51. *The sequence* $N^{p/d}\mathcal{F}_N(\rho,Q)$*, with* $Q = \left(-\frac{1}{2},\frac{1}{2}\right)^d$ *a d-dimensional cube, has limit* $C_{p,d} > 0$ *as* $N \to +\infty$*.*

Proof. Observing first that $\mathcal{F}_N$ is decreasing, it is suffice to prove that the subsequence $N^{p/d}\mathcal{F}_N(1,Q)$ converges. With change of variable $N \to N^{p/d}$, we have:

$$N\varphi_{N^d}(1,Q) = \frac{1}{N^{p/d}}\inf\left\{\int_{[0,N)^d} \mathrm{dist^p(x,\Sigma)\,dx} \,:\, \sharp(\Sigma) = \mathrm{N^{p/d}}\right\} = \frac{S([0,N)^d)}{N^{p/d}}.$$

Here $S(\cdot)$ is the set function defined by $S(A) = \inf\left\{\int_A d^p(x,\Sigma)\,dx \,:\, \sharp(\Sigma) \le |A|\right\}$ which is translation invariant and sub-additive with respect to inclusion (for $A \cap B = \emptyset$). Then the existence of the limit is a classical result. $\square$

Proof of Theorem 1.49. In order to obtain the Γ-limit result, we have to achieve the $\Gamma - \liminf$ and $\Gamma - \limsup$ inequalities, that is, $\forall(\nu,\mu) \in (\mathcal{M}_+(\overline{\Omega}))^2$, $\forall(\nu_\varepsilon,\mu_\varepsilon) \overset{*}{\rightharpoonup} (\nu,\mu)$ it follows:

$$\liminf_{\varepsilon\to 0}\varphi_\varepsilon(\nu_\varepsilon,\mu_\varepsilon) \ge \varphi(\nu,\mu),$$

and, $\forall(\nu,\mu) \in (\mathcal{M}_+(\overline{\Omega}))^2$ it exists $(\nu_\varepsilon,\mu_\varepsilon) \overset{*}{\rightharpoonup} (\nu,\mu)$ such that

$$\limsup_{\varepsilon\to 0}\varphi_\varepsilon(\nu_\varepsilon,\mu_\varepsilon) \le \varphi(\nu,\mu).$$

Step 1: $\Gamma - \liminf$ *inequality:* we take $(\nu_\varepsilon,\mu_\varepsilon) \overset{*}{\rightharpoonup} (\nu,\mu)$ and, without loss of generality, we can suppose that $\varphi_\varepsilon(\nu_\varepsilon,\mu_\varepsilon)$ is bounded, so

$$W_p(\rho,\nu_\varepsilon) \le C\varepsilon \qquad \mu_\varepsilon = \varepsilon^d G(\nu_\varepsilon,\cdot) \tag{1.28}$$

It follows, from l.s.c. of W_p on $(\mathcal{M}_+(\overline{\Omega}))^2$:

$$0 \le W_p(\rho,\nu) \le \liminf_{\varepsilon\to 0} W_p(\rho,\nu_\varepsilon) \le 0$$

and so $\rho = \nu$. Denoting with $\Sigma_\varepsilon = \operatorname{spt}\nu_\varepsilon$, we have:

$$\varphi_\varepsilon(\nu_\varepsilon, \mu_\varepsilon) \geq \frac{1}{\varepsilon}\int_\Omega \operatorname{dist}^p(x, \Sigma_\varepsilon)\rho(x)\, dx$$

Moreover, thanks to (1.28), the sequence $\frac{1}{\varepsilon}\operatorname{dist}^p(x, \Sigma_\varepsilon)$ is bounded in $L^1(\Omega)$ (up to subsequences), so it exists $m \in \mathcal{M}_+(\overline{\Omega})$ such that

$$\frac{1}{\varepsilon}\operatorname{dist}^p(x, \Sigma_\varepsilon)\mathcal{L}^d \stackrel{*}{\rightharpoonup} m.$$

By contradiction, $d^p(x, \Sigma_\varepsilon) \to 0$ uniformly on Ω, so:

$$\liminf_{\varepsilon\to 0}\varphi_\varepsilon(\nu_\varepsilon, \mu_\varepsilon) \geq m(\bar{\Omega}) \geq \int_\Omega \frac{dm}{dx}\, dx.$$

To conclude $\Gamma - \liminf$ inequality, it is sufficient to prove

$$m_a(x) = \frac{dm}{dx} \geq C_{p,d}\frac{\rho(x)}{(\mu_a(x))^{p/d}} \ \ \forall x \in \Omega.$$

Fix $x_0 \in \Omega$, $\delta > 0$ and set $N_\varepsilon = \sharp(\Sigma_\varepsilon \cap Q(x_0, \delta))$. We can suppose that:

$$\lim_{\varepsilon\to 0}\varepsilon^{p/d}N_\varepsilon = \lim_{\varepsilon\to 0}\mu_\varepsilon(Q(x_0, \delta)) = \mu(Q(x_0, \delta)),$$

$$\lim_{\varepsilon\to 0}\int_{Q(x_0,\delta)}\frac{1}{\varepsilon}\operatorname{dist}(x, \Sigma_\varepsilon)\rho(x)\, dx = m(Q(x_0, \delta)).$$

By convergence $d^p(x, \Sigma_\varepsilon) \to 0$, we have:

$$\begin{aligned}\int_{Q(x_0,\delta)}\operatorname{dist}^p(x, \Sigma_\varepsilon)\, dx &\geq \int_{Q(x_0,\delta)}\operatorname{dist}^p(x, \Sigma_\varepsilon \cap Q(x_0, \delta))\, dx \\ &\geq \varphi_{N_\varepsilon}(1, Q(x_0, \delta)) = (\delta)^{d+1}\varphi_{N_\varepsilon}(1, Q).\end{aligned} \tag{1.29}$$

Using Lemma 1.51, as $\delta \to 0$, we get the claim.

Step 2: $\Gamma - \limsup$ *inequality:* Denoting $\varphi^+(\nu, \mu)$ the upper Γ-limit, we can suppose $\nu = \rho$ and, with a diagonalization argument, it will be sufficient to prove, $\forall \mu \in \mathcal{M}_+(\overline{\Omega})$,

$$\varphi^+(\nu, \mu) \leq C_d\int_\Omega \frac{\rho(x)}{(\mu_a(x))^{1/d}}\, dx. \tag{1.30}$$

Assuming (1.30) for $\mu = u(x)dx$ with $u(x) \in L^1(\Omega)$, fix a general $\mu \in \mathcal{P}_+(\Omega)$. Using a representation formula for integral defined on measures (see [12]) and the l.s.c. of ρ, there exists a sequence $\{u_h\}_h \in L^1(\Omega)$, $u_h \stackrel{*}{\rightharpoonup} \mu$ such that

$$\lim_{h\to\infty}\int_\Omega \frac{\rho(x)}{(u_h(x))^{p/d}}\, dx = \int_\Omega \frac{\rho(x)}{(\mu_a(x))^{p/d}}\, dx.$$

To obtain (1.30) it is sufficient to observe that $\liminf\limits_{h\to\infty}\varphi^+(\rho, u_h\mathcal{L}^d\llcorner\Omega) \geq \varphi^+(\rho, \mu)$.

It remains to prove (1.30) in the case $\mu = u(x)dx$ with $u(x) \in L^1(\Omega)$. To achieve this case, fix $\lambda > 0$, consider a partition of Ω in little cubes of dimension $\varepsilon' = \lambda\varepsilon$ and a discretization of functions ρ and u on the considered partition. Considering a discretization of $(\nu_\varepsilon, \mu_\varepsilon)$, it is possible to obtain the $\Gamma - \limsup$ inequality passing to the limit as $\lambda \to 0$. Here we do not give all details but we refer to original work [11]. $\square$

The constant $C_{p,d}$ in (1.26) can be explicitly computed only in few cases. In particular, it is known that in case $d = 2$ we have:

$$C_{p,2} = \int_H |x|^p \, dx$$

where H is the regular hexagon of unitary area centered in the origin. For instance, one has $C_{1,2} \sim 0.377$ and $C_{2,2} \sim 0.16$. A plot of the values of $C_{p,2}$ for $p \in [0, 2]$ is given in Figure 1.2.

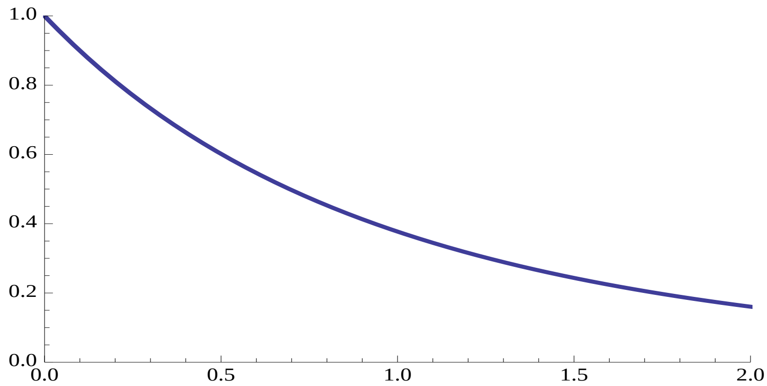


Figure 1.2: Plot of the value of $C_{p,2}$ for $p \in [0, 2]$.

As a conclusion of this paragraph, observe that in the literature the optimal location problem was intensively studied and a lot of generalizations have been done. For example, in [16], the analysis had a different point of view. Since problems of optimal location would model as far as possible real situations, another question could be interesting: if we are planning economic activities, it is reasonable that after some years the situation is different and it is necessary to re-plan or to change something. We can think to a long-term planner that optimizes by thinking to a large time horizon, and to a short-term planner that has in mind to modify his strategy step by step. With the short-term point of view in mind, Problem 1.45 can be compare with

Problem 1.52 (Short-term). *Let Ω be a region with density ρ. The density of facilities will be modeled by a discrete set $\Sigma_k \subset \Omega$ such that $\sharp\Sigma \le N$ at time $k \in \mathbb{N}$. Under the additional constraint $\Sigma_{k-1} \subset \Sigma_k \subset \Omega$, minimize $\varphi_N(\rho, \Omega)$.*

The asymptotic analysis for the short-term case is similar to the long-term and denoting by s_n the minimal value, the asymptotic estimate is $s_n \tilde{n}^{-1/d}$.

1.5 Irrigation problem

In this paragraph we study the so-called *irrigation problem* and we refer in particular to Buttazzo, Oudet, Stepanov [25] for one of the first works on the subject, and to Mosconi, Tilli [53] for an asymptotic result. Many other works face with this problem as, for example, [30], [31], [26].

Let Ω be a bounded connected open set with Lipschitz boundary in $\mathbb{R}^d$, $d \geq 2$ and let $S(\Omega)$ be the class of all compact connected sets $\Sigma \subset \overline{\Omega}$ of finite one-dimensional Hausdorff measure $\mathcal{H}^1(\Sigma)$. The irrigation problem consists in minimizing the integral of the distance function to Σ, that is $\int_\Omega \operatorname{dist}(x,\Sigma)\,dx$, among all $\Sigma \in S(\Omega)$ of prescribed length $\mathcal{H}^1(\Sigma) = l$.

This problem has a strong connection with mass transportation. Consider the Monge-Kantorovich problem with cost $c(x,y)$ of the form $c(|x-y|)$ in Ω. Observe that if the domain is not convex, then the Euclidean distance has to be replaced by the geodesic distance $\operatorname{dist}_\Omega(x,y)$ on Ω, given by:

$$\operatorname{dist}_\Omega(x,y) = \inf\left\{\int_0^1 |\gamma^{'}(t)\,dt| \;:\; \gamma \in \operatorname{Lip}([0,1],\bar{\Omega}), \gamma(0)=x, \gamma(1)=y\right\}$$

Consider also a so-called *Dirichlet region* $\Sigma \in S(\Omega)$, which represent a zone where the cost of trasportation vanishes. Presence of Σ modifies the "distance" from two points in the transport in the sense that, if it is possible, it is better pass throught Σ. For a fixed set Σ, the semi-distance

$$\operatorname{dist}_{\Omega,\Sigma}(x,y) = \inf\{\operatorname{dist}_\Omega(x,y) \wedge (\operatorname{dist}_\Omega(x,\xi_1) + \operatorname{dist}_\Omega(\xi_2,y)) \;:\; \xi_1,\xi_2 \in \Sigma\}$$

models cost distance in the "free of charge zone". We generalize the notion of a transport plan for the case of the presence of a nonempty Dirichlet region $\Sigma \subset \overline{\Omega}$, saying that a Borel measure π over $\overline{\Omega} \times \overline{\Omega}$ is a transport plan of ρ_0 into ρ_1, if

$$(\pi_1)_\sharp \pi - (\pi_2)_\sharp \pi = \rho_0 - \rho_1 \quad \text{on } \overline{\Omega} \setminus \Sigma.$$

Consider then the quantity:

$$MK(\Sigma) = \min\left\{\int_\Omega c(\operatorname{dist}_{\Omega,\Sigma}(x,y))\,d\pi(x,y) \;:\; \pi \in \Pi(\rho_0,\rho_1)\right\} \tag{1.31}$$

that is the total cost of transport associated to a given set $\Sigma \in S(\Omega)$. Note that in this problem, in view of the generalized definition of a transport plan, it is not necessary to assume that ρ_0 and ρ_1 have the same mass.

In general case, the optimization problem we faced with is to find the best Dirichlet region $\Sigma \subset \Omega$ subject to certain constraints, that is, optimizing (1.31). So, the irrigation problem can be formulated in the following form:

$$\min\{MK(\Sigma) \;:\; \Sigma \subset \Omega \text{ closed}, \; \mathcal{H}^1(\Sigma) \leq l\} \tag{1.32}$$

From now, we will make some simplification hypothesis in (1.32). For the moment, suppose the cost function c to be the identity, we postpone the other cases to the end of the section. Moreover, we concentrate on the simpler case when Ω is convex and $\rho_0 = \rho_0(x)dx$ and $\rho_1 = 0$. These assumptions reduce (1.32) to minimize the average distance functional

$$\min\left\{\int_\Omega \mathrm{dist}(x,\Sigma)\rho_0(x)\,dx\ :\ \Sigma\subset\Omega\,\text{closed}, \mathcal{H}^1(\Sigma)\le l\right\}$$

As we said, the minimum in (1.31) is computed in the class of the closed subsets of Ω with finite length, that is the sets Σ with $\mathcal{H}^1(\Sigma) \le l$. This constraint seems very natural, if one thinks to a transport problem with the presence of finite resources. Observe also that in the limit case $\mathcal{H}^1(\Sigma) = 0$, the above problem is precisely the location of finite number of points, as seen in the previous section.

It is not difficult to prove the existence of solutions for (1.32). To achieve existence, some preliminary results are useful, whose proof can be found in [6].

Theorem 1.53 (Golab)**.** *If $\{\Sigma_n\}_n$ is a sequence of connected closed subsets of Ω which Hausdorff converges to a set Σ, then*

$$\mathcal{H}^1(\Sigma)\le\liminf_{n\to+\infty}\mathcal{H}^1(\Sigma_n)$$

We remark that the connectedness assumption in the Golab theorem is necessary, otherwise it is easy to find counterexamples.

Theorem 1.54 (Blaschke)**.** *Given a metric space E we denote by $\mathcal{C}_E$ the family of non-empty closed subset of E. If E is compact, then $\mathcal{C}_E$ wiht the Hausdorff distance is a compact metric space.*

Then, (1.32) is well posed:

Theorem 1.55. *If the function c is continuous,* (1.32) *admits a solution.*

Proof. Let Σ_n be a minimizing sequence for (1.32) satisfying the condition $\mathcal{H}^1(\Sigma_n) \le l$ for all $n \in \mathbb{N}$. Applying the Blaschke compactness theorem, there exists a closed connected Σ such that, up to subsequences, $\Sigma_n \to \Sigma$ in the sense of Hausdorff. Furthermore, applying the Golab theorem, it follows $\mathcal{H}^1(\Sigma) \le l$. Since Hausdorff convergence implies $\mathrm{dist}_\Omega(x,\Sigma_n) \to \mathrm{dist}_\Omega(x,\Sigma)$ for all $x \in \Omega$, we have

$$\mathrm{dist}_{\Omega,\Sigma_n}(x,y)\to\mathrm{dist}_{\Omega,\Sigma}(x,y)$$

for all $(x,y) \in \Omega\times\Omega$. Using the fact that $\mathrm{dist}_{\Omega,\Sigma_n}$ are Lipschitz continuous with the same constant, the convergence is uniform. Now, denoting with π_h the respective optimal plan in (1.31), up to subsequences, $\pi_h \rightharpoonup \pi$ in the weak* convergence of measure, with $\pi \in \Pi(\rho_0,\rho_1)$. Then

$$MK(\Sigma)\le\int c(\mathrm{dist}_{\Omega,\Sigma}(x,y))\,d\pi(x,y)\le\lim_h\int c(\mathrm{dist}_{\Omega,\Sigma_h}(x,y))\,d\pi_h(x,y)$$

which shows that Σ is a minimizer. $\square$

Besides existence, some considerations on the qualitative and regularity properties of an optimal set Σ_0 can be done. Precisely, the main facts that can be proved, at least in dimension $d = 2$, are:

- an optimal set Σ_0 cannot contain a cross (i.e. the union of two transversal curves);
- in the bifurcation points of Σ_0, curves make angles of $120°$;
- the optimal set does not form closed loops in Ω, in dimension 2, it is equivalent to say that $\mathbb{R}^2 \setminus \Sigma$ is connected;
- the optimal solution cannot touch the boundary of Ω.

We briefly analyze and comment the properties listed above. The proof of the first point proceeds by contradiction: centering the cross in the origin and making a Steiner construction with the four points resulting by the intersection between the cross and the border of a ball of sufficiently small radius, contradicts the optimality. A similar argument can be used to prove the second point if the optimal set is sufficiently smooth regular. Regarding the third point, it is true that the set obtained taking off a ball of small radius from the optimal set Σ_0 is disconnected and this implies that $\Omega \setminus \Sigma_0$ cannot contain loops. Finally, it can be proved that the intersection $\Sigma_0 \cap \partial\Omega$ cannot have positive $\mathcal{H}^1$ measure. A stronger result holds for small length:

Theorem 1.56. *There exist $l_0 > 0$ and $d_0 > 0$ depending on Ω and d such that for all $l < l_0$ the optimal set Σ_0 satisfies $\operatorname{dist}(\Sigma_0, \partial\Omega) > d_0$. In particular $\Sigma_0 \cap \partial\Omega = \emptyset$.*

Denoting by Σ_l the optimal set for (1.32) for fixed length l, it is interesting to study the asymptotic behavior of Σ_l as $l \to +\infty$, analogously as in the case of location of N points. This analysis has been done in [53] using the theory of Γ-convergence. Also in this case, it is convenient to associate at every Σ_l the probability measure $\mu = \frac{1}{l}\mathcal{H}^1 \llcorner \Sigma_l$ (that is, the normalized Hausdorff measure restricted to Σ_l). We denote by $F_l(\mu)$ the functionals

$$F_l(\mu) = \begin{cases} l^{\frac{p}{d-1}} \int_\Omega \operatorname{dist}(x, \Sigma)^p \rho(x)\, dx & \text{if } \mu = \frac{1}{l}\mathcal{H}^1 \llcorner \Sigma_l \\ +\infty & \text{otherwise} \end{cases} \tag{1.33}$$

In (1.33), the term $l^{\frac{p}{d-1}}$ is a normalization that prevents the degeneration of the functional. The connection to mass transportation is the same as in location problem:

$$\int_\Omega \big(\operatorname{dist}(x, \Sigma_l)\big)^p \rho(x)\, dx = \inf\left\{W_p^p(\rho, \nu) : \nu \in \mathcal{M}_+(\overline{\Omega})\,,\ \operatorname{spt}(\nu) \subset \Sigma_l\right\}.$$

The asymptotic analysis performed in [53] shows:

Theorem 1.57. *The sequence $F_l(\mu)$, defined in (1.33), Γ-converges, with respect to the weak* topology on $\mathcal{M}(\Omega)$, to the functional*

$$F_\infty(\mu) = C_{p,d} \int_\Omega \frac{\rho(x)}{\mu_a(x)^{\frac{p}{d-1}}}\, dx. \tag{1.34}$$

Here μ_a is the Radon-Nykodym derivative with respect to Lebesgue measure and $C_{p,d}$ is a constant depending only on p and d. The proof of Theorem 1.57 uses techniques quite different from the analogous asymptotic theorem of the previous section. We give here a sketch of the proof that does not stress on the technical issues necessary for a detailed argument.

Sketch of the proof of Theorem 1.57. To achieve Γ-limit, as usual, we prove $\Gamma - \liminf$ and $\Gamma - \limsup$ inequalities, in the next two steps.

Step 1: $\Gamma - \liminf$ *inequality:* we have to prove that for all $\mu \in \mathcal{P}(\bar{\Omega})$, $\mu_n \overset{*}{\rightharpoonup} \mu$ and $l_n \to \infty$

$$\Gamma - \liminf F_{l_n}(\mu_n) \geq F_\infty(\mu)$$

where we can assume $\mu_n = \frac{1}{l_n}\mathcal{H}^1 \llcorner \Sigma_n$ and $\mathcal{H}^1(\Sigma_n) = l_n$. First we prove

$$\liminf_n l_n \int_Q \rho(x) \mathrm{dist}(x, \Sigma_n)^p \geq \int C_{p,d} \int_Q \frac{\rho}{\mu_a^{\frac{p}{d-1}}}$$

for every cube $Q \subset \Omega$. To reach this inequality we argue by contradiction. Then we decompose Ω with a finite family of disjoint cubes $\{Q_j\}$ and using previous result on each cube, we conclude.

Step 2: $\Gamma - \limsup$ *inequality:* we have to prove that for all $\mu \in \mathcal{P}(\bar{\Omega})$ there exists a sequence $\mu_n \overset{*}{\rightharpoonup} \mu$ and $l_n \to \infty$ such that

$$\Gamma - \limsup F_{l_n}(\mu_n) \leq F_\infty(\mu)$$

First the claim is proved under the assumption that μ is absolutely continuous, positive and piecewise constant using technical preliminaries that we have omitted here. Taking a general $\mu \in \mathcal{P}(\bar{\Omega})$, arguing by density, there exist $\mu_k \in \mathcal{P}(\bar{\Omega})$ such that $\mu_k \overset{*}{\rightharpoonup} \mu$ and $F_\infty(\mu_k) \to F_\infty(\mu)$. Then

$$\Gamma - \limsup(\mu_k) \leq \liminf \left(\Gamma - \limsup(\mu_k)\right) \leq \liminf F_\kappa(\mu_k) \leq F_\infty(\mu),$$

that concludes for a general μ.

Step 1 and Step 2 prove the Γ-limit. □

1.6 Transport with congestion

In the field of traffic modelling, congestion effects often play an important role. In fact, as in the daily experience, when a large number of vehicles have to go from one location to another, the choice of the "best" road takes into account the total flow of vehicles in each road. In this case, we can think to the "cost" for every vehicle in terms of time spent to reach the destination that is of course increasingly dependent on the total number of vehicles present on some road.

The first models in this direction were developed in 50's by Wardrop [58] and Beckmann [7]. Wardrop's approach was based on two considerations: all paths connecting two points which are actually followed by some vehicles must provide the same traveling time (which depends on their length as well as congestion) and all other paths provide much time. This means that we use only the geodesic paths for a metric that is induced by the use of the paths themselves. This gives an equilibrium problem, that can be seen as a fixed point and that has a variational characterization discovered by Beckmann (see [8]).

In the classical Monge-Kantorovich transportation problem, the cost of transporting a unit of mass from x to y, say $c(x, y)$, does not depend on the path followed by the mass from x to y. In order to consider also congestion effects, the total transportation cost will depend on the transportation plan between initial and final distributions and on the way used by travelers.

The results and models describes in this paragraph are principally based on [32–34].

1.6.1 A continuous model based on Wardrop equilibria

The problem setting is a region Ω, say modelling a city, which is an open bounded subset of $\mathbb{R}^d$ and two probability measure f^+ and f^- representing the distribution of residents and services in Ω. As in the independent-path model, the set of transportation plans with marginals f^+ and f^- is denoted by $\Pi(f^+, f^-)$.

The introduction of congestions in the model requests to consider some notations regarding path space and curves. In the following we denote with $C^{x,y}$ the subset of $C = W^{1,\infty}([0, 1], \overline{\Omega})$, endowed with uniform topology, of continuous path from x to y:

$$C^{x,y} = \{\tau \in C \,:\, \tau(0) = x \,,\, \tau(1) = y\}.$$

For every $\tau \in C$ we denote with $\tilde{\tau}$ the constant speed reparametrization of $\tau \in C$, i.e. $\|\dot{\tilde{\tau}}(t)\| = \lambda(\tau) = \int_0^1 \|\dot{\tau}(s)|ds$. Recall that the classical form of Monge-Kantorovich problem for a given cost function c is given by:

$$\inf \left\{ \int_{\overline{\Omega}\times\overline{\Omega}} c(x, y)\, d\gamma(x, y) \;:\; \gamma \in \Pi(f^+, f^-) \right\} \tag{1.35}$$

Since we want to take into account congestion effects, the transportation cost will depend on the transportation plan $\gamma \in \Pi(f^+, f^-)$ and on the possible paths $\tau \in C^{x,y}$ followed by the mass transported. We introduce a probability measure $p^{x,y}$ on $C^{x,y}$ with the following meaning: $p^{x,y}(\Sigma)$ will be the proportion of travelers from x to y using a path $\tau \in \Sigma \subset C^{x,y}$. Then, in order to put together these two contributions in optimization problem, we give the following:

Definition 1.58. Given a plan $\gamma \in \Pi(f^+, f^-)$ and a family of probability measures $(p^{x,y})$ on C such that $p^{x,y}(C^{x,y}) = 1$, we call transportation strategy the pair (γ, p).

We introduce the notation, for all $\varphi \in C^0(\overline{\Omega}, \mathbb{R}^+)$:

$$L_\varphi(\tau) = \int_0^1 \varphi(\tau(t))|\dot{\tau}(t)|\, dt.$$

For a given transportation strategy (γ, p), it is possible to define the probability measure $I_{\gamma,p} \in \mathcal{M}_+(\overline{\Omega})$ of total traffic intensity. For every $\varphi \in C^0(\overline{\Omega}, \mathbb{R}^+)$, $I_{\gamma,p}$ is defined by

$$\int_{\overline{\Omega}} \varphi(x)\, dI_{\gamma,p}(x) = \int_{\overline{\Omega}\times\overline{\Omega}} \Big(\int_{C^{x,y}} \Big(\int_0^1 \varphi(\tau(t))|\dot{\tau}(t)|\, dt \Big) dp^{x,y}(\tau) \Big) d\gamma(x,y).$$

The finiteness of traffic intensity $I_{\gamma,p}$ is guaranteed by Lemma 2.7 in [17], under some additional hypothesis detailed later.

We consider the probability measure $Q_{\gamma,p} = p^{x,y} \otimes \gamma \in \mathcal{M}^1_+(C)$ defined by

$$\int_C F(\tau)\, dQ_{\gamma,p}(\tau) = \int_{\overline{\Omega}\times\overline{\Omega}} \Big(\int_{C^{x,y}} F(\tau)\, dp^{x,y}(\tau) \Big) d\gamma(x,y) \qquad \forall F \in C^0(C, \mathbb{R})$$

that represents the total number of travelers using a path $\tau \in \Sigma$, for a given the global transportation strategy (γ, p).

Observe that if we set $Q = Q_{\gamma,p} \in \mathcal{M}^1_+(C)$, then $I_{\gamma,p}$ depends only on Q and we can write $I_{\gamma,p} = i_Q$, where i_Q is defined by:

$$\int_{\overline{\Omega}} \varphi(x)\, di_Q(x) = \int_C L_{\varphi(\tau)}\, dQ(\tau), \qquad \forall \varphi \in C^0(\overline{\Omega}, \mathbb{R}^+).$$

The measure i_Q, the traffic intensity associated to Q, is a generalization of the notion of transport density: for a region A, $i_Q(A)$ is the total traffic in A induced by Q.

We can use the concept of transportation strategy introduced before to define a model in which traffic congestion is involved, considering the following quantity as cost for unit of mass transported from x to y:

$$c_{\gamma,p}(x,y) = \int_{C^{x,y}} L_{G_{I_{\gamma,p}}(\tau)}\, dp^{x,y}(\tau)$$

where $G_{I_{\gamma,p}}$ is a nonnegative function which depends on the traffic intensity $I_{\gamma,p}$. Observe that, in the definition above, $G_{I_{\gamma,p}}$ has to be continuous.

Now, we can write a traffic congestion model in Monge-Kantorovich's form, with the new path-dependent cost function:

$$\inf \Big\{ \int_{\overline{\Omega}\times\overline{\Omega}} c_{\gamma,p}(x,y)\, d\gamma(x,y) \ : \ (\gamma,p) \text{ transp. strategy} \Big\} \tag{1.36}$$

Once observed the link between the traffic intensity $I_{\gamma,p}$ and i_Q, it will be convenient to formulate the optimization problem 1.36 in terms of $Q = Q_{\gamma,p}$ rather than in terms of the optimization strategy (γ, p). A simple lemma in [32] is useful to the scope:

Lemma 1.59. *Let us define*

$$\mathcal{Q}(f^+, f^-) = \{Q_{\gamma,p} \,:\, (\gamma,p) \text{ transportation strategy}\}.$$

Then

$$\mathcal{Q}(f^+, f^-) = \{Q \in \mathcal{M}^1_+(C) \,:\, (e_0\sharp)Q = f^+,\ (e_1\sharp)Q = f^-\}$$

where e_i is the projection on the $i-th$ component.

Setting $Q = Q_{\gamma,p}$ and using formally the definitions above, we can rewrite:

$$\int_{\overline{\Omega}\times\overline{\Omega}} c_{\gamma,p}(x,y)\,d\gamma(x,y) = \int_C L_{G_{i_Q}}(\tau)\,dQ(\tau) = \int_{\overline{\Omega}} G_{i_Q}(x)\,di_Q(x),$$

where here G can assume a more general form than the continuous one above. Thanks to Lemma 1.59, we can reformulate (1.36) in terms of Q only:

$$\inf\left\{\int_{\overline{\Omega}} G_{i_Q}(x)\,di_Q(x)\ :\ Q\in\mathcal{Q}(f^+,f^-)\right\}. \tag{1.37}$$

Assume now that G_i has the following form

$$G_i(x) = g\left(\frac{di}{d\mathcal{L}^2}(x)\right),$$

where $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function such that the function $H : \mathbb{R}^+ \to \mathbb{R}^+$, given by $H(z) = zg(z)$, is convex and superlinear (i.e., $\lim_{z\to+\infty}\frac{H(z)}{|z|} \to +\infty$).

The optimization problem then becomes

$$\inf\left\{\begin{array}{ll} \int_{\overline{\Omega}} H(i_Q(x))\,dx & \text{if } i_Q \ll \mathcal{L}^2 \\ +\infty & \text{otherwise} \end{array}\quad :\ \text{for}\ \ Q\in\mathcal{Q}(f^+,f^-)\right\}. \tag{1.38}$$

We say that a transportation strategy (γ,p) is optimal if the associated $Q_{\gamma,p}$ solves (1.38).

We briefly discuss now on the regularity issues involved to prove an existence theorem and to make the following definition rigorous. In any case, for the details we refer to the original work [32] and successive [33].

We assume for function H to be convex and nondecreasing on $\mathbb{R}^+$, with $H(0) = 0$. On the growth and regularity of H, we assume:

- there exist $q > 1$ and positive constants a and b such that

$$az^q \le H(z) \le b(z^q + 1) \qquad \forall\ z\in\mathbb{R}^+;$$

- H is differentiable on $\mathbb{R}^+$ and there exists a positive constant d such that

$$0 \le H^{'}(z) \le d(z^{q-1}+1) \qquad \forall\ z\in\mathbb{R}^+.$$

Moreover, the admissible Q satisfy $i_Q \ll \mathcal{L}^2$, $\frac{di_Q}{d\mathcal{L}^2} \in L^p$ and belong to the set

$$\mathcal{Q}^p(f^+,f^-) = \{Q\in\mathcal{Q}(f^+,f^-)\ :\ i_Q\in L^p\}.$$

The assumption $\mathcal{Q}^p(f^+,f^-) \neq$ can be easily proved assuming f^+ and f^- to have finite support. The existence of a $Q\in\mathcal{Q}(f^+,f^-)$ such that $i_Q\in L^p$ can be proved also assuming that f^+ and f^- are in L^p, a hypothesis of a non trivial regularity result of De Pascale and Pratelli in [39] and [40].

Observe now that, if H satisfies these conditions, then we can rewrite (1.38) as:

$$\inf_{Q \in \mathcal{Q}^p(f^+,f^-)} \int_\Omega H(i_Q(x))\, dx. \tag{1.39}$$

Using the convexity of H, it is not difficult to see that an optimal $\overline{Q}$ solving (1.38) is characterized by:

$$\overline{Q} \text{ solves } (1.39) \;\Leftrightarrow\; \int_\Omega \overline{\xi} i_{\overline{Q}} = \inf\{\int_\Omega \overline{\xi} i_Q \;:\; Q \in \mathcal{Q}^q(f^+, f^-)\} \tag{1.40}$$

where $\overline{\xi} = H^{'}(i_{\overline{Q}})$.

Characterization (1.40) above, after some formal manipulations can be expressed in terms of transportation strategy. Let us call $(\overline{\gamma}, \overline{p})$ the optimal transportation strategy associated to $\overline{Q} = Q_{\overline{\gamma},\overline{p}}$ solving (1.40). Then formally:

$$\begin{aligned}
\int_\Omega \overline{\xi} i_{\overline{Q}} = \int_C L_{\overline{\xi}}(\tau)\, dQ(\tau) &= \int_{\overline{\Omega}\times\overline{\Omega}} \Big(\int_{C^{x,y}} L_{\overline{\xi}}(\tau) d\overline{p}^{x,y}(\tau) \Big) d\overline{\gamma}(x,y) \\
&= \inf_{(\gamma,p)} \int_{\overline{\Omega}\times\overline{\Omega}} \Big(\int_{C^{x,y}} L_{\overline{\xi}}(\tau)\, dp^{x,y}(\tau) \Big) d\gamma(x,y) \\
= \inf_{\gamma \in \Pi(f^+,f^-)} \int_{\overline{\Omega}\times\overline{\Omega}} & \Big(\inf_{p \in \mathcal{M}^1(C^{x,y})} \int_{C^{x,y}} L_{\overline{\xi}}(\tau)\, dp^{x,y}(\tau) \Big) d\gamma(x,y) \\
&= \inf_{\gamma \in \Pi(f^+,f^-)} \int_{\overline{\Omega}\times\overline{\Omega}} \Big(\inf_{\tau \in C^{x,y}} L_{\overline{\xi}}(\tau) \Big)\, d\gamma(x,y).
\end{aligned} \tag{1.41}$$

Defining

$$c_{\overline{\xi}}(x,y) = \inf_{\tau \in C^{x,y}} L_{\overline{\xi}}(\tau), \tag{1.42}$$

it follows

$$\int_{\overline{\Omega}\times\overline{\Omega}} c_{\overline{\xi}}(x,y)\, d\overline{\gamma}(x,y) \leq \int_C L_{\overline{\xi}}\, d\overline{Q} = \inf_{\gamma \in \Pi(f^+,f^-)} c_{\overline{\xi}}(x,y) d\gamma(x,y)$$

and then $\overline{\gamma}$ solves

$$\inf_{\gamma \in \Pi(f^+,f^-)} \int_{\overline{\Omega}\times\overline{\Omega}} c_\xi(x,y)\, d\gamma(x,y). \tag{1.43}$$

Moreover we have:

$$\int_C L_{\overline{\xi}}(\tau)\, d\overline{Q}(\tau) = \int_{\overline{\Omega}\times\overline{\Omega}} c_{\overline{\xi}}(x,y)\, d\overline{\gamma}(x,y) = \int_C c_{\overline{\xi}}(\tau(0), \tau(1))\, d\overline{Q}(\tau)$$

that, since $L_{\overline{\xi}}(\tau) \geq c_{\overline{\xi}}(\tau(0), \tau(1))$, lead to

$$L_{\overline{\xi}}(\tau) = c_{\overline{\xi}}(\tau(0), \tau(1)).$$

This discussion permits us to give the following definition:

Definition 1.60. A Wardrop equilibrium is a $Q \in \mathcal{Q}(f^+, f^-)$ such that

$$Q\Big(\{\tau \ : \ L_{\overline{\xi}}(\tau) \geq c_{\overline{\xi}}(\tau(0), \tau(1)) \ \}\Big) = 1,$$

$\xi = H^{'}(i_Q)$ and $(e_0, e_1)_\sharp Q$ solves (1.43).

In this discussion we have performed all manipulations in a formal way, without care on the regularity necessary for rigorous proofs.

Under regularity assumptions on function H and using some technical lemmata, the main result in [32] can be summarized in:

Theorem 1.61. *Problem* (1.43) *admits a solution. Moreover a Wardrop equilibrium exists since* $\overline{Q} \in \mathcal{Q}(f^+, f^-)$ *solves* (1.43) *if and only if* $\overline{Q}$ *is a Wardrop equilibria.*

1.6.2 A more tractable form

We will see that Wardrop equilibria have a more tractable form, preferable to the definition. It is a variational form equivalent to a minimal flow problem, following a model of Beckmann type (see [7]). Moreover, using a Moser type approach, it reduces to solve a non linear elliptic PDE.

For every $Q \in \mathcal{Q}(f^+, f^-)$, we define the vector field σ_Q:

$$\int_{\overline{\Omega}} X(x)\sigma_Q(x)\, dx = \int_{C([0,1],\overline{\Omega})} \Big(\int_0^1 X(\tau(t))\dot{\tau}(t)dt\Big)\, dQ(\tau) \qquad \forall\ X \in C(\overline{\Omega}, \mathbb{R}^d)$$

Taking X in gradient form, say $X = \nabla u$, the previous equation reduces to

$$\int_{\overline{\Omega}} \nabla u \sigma_Q = \int_{C([0,1],\overline{\Omega})} (u(\sigma(0)) - u(\sigma(1)))dQ(\gamma) = \int_{\overline{\Omega}} u(f^+ - f^-),$$

that is

$$\nabla \cdot \sigma = f^+ - f^-.$$

It is easy to check that $|\sigma_Q| \leq i_Q$.

From now, we refer to the following problems as scalar problem and problem of Beckmann type respectively:

$$\inf_{Q\in\mathcal{Q}(f^+,f^-)} \int_\Omega H(i_Q(x))\, dx \tag{1.44}$$

$$\inf_{\nabla\cdot\sigma=f^+-f^-} \int_\Omega H(|\sigma(x)|)\, dx \tag{1.45}$$

First, since H is an increasing function, the value of the scalar problem (1.44) is larger than the value of (1.45). Conversely, taking σ minimizing (1.45), if we construct $Q \in \mathcal{Q}(f^+, f^-)$ such that $i_Q = |\sigma|$, then Q solves also (1.44). In order to construct such a Q, we will use a construction following a Moser approach as in [36] and [54]. Once again we will use a formal argument: let σ be smooth, f^+, f^- be absolutely continuous and sufficiently nice and consider:

$$\begin{cases} \dot{X}(t,x) = \frac{\sigma(X(t,x))}{(1-t)f^+(X(t,x))+tf^-(X(t,x))} \\ X(0,x) = x \end{cases}$$

Define Q by

$$Q = \delta_{X(\cdot,x)} \otimes f^+$$

and consider

$$v(t,x) = \frac{\sigma(x)}{f^t(x)}$$

where $f^t(x) = (1-t)f^+ + tf^-$. By construction f^t solves the continuity equation:

$$\partial_t f^t + \nabla \cdot (f^t v) = 0$$

and $e_{0\sharp}\overline{Q} = f^+$. Notice also that, by uniqueness of solution of continuity equation (recall Theorem 1.29), we have $X(t,\cdot)_\sharp f^+ = f^t$ and then $X(1,\cdot) = f^-$ that implies $e_{1\sharp}\overline{Q} = f^-$. So $\overline{Q} \in \mathcal{Q}(f^+, f^-)$. Moreover, for every test function φ:

$$\begin{aligned} \int_\Omega \varphi di_{\overline{Q}} &= \int_\Omega \int_0^1 \varphi(X(t,x))|v(t,X(t,x))|\,dt\,df^+(x) \\ &= \int_0^1 \int_\Omega \varphi(x)|v(t,x)|f^t(x)\,dx\,dt = \int_\Omega \varphi(x)|\sigma(x)|\,dx \end{aligned} \tag{1.46}$$

that is $i_{\overline{Q}} = |\sigma|$ and then $\overline{Q}$ is optimal. It is clear that, again as in the previous paragraph, this argument works under suitable regularity of σ.

Chapter 2

Optimal location problems with routing cost

This chapter is based on the paper [23], written in collaboration with Giuseppe Buttazzo and Fabrizio Oliviero.

Locating a given number of points in a region, in order to fulfill a given optimization criterion, is a widely studied problem, and a large number of references on the field is available (see Chapter 1), with many of them devoted to several applications to economy, urban planning, electronics, communication systems.

As said in the preliminar, location problem can be describes as follow: a given bounded and closed region $\Omega \subset \mathbb{R}^d$ is considered, together with a given nonnegative function $\rho : \Omega \to \mathbb{R}^+$ which represents the distribution density of resources in Ω. The goal is to concentrate the resources into a given number N of points $x_1, \dots, x_N$ in an optimal way; assuming that the cost to move a unit mass from x to y is proportional to a suitable power $|x-y|^p$ of the distance, allows us to write the optimization problem as

$$\min\Big\{\int_\Omega \big(\operatorname{dist}(x,\Sigma)\big)^p \rho(x)\,dx \ :\ \Sigma \subset \Omega,\ \#\Sigma = N\Big\}. \tag{2.1}$$

Here Σ is the unknown set of N points to be determined, $\#\Sigma$ is the cardinality of Σ, and $\operatorname{dist}(x,\Sigma)$ is the distance function

$$\operatorname{dist}(x,\Sigma) = \min\big\{|x-y| \ :\ y \in \Sigma\big\}.$$

We already presented problems of the form (2.1) for which the existence of an optimal configuration is straightforward. On the contrary, in spite of its simplicity, the numerical computation of an optimal set Σ, when the number N is large, presents big difficulties, essentially due to the fact that the cost in (2.1) admits a huge number of local minima, which prevents the use of fast gradient methods and makes necessary the implementation of global optimization methods that are in general much slower.

The asymptotic analysis, as $N \to +\infty$, has been performed (see for instance Chapter 1, [11, 28] and references therein) for problem (2.1) and gives important informations about the limit density of optimal points $x_i \in \Sigma$. In Section 1.4 we have recalled the main results about this subject.

The problem we deal with in this chapter is concerned with the optimal location of a given number N of airports in a region Ω. The airports collect the resources that are distributed in Ω with a known density $\rho(x)$; moreover, the goods travel between airports on a point-to-point basis, which provides an additional cost, called *routing cost*. The complete problem that comes out by adding location and routing costs will be discussed in Section 2.1. When the number N of airports is large, we replace the location cost by its asymptotic counterpart and we discuss the corresponding first order necessary conditions of optimality. Finally, in Section 2.3 some numerical simulations are shown.

The density of population will be modeled by a given Borel probability measure ρ in Ω and the configuration of facilities will be modeled by a set $\Sigma \subset \Omega$ consisting of at most n points. The simplest way to measure the optimality of a distribution of facilities, namely to find the optimal Σ is to consider the average distance that the people have to cover to reach the nearest facility and minimize it. As described in the preliminaries, if we suppose that the cost to move a unit of mass along the distance l is Al^p, the total cost of transport is given by

$$A \int_\Omega \big(\mathrm{dist}(x, \Sigma)\big)^p \rho(x)\, dx \tag{2.2}$$

where Σ is the unknown set of N points to be determined. The most efficient choice of the positions is then obtained by solving the minimization problem (2.1).

When the number N tends to $+\infty$, instead of looking at the precise positions x_i in Ω of the airports, one will simply target at determining the limit density μ of the points x_i. In order to do it, we identify each set $\Sigma \subset \Omega$ of N points with the measure

$$\mu_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}. \tag{2.3}$$

If we assume that, up to a normalization, the density ρ has a unitary total mass, the location cost (2.2) is proportional to the p-th power of the Wasserstein distance between the probabilities $\rho\, dx$ and μ_N. The asymptotic analysis of the cost above has been performed (see for instance [11, 28] and references therein) and we summarize here below the available results that, to be correctly stated, require the use of the Γ-convergence, a variational theory developed by De Giorgi and his school starting from the seventies.

When $N \to +\infty$ the cost (2.2) is asymptotically equivalent to the limit cost

$$A\, C_{p,d}\, N^{-p/d} \int_\Omega \frac{\rho(x)}{\big(\mu(x)\big)^{p/d}}\, dx \tag{2.4}$$

expressed in terms of the limit density μ of points, where $C_{p,d}$ is a constant depending on the exponent p and on the dimension d. It has to be noticed that in the integral above only the absolutely continuous part of μ has to be taken into account, neglecting the singular part. The constant $C_{p,d}$ can be explicitly computed only in few cases, as explained in Section 1.4.

On the other hand, if we are interested not only in the location x_i of the i-th airport but also in the mass m_i that is there concentrated, instead of the measures μ_N above we have to consider the measures

$$\nu_N = \sum_{i=1}^{N} m_i \delta_{x_i}$$

and the optimization problem is written in terms of the p-Wasserstein distance as

$$\min\left\{W_p^p(\rho,\nu)\ :\ \#(\operatorname{spt}\nu) = N\right\}. \tag{2.5}$$

We notice that, without the normalization $\int \rho\, dx = 1$, passing to the probability $\rho(x)/\int \rho\, dx$, the optimization problems (2.4) and (2.5) remain of the same form.

2.1 A new model with routing costs

In this subsection we assume that the mass m_i concentrated at the point x_i is dispatched to the remaining points x_j proportionally to the masses m_j; moreover, we assume that the cost to move a unit mass from a point x to a point y is proportional to $|x-y|^q$ for a suitable power q. Therefore, the cost to move the entire mass m_i is

$$B \sum_j m_i \frac{m_j}{m} |x_i - x_j|^q$$

where B is a proportionality constant and $m = \sum_j m_j = \int \rho\, dx$. Finally, the total routing cost is

$$\frac{B}{m} \sum_{i,j} m_i m_j |x_i - x_j|^q.$$

If we write the routing cost in terms of the measure ν_N we obtain

$$\frac{B}{m} \int_\Omega \int_\Omega |x-y|^q\, d\nu_N(x)\, d\nu_N(y) = \frac{B}{m} \int_{\Omega\times\Omega} V(x-y)\, d(\nu_N \otimes \nu_N)$$

and the total cost taking into account location and routing terms gives the optimization problem

$$\min\left\{AW_p^p(\rho,\nu) + \frac{B}{m}\int_{\Omega\times\Omega} V(x-y)\, d(\nu\otimes\nu)\ :\ \#(\operatorname{spt}\nu) = N\right\} \tag{2.6}$$

The characterization of the limit problem as $N \to \infty$ in this case is easy and we can write it as

$$\min\left\{AW_p^p(\rho,\nu) + \frac{B}{m}\int_{\Omega\times\Omega} V(x-y)\, d(\nu\otimes\nu)\right\} \tag{2.7}$$

where the minimization above is intended in the class of all measures ν having the same total mass as ρ.

The necessary conditions of optimality for the optimization problem (2.7) can be obtained by differentiating the Wasserstein distance term (see [27]) and the routing cost; we obtain

$$A\phi + \frac{2B}{m} V * \nu = c \qquad \nu\text{-a.e.} \tag{2.8}$$

where ϕ is the Kantorovich potential for the transport from ρ to ν and c is a constant playing the role of the Lagrange multiplier of the mass constraint on ν. In (2.8) the measure ν appears in a very implicit way and can be determined only numerically. One connection between the Kantorovich pontential ϕ and the transport map T from ρ to ν is given by the Monge-Ampère equation

$$\rho = \nu(T) \det(\nabla T).$$

Differentiating in (2.8) we obtain

$$A\nabla\phi + \frac{2B}{m} \nabla V * \nu = 0$$

and $T(x) = x - \nabla\phi(x)$. Therefore we have the system

$$\begin{cases} A(x - T(x)) + \dfrac{2B}{m} \nabla V * \nu = 0 \\ \rho = \nu(T) \det(\nabla T). \end{cases} \tag{2.9}$$

In dimension 1 we can proceed by an iterative scheme, fixing an initial ν_0 and obtaining T_0 from the first equation in (2.9). Then we can recover ν_1 by the second equation

$$\nu_1(T_0(x)) = \frac{\rho(x)}{T_0'(x)}$$

and, assuming T_0 invertible,

$$\nu_1(y) = \frac{\rho(T_0^{-1}(y))}{T_0'(T_0^{-1}(y))}.$$

We can now proceed by iterating the scheme above.

Example 2.1. In this particular example we can find an explicit solution taking $d = 1$, $p = 2$, and $V(s) = |s|^2$. If we suppose that the barycenter of ν is in the origin, we obtain:

$$V * \nu = mx^2 + \int y^2 \, d\nu(y)$$

so that

$$A\phi'(x) + 4Bx = 0$$

which gives

$$\phi'(x) = -\frac{4B}{A} x \qquad \text{and} \qquad T(x) = \Big(1 + \frac{4B}{A}\Big) x.$$

Putting this in the 1-dimensional Monge-Ampère equation and indicating by v the density of ν, we obtain

$$\rho(x) = v((1+4B/A)x)\Big(1+\frac{4B}{A}\Big),$$

and changing variables,

$$v(y) = \frac{1}{1+4B/A}\rho\Big(\frac{y}{1+4B/A}\Big).$$

2.1.1 Mass independent routing costs

In this subsection we assume that the cost to connect the airports located at the points x_i and x_j does not depend on the transported mass and amounts simply to $K|x_i - x_j|^q$ where now the constant K is the cost of flying along a unit distance. In this case it is more convenient to use the probability measures μ_N introduced in (2.3) which provides the routing cost in the form

$$K\sum_{i,j}|x_i - x_j|^q = KN^2\int_{\Omega\times\Omega} V(x-y)\,d(\mu_N\otimes\mu_N).$$

Taking into account the asymptotic expression of the location cost given in (2.4), we obtain the optimization problem

$$\min\Big\{A\,C_{p,d}\,N^{-p/d}\int_\Omega \frac{\rho(x)}{\big(\mu(x)\big)^{p/d}}\,dx + KN^2\int_{\Omega\times\Omega} V(x-y)\,d(\mu\otimes\mu)\Big\}$$

where now μ runs in the class of all probabilities on Ω. Setting $\varepsilon = A\,C_{p,d}\,N^{-2-p/d}/K$ we are now faced with the problem

$$\min\Big\{F_\varepsilon(\mu) := \varepsilon\int_\Omega \frac{\rho(x)}{\big(\mu(x)\big)^{p/d}}\,dx + \int_{\Omega\times\Omega} V(x-y)\,d(\mu\otimes\mu)\Big\}. \tag{2.10}$$

The necessary conditions of optimality for problem (2.10) simply follow by differentiation of the cost functional and give:

$$\varepsilon\rho\,\frac{p}{d}\,\mu^{-1-p/d} + 2V*\mu = c \tag{2.11}$$

where $*$ denotes the convolution operator and c is a constant coming from the mass constraint on μ.

When $\varepsilon \to 0$ the optimal densities μ_ε of problem (2.10) tend to a Dirac mass δ_{x_0} for a suitable point x_0. In order to identify the limit problem as $\varepsilon \to 0$, and so to identify the point x_0 around which the optimal densities μ_ε concentrate (it can be seen as the *main hub* of the airports system), it is convenient to rescale the cost above dividing it by its minimum value. Considering the measures

$$\mu = \delta\frac{1}{|\Omega|} + (1-\delta)\frac{1_{B_r(x_0)}}{|B_r(x_0)|} \qquad \text{with } r^q \ll \delta,$$

a simple calculation provides for the minimal cost of problem (2.10):

$$\min F_\varepsilon \sim C\varepsilon\delta^{-p/d} + \delta$$

for a suitable constant C. Optimizing with respect to δ the quantity above, we obtain $\delta \sim \varepsilon^{1/(1+p/d)}$, so that

$$\min F_\varepsilon \sim \varepsilon^{1/(1+p/d)}.$$

Therefore the rescaled functionals become

$$G_\varepsilon(\mu) = \varepsilon^{(p/d)/(1+p/d)} \int_\Omega \frac{\rho(x)}{\big(\mu(x)\big)^{p/d}}\,dx + \varepsilon^{-1/(1+p/d)} \int_{\Omega\times\Omega} V(x-y)\,d(\mu\otimes\mu).$$

Note that the optimal measures for F_ε and for G_ε are the same.

In order to characterize the asymptotic behavior of the minimizing sequences (μ_ε) we will compute in the next section the Γ-limit of the sequence of functionals (G_ε). The general theory of Γ-convergence (see for instance Chapter 1 and [37]) then provides the identification of the main hub x_0 around which the measures μ_ε tend to concentrate.

2.2 The Γ-convergence result

First of all we notice that, due to the presence of the coefficient $\varepsilon^{-1/(1+p/d)}$ in front of the routing term, the Γ-limit on a measure μ will be $+\infty$ whenever $\int_{\Omega\times\Omega} V(x-y)\,d(\mu\otimes\mu) \neq 0$. Therefore, we may limit ourselves to analyze only the measures for which the routing term vanishes, i.e. the Dirac masses $\mu = \delta_{x_0}$.

It is convenient to set

$$\alpha = \frac{p/d}{1+p/d}\,, \qquad \beta = \frac{1}{1+p/d}\,;$$

notice that $\alpha + \beta = 1$ and that $\alpha = \beta p/d$. We will show that the Γ-limit of the sequence of functionals G_ε, computed on the Dirac mass δ_{x_0} and with respect to the weak* convergence of measures, coincides with the functional

$$H(\delta_{x_0}) = A\int_\Omega \big(\rho(x)\big)^\beta |x-x_0|^{\alpha q}\,dx \qquad \text{where } A = \Big(1+\frac{p}{d}\Big)\Big(\frac{2d}{p}\Big)^\alpha.$$

2.2.1 The Γ-limsup inequality

In order to obtain a Γ-limsup inequality, we have to choose a suitable sequence $\mu_\varepsilon \rightharpoonup \delta_{x_0}$ and to compute the limit of $G_\varepsilon(\mu_\varepsilon)$. We take

$$\mu_\varepsilon = \varepsilon^\beta\phi + \Big(1-\varepsilon^\beta\int_\Omega \phi\,dx\Big)\delta_{x_0}$$

where the function ϕ will be chosen later. Then μ_ε is a probability measure and we have

$$\begin{aligned}
G_\varepsilon(\mu_\varepsilon) &= \varepsilon^\alpha \int_\Omega \frac{\rho(x)}{\varepsilon^{\beta p/d}\phi^{p/d}}\,dx + \varepsilon^{-\beta}\int_{\Omega\times\Omega} \varepsilon^{2\beta} V(x-y)\phi(x)\phi(y)\,dxdy \\
&\qquad + \varepsilon^{-\beta}\int_\Omega 2\varepsilon^\beta\Big(1-\varepsilon^\beta\int_\Omega \phi\,dx\Big)V(x-x_0)\phi(x)\,dx \\
&= \int_\Omega \Big[\frac{\rho(x)}{\phi^{p/d}} + 2V(x-x_0)\phi\Big]\,dx + \varepsilon^\beta\int_{\Omega\times\Omega} V(x-y)\phi(x)\phi(y)\,dxdy \\
&\qquad - 2\varepsilon^\beta\int_\Omega \phi\,dx\int_\Omega V(x-x_0)\phi(x)\,dx
\end{aligned}$$

which gives

$$\lim_{\varepsilon\to 0} G_\varepsilon(\mu_\varepsilon) = \int_\Omega \Big[\frac{\rho(x)}{\phi^{p/d}} + 2V(x-x_0)\phi\Big]\,dx.$$

We choose now ϕ in order to minimize the quantity at the right-hand side. An easy computation gives

$$\phi(x) = \Big(\frac{p}{2d}\frac{\rho(x)}{V(x-x_0)}\Big)^\beta$$

which implies

$$\lim_{\varepsilon\to 0} G_\varepsilon(\mu_\varepsilon) = H(\delta_{x_0}).$$

2.2.2 The Γ-liminf inequality

In order to conclude that the Γ-limit of the functionals G_ε is the functional H it remains to show the Γ-liminf inequality, which amounts to prove that for every sequence $\mu_\varepsilon \rightharpoonup \delta_{x_0}$ we have

$$\liminf_{\varepsilon\to 0} G_\varepsilon(\mu_\varepsilon) \ge H(\delta_{x_0}). \tag{2.12}$$

The following lemma will be useful.

Lemma 2.2. *Let μ be a measure on Ω that is singular with respect to the Lebesgue measure and let $\mu_n \rightharpoonup \mu$. Then there exists a sequence of open sets (A_n) such that:*

i) $|A_n| \to 0$*;*

ii) $\mu_n(\Omega \setminus A_n) \to 0$ *(hence $\mu_n \llcorner A_n \rightharpoonup \mu$).*

Proof. Since μ is singular, it is concentrated on a measurable set S with $|S| = 0$ and there is a sequence of open sets (A_k) containing S and such that $|A_k| \to 0$. Since $\Omega\setminus A_k$ are closed sets, we have for every k

$$\limsup_{n\to\infty} \mu_n(\Omega \setminus A_k) \le \mu(\Omega \setminus A_k) = 0.$$

So it exists a subsequence n_k such that $\mu_{n_k}(\Omega \setminus A_k) \to 0$. We can assume $n_k < n_{k+1}$. We now define $A_n = A_k$ for $n \in [n_k, n_{k+1}]$. Then $\mu_n(\Omega \setminus A_n) = \mu_n(\Omega \setminus A_k) \le \frac{1}{2^k}$ for $n \in [n_k, n_{k+1}]$.

We thank D.Bucur for his help to find a precise proof of the lemma above. □

Take now a generic sequence $\mu_\varepsilon \rightharpoonup \delta_{x_0}$, denote by $u_\varepsilon(x)$ the density of the absolutely continuous part of μ_ε with respect to the Lebesgue measure, and let A_ε be the open sets provided by Lemma 2.2. Define

$$\mu^1_\varepsilon = \mu_\varepsilon \llcorner A_\varepsilon, \qquad \mu^2_\varepsilon = \mu_\varepsilon \llcorner A^c_\varepsilon.$$

We have

$$\begin{aligned} G_\varepsilon(\mu_\varepsilon) &= \varepsilon^\alpha \int_\Omega \frac{\rho}{u_\varepsilon^{p/d}}\,dx + \varepsilon^{-\beta}\Big[\int_{\Omega\times\Omega} V(x-y)\,d(\mu^1_\varepsilon\otimes\mu^1_\varepsilon) \\ &+ \int_{\Omega\times\Omega} V(x-y)\,d(\mu^2_\varepsilon\otimes\mu^2_\varepsilon) + \int_{\Omega\times\Omega} 2V(x-y)\,d(\mu^1_\varepsilon\otimes\mu^2_\varepsilon)\Big] \\ &\geq \int_\Omega \Big[\varepsilon^\alpha \frac{\rho}{u_\varepsilon^{p/d}} + \varepsilon^{-\beta} 2(V*\mu^1_\varepsilon)u_\varepsilon 1_{A^c_\varepsilon}\Big]\,dx \end{aligned}$$

where we used the fact that $\int_{\Omega\times\Omega} V(x-y)\,d(\nu\otimes\nu)\geq 0$ for every measure ν and that $\mu^2_\varepsilon \geq u_\varepsilon 1_{A^c_\varepsilon}\,dx$. Using the Young inequality

$$X\varepsilon^\alpha + Y\varepsilon^{-\beta} \geq \frac{X^\beta Y^\alpha}{\alpha^\alpha\beta^\beta} \qquad \text{for } \alpha+\beta=1,$$

we obtain

$$\begin{aligned} G_\varepsilon(\mu) &\geq \int_\Omega \frac{1}{\alpha^\alpha\beta^\beta}\left(\frac{\rho}{u_\varepsilon^{p/d}}\right)^\beta \big(2(V*\mu^1_\varepsilon)u_\varepsilon 1_{A^c_\varepsilon}\big)^\alpha\,dx \\ &= A\int_{A^c_\varepsilon} \rho^\beta (V*\mu^1_\varepsilon)^\alpha\,dx. \end{aligned}$$

Since $|A_\varepsilon|\to 0$ and $(V*\mu^1_\varepsilon)(x)\to V(x-x_0)$ by Lemma 2.2 (which guarantees that $\mu_\varepsilon(\Omega\setminus A_\varepsilon)\to 0$), we finally obtain (2.12).

As a conclusion, the Γ-limit computation is achieved and the optimal *main hub* for the limit location-routing problem of (2.10) is located at the point x_0 which minimizes the quantity

$$\int_\Omega \big(\rho(x)\big)^{1/(1+p/d)}|x-x_0|^{q(p/d)/(1+p/d)}\,dx. \tag{2.13}$$

Note that this minimization problem for x_0 is of the form of a Torricelli optimal location problem with suitable exponents.

2.3 Some numerical simulations

In this section, we perform some numerical simulations, based on the results of the previous section, that can be applied to real cases. In the first subsection, some 1-dimension and 2-dimension examples will be presented for different routing costs and density functions and varying ε. The last subsection introduces an application of the model to the USA airfreight system in order to compare the results with the current location of US airfreight hubs.

We are interested to find minimizer of the functional representing the sum of location and routing costs and the x_0 minimizing the Γ-limit functional $H(\delta_{x_0})$. In fact, one of the properties of Γ-convergence is the convergence of minima, so if H is the Γ-limit of G_ε, the limit of minimizers of G_ε is a minimizer of H. We will find numerically the optimal μ_ε and we observe that for small ε they are close to a Dirac mass at a suitable point x_0, according to the result of the previous section.

In Section 2.1 we found an optimality condition for the minimizers of F_ε. Unfortunately, condition (2.11) does not admit an explicit solution, so we approximate it numerically. More specifically, we approximate the minimizer via an iterative scheme: we start from the uniform distribution with total mass 1 and then we define the iteration term according to the necessary condition:

$$\begin{cases} \mu_0 = \mathcal{U}(\Omega) \\ \mu_{n+1} = \left(\frac{\varepsilon\rho}{c+V*\mu_n}\right)^{p/d+1} \end{cases}. \tag{2.14}$$

Here the Lagrange multiplier c, according to (2.11), is proportional to $\varepsilon^{1/(1+p/d)}$. The numerical scheme (2.14) permits us to do some interesting numerical simulation although we are not able to prove a convergence result, stopping the simulations after a certain number of iterations. Moreover we can observe that functional (2.13) may be non convex and then it could has several minima.

2.3.1 1-D and 2-D examples

In the one-dimensional case, the domain Ω is the interval $[-1, +1]$ that is discretized in order to solve numerically the problem. Consequently, both the functions ρ and μ are expressed through an array of values in correspondence of the discretization points. At each step of the convergence procedure shown in Figure 2.1, μ_{n+1} is obtained from the relationship (2.14) and then normalized to a probability measure.

The first simulation, reported in Figure 2.2, is related to a non-symmetric distribution of population density ρ and a quadratic routing cost function:

$$\rho(x) = \begin{cases} 2 & \text{if } x \in [-1, 0] \\ 1 & \text{if } x \in [0, +1], \end{cases} \qquad V = |x-y|^2, \qquad p = 1.$$

We assume that the convergence is reached when the maximum error between the values of μ_n and μ_{n+1} is less than 2%. In this conditions, about 10 iterations are requested to solve the problem and the computational time results to be proportional to the number of point used to discretized the domain; when it is divided into 200 steps, the calculation time is about 100 sec.

The results for different values of ε coefficient show that, as the ε decreases, the limit density μ tends to have a concentration centered on a single point. At the limit as $\varepsilon \to 0$, the density μ becomes a Dirac mass located at the point x_0 that minimizes the functional (2.13). The convergence towards the limit conditions of $\varepsilon \to 0$ is slow and it cannot be reached numerically because the onset of numerical errors below the value of

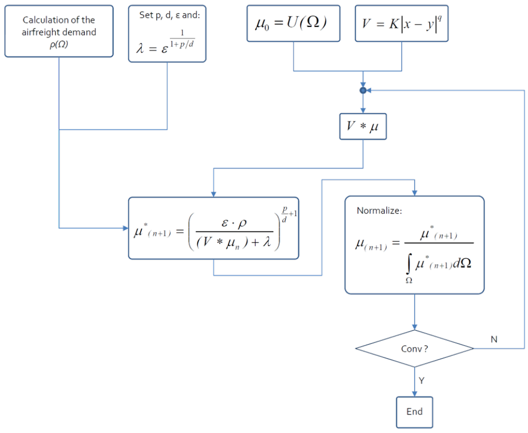


Figure 2.1: Numeric procedure for the determination of probability distribution μ.

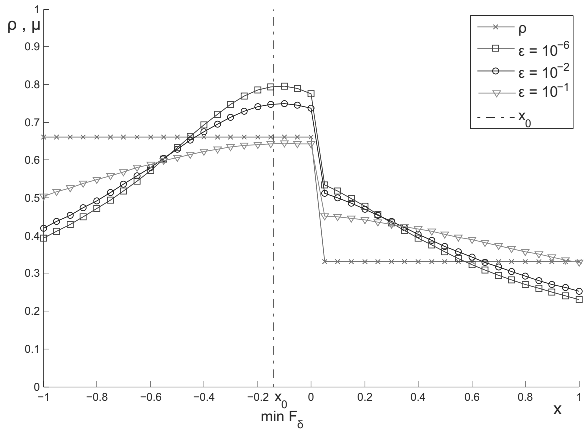


Figure 2.2: Results of the first simulation (asymmetric population).

$\varepsilon \simeq 10^{-4}$. Therefore, the routine can be completed by calculating also the value of the functional $H(\delta_{x_0})$ reported in (2.13), and find the point x_0 of minimum. In this case, the minimum of the functional (2.13) can be found explicitly:

$$
\begin{aligned}
H(\delta_{x_0}) &= \int_{-1}^{1} \sqrt{\rho(x)}|x - x_0|\, dx \\
&= \sqrt{2}\int_{-1}^{x_0} (x_0 - x)\, dx + \sqrt{2}\int_{x_0}^{0} (x - x_0)\, dx + \int_{0}^{1} (x - x_0)\, dx \\
&= \sqrt{2}\, x_0^2 + x_0(\sqrt{2} - 1) + \frac{1}{2}(\sqrt{2} + 1)
\end{aligned}
\tag{2.15}
$$

which gives

$$x_0 = \frac{\sqrt{2}-2}{4} \simeq -0.146.$$

The analytical solution equals to the value determined by the numerical procedure that is also reported in Figure 2.2.

The population ρ often can have an uneven distribution among the domain, and therefore an adequate function is requested in order to model correctly this aspect. A first solution can be provided by treating this distribution as a sum of M Gaussian functions:

$$\rho(x) = \sum_{j=1}^{M} A_j e^{-B_j|X_j - x|^2} \tag{2.16}$$

where the coefficients A_j,B_j,X_j are used to set respectively the height, the width and the position of the j-th peak.

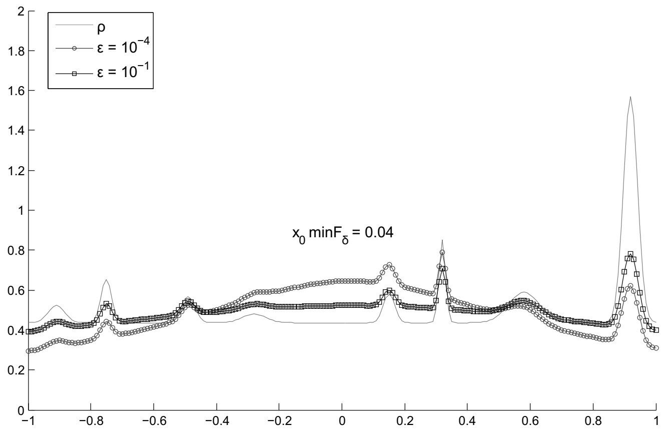


Figure 2.3: Second simulation: population modeled through a sum of Gaussian function.

The results reported in Figure 2.3 refer to the case of a population ρ with 8 peaks of different position, height, and area of influence (width); also in this case the simulations have been conducted with two different values of the coefficient ε.

The density of probability μ is highly dependent by the values of the coefficients; the point x_0 is in this case numerically determined:

$$x_0 \simeq +0.044.$$

As the ε decrease, the influence of the routing cost becomes larger on despite of the location ones so that the system tends to minimize the airport distance. The large differences between the solutions remarks the importance in choosing a value of the coefficient as realistic as possible. Moreover, one can note that the computational time is not affected by the "complication level" of the population function but only by the used discretization step. The effective decisional process related to the facilities (the airports) location, can be done in a post-processing phase: in this way, we can decide how many airports can be located in a given region, proportionally to the area limited by the density distribution; for example, the numbers on the X-axis of Figure 2.3 equals to the airport on each step (each step length is 0.2).

The routine has been applied also in the 2-D case, considering a correspondent peaks distribution shown in Figure 2.4. We remember the Gaussian function in the case of two variables and also both the routing cost function and the exponent of location:

$$\rho(x,y)=\sum_{j=1}^{M}A_j e^{-B_j(|X_j-x|^2+|Y_j-y|^2)}, \qquad V(x)=|x|^{0.5}, \qquad p=1.$$

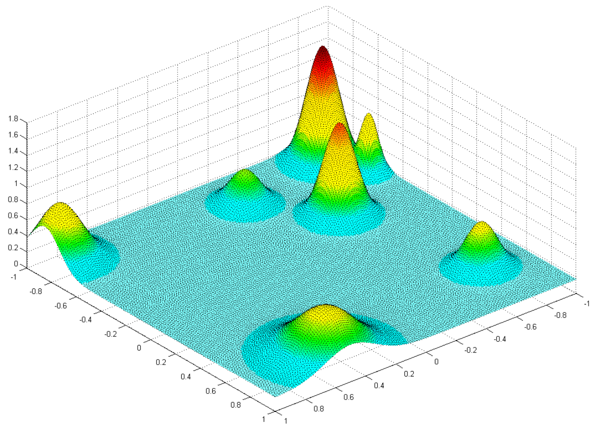


Figure 2.4: 2-D simulation: population ρ.

Although the calculation procedure does not change, the computational time results much higher than in the previous cases because of the great number of points requested to discretize properly the domain. Neverthless, it remains notably lower than the ones of the common Operating Research models (about 2200 sec. when the domain is divided into 1600 cells).

The result shows that the probability density follows the shape of the initial population $\rho(x,y)$ (we can note that the exponent q=0.5 determines the minor importance of the routing costs on despite of location ones) the point of maxima can be observed near the central peaks where the effect of both the location and routing terms are summed.

When the population distribution becomes irregular the position x_0 of the "main hub" cannot be estimated immediately but the functional (2.13) can be easily computed and its minimum can be found. For the same 2-D case, the values of the functional are shown in Figures 2.5 and 2.6 while its minimum point is depicted in Figure 2.6 together with the level curves of the population ρ.

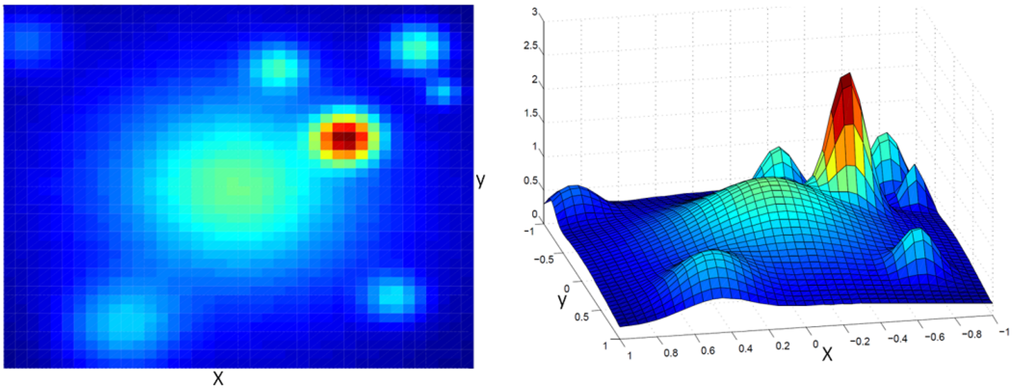


Figure 2.5: 2-D simulation: result.

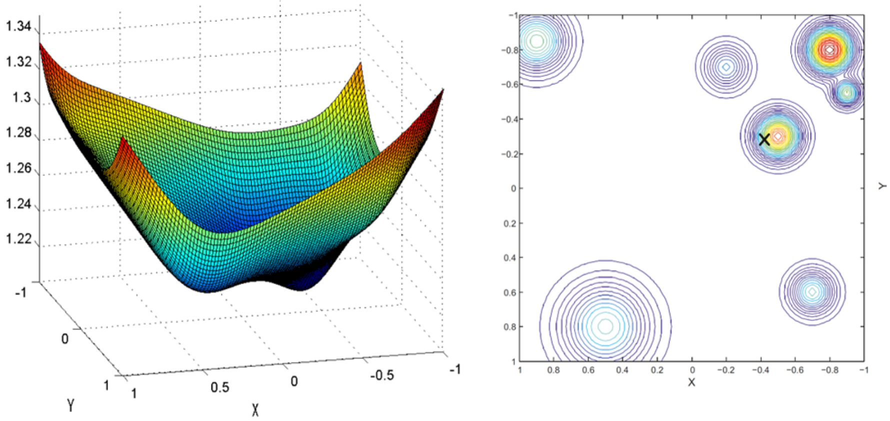


Figure 2.6: 2-D main hub result.

2.3.2 Application to the US airfreight system

Two main problems will be faced in order to apply location-routing models to real cases:

- Location and Routing terms are related with ground and air transportation costs respectively. Preliminarily, we can suppose a linear dependence of ground cost with the transport distance but the same assumption becomes not valid in the case of air transportation.

- The distribution of population ρ identifies the airfreight demand among the domain; data are directly available only for some areas (occidental countries) while in most cases an extrapolation from some socio-economic data is needed.

Therefore, the aim of the present section is to set the coefficients and exponents appearing in (2.10) in such the way the terms of cost functional reflect as realistic as possible the dynamics of the real world. The numerical routine will be finally applied on the US domain that will be represented as a polygon on a Cartesian system.

Routing and airfreight cost

In the functional (2.13), the routing costs can be modelled as a function of the transport distance, through the general power relationship $V(x) = K|x|^q$ by simply setting the coefficient K and the exponent q. In the case of airfreight, most of cost terms depend strongly by the *economies of scale* in which the company operates (countries connected, commercial accordances, kind of service done, aircraft used) so that it often is not possible to determine a general function that could be valid in every case. Nevertheless, if we cannot determine an explicit function, we can determine its "shape" by supposing that the part of cost variable with the transport distance, is mainly related to the fuel consumption during flight. In a first approximation, the amount of fuel required for a given mission can be determined (for example in the case of constant power aircraft) by using the so-called Breguet relations:

$$Cost_{fuel} \propto W_{fuel} = 1 - e^{\frac{-Range * k_c}{\eta_p E}}, \tag{2.17}$$

where the *Range* equals to the transport distance, E is the aerodynamic efficiency of the considered aircraft, and k_c and η_p are respectively the Specific Fuel Consumption and the propeller efficiency: since these parameters are all known for each aircraft and engine, the amount of fuel and its cost can be calculated in dependence of the Range flown. The operating costs are usually reported in terms of *Costs per Unit of carried mass and flown distance,* $Cost/(Ton \cdot Km)$(simply by dividing by the total payload and the transport distance) and the results of this procedure deriving from Breguet, has been compared in Figure 2.7 with some statistical models ([47] and [46]) that use a regression of both historical data about existing freighter and data collection of the financial report of transport companies.

The Curves in Figure 2.7 have a similar shape and they differ only by a translating coefficient that can be related to the different economies of scale which data are extrapolated from. Moreover, the $Cost/Ton$ can be determined by integrating the curves in Figure 2.7 so that finally a suitable value of the exponent q is determined:

$$q = 0.7$$

The value results lower than 1 because, conceptually, the air cannot be considered as a constant mass transport: in the case of existing aircraft in fact the Weight of embarked fuel represents until 30-40% of the total mass so that the flight condition and, consequently, also the fuel vary notably during cruise (as the aircraft is lightening, the burn fuel decreases).

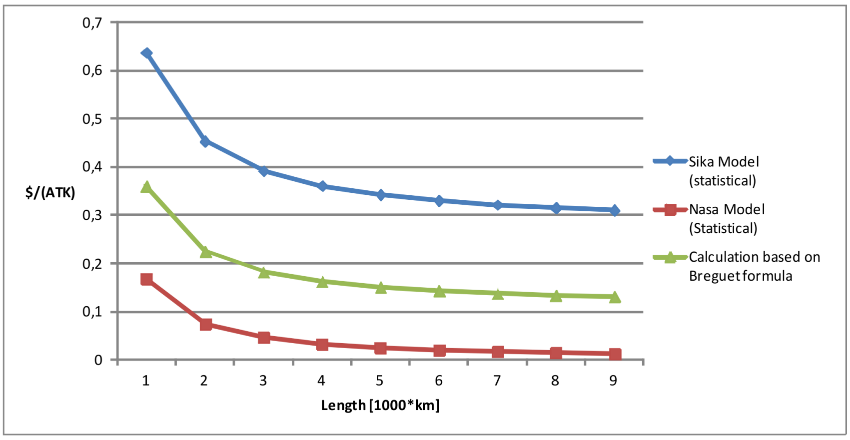


Figure 2.7: Variation of air cost with transport distance

Modeling the airfreight demand

The airfreight demand is often not directly measurable so that also the initial population ρ has to be properly modelled. The identification of the socio-economic parameters affecting the airfreight demand, results very difficult; also in this case the models are highly affected by the economies of scale and the geographical region on which the air transport is operated. In the present study, we assume that the airfreight depends by some socio-economic parameters in a likely linear regression as proposed in [2]. In the model proposed, the airfreight demand in some point of the domain is obtained by the following expression:

$$ln(AF) = C_0 + C_1PC + C_2TSE + C_3TSL + C_4MD + C_5HT \tag{2.18}$$

where

- $C_0, .., C_5$: coefficient coming from linear regression of economic data;
- AF: volume of airfreight demand (TON);
- PC: per capita personal income ($1,000);
- TSE: traffic shadow effect. In first approximation, this parameter will not considered; in order to avoid any iterative process also for the input data;
- TSL: transportation-shipping-logistics employment market share (%);
- MD: number of medical diagnostic establishments;
- HT: average high-tech employee wage ($1,000);

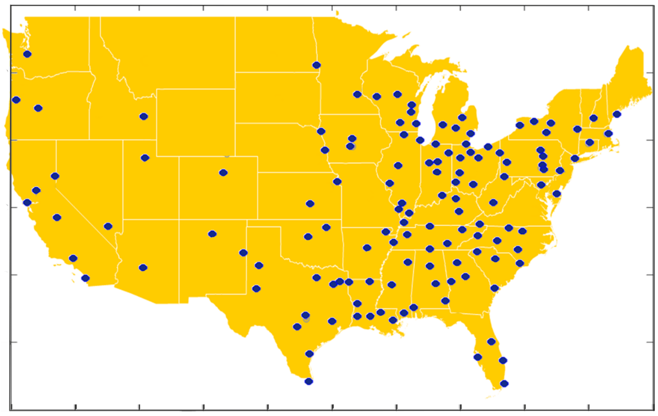


Figure 2.8: Centroids of MSAs among the US.

An airport has a relatively small catchment region (cities or districts), so that on the "ground side", the airfreight demand has influence on a very small area on despite of the worldwide dimension of the air transport; for this reason the airfreight function must to be refined also if the domain is very large and, consequently, the required discretization step is small. In this contest,the socio-economic data in (2.18) are extrapolated from common statistical reports (National Bureau for USA or Eurostat for E.U.) for each metropolitan districts, so that a detailed function can be easily determined.

The points used to define the spatial distribution of airfreight demand, are reported in Figure 2.8 in blue dots, and their position refers to the centroids of the so called *metropolitan statistical area* of the US territory.

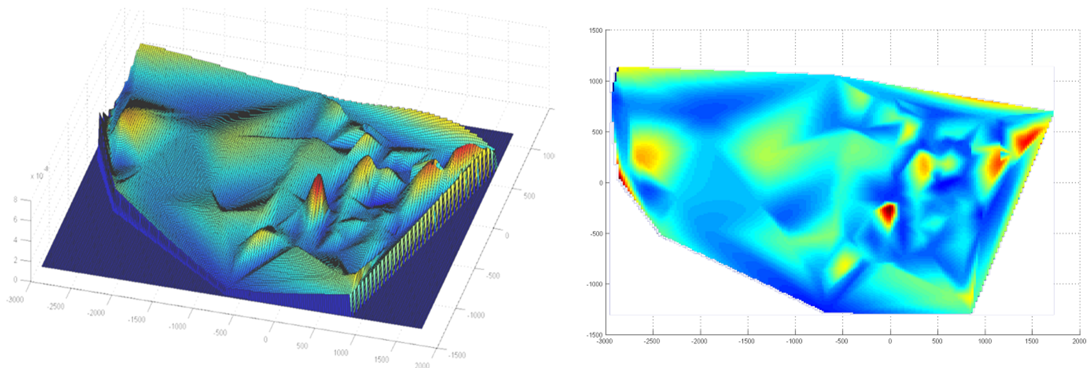


Figure 2.9: Model of the airfreight demand in Usa

Since data are known in correspondence of these points, the ρ function is then obtained through a cubic interpolation with a matlab routine. In Figure 2.9 it is reported the ρ obtained by this procedure: it has an uneven distribution, peaks are concentrated in very rich or very populated regions and their area of influence is relatively small.

Results

The Figure 2.10 shows the level curves of the μ function as result of the real case study in which the coefficient ε is set $\varepsilon \simeq 10^{-1}$.

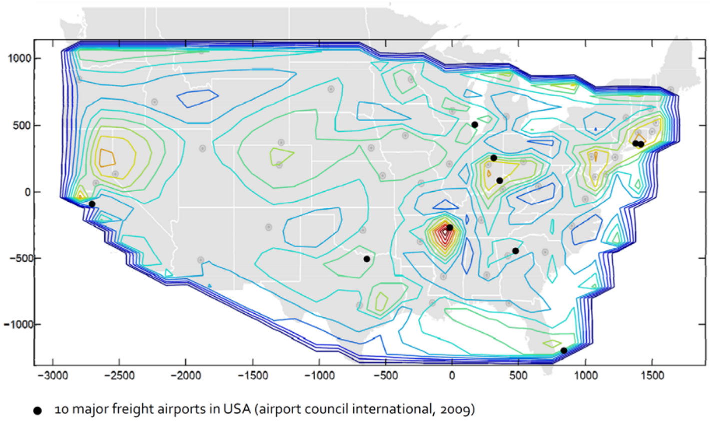


Figure 2.10: Density of probability μ among the US domain, level curves

The black dots of Figure 2.10 indicate the positions of the 10 major cargo airports in US. The global maxima of the μ density results very close to the Memphis airport which is the busiest center of airfreight transport and the hub of the FedEx: its airfreight volumes are doubled respect to the other airports. Also the other local maxima are located near the effective position of the other airports: largest errors are appreciated along the boundary areas where also the used projection method presents the largest errors.

The level curves of the functional $H(\delta_{x_0})$ are plotted in Figure 2.11. Although the exponent q results lower than the unity, the effects of the routing costs tend to predominate the location ones and consequently the importance of initial population ρ is reduced on despite of the distance power relationship.

The position of the minimum point (the "main hub") differs from the global maxima of the limit density μ displayed in Figure 2.10. In this case, the difference is due to the value of the coefficient ε used to determine the limit density μ, which is relatively high so that the results of the functional (2.10) and (2.13) are not coincidents.

Some considerations can be done. Our initial problem was to determine the optimal position of a certain number N of airports into a domain with location and routing cost condition. This problem is hard if we try to solve it with a direct approach because of its intrinsic complexity. After the modelling phase we concentrate to mass independent routing cost and we characterized the asymptotic behavior of the total cost problem.

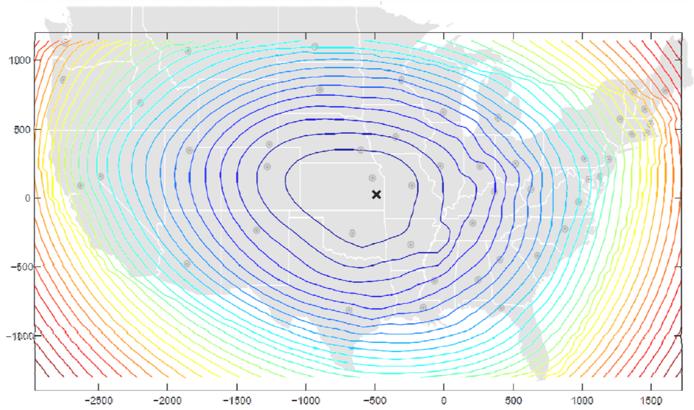


Figure 2.11: Level curves of the functional $H(\delta_{x_0})$

This means that, instead of finding the exact position of the N airports, we compute a probability density that represents the “importance” of a certain point in the area taken as domain. Moreover, Γ-limit result allow us to find the position of the optimal main hub minimizing the functional $H(\delta_{x_0})$.

Supported by the examples of the one and two dimensional cases, we observe that these two limit problems are very “easy”, in terms of computational costs. So, instead of looking at the initial problem it is more convenient be reduce at the other two. This makes possible to apply the procedure described to real cases, as in the USA airfreight system.

Chapter 3

Optimal region for congested transport

This Chapter is based on the work done in collaboration with Giuseppe Buttazzo and Guillaume Carlier (see [22]). In the present Chapter, we consider a very simplified model in which the densities of residents and of working places are known, represented by two probability measures f^+ and f^-. Congestion effects have been very much studied in the literature and as describe in Paragraph (1.6), denoting by f the difference $f = f^+ - f^-$ and by σ the traffic flux, the model, in the stationary regime, reduces to a minimization problem of the form

$$\min\left\{\int_\Omega H(\sigma)\,dx\ :\ -\operatorname{div}\sigma = f \text{ in } \Omega,\ \sigma\cdot n = 0 \text{ on } \partial\Omega\right\}. \tag{3.1}$$

Here Ω is the urban region under consideration, a bounded Lipschitz domain of $\mathbb{R}^d$, the boundary conditions at $\partial\Omega$ are usually taken imposing zero normal flux $\sigma \cdot n = 0$, and $H : \mathbb{R}^d \to [0, +\infty]$ is the *congestion function*, a convex nonnegative function with $\lim_{|s|\to+\infty} H(s) = +\infty$. In the isotropic case where $H(s)$ only depends on $|s|$, the interpretation of H (see [18, 32] and [17] for anisotropic extensions), is that its derivative represents the congested metric that is the commuting time per unit of length as a function of the traffic intensity $|s|$, since transport cannot occur at infinite speed even when there is no traffic, $H(s)$ typically behaves like $|s|$ close to 0 and is superlinear when $|s|$ is large. The first order PDE

$$-\operatorname{div}\sigma = f \text{ in } \Omega, \qquad \sigma\cdot n = 0 \text{ on } \partial\Omega$$

has to be intended in the weak sense

$$\langle\sigma, \nabla\phi\rangle = \langle f, \phi\rangle \qquad \text{for every } \phi \in C^\infty(\overline{\Omega})$$

and it captures the equilibrium between the traffic flux σ and the difference between supply and demand f.

In the case $H(s) = |s|$ no congestion effect occurs, and the transport problem reduces to the Monge's transport, where mass particles travel along geodesics (segments in the Euclidean case). As it is well known, in the Monge's case the integral cost above is finite for every choice of the probabilities f^+ and f^-. On the contrary, when H is superlinear, that is

$$\lim_{|s|\to+\infty} \frac{H(s)}{|s|} = +\infty,$$

congestion effects may occur and the mass particles trajectories follow more complicated paths. In this case the integral cost can be $+\infty$ if the source and target measures f^+ and f^- are singular. For instance, if the congestion function H has a quadratic growth, in order to have a finite cost it is necessary that the signed measure $f = f^+ - f^-$ be in the dual Sobolev space H^{-1}; thus, if $d > 1$ and the measures f^+ or f^- contain some Dirac mass, the minimization problem (3.1) is meaningless. In other words, superlinear congestion costs prevent too high concentrations.

In the present Chapter, we aim to address the efficient design of low-congestion regions; more precisely, two congestion functions H_1 and H_2 are given, with $H_1 \leq H_2$, and the goal is to find an optimal region $C \subset \Omega$ where we enforce a traffic congestion reduction. Since reducing the congestion in a region C is costly (because of roads improvement, traffic devices, ...), a term $m(C)$ will be added, to describe the cost of improving the region C, then penalizing too large low-congestion regions. On the region $\Omega \setminus C$ we then have the normally congested traffic governed by the function H_2, while on the low-congestion region C the traffic is governed by the function H_1. Throughout the Chapter, we will assume that H_1 and H_2 are two continuous convex functions such that $0 \leq H_1 \leq H_2$ and

$$\lim_{|s|\to+\infty} \frac{H_i(s)}{|s|} = +\infty, \ i = 1, 2.$$

For every region C we may consider the cost function

$$F(C) = \min\left\{ \int_{\Omega\setminus C} H_2(\sigma)\,dx + \int_C H_1(\sigma)\,dx \ : \ \sigma \in \Gamma_f \right\}, \tag{3.2}$$

where

$$\Gamma_f = \big\{\sigma \in L^1(\Omega;\mathbb{R}^d) \ : \ -\operatorname{div}\sigma = f \text{ in } \Omega,\ \sigma\cdot n = 0 \text{ on } \partial\Omega\big\}.$$

Therefore the optimal design of the low-congestion region amounts to the minimization problem

$$\min\big\{F(C) + m(C) \ : \ C \subset \Omega\big\}. \tag{3.3}$$

Several cases will be studied in the sequel, according to the various constraints on the low-congestion region C and the corresponding penalization/cost $m(C)$.

We also point out that a similar problem arises in some models for the mechanics of damage, see for instance [15].

3.1 Perimeter constraint on the low-congestion region

In this section we consider the minimum problem (3.3), where the cost $F(C)$ is given by (3.2) and $m(C) = k\,\mathrm{Per}(C)$, being $k > 0$ and $\mathrm{Per}(C)$ the perimeter of the set C in the sense of De Giorgi (see for instance [4]). Thanks to the coercivity properties of the perimeter with respect to the L^1 convergence of the characteristic functions (that we still call L^1 convergence of sets), we have the following existence result.

Theorem 3.1. *Assume that the cost $F(C)$ is finite for at least a subset C of $\overline{\Omega}$ with finite perimeter and that $m(C) = k\,\mathrm{Per}(C)$ with $k > 0$. Then there exists at least an optimal set C_{opt} for problem* (3.3)*.*

Proof. Let $(C_n)_{n\in\mathbb{N}}$ be a minimizing sequence for the optimization problem (3.3); then the sequence $\mathrm{Per}(C_n)$ is bounded. Thanks to the compactness of the embedding of BV into L^1, we may extract a (not relabeled) subsequence converging in L^1 to a subset C of Ω. We claim that this set C is an optimal set for the problem (3.3). Indeed, for the properties of the perimeter we have

$$\mathrm{Per}(C) \leq \liminf_n \mathrm{Per}(C_n).$$

Moreover, if we denote by $\sigma_n \in \Gamma_f$ an optimal (or asymptotically optimal) function for

$$F(C_n) = \int_{\Omega\setminus C_n} H_2(\sigma_n)\,dx + \int_{C_n} H_1(\sigma_n)\,dx,$$

by the superlinearity assumption on the congestion functions H_1 and H_2, and by the De La Vallée Poussin compactness theorem, we have that $(\sigma_n)_{n\in\mathbb{N}}$ is compact for the weak L^1 convergence and so we may assume that σ_n weakly converges in $L^1(\Omega)$ to a suitable function σ. This function σ still verifies the condition $\sigma \in \Gamma_f$. Thanks to the convexity of H_1 and H_2 the function

$$\Phi(\eta, \sigma) = (1-\eta)H_2(\sigma) + \eta H_1(\sigma)$$

satisfies the assumptions of the strong-weak lower semicontinuity theorem for integral functionals (see for instance [21]), so that we have

$$F(C) = \int_\Omega \Phi(1_C, \sigma)\,dx \leq \liminf_n \int_\Omega \Phi(1_{C_n}, \sigma_n)\,dx = \liminf_n F(C_n).$$

Therefore the set C is optimal and the proof is concluded. □

Our aim now is to establish optimality conditions not only on an optimal flow σ but also on the corresponding optimal low-congestion regions C. Optimality conditions for σ can be directly derived from the duality formula:

$$\begin{aligned} F(C) &= \inf_{\sigma\in\Gamma_f} \int_C H_1(\sigma)\,dx + \int_{\Omega\setminus C} H_2(\sigma)\,dx \\ &= -\inf_u \Big\{ \int_C H_1^*(\nabla u)\,dx + \int_{\Omega\setminus C} H_2^*(\nabla u)\,dx - \int_\Omega uf\,dx \Big\}, \end{aligned}$$

from which one easily infers that

$$\sigma = \begin{cases} \sigma_{\text{int}} \text{ in } C, \\ \sigma_{\text{ext}} \text{ in } \Omega \setminus C \end{cases}$$

where

$$\sigma_{\text{int}} = \nabla H_1^*(\nabla u_{\text{int}}) \text{ in } C, \qquad \sigma_{\text{ext}} = \nabla H_2^*(\nabla u_{\text{ext}}), \text{ in } \Omega \setminus C$$

the minimizer u in the dual is then given by:

$$u = \begin{cases} u_{\text{int}} \text{ in } C, \\ u_{\text{ext}} \text{ in } \Omega \setminus C \end{cases} .$$

We have used the notations σ_{int}, σ_{ext}, u_{int} and u_{ext} to emphasize the fact that σ and ∇u may have a discontinuity when crossing ∂C. It is reasonable (by elliptic regularity and assuming smoothness of C) to assume that σ and ∇u are Sobolev on C and $\Omega \setminus C$ separately but they are a priori not Sobolev on the whole of Ω (see the quadratic example below). The functions u_{int} and u_{ext} are then at least formally characterized by the Euler-Lagrange equations

$$-\operatorname{div}\Big(\nabla H_1^*(\nabla u_{\text{int}})\Big) = f \text{ in } C, \qquad -\operatorname{div}\Big(\nabla H_2^*(\nabla u_{\text{ext}})\Big) = f, \text{ in } \Omega \setminus C$$

together with

$$\nabla H_1^*(\nabla u_{\text{int}}) \cdot n = 0, \text{ on } \partial\Omega \cap C, \qquad \nabla H_2^*(\nabla u_{\text{ext}}) \cdot n = 0, \text{ on } \partial\Omega \cap \overline{\Omega} \setminus C,$$

and (assuming that f does not give mass to ∂C) the continuity of the normal component of σ across ∂C:

$$\Big(\nabla H_1^*(\nabla u_{\text{int}}) - \nabla H_2^*(\nabla u_{\text{ext}})\Big) \cdot n_C = 0, \text{ on } \partial C \cap \Omega,$$

where n_C denotes the exterior unit vector to C.

Now, we wish to give an extra optimality condition on C itself assuming that is smooth. To do so, we take a smooth vector field V such that $V \cdot n = 0$ on $\partial\Omega$, and we set $C_t = \varphi_t(C)$, where φ_t denotes the flow of V (i.e. $\varphi_0 = \text{id}$, $\partial_t \varphi_t(x) = V(\varphi_t(x))$). For $t > 0$, we then have

$$0 \leq \frac{1}{t}[F(C_t) - F(C) + k \operatorname{Per}(C_t) - k \operatorname{Per}(C)]. \tag{3.4}$$

As for the perimeter term, it is well-known (see for instance [45]) that the first-variation of the perimeter involves the mean curvature $\mathcal{H}$ of ∂C, more precisely, we have:

$$\frac{d}{dt} \operatorname{Per}(C_t)\big|_{t=0} = \int_{\partial C} \mathcal{H} V \cdot n_C \, d\mathcal{H}^{d-1}. \tag{3.5}$$

For the term involving H, we observe that

$$F(C_t) - F(C) \leq \int_{C_t} H_1(\sigma)\,dx - \int_C H_1(\sigma)\,dx + \int_{\Omega\setminus C_t} H_2(\sigma)\,dx - \int_{\Omega\setminus C} H_2(\sigma)\,dx$$

where $\sigma \in \Gamma_f$ is such that

$$F(C) = \int_C H_1(\sigma)\,dx + \int_{\Omega\setminus C} H_2(\sigma)\,dx.$$

At this point, we have to be a little bit careful because of the discontinuity of σ at ∂C, but distinguishing the part of ∂C on which $V \cdot n_C > 0$ that is moved outside C by the flow, and that on which $V \cdot n_C < 0$ that is moved inside C by the flow, and arguing as in Theorem 5.2.2 of [45], we arrive at:

$$\begin{aligned}\limsup_{t\to 0} \frac{F(C_t) - F(C)}{t} \le \int_{\partial C} \Big(&\big(H_1(\sigma_{\text{ext}}) - H_2(\sigma_{\text{ext}})\big)(V \cdot n_C)_+ \\ &+ \big(H_2(\sigma_{\text{int}}) - H_1(\sigma_{\text{int}})\big)(V \cdot n_C)_- \Big)\, d\mathcal{H}^{d-1}.\end{aligned} \tag{3.6}$$

Combining (3.4), (3.5) and (3.6), we obtain

$$0 \le \int_{\partial C} \Big(\big(H_1(\sigma_{\text{ext}}) - H_2(\sigma_{\text{ext}}) + k\mathcal{H}\big)(V \cdot n_C)_+ + \big(H_2(\sigma_{\text{int}}) - H_1(\sigma_{\text{int}}) - k\mathcal{H}\big)(V \cdot n_C)_- \Big)\, d\mathcal{H}^{d-1}.$$

But since V is arbitrary, we obtain the extra optimality conditions:

$$H_2(\sigma_{\text{int}}) - H_1(\sigma_{\text{int}}) \ge k\mathcal{H} \ge H_2(\sigma_{\text{ext}}) - H_1(\sigma_{\text{ext}}) \qquad \text{on } \partial C \cap \Omega$$

which, since $H_2 \ge H_1$, in particular implies that ∂C has nonnegative mean curvature.

The regularity of ∂C is an interesting open question. Note that when $d = 2$ and Ω is convex, replacing C by its convex hull diminishes the perimeter and also the congestion cost, so that optimal regions C are convex, this is a first step towards regularity, note also that convexity of optimal regions is consistent with the curvature inequality above.

Let us illustrate the previous conditions on the simple quadratic case where $H_1(\sigma) = \frac{a}{2}|\sigma|^2$, $H_2(\sigma) = \frac{b}{2}|\sigma|^2$ with $0 < a < b$. The optimality conditions for the pair u, σ then read as

$$\begin{cases} -a\Delta u_{\text{int}} = f & \text{in } C \\ -b\Delta u_{\text{ext}} = f & \text{in } \Omega \setminus C, \end{cases} \qquad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \qquad \begin{cases} \sigma_{\text{int}} = \frac{\nabla u_{\text{int}}}{a} \\ \sigma_{\text{ext}} = \frac{\nabla u_{\text{ext}}}{b}, \end{cases}$$

together with

$$\Big(\frac{\nabla u_{\text{int}}}{a} - \frac{\nabla u_{\text{ext}}}{b}\Big) \cdot n_C = 0 \qquad \text{on } \partial C \cap \Omega$$

(which shows that there is a priori a jump in the normal component of ∇u across ∂C) and

$$\frac{b-a}{2}|\sigma_{\text{int}}|^2 = \frac{b-a}{2a^2}|\nabla u_{\text{int}}|^2 \ge k\mathcal{H} \ge \frac{b-a}{2}|\sigma_{\text{ext}}|^2 = \frac{b-a}{2b^2}|\nabla u_{\text{ext}}|^2 \qquad \text{on } \partial C \cap \Omega$$

where $\mathcal{H}$ again denotes the mean curvature of ∂C.

3.2 Relaxed formulation for the measure penalization

In this section we consider the case when the penalization on the low-congestion region is proportional to the Lebesgue measure, that is $m(C) = k|C|$ with $k > 0$. The minimization problem we are dealing with then becomes

$$\min_{\sigma,C}\left\{\int_C H_1(\sigma)\,dx + \int_{\Omega\setminus C} H_2(\sigma)\,dx + k|C| \ :\ \sigma\in\Gamma_f\right\}. \tag{3.7}$$

Passing from sets C to density functions θ with $0 \le \theta(x) \le 1$ we obtain the relaxed formulation of (3.7)

$$\min_{\sigma,\theta}\left\{\int_\Omega \theta H_1(\sigma)\,dx + \int_\Omega (1-\theta)H_2(\sigma)\,dx + k\int_\Omega \theta\,dx \ :\ \sigma\in\Gamma_f\right\}. \tag{3.8}$$

Writing the quantity to be minimized as

$$\int_\Omega H_2(\sigma) + \theta\big(H_1(\sigma) + k - H_2(\sigma)\big)\,dx,$$

the minimization with respect to θ is straightforward; in fact, for a fixed $\sigma \in \Gamma_f$, if $H_1(\sigma) + k > H_2(\sigma)$ we take $\theta = 0$, while if $H_1(\sigma) + k < H_2(\sigma)$ we take $\theta = 1$. In the region where $H_1(\sigma) + k = H_2(\sigma)$ the choice of θ is irrelevant. In other words, for a fixed $\sigma \in \Gamma_f$ we have taken

$$\theta = 1_{\{H_1(\sigma)+k<H_2(\sigma)\}},$$

which gives

$$H_2 + \theta\big(H_1 + k - H_2\big) = H_2 - \big(H_1 + k - H_2\big)^- = H_2 \wedge \big(H_1 + k\big).$$

Therefore, in the relaxed problem (3.8) the variable θ can be eliminated and the problem reduces to

$$\min\left\{\int_\Omega H_2(\sigma)\wedge\big(H_1(\sigma)+k\big)\,dx \ :\ \sigma\in\Gamma_f\right\}. \tag{3.9}$$

Clearly the infimum in (3.9) coincides with that of (3.7) but since the new integrand $H_2 \wedge \big(H_1 + k\big)$ is not convex, a further relaxation with respect to σ is necessary. This relaxation issue with a divergence constraint has been studied in [14], where it is shown that the relaxation procedure amounts to convexify the integrand. We then end up with the minimum problem

$$\min\left\{\int_\Omega \Big(H_2(\sigma)\wedge\big(H_1(\sigma)+k\big)\Big)^{**}\,dx \ :\ \sigma\in\Gamma_f\right\} \tag{3.10}$$

where $**$ indicates the convexification operation. Recalling that H_1 and H_2 are superlinear, and indicating by $\overline{\sigma}$ an optimal solution to (3.10), we have that:

- in the region where
$$\Big(H_2 \wedge \big(H_1 + k\big)\Big)^{**}(\overline{\sigma}) = H_2(\overline{\sigma})$$
we take $\theta = 0$. In other words, in this region, it is better not to spend resources for improving the traffic congestion;

- in the region where
$$\Big(H_2 \wedge \big(H_1 + k\big)\Big)^{**}(\overline{\sigma}) = H_1(\overline{\sigma}) + k$$
we take $\theta = 1$. In other words, in this region, it is necessary to spend a lot of resources for improving the traffic congestion;

- in the region where
$$\Big(H_2 \wedge \big(H_1 + k\big)\Big)^{**}(\overline{\sigma}) < \Big(H_2 \wedge \big(H_1 + k\big)\Big)(\overline{\sigma})$$
we have $0 < \theta(x) < 1$ so that there is some mixing between the low and the high congestion functions. In other words, in this region the resources that are spent for improving the traffic congestion are proportional to θ.

The previous situation is better illustrated in the case where both functions H_1 and H_2 depend on $|\sigma|$ and $H_2 - H_1$ increases with $|\sigma|$. In this case, we denote by r_1 the maximum number such that
$$\Big(H_2 \wedge \big(H_1 + k\big)\Big)^{**}(r) = H_2(r)$$
and by r_2 the minimum number such that
$$\Big(H_2 \wedge \big(H_1 + k\big)\Big)^{**}(r) = H_1(r) + k,$$
then we have
$$\theta(x) = \frac{|\sigma| - r_1}{r_2 - r_1} \qquad \text{whenever } r_1 < |\sigma| < r_2.$$
In this case, for small values of the traffic flow ($|\sigma| \le r_1$), it is optimal not to spend any resource to diminish congestion, on the contrary when traffic becomes large ($|\sigma| \ge r_2$), it becomes optimal to reduce the congestion to H_1. Finally, for intermediate values of the traffic, mixing occurs with the coefficient θ above as a result of the relaxation procedure.

Also, problem (3.10) is of type (3.1) and it is well-known, by convex analysis, that we have the dual formulation
$$\begin{aligned}\min\Big\{\int_\Omega H(\sigma)\,dx \ :\ \sigma \in \Gamma_f\Big\} &= \sup\Big\{\int_\Omega u\,df - \int_\Omega H^*(\nabla u)\,dx\Big\}\\ &= -\inf\Big\{\int_\Omega H^*(\nabla u)\,dx - \int_\Omega u\,df\Big\},\end{aligned} \tag{3.11}$$

where $H(\sigma) = (H_2(\sigma) \wedge (H_1(\sigma) + k))^{**}$. Notice that the Euler-Lagrange equation of problem (3.11) is formally written as

$$\begin{cases} -\operatorname{div} \nabla H^*(\nabla u) = f & \text{in } \Omega \\ \nabla H^*(\nabla u) \cdot \nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.12}$$

Moreover, the link between the flux σ and the dual variable u is

$$\sigma = \nabla H^*(\nabla u).$$

In our case, the Fenchel tranform is easy computed and we have:

$$H^*(\xi) = H_2^*(\xi) \vee (H_1^*(\xi) - k).$$

As a conclusion of this paragraph, we observe that the treatment above is similar to the analysis of two-phase optimization problems. This consists in finding an optimal design for a domain that is occupied by two constituent media with constant conductivities α and β with $0 < \alpha < \beta < +\infty$, under an objective function and a state equation that have a form similar to (3.11) and (3.12). We refer to [20] (and references therein) for a general presentation of shape optimization problems and to [1] for a complete analysis of two-phase optimization problems together with numerical methods to treat them.

3.3 Low-congestion transportation networks

In this section, our main unknown is a one-dimensional subset Σ of Ω; we consider a fixed number $r > 0$ and the low-congestion regions of the form

$$C_{\Sigma,r} = \big\{x \in \overline{\Omega} \ : \ \operatorname{dist}(x, \Sigma) \le r\big\} = \Sigma^r \cap \overline{\Omega}, \ \text{where } \Sigma^r := \Sigma + B_r(0).$$

and Σ is required to be a closed subset of $\overline{\Omega}$ such that $\mathcal{H}^1(\Sigma) < +\infty$. The penalization term $m(C_{\Sigma,r})$ is taken proportional to the Lebesgue measure of $C_{\Sigma,r}$, so that our optimization problem becomes

$$\min_{\sigma,\Sigma} \left\{ \int_{C_{\Sigma,r}} H_1(\sigma)\,dx + \int_{\Omega\setminus C_{\Sigma,r}} H_2(\sigma)\,dx + k|C_{\Sigma,r}| \ : \ \sigma \in \Gamma_f \right\} \tag{3.13}$$

with $k > 0$. A key point in the existence proof below consists in remarking that the perimeter of an r-enlarged set Σ^r can be controlled by its measure (see Proposition 3.2). It also worth remarking that Σ^r has the *uniform interior ball of radius r property*; for every $x \in \Sigma^r$ there exists $y \in \mathbb{R}^d$ such that $|x - y| \le r$ and $B_r(y) \subset \Sigma^r$. Clearly, r-enlarged sets have the uniform interior ball of radius r property and sets with this property are r-enlarged sets (i.e. can be written as the sum of a closed set and $B_r(0)$), we refer to [3] for more on sets with the uniform interior ball property, and in particular estimates on their perimeter.

For the sake of completeness we show the following result.

Proposition 3.2. *For every set* $E \subset \mathbb{R}^d$ *and for every* $r > 0$*, setting* $E_r = \{x \in \mathbb{R}^d : \mathrm{dist}(x, E) < r\}$*, we have*

$$\mathrm{Per}(E_r) \leq \frac{d}{r}|E_r|. \tag{3.14}$$

Proof. The inequality above can be deduced from the results in the appendix of [28]; the present proof was obtained during a discussion with Giovanni Alberti, that we thank for his help.

Since the set E_r only depends on the closure of E, we may assume that E is closed; moreover, approximating E by smooth sets (for instance by the sets E_s with $s \to 0$), we may also assume that E is smooth.

Consider now the function

$$f(r) = d|E_r| - r\,\mathrm{Per}(E_r);$$

proving (3.14) amounts to show that $f(r) \geq 0$ for every $r > 0$. Since E is assumed smooth, we have

$$\lim_{r\to 0} |E_r| = |E|, \qquad \lim_{r\to 0} \mathrm{Per}(E_r) = \mathrm{Per}(E),$$

so that

$$\lim_{r\to 0} f(r) = d|E| \geq 0.$$

By the coarea formula we have for all $r < s$

$$|E_s| - |E_r| = \int_{E_s \setminus E_r} |\nabla\,\mathrm{dist}(x, E)|\,dx = \int_r^s \mathrm{Per}(E_t)\,dt$$

so that, indicating by $'$ the derivation with respect to r,

$$\big(|E_r|\big)' = \mathrm{Per}(E_r).$$

Denoting by $h(x)$ the mean curvature of ∂E_r at x, and taking into account the definition of E_r, we have $h(x) \leq (d-1)/r$, so that

$$\big(\mathrm{Per}(E_r)\big)' = \int_{\partial E_r} h(x)\,d\mathcal{H}^{d-1} \leq \frac{d-1}{r}\,\mathrm{Per}(E_r).$$

Therefore,

$$f'(r) = d\big(|E_r|\big)' - \mathrm{Per}(E_r) - r\big(\mathrm{Per}(E_r)\big)' \geq 0,$$

which implies that $f(r) \geq 0$ for every $r > 0$. $\square$

Proposition 3.3. *Ler* $r > 0$ *be fixed,* $d = 2$ *and assume that* $F(C_{\Sigma,r}) < +\infty$ *for some closed one-dimensional subset* Σ *of* $\overline{\Omega}$*. Then the optimization problem* (3.13) *admits a solution.*

Proof. The sets $C_{\Sigma,r}$ satisfy the inequality (see for instance Proposition 3.2)

$$\operatorname{Per}(C_{\Sigma,r}) \leq \frac{K}{r}|C_{\Sigma,r}|$$

for a suitable constant K depending only on the dimension d. Therefore, for a minimizing sequence $(\Sigma_n)_{n\in\mathbb{N}}$, the sets $C_n := C_{\Sigma_n,r} = \Sigma_n^r \cap \overline{\Omega}$ are compact in the strong L^1 convergence, we can thus extract a (not relabeled) subsequence such that C_n converges strongly in L^1 (and a.e.) to some C. One can then repeat the proof of Theorem 3.1, to obtain

$$F(C) + k|C| \leq \inf (3.13).$$

It only remains to show that C can be obtained as $C = C_{\Sigma,r}$ (up to a negligible set) for some closed subset of $\overline{\Omega}$, Σ such that $\mathcal{H}^1(\Sigma) < +\infty$. Up to extracting a subsequence from (Σ_n), one can assume that Σ_n^r converges for the Hausdorff distance to some compact set E (which also satisfies the uniform interior ball property of radius r). Let us first check that $C = E \cap \overline{\Omega}$ (up to a negligible set), the inclusion $C \subset E \cap \overline{\Omega}$ is standard (see for instance [45]). To prove the converse inclusion, it is enough to show that $|C| = |E \cap \overline{\Omega}|$ i.e. $|C_n| \to |E \cap \overline{\Omega}|$ as $n \to \infty$. For this, we observe that

$$\big||C_n| - |E \cap \overline{\Omega}|\big| \leq |\Sigma_n^r \setminus E| + |E \setminus \Sigma_n^r|.$$

The convergence of $|\Sigma_n^r \setminus E|$ to 0 easily follows from the Hausdorff convergence of Σ_n^r to E and the fact that E is closed (see [45] for details). As for the convergence of $|E \setminus \Sigma_n^r|$ to 0, we proceed as follows: let $\varepsilon > 0$ and n be large enough so that $E \subset \Sigma_n^r + B_\varepsilon(0) = \Sigma_n^{r+\varepsilon}$. Thanks to Proposition 3.2, there is a constant M such for any $s \in [r, r+\varepsilon]$ and any n, Σ_n^s has a perimeter bounded by M, by the coarea formula, we then get that for n large enough:

$$|E \setminus \Sigma_n^r| \leq |\Sigma_n^{r+\varepsilon} \setminus \Sigma_n^r| = \int_{\Sigma_n^{r+\varepsilon} \setminus \Sigma_n^r} |\nabla \operatorname{dist}(x, \Sigma_n^r)| dx = \int_r^{r+\varepsilon} \operatorname{Per}(\Sigma_n^s) ds \leq M\varepsilon.$$

We thus have proved that $C = E \cap \overline{\Omega}$ (up to a negligible set). Let us finally denote by dist the distance to $\mathbb{R}^2 \setminus E$ and set

$$\Sigma := \bigcup_{l=1}^{L} \operatorname{dist}^{-1}(\{lr\})$$

where L is the integer part of $r^{-1} \max \operatorname{dist}$. It is then not difficult to check that $\mathcal{H}^1(\Sigma) < +\infty$ and $\Sigma^r = E$ because E satisfies the uniform interior ball property of radius r so that $C = C_{\Sigma,r}$, which ends the proof. □

Remark 3.4. We have used the assumption that $d = 2$ only in the last step that is to prove that $C = C_{\Sigma,r}$ for some *one-dimensional* Σ. In higher dimensions, the same proof works if one requires $\mathcal{H}^{d-1}(\Sigma) < +\infty$ (however we believe the result remains true for one-dimensional sets in any dimension).

Remark 3.5. If the admissible sets Σ are supposed connected (in this case we call them *networks*), or with an a priori bounded number of connected components, then the penalization term $|C_{\Sigma,r}|$ can be replaced by the one-dimensional Hausdorff measure $\mathcal{H}^1(\Sigma)$. In fact, for such sets we have

$$|C_{\Sigma,r}| \leq M\big(1 + \mathcal{H}^1(\Sigma)\big)$$

where the constant M depends on the dimension d, on r, and on the number of connected components of Σ. Therefore the argument of Proposition 3.3 applies, providing the existence of an optimal solution.

We deal now with the case when the low-congestion region is a one-dimensional set Σ. We assume Σ connected (or with an a priori bounded number of connected components) and we take $m(\Sigma)$ proportional to the one-dimensional Hausdorff measure $\mathcal{H}^1(\Sigma)$. The integral on the low-congestion region has to be modified accordingly and we have to consider the problem formally written as

$$\min_{\sigma,\Sigma} \left\{ \int_\Sigma H_1(\sigma)\, d\mathcal{H}^1 + \int_\Omega H_2(\sigma)\, dx + k\mathcal{H}^1(\Sigma) \ : \ \sigma \in \Gamma_f \right\} \tag{3.15}$$

with $k > 0$. Notice that, in view of the superlinearity assumption on the congestion functions H_1 and H_2, the admissible fluxes u have to be assumed absolutely continuous measures with respect to $\mathcal{L}^d\lfloor\Omega + \mathcal{H}^1\lfloor\Sigma$. Subsequently, the integral terms in the cost expression have to be intended as:

$$\int_\Sigma H_1\Big(\frac{d\sigma}{d\mathcal{H}^1}\Big)\, d\mathcal{H}^1 + \int_\Omega H_2\Big(\frac{d\sigma}{d\mathcal{L}^d}\Big)\, dx.$$

By an abuse of notation, when no confusion may arise, we continue to write the terms above as $\int_\Sigma H_1(\sigma)\, d\mathcal{H}^1 + \int_\Omega H_2(\sigma)\, dx$.

Remark 3.6. At least formally, (3.15) can be thought of as a limit case of (3.13) as $r \to 0^+$ when in (3.13) one replaces H_1 by $r^{1-d}H_1(r^{d-1}\sigma)$ and k by kr^{1-d}. A rigorous Γ-convergence derivation of (3.15) by letting $r \to 0^+$ in (3.13) is an interesting issue even though it is beyond the scope of this work. Also, one should emphasize that the network model (3.15) is very different from the ones considered in Sections 3.1 and 3.2 because the traffic density on the network Σ is computed with respect to $\mathcal{H}^1$. In some sense, this means that the congestion effect is much weaker on Σ whatever the congestion functions H_1 and H_2 are, in particular it is not really meaningful in the context of network models to assume that $H_1 \leq H_2$.

In general, the optimization problem (3.15) does not admit a solution Σ_{opt}, because the limits of minimizing sequences Σ_n may develop multiplicities, providing as an optimum a relaxed solution made by a one-dimensional set Σ_{opt} and function $a \in L^1(\Sigma_{opt})$ with $a(x) \geq 1$. The relaxed version of problem (3.15), taking into account these multiplicities, becomes

$$\min_{\sigma,\Sigma,a} \left\{ \int_\Sigma H_1(\sigma/a)a\, d\mathcal{H}^1 + \int_\Omega H_2(\sigma)\, dx + k\int_\Sigma a\, d\mathcal{H}^1 \ : \ \sigma \in \Gamma_f \right\}. \tag{3.16}$$

The optimization with respect to a is easy: consider for simplicity the case

$$H_1(\sigma) = \alpha|\sigma|^p \qquad \text{with } \alpha > 0,\ p > 1;$$

then we have

$$\min_{a\ge1}\left(ka + \alpha\frac{|\sigma|^p}{a^{p-1}}\right) = H(\sigma) = \begin{cases} \alpha|\sigma|^p + k & \text{if } |\sigma|^p \le \dfrac{k}{\alpha(p-1)} \\ |\sigma|\alpha^{1/p}p\Big(\dfrac{k}{p-1}\Big)^{1-1/p} & \text{if } |\sigma|^p \ge \dfrac{k}{\alpha(p-1)}. \end{cases}$$

Therefore the relaxed problem (3.16) can be rewritten as

$$\min_{\sigma,\Sigma}\left\{\int_\Sigma H(\sigma)\,d\mathcal{H}^1 + \int_\Omega H_2(\sigma)\,dx \ :\ \sigma\in\Gamma_f\right\}$$

and the multiplicity density $a(x)$ on Σ (that can be interpreted as the width of the road Σ at the point x) is given by

$$a(x) = 1 \vee |\sigma(x)|\Big(\frac{\alpha(p-1)}{k}\Big)^{1/p}. \tag{3.17}$$

To illustrate the necessity of relaxation, let us consider the (somehow extreme) special case where $H_2(0) = 0$ and $H_2 = +\infty$ elsewhere, f^+ and f^- are Dirac masses at two distinct points x^+ and x^- and H_1 is the power function above. Let then Σ and σ be optimal (with σ identified with its density with respect to the one dimensional measure on Σ). We claim that $|\sigma|$ has to be larger than 1, somewhere because otherwise taking the distance to x^- as a test-function in the divergence constraint we would get $|x^+ - x^-| < \mathcal{H}^1(\Sigma)$. But when $|\sigma| \ge 1$, (3.17) gives $a > 1$ as soon as k is small enough, this means that multiplicity may occur at least when the cost for the length of the network is small.

3.4 Numerical simulations

Here we wish to give a numerical example which clarifies and confirms what we expected from the analysis done in Section 3.2. In our examples, we mainly focus on the problem in the form (3.11):

$$\min\Big\{\int_\Omega H^*(\nabla u)\,dx - \int_\Omega fu\,dx\Big\}.$$

The numerical simulation is based on a very simple situation that however seems quite reasonable. The two congestion function considered are both quadratic but with a different coefficient, say $H_1(\sigma) = a|\sigma|^2$ and $H_2(\sigma) = b|\sigma|^2$ with $a < b$. Then, in this case, the function H^* involved in (3.11) is easy to compute:

$$H^*(\xi) = \Big(\frac{\xi^2}{4b}\Big) \vee \Big(\frac{\xi^2}{4a} - k\Big)$$

Before we start illustrating the numerical result, it is useful to do some considerations that justify the choice of some parameters in the following. The dual variable u has to be thought as a price system for a company handling the transport in a congested situation. An optimizer u then gives the price system which maximizes the profit of the company. When you take into account a congested transport between sources (here called f^+ and f^-), the total mass $\int df^+ = \int df^-$ plays an important role: as observed in [17], in the case of a small mass, the congestion effects are negligible. Therefore we may expect for highly concentrated sources a distribution of the low-congestion region around the sources. On the contrary, for sources with a low concentration, we may expect a distribution of the low-congestion region also between f^+ and f^-.

In the following examples, we consider as sources f^+ and f^- two Gaussian distributions with variance λ, centered at two points x_0 and x_1

$$f^+(x) = \frac{1}{\sqrt{2\pi\lambda}} e^{-|x-x_0|^2/(2\lambda)}, \qquad f^-(x) = \frac{1}{\sqrt{2\pi\lambda}} e^{-|x-x_1|^2/(2\lambda)}.$$

In this case, a large value of λ means less concentration (and, on the contrary, a small λ captures more concentration). The total mass is taken equal to one and, to capture the influence of the total quantity of available resources, we use a Lagrange multiplier k that penalizes the measure $|C|$. Hence, a large value of the penalization parameter k corresponds to a small quantity of available resources. Ending this consideration on parameters involved, we note that the traffic congestion parameters a, b and the "construction cost" parameter k are linked: we will change value of k according to a suitable choice of ratio $\frac{a}{b}$, for fixed λ. Now, concerning the choice of the coefficients a, b we take $a = 1$ and $b = 4$, which means that the velocity in the low-congestion region is, at equal traffic density, four times the one in the region with normal congestion.

Using the equivalent dual formulation (3.11) of problem (3.8), we find numerically the solution u, hence the flux σ and the optimal density θ.

Now, using the dual formulation of the problem, we find numerically the solution u of (3.11) and we obtain the flux σ as explained in Section 3.2. The numerical procedure to find u uses a *Quasi-Newton* method that updates an approximation of the Hessian matrix at each iteration (see [48] and reference therein). First we generate a finite element space with respect to a square grid. Then we implement the BFGS method, using a routine included in the packages of software `FreeFem3D` (available at *http://www.freefem.org/ff3d*) that has the follow structure:

BFGS(J,dJ,u,eps=1.e-6,nbiter=20)

The routine above means: *find the optimal "u" for the functional J.* The necessary parameters are the functional J, the gradient dJ and the u variable. The value `eps` of the stop test and the number `nbiter` of iterations are fixed.

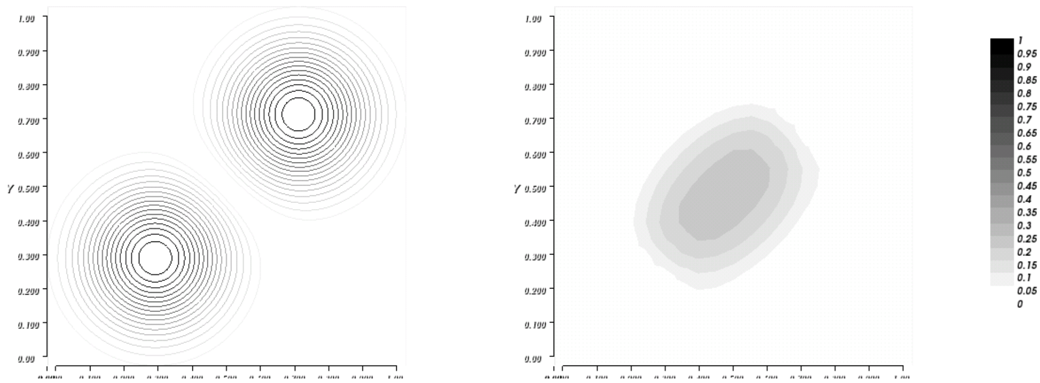
Example 3.7. The common setting of the simulation is a transportation domain $\Omega = [0, 1]^2$ with a 30×30 grid; we consider as initial and final distribution of resources two Gaussian approximations (with common variance λ) of Dirac delta function f^- and f^+

respectively centered at $x_0 = (0.3, 0.3)$ and $x_1 = (0.7, 0.7)$. In the examples below we take different values of the parameters k and λ according to the considerations above, to show how the optimal distributions of the low-congestion regions may vary. Using the same notation as in Section 3.2, there are black and white region (respectively $\theta = 1$ and $\theta = 0$), passing through grey levels for the intermediate congestion.

In Figure 3.1 we take the variance parameter $\lambda = 0.02$, which provides the initial and final mass distributions not too concentrated, as depicted in Figure 3.1 (a). In Figure 3.1 (b) we take the penalization parameter $k = 0.4$; we see that in this case, due to the low concentration of the initial and final mass distributions, the optimal density θ is higher in the region between x_0 and x_1.

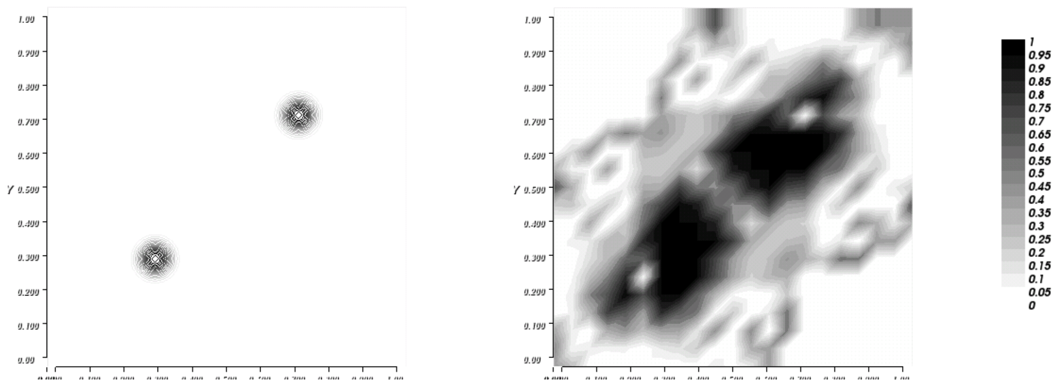
In Figure 3.2 we take the variance parameter $\lambda = 0.001$, which provides the initial and final mass distributions rather concentrated, as depicted in Figure 3.2 (a). In Figure 3.2 (b) we take the penalization parameter $k = 0.01$; we see that in this case, due to the high concentration of the initial and final mass distributions, the optimal density θ is high also in the region around x_0 and x_1.

The computational time results to be proportional to the number of point used to discretize the domain: when it is divided into a grid 30×30, the calculation time on a standard portable PC is about 10 sec.



(a) $\lambda = 0.02$ (b) $k = 0.4$

Figure 3.1:



(a) $\lambda = 0.001$ (b) $k = 0.01$

Figure 3.2:

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