Some fine properties of BV functions on Wiener spaces

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Abstract

In this paper we define jump set and approximate limits for BV functions on Wiener spaces and show that the weak gradient admits a decomposition similar to the finite dimensional case. We also define the SBV class of functions of special bounded variation and give a characterisation of SBV via a chain rule and a closure theorem. We also provide a characterisation of BV functions in terms of the short-time behaviour of the Ornstein-Uhlenbeck semigroup following an approach due to Ledoux.

1 Introduction

In this paper we shall continue the investigation of the properties of functions with bounded variation in infinite dimensional spaces, that is the setting of abstract Wiener spaces. The theory started with the papers [14], [15] where essentially a probabilistic approach was given and has been subsequently developed in [17], [5] with a more analytic approach.

Motivations for considering functions with bounded variation come from stochastic analysis, see e.g. [18], [28], [24], [25]; recently, in [23], properties of sets with finite Gaussian perimeter have been linked to some application in information technology. We point out also [26], for an application of BV functions to Lagrangian flows in Wiener spaces.

The aim of this paper is twofold; in Section 3 we give an equivalent characterisation of functions with bounded variation following an approach suggested by Ledoux in [19] and subsequently generalised in Euclidean spaces in [22]. In addition, in Section 4, following [6], [2], [3], we discuss the decomposition of the gradient of a BV function into absolutely continuous, jump and Cantor part.

We close the paper by introducing the notion of special function of bounded variation: the definition coincides with the original one given by De Giorgi and Ambrosio in [11]. We also give the characterization based on the chain rule proposed by Alberti and Mantegazza [1]; such characterization turns out to be very useful when giving closure and compactness results. We point out that for the compactness theorem, the only difference with respect to the Euclidean case, is that we have to assume a priori the strong convergence in L^2, from which we deduce the separate weak convergence of the two parts of the total variation measure. Given a set $E \subset X$ with finite perimeter, we deduce from this result the compactness w.r.t. the weak topology of $L^p(E,\gamma)$ of bounded and closed subsets of the Sobolev $H^{1,p}(E,\gamma)$, $1 < p < \infty$. This Sobolev space, defined in (5.4) below, consists of all $L^p(E,\gamma)$ functions whose null extension to $X \setminus E$ belongs to $SBV(X,\gamma)$ and has absolutely continuous part of the derivative in $L^p(E;H)$. This is related to the general problem of extension and traces of weakly differentiable functions defined on subsets of the Wiener space, see also [8], [10].

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2 Notation and preliminary results

Let us describe our framework. X is a separable Banach space endowed with a Gaussian probability measure $\gamma = \mathscr{N}(0, Q)$ on $\mathscr{B}(X)$, the Borel σ-algebra of X. We assume that γ is nondegenerate (i.e., all closed proper subspaces of X are γ-negligible) and centred (i.e., $\int_X x d\gamma = 0$). We denote by X^* the topological dual of X and by H the Cameron-Martin subspace of X, that is

$$H = \Big\{ \int_X f(x) x d\gamma(x) : \ f \in \mathscr{H} \Big\},$$

where $\mathscr{H}$ is the closure of X^* in $L^2(X, \gamma)$ and, for $h \in H$, we denote by $\hat{h} \in \mathscr{H}$ the Fomin derivative of γ along h, namely the function $\hat{h}$ verifying

$$\int_X \partial_h \varphi d\gamma = -\int_X \hat{h} \varphi d\gamma \qquad \forall\ \varphi \in \mathcal{F}C^1_b(X).$$

Here and in the sequel $\mathcal{F}C^1_b(X)$ denotes the space of continuously differentiable cylindrical functions in X, bounded and with a bounded gradient, i.e., the functions $\varphi : X \to \mathbb{R}$ such that there are $m \in \mathbb{N}$ and $v \in C^1_b(\mathbb{R}^m)$ such that $\varphi(x) = v(\langle x, x_1^*\rangle, \ldots, \langle x, x_m^*\rangle)$ for some $x_1^*, \ldots, x_m^* \in X^*$; the space $\mathcal{F}C^1_b(X, H)$ is the space of cylindrical C^1_b functions with values in H and finite dimensional image. The space H is endowed with the inner product $[\cdot, \cdot]_H$ and the norm $|\cdot|_H$ such that the map $\hat{h} \mapsto h$ is an isometry with respect to the $L^2(X, \gamma)$ Hilbert structure. Notice that the embedding $H \hookrightarrow X$ is compact. This framework has been introduced by L. Gross, see [16] and P. Malliavin, see [20] and also [7]. A summary of what we need here can be found in [21].

With a slight abuse of notation, we consider X^* as a subset of H, the subset of vectors of the form

$$\int_X \langle x, x^*\rangle x d\gamma(x), \qquad x^* \in X^*,$$

which is a dense (even w.r.t. to the Hilbertian norm) subspace of H. For $h = x^* \in X^*$, the corresponding $\hat{h}$ is precisely the map $x \mapsto \langle x, x^*\rangle$.

A relevant role in the sequel is played by the Ornstein-Uhlenbeck semigroup T_t, pointwise defined for $u \in L^1(X, \gamma)$ by Mehler's formula

$$T_t u(x) = \int_X u\left(e^{-t}x + \sqrt{1 - e^{-2t}} y\right) d\gamma(y), \quad t > 0. \tag{2.1}$$

Given an n-dimensional subspace $F \subset X^*$, we frequently consider an orthonormal basis $\{h_1, \ldots, h_n\}$ of F and the factorization $X = F \oplus ker(\pi_F)$, π_F is the continuous linear map

$$X \ni x \mapsto \pi_F(x) = \sum_{i=1}^n \hat{h}_i(x) h_i \in F.$$

The decomposition $x = \pi_F(x) + (x - \pi_F(x))$ is well defined because $\pi_F \circ \pi_F = \pi_F$ and so $x - \pi_F(x) \in ker(\pi_F)$; in turn, this follows from $\hat{h}_i(h_j) = \delta_{ij}$. Thanks to the fact that $|h_i|_H = 1$, this induces a factorization $\gamma = \gamma_F \otimes \gamma_F^\perp$, with γ_F the standard Gaussian in F (endowed with the metric inherited from H) and $\gamma_F^\perp$ Gaussian in $ker(\pi_F)$ with Cameron–Martin space $F^\perp$. Let us define the space of functions of bounded variation in X. First, let us recall the definition of the Orlicz space $L\log^{1/2}L(X, \gamma)$:

$$L\log^{1/2}L(X, \gamma) := \Big\{ u : X \to \mathbb{R} \text{ measurable} : A_{1/2}(\lambda |u|) \in L^1(X, \gamma) \ \text{ for some } \lambda > 0 \Big\},$$

endowed with the Luxemburg norm

$$\|u\|_{L\log^{1/2}L(X,\gamma)} := \inf\left\{\lambda > 0 :\ \int_X A_{1/2}(|u|/\lambda)\,d\gamma \le 1\right\}, \quad A_{1/2}(t) := \int_0^t \log^{1/2}(1+s)ds.$$

Given $h \in H$ and $f \in C_b^1(X)$, beside the directional derivative of f along h, denoted $\partial_h f$, we define the formal adjoint differential operator $\partial_h^* f = \partial_h f - \hat{h} f$ and, for $\varphi \in \mathcal{F}C_b^1(X,H)$, we define the divergence as follows: $\mathrm{div}_H \varphi = \sum_j \partial_{h_j}^* \varphi_j$, $\varphi_j = [\varphi, h_j]_H$.

Definition 2.1 *A function u is said to be of bounded variation, $u \in BV(X,\gamma)$, if $u \in L\log^{1/2}L(X,\gamma)$ and there exists $D_H u \in \mathscr{M}(X,H)$ (the space of H-valued Borel measures in X with finite total variation) for which*

$$\int_X u \mathrm{div}_H \varphi d\gamma = -\int_X [\varphi, dD_H u]_H, \qquad \forall \varphi \in \mathcal{F}C_b^1(X,H).$$

If we fix $h \in H$, we denote by μ_h the measure $[D_H u, h]_H$ defined as

$$\mu_h(B) = [D_H u(B), h]_H.$$

If in particular $u = \chi_E$ is the characteristic function of a measurable set E and $u \in BV(X,\gamma)$ we say that E has finite perimeter and set $P_H(E,\cdot) = |D_H u|(\cdot)$.

The study of $BV(X,\gamma)$ functions has been mainly developed so far for finite perimeter sets, see [17], [6], [2], [3], [9] and the first question that has been addressed is the identification of the subset of the topological boundary of E where the perimeter measure is concentrated. It is known that

$$|D_H \chi_E| = \mathcal{S}_{\mathcal{F}}^{\infty-1} \llcorner \partial_{\mathcal{F}}^* E = \mathcal{S}^{\infty-1} \llcorner E^{1/2}. \tag{2.2}$$

Let us explain the meaning of the above symbols. For an n-dimensional subspace $F \subset X^*$ as before, and for $y \in \mathrm{Ker}(\pi_F)$, we denote by

$$B_y := \{z \in F :\ y + z \in B\} \tag{2.3}$$

the section of $B \subset X$. Moreover, denoting by

$$G_n(z) := (2\pi)^{-n/2} \exp(-|z|^2/2)$$

the n-dimensional Gaussian kernel, we take advantage of the above described decomposition and define the pre-Hausdorff measures in X induced by F setting

$$\mathcal{S}_F^{\infty-1}(B) := \int^*_{\mathrm{Ker}(\pi_F)} \int_{B_y} G_n(z)\, d\mathcal{S}^{n-1}(z)\, d\gamma_F^\perp(y) \qquad \forall B \subset X. \tag{2.4}$$

Fixing an increasing family $\mathcal{F} = \{F_n\}_{n\ge 1}$ of finite-dimensional subspaces of X^*, whose union is dense in H, we define the $(\infty-1)$-dimensional Hausdorff spherical measures $\mathcal{S}_{\mathcal{F}}^{\infty-1}$basically introduced in [13], see also [17], [6], by setting:

$$\mathcal{S}_{\mathcal{F}}^{\infty-1} = \sup_n \mathcal{S}_{F_n}^{\infty-1}.$$

In the same vein, if E is a set with finite perimeter, we define the *cylindrical essential boundary* $\partial_{\mathcal{F}}^* E$ in the first equality in (2.2), by

$$\partial_{\mathcal{F}}^* E := \bigcup_{n\in\mathbb{N}} \bigcap_{k\ge n} \partial_{F_k}^* E, \tag{2.5}$$

where, with the usual notation,

$$\partial_F^* E := \{y + z : \ y \in \mathrm{Ker}(\pi_F), \ z \in \partial^* E_y\}$$

and $\partial^* E_y$ is the essential boundary of the section E_y in finite dimensions. The measure $\mathcal{S}^{\infty-1}$ is defined by taking the supremum of $\mathcal{S}_F^{\infty-1}$ when F runs along all the finite dimensional subspaces of X^*. For a comparison of the two approaches we refer to [2, Remark 2.6]. The set $E^{1/2}$ of points of density 1/2 is defined in [2] by using the semigroup T_t introduced in (2.1). Let $(t_i) \downarrow 0$ be such that $\sum_i \sqrt{t_i} < \infty$ and

$$\sum_{i=1}^{\infty} \int_X \Big| T_{t_i}\chi_E - \frac{1}{2} \Big| \, d|D_H \chi_E| < \infty.$$

We denote by $E^{1/2}$ the set

$$E^{1/2} = \Big\{ x \in X : \ \lim_{i\to\infty} T_{t_i}\chi_E(x) = \frac{1}{2} \Big\},$$

where we apply the semigroup T_t to the Borel representative of the set E, still denoted by E. Notice that the representation in the last term in (2.2) has the advantage of being coordinate-free. Thanks to (2.2), in all the statements that hold up to $|D\chi_E|$ negligible sets we may use both representations indifferently. Let us recall the main result of [3]. For $h \in H$, we define the halfspace having h as its "inner normal" by

$$S_h = \{x \in X : \hat{h}(x) > 0\}$$

and for E with finite perimeter we write $D_H\chi_E = \nu_E |D_H \chi_E|$. Then, see [3, Theorem 1.1], we can state the following result.

Theorem 2.2 *Let E be a set of finite perimeter in X and let $S(x) = S_{\nu_E(x)}$ be the halfspace determined by $\nu_E(x)$. Then*

$$\lim_{t\downarrow 0} \int_X \int_X \Big| \chi_E(e^{-t}x + \sqrt{1-e^{-2t}}y) - \chi_{S(x)}(y) \Big| d\gamma(y) \, d|D_H\chi_E|(x) = 0.$$

Let us draw a consequence that is useful later.

Corollary 2.3 *Given two finite perimeter sets E, F, the equality $\nu_E = \pm \nu_F$ holds $\mathcal{S}^{\infty-1}$-a.e. in $\partial_{\mathcal{F}}^* E \cap \partial_{\mathcal{F}}^* F$.*

Proof. If $E \subset F$ then this is an easy consequence of Theorem 2.2. Indeed, for $\mathcal{S}^{\infty-1}$-a.e. x the rescaled sets

$$E_{x,t} = \frac{E - e^{-t}x}{\sqrt{1-e^{-2t}}}, \qquad F_{x,t} = \frac{F - e^{-t}x}{\sqrt{1-e^{-2t}}}$$

converge in $L^2(X,\gamma)$ to two halfspaces $S_1(x)$, $S_2(x)$ respectively; the inclusion $E_{x,t} \subset F_{x,t}$ implies that

$$\gamma(S_1(x) \setminus S_2(x)) = \lim_{t\to 0} \gamma(E_{x,t} \setminus S_2(x)) \le \lim_{t\to 0} \gamma(F_{x,t} \setminus S_2(x)) = 0,$$

so that $S_1(x) \subset S_2(x)$ and then $S_1(x) = S_2(x)$ $\mathcal{S}^{\infty-1}$-a.e. $x \in \partial_{\mathcal{F}}^* E \cap \partial_{\mathcal{F}}^* F$. Since each halfspace is determined by the normal unit vector we get the thesis.
For the general case, notice that

$$\partial_{\mathcal{F}}^* E \subset \partial_{\mathcal{F}}^*(E \cup F) \cup \partial_{\mathcal{F}}^*(F \cap E^c),$$

whence, using the equality $\partial^*_{\mathcal{F}} G = \partial^*_{\mathcal{F}}(G^c)$, we deduce

$$\begin{aligned}\partial^*_{\mathcal{F}} E \cap \partial^*_{\mathcal{F}} F &\subset \big(\partial^*_{\mathcal{F}}(E\cup F)\cap \partial^*_{\mathcal{F}} F\big) \cup \big(\partial^*_{\mathcal{F}}(F\cap E^c)\cap \partial^*_{\mathcal{F}} F\big)\\ &= \big(\partial^*_{\mathcal{F}}(E\cup F)\cap \partial^*_{\mathcal{F}} F\big) \cup \big(\partial^*_{\mathcal{F}}(E\cup F^c)\cap \partial^*_{\mathcal{F}} F^c\big).\end{aligned}$$

Therefore, if $x \in \partial^*_{\mathcal{F}} E \cap \partial^*_{\mathcal{F}} F$ then either $x \in \partial^*_{\mathcal{F}}(E\cup F)$ or $x \in \partial^*_{\mathcal{F}}(E\cup F^c)$. Since E, $F \subset E\cup F$ and E, $F^c \subset E\cup F^c$ in both cases the equality $\nu_E(x) = \nu_F(x)$ (up to the sign) follows from the case $E \subset F$. QED

Remark 2.4 Notice that, by definition, if E has finite perimeter and $x \in \partial^*_{\mathcal{F}} E$ then there is $n \in \mathbb{N}$ such that $x \in \partial^*_{F_k} E$ for all $k \geq n$. Conversely, if there is $n \in \mathbb{N}$ such that $x \in \partial^*_{F_k} E$ for all $k \geq n$, then, by monotonicity, $x \in \partial^*_{\mathcal{F}} E$ as well. Therefore,

$$\text{if } u \in BV(X,\gamma) \text{ and } x \in X, \text{ the set } J_x = \{t \in \mathbb{R}:\ x \in \partial^*_{\mathcal{F}}\{u > t\}\} \text{ is a real interval.}$$

Indeed, if $s < t,\ s,t \in J_x$ then there is $n \in \mathbb{N}$ such that $x \in \partial^*_{F_k}\{u > s\} \cap \partial^*_{F_k}\{u > t\}$ for every $k \geq n$. But then $x \in \partial^*_{F_k}\{u > \tau\}$ for every $\tau \in [s,t]$ and $k \geq n$ and $x \in \partial^*_{\mathcal{F}}\{u > \tau\}$, which proves that J_x is an interval.

3 BV functions and the short time behaviour of T_t

The following characterisation of $BV(X,\gamma)$ functions in terms of the short-time behaviour of T_t is by now well-known:

$$u \in BV(X,\gamma) \quad \Longleftrightarrow \quad u \in L\log^{1/2}L(X,\gamma) \text{ and } \limsup_{t\to 0} \int_X |\nabla_H T_t u|_H \, d\gamma < \infty.$$

In this Section we present a second way to characterise sets and functions of bounded variation in terms of the semigroup; this approach was suggested, using the heat semigroup, by Ledoux [19] and subsequently investigated in [22]. Even though the results in this section are not necessary in the sequel of this paper, they seem to be worth presenting here, also in view of different applications, see e.g. [23]. If $E \subset X$ has finite perimeter, then

$$\lim_{t\to 0} \sqrt{\frac{\pi}{t}} \int_{E^c} T_t\chi_E(x) d\gamma(x) = P_H(E,X). \tag{3.1}$$

In addition, if

$$\liminf_{t\to 0} \frac{1}{\sqrt{t}} \int_{E^c} T_t\chi_E(x) d\gamma(x) < +\infty,$$

then E has finite perimeter and the limiting formula (3.1) holds. We notice that we can equivalently write

$$\begin{aligned}\int_{E^c} T_t\chi_E(x)d\gamma(x) &= \frac{1}{2}\|T_t\chi_E - \chi_E\|_{L^1(X,\gamma)} \qquad (3.2)\\ &= \frac{1}{2}\int_{X\times X} |\chi_E(e^{-t}x + \sqrt{1-e^{-2t}}y) - \chi_E(x)| d\gamma\otimes\gamma(x,y);\end{aligned}$$

let us also define the function

$$c_t = \sqrt{\frac{2}{\pi}} \int_0^t \frac{e^{-\tau}}{\sqrt{1-e^{-2\tau}}} d\tau = \sqrt{\frac{2}{\pi}} \Big(\frac{\pi}{2} - \arcsin(e^{-t})\Big) = \sqrt{\frac{2}{\pi}} \arccos(e^{-t}). \tag{3.3}$$

Since

$$\lim_{t\to 0}\frac{c_t}{\sqrt{t}}=\frac{2}{\sqrt{\pi}},$$

the characterisation (3.1) of sets with finite perimeter following the Ledoux approach is a consequence of Theorem 3.2 below. In the proof we need some properties of the conditional expectation and of the semigroup that are likely known. We prove them for the convenience of the reader, as we did not find a reference.

We denote by $\pi_n : X \to \mathbb{R}^n$ and $\Pi_n = \pi_n \times \pi_n : X \times X \to \mathbb{R}^{2n}$ the finite dimensional projections and by $\mathcal{F}_n$ and $\mathcal{F}_n \times \mathcal{F}_n$ the induced σ-algebras. By $p_1 : X \times X \to X$ and $p_1^n : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ we denote the projections on the first components and by $R_t : X \times X \to X \times X$ and $R_t^n : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ the rotations

$$(x,y)\mapsto (e^{-t}x+\sqrt{1-e^{-2t}}y, -\sqrt{1-e^{-2t}}x+e^{-t}y). \tag{3.4}$$

Lemma 3.1 *Let π_n, Π_n, $\mathcal{F}_n$ and $\mathcal{F}_n \times \mathcal{F}_n$ be as before; then*

1) *for any $F \in L^1(X\times X, \gamma\otimes\gamma)$,*

$$\mathbb{E}\Big(F\circ R_t\Big|\mathcal{F}_n\times\mathcal{F}_n\Big)=\mathbb{E}\Big(F\Big|\mathcal{F}_n\times\mathcal{F}_n\Big)\circ R_t;$$

2) *for any $f\in L^1(X,\gamma)$,*

$$\mathbb{E}\Big(f\circ p_1\Big|\mathcal{F}_n\times\mathcal{F}_n\Big)=\mathbb{E}\Big(f\Big|\mathcal{F}_n\Big)\circ p_1.$$

In addition, if T_s^{2n} and T_s^n denote the Ornstein–Uhlenbeck semigroups on $\mathbb{R}^{2n}$ and $\mathbb{R}^n$ respectively, then

3) *for any $F\in L^1(\mathbb{R}^{2n},\gamma_{2n})$,*

$$T_s^{2n}(F\circ R_t^n)=T_s^{2n}(F)\circ R_t^n;$$

4) *for any $f\in L^1(\mathbb{R}^n,\gamma_n)$ and any $(x,y)\in\mathbb{R}^{2n}$ there holds*

$$T_s^{2n}(f\circ p_1^n)(x,y)=T_s^n(f)(x),$$

where indeed the function $T_s^{2n}(f\circ p_1^n)$ depends only on the first n variables in $\mathbb{R}^{2n}$.

PROOF.

1) We fix $A\in\mathcal{F}_n\times\mathcal{F}_n$, that is $A=\Pi_n^{-1}(A_n)$ with $A_n\subset\mathbb{R}^{2n}$; since $R_t(A)=\Pi_n^{-1}(R_t^n(A_n))\in\mathcal{F}_n\times\mathcal{F}_n$, we obtain

$$\begin{aligned}\int_A \mathbb{E}\Big(F\circ R_t\Big|\mathcal{F}_n\times\mathcal{F}_n\Big)d\gamma\otimes\gamma &= \int_A F\circ R_t d\gamma\otimes\gamma\\ &=\int_{R_t(A)} Fd\gamma\otimes\gamma=\int_{R_t(A)}\mathbb{E}\Big(F\Big|\mathcal{F}_n\times\mathcal{F}_n\Big)d\gamma\otimes\gamma\\ &=\int_A \mathbb{E}\Big(F\Big|\mathcal{F}_n\times\mathcal{F}_n\Big)\circ R_t d\gamma\otimes\gamma.\end{aligned}$$

2) We take $A\times B\in\mathcal{F}_n\times\mathcal{F}_n$ with $A=\pi_n(A_n)$, $B=\pi_n^{-1}(B_n)$, $A_n,B_n\subset\mathbb{R}^n$. Then

$$\begin{aligned}\int_{A\times B}\mathbb{E}\Big(f\circ p_1\Big|\mathcal{F}_n\times\mathcal{F}_n\Big)d\gamma\otimes\gamma &=\int_{A\times B} f\circ p_1 d\gamma\otimes\gamma=\gamma(B)\int_A f d\gamma\\ &=\gamma(B)\int_A \mathbb{E}\Big(f\Big|\mathcal{F}_n\Big)d\gamma=\int_{A\times B}\mathbb{E}\Big(f\Big|\mathcal{F}_n\Big)\circ p_1 d\gamma\otimes\gamma.\end{aligned}$$

The general statement follows since the sets of the form $A\times B$ form a basis for the σ–algebra $\mathcal{F}_n\times\mathcal{F}_n$.

3) Fix $F \in L^1(\mathbb{R}^{2n}, \gamma_{2n})$, then

$$
\begin{aligned}
T_s^{2n} F \circ R_t^n(x,y) = & \int_{\mathbb{R}^{2n}} F \circ R_t^n(e^{-s}(x,y) + \sqrt{1-e^{-2s}}(\bar{x},\bar{y})) d\gamma_{2n}(\bar{x},\bar{y}) \\
= & \int_{\mathbb{R}^{2n}} F\Big(e^{-t}(e^{-s}x + \sqrt{1-e^{-2s}}\bar{x}) + \sqrt{1-e^{-2t}}(e^{-s}y + \sqrt{1-e^{-2s}}\bar{y}), \\
& -\sqrt{1-e^{-2t}}(e^{-s}x + \sqrt{1-e^{-2s}}\bar{x}) + e^{-t}(e^{-s}y + \sqrt{1-e^{-2s}}\bar{y})\Big) d\gamma(\bar{x},\bar{y}) \\
= & \int_{\mathbb{R}^{2n}} F\Big(e^{-s}(e^{-t}x + \sqrt{1-e^{-2t}}y) + \sqrt{1-e^{-2s}}(e^{-t}\bar{x} + \sqrt{1-e^{-2t}}\bar{y}), \\
& e^{-s}(-\sqrt{1-e^{-2t}}x + e^{-t}y) + \sqrt{1-e^{-2s}}(-\sqrt{1-e^{1-2t}}\bar{x} + e^{-t}\bar{y})\Big) d\gamma(\bar{x},\bar{y}) \\
= & \int_{\mathbb{R}^{2n}} F\Big(e^{-s}(e^{-t}x + \sqrt{1-e^{-2t}}y, -\sqrt{1-e^{-2t}}x + e^{-t}y) + \sqrt{1-e^{-2s}}(\bar{x},\bar{y})\Big) d\gamma(\bar{x},\bar{y}) \\
= & T_s^{2n}(F) \circ R_t(x,y).
\end{aligned}
$$

4) Let $f \in L^1(\mathbb{R}^n, \gamma_n)$; then

$$
\begin{aligned}
T_s^{2n}(f \circ p_1^n)(x,y) = & \int_{\mathbb{R}^{2n}} f \circ p_1^n(e^{-s}(x,y) + \sqrt{1-e^{-2s}}(\bar{x},\bar{y})) d\gamma_{2n}(\bar{x},\bar{y}) \\
= & \int_{\mathbb{R}^{2n}} f(e^{-s}x + \sqrt{1-e^{-2s}}\bar{x}) d\gamma_{2n}(\bar{x},\bar{y}) \\
= & \int_{\mathbb{R}^{n}} f(e^{-s}x + \sqrt{1-e^{-2s}}\bar{x}) d\gamma_{n}(\bar{x}) = T_s^n(f)(x).
\end{aligned}
$$

QED

Theorem 3.2 *Let $u \in BV(X,\gamma)$ and c_t defined in* (3.3)*; then for any $t > 0$*

$$
\int_{X\times X} |u(e^{-t}x + \sqrt{1-e^{-2t}}y) - u(x)| d\gamma \otimes \gamma(x,y) \le c_t |D_H u|(X). \tag{3.5}
$$

Conversely, given $u \in L\log^{1/2} L(X,\gamma)$, if there exists $C > 0$ such that

$$
\int_{X\times X} |u(e^{-t}x + \sqrt{1-e^{-2t}}y) - u(x)| d\gamma \otimes \gamma(x,y) \le C c_t, \quad \forall t > 0, \tag{3.6}
$$

then $u \in BV(X,\gamma)$. In addition, the following limit holds

$$
\lim_{t\to 0} \frac{1}{c_t} \int_{X\times X} |u(e^{-t}x + \sqrt{1-e^{-2t}}y) - u(x)| d\gamma \otimes \gamma(x,y) = |D_H u|(X). \tag{3.7}
$$

Proof. The first part of the proof is based on [2, Lemma 2.3]. We start by considering a function $v \in C_b^1(\mathbb{R}^n)$; then denoting by γ_n and γ_{2n} the standard Gaussian measures on $\mathbb{R}^n$ and $\mathbb{R}^{2n}$ respectively, and using the rotation invariance of the Gaussian measure, that is the fact that $R_{\tau\sharp}\gamma_{2n} = \gamma_{2n}$, where $R_\tau : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is the map defined in (3.4) and $R_{\tau\sharp}$ is the push-forward operator, we obtain the following estimate:

$$
\begin{aligned}
\int_{\mathbb{R}^{2n}} |v(e^{-t}x + \sqrt{1-e^{-2t}}y) - v(x)| d\gamma_{2n}(x,y) = & \int_{\mathbb{R}^{2n}} \left| \int_0^t \frac{d}{d\tau} v(e^{-\tau}x + \sqrt{1-e^{-2\tau}}y) d\tau \right| d\gamma_{2n}(x,y) \\
= \int_{\mathbb{R}^{2n}} \left| \int_0^t \frac{e^{-\tau}}{\sqrt{1-e^{-2\tau}}} \nabla v(e^{-\tau}x + \sqrt{1-e^{-2\tau}}y) \cdot (-\sqrt{1-e^{-2\tau}}x + e^{-\tau}y) d\tau \right| & d\gamma_{2n}(x,y)
\end{aligned}
$$

$$
\begin{aligned}
&\le \int_0^t \frac{e^{-\tau}}{\sqrt{1-e^{-2\tau}}} \int_{\mathbb{R}^{2n}} \left| \nabla v(e^{-\tau}x + \sqrt{1-e^{-2\tau}}y) \cdot (-\sqrt{1-e^{-2\tau}}x + e^{-\tau}y) \right| d\gamma_{2n}(x,y) d\tau \\
&= \int_{\mathbb{R}^{2n}} \int_0^t \frac{e^{-\tau}}{\sqrt{1-e^{-2\tau}}} d\tau \, |\nabla v(x) \cdot y| \, d\gamma_{2n}(x,y) \\
&= c_t \int_{\mathbb{R}^n} |\nabla v(x)| d\gamma_n(x).
\end{aligned}
\tag{3.8}
$$

Here we have used the equality $\int_{\mathbb{R}^n} |\nabla v(x) \cdot y| d\gamma_n(y) = |\nabla v(x)| \sqrt{2/\pi}$. Notice that taking a sequence of $\mathcal{F}C_b^1$ functions that converges in variation, i.e., $v_k \to v$ in $L^1(X,\gamma)$ such that $|D_H v_k|(X) \to |D_H v|(X)$, the above estimate holds for every $v \in BV(X,\gamma)$. We now show that

$$
\lim_{t\to 0} \frac{1}{c_t} \int_{\mathbb{R}^{2n}} |v(e^{-t}x + \sqrt{1-e^{-2t}}y) - v(x)| d\gamma_{2n}(x,y) = \int_{\mathbb{R}^n} |\nabla v(x)| d\gamma_n(x). \tag{3.9}
$$

Indeed, the linear functionals on $C_b(\mathbb{R}^{2n})$

$$
L_t\varphi := \frac{1}{c_t} \int_{\mathbb{R}^{2n}} \varphi(x,y)(v(e^{-t}x + \sqrt{1-e^{-2t}}y) - v(x)) d\gamma_{2n}(x,y)
$$

have, thanks to (3.8), norm uniformly bounded by

$$
\|L_t\| \le \int_{\mathbb{R}^n} |\nabla v(x)| d\gamma_n(x).
$$

In addition

$$
\begin{aligned}
&\lim_{t\to 0} L_t\varphi = \\
&= \lim_{t\to 0} \int_{\mathbb{R}^{2n}} \frac{1}{c_t} \int_0^t \frac{e^{-\tau}}{\sqrt{1-e^{-2\tau}}} \varphi(e^{-\tau}x - \sqrt{1-e^{-2\tau}}y, \sqrt{1-e^{-2\tau}}x + e^{-\tau}y) d\tau \nabla v(x) \cdot y d\gamma_{2n}(x,y) \\
&= \int_{\mathbb{R}^{2n}} \varphi(x,y) \nabla v(x) \cdot y d\gamma_{2n}(x,y) := L_0\varphi.
\end{aligned}
$$

Then the functionals L_t weakly* converge to the functional L_0 and

$$
\|L_0\| = \int_{\mathbb{R}^n} |\nabla v(x)| d\gamma_n(x) \le \liminf_{t\to 0} \|L_t\| \le \int_{\mathbb{R}^n} |\nabla v(x)| d\gamma_n(x).
$$

If now $u \in BV(X,\gamma)$, we can consider a cylindrical approximation $u_j = v_j \circ \pi_j$, with $v_j \in C_b^1(\mathbb{R}^{n_j})$ and $\pi_j : X \to \mathbb{R}^{n_j}$ a projection induced by orthonormal elements $h_1, \ldots, h_{n_j} \in H$; the cylindrical approximation can be chosen in such a way that

$$
\lim_{j\to+\infty} \|u_j - u\|_{L^1(X,\gamma)} = 0, \qquad \lim_{j\to+\infty} \int_X |\nabla_H u_j|_H d\gamma = |D_H u|(X).
$$

The convergence in $L^1(X,\gamma)$ implies that

$$
\begin{aligned}
&\int_{X\times X} |u(e^{-t}x+\sqrt{1-e^{-2t}}y)-u(x)|d\gamma\otimes\gamma(x,y)\\
&\quad=\lim_{j\to+\infty}\int_{X\times X} |u_j(e^{-t}x+\sqrt{1-e^{-2t}}y)-u_j(x)|d\gamma\otimes\gamma(x,y)\\
&\quad=\lim_{j\to+\infty}\int_{\mathbb{R}^{2n_j}} |v_j(e^{-t}x+\sqrt{1-e^{-2t}}y)-v_j(x)|d\gamma_{2n_j}(x,y)\\
&\quad\le c_t\lim_{j\to+\infty}\int_{\mathbb{R}^{n_j}} |\nabla v_j(x)|d\gamma_{n_j}(x)\\
&\quad=c_t\lim_{j\to+\infty}\int_{X} |\nabla_H u_j(x)|_H d\gamma(x)\\
&\quad=c_t|D_H u|(X),
\end{aligned}
$$

which proves the inequality in (3.5).

Let now fix $u\in L\log^{1/2}L(X,\gamma)$ and assume (3.6), that we can rewrite as

$$
\int_{X\times X} |u\circ p_1\circ R_t(x,y)-u\circ p_1(x,y)|d\gamma\otimes\gamma(x,y)\le Cc_t,
$$

where $p_1 : X\times X\to X$ is the projection $p_1(x,y)=x$ and R_t is the rotation defined in (3.4). Then, if $\mathcal{F}_n$ is the σ–algebra induced by the projection $\pi_n : X\to\mathbb{R}^n$ and $\mathcal{F}_n\times\mathcal{F}_n$ the σ–algebra induced by $\Pi_n : X\times X\to\mathbb{R}^{2n}$, $\Pi_n(x,y)=(\pi_n(x),\pi_n(y))$, we have that

$$
\begin{aligned}
\int_{X\times X} |u\circ p_1\circ R_t-u\circ p_1|d\gamma\otimes\gamma &=\int_{X\times X}\mathbb{E}\Big(|u\circ p_1\circ R_t-u\circ p_1|\Big|\mathcal{F}_n\times\mathcal{F}_n\Big)d\gamma\otimes\gamma\\
&\ge\int_{X\times X}\Big|\mathbb{E}\Big(u\circ p_1\circ R_t-u\circ p_1\Big|\mathcal{F}_n\times\mathcal{F}_n\Big)\Big|d\gamma\otimes\gamma\\
&=\int_{X\times X}\Big|\mathbb{E}\Big(u\Big|\mathcal{F}_n\Big)\circ p_1\circ R_t-\mathbb{E}\Big(u\Big|\mathcal{F}_n\Big)\circ p_1\Big|d\gamma\otimes\gamma,
\end{aligned}
$$

where we have used the properties (1), (2) of the conditional expectation stated in Lemma 3.1. So we may assume that $v\in L\log^{1/2}L(\mathbb{R}^n,\gamma_n)$ is a function such that

$$
\int_{\mathbb{R}^{2n}} |v\circ p_1\circ R_t-v\circ p_1|d\gamma_{2n}\le Cc_t
$$

and we prove that $v\in BV(\mathbb{R}^n,\gamma_n)$. If we denote by $(T^n_s)_s$ and $(T^{2n}_s)_s$ the Ornstein–Uhlenbeck semigroups on $\mathbb{R}^n$ and $\mathbb{R}^{2n}$ respectively, then since they are mass preserving, we have that

$$
\begin{aligned}
\int_{\mathbb{R}^{2n}} |v\circ p_1\circ R_t-v\circ p_1|d\gamma_{2n} &=\int_{\mathbb{R}^{2n}} T^{2n}_s|v\circ p_1\circ R_t-v\circ p_1|d\gamma_{2n}\\
&\ge\int_{\mathbb{R}^{2n}} |T^{2n}_s(v\circ p_1\circ R_t)-T^{2n}_s(v\circ p_1)|d\gamma_{2n}\\
&=\int_{\mathbb{R}^{2n}} |T^{n}_s(v)\circ p_1\circ R_t-T^{n}_s(v)\circ p_1|d\gamma_{2n}.
\end{aligned}
$$

Thanks to the previous arguments, we obtain that

$$
\int_{\mathbb{R}^n} |\nabla T^n_s v|d\gamma_n\le C,
$$

which implies $v \in BV(\mathbb{R}^n, \gamma_n)$ with

$$|D_{\mathbb{R}^n} v|(\mathbb{R}^n) \le \liminf_{s\to 0} \int_{\mathbb{R}^n} |\nabla T^n_s v| d\gamma_n \le C.$$

The same conclusion holds for $u \in L\log^{1/2} L(X,\gamma)$, by taking conditional expectations. The limiting formula

$$|D_H u|(X) = \lim_{t\to 0} \frac{1}{c_t} \int_{X\times X} |u\circ p_1 \circ R_t - u\circ p_1| d\gamma\otimes\gamma, \qquad \forall u \in BV(X,\gamma),$$

follows from the inequalities

$$\begin{aligned}\int_{X\times X} |u\circ p_1\circ R_t - u\circ p_1| d\gamma\otimes\gamma &\ge \int_{X\times X} |u_n\circ p_1\circ R_t - u_n\circ p_1| d\gamma\otimes\gamma \\ &\ge \int_{\mathbb{R}^{2n}} |T^n_s v_n \circ p_1\circ R_t - T^n_s v_n\circ p_1| d\gamma_{2n},\end{aligned}$$

where $u_n = \mathbb{E}\Big(u\Big|\mathcal{F}_n\Big) = v_n\circ\pi_n$. Indeed

$$\liminf_{t\to 0} \frac{1}{c_t} \int_{\mathbb{R}^{2n}} |T^n_s v_n\circ p_1\circ R_t - T^n_s v_n\circ p_1| d\gamma_{2n} = \int_{\mathbb{R}^n} |\nabla T^n_s v_n| d\gamma_n = \int_X |\nabla_H T_s u_n| d\gamma$$

which is true for any $n\in\mathbb{N}$ and $s>0$; the result then follows by letting $n\to+\infty$ and $s\to 0$. QED

Proposition 3.3 *For every $E\subset X$ such that $0<\gamma(E)<1$ the following isoperimetric type inequality holds:*

$$\frac{P_H(E,X)}{\gamma(E)\gamma(E^c)} \ge \frac{2\sqrt{2}}{\sqrt{\pi}}. \tag{3.10}$$

Moreover, equality holds if and only if E is a halfspace.

Proof. Starting from (3.2) and (3.5) we get

$$\int_{E^c} T_t\chi_E d\gamma \le \frac{c_t}{2} P_H(E,X), \tag{3.11}$$

whence, taking into account that

$$\lim_{t\to+\infty} c_t = \sqrt{\frac{\pi}{2}}, \quad \lim_{t\to+\infty} T_t\chi_E(x) = \gamma(E),$$

we deduce (3.10) by taking the limit as $t\to\infty$ in (3.11). Let us prove that the only set attaining equality in (3.10) is a halfspace

$$E^h_\alpha = \{\hat{h}\le\alpha\}, \qquad h\in H,$$

with $\alpha=0$. Indeed, by Ehrhard symmetrisation, if E is a set with finite perimeter, then

$$P_H(E,X) \ge P_H(E^s,X),$$

where E^s is a halfspace with $\gamma(E)=\gamma(E^s)$; in addition, equality holds in the last estimate if and only if E is equivalent to a halfspace. Then

$$\frac{P_H(E,X)}{\gamma(E)\gamma(E^c)} \ge \frac{P_H(E^s,X)}{\gamma(E^s)\gamma((E^s)^c)} \ge \frac{2\sqrt{2}}{\sqrt{\pi}}.$$

The first inequality is an equality if and only E is a halfspace; for the second inequality, if we take $E^s = E^h_\alpha$ with $|h|_H = 1$, then an explicit computation yields

$$\frac{P_H(E^h_\alpha, X)}{\gamma(E^h_\alpha)\gamma((E^h_\alpha)^c)} = \frac{\sqrt{2\pi}e^{-\frac{\alpha^2}{2}}}{\int_{-\infty}^{\alpha} e^{-\frac{t^2}{2}}dt \int_{\alpha}^{+\infty} e^{-\frac{t^2}{2}}dt} =: F(\alpha).$$

We have then shown that $F(\alpha) \geq \frac{2\sqrt{2}}{\sqrt{\pi}}$; a direct computation shows that the only solution of the equation

$$F(\alpha) = \frac{2\sqrt{2}}{\sqrt{\pi}}$$

is $\alpha = 0$. QED

Remark 3.4 For the halfspace $E_0 = E^h_0$, $|h|_H = 1$, the following equality holds:

$$\int_{E_0^c} T_t\chi_{E_0}(x)d\gamma(x) = \frac{c_t}{2}P_H(E_0, X), \qquad \forall t \geq 0.$$

Moreover, it is the only set with finite perimeter with this property.
First, let us explicitly compute the quantity

$$\int_{E_0^c} T_t\chi_{E_0}(x)d\gamma(x).$$

Using the fact that the γ-measurable linear functional $x \mapsto \hat{h}(x)$ has Gaussian law, by writing $\chi_{E_0}(x) = \chi_{(-\infty,0)}(\hat{h}(x))$, we may write

$$\begin{aligned}\int_{E_0^c} T_t\chi_{E_0}(x)d\gamma(x) =& \frac{1}{\sqrt{2\pi}}\int_{E_0^c}\int_{\mathbb{R}} \chi_{(-\infty,0)}(e^{-t}\hat{h}(x) + \sqrt{1-e^{-2t}}b)e^{-\frac{b^2}{2}}db \\ =& \frac{1}{2\pi}\int_{\mathbb{R}^2} \chi_{(0,+\infty)}(a)\chi_{(-\infty,0)}(e^{-t}a + \sqrt{1-e^{-2t}}b)e^{-\frac{a^2+b^2}{2}}dadb \\ =& \frac{c_t}{2}P_H(E_0, X),\end{aligned}$$

as claimed. For the uniqueness, if E is such that

$$\int_{E^c} T_t\chi_E d\gamma = \frac{c_t}{2}P_H(E, X), \qquad \forall t > 0,$$

passing to the limit as $t \to +\infty$, we find that E is such that

$$\frac{P_H(E, X)}{\gamma(E)\gamma(E^c)} = \frac{2\sqrt{2}}{\sqrt{\pi}}$$

and then $E = E^h_0$ for some $h \in H$. Analogously, if a function u satisfies

$$\int_{X\times X} |u(e^{-t}x + \sqrt{e^{-2t}}y)|d\gamma\otimes\gamma(x, y) = c_t|D_H u|(X), \qquad \forall t > 0,$$

then by coarea formula, for almost every $\tau \in \mathbb{R}$, the sets $E_\tau = \{u > \tau\}$ are sets with finite perimeter such that

$$\int_{E_\tau^c} T_t\chi_{E_\tau} d\gamma = \frac{c_t}{2}P_H(E_\tau, X), \qquad \forall t > 0,$$

and then $u = a + b\chi_{E^h_0}$ for some $a, b \in \mathbb{R}$ and $h \in H$.

4 Decomposition of the gradient and chain rule

In this section we discuss a few finer properties of a function u with bounded variation. If we fix $h \in H$, recall that we are denoting by μ_h the measure $[D_H u, h]_H$ defined as

$$\mu_h(B) = [D_H u(B), h]_H.$$

The measure $D_H u$ can be decomposed into an absolutely continuous part $D_H^a u$ with respect to γ, whose density is denoted by $\nabla_H u$, and a singular part $D_H^\sigma u$, as follows

$$D_H u = D_H^a u + D_H^\sigma u = \nabla_H u \gamma + D_H^\sigma u. \tag{4.1}$$

In this way the measure μ_h admits the Radon-Nikodym decomposition

$$\mu_h = \partial_h u \gamma + \mu_h^\sigma,$$

where $\mu_h^\sigma = [D_H^\sigma u, h]_H \perp \gamma$ and we have used the equality $\partial_h u = [\nabla_H u, h]_H$.

We recall also that if we write $X = F \oplus F^\perp$ with $F = \mathrm{span}\{h\}$ and $F^\perp = \ker \pi_F$, then by defining $u_y(t) = u(y + th)$ we have that for $\gamma_F^\perp$-a.e. $y \in F^\perp$, $u_y \in BV(F, \gamma_F)$; we define

$$B_F = \{y \in F^\perp : u_y \in BV(F, \gamma_F)\},$$

with $\gamma_F^\perp(B_F) = 1$. For all $y \in B_F$ we write

$$D_F u_y = u_y' \gamma_F + D_F^\sigma u_y.$$

We recall the formula

$$[D_H u, h]_H = (\gamma_F^\perp \llcorner B_F) \otimes D_F u_y \tag{4.2}$$

(a simple consequence of Fubini's theorem) and we analyse in the next lemma the effect of the decomposition in absolutely continuous and singular part.

Lemma 4.1 *Let $u \in BV(X, \gamma)$ and $h \in H$; then, for $\gamma_F^\perp$-a.e. $y \in F^\perp$,*

$$\mu_h^a = (\gamma_F^\perp \llcorner B_F) \otimes (u_y' \gamma_F) \tag{4.3}$$

and

$$\mu_h^\sigma = (\gamma_F^\perp \llcorner B_F) \otimes (D_F^\sigma u_y). \tag{4.4}$$

As a consequence, for $\gamma_F^\perp$-a.e. $y \in F^\perp$ there holds $u_y'(z) = (\partial_h u)_y(z)$ for γ_F-a.e. $z \in F$.

Proof. Obviously, $(\gamma_F^\perp \llcorner B_F) \otimes (u_y' \gamma_F) \ll \gamma$. Let us prove that $(\gamma_F^\perp \llcorner B_F) \otimes (D_F^\sigma u_y) \perp \gamma$. Notice that for $\gamma_F^\perp$-a.e. $y \in F^\perp$ the measure $D_F^\sigma u_y$ is singular with respect to γ_F, so that we may define the γ_F-negligible Borel set

$$A_y = \left\{ t \in \mathbb{R} : \lim_{r \to 0} \frac{|Du_y|(t - r, t + r)}{r} = +\infty \right\},$$

and we have $D_F^\sigma u_y(B) = D_F^\sigma u_y(A_y \cap B)$, see [4, section 3.2]. Setting $A_y = \mathbb{R}$ if $y \in F^\perp \setminus B_F$, define

$$A = \{x = y + z,\ y \in F^\perp,\ z \in A_y\}$$

and observe that

$$\gamma(A) = \int_{F^\perp} \gamma_F(A_y) d\gamma_F^\perp(y) = 0.$$

Since $\mu_h^\sigma(B) = \mu_h^\sigma(B \cap A)$, we deduce that $\mu_h^\sigma \perp \gamma$. By (4.2) and the uniqueness of the Radon-Nikodym decomposition we get that μ_h^a and μ_h^σ are given by (4.3), (4.4). The fact that $u_y' = (\partial_h u)_y$ $\gamma_F^\perp$-a.e. for $\gamma_F^\perp$ a.e. $y \in F^\perp$ is then an easy consequence. QED

Lemma 4.2 *Let $\{\lambda_h : \ h \in H\}$ be a family of signed measures on X linear in h, i.e., such that $\lambda_{ah_1+bh_2} = a\lambda_{h_1} + b\lambda_{h_2}$, and continuous, i.e., for every $\varepsilon > 0$ there is $\delta > 0$ such that $|h|_H < \delta$ implies $|\lambda_h|(X) < \varepsilon$. Assuming*

$$\bigvee_{\{|h|_H=1\}} \lambda_h(X) := \sup\Big\{\sum_{j=1}^{\infty} |\lambda_{k_j}(A_j)| : \ X = \bigcup_{j=1}^{\infty} A_j, \ (A_j) \text{ pairwise disjoint}, \ |k_j| = 1\Big\} < \infty, \quad (4.5)$$

for every orthonormal basis (h_j) of H the set function

$$\lambda(B) = \sum_{j=1}^{\infty} \lambda_{h_j}(B) h_j$$

belongs to $\mathscr{M}(X, H)$. In particular, the families of measures

$$h \mapsto \lambda_h^J(B) = \int_{h^\perp} D^J_{\gamma_1} u_y(B_y)\, d\gamma_F^\perp(y), \qquad B \in \mathcal{B}(X) \quad (4.6)$$

$$h \mapsto \lambda_h^C(B) = \int_{h^\perp} D^C_{\gamma_1} u_y(B_y)\, d\gamma_F^\perp(y), \qquad B \in \mathcal{B}(X) \quad (4.7)$$

define two H-valued measures denoted by $D^J_H u$ and $D^C_H u$.

Proof. Notice that

$$\begin{aligned} |\lambda|(X) &= \sup\Big\{\sum_{j=1}^{\infty} |\lambda(B_j)|_H : \ X = \bigcup_{j=1}^{\infty} B_j, \ (B_j) \text{ pairwise disjoint}\Big\} \\ &= \sup\Big\{\lim_{N\to\infty} \sum_{j=1}^{N} |\lambda(B_j)|_H : \ X = \bigcup_{j=1}^{\infty} B_j, \ (B_j) \text{ pairwise disjoint}\Big\}; \end{aligned}$$

but by the linearity if (B_j) is a countable partition of X such that $\lambda(B_j) \neq 0$ for all $j \in \mathbb{N}$, setting $k_j = \frac{\lambda(B_j)}{|\lambda(B_j)|_H}$, for every $N \in \mathbb{N}$ we have

$$\sum_{j=1}^{N} |\lambda(B_j)|_H = \sum_{j=1}^{N} |[\lambda(B_j), k_j]_H k_j|_H = \sum_{j=1}^{N} |\lambda_{k_j}(B_j)| \le \bigvee_{\{|h|_H=1\}} \lambda_h(X) < \infty.$$

It follows that $\lambda(B) \in H$ for any $B \in \mathscr{B}(X)$, that λ belongs to $\mathscr{M}(X, H)$ and the definition is independent of the basis.
Let us show that the set functions defined in (4.6), (4.7) verify the hypotheses of linearity and continuity with respect to h and also the boundedness assumption (4.5). The linearity of λ_h^J, λ_h^C follows basically arguing as in [5, Proposition 4.8], see also [4, Theorem 3.108]. Indeed, take $h \in F = \text{span}\{h_1, h_2\}$. Then,

$$\lambda_h^J(B) = \int_{h^\perp} D^J_h u_y(B_y)\, d\gamma_h^\perp(y) = \int_{F^\perp} [D^J_F u_y(B_y), h]_H, d\gamma_F^\perp(y)$$

by the finite dimensional result. Moreover, the boundedness follows from

$$\bigvee_{\{|h|_H=1\}} \lambda_h^J(X), \quad \bigvee_{\{|h|_H=1\}} \lambda_h^C(X) \le |D_H u|(X) < \infty.$$

QED

According to Lemma 4.2, we define the jump and the Cantor parts of $D_H u$ as

$$D_H^J u = \sum_{j=1}^{\infty} \lambda_{h_j}^J h_j, \tag{4.8}$$

$$D_H^C u = \sum_{j=1}^{\infty} \lambda_{h_j}^C h_j, \tag{4.9}$$

so that $D_H^\sigma u = D_H^J u + D_H^C u$ and

$$D_H u = D_H^a u + D_H^\sigma u = D_H^a u + D_H^J u + D_H^C u. \tag{4.10}$$

In the sequel, we use the notation $D_H^d u = \nabla_H u d\gamma + D_H^C u$ for the part of the measure $D_H u$ out of the jump set. The next goal is to identify the sets where such measures are concentrated. Indeed, the jump part is concentrated on a set which is σ-finite with respect to the $\mathcal{S}^{\infty-1}$ measure (the discontinuity set), whereas the Cantor part is concentrated on a bigger set.

Lemma 4.3 *If $B \subset X$ is a Borel set with $\mathcal{S}^{\infty-1}(B) < \infty$ then $|D_H^C u|(B) = 0$.*

Proof. Fix $h \in H$ and write

$$D_H^C u(B) = \int_{h^\perp} D_h^C u_y(B_y)\, d\gamma_h^\perp(y) = 0$$

because B_y is a σ-finite set for $\gamma^\perp$-a.e. $y \in h^\perp$ and $D_h^C u_y(B_y) = 0$ by the analogous one-dimensional property. QED

Let us consider the jump part. In the following definition we think of $\mathcal{F}$ as a fixed increasing sequence of finite dimensional subspaces of X^*, as explained in Section 2.

Definition 4.4 *Let $u \in BV(X, \gamma)$ and let $\mathcal{D} \subset \mathbb{R}$ be a countable dense set such that $\{u > t\}$ has finite perimeter for all $t \in \mathcal{D}$. Define the discontinuity set of u as*

$$S(u) = \bigcup_{s,\, t\in\mathcal{D},\, s\neq t} \Big(\partial_{\mathcal{F}}^*\{u > s\} \cap \partial_{\mathcal{F}}^*\{u > t\}\Big).$$

By definition, $S(u)$ is σ-finite with respect to $\mathcal{S}^{\infty-1}$. Let us show that $D_H^J u$ is concentrated on $S(u)$.

Theorem 4.5 *Let $u \in BV(X, \gamma)$. Then, the measure $D_H^J u$ is absolutely continuous with respect to $\mathcal{S}^{\infty-1} \llcorner S(u)$ and there is a $D_H u$-measurable unit vector field ν_u on $S(u)$ such that*

$$D_H^J u(B) = \int_{B\cap S(u)} (u^\vee(x) - u^\wedge(x))\nu_u(x)\, d\mathcal{S}^{\infty-1}(x) \tag{4.11}$$

for all $B \in \mathscr{B}(X)$. Moreover, if $v \in BV(X, \gamma)$ then $\nu_u = \nu_v$ $\mathcal{S}^{\infty-1}$-a.e. on $S(u) \cap S(v)$.

Proof. Notice that there is a $|D_H u|$-measurable unit vector field ν_u on $S(u)$ such that the equality $\nu_u(x) = \nu_{\{u>t\}}(x)$ holds $\mathcal{S}^{\infty-1}$-a.e. on $S(u) \cap \partial_{\mathcal{F}}^*\{u > t\}$, for all t such that $\{u > t\}$ has finite perimeter. Indeed, this is an obvious consequence of the inclusion $\{u > t\} \subset \{u > s\}$ for $t \ge s \in \mathcal{D}$ and Corollary 2.3.
Fixed $h \in H \setminus \{0\}$, for every $y \in h^\perp$ the equality

$$\bigcup_{s,\, t\in\mathcal{D}, s\neq t} \Big(\partial^*\{u_y > s\} \cap \partial^*\{u_y > t\}\Big) = \Big(\bigcup_{s,t\in\mathcal{D}} \Big(\partial_{\mathcal{F}}^*\{u > s\} \cap \partial_{\mathcal{F}}^*\{u > t\}\Big)\Big)_y,$$

holds, hence $S(u_y) = (S(u))_y$ and for every $B \subset X \setminus S(u)$ we have $B_y \cap (S(u))_y = B_y \cap S(u_y) = \emptyset$. As a consequence, for any $h \in H$

$$|[D_H^J u, h]_H|(B) \leq \int_{h^\perp} |D_h^J u_y|(B_y)\, d\gamma_h^\perp(y) = 0$$

and therefore $|D_H^J u|(B) = 0$. Finally, by the coarea formula and Corollary 2.3, for every Borel set B we have

$$\begin{aligned}
D_H^J u(B) = D_H u(B \cap S(u)) &= \int_{\mathbb{R}} D_H \chi_{\{u>t\}}(B \cap S(u))\, dt \\
&= \int_{\mathbb{R}} dt \int_{B \cap S(u)} \chi_{\partial_{\mathcal{F}}^*\{u>t\}}(x) \nu_{\{u>t\}}(x)\, d\mathcal{S}^{\infty-1}(x) \\
&= \int_{B \cap S(u)} d\mathcal{S}^{\infty-1}(x) \int_{\mathbb{R}} \chi_{\partial_{\mathcal{F}}^*\{u>t\}}(x) \nu_{\{u>t\}}(x)\, dt \\
&= \int_{B \cap S(u)} (u^\vee(x) - u^\wedge(x)) \nu_u(x)\, d\mathcal{S}^{\infty-1}(x).
\end{aligned}$$

Let now u, $v \in BV(X,\gamma)$. For $x \in S(u) \cap S(v)$ there are $s, t \in \mathcal{D}$ such that $x \in \partial_{\mathcal{F}}^*\{u > t\} \cap \partial_{\mathcal{F}}^*\{v > s\}$, the unit vector fields ν_u and ν_v coincide $\mathcal{S}^{\infty-1}$-a.e. on $S(u) \cap S(v)$ again by Corollary 2.3. QED

Lemma 4.6 *Let $u \in BV(X,\gamma)$. Then, for $|D_H u|$-a.e. $x \in X \setminus S(u)$ there is a unique $t = \tilde{u}(x) \in \mathbb{R}$ such that $x \in \partial_{\mathcal{F}}^*\{u > t\}$.*

Proof. The coarea formula and [6, Theorem 5.2] yield

$$|D_H u|(B) = \int_{\mathbb{R}} P_H(\{u > t\}, B)\, dt = \int_{\mathbb{R}} \mathcal{S}^{\infty-1}(\partial_{\mathcal{F}}^*\{u > t\} \cap B)\, dt$$

for every Borel set B. Therefore, for

$$B_u = \Big\{x \in X \setminus S(u) :\ \nexists t \in \mathbb{R} \text{ such that } x \in \partial_{\mathcal{F}}^*\{u > t\}\Big\} \tag{4.12}$$

we get $|D_H u|(B_u) = 0$ and the existence of $\tilde{u}(x)$ for $|D_H u|$-a.e. $x \in X \setminus S(u)$. Let us prove the uniqueness: if there were $s \neq t \in \mathbb{R}$ such that $x \in \partial_{\mathcal{F}}^*\{u > t\} \cap \partial_{\mathcal{F}}^*\{u > s\}$ then the set J_x of such numbers, according to Remark 2.4, would be an interval containing a pair $s' \neq t' \in \mathcal{D}$, whence we would get the contradiction $x \in S(u)$. QED

According to Lemma 4.6, for $|D_H u|$-a.e. $x \in X \setminus S(u)$ we may define

$$\tilde{u}(x) = t, \tag{4.13}$$

where $t \in \mathbb{R}$ is the unique value such that $x \in \partial_{\mathcal{F}}^*\{u > t\}$ and we call $\tilde{u}(x)$ the *approximate limit of u at x*.

Proposition 4.7 *Let $u \in BV(X,\gamma)$. For every $\psi \in C^1(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R})$, the function $\psi \circ u$ belongs to $BV(X,\gamma)$ and the equality*

$$D_H(\psi \circ u) = \psi'(\tilde{u}) \nabla_H u d\gamma + \psi'(\tilde{u}) D_H^C u + (\psi(u^\vee) - \psi(u^\wedge)) \nu_u \mathcal{S}^{\infty-1} \llcorner S(u) \tag{4.14}$$

holds.

Proof. Let us first show that $v = \psi \circ u$ belongs to $BV(X,\gamma)$. To this end, notice first that v has at most a linear growth, hence it belongs to $L \log^{1/2} L(X,\gamma)$. Moreover, if u_m are the canonical cylindrical approximations of u we have that $\psi \circ u_m \in BV(X,\gamma)$, $\psi \circ u_m \to v$ in $L^2(X,\gamma)$ and

$$|D_H v|(X) \leq \liminf_{m\to\infty} |D_H(\psi \circ u_m)|(X) \leq \|\psi'\|_\infty \liminf_{m\to\infty} |D_H u_m|(X)$$

by lower semicontinuity.
Next, we prove that $S(v) \subset S(u)$. Indeed, if $x \in S(v)$ then there are $s, t \in \mathcal{D}$, $s < t$, such that $x \in \partial^*_{\mathcal{F}}\{v > s\} \cap \partial^*_{\mathcal{F}}\{v > t\}$. By definition of cylindrical essential boundary, there are two finite dimensional subspaces $F_1, F_2 \in \mathcal{F}$ such that for any $G \in \mathcal{F}$ containing both F_1 and F_2 we have $x \in \partial^*_G\{v > s\} \cap \partial^*_G\{v > t\}$. For every such G we may write $x = y + z$, with $y \in \ker \pi_G$ and $z \in \partial^*\{v > s\}_y \cap \partial^*\{v > t\}_y$. By the finite dimensional case, see [4, Theorem 3.96], $z \in S(v)_y$ implies $z \in S(u)_y$, and therefore $x = y + z \in S(u)$.
Let $B \subset S(u)$ and assume that ψ is increasing. In this case, $[v^\wedge(x), v^\vee(x)] = [\psi(u^\wedge(x)), \psi(u^\vee(x))]$ for any $x \in S(u)$ and by the coarea formula we get

$$|D_H(\psi \circ u)|(B) = \int_{\mathbb{R}} P_H(\{\psi(u) > t\}, B)\, dt,$$

but then, if we set $t = \psi(s)$, we get $\{\psi(u) > t\} = \{u > s\}$, and thus

$$\begin{aligned}|D_H(\psi \circ u)|(B) &= \int_{\mathbb{R}} \psi'(s)\, P_H(\{u > s\}, B)\, ds = \int_B \int_{u^\wedge(x)}^{u^\vee(x)} \psi'(s)\, ds\, d\mathcal{S}^{\infty-1}(x) \\ &= \int_B (\psi(u^\vee(x)) - \psi(u^\wedge(x)))\, d\mathcal{S}^{\infty-1}(x).\end{aligned}$$

In particular, $|D_H v|(S(u) \setminus S(v)) = 0$ and $D_H v = D^J_H v$ in $S(v)$. If $B \subset X \setminus S(u)$ and $x \in B$, then $x \in \partial^*_{\mathcal{F}}\{u > t\}$ only if $t = \tilde{u}(x)$, see Lemma 4.6 and (4.13), and then arguing as before we find

$$\begin{aligned}|D^d_H(\psi \circ u)|(B) = &|D_H(\psi \circ u)|(B) = \int_{\mathbb{R}} P_H(\{\psi \circ u > t\}, B)\, dt = \int_{\mathbb{R}} \psi'(s) P_H(\{u > s\}, B)\, ds \\ &= \int_{\mathbb{R}} \int_B \psi'(\tilde{u}(x)) d|D_H \chi_{\{u>s\}}|(x)\, ds = \int_B \psi'(\tilde{u})\, d|D^d_H u|.\end{aligned}$$

Finally, for $\psi \in C^1(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R})$, if we fix $L > \|\psi'\|_\infty$, we may apply the previous result to the strictly increasing function $\psi(t) + Lt$. QED

A useful consequence of Lemma 4.6, Lemma 4.5 and Proposition 4.7 is the following Leibniz rule for the derivative of the product of two bounded BV functions.

Proposition 4.8 *Let $u, v \in BV(X, \gamma) \cap L^\infty(X, \gamma)$; then, $uv \in BV(X, \gamma) \cap L^\infty(X, \gamma)$ and there is a pair of functions $\tilde{u}$, $\tilde{v}$ such that $\tilde{u}$ (resp. $\tilde{v}$) coincides with the approximate limit of u (resp. of v) $|D^d_H u|$-a.e. (resp. $|D^d_H v|$-a.e.) in its domain such that the following formula holds:*

$$D_H(uv) = \tilde{u} D^d_H v + \tilde{v} D^d_H u + \big((uv)^\vee - (uv)^\wedge\big) \nu \mathcal{S}^{\infty-1} \llcorner \big(S(u) \cup S(v)\big). \tag{4.15}$$

Here $\nu = \nu_u$ on $S(u)$ and $\nu = \nu_v$ on $S(v)$, which is well defined in view of Lemma 4.5. In particular, if $E \subset X$ has finite perimeter then $u\chi_E \in BV(X, \gamma)$.

Proof. Possibly adding a constant which is irrelevant for our purposes, we may assume that u, v are positive. Set $w = uv$ and define the positive measure $\lambda = |D^d_H u| + |D^d_H v|$. Since $D^d_H u$, $D^d_H v \ll \lambda$ there are Borel functions f, g such that $D^d_H u = f\lambda$, $D^d_H v = g\lambda$ and $|f| + |g| = 1$ λ-a.e. in X. Notice also that by definition the approximate limit $\tilde{w}(x)$ exists and coincides with $\tilde{u}(x)\tilde{v}(x)$ wherever the approximate limits of u and v exist. Moreover, setting

$$E = \{x \in X :\ \min\{|f(x)|, |g(x)|\} > 0\},$$

the measures λ, $D^d_H u$, $D^d_H v$ are all equivalent in E, hence $\tilde{w}$ exists and coincides with $\tilde{u}\tilde{v}$ λ-a.e. in E. In the following computation we fix a pointwise defined Borel function $\tilde{w}$ that coincides with the approximate limit of w wherever it exists. By the chain rule (notice that the function log is Lipschitz continuous on the range of u and v, which are supposed to be positive and bounded) we get

$$\frac{D^d_H w}{\tilde{w}} = D^d_H(\log\, w) = D^d_H(\log\, u) + D^d_H(\log\, v) = \frac{D^d_H u}{\tilde{u}} + \frac{D^d_H v}{\tilde{v}} = \Big(\frac{f}{\tilde{u}} + \frac{g}{\tilde{v}}\Big)\lambda,$$

on Borel subsets of E, whence

$$D^d_H w = (\tilde{v} f + \tilde{u} g)\lambda = \tilde{v} D^d_H u + \tilde{u} D^d_H v \qquad \text{on Borel subsets of } E.$$

If we consider the sets $E_1 = \{f = 0\} \cap \{g > 0\}$ and $E_2 = \{g = 0\} \cap \{f > 0\}$, the same computation gives $D^d_H w/\tilde{w} = g\lambda/\tilde{v}$ on Borel subsets of E_1 and $D^d_H w/\tilde{w} = f\lambda/\tilde{u}$ on Borel subsets of E_2. Therefore, we may define $\tilde{u} = \tilde{w}/\tilde{v}$ on E_1 and $\tilde{v} = \tilde{w}/\tilde{u}$ on E_2. Then, (4.15) follows from the decomposition of the gradient of general BV functions. QED

5 Special functions of bounded variation

In this section we introduce the space of special functions with bounded variation and investigate some of their properties; in particular, we give a characterisation of them in terms of the chain rule and study the closedness under the weak convergence in BV.

Definition 5.1 *A function $u \in BV(X,\gamma)$ is called a* special function of bounded variation, *$u \in SBV(X,\gamma)$, if $|D^C_H u| = 0$, i.e., the equality $D_H u = \nabla_H u d\gamma + (u^\vee - u^\wedge)\nu_u \mathcal{S}^{\infty-1} \llcorner S(u)$ holds.*

Remark 5.2 A $BV(X,\gamma)$ function u is $SBV(X,\gamma)$ if and only if

$$\int_X |\nabla_H u|\, d\gamma = \inf\Big\{|D_H u|(X \setminus K) :\ K \text{ compact}, \mathcal{S}^{\infty-1}(K) < \infty\Big\}.$$

Indeed, the equality

$$|D^d_H u|(X) = |D^d_H u|(X \setminus S(u)) = \inf\Big\{|D_H u|(X \setminus K) :\ K \text{ compact}, \mathcal{S}^{\infty-1}(K) < \infty\Big\}.$$

follows from Theorem 4.5 and the σ-finiteness of $S(u)$ with respect to $\mathcal{S}^{\infty-1}$. Therefore, if $u \in SBV(X,\gamma)$ the statement is obvious. The opposite implication follows from Lemma 4.3.

SBV functions can be characterised by using the chain rule (4.14) as well. To this end, let us fix an increasing concave function $\theta : [0,+\infty) \to [0,+\infty)$ such that

$$\lim_{t\to 0^+} \frac{\theta(t)}{t} = +\infty \tag{5.1}$$

and the class of related test functions

$$\mathcal{C}(\theta) = \left\{\Phi \in C^1_b(\mathbb{R}) : \|\Phi\|_\theta := \sup_{s,t\in\mathbb{R}, s\neq t} \frac{|\Phi(t) - \Phi(s)|}{\theta(|t-s|)} < \infty\right\}.$$

Proposition 5.3 *Consider $u \in BV(X,\gamma)$ and θ as in (5.1); then, if there is a measure $\lambda \in \mathscr{M}(X,H)$ with $|\lambda|(S(u)) = 0$ and a positive functional $\Lambda \in (C_b(X))'$ s.t.*

$$|D_H \Phi(u) - \Phi'(\tilde{u})\lambda| \le \|\Phi\|_\theta \Lambda, \qquad \forall\, \Phi \in \mathcal{C}(\theta),$$

then $\lambda = D^d_H u$ and

$$\Lambda \ge \theta(u^\vee(x) - u^\wedge(x))\mathcal{S}^{\infty-1} \llcorner S(u). \tag{5.2}$$

In particular, $u \in SBV(X,\gamma)$ if and only if there is $g \in L^1(X,H)$ such that

$$|D_H \Phi(u) - \Phi'(\tilde{u}) g\gamma| \le \|\Phi\|_\theta \Lambda, \qquad \forall \Phi \in \mathcal{C}(\theta).$$

Proof. We want to show that the measure $\mu = D_H^d u - \lambda$ vanishes, i.e., $|\mu|(X) = 0$. First, notice that $|\mu|(S(u)) = 0$, as $|D_H^d u|(X) = |\lambda|(S(u)) = 0$. Therefore, setting $B = X \setminus S(u)$, we have $|\mu|(X) = |\mu|(B)$ and the measures μ and $\mathcal{S}^{\infty-1} \llcorner S(u)$ are mutually singular. In particular, the relationship

$$|\Phi'(\tilde{u})\mu| = |\Phi'(\tilde{u}) D_H^d u - \Phi'(\tilde{u})\lambda| = \chi_{(X \setminus S(u))} |D_H \Phi(u) - \Phi'(\tilde{u})\lambda|$$

holds. It follows that

$$\int_X |\Phi'(\tilde{u}(x))| d|\mu|(x) \leq |D_H \Phi(u) - \Phi'(\tilde{u})\lambda|(X) \leq \|\Phi\|_\theta \Lambda(1)$$

for any $\Phi \in \mathcal{C}(\theta)$. Taking now $\Phi_{1,\varepsilon}(t) = \sin(t/\varepsilon)$ and $\Phi_{2,\varepsilon}(t) = \cos(t/\varepsilon)$ in the previous inequality and summing up, as $|\sin t| + |\cos t| \geq 1$ we get

$$|\mu|(X) \leq \varepsilon(\|\Phi_{1,\varepsilon}\|_\theta + \|\Phi_{2,\varepsilon}\|_\theta)\Lambda(1).$$

Letting $\varepsilon \to 0$ and applying [4, Lemma 4.10], we deduce that $|\mu|(X) = 0$. It then follows from Proposition 4.7 that

$$|(\Phi(u^\vee) - \Phi(u^\wedge))\nu_u \mathcal{S}^{\infty-1} \llcorner S(u)| = |D_H(\Phi \circ u) - \Phi'(\tilde{u})\lambda| \leq \|\Phi\|_\theta \Lambda$$

for all $\Phi \in \mathcal{C}(\theta)$. Therefore, taking into account that by [4, Lemma 4.11] we have

$$\theta(|s-t|) = \sup_\Theta \Big\{ \frac{|\Phi(s) - \Phi(t)|}{\|\Phi\|_\theta} \Big\}$$

where Θ is a countable dense set in $\{\Phi \in \mathcal{C}(\theta) : \ \Phi' \in C_c(\mathbb{R}),\ \Phi \ \text{ non constant}\}$, and using also [4, Remark 1.69] we deduce (5.2):

$$\begin{aligned} \Lambda &\geq \bigvee_{\Phi \in \Theta} \frac{|\Phi(u^\vee) - \Phi(u^\wedge)|}{\|\Phi\|_\theta} \mathcal{S}^{\infty-1} \llcorner S(u) \\ &= \left[\sup_{\Phi \in \Theta} \frac{|\Phi(u^\vee) - \Phi(u^\wedge)|}{\|\Phi\|_\theta} \right] \mathcal{S}^{\infty-1} \llcorner S(u) = (\theta(u^\vee) - u^\wedge)) \mathcal{S}^{\infty-1} \llcorner S(u). \end{aligned}$$

QED

The following closure-compactness theorem follows.

Theorem 5.4 *Let $\psi : [0, +\infty) \to [0, +\infty]$ and $\theta : (0, +\infty) \to (0, +\infty]$ be two continuous and increasing functions such that*

$$\lim_{t \to +\infty} \frac{\psi(t)}{t} = +\infty, \quad \lim_{t \to 0} \frac{\theta(t)}{t} = +\infty,$$

and let $(u_k)_k \subset SBV(X, \gamma)$ be a sequence such that $\sup_k \|u_k\|_\infty < \infty$ and

$$\sup_{k \in \mathbb{N}} \Big\{ \int_X \psi(|\nabla_H u_k|_H) d\gamma + \int_{S(u_k)} \theta(u_k^\vee - u_k^\wedge) d\mathcal{S}^{\infty-1} \Big\} < +\infty.$$

If $u_k \to u$ in measure, then $u \in SBV(X, \gamma)$, $\nabla_H u_k$ converges to $\nabla_H u$ weakly in $L^1(X, H)$ and $D_H^J u_k$ weakly converges to $D_H^J u$ in $\mathscr{M}(X, H)$. Moreover, if ψ is convex and θ is concave, then*

$$\int_X \psi(|\nabla_H u|_H) d\gamma \leq \liminf_{k \to +\infty} \int_X \psi(|\nabla_H u_k|_H) d\gamma, \tag{5.3}$$

and

$$\int_{S(u)} \theta(u^\vee - u^\wedge) d\mathcal{S}^{\infty-1} \leq \liminf_{k \to +\infty} \int_{S(u_k)} \theta(u_k^\vee - u_k^\wedge) d\mathcal{S}^{\infty-1}.$$

PROOF. Possibly considering $\theta \wedge M = \min\{\theta, M\}$ in place of θ, we may suppose that θ is bounded. First, notice that the functions $\nabla_H u_k$ are equi–integrable, so that, up to subsequences which we don't relabel, they weakly converge in $L^1(X, H)$ to some $g \in L^1(X, H)$. For $\Phi \in \mathcal{C}(\theta)$, let us show that $\Phi'(u_k)\nabla_H u_k$ converges to $\Phi'(u)g$ weakly in $L^1(X, H)$. To this aim, consider $f \in L^\infty(X, H)$ and notice that

$$\begin{aligned}\lim_{k\to+\infty}\int_X \Phi'(u_k)[f, \nabla_H u_k]_H d\gamma &= \lim_{k\to+\infty}\biggl(\int_X (\Phi'(u_k) - \Phi'(u))[f, \nabla_H u_k]_H d\gamma \\ &\quad + \int_X \Phi'(u)[f, \nabla_H u_k]_H d\gamma\biggr) \\ &= \lim_{k\to+\infty}\int_X \Phi'(u)[f, \nabla_H u_k]_H d\gamma = \int_X \Phi'(u)[f, g]_H d\gamma\end{aligned}$$

because $|\Phi'(u_k(x)) - \Phi'(u(x))||\nabla_H u_k(x)|_H \leq 2\|\Phi'\|_\infty |\nabla_H u_k(x)|_H$ and the functions ∇u_h are equi–integrable, so that we may apply Vitali theorem (see e.g. [12, Theorem III.6.15]) and deduce

$$\Bigl|\int_X (\Phi'(u_k) - \Phi'(u))[f, \nabla_H u_k]_H d\gamma\Bigr| \leq \|f\|_\infty \int_X |\Phi'(u_k) - \Phi'(u)||\nabla_H u_k|_H \, d\gamma \to 0.$$

From (4.14) we infer that $D_H\Phi(u_k)$ weakly* converges to $D_H\Phi(u)$ in the duality with respect to $\mathcal{F}C_b(X)$, and then

$$\lim_{k\to+\infty}\Bigl(D_H\Phi(u_k) - \Phi'(u_k)\nabla_H u_k\gamma\Bigr) = D_H\Phi(u) - \Phi'(u)g\gamma.$$

Moreover, the measures

$$\mu_k = \theta(u_k^\vee - u_k^\wedge)\mathcal{S}^{\infty-1} \llcorner S(u_k)$$

have bounded total variation, hence, up to subsequences again, they are weakly* converging in the duality with $C_b(X)$ to a positive functional Λ on $C_b(X)$. Since $\Phi \in \mathcal{C}(\theta)$, we get

$$|D_H\Phi(u_k) - \Phi'(u_k)\nabla_H u_k\gamma| \leq \|\Phi\|_\theta \mu_k$$

and letting $k \to \infty$, by the lower semicontinuity of the total variation,

$$|D_H\Phi(u) - \Phi'(u)g\gamma| \leq \|\Phi\|_\theta \Lambda,$$

hence by Proposition 5.3, $u \in SBV(X, \gamma)$ and $\nabla_H u = g$. By the previous argument, $\nabla_H u_k$ weakly converges to $\nabla_H u$ and as a consequence $D_H^J u_k \to D_H^J u$. QED

If $E \subset X$ is a set with finite perimeter, for every $u : E \to \mathbb{R}$ we define $u^* : X \to \mathbb{R}$ the zero extension of u out of E, and for $1 < p < \infty$ the space

$$H^{1,p}(E, \gamma) = \Bigl\{u \in L^p(E, \gamma) : \ u^* \in SBV(X, \gamma), \ \nabla_H u^* \in L^p(X, \gamma), \ D_H^\sigma u^*(X \setminus \partial_{\mathcal{F}}^* E) = 0\Bigr\}, \quad (5.4)$$

endowed with the norm

$$\|u\|_{H^{1,p}(E\gamma)} := \|u^*\|_p + \|\nabla_H u^*\|_p.$$

As an application of the previous result, we deduce a compactness theorem for the space $H^{1,p}(E, \gamma)$, in the spirit of [27, Chapter 5, Section 3].

Theorem 5.5 *The bounded closed subsets of $H^{1,p}(E, \gamma)$, $1 < p < \infty$, are weakly compact in $L^p(E, \gamma)$.*

Proof. Let (u_k) be a bounded sequence in $H^{1,p}(E,\gamma)$. Since a function u belongs to $SBV(X,\gamma)$ if and only if all its truncations $u_K = (u \wedge K) \vee (-K)$ are SBV, we may suppose that the u_k are equibounded. Eventually, a diagonal argument allows us to remove this hypothesis. By the boundedness in $L^p(E,\gamma)$ we infer that a subsequence (which we don't relabel) is weakly converging to a function u in $L^p(E,\gamma)$. Let us show that $u \in H^{1,p}(E,\gamma)$. To this aim, notice first that by Mazur's lemma a suitable sequence (v_k) of convex combinations of the (u_k) converges strongly to u in $L^p(E,\gamma)$ and that the null extensions v_k^* still belong to $H^{1,p}(E,\gamma)$ because, as for the (u_k^*), $D_H^\sigma v_k^*(X \setminus \partial_{\mathcal{F}}^* E) = 0$. Therefore, we may apply Theorem 5.4 to the sequence $(v_k^*) \subset SBV(X,\gamma)$ with $\psi(t) = |t|^p$ and $\theta(t) = 1$ and conclude that $u^* \in SBV(X,\gamma)$. Finally, since $D_H^\sigma u^*(X \setminus \partial_{\mathcal{F}}^* E) = 0$ by the weak convergence of $D^J v_k^*$ and $\nabla_H u^* \in L^p(E,\gamma)$ by (5.3), the proof is complete. QED

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