Maximal regularity for gradient systems with boundary degeneracy

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Abstract

We study a class of elliptic operators L that degenerate at the boundary of a bounded open set $\mathscr{O} \subset \mathbb{R}^d$ and possess a symmetrizing invariant measure μ. Such operators are associated with diffusion processes in $\mathscr{O}$ which are invariant for time reversal. After showing that the corresponding elliptic equation $\lambda\varphi - L\varphi = f$ has a unique weak solution for any $\lambda > 0$ and $f \in L^2(\mathscr{O}, \mu)$, we obtain new results for the characterization of the domain of L.

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1 Introduction and setting of the problem

Let $\mathscr{O}$ be a bounded open subset of $\mathbb{R}^d$ with closure $\overline{\mathscr{O}}$ and boundary $\partial\mathscr{O}$ of class C^1. We are concerned with the following elliptic operator in $\mathscr{O}$

$$L\varphi = \frac{1}{2}\text{Tr}\,[\sigma\sigma^* D^2\varphi] + \langle b, D\varphi\rangle, \quad \varphi \in C^2(\mathscr{O})\,, \tag{1.1}$$

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where $b:\overline{\mathscr{O}}\to\mathbb{R}^d$ is of class C^1, $\sigma:\overline{\mathscr{O}}\to L(\mathbb{R}^d)$ is continuous on $\overline{\mathscr{O}}$, of class $C^1(\mathscr{O})$ and such that, setting $a=\sigma\sigma^*$,

$$\det a(x)>0,\quad \forall\, x\in\mathscr{O}\,. \tag{1.2}$$

It is well known that L is the Kolmogorov operator associated with the diffusion process described by the stochastic differential equation

$$\begin{cases} dX(t)=b(X(t))dt+\sigma(X(t))dW(t),\\ \\ X(0)=x\in\mathscr{O}. \end{cases} \tag{1.3}$$

A unique solution of (1.3) exists in a suitable random interval $[0,\tau_x)$, where τ_x is the first time when $X(t)$ reaches $\partial\mathscr{O}$.

The formal adjoint of L reads as follows

$$L^*\rho=\frac{1}{2}\text{div}\,[aD\rho+(g-2b)\rho], \tag{1.4}$$

where g is the vector field

$$g_j=\sum_{i=1}^d D_i a_{ij},\quad j=1,...,d. \tag{1.5}$$

The role of the above operator in the characterization of the invariant measures for X, that are absolutely continuous with respect to the Lebesgue measure, is well known.

Let $X(t),t\in[0,1]$, be a solution of the equation

$$dX(t)=b(X(t))dt+\sigma(X(t))dW(t)\,,$$

with coefficients defined on $\mathbb{R}^d$. We call X *reversible* if $Y(t):=X(1-t)$ is a solution of the same stochastic differential equation (with exactly the same coefficients b and σ as X but a different Brownian motion). From a general result by Haussmann and Pardoux [13] it follows that Y is a diffusion process that satisfies the equation

$$dY(t)=\overline{b}(t,Y(t))dt+\sigma(Y(t))d\overline{W}(t)$$

for some Brownian motion $\overline{W}$, where we have set, for $(t,x)\in[0,1]\times\mathbb{R}^d$,

$$\overline{b}_i(t,x)=-b_i(x)+[p(1-t,x)]^{-1}\sum_{j=1}^d D_j[a_{ij}(x)p(1-t,x)] \tag{1.6}$$

with $p(t,x)dx$ the law of $X(t)$. Now, suppose there exists $\rho \in C^1(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ such that

$$aD\rho + (g - 2b)\rho = 0\,. \tag{1.7}$$

Then $L^*\rho = 0$, and $\mu(dx) = \rho(x)dx$ is an invariant measure for X. So, $p(t,x) = \rho(x)$ is independent of t, and (1.6) reduces to

$$\begin{aligned}\bar{b}_i(t,x) &= -b_i(x) + \rho(x)^{-1}\sum_{j=1}^{d} D_j[a_{ij}(x)\rho(x)] \\ &= -b_i(x) + g_i(x) + \rho(x)^{-1}\big(a(x)D\rho(x)\big)_i = b_i(x)\,.\end{aligned}$$

Therefore, X is reversible. Conversely, one can show that if X is reversible and $\mu(dx) = \rho(x)dx$ is an invariant measure for X, then condition (1.7) is satisfied. We are thus led to regard (1.7), in some sense, as a quantitative characterization of reversibility.

By analogy, we say that the operator L in (1.1), with coefficients defined in $\mathscr{O}$, is of *gradient form* if there exists $\rho \in C^1(\mathscr{O}) \cap L^1(\mathscr{O})$ such that (1.7) holds in $\mathscr{O}$. Such a terminology is justified by the observation that, in this case, L can be recast (at least formally) as

$$L\varphi = \frac{1}{2\rho}\mathrm{div}\,\big(\rho a D\varphi\big)\,. \tag{1.8}$$

Consequently, L is symmetric on $L^2(\mathscr{O},\mu)$, where μ is the finite measure

$$\mu(dx) = \rho(x)dx\,. \tag{1.9}$$

We shall proceed as follows. In Section 2, we will show that, under general hypotheses, for any $\lambda > 0$ and $f \in L^2(\mathscr{O},\mu)$ the equation

$$\lambda\varphi - L\varphi = f \tag{1.10}$$

has a unique distributional solution $\varphi \in W^{1,2}(\mathscr{O},\mu)$. This simple result, obtained by a standard variational method, will allow us to define rigorously the variational operator L in $L^2(\mathscr{O},\mu)$ with domain $D(L)$.

Then, as a first step in the direction of studying regularity properties of the operator $(L, D(L))$, we will focus on a special class of gradient operators, namely operators of the form

$$L\varphi = \frac{1}{2}\,\alpha\Delta\varphi + \langle b, D\varphi\rangle\,. \tag{1.11}$$

under the following assumptions.

Hypothesis 1.1. *(i) $\mathscr{O}$ is C^1-regular, $0 \leqslant \alpha \in C^1(\overline{\mathscr{O}})$, and*

$$\alpha(x) = 0 \iff x \in \partial\mathscr{O}\,.$$

(ii) $b \in C^1(\overline{\mathscr{O}})$ and

$$b(x) \neq 0 \quad \forall\, x \in \partial\mathscr{O}\,.$$

(iii) There exists $\rho \in C^1(\mathscr{O})$ such that $\rho \in L^1(\mathscr{O})$ and

$$\alpha(x) D \log \rho(x) + D\alpha(x) = 2b(x) \quad \forall x \in \mathscr{O}. \tag{1.12}$$

Observe that, in view of condition (i) above, L degenerates on the whole boundary of $\mathscr{O}$. On the other hand, condition (ii) ensures that b is nondegenerate on $\partial\mathscr{O}$. Moreover, condition (1.12) is nothing but (1.7), restricted to $\mathscr{O}$, for the operator L in (1.11).

In section 3, we will show that

$$D(L) = \left\{\varphi \in W^{2,2}_{loc}(\mathscr{O},\mu) \ : \ D\varphi \in L^2(\mathscr{O},\mu;\mathbb{R}^d),\ \alpha\Delta\varphi \in L^2(\mathscr{O},\mu)\right\} \tag{1.13}$$

under the further assumption

Hypothesis 1.2. *Besides the conditions in Hypothesis 1.1, assume that $\mathscr{O}$ is C^∞-regular, and b, α, and ρ belong to $C^\infty(\overline{\mathscr{O}})$.*

The characterization of the domain, even for more general operators but only in L^2 with respect to the Lebesgue measure, is proved by different methods in [2]. The L^p case, $1 < p \leqslant \infty$, with Dirichlet boundary condition and first order global degeneracy of the diffusion coefficients at the boundary is treated in [10]. It is worth pointing out that, in the literature on degenerate parabolic equations (see, e.g., [1], [6],[7], and [8]), one can generally guarantee that

$$\alpha^{1/2} D\varphi \in L^2(\mathscr{O};\mathbb{R}^d)\,,$$

though exceptions to such a tradition are known in low space dimension (see, e.g., [9] and [5]). Here, however, we obtain stronger integrability for the gradient, namely

$$\int_{\mathscr{O}} |D\varphi|^2 d\mu \leqslant C(\lambda) \int_{\mathscr{O}} |f|^2 d\mu, \tag{1.14}$$

(where $C(\lambda)$ is independent of φ) in a multidimensional setting. This can be explained recalling the nondegenerate contribution of b at the boundary.

Indeed, the above statement (1.13) turns out to be a maximal regularity result for L. Here, the key technical tool will be a regularity result from [14] ensuring that $\varphi \in C^\infty(\overline{\mathscr{O}})$ for $f \in C^\infty(\overline{\mathscr{O}})$. It is for this point that we need Hypothesis 1.2 in its full strength.

Finally, in Section 4, under weaker regularity hypotheses on the coefficients, we shall consider operators with the even more restrictive structure

$$L\varphi = \frac{1}{2}\alpha\Delta\varphi + \frac{k}{2}\langle D\alpha, D\varphi\rangle, \qquad k \geqslant 1. \tag{1.15}$$

In this case, we introduce suitable approximating operators L_ϵ, that are still reversible and satisfy the main estimates leading to the domain characterization, and then pass to the limit as $\epsilon \to 0$ in order to recover the characterization of the domain of L.

An important feature of our analysis is that it just requires α to be regular in $\overline{\mathscr{O}}$ and positive in $\mathscr{O}$, so that α in general (for example in the case of first-order degeneracy) it does not possess any smooth extension outside $\overline{\mathscr{O}}$ which is still nonnegative.

2 Weak solutions

In this section, we start our analysis of operator (1.8) under Hypothesis 1.1. Let us denote by $B(\mathscr{O})$ the Borel σ-algebra in $\mathscr{O}$, and by μ the finite measure on $(\mathscr{O}, \mathscr{B}(\mathscr{O}))$ defined by (1.9), where ρ is given by (1.12).

Now, let us define the Sobolev space $W^{1,2}_a(\mathscr{O},\mu)$ as follows

$$W^{1,2}_a(\mathscr{O},\mu) = \left\{ u \in W^{1,2}_{loc}(\mathscr{O}) : \ \|u\|_{W^{1,2}_a(\mathscr{O},\mu)} < \infty \right\}, \tag{2.1}$$

where $\|\cdot\|_{W^{1,2}_a(\mathscr{O},\mu)}$ is the norm which is associated with the inner product

$$\langle \varphi, \psi\rangle_{W^{1,2}_a(\mathscr{O},\mu)} := \langle \varphi,\psi\rangle_{L^2(\mathscr{O},\mu)} + \int_{\mathscr{O}} \langle aD\varphi, D\psi\rangle \, d\mu.$$

We now proceed to showing that the space $W^{1,2}_a(\mathscr{O},\mu)$ is a Hilbert space, the only point which needs to be checked being completeness. Let (u_j) be a Cauchy sequence in $W^{1,2}_a(\mathscr{O},\mu)$. Then, (u_j) is convergent in $L^2(\mathscr{O},\mu)$ and $W^{1,2}_{loc}(\mathscr{O})$, because a is nondegenerate in $\mathscr{O}$. Moreover, there is a subsequence such that $u_{j_k} \to u$ and $Du_{j_k} \to Du$ a.e. in $\mathscr{O}$. In order to verify that $u_{j_k} \to u$ in $W^{1,2}_a(\mathscr{O},\mu)$, let us fix ϵ, and let $\nu \in \mathbb{N}$ be such that

$$\|u_{j_k} - u_{j_n}\|_{W^{1,2}_a(\mathscr{O},\mu)} < \epsilon \qquad \forall\, k,n > \nu.$$

Letting $n \to \infty$, by Fatou's Lemma we get

$$\|u_{j_k} - u\|_{W_a^{1,2}(\mathscr{O},\mu)} < \epsilon \qquad \forall\, k > \nu\,.$$

Since ϵ is arbitrary, the convergence of u_{j_k} to u in $W_a^{1,2}(\mathscr{O},\mu)$ follows, together with the convergence of the whole sequence.

Next, consider the Dirichlet form

$$A(\varphi,\psi) := \frac{1}{2}\int_{\mathscr{O}} \langle aD\varphi, D\psi\rangle\; d\mu, \tag{2.2}$$

which is obviously symmetric, coercive and continuous on $W_a^{1,2}(\mathscr{O},\mu)$. Given $\lambda > 0$ and $f \in L^2(\mathscr{O},\mu)$, by the Lax–Milgram theorem there is a unique $\varphi \in W_a^{1,2}(\mathscr{O},\mu)$ such that

$$\lambda \int_{\mathscr{O}} \varphi\,\psi\, d\mu + \frac{1}{2}\int_{\mathscr{O}} \langle aD\varphi, D\psi\rangle\; d\mu = \int_{\mathscr{O}} f\,\psi\, d\mu, \quad \forall\, \psi \in W_a^{1,2}(\mathscr{O},\mu). \tag{2.3}$$

Let us recall the definition of the variational operator, L_v, associated with (2.2). We say that $\varphi \in W_a^{1,2}(\mathscr{O},\mu)$ belongs to the the domain, $D(L_v)$, of L_v if there exists a constant $K(\varphi) \geqslant 0$ such that

$$\left|\int_{\mathscr{O}} \langle aD\varphi, D\psi\rangle\; d\mu\right| \leqslant K(\varphi)\|\psi\|_{L^2(\mathscr{O},\mu)}, \quad \forall\, \psi \in W_a^{1,2}(\mathscr{O},\mu). \tag{2.4}$$

In this case, by the Riesz theorem, there is a unique element in $L^2(\mathscr{O},\mu)$, that will be denoted by $L_v\varphi$, such that

$$-\frac{1}{2}\int_{\mathscr{O}} \langle aD\varphi, D\psi\rangle\; d\mu = \int_{\mathscr{O}} L_v\varphi\,\psi\; d\mu, \quad \forall\, \psi \in W_a^{1,2}(\mathscr{O},\mu). \tag{2.5}$$

Equivalently, by (2.3) and (2.5), for any $\lambda > 0$ and $f \in L^2(\mathscr{O},\mu)$ there exists a unique $\varphi \in D(L_v)$ such that

$$\lambda\varphi - L_v\varphi = f.$$

3 A characterization of $D(L_v)$

In this section, we characterize the domain of the variational operator L_v in the "diagonal" case (1.11). Our first step is the following identity satisfied by the operator L introduced in (1.1) under Hypotesis 1.1.

Proposition 3.1. *Under Hypotesis 1.1, for any $\varphi,\psi \in C^2(\overline{\mathscr{O}})$ we have*

$$\int_{\mathscr{O}} L\varphi\,\psi\,\mu(dx) = -\frac{1}{2}\int_{\mathscr{O}} \alpha\langle D\varphi, D\psi\rangle\,\mu(dx). \tag{3.1}$$

Proof. Integrating by parts we find, recalling that α vanishes on $\partial\Omega$,

$$\frac{1}{2}\int_{\mathscr{O}} \alpha\,\Delta\varphi\,\psi\, d\mu = -\frac{1}{2}\int_{\mathscr{O}} \alpha\langle D\varphi\, D\psi\rangle\, d\mu - \frac{1}{2}\int_{\mathscr{O}} \langle D\varphi\, D\alpha + \alpha D\log\rho\rangle\,\psi\, d\mu.$$

Since $D\alpha + \alpha D\log\rho = 2b$, the conclusion follows. □

Let us now prove some classical estimates.

Lemma 3.2. *Let $\lambda > 0$, $\varphi \in C^2(\overline{\mathscr{O}})$ and set*

$$f = \lambda\varphi - L\varphi. \tag{3.2}$$

Then we have

$$\int_{\mathscr{O}} |\varphi|^2 d\mu \leqslant \frac{1}{\lambda^2} \int_{\mathscr{O}} |f|^2 d\mu \tag{3.3}$$

and

$$\int_{\mathscr{O}} \alpha |D\varphi|^2\, d\mu \leqslant \frac{1}{2\lambda} \int_{\mathscr{O}} |f|^2 d\mu. \tag{3.4}$$

Proof. Multiplying both sides of (3.2) by φ and taking into account (3.1) yields

$$\lambda \int_{\mathscr{O}} |\varphi|^2 d\mu + \frac{1}{2}\int_{\mathscr{O}} \alpha|D\varphi|^2\, d\mu = \int_{\mathscr{O}} \varphi\, f d\mu.$$

The conclusion follows via standard arguments that use the Cauchy-Schwarz inequality. □

Lemma 3.3. *Let $\varphi \in C^2(\overline{\mathscr{O}})$. Then the following identity holds*

$$\begin{aligned}
&\int_{\mathscr{O}} L\varphi\,\langle b, D\varphi\rangle\, d\mu \\
&= \frac{1}{4} \int_{\mathscr{O}} \alpha\langle(\operatorname{div} bI - 2Db)\, D\varphi, D\varphi\rangle d\mu + \frac{1}{2} \int_{\mathscr{O}} |D\varphi|^2\, |b|^2 d\mu.
\end{aligned} \tag{3.5}$$

Proof. By (3.1) we have

$$\begin{aligned}
\int_{\mathscr{O}} L\varphi\,\langle b, D\varphi\rangle\, d\mu &= -\frac{1}{2} \int_{\mathscr{O}} \alpha\langle D\varphi, D\langle b, D\varphi\rangle\rangle d\mu \\
&= -\frac{1}{2} \int_{\mathscr{O}} \alpha\langle Db\, D\varphi, D\varphi\rangle d\mu - \frac{1}{2} \int_{\mathscr{O}} \alpha\langle D^2\varphi\, D\varphi, b\rangle d\mu\,.
\end{aligned} \tag{3.6}$$

Now, let us compute the rightmost term above as follows:

$$\int_{\mathscr{O}} \alpha\langle D^2\varphi\, D\varphi, b\rangle d\mu = \sum_{h,k=1}^{d} \int_{\mathscr{O}} D_h D_k\varphi\, D_k\varphi\, \alpha\, b_h\, \rho\, dx$$

$$= \frac{1}{2}\sum_{h,k=1}^{d} \int_{\mathscr{O}} D_h[(D_k\varphi)^2]\alpha\, b_h\, \rho\, dx = -\frac{1}{2}\sum_{h,k=1}^{d} \int_{\mathscr{O}} (D_k\varphi)^2 D_h(\alpha\, b_h\, \rho)\, dx$$

$$= -\frac{1}{2}\int_{\mathscr{O}} |D\varphi|^2 \operatorname{div}(\alpha\rho\, b) dx = -\frac{1}{2}\int_{\mathscr{O}} |D\varphi|^2(\alpha\rho \operatorname{div} b + \langle b, D\,(\alpha\,\rho)) dx$$

$$= -\frac{1}{2}\int_{\mathscr{O}} |D\varphi|^2\,(\alpha \operatorname{div}\, b + \langle b, \alpha D\log\rho + D\alpha\rangle) d\mu$$

$$= -\frac{1}{2}\int_{\mathscr{O}} |D\varphi|^2\,(\alpha \operatorname{div}\, b + 2|b|^2) d\mu.$$

(in the integration by parts we have used the fact that $\rho\alpha = 0$ on $\partial\mathscr{O}$). Therefore, substituting in (3.6) yields

$$\int_{\mathscr{O}} L\varphi\,\langle b, D\varphi\rangle\, d\mu$$
$$= -\frac{1}{2}\int_{\mathscr{O}} \alpha\langle Db\, D\varphi, D\varphi\rangle d\mu + \frac{1}{4}\int_{\mathscr{O}} |D\varphi|^2\,(\alpha \operatorname{div}\, b + 2|b|^2) d\mu,$$

which implies (3.5). □

Lemma 3.4. *Assume Hypotesis 1.1, let $\lambda > 0$ and $\varphi \in C^2(\overline{\mathscr{O}})$, and set $f = \lambda\varphi - L\varphi$. Then, for any $\epsilon > 0$ there is $C(\epsilon) > 0$ such that*

$$\int_{\mathscr{O}} |D\varphi|^2\, |b|^2\, d\mu \leqslant \epsilon \int_{\mathscr{O}} |D\varphi|^2\, d\mu + C(\epsilon)\int_{\mathscr{O}} |f|^2\, d\mu. \tag{3.7}$$

Proof. Multiplying both sides of (3.2) by $\langle b, D\varphi\rangle$ and integrating over $\mathscr{O}$ yields

$$\lambda\int_{\mathscr{O}} \varphi\langle b, D\varphi\rangle d\mu - \int_{\mathscr{O}} L\varphi\langle b, D\varphi\rangle d\mu = \int_{\mathscr{O}} f\langle b, D\varphi\rangle d\mu.$$

Taking into account (3.5), we obtain

$$\frac{1}{2}\int_{\mathscr{O}} |D\varphi|^2\,|b|^2 d\mu = \lambda\int_{\mathscr{O}} \varphi\,\langle b, D\varphi\rangle\, d\mu$$
$$-\frac{1}{4}\int_{\mathscr{O}} \alpha\langle(\operatorname{div}\, bI - 2Db)\, D\varphi, D\varphi\rangle d\mu - \int_{\mathscr{O}} f\langle b, D\varphi\rangle d\mu.$$

It follows that

$$\frac{1}{2}\int_{\mathscr{O}}|D\varphi|^2\,|b|^2 d\mu \leqslant \lambda\|b\|_\infty \int_{\mathscr{O}}|\varphi|\,|D\varphi|\,d\mu$$
$$+\frac{1}{4}\left(\|\text{div } b\|_\infty + \|Db\|_\infty\right)\int_{\mathscr{O}}\alpha|D\varphi|^2\,d\mu + \|b\|_\infty\int_{\mathscr{O}}|f|\,|D\varphi|\,d\mu,$$

which in turn yields

$$\int_{\mathscr{O}}|D\varphi|^2\,|b|^2 d\mu \leqslant \frac{\epsilon}{2}\int_{\mathscr{O}}|D\varphi|^2\,d\mu + \frac{8\lambda^2}{\epsilon}\|b\|_\infty^2\int_{\mathscr{O}}|\varphi|^2\,d\mu$$
$$+\frac{1}{2}\left(\|\text{div } b\|_\infty + \|Db\|_\infty\right)\int_{\mathscr{O}}\alpha|D\varphi|^2\,d\mu$$
$$+\frac{\epsilon}{2}\int_{\mathscr{O}}|D\varphi|^2\,d\mu + \frac{4}{\epsilon}\|b\|_\infty^2\int_{\mathscr{O}}|f|^2\,d\mu\,.$$

So,

$$\int_{\mathscr{O}}|D\varphi|^2\,|b|^2 d\mu \leqslant \epsilon\int_{\mathscr{O}}|D\varphi|^2\,d\mu + \frac{8\lambda^2}{\epsilon}\|b\|_\infty^2\int_{\mathscr{O}}|\varphi|^2\,d\mu$$
$$+\frac{1}{2}\left(\|\text{div } b\|_\infty + \|Db\|_\infty\right)\int_{\mathscr{O}}\alpha|D\varphi|^2\,d\mu + \frac{4}{\epsilon}\|b\|_\infty^2\int_{\mathscr{O}}|f|^2\,d\mu.$$

Now, using (3.3) and (3.4), we have

$$\int_{\mathscr{O}}|D\varphi|^2\,|b|^2 d\mu \leqslant \epsilon\int_{\mathscr{O}}|D\varphi|^2\,d\mu + \frac{8}{\epsilon}\|b\|_\infty^2\int_{\mathscr{O}}|f|^2\,d\mu$$
$$+\frac{1}{\lambda}\left(\|\text{div } b\|_\infty + \|Db\|_\infty\right)\int_{\mathscr{O}}|f|^2\,d\mu + \frac{4}{\epsilon}\|b\|_\infty^2\int_{\mathscr{O}}|f|^2\,d\mu.$$

The conclusion follows. □

Remark 3.5. As is clear from the above proof, the constant $C(\epsilon)$ in estimate (3.7) is independent of α. On the other hand, it depends on $\lambda, \|b\|_\infty$, and $\|Db\|_\infty$, and is bounded if $\|b\|_\infty$ and $\|Db\|_\infty$ stay bounded.

The condition $b \neq 0$ on $\partial\mathscr{O}$, which has not been used so far, will hereafter become essential for it implies, for some real number $\delta > 0$ and compact set $\mathscr{K} \subset \mathscr{O}$,

$$\alpha(x) \geqslant \delta \quad \forall x \in \mathscr{K} \qquad \text{and} \qquad |b(x)| \geqslant \delta \quad \forall x \in \mathscr{O}\setminus\mathscr{K}\,. \tag{3.8}$$

Indeed, fix $0 < \delta' < \min_{\partial\mathscr{O}}|b|$ and let $\mathscr{K} \subset \mathscr{O}$ be a compact such that

$$|b(x)| > \delta' \quad \forall x \in \mathscr{O}\setminus\mathscr{K}\,.$$

If $\min_{\mathscr{K}}\alpha \geqslant \delta'$, then (3.8) follows choosing $\delta = \delta'$. Otherwise, since $\min_{\mathscr{K}}\alpha$ is positive, it suffices to take δ equal to such a minimum.

Proposition 3.6. *Assume Hypothesis 1.1 and let δ be a positive number satisfying* (3.8). *Moreover, let $\lambda > 0$ and $\varphi \in C^2(\overline{\mathscr{O}})$, and set $f = \lambda\varphi - L\varphi$. Then there is a constant $C_1 = C_1(\delta, \lambda, \|b\|_\infty, \|Db\|_\infty) > 0$ such that*

$$\int_{\mathscr{O}} |D\varphi|^2 \, d\mu \leqslant C_1 \int_{\mathscr{O}} |f|^2 \, d\mu. \tag{3.9}$$

Proof. Let $\mathscr{K} \subset \mathscr{O}$ be the compact set associated with δ in (3.8). Then, by (3.4) and (3.7) we deduce that for any $\epsilon > 0$ there exists $C(\epsilon) > 0$ such that

$$\begin{aligned}\int_{\mathscr{O}} |D\varphi|^2 \, d\mu| &\leqslant \frac{1}{\delta}\Big\{\int_{\mathscr{O}\setminus\mathscr{K}} |b||D\varphi|^2 \, d\mu + \int_{\mathscr{K}} \alpha|D\varphi|^2 \, d\mu\Big\} \\ &\leqslant \frac{1}{\delta}\Big\{\epsilon\int_{\mathscr{O}} |D\varphi|^2 \, d\mu + \Big(C(\epsilon) + \frac{1}{2\lambda}\Big)\int_{\mathscr{O}} |f|^2 \, d\mu\Big\}.\end{aligned}$$

Now, choosing $\epsilon = \delta/2$ the conclusion follows. □

We are now in a position to chercterize the domain of L_v in $L^2(\mathscr{O}, \mu)$, assuming more regularity on the coefficients.

Theorem 3.7. *Assume Hypotheses 1.1 and 1.2. The domain of the variational operator L_v defined in* (2.5) *is characterized as follows:*

$$D(L_v) = \{\varphi \in W^{2,2}_{loc}(\mathscr{O}) : \ D\varphi \in L^2(\mathscr{O}, \mu; \mathbb{R}^d), \ \alpha\Delta\varphi \in L^2(\mathscr{O}, \mu)\}. \tag{3.10}$$

Proof. Let W denote the right-hand side of (3.10). We have to show that every weak solution $\varphi \in W^{1,2}_\alpha(\mathscr{O}, \mu)$ of $f = \lambda\varphi - L_v\varphi$ belongs to W. Let $(f_h) \subset C^\infty(\overline{\mathscr{O}})$ be a sequence converging to f in $L^2(\mathscr{O}, \mu)$, and, for every $h \in \mathbb{N}$, let $\varphi_h \in D(L_v)$ be such that $\lambda\varphi_h - L_v\varphi_h = f_h$. Then, by [14] we know that $\varphi_h \in C^\infty(\overline{\mathscr{O}})$ and therefore estimates (3.3), (3.9) hold (L_v coincides with L on smooth functions). In particular, (3.3) implies that $\varphi_h \to \varphi$ in $L^2(\mathscr{O}, \mu)$, so that (3.9) holds for φ as well. Thus, $\varphi \in W$ and the proof is complete. □

In the particular case when $\varrho = 1$, that is μ is the Lebesgue measure, we can further specialize $D(L_v)$, as shown below. Observe that $\varrho = 1$ if and only if $2b = D\alpha$.

Proposition 3.8. *Assume Hypotheses 1.1 and 1.2 and moreover that $2b = D\alpha$. Then $d\mu = dx$ and the domain of the variational operator L_v defined in* (2.5) *is characterized as follows:*

$$D(L_v) = \{\varphi \in W^{2,2}_{loc}(\mathscr{O}) : \ D\varphi \in L^2(\mathscr{O}, \mathbb{R}^d), \ \alpha D^2\varphi \in L^2(\mathscr{O}, \mathbb{R}^{d\times d})\}. \tag{3.11}$$

Proof. Arguing as in the proof of Theroem 3.7, it is sufficient to show the existence of a constant $C > 0$ such that for every $\varphi \in C^\infty(\overline{\mathscr{O}})$, setting $f = \lambda\varphi - L_v\varphi$, the inequality $\|\alpha D_{hk}\varphi\|_2 \leqslant C\|f\|_2$ holds. Since $\alpha\varphi$ is a smooth funcion vanishing at the boundary, we can apply the classical elliptic estimates $\|D_{hk}(\alpha\varphi)\|_2 \leqslant C\|\Delta(\alpha\varphi)\|_2$, which yield

$$\begin{aligned}\|\alpha D_{hk}\varphi\|_2 &\leqslant C_1\left(\|\alpha\Delta\varphi\|_2 + \|D\alpha\|_\infty\|\|D\varphi\|_2 + \|D^2\alpha\|_\infty\|\varphi\|_2\right)\\ &\leqslant C_2\left(\|(\lambda - L_v)\varphi\|_2 + \|D\alpha\|_\infty\|\|D\varphi\|_2 + \|D^2\alpha\|_\infty\|\varphi\|_2\right).\end{aligned}$$

The thesis now follows from estimates (3.3), (3.9). □

We do not know whether the above Proposition holds in more general contexts.

4 A special case with less regularity

In this section, we study operator L relaxing the regularity assumptions on the coefficients but restricting the analysis to the special structure

$$L\varphi = \frac{1}{2}\,\alpha\Delta\varphi + \frac{k}{2}\,\langle D\alpha, D\varphi\rangle \tag{4.1}$$

with $k \geqslant 1$ (note that if $k = 1$ then $L\varphi = \operatorname{div}[\alpha D\varphi]$). We assume that $\alpha \in C^3(\overline{\mathscr{O}})$, $\mathscr{O} = \{\alpha > 0\}$, $\partial\mathscr{O} = \{\alpha = 0\}$ and $D\alpha \neq 0$ on $\partial\mathscr{O}$, and we argue by approximation. For any $\epsilon > 0$ we set

$$L_\epsilon\varphi = \frac{1}{2}\,\alpha_\epsilon(x)\Delta\varphi + \frac{k}{2}\langle D\alpha, D\varphi\rangle, \tag{4.2}$$

where

$$\alpha_\epsilon(x) := \left(\sqrt{\alpha(x)+\epsilon} - \sqrt{\epsilon}\right)^2, \quad x \in \overline{\mathscr{O}}. \tag{4.3}$$

Notice that

$$\lim_{\epsilon\to 0}\alpha_\epsilon(x) = \alpha(x)$$

and

$$0 \leqslant \alpha_\epsilon(x) \leqslant \alpha(x), \quad x \in \overline{\mathscr{O}}.$$

We set moreover $b = \frac{k}{2}D\alpha$ and

$$\sigma_\epsilon(x) := \sqrt{\alpha(x)+\epsilon} - \sqrt{\epsilon}, \quad x \in \overline{\mathscr{O}}. \tag{4.4}$$

Then, $\sigma_\epsilon \in C^3(\overline{\mathscr{O}})$. The stochastic differential equation

$$\begin{cases} dX(t) = b(X(t))dt + \sigma_\epsilon(X(t))dW(t), \\ \\ X(0) = x \in \mathscr{O}, \end{cases} \tag{4.5}$$

has a unique regular solution $X_\epsilon(t,x)$, which is global in time because the set $\overline{\mathscr{O}}$ is invariant under the flow, as can be checked using the criteria in [11], [3] (here is where we need the C^3 regularity). Therefore, for any $\lambda > 0$ and any $f \in C^2(\overline{\mathscr{O}})$ the problem

$$\lambda\varphi_\epsilon - L_\epsilon\varphi_\epsilon = f \tag{4.6}$$

has a unique solution $\varphi_\epsilon \in C^2(\overline{\mathscr{O}})$ given by the probabilistic formula

$$\varphi_\epsilon(x) = \int_0^\infty e^{-\lambda t}\mathbb{E}[f(X_\epsilon(t,x))]dt. \tag{4.7}$$

For the operator (4.1) a solution ρ of (1.12) is given by

$$\rho(x) = \alpha^{k-1}(x). \tag{4.8}$$

Now we show that L_ϵ is of gradient form, that is there exists a solution of the equation

$$\alpha_\epsilon D\log\rho_\epsilon + D\alpha_\epsilon = kD\alpha. \tag{4.9}$$

Lemma 4.1. *A solution of* (4.9) *is given by*

$$\rho_\epsilon(x) = \frac{B^k\alpha_\epsilon^{k-1}(x)}{(\sqrt{B+\epsilon}-\sqrt{\epsilon})^{2k}}\exp\Big\{-2k\sqrt{\epsilon}\left(\frac{\sqrt{B+\epsilon}-\sqrt{\alpha+\epsilon}}{\sigma_\epsilon(\sqrt{B+\epsilon}-\sqrt{\epsilon})}\right)\Big\}, \tag{4.10}$$

where $B = \sup\alpha$.

Proof. By (4.9) we have

$$D\log\rho_\epsilon = k\,\frac{D\alpha}{\alpha_\epsilon} - D\log\alpha_\epsilon \tag{4.11}$$

Let us write $\frac{D\alpha}{\alpha_\epsilon} = DG_\epsilon(\alpha)$, where

$$\begin{aligned} G_\epsilon(r) &= \int_B^r \frac{1}{(\sqrt{s+\epsilon}-\sqrt{\epsilon})^2}\,ds \\ &= 2\log\frac{\sqrt{r+\epsilon}-\sqrt{\epsilon}}{\sqrt{B+\epsilon}-\sqrt{\epsilon}} - 2\sqrt{\epsilon}\,\frac{\sqrt{B+\epsilon}-\sqrt{r+\epsilon}}{(\sqrt{r+\epsilon}-\sqrt{\epsilon})(\sqrt{B+\epsilon}-\sqrt{\epsilon})}. \end{aligned} \tag{4.12}$$

Then we have

$$D \log \rho_\epsilon = k\, DG_\epsilon(\alpha) - D \log \alpha_\epsilon,$$

so that

$$\rho_\epsilon = \frac{B^k}{\alpha_\epsilon} e^{k\, G_\epsilon(\alpha)}$$

Now the conclusion follows in view of (4.12). □

Remark 4.2. Since there is $\epsilon_0 > 0$ such that

$$\sqrt{B+\epsilon} - \sqrt{\epsilon} > \frac{\sqrt{B}}{2},$$

we have

$$\rho_\epsilon(x) \leqslant 2^k \rho(x) = 2^k \alpha^{k-1}(x), \quad x \in \mathscr{O}$$

for $0 < \epsilon \leqslant \epsilon_0$. Moreover, ρ_ϵ converges to ρ uniformly on the compact subsets of $\mathscr{O}$ and then also in $L^1(\mathscr{O})$. As a consequence, estimates (3.3), (3.9) hold in the form

$$\int_{\mathscr{O}} |\varphi_\epsilon|^2 \, d\mu_\epsilon \leqslant \frac{2^k}{\lambda^2} \int_{\mathscr{O}} |f|^2 \, d\mu \tag{4.13}$$

$$\int_{\mathscr{O}} |D\varphi_\epsilon|^2 \, d\mu_\epsilon \leqslant 2^k C_1 \int_{\mathscr{O}} |f|^2 \, d\mu \tag{4.14}$$

for the solutions φ_ϵ of equation (4.6) with $f \in L^2(\mathscr{O}, \mu)$. Indeed, $\varphi_\epsilon \in C^2(\overline{\mathscr{O}})$ by (4.7). Notice that the constant C_1 can be taken independent of $\epsilon \leq \epsilon_0$. In fact, it suffices to fix $\mathscr{O}_\delta$ in the proof of Proposition 3.6 in such a way that $\alpha_\epsilon \geq \delta$ for all $\epsilon \leqslant \epsilon_0$. The other constants involved are independent of α, as pointed out in Remark 3.5.

From the previous remark it is clear that if we are able to pass to the limit as ϵ goes to 0 in (4.13), (4.14) then the domain characterization (3.10) follows. This is done in the following

Theorem 4.3. *Under the above hypotheses on α, the domain $D(L)$ of L is given by*

$$D(L) = \{\varphi \in W^{2,2}_{loc}(\mathscr{O}) : \; D\varphi \in L^2(\mathscr{O}, \mu; \mathbb{R}^d), \; \alpha\Delta\varphi \in L^2(\mathscr{O}, \mu)\}. \tag{4.15}$$

Proof. Set $d\mu_\epsilon = \rho_\epsilon\, dx$ and define for $r > 0$

$$B(r) = \{x \in \mathscr{O} : \ \alpha(x) > r\} \tag{4.16}$$
$$\delta(r) = \inf\{\rho(x) : \ x \in B(r)\}\,. \tag{4.17}$$

Observe that $\rho_\epsilon \geqslant \delta(r)/2$ for all $x \in \overline{B(r)}$ and $\epsilon > 0$ small enough. So, owing to (4.14),

$$\int_{B(r)} |D\varphi_\epsilon|^2\, dx \leqslant \frac{2}{\delta(r)} \int_{B(r)} |D\varphi_\epsilon|^2\, d\mu_\epsilon \leqslant \frac{2^{k+1}C_1}{\delta(r)} \int_{\mathscr{O}} |f|^2\, d\mu =: K(r) \tag{4.18}$$

and then (φ_ϵ) is weakly compact in $H^1_{loc}(\mathscr{O})$, i.e., there is a sequence $\varphi_h = \varphi_{\epsilon_h}$ srongly convergent to a function φ in $L^2_{loc}(\mathscr{O})$ with $D\varphi_h$ weakly convergent to $D\varphi$ in $L^2_{loc}(\mathscr{O})$. In particular, $\varphi_h \to \varphi$ a.e. and then $|\varphi_h|^2\rho_\epsilon \to |\varphi|^2\rho$ a.e. and in $L^1(\mathscr{O})$. Using the Fatou Lemma in (4.13) we get

$$\int_{\mathscr{O}} |\varphi|^2\, d\mu \leqslant \frac{2}{\lambda^2} \int_{\mathscr{O}} |f|^2. \tag{4.19}$$

Coming to the gradient estimates, observe that

$$\int_{\mathscr{O}} |D\varphi|^2 \rho\, dx = \lim_{r\to 0} \int_{B(r)} |D\varphi|^2 \rho\, dx\,. \tag{4.20}$$

Moreover, we claim that

$$\int_{B(r)} |D\varphi|^2 \rho\, dx \leqslant \liminf_{h\to\infty} \int_{B(r)} |D\varphi_h|^2 \rho_h\, dx \tag{4.21}$$

(where $\rho_h = \rho_{\epsilon_h}$) for all $r > 0$. Indeed, writing

$$\int_{B(r)} |D\varphi_h|^2 \rho\, dx = \int_{B(r)} |D\varphi_h|^2 \rho_h\, dx + \int_{B(r)} |D\varphi_h|^2 (\rho - \rho_h)\, dx\,,$$

we have that the last integral converges to 0 by the uniform convergence of the ρ_ϵ to ρ in $B(r)$. Then, (4.21) follows by the weak lower semicontinuity of the norm invoking the L^2_{loc}-weak convergence of $D\varphi_h$.

Finally, from (4.14), (4.21) and (4.20) we deduce the estimate

$$\int_{\mathscr{O}} |D\varphi|^2 d\mu \leqslant C \int_{\mathscr{O}} |f|^2\, d\mu$$

and the proof is complete. □

Remark 4.4. The special feature of operator (4.1) is that the gradient structure is preserved under the approximation. In the general case of (1.11), on can either approximate the coefficient α as we did, keeping b fixed, or modify both α and b in order to keep the gradient structure. In the first case the gradient structure is destroyed in general, while in the second one, which leads e.g. to $2b_\epsilon = \alpha_\epsilon D \log \rho + D\alpha_\epsilon$, where ρ is defined by (1.12), the derivative of b_ϵ is not bounded in general (this is needed to get an estimate as (3.7) uniform in ϵ).

We conclude the paper by giving an example.

Example 4.5. We take $\mathscr{O} = B_1$, the ball of center 0 and radius 1. In this case the operator L reads as follows

$$L\varphi(x) = \frac{1}{2}\,(1-|x|^2)\Delta\varphi - k\langle x, D\varphi\rangle,$$

where $k \geq 1$, and the density of the invariant measure μ is

$$\rho(x) = (1-|x|^2)^{k-1}, \quad \forall\, x \in B_1.$$

By (4.15) the domain of L in $L^2(B_1,\mu)$ is given by

$$D(L) = \{\varphi \in W^{2,2}_{loc}(B_1):\ D\varphi \in L^2(B_1,\mu;\mathbb{R}^d),\ (1-|x|^2)\Delta\varphi \in L^2(B_1,\mu)\}.$$

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