Density results relative to the Dirichlet energy of mappings into a manifold

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Abstract. *Let $\mathcal{Y}$ be a smooth compact oriented Riemannian manifold without boundary. Weak limits of graphs of smooth maps $u_k : B^n \to \mathcal{Y}$ with equibounded Dirichlet integral give rise to elements of the space $\mathrm{cart}^{2,1}(B^n \times \mathcal{Y})$. Assume that $\mathcal{Y}$ is 1-connected and that its 2-homology group has no torsion. In any dimension n we prove that every element T in $\mathrm{cart}^{2,1}(B^n \times \mathcal{Y})$ with no singular vertical part can be approximated weakly in the sense of currents by a sequence of graphs of smooth maps $u_k : B^n \to \mathcal{Y}$ with Dirichlet energies converging to the energy of T.*

Let B^n be the unit ball in $\mathbb{R}^n$ and let $\mathcal{Y}$ be a smooth oriented Riemannian manifold of dimension $M \geq 2$, isometrically embedded in $\mathbb{R}^N$ for some $N \geq 3$. We shall assume that $\mathcal{Y}$ is compact, connected, without boundary. In addition, we assume that $\mathcal{Y}$ is 1-connected, i.e., $\pi_1(\mathcal{Y}) = 0$, and that its integral 2-homology group $H_2(\mathcal{Y}) := H_2(\mathcal{Y}; \mathbb{Z})$ has no torsion. We also notice that $H_2(\mathcal{Y}; X) = H_2(\mathcal{Y}) \otimes X$ for $X = \mathbb{R}, \mathbb{Q}$.

In this paper we consider sequences of smooth maps $u_k : B^n \to \mathcal{Y}$ with

$$\sup_k \mathbf{D}(u_k) < +\infty \,,$$

where $\mathbf{D}(u)$ is the Dirichlet integral

$$\mathbf{D}(u) := \frac{1}{2} \int_{B^n} |Du|^2 \, dx \,.$$

Modulo passing to a subsequence the $(n,2)$-currents G_{u_k}, integration over the graphs of u_k of n-forms with at most two vertical differentials, converge to a current $T \in \mathrm{cart}^{2,1}(B^n \times \mathcal{Y})$, see [11] [8] [13] and Sec. 1 below. In order to discuss these currents, we recall the following

Definition 0.1 *We say that an integral 2-cycle $C \in \mathcal{Z}_2(\mathcal{Y})$ is* of spherical type *if its homology class contains a Lipschitz image of the 2-sphere S^2. More precisely, if there exist $Z \in \mathcal{Z}_2(\mathcal{Y})$, $R \in \mathcal{R}_3(\mathcal{Y})$ and a Lipschitz function $\phi : S^2 \to \mathcal{Y}$ such that*

$$C - Z = \partial R \,, \qquad \phi_\# [\![S^2]\!] = Z \,.$$

We also denote

$$H_2^{sph}(\mathcal{Y}) := \{ [\gamma] \in H_2(\mathcal{Y}) \mid \exists \, \phi \in \mathrm{Lip}(S^2, \mathcal{Y}) \, : \, \phi_\# [\![S^2]\!] \in [\gamma] \}$$

and we shall also assume that $H_2(\mathcal{Y}) / H_2^{sph}(\mathcal{Y})$ has no torsion.

Currents $T \in \mathrm{cart}^{2,1}(B^n \times \mathcal{Y})$ have the form

$$T = G_{u_T} + \sum_{q \in H_2^{sph}(\mathcal{Y})} \mathbb{L}_q \times R_q + S_{T,sing} \,,$$

where u_T is the weak $W^{1,2}$ limit of the u_k's, $\mathbb{L}_q$ is an integer multiplicity rectifiable current of dimension $n-2$ in B^n, R_q is a 2-cycle in q and $S_{T,sing}$ is nonzero only on forms ω with exactly two vertical differentials and such that $d_y \omega \neq 0$. Apart from being completely vertical and homologically trivial, otherwise the current $S_{T,sing}$ can be very wild and essentially any measure in $B^n \times \mathcal{Y}$, compare [10].

For every $T \in \mathrm{cart}^{2,1}(B^n \times \mathcal{Y})$, the Dirichlet integral of T, $\mathbf{D}(T)$, turns out to be well defined, see below; in particular $\mathbf{D}(G_u) = \mathbf{D}(u)$ if $T = G_u$. Moreover, see [8] [13], in the *vertical homology class* of each T in $\mathrm{cart}^{2,1}(B^n \times \mathcal{Y})$ there is a representative that minimizes the Dirichlet integral and has the form

$$T = G_{u_T} + \sum_{q \in H_2^{sph}(\mathcal{Y})} \mathbb{L}_q \times R_q \,, \tag{0.1}$$

where R_q is an integral cycle in q. In this case the Dirichlet energy is given by

$$\mathbf{D}(T) = \frac{1}{2}\int_{B^n} |Du|^2\, dx + \sum_{q\in H_2^{sph}(\mathcal{Y})} \mathbf{M}(\mathbb{L}_q)\cdot \mathbf{M}(R_q)\,.$$

In this paper we show that every current $T \in \mathrm{cart}^{2,1}(B^n \times \mathcal{Y})$ which has the form (0.1) is the weak limit of a sequence of graphs of smooth maps $u_k : B^n \to \mathcal{Y}$, i.e., $G_{u_k} \rightharpoonup T$, moreover

$$\mathbf{D}(T) = \inf\{ \liminf_{k\to+\infty} \mathbf{D}(u_k) \mid \{u_k\} \subset C^1(B^n, \mathcal{Y})\,, \quad G_{u_k} \rightharpoonup T\}$$

and actually there is a sequence of smooth maps $u_k : B^n \to \mathcal{Y}$ such that

$$G_{u_k} \rightharpoonup T \qquad \text{and} \qquad \frac{1}{2}\int_{B^n} |Du_k|^2\, dx \to \mathbf{D}(T)\,.$$

In the case $\mathcal{Y} = S^2$, the unit 2-sphere, this density result was first proved in [9] in the case $n \leq 3$, see also [11, Vol. II], and it is new, if $n \geq 4$, even in the case $\mathcal{Y} = S^2$. Moreover, it was proved in [12], in the case of dimension $n = 2$, with no additional hypothesis on the first homotopy group $\pi_1(\mathcal{Y})$, and in [13], in the case $n = 3$, by assuming the slightly weaker hypothesis that the Hurewicz homomorphism $\pi_2(\mathcal{Y}; y_0) \to H_2(\mathcal{Y}; \mathbb{Q})$ is injective for every $y_0 \in \mathcal{Y}$. Note that this injectivity condition automatically holds if $\mathcal{Y}$ is 1-connected, by the Hurewicz theorem [17]. We remark that if $n \geq 3$ the injectivity assumption on the Hurewicz map cannot be avoided. In fact, in [13] we showed that if the Hurewicz map is not injective there exist currents T in $\mathrm{cart}^{2,1}(B^3 \times \mathcal{Y})$ of the type $T = G_u$, where $u(x) = \varphi(x/|x|)$ for a suitable smooth map $\varphi : \partial B^3 \to \mathcal{Y}$, for which there is a positive constant C such that

$$\mathbf{D}(u) + C \leq \liminf_{k\to+\infty} \mathbf{D}(u_k)$$

for any sequence of smooth maps $u_k : B^3 \to \mathcal{Y}$ with $G_{u_k} \rightharpoonup G_u$.

1 Notation and preliminary results

In this section we recall some facts from the theory of Cartesian currents with finite Dirichlet energy. We refer to [11] and [8] for proofs and details.

Homological facts. Since $H_2(\mathcal{Y})$ has no torsion, a condition which automatically holds if $\dim \mathcal{Y} = 2$, there are generators $[\gamma_1], \ldots, [\gamma_{\overline{s}}]$, i.e. integral cycles in $\mathcal{Z}_2(\mathcal{Y})$, such that

$$H_2(\mathcal{Y}) = \left\{ \sum_{s=1}^{\overline{s}} n_s\, [\gamma_s] \mid n_s \in \mathbb{Z} \right\}\,,$$

see e.g. [11], Vol. I, Sec. 5.4.1. Moreover, since $H_2(\mathcal{Y})/H_2^{sph}(\mathcal{Y})$ has no torsion, we may and do choose the γ_s's in such a way that $[\gamma_1], \ldots, [\gamma_{\widetilde{s}}]$ generate the spherical homology classes in $H_2^{sph}(\mathcal{Y})$ for some $\widetilde{s} \leq \overline{s}$. By de Rham's theorem the second real homology group is in duality with the second cohomology group $H^2_{dR}(\mathcal{Y})$, the duality being given by the natural pairing

$$\langle [\gamma], [\omega] \rangle := \gamma(\omega) = \int_\gamma \omega\,, \qquad [\gamma]_{\mathbb{R}} \in H_2(\mathcal{Y}; \mathbb{R})\,, \quad [\omega] \in H^2_{dR}(\mathcal{Y})\,.$$

We will then denote by $[\omega^1], \ldots, [\omega^{\overline{s}}]$ a dual basis in $H^2_{dR}(\mathcal{Y})$ so that $\gamma_s(\omega^r) = \delta_{sr}$, where δ_{sr} denotes the Kronecker symbols. Also, we may and do assume that ω^s is the harmonic form in its cohomology class.

$\mathcal{D}_{n,2}$-currents. Every differential k-form $\omega \in \mathcal{D}^k(B^n \times \mathcal{Y})$ splits as a sum $\omega = \sum_{j=0}^{\underline{j}} \omega^{(j)}$, $\underline{j} := \min(k, M)$, where the $\omega^{(j)}$'s are the k-forms that contain exactly j differentials in the vertical $\mathcal{Y}$ variables. We denote by $\mathcal{D}^{k,2}(B^n\times\mathcal{Y})$ the subspace of $\mathcal{D}^k(B^n\times\mathcal{Y})$ of k-forms of the type $\omega=\sum_{j=0}^2\omega^{(j)}$, and by $\mathcal{D}_{k,2}(B^n\times\mathcal{Y})$ the dual space of $\mathcal{D}^{k,2}(B^n\times\mathcal{Y})$. Every $(k,2)$-current $T\in\mathcal{D}_{k,2}(B^n\times\mathcal{Y})$ splits as $T=\sum_{j=0}^2 T_{(j)}$, where $T_{(j)}(\omega):=T(\omega^{(j)})$. For example, if $u\in W^{1,2}(B^n,\mathcal{Y})$, i.e., $u\in W^{1,2}(B^n,\mathbb{R}^N)$ with $u(x)\in\mathcal{Y}$ for a.e. $x\in B^n$, then G_u is an $(n,2)$-current in $\mathcal{D}_{n,2}(B^n\times\mathcal{Y})$, where in an approximate sense $G_u:=(Id\bowtie u)_\#[\![B^n]\!]$, $(Id\bowtie u)(x):=(x,u(x))$, compare [11].

D-NORM. For any $\omega\in\mathcal{D}^{n,2}(B^n\times\mathcal{Y})$ and $T\in\mathcal{D}_{n,2}(B^n\times\mathcal{Y})$ we set

$$\|\omega\|_{\mathbf{D}}:=\max\Big\{\sup_{x,y}\frac{|\omega^{(0)}(x,y)|}{1+|y|^2}\,,\int_{B^n}\sup_y|\omega^{(1)}(x,y)|^2\,dx\,,\int_{B^n}\sup_y|\omega^{(2)}(x,y)|\,dx\Big\}$$
$$\|T\|_{\mathbf{D}}:=\sup\Big\{T(\omega)\mid\omega\in\mathcal{D}^{n,2}(B^n\times\mathcal{Y})\,,\ \|\omega\|_{\mathbf{D}}\le 1\Big\}.$$

It is easily checked that $\|T\|_{\mathbf{D}}$ is a norm on $\{T\in\mathcal{D}_{n,2}(B^n\times\mathcal{Y})\mid\|T\|_{\mathbf{D}}<+\infty\}$.

WEAK $\mathcal{D}_{n,2}$-CONVERGENCE. If $\{T_k\}\subset\mathcal{D}_{n,2}(B^n\times\mathcal{Y})$, we say that $\{T_k\}$ converges weakly in $\mathcal{D}_{n,2}(B^n\times\mathcal{Y})$, $T_k\rightharpoonup T$, if $T_k(\omega)\to T(\omega)$ for every $\omega\in\mathcal{D}^{n,2}(B^n\times\mathcal{Y})$. The class $\mathcal{D}_{n,2}(B^n\times\mathcal{Y})$ is closed under weak convergence and $\|\cdot\|_{\mathbf{D}}$ is weakly lower semicontinuous. Moreover, if $\sup_k\|T_k\|_{\mathbf{D}}<+\infty$ there is a subsequence which weakly converges to some $T\in\mathcal{D}_{n,2}(B^n\times\mathcal{Y})$ with $\|T\|_{\mathbf{D}}<+\infty$.

BOUNDARIES. The exterior differential d splits into a horizontal and a vertical differential $d=d_x+d_y$. Of course $\partial_x T(\omega):=T(d_x\omega)$ defines a boundary operator $\partial_x:\mathcal{D}_{n,2}(B^n\times\mathcal{Y})\to\mathcal{D}_{n-1,2}(B^n\times\mathcal{Y})$. Now, for any $\omega\in\mathcal{D}^{n-1,2}(B^n\times\mathcal{Y})$, $d_y\omega$ belongs to $\mathcal{D}^{n,2}(B^n\times\mathcal{Y})$ if and only if $d_y\omega^{(2)}=0$. Therefore, $\partial_y T$ makes sense only as an element of the dual space of $\mathcal{Z}^{n-1,2}(B^n\times\mathcal{Y})$, where

$$\mathcal{Z}^{k,2}(B^n\times\mathcal{Y}):=\{\omega\in\mathcal{D}^{k,2}(B^n\times\mathcal{Y})\mid d_y\omega^{(2)}=0\}\,.$$

D-GRAPHS. The study of weak limits of sequences of maps with equibounded Dirichlet energy, minimization problems and concentration phenomena, see [11], draw the authors of [10] to introduce the subclass $\mathbf{D}$-graph$(B^n\times\mathcal{Y})$ given by the $(n,2)$-currents $T\in\mathcal{D}_{n,2}(B^n\times\mathcal{Y})$ with $\|T\|_{\mathbf{D}}<+\infty$ and such that

$$T=G_{u_T}+S_T \tag{1.1}$$

for some function $u_T\in W^{1,2}(B^n,\mathcal{Y})$ and some $S_T\in\mathcal{D}_{n,2}(B^n\times\mathcal{Y})$ with $S_{T(0)}=S_{T(1)}=0$, i.e. S_T completely vertical, so that

$$\partial_x T=0\quad\text{on }\mathcal{D}^{n-1,2}(B^n\times\mathcal{Y})\,,\qquad\partial_y T=0\quad\text{on }\mathcal{Z}^{n-1,2}(B^n\times\mathcal{Y})\,. \tag{1.2}$$

THE 2-DIMENSIONAL CASE. If $n=2$, obviously $\mathcal{D}_{n,2}(B^n\times\mathcal{Y})=\mathcal{D}_2(B^2\times\mathcal{Y})$ and ∂T is the usual boundary of currents, whereas $\mathbf{M}(T)\le c\,(\|T\|_{\mathbf{D}}+1)$ for some absolute constant. Consequently, weak limits of smooth graphs with equibounded Dirichlet energy are integer multiplicity (shortly i.m.) rectifiable currents in $\mathcal{R}_2(B^2\times\mathcal{Y})$, and $\mathbf{D}$-graph$(B^2\times\mathcal{Y})\cap\mathcal{R}_2(B^2\times\mathcal{Y})$ is closed under weak convergence with equibounded $\mathbf{D}$-norm. As proved in [10] and [11], every T in $\mathbf{D}$-graph$(B^2\times\mathcal{Y})\cap\mathcal{R}_2(B^2\times\mathcal{Y})$ decomposes as

$$T=G_{u_T}+S_T\,,\qquad S_T=\sum_{i=1}^I\delta_{x_i}\times C_i+S_{T,sing}\,, \tag{1.3}$$

where δ_x is the Dirac mass in x, $x_i\in B^2$, $C_i\in\mathcal{Z}_2(\mathcal{Y})$ are integral cycles with non trivial homology and $S_{T,sing}$ is a completely vertical, homologically trivial, i.m. rectifiable current supported on a set not containing $\{x_i\}\times\mathcal{Y}$, $i=1,\dots,I$. More precisely, $S_{T,sing}(\omega)\neq 0$ only on forms $\omega\in\mathcal{D}^2(B^2\times\mathcal{Y})$ such that $d_y\omega^{(2)}\neq 0$. Moreover, see [10] [11], if T is in the sequential weak closure of smooth graphs with equibounded Dirichlet energies, then every C_i is of spherical type, see Definition 0.1. These facts lead to

Definition 1.1 *If $n=2$, $\mathrm{cart}^{2,1}(B^2\times\mathcal{Y})$ denotes the class of i.m. rectifiable currents T in* $\mathbf{D}$-graph$(B^2$ $\times\mathcal{Y})$ *which decompose as in* (1.3)*, where the C_i's are of spherical type.*

It turns out, see [10] [8], that $\text{cart}^{2,1}(B^2\times\mathcal{Y})$ is closed under weak convergence, with equibounded **D**-norm, and contains the weak limits of sequences of smooth graphs with equibounded **D**-norm.

THE n-DIMENSIONAL CASE. Denote by $\pi:\mathbb{R}^n\times\mathbb{R}^N\to\mathbb{R}^n$ and $\widehat{\pi}:\mathbb{R}^n\times\mathbb{R}^N\to\mathbb{R}^N$ the orthogonal projections onto the first and the second factor, respectively. Let P be an oriented 2-plane in $\mathbb{R}^n$, and $P_t:=P+\sum_{i=1}^{n-2}t_i\nu_i$ the family of oriented 2-planes parallel to P, $t=(t_1,\dots,t_{n-2})\in\mathbb{R}^{n-2}$, where $\text{span}(\nu_1,\dots,\nu_{n-2})$ is the orthogonal subspace to P. Similarly to the case of normal currents, for every $T\in\mathcal{D}_{n,2}(B^n\times\mathcal{Y})$ with $\|T\|_{\mathbf{D}}<+\infty$ and for $\mathcal{H}^{n-2}$-a.e. t the *slice* $T\llcorner\pi^{-1}(P_t)$ of T over $\pi^{-1}(P_t)$ is a well defined current in $\mathcal{D}_2((B^n\cap P_t)\times\mathcal{Y})$ with finite **D**-norm. Moreover, if $T_k\rightharpoonup T$ with equibounded **D**-norm, for $\mathcal{H}^{n-2}$-a.e. t, passing to a subsequence we have $T_k\llcorner\pi^{-1}(P_t)\rightharpoonup T\llcorner\pi^{-1}(P_t)$ with equibounded **D**-norm. Finally, if $T\in\mathbf{D}\text{-graph}(B^n\times\mathcal{Y})$, for $\mathcal{H}^{n-2}$-a.e. t we have $T\llcorner\pi^{-1}(P_t)\in\mathbf{D}\text{-graph}((B^n\cap P_t)\times\mathcal{Y})$. Therefore in any dimension n it was introduced in [8] the following

Definition 1.2 *T is said to be in $\text{cart}^{2,1}(B^n\times\mathcal{Y})$ if $T\in\mathbf{D}\text{-graph}(B^n\times\mathcal{Y})$ and for any 2-plane P and for $\mathcal{H}^{n-2}$-a.e. t the 2-dimensional current $T\llcorner\pi^{-1}(P_t)$ belongs to $\text{cart}^{2,1}((B^n\cap P_t)\times\mathcal{Y})$.*

It turns out that the class $\text{cart}^{2,1}(B^n\times\mathcal{Y})$ is closed under weak convergence with equibounded **D**-norms.

STRUCTURE OF $\text{cart}^{2,1}(B^n\times\mathcal{Y})$. If $T\in\mathbf{D}\text{-graph}(B^n\times\mathcal{Y})$, for any closed 2-form $\omega\in\mathcal{Z}^2(\mathcal{Y})$ define the $(n-2)$-currents $\mathbb{D}(T;\omega)$ and $\mathbb{L}(T;\omega)$ in B^n by setting

$$\mathbb{D}(T;\omega)(\phi):=G_{u_T}(\pi^{\#}\phi\wedge\widehat{\pi}^{\#}\omega)\,,\qquad \mathbb{L}(T;\omega)(\phi):=S_T(\pi^{\#}\phi\wedge\widehat{\pi}^{\#}\omega)$$

for every $\phi\in\mathcal{D}^{n-2}(B^n)$. Then, compare [10] and [11], $\mathbb{D}(T;\omega)$ and $\mathbb{L}(T;\omega)$ have finite mass, $\partial(\mathbb{D}(T;\omega)+\mathbb{L}(T;\omega))=0$ in $\mathcal{D}^{n-3}(B^n)$ and $\mathbb{L}(T;\omega)$ only depends on the cohomology class of ω. Finally, setting for $s=1,\dots,\overline{s}$

$$\mathbb{D}_s(T):=\mathbb{D}(T;\omega^s)\,,\qquad \mathbb{L}_s(T):=\mathbb{L}(T;\omega^s)\,,$$

we have

$$T=G_{u_T}+\sum_{s=1}^{\overline{s}}\mathbb{L}_s(T)\times\gamma_s\qquad\text{on}\quad \mathcal{Z}^{n,2}(B^n\times\mathcal{Y})$$

for some function $u_T\in W^{1,2}(B^n,\mathcal{Y})$. Moreover, if $T\in\text{cart}^{2,1}(B^n\times\mathcal{Y})$, it is proved in [8] that the $\mathbb{L}_s(T)$'s are i.m. rectifiable currents in $\mathcal{R}_{n-2}(B^n)$, with $\mathbb{L}_s(T)=0$ for $s=\widetilde{s}+1,\dots,\overline{s}$. As a consequence, every $T\in\text{cart}^{2,1}(B^n\times\mathcal{Y})$ decomposes as

$$T=G_{u_T}+\sum_{q\in H_2^{sph}(\mathcal{Y})}\mathbb{L}_q\times R_q\qquad\text{on}\quad \mathcal{Z}^{n,2}(B^n\times\mathcal{Y})\,,\tag{1.4}$$

where $\mathbb{L}_q$ is an i.m. rectifiable current in $\mathcal{R}_{n-2}(B^n)$ and $R_q\in\mathcal{Z}^2(\mathcal{Y})$ is an integral 2-cycle of spherical type in the homology class q. More precisely, if $q=\sum_{s=1}^{\widetilde{s}}n_s[\gamma_s]$, $n_s\in\mathbb{Z}$, then

$$R_q=\sum_{s=1}^{\widetilde{s}}n_s\gamma_s\qquad\text{and}\qquad \mathbb{L}_q=\tau(\mathcal{L}_q,1,\overrightarrow{\mathcal{L}})\,,\tag{1.5}$$

where $\overrightarrow{\mathcal{L}}$ is a unit simple orienting $(n-2)$-vector field and the rectifiable sets $\mathcal{L}_q$ are pairwise disjoint.

We finally remark that though the *singular vertical part*

$$S_{T,sing}:=T-G_{u_T}-\sum_{s=1}^{\widetilde{s}}\mathbb{L}_s(T)\times\gamma_s$$

is completely vertical and homologically trivial, i.e., $S_{T,sing}(\omega)=0$ if $\omega^{(2)}=0$ or $\omega\in\mathcal{Z}^{n,2}(B^n\times\mathcal{Y})$, it is in general non zero on forms $\omega\in\mathcal{D}^{n,2}(B^n\times\mathcal{Y})$ for which $d_y\omega^{(2)}\neq 0$. However, in [8] it was shown that, in any dimension n, solutions of minimum problems "have" no singular part and, in fact, for every $T \in \mathrm{cart}^{2,1}(B^n \times \mathcal{Y})$,

$$T = G_{u_T} + \sum_{q \in H_2^{sph}(\mathcal{Y})} \mathbb{L}_q \times S_q + S_{T,sing}\,,$$

where $u_T \in W^{1,2}(B^n, \mathcal{Y})$, $\mathbb{L}_q \in \mathcal{R}_{n-2}(B^n)$, and $S_q \in \mathcal{Z}_2(\mathcal{Y})$, in the vertical homological class $[T]$, see below, there is a current of the form

$$T = G_{u_T} + \sum_{q \in H_2^{sph}(\mathcal{Y})} \mathbb{L}_q \times R_q\,,$$

where $R_q \in \mathcal{Z}_2(\mathcal{Y})$ is an integral cycle in q, with finite mass.

Vertical homology classes. Let $u \in W^{1,2}(B^n, \mathcal{Y})$. We have $\partial G_u(\omega) = 0$ if $\omega \in \mathcal{D}^{n-1,2}(B^n \times \mathcal{Y})$ with $\omega^{(2)} = 0$ or $d_y \omega = 0$. Setting

$$\mathcal{B}^{k,2}(B^n \times \mathcal{Y}) := \{\omega \in \mathcal{D}^{k,2}(B^n \times \mathcal{Y}) \mid \exists \eta \in \mathcal{D}^{k-1,1}(B^n \times \mathcal{Y})\ :\ \omega^{(2)} = d_y \eta\}$$

and

$$\mathcal{H}^{k,2}(B^n \times \mathcal{Y}) := \frac{\mathcal{Z}^{k,2}(B^n \times \mathcal{Y})}{\mathcal{B}^{k,2}(B^n \times \mathcal{Y})}\,,$$

then $\partial G_u = 0$ on $\mathcal{B}^{n-1,2}(B^n \times \mathcal{Y})$ and $\partial_y \partial G_u = 0$, whence $\partial G_u(\omega)$ depends only on the cohomology class of $\omega \in \mathcal{Z}^{n-1,2}(B^n \times \mathcal{Y})$. As a consequence ∂G_u induces a functional $(\partial G_u)_\star$ on $\mathcal{H}^{n-1,2}(B^n \times \mathcal{Y})$ given by

$$(\partial G_u)_\star(\omega + \mathcal{B}^{n-1,2}) := \partial G_u(\omega + \mathcal{B}^{n-1,2}) = \partial G_u(\omega)\,, \qquad \omega \in \mathcal{Z}^{n-1,2}\,,$$

compare [11], Vol. II, Sec. 5.4.1. Therefore, since

$$\mathcal{H}^{k,2}(B^n \times \mathcal{Y}) \simeq \mathcal{D}^{k-2}(B^n) \otimes H^2_{dR}(\mathcal{Y})\,,$$

the homology map $(\partial G_u)_\star$ is uniquely represented as an element of the class $\mathcal{D}_{n-3}(B^n; H_2(\mathcal{Y}; \mathbb{R}))$. More explicitly, if $\phi \in \mathcal{D}^{n-3}(B^n)$, we have $[(\partial G_u)_\star(\phi)] \in H_2(\mathcal{Y}; \mathbb{R})$ and for $s = 1, \ldots, \overline{s}$

$$\langle (\partial G_u)_\star(\phi), [\omega^s] \rangle = \partial G_u(\pi^\# \phi \wedge \widehat{\pi}^\# \omega^s)\,,$$

$\langle , \rangle$ denoting the de Rham duality between $H_2(\mathcal{Y}; \mathbb{R})$ and $H^2_{dR}(\mathcal{Y})$: in general $(\partial G_u)_\star$ is non-trivial.

Singularities of Sobolev maps. Following [11], Vol. II, Sec. 5.4.2, we set

$$\mathbb{P}(u) := (\partial G_u)_\star \in \mathcal{D}_{n-3}(B^n; H_2(\mathcal{Y}; \mathbb{R}))\,,$$

for each $\omega \in [\omega] \in H^2_{dR}(\mathcal{Y})$ we define the current $\mathbb{P}(u; \omega) := \partial \pi_\#(G_u \llcorner \widehat{\pi}^\# \omega)$ in $\mathcal{D}_{n-3}(B^n)$, so that

$$\mathbb{P}(u; \omega)(\phi) = \partial G_u(\pi^\# \phi \wedge \widehat{\pi}^\# \omega) \qquad \forall\, \phi \in \mathcal{D}^{n-3}(B^n)\,,$$

and for every $\omega \in \mathcal{Z}^2(\mathcal{Y})$ the current $\mathbb{D}(u; \omega) := \pi_\#(G_u \llcorner \widehat{\pi}^\# \omega)$ in $\mathcal{D}_{n-2}(B^n)$, so that

$$\mathbb{D}(u; \omega)(\phi) = G_u(\pi^\# \phi \wedge \widehat{\pi}^\# \omega) \qquad \forall\, \phi \in \mathcal{D}^{n-2}(B^n)\,.$$

The following facts hold:

(i) for $s = 1, \ldots, \widetilde{s}$

$$\mathbb{P}(u; \omega^s)(\phi) = \langle \mathbb{P}(u)(\phi), [\omega^s] \rangle\,,$$

i.e., $\mathbb{P}(u; \omega^s)$ does not depend on the representative in the cohomology class $[\omega^s]$;

(ii) $\partial\, \mathbb{P}(u) = 0$ and $\mathbb{P}(u) = \sum_{s=1}^{\widetilde{s}} \mathbb{P}(u; \omega^s) \otimes [\gamma_s]$, hence it does not depend on the choice of $\gamma_1, \ldots, \gamma_{\widetilde{s}}$;

(iii) $\partial\,\mathbb{D}(u;\omega)(\phi) = \langle \mathbb{P}(u)(\phi), [\omega]\rangle$ and hence $\partial\,\mathbb{D}(u;\omega^s) = \mathbb{P}(u;\omega^s)$ for each representative ω^s in $[\omega^s]$.

Therefore, for any $s = 1,\dots,\widetilde{s}$ we can set

$$\mathbb{D}_s(u) := \mathbb{D}(u;\omega^s)\,, \qquad \mathbb{P}_s(u) := \mathbb{P}(u;\omega^s) = \partial\,\mathbb{D}_s(u) \llcorner B^n\,. \tag{1.6}$$

Note that if $T = G_{u_T} + S_T \in \mathrm{cart}^{2,1}(B^n \times \mathcal{Y})$, then $\mathbb{D}_s(u_T) = \mathbb{D}_s(T)$ and

$$\mathbb{P}_s(u_T) = -\partial\,\mathbb{L}_s(T) \llcorner B^n \qquad \forall\, s = 1,\dots,\widetilde{s}\,. \tag{1.7}$$

Finally, we clearly have $\mathbb{P}(u) = 0$ if u is smooth, say Lipschitz, or at least in $W^{1,3}(B^n,\mathcal{Y})$.

The Dirichlet energy. Following [11], Vol. II, Sec. 1.2, we recall that the parametric polyconvex l.s.c. envelope of the Dirichlet integrand is the function $F : \Lambda_n\mathbb{R}^{n+N} \to \overline{\mathbb{R}}^+$ given by

$$\begin{aligned} \|\xi\|_{\mathcal{D}} := \sup\{\phi(\xi) \quad | \quad & \phi : \Lambda_n\mathbb{R}^{n+N} \to \overline{\mathbb{R}}^+,\ \phi \text{ linear}, \\ & \phi(M(G)) \le \frac{1}{2}|G|^2 \quad \forall\, G \in M(N,n)\}\,. \end{aligned} \tag{1.8}$$

Here $M(N,n)$ denotes the set of $N \times n$ real valued matrices G and $M(G)$ is the n-vector in $\Lambda_n\mathbb{R}^{n+N}$ given by

$$M(G) := \Big(e_1 + \sum_{j=1}^N G_1^j\,\varepsilon_j\Big) \wedge \cdots \wedge \Big(e_n + \sum_{j=1}^N G_n^j\,\varepsilon_j\Big)\,, \qquad G = \Big(G_i^j\Big)_{i,j=1}^{n,N}$$

$(e_1,\dots,e_n)$, $(\varepsilon_1,\dots,\varepsilon_N)$ being the standard basis in $\mathbb{R}^n$ and $\mathbb{R}^N$, respectively, so that $M(G)$ identifies the n-plane graph of G in $\mathbb{R}^{n+N}$, and in fact yields an orientation to it. The Dirichlet density for maps $u : B^n \to \mathcal{Y}$ is then defined by the function $F(y,\xi) : \mathbb{R}^N \times \Lambda_n\mathbb{R}^{n+N} \to \overline{\mathbb{R}}^+$ given by

$$F(y,\xi) := \begin{cases} \|\xi\|_{\mathcal{D}} & \text{if} \quad y \in \mathcal{Y} \quad \text{and} \quad \xi \in \Lambda_n(\mathbb{R}^n \times T_y\mathcal{Y}) \\ +\infty & \text{otherwise} \end{cases} \tag{1.9}$$

where $T_y\mathcal{Y}$ is the tangent space to $\mathcal{Y}$ at y. The Dirichlet integral is then extended to currents T in $\mathbf{D}$-graph$(B^n \times \mathcal{Y})$ by

$$\mathbf{D}(T) := \int F(y,\overrightarrow{T})\,d\|T\|_{\mathbf{D}}\,,$$

$\overrightarrow{T}$ being the Radon-Nikodym derivative $dT/d\|T\|_{\mathbf{D}}$. If (1.1) holds one has

$$\mathbf{D}(T) = \frac{1}{2}\int_{B^n} |Du_T|^2\,dx + \int_{B^n\times\mathcal{Y}} F(y,\overrightarrow{S_T})\,d\|S_T\|_{\mathbf{D}}\,. \tag{1.10}$$

In particular

$$\|T\|_{\mathbf{D}} \le c\,\mathbf{D}(T)$$

for some absolute constant $c = c(n)$. Finally, if $A \subset B^n$ is a Borel set we will denote

$$\mathbf{D}(T, A\times\mathcal{Y}) := \mathbf{D}(T \llcorner A \times \mathcal{Y})$$

and if $u \in W^{1,2}(B^n,\mathcal{Y})$

$$\mathbf{D}(u,A) := \frac{1}{2}\int_A |Du|^2\,dx = \mathbf{D}(G_u, A\times\mathcal{Y})\,, \qquad \mathbf{D}(u) := \mathbf{D}(u,B^n)\,.$$

In case $n = 2$, if (1.3) holds we have

$$\mathbf{D}(T) = \frac{1}{2}\int_{B^2} |Du_T|^2\,dx + \sum_{i=1}^I \mathbf{M}(C_i) + \mathbf{M}(S_{T,sing})\,.$$

In case $n \geq 3$, as remarked in the introduction, a part from the case of energy minimizing currents, or of homological representatives, we do not have an explicit formula for the second term on the righthand side of (1.10). However, if $T = G_{u_T} + \sum_{q\in H_2^{sph}(\mathcal{Y})} \mathbb{L}_q \times R_q$, where R_q is an integral cycle in q, we have, see [8],

$$\mathbf{D}(T) = \frac{1}{2}\int_{B^n} |Du_T|^2\,dx + \sum_{q\in H_2^{sph}(\mathcal{Y})} \mathbf{M}(\mathbb{L}_q)\cdot \mathbf{M}(R_q)\,. \tag{1.11}$$

Therefore, writing T as $T = G_{u_T} + \sum_{s=1}^{\widetilde{s}} \mathbb{L}_s(T)\times\gamma_s$, we have

$$\mathbf{D}(T) = \frac{1}{2}\int_{B^n} |Du_T|^2\,dx + \sum_{s=1}^{\widetilde{s}} \mathbf{M}(\mathbb{L}_s(T))\cdot \mathbf{M}(\gamma_s)\,. \tag{1.12}$$

2 Density of smooth graphs

In this section we prove a strong density result for the Dirichlet energy of maps from n-dimensional domains into $\mathcal{Y}$. We assume that $\mathcal{Y}$ is a smooth, compact, boundaryless, connected, oriented Riemannian manifold of dimension $M \geq 2$, isometrically embedded in $\mathbb{R}^N$, $N \geq 3$.

Boundary data. Let $\widetilde{B}^n$ be a bounded domain in $\mathbb{R}^n$ such that $B^n \subset\subset \widetilde{B}^n$, e.g. $\widetilde{B}^n := B^n(0,2)$, and $\varphi : \widetilde{B}^n \to \mathcal{Y}$ be a given smooth $W^{1,2}$ function. In the sequel we will denote

$$\begin{aligned} W^{1,2}_\varphi(\widetilde{B}^n,\mathcal{Y}) &:= \{u \in W^{1,2}(\widetilde{B}^n,\mathcal{Y}) \mid u = \varphi \quad \text{on} \quad \widetilde{B}^n \setminus \overline{B}^n\} \\ C^1_\varphi(\widetilde{B}^n,\mathcal{Y}) &:= \{u \in C^1(\widetilde{B}^n,\mathcal{Y}) \mid u = \varphi \quad \text{on} \quad \widetilde{B}^n \setminus \overline{B}^n\} \\ \mathrm{cart}^{2,1}_\varphi(\widetilde{B}^n\times\mathcal{Y}) &:= \{T \in \mathrm{cart}^{2,1}(\widetilde{B}^n\times\mathcal{Y}) \mid \\ &\qquad (T - G_\varphi) \llcorner (\widetilde{B}^n \setminus \overline{B}^n)\times\mathbb{R}^N = 0\}\,. \end{aligned}$$

Theorem 2.1 (Approximation by smooth graphs with prescribed boundary data) *Assume that $\mathcal{Y}$ is 1-connected, i.e., $\pi_1(\mathcal{Y}) = 0$, and that the integral 2-homology group $H_2(\mathcal{Y};\mathbb{Z})$ has no torsion. Let $\varphi : \widetilde{B}^n \to \mathcal{Y}$ be a given smooth map. Also, let $T \in \mathrm{cart}^{2,1}_\varphi(\widetilde{B}^n\times\mathcal{Y})$ be such that*

$$T = G_{u_T} + \sum_{q\in H_2^{sph}(\mathcal{Y})} \mathbb{L}_q \times R_q\,, \tag{2.1}$$

where R_q is an integral cycle in q. Then there exists a sequence of smooth maps $\{u_k\} \subset C^1_\varphi(\widetilde{B}^n,\mathcal{Y})$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{D}_{n,2}(\widetilde{B}^n\times\mathcal{Y})$ and

$$\lim_{k\to+\infty} \mathbf{D}(u_k,\widetilde{B}^n) = \mathbf{D}(T,\widetilde{B}^n\times\mathcal{Y})\,.$$

Theorem 2.2 (Approximation by smooth graphs) *Let $\mathcal{Y}$ be as in Theorem 2.1 and $T \in \mathrm{cart}^{2,1}(B^n \times \mathcal{Y})$ be given by (2.1), where R_q is an integral cycle in q. There exists a sequence of smooth maps $\{u_k\} \subset C^1(B^n,\mathcal{Y})$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{D}_{n,2}(B^n\times\mathcal{Y})$ and*

$$\lim_{k\to+\infty} \mathbf{D}(u_k,B^n) = \mathbf{D}(T,B^n\times\mathcal{Y})\,.$$

As already mentioned, Theorems 2.1 and 2.2 have been proved in [13] in case of dimension $n = 3$, by adapting the proofs of Theorem 1 in Sec. 4.2.5, and of Theorem 1 in Sec. 4.2.6, respectively, of [11, Vol. II], where these results are proved in the case $n = 3$ and $\mathcal{Y} = S^2$. We will only give the proof of Theorem 2.1, since Theorem 2.2 is obtained in a similar way, arguing e.g. as in [13, Thm. 3.2].

Remark 2.3 Arguing as in [11], Vol. II, Sec. 4.2.5, from Theorems 2.1 and 2.2 we obtain the sequential weak density of smooth maps in $W^{1,2}(B^n,\mathcal{Y})$ and in $W^{1,2}_\varphi(\widetilde{B}^n,\mathcal{Y})$; compare [18] for a more general result. Moreover, due to the lower semicontinuity of the Dirichlet energy w.r.t. the weak convergence in $\mathcal{D}_{n,2}$, we immediately obtain the characterization of the lower semicontinuous envelope of the Dirichlet integral of mappings into a manifold, already proved in [15].

Minimal connections. Before giving the proof of Theorem 2.1, we recall some further facts.

Definition 2.4 *For every* $n\geq 3$ *and* $\Gamma\in\mathcal{D}_{n-3}(\widetilde{B}^n)$ *with* $\operatorname{spt}\Gamma\subset\overline{B}^n$, *we denote by*

$$m_i(\Gamma):=\inf\{\mathbf{M}(L)\mid L\in\mathcal{R}_{n-2}(\widetilde{B}^n)\,,\quad \operatorname{spt}L\subset\overline{B}^n\,,\quad \partial L=\Gamma\}$$

the integral mass *of* Γ *and by*

$$m_r(\Gamma):=\inf\{\mathbf{M}(D)\mid D\in\mathcal{D}_{n-2}(\widetilde{B}^n)\,,\quad \operatorname{spt}D\subset\overline{B}^n\,,\quad \partial D=\Gamma\}$$

the real mass *of* Γ. *Moreover, in case* $m_i(\Gamma)<+\infty$, *we say that an i.m. rectifiable current* $L\in\mathcal{R}_{n-2}(\widetilde{B}^n)$ *is an* integral minimal connection *of* Γ *if* $\operatorname{spt}L\subset\overline{B}^n$, $\partial L=\Gamma$, *and* $\mathbf{M}(L)=m_i(\Gamma)$.

We recall that by Federer's theorem [6] if Γ has dimension zero we have $m_r(\Gamma)=m_i(\Gamma)$.

Density results for Sobolev maps. Let $R^\infty_{2,\varphi}(\widetilde{B}^n,\mathcal{Y})$ denote the set of all the maps $u\in W^{1,2}_\varphi(\widetilde{B}^n,\mathcal{Y})$ which are smooth except on a singular set $\Sigma(u)$ of the type

$$\Sigma(u)=\bigcup_{i=1}^r\Sigma_i\,,\qquad r\in\mathbb{N}\,, \tag{2.2}$$

where Σ_i is a smooth $(n-3)$-dimensional subset of B^n with smooth boundary, if $n\geq 4$, and Σ_i is a point if $n=3$. The following density result appears in [3], see also [16].

Theorem 2.5 *For every* $n\geq 3$ *the class* $R^\infty_{2,\varphi}(\widetilde{B}^n,\mathcal{Y})$ *is dense in* $W^{1,2}_\varphi(\widetilde{B}^n,\mathcal{Y})$.

By (2.1) every $T\in\operatorname{cart}^{2,1}_\varphi(\widetilde{B}^n\times\mathcal{Y})$ as in Theorem 2.1 decomposes as

$$T=G_{u_T}+\sum_{s=1}^{\widetilde{s}}\mathbb{L}_s(T)\times\gamma_s\qquad\text{on}\quad \mathcal{D}^{n,2}(\widetilde{B}^n\times\mathcal{Y})\,. \tag{2.3}$$

If $\{u_k\}\subset R^\infty_{2,\varphi}(\widetilde{B}^n,\mathcal{Y})$ is such that $u_k\to u_T$ in $W^{1,2}(\widetilde{B}^n,\mathbb{R}^N)$, it readily follows that

$$\lim_{k\to+\infty}\mathbf{M}(\mathbb{D}_s(u_T)-\mathbb{D}_s(u_k))=0\,,$$

compare [11], Vol. II, Sec. 4.2.5 and Sec. 5.4.2, so that by (1.6) we infer

$$\lim_{k\to+\infty}m_r(\mathbb{P}_s(u_T)-\mathbb{P}_s(u_k))=0\qquad\forall\, s=1,\dots,\widetilde{s}\,.$$

As a consequence, since $\mathbb{P}(u_k)$ belongs to $\mathcal{R}_{n-3}(\widetilde{B}^n;H^{sph}_2(\mathcal{Y}))$, it follows that $\mathbb{P}(u)$ is an $(n-3)$-dimensional *real flat chain*, being the real flat limit of the currents $\mathbb{P}(u_k)$, and

$$\mathbb{P}(u)(\phi)=\sum_{s=1}^{\widetilde{s}}\mathbb{P}_s(u)(\phi)\,[\gamma^s]\in H^{sph}_2(\mathcal{Y};\mathbb{R})\qquad\forall\,\phi\in\mathcal{D}^{n-3}(\widetilde{B}^n)\,, \tag{2.4}$$

where $H^{sph}_2(\mathcal{Y};\mathbb{R}):=H^{sph}_2(\mathcal{Y})\otimes\mathbb{R}$. Therefore, in case of dimension $n=3$, we obtain that $\mathbb{P}_s(u_T)$ is a zero dimensional *integral flat chain*, since by Federer's theorem [6]

$$\lim_{k\to+\infty}m_i(\mathbb{P}_s(u_T)-\mathbb{P}_s(u_k))=0\qquad\forall\, s=1,\dots,\widetilde{s}\,. \tag{2.5}$$

This is one of the crucial points of the proof in the case $n=3$, see [9], [13]. If $n\geq 4$, we argue in a different way, making use of arguments from [15], which go back to [18] and [21]. To this aim, we first recall the following result of Pakzad-Rivière [18].

Proposition 2.6 *Let $\mathcal{Y}$ be 1-connected and let $u \in R^\infty_{2,\varphi}(\widetilde{B}^n,\mathcal{Y})$. Then for every $s=1,\dots,\widetilde{s}$ there exists an integral current $L_s \in \mathcal{R}_{n-2}(\widetilde{B}^n)$, with $\operatorname{spt} L_s \subset \overline{B}^n$, such that*

$$\partial L_s = \mathbb{P}_s(u) \qquad \textit{and} \qquad \mathbf{M}(L_s) \le C \int_{B^n} |Du|^2\,dx$$

for some absolute constant $C>0$ independent of u.

In case $\mathcal{Y}=S^2$ this property goes back to [4] and is proved in [1] by means of the coarea formula. In [18] the result is given in terms of polyhedral chains with coefficients in the homotopy group $\pi_2(\mathcal{Y})$. However, since $\mathcal{Y}$ is 1-connected, by the Hurewicz theorem $\pi_2(\mathcal{Y}) \approx H_2(\mathcal{Y};\mathbb{R})$ and hence it can be re-stated in terms of currents in $\mathcal{D}_{n-2}(\widetilde{B}^n; H_2(\mathcal{Y};\mathbb{R}))$. Moreover, in [15] we proved the following local version, see [21] for the case $\mathcal{Y}=S^2$.

Proposition 2.7 *Let W be a relatively open subset of $\overline{B}^n$ such that $\mathcal{L}^n(\partial W)=0$. Let $u,v \in R^\infty_{2,\varphi}(\widetilde{B}^n,\mathcal{Y})$ be such that $u=v$ a.e. on $\overline{B}^n \setminus W$. For every $s=1,\dots,\widetilde{s}$ there exists an i.m. rectifiable current $L_s \in \mathcal{R}_{n-2}(B^n)$ with $\operatorname{spt} L_s \subset \overline{W}$ such that*

$$\partial L_s = \mathbb{P}_s(u) - \mathbb{P}_s(v) \qquad \textit{and} \qquad \mathbf{M}(L_s) \le C\,\big(\mathbf{D}(u,W) + \mathbf{D}(v,W)\big)\,.$$

We now begin with the proof of Theorem 2.1, that we divide in three steps.

Step 1: a strong density result. In the next section we will prove the following density result, which is the main new contribution of this paper. We denote by $\operatorname{sing} v$ the closure of the discontinuity set of a map v.

Theorem 2.8 *Under the hypotheses of Theorem 2.1, there exists a sequence $\{T_k\}$ in $\operatorname{cart}^{2,1}_\varphi(\widetilde{B}^n \times \mathcal{Y})$ weakly converging to T in $\mathcal{D}_{n,2}(\widetilde{B}^n \times \mathcal{Y})$, with energy $\mathbf{D}(T_k,\widetilde{B}^n\times\mathcal{Y}) \to \mathbf{D}(T,\widetilde{B}^n\times\mathcal{Y})$ as $k\to\infty$, such that every T_k decomposes as*

$$T_k = G_{u_k} + \sum_{s=1}^{\widetilde{s}} \mathbb{L}_s(T_k) \times \gamma_s \qquad \textit{on} \quad \mathcal{D}^{n,2}(\widetilde{B}^n \times \mathcal{Y})\,,$$

where every u_k is smooth outside a singular set of zero Lebesgue measure and

$$\mathcal{L}^n(\operatorname{sing} u_k) = 0\,.$$

Step 2: reduction to finite mass singularities. By Theorem 2.8, we may and will assume that T satisfies (2.3), where $u_T \in W^{1,2}_\varphi(\widetilde{B}^n,\mathcal{Y})$ is smooth outside a set of zero Lebesgue measure, $\mathcal{L}^n(\operatorname{sing} u_T)=0$.

Using the same argument as in [21], there exists a sequence of relatively open sets $W_k \subset \overline{B}^n$ such that $\mathcal{L}^n(\partial W_k)=0$, $\mathcal{L}^n(W_k)<1/k$ and

$$\operatorname{sing}(u_T) \subset \ldots \subset W_{k+1} \subset \overline{W}_{k+1} \subset W_k \subset \ldots \subset W_1\,.$$

Setting $V_k := B^n \setminus W_k$, then V_{k+1} is a neighborhood of V_k and u_T is smooth on V_{k+1}. Therefore, applying a refined version of Bethuel's density result, Theorem 2.5, compare [21, Thm. 4], we find the existence of a sequence $\{u_k\} \subset R^\infty_{2,\varphi}(\widetilde{B}^n,\mathcal{Y})$, strongly converging to u_T in $W^{1,2}(\widetilde{B}^n,\mathbb{R}^N)$, such that for every k

$$u_k = u_T \quad \text{on} \quad V_k \quad \text{and} \quad \int_{\widetilde{B}^n} (|u_k-u_T|^2 + |Du_k - Du_T|^2)\,dx < \frac{1}{k}\,.$$

By applying Proposition 2.7 with $u=u_k$, $v=u_{k+1}$ and $W=W_k$, for every s we find $L^{(k)}_s \in \mathcal{R}_{n-2}(B^n)$ with $\operatorname{spt} L^{(k)}_s \subset \overline{W}_k$ such that

$$\partial L^{(k)}_s = \mathbb{P}_s(u_k) - \mathbb{P}_s(u_{k+1}) \quad \text{and} \quad \mathbf{M}(L^{(k)}_s) \le C\,\big(\mathbf{D}(u_k,W_k) + \mathbf{D}(u_{k+1},W_k)\big)\,.$$

Since $\mathcal{L}^n(W_k)\to 0$ and $\{u_k\}$ strongly converges to u_T, possibly passing to a subsequence we may and will assume that $\mathbf{M}(L^{(k)}_s) \le 2^{-k}$ for every k and s. Setting then

$$L^s_{u_k,u_T} := -\sum_{j=k}^{+\infty} L^{(j)}_s\,, \qquad s=1,\dots,\widetilde{s}\,,$$

since $\mathbb{P}_s(u_k) \rightharpoonup \mathbb{P}_s(u_T)$, we have

$$\partial L^s_{u_k,u_T} = \mathbb{P}_s(u_T) - \mathbb{P}_s(u_k) \qquad \text{and} \qquad \lim_{k\to+\infty} \mathbf{M}(L^s_{u_k,u_T}) = 0\,,$$

so that (2.5) holds true and $\mathbb{P}_s(u_T)$ is an integral flat chain. Now, since T satisfies (2.3), we have

$$\mathbb{P}_s(u_T) = -\partial\,\mathbb{L}_s(T) \qquad \forall\, s = 1,\ldots,\widetilde{s}\,.$$

Therefore, setting

$$T_k := G_{u_k} + \sum_{s=1}^{\widetilde{s}} (L^s_{u_k,u_T} + \mathbb{L}_s(T)) \times \gamma_s\,,$$

we obtain that $\partial(L^s_{u_k,u_T} + \mathbb{L}_s(T)) = \mathbb{P}_s(u_k)$ and hence that $T_k \in \mathrm{cart}^{2,1}_\varphi(\widetilde{B}^n \times \mathcal{Y})$, with

$$\mathbf{M}\big(\partial(L^s_{u_k,u_T} + \mathbb{L}_s(T))\big) = \mathbf{M}(\mathbb{P}_s(u_k)) < +\infty$$

for every s and k, whereas $T_k \rightharpoonup T$ weakly in $\mathcal{D}_{n,2}(\widetilde{B}^n \times \mathcal{Y})$ and $\mathbf{D}(T_k) \to \mathbf{D}(T)$, by (1.12), as $k \to +\infty$.

Step 3: smooth approximating sequence. Since $\partial(L^s_{u_k,u_T} + \mathbb{L}_s(T)) = \mathbb{P}_s(u_k)$ has finite mass, hence is rectifiable, we can assume that $\mathbb{P}_s(u_T)$ is rectifiable for every $s = 1,\ldots,\widetilde{s}$. This yields that we may and do assume

$$T = G_{u_T} + \sum_{q\in H_2^{sph}(\mathcal{Y})} \mathbb{L}_q \times R_q \tag{2.6}$$

and by (1.11)

$$\mathbf{D}(T) = \mathbf{D}(u_T, \widetilde{B}^n) + \sum_{q\in H_2^{sph}(\mathcal{Y})} \mathbf{M}(\mathbb{L}_q)\cdot \mathbf{M}(R_q) < +\infty\,, \tag{2.7}$$

where the $\mathbb{L}_q$'s are i.m. rectifiable currents in $\mathcal{R}_{n-2}(B^n)$ with pairwise disjoint supports and finite boundary mass

$$\sum_{q\in H_2^{sph}(\mathcal{Y})} \mathbf{M}(\partial\,\mathbb{L}_q) < +\infty\,.$$

In particular, by the boundary rectifiability theorem [5], the $\partial\,\mathbb{L}_q$'s are i.m. rectifiable currents.

Following [11], see also [13], by applying Federer's strong polyhedral approximation theorem [5], we approximate T by a sequence of currents as in (2.6), where this time the $\mathbb{L}_q$'s are $(n-2)$-dimensional polyhedral chains. As a consequence, one reduces to approximate dipoles of the type $[\![\,\Delta\,]\!] \times R_q$, where $[\![\,\Delta\,]\!]$ is the current integration over an $(n-2)$-dimensional simplex of B^n. We omit writing the details of this final part, since it is an adaptation of similar arguments from [14]. We only sketch the main steps.

We first recall from [12, Prop. 4.5] how to approximate spherical type cycles.

Proposition 2.9 *Let $C \in \mathcal{Z}_2(\mathcal{Y})$ be an integral 2-cycle of spherical type and $P \in \mathcal{Y}$ be a given point. There exists a family of Lipschitz functions $f_\varepsilon : B^2 \to \mathcal{Y}$ such that $f_{\varepsilon|\partial B^2} \equiv P$, $f_{\varepsilon\#}[\![\,B^2\,]\!] \rightharpoonup C$ weakly in $\mathcal{D}_2(\mathcal{Y})$ as $\varepsilon \to 0$ and*

$$\mathbf{D}(f_\varepsilon, B^2) \le \mathbf{M}(C) + \varepsilon\,.$$

Finally, the 2-cycle $C_\varepsilon := f_{\varepsilon\#}[\![\,B^2\,]\!]$ does not depend on the choice of $P \in \mathcal{Y}$.

Let now Δ be e.g. given by the convex hull

$$\Delta := \mathrm{co}(\{0_{\mathbb{R}^n}, l\,e_1, l\,e_2, \ldots, l\,e_{n-2}\})\,, \qquad 0 < l \ll 1\,.$$

Also, for $\delta > 0$ and $0 < m \ll 1$, let

$$\phi^m_\delta(x) := (\widetilde{x}, \varphi^m_\delta(y(\widetilde{x}))\widehat{x})\,, \qquad \widetilde{x} := (x_1,\ldots,x_{n-2})\,, \quad \widehat{x} := (x_{n-1}, x_n)\,,$$

where $\varphi^m_\delta(y) := \min\{my, \delta\}$, $y \ge 0$, and $y(\widetilde{x}) := \mathrm{dist}(\widetilde{x}, \partial\Delta)$ is the distance of $\widetilde{x}$ from the boundary of the $(n-2)$-simplex Δ. Therefore, $\Omega^m_\delta := \phi^m_\delta(\Delta)$ is a small neighborhood of the simplex Δ in B^n, and, arguing in a way similar to [14], and making use of Proposition 2.9, we obtain the following

Proposition 2.10 *Let* $u : B^n \to \mathcal{Y}$ *be a* $W^{1,2}$ *map which is smooth in the interior of* $\Omega^{m_0}_{\delta_0}$, *for some fixed small* $m_0, \delta_0 > 0$. *Let* $C \in \mathcal{Z}_2(\mathcal{Y})$ *be an integral cycle of spherical type. For every* $\varepsilon > 0$, $0 < \delta < \delta_0$, *and* $0 < m < m_0$, *there exists a map* $u_\varepsilon : B^n \to \mathcal{Y}$ *such that* $G_{u_\varepsilon} \rightharpoonup G_u + [\![\Delta]\!] \times C$ *weakly in* $\mathcal{D}_{n,2}(B^n \times \mathcal{Y})$ *as* $\varepsilon \to 0^+$ *and*

$$\mathbf{D}(u_\varepsilon, B^n) \le \mathbf{D}(u, B^n) + \mathcal{H}^{n-2}(\Delta) \cdot \mathbf{M}(C) + \varepsilon\,.$$

Moreover, u_ε *is smooth in the closure of* Ω^m_δ, *except for the* $(n-3)$*-skeleton of a triangulation of* Δ, *and* $u_\varepsilon \equiv u$ *outside the closure of* Ω^m_δ.

Applying Proposition 2.10 to any $(n-2)$-simplex Δ of $\mathbb{L}_q$, with $C = R_q$, we find an approximating sequence of graphs of maps $\{u_\varepsilon\} \subset R^\infty_{2,\varphi}(\widetilde{B}^n, \mathcal{Y})$ which are smooth outside a singular set Σ_ε given by the $(n-3)$-skeleton of a triangulation of a polyhedral $(n-2)$-chain of B^n. To remove the singular set Σ_ε, we make use of a variant of a result from [14] that states that in these circumstances, for $\varepsilon > 0$ small enough there exists a sequence of smooth maps $\{u^{(\varepsilon)}_m\} \subset C^1_\varphi(\widetilde{B}^n, \mathcal{Y})$ which converges to u_ε strongly in $W^{1,2}$ as $m \to +\infty$. This is the point where, even in dimension $n = 3$, we make use of the injectivity hypothesis on the Hurewicz map $\pi_2(\mathcal{Y}) \to H_2(\mathcal{Y};\mathbb{Q})$, together with the condition

$$\partial_x G_{u_\varepsilon} = 0 \quad \text{on} \quad \mathcal{D}^{n-1,2}(\widetilde{B}^n \times \mathcal{Y})\,, \qquad \partial_y G_{u_\varepsilon} = 0 \quad \text{on} \quad \mathcal{Z}^{n-1,2}(\widetilde{B}^n \times \mathcal{Y})\,,$$

to remove the singular set Σ_ε, see [13].

3 A strong density result

In this section we prove Theorem 2.8, concluding the proof of Theorem 2.1.

Slicing properties. Denote by $B_r(x_0)$ the n-dimensional ball centered at x_0 and of radius r. Let $T \in \mathrm{cart}^{2,1}_\varphi(\widetilde{B}^n \times \mathcal{Y})$ be as in Theorem 2.8. For every point $x_0 \in B^n$ and for a.e. radius $r \in (0, r_0)$, where $r_0 := \mathrm{dist}(x_0, \partial B^n)$, the slice

$$\langle T, d_{x_0}, r\rangle\,,$$

where $d_{x_0}(x,y) := |x - x_0|$, is a Cartesian current in $\mathrm{cart}^{2,1}(\partial B_r(x_0) \times \mathcal{Y})$. Moreover, due to (1.2) we have

$$\partial(T \llcorner B_r(x_0) \times \mathcal{Y}) = \langle T, d_{x_0}, r\rangle\,, \tag{3.1}$$

where the boundary of T is to be intended in the sense of Sec. 1. In this case we will say that r is a *good radius* for T at x_0.
Proof: [Proof of Theorem 2.8] We divide it in six steps.

Step 1: definition of the fine cover $\mathcal{F}_m$. We define a suitable dense subset A of B^n and, for every $m \in \mathbb{N}$, a *fine cover* $\mathcal{F}_m$ of A consisting of closed balls with radii smaller than $1/m$. To this aim, in the sequel we will denote by μ_s the finite Radon measure on B^n given for every Borel set $A \subset B^n$ by

$$\mu_s(A) := \sum_{q \in H_2^{sph}(\mathcal{Y})} \mathbf{M}(R_q) \cdot \mathcal{H}^{n-2}(A \cap \mathcal{L}_q)\,, \tag{3.2}$$

the $\mathcal{L}_q$'s being the $(n-2)$-rectifiable sets such that $\mathbb{L}_q = \tau(\mathcal{L}_q, 1, \vec{\mathcal{L}})$, see (1.5). Since μ_s is a finite Radon measure concentrated in the $(n-2)$-rectifiable set $\mathcal{L}$,

$$\mathcal{L} := \bigcup_{q \in H_2^{sph}(\mathcal{Y})} \mathcal{L}_q\,,$$

by inner regularity, for every $m \in \mathbb{N}$ we find a closed subset $J_m \subset \mathcal{L}$ such that

$$\mu_s(\mathcal{L} \setminus J_m) < \frac{1}{m}\,. \tag{3.3}$$

Let A be the set of points x_0 in B^n such that

$$\lim_{r\to 0} \frac{\mu_s(B_r(x_0))}{r^{n-2}} = 0 .$$

We readily infer that A is a dense subset of B^n. Otherwise, the set $B^n \setminus A$ would contain a nonempty open set, a contradiction since $\mathcal{H}^{n-2+\varepsilon}(B^n \setminus A) = 0$ for all $\varepsilon > 0$. Moreover, since every point $x_0 \in A$ does not belong to $\mathcal{L}$, and J_m is a closed subset of $\mathcal{L}$, there exists a positive radius $r(x_0)$, smaller than the distance of x_0 to the boundary ∂B^n, such that for every $0 < r < r(x_0)$

$$\overline{B}_r(x_0) \cap J_m = \emptyset . \tag{3.4}$$

We then denote by $\mathcal{F}_m$ the union of all the closed balls centered at points $x_0 \in A$ and with good radii $0 < r < r(x_0)$.

Step 2: covering argument. We apply the following extension of the classical Vitali-Besicovitch covering theorem, with A and $\mathcal{F} = \mathcal{F}_m$ as in Step 1, with respect to the Lebesgue measure $\mathcal{L}^n$.

Theorem 3.1 *Let $A \subset B^n$ be a dense subset of B^n and let $\mathcal{F}$ be a fine cover of A made of closed balls contained in B^n. There is a disjoint countable family $\mathcal{G}$ of $\mathcal{F}$ such that*

$$\mathcal{L}^n\Big(B^n \setminus \bigcup \mathcal{G}\Big) = 0 .$$

Step 3: approximation on the balls of $\mathcal{G}_m$. In Step 2 we have obtained for every m a disjoint countable family $\mathcal{G}_m = \bigcup_{j=1}^{+\infty} B_j$ of closed balls $B_j \subset B^n$ with centers in A. Moreover, if $B_j = \overline{B}_r(x_0)$, then (3.4) and (3.1) hold true, being r a good radius for T at x_0.

Following an idea from [15], for any $j \in \mathbb{N}$ we let

$$\mathcal{T}_j := \{\widetilde{T} \in \text{cart}^{2,1}(\text{int}(B_j) \times \mathcal{Y}) \;:\; \partial \widetilde{T} = \langle T, d_{x_0}, r\rangle\}$$

denote the class of Cartesian currents in $\text{int}(B_j) \times \mathcal{Y}$ with boundary equal to the boundary of the restriction $T \llcorner \text{int}(B_j) \times \mathcal{Y}$, see (3.1). For any $\widetilde{T} \in \mathcal{T}_j$ let

$$\mathbf{D}(\widetilde{T}) := \mathbf{D}(\widetilde{T}, \text{int}(B_j) \times \mathcal{Y}) .$$

Moreover, we let $u_j := u_{T|B_j}$ denote the restriction to B_j of the function $u_T \in W^{1,2}_{\varphi}(\widetilde{B}^n, \mathcal{Y})$ corresponding to T.

For any $\varepsilon > 0$ we consider the minimum problem

$$\inf\{\, \mathbf{D}_\varepsilon(\widetilde{T}) \mid \widetilde{T} \in \mathcal{T}_j \} , \tag{3.5}$$

where

$$\mathbf{D}_\varepsilon(\widetilde{T}) := \mathbf{D}(\widetilde{T}) + \frac{1}{\varepsilon} \int_{B_j} |u_{\widetilde{T}} - u_j|^2 \, dx ,$$

$u_{\widetilde{T}} \in W^{1,2}(B_j, \mathcal{Y})$ being the $W^{1,2}$-function corresponding to $\widetilde{T}$. By the closure of the class $\mathcal{T}_j$ under the weak convergence in $\mathcal{D}_{n,2}(\text{int}(B_j) \times \mathcal{Y})$, and by the lower semicontinuity of $\widetilde{T} \mapsto \mathbf{D}_\varepsilon(\widetilde{T})$, we infer that (3.5) has a solution $T^j_\varepsilon \in \mathcal{T}_j$ such that $u_{T^j_\varepsilon} \to u_j$ in $L^2(B_j, \mathbb{R}^N)$ as $\varepsilon \to 0$. By [8] we may and do assume that the singular part $S_{T^j_\varepsilon, sing} = 0$ and that T^j_ε decomposes as

$$T^j_\varepsilon := G_{u^j_\varepsilon} + \sum_{s=1}^{\widetilde{s}} \mathbb{L}_s(T^j_\varepsilon) \times \gamma_s ,$$

where $\mathbb{L}_s(T^j_\varepsilon) \in \mathcal{R}_{n-2}(\text{int}(B_j))$. Possibly taking a sequence $\varepsilon_k \searrow 0$, we find that T^j_ε weakly converges in $\mathcal{D}_{n,2}(\text{int}(B_j) \times \mathcal{Y})$ to a current $T_j \in \mathcal{T}_j$. The T^j_ε's being minimizers, we also infer that T_j satisfies

$$T_j := G_{u_j} + \sum_{s=1}^{\widetilde{s}} \mathbb{L}_s(T_j) \times \gamma_s , \tag{3.6}$$

where $\mathbb{L}_s(T^j_\varepsilon) \in \mathcal{R}_{n-2}(\mathrm{int}(B_j))$. In particular, $u_{T_j} = u_j$. Moreover, since

$$\mathbf{D}(T^j_\varepsilon) \le \mathbf{D}_\varepsilon(T^j_\varepsilon) \le \mathbf{D}_\varepsilon(T_j) = \mathbf{D}(T_j)\,,$$

by the lower semicontinuity of the Dirichlet energy we have

$$\mathbf{D}(T_j) \le \liminf_{\varepsilon\to 0^+} \mathbf{D}(T^j_\varepsilon) \le \limsup_{\varepsilon\to 0^+} \mathbf{D}(T^j_\varepsilon) \le \mathbf{D}(T_j)\,,$$

so that $\mathbf{D}(T^j_\varepsilon) \to \mathbf{D}(T_j)$. On the other hand, if $\widetilde{T}$ is any Cartesian current in $\mathcal{T}_j$ such that $u_{\widetilde{T}} = u_j$, we clearly have

$$\mathbf{D}(T^j_\varepsilon) \le \mathbf{D}_\varepsilon(T^j_\varepsilon) \le \mathbf{D}_\varepsilon(\widetilde{T}) = \mathbf{D}(\widetilde{T})\,,$$

which yields $\mathbf{D}(T_j) \le \mathbf{D}(\widetilde{T})$, letting $\varepsilon \to 0^+$. As a consequence, taking $\widetilde{T} := T \llcorner \mathrm{int}(B_j) \times \mathcal{Y}$, we obtain that

$$\sum_{s=1}^{\widetilde{s}} \mathbf{M}(\mathbb{L}_s(T_j)) \cdot \mathbf{M}(\gamma_s) \le \sum_{s=1}^{\widetilde{s}} \mathbf{M}(\mathbb{L}_s(T \llcorner \mathrm{int}(B_j) \times \mathcal{Y})) \cdot \mathbf{M}(\gamma_s)\,. \tag{3.7}$$

Step 4: approximation on the whole of B^n. By paying a small amount of energy, we may and do modify the current T^j_ε in Step 3 in such a way that T^j_ε weakly converges in $\mathcal{D}_{n,2}(\mathrm{int}(B_j) \times \mathcal{Y})$ to the restriction $T \llcorner \mathrm{int}(B_j) \times \mathcal{Y}$.

In fact, we notice that by (1.7)

$$-\partial\, \mathbb{L}_s(T_j) \llcorner \mathrm{int}(B_j) = \mathbb{P}_s(u_j) = -\partial\, \mathbb{L}_s(T \llcorner \mathrm{int}(B_j) \times \mathcal{Y})\,,$$

whereas by the membership to $\mathcal{T}_j$

$$\partial T_j = \partial(T \llcorner \mathrm{int}(B_j) \times \mathcal{Y})\,.$$

Therefore, since $T^j_\varepsilon \rightharpoonup T_j$, letting

$$\widetilde{T}^j_\varepsilon := T^j_\varepsilon + \sum_{s=1}^{\widetilde{s}} \big(\mathbb{L}_s(T \llcorner \mathrm{int}(B_j) \times \mathcal{Y}) - \mathbb{L}_s(T_j)\big) \times \gamma_s\,,$$

we readily infer that $\widetilde{T}^j_\varepsilon$ belongs to $\mathcal{T}_j$ for every ε and that $\widetilde{T}^j_\varepsilon$ weakly converges to $T \llcorner \mathrm{int}(B_j) \times \mathcal{Y}$ in $\mathcal{D}_{n,2}(\mathrm{int}(B_j) \times \mathcal{Y})$ along a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$, possibly depending on j, with $\varepsilon_k \searrow 0$. Moreover, on account of (3.7), since by (3.2)

$$\sum_{s=1}^{\widetilde{s}} \mathbf{M}(\mathbb{L}_s(T \llcorner \mathrm{int}(B_j) \times \mathcal{Y})) \cdot \mathbf{M}(\gamma_s) = \mu_s(\mathrm{int}(B_j))\,,$$

whereas $\mathbf{D}(T^j_\varepsilon) \to \mathbf{D}(T_j)$ and $\mathbf{D}(T_j) \le \mathbf{D}(T, \mathrm{int}(B_j) \times \mathcal{Y})$, we also obtain that

$$\limsup_{\varepsilon\to 0^+} \mathbf{D}(\widetilde{T}^j_\varepsilon) \le \mathbf{D}(T, \mathrm{int}(B_j) \times \mathcal{Y}) + 2\,\mu_s(\mathrm{int}(B_j))\,. \tag{3.8}$$

We finally define for every k

$$T^m_k := T \llcorner (\widetilde{B}^n \setminus \bigcup \mathcal{G}^m) \times \mathcal{Y} + \sum_{j=1}^{+\infty} \widetilde{T}^j_{\varepsilon_k}\,.$$

Step 5: strong convergence. Due to the membership of $\widetilde{T}^j_{\varepsilon_k}$ to $\mathcal{T}_j$, we readily infer that T^m_k belongs to the class $\mathrm{cart}^{2,1}_\varphi(\widetilde{B}^n \times \mathcal{Y})$. The weak convergence of $\widetilde{T}^j_{\varepsilon_k}$ to $T \llcorner \mathrm{int}(B_j) \times \mathcal{Y}$ yields that T^m_k weakly converges to T in $\mathcal{D}_{n,2}(\widetilde{B}^n \times \mathcal{Y})$. Moreover, the energy estimate (3.8) yields that

$$\limsup_{k\to+\infty} \mathbf{D}(T^m_k, \widetilde{B}^n \times \mathcal{Y}) \le \mathbf{D}(T, \widetilde{B}^n \times \mathcal{Y}) + 2 \sum_{j=1}^{+\infty} \mu_s(\mathrm{int}(B_j))$$

and hence, by (3.3) and (3.4),

$$\limsup_{k\to+\infty} \mathbf{D}(T_k^m) \le \mathbf{D}(T) + \frac{2}{m} \qquad \forall\, m \in \mathbb{N}\,.$$

Step 6: conclusion. Now, the $W^{1,2}$-function u_k^m corresponding to T_k^m belongs to $W^{1,2}_\varphi(\widetilde{B}^n \times \mathcal{Y})$ and coincides with u_k^j on each ball B_j, where u_k^j denotes the function in $W^{1,2}(B_j, \mathcal{Y})$ corresponding to $T^j_{\varepsilon_k}$. Arguing similarly to when proving partial regularity results in [11, Vol. II, Sect. 4.2.9] or [8] to the minimum problem (3.5), with $\varepsilon = \varepsilon_k$, since $\int_{B_j} |u_{\widetilde{T}} - u_j|^2\, dx$ is a lower order term, it follows that the Sobolev maps u_k^j satisfy the condition

$$\mathcal{L}^n(\operatorname{sing} u_k^j) = 0\,,$$

where $\operatorname{sing} u_k^j$ denotes the closure of the the discontinuity set of u_k^j. As a consequence, the functions u_k^m are smooth outside the closed subset of $\widetilde{B}^n$ given by $B^n \setminus \cup\mathcal{G}_m$, which has zero full measure, plus the union of the singular sets $\operatorname{sing} u_k^j$. Definitively, $\operatorname{sing} u_k^m$ is a closed subset of B^n of null measure $\mathcal{L}^n(\operatorname{sing} u_k^m) = 0$. In conclusion, letting $m \to +\infty$, a diagonal argument yields the assertion. □

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