The Laplacian with Wentzell-Robin Boundary Conditions on Spaces of Continuous Functions*

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Dedicated to Jerry Goldstein on the occasion of his 60th birthday

Abstract

We investigate the Laplacian Δ on a smooth bounded open set $\Omega \subset \mathbf{R}^n$ with Wentzell-Robin boundary condition $\beta u + \frac{\partial u}{\partial \nu} + \Delta u = 0$ on the boundary Γ. Under the assumption $\beta \in C(\Gamma)$ with $\beta \geq 0$, we prove that Δ generates a differentiable positive contraction semigroup on $C(\bar{\Omega})$ and study some monotonicity properties and the asymptotic behaviour.

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Introduction.

The aim of this article is to show that the Laplacian Δ with Wentzell-Robin boundary condition

$$\beta u + \frac{\partial u}{\partial \nu} + \Delta u = 0 \text{ on } \Gamma \tag{1}$$

generates a positive contraction semigroup T on $C(\bar{\Omega})$. Here Ω is a bounded open subset of $\mathbf{R}^n$ with smooth boundary Γ and $0 \leq \beta \in C(\Gamma)$. Note that (1) is a dynamic boundary condition. In fact, let f be an element of $C(\bar{\Omega})$ and $u(t) = T(t)f$. Then $u'(t) = \Delta u(t)$. Introducing this in (1) we obtain

$$\frac{d}{dt}u(t) = -\beta u(t) - \frac{\partial}{\partial \nu}u(t) \qquad \text{on} \quad \Gamma.$$

We also establish monotonicity properties of this semigroup with respect to β. Also the asymptotic behaviour for $t \to \infty$ is studied. The boundary condition (1) was first studied in [9] in the space $C([0,1])$ and then in [10] in the space $L^p(\Omega) \oplus L^p(\Gamma)$ for $1 \leq p < \infty$ and in $C(\bar{\Omega})$ by a direct energy method and the Lumer-Phillips theorem. The semigroup is shown to be holomorphic on $L^p(\Omega) \oplus L^p(\Gamma)$ for $1 < p < \infty$ and also in $H^1(\Omega)$, as has been shown in a subsequent paper [11].

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Here we follow another path: the semigroup is constructed on $L^2(\Omega)\oplus L^2(\Gamma)$ by form methods and extended to $L^p(\Omega)\oplus L^p(\Gamma)$ by the Beurling-Deny criterion. Finally, using Schauder estimates, it is shown that $C(\bar{\Omega})\oplus C(\Gamma)$ is an invariant subspace, and this leads to a Feller semigroup on $C(\bar{\Omega})$, maybe the most natural space for such boundary conditions. The semigroup in $C(\overline{\Omega})$ is even more regular. In fact, K.J. Engel (see [7]) has proved very recently, using a completely different approach, that it is analytic.

The idea to incorporate boundary conditions into a product space goes back to Greiner [16] and has also been used by Amann-Escher [1] and in [2, Chapter 6]. Robin boundary conditions

$$\frac{\partial u}{\partial \nu} + \beta u = 0 \quad \text{on } \Gamma \tag{2}$$

have already been treated by form methods (see [3] and [5]), whereas for the Wentzell-Robin conditions (1) this seems to be new. Concerning generation theorems for elliptic operators (possibly degenerate) with pure Wentzell boundary conditions (i.e., $\Delta u_{|\Gamma} = 0$) in spaces of continous functions we refer to pioneer work of Feller [13] (in dimension one) and subsequent results by Clément-Timmermans [4], Goldstein-Lin [15] and Taira, Favini and Romanelli [24] among others. Concerning regularity properties and holomorphy of the generated semigroup in the case of pure Wentzell conditions see Vespri [25], Favini and Romanelli [12], Metafune [20], Engel-Nagel [8, Chapter VI Section 4], and most recently Warma [26] for Wentzell-Robin boundary conditions in $C([0,1])$.

1 Beurling-Deny criteria and ultracontractivity

In this section we recall some results on positive forms which we will use in the sequel refering essentially to Davies [6] for the proofs.

Let $(H, (\ |\ )_H)$ be a real Hilbert space. By a **positive form** on H we mean a bilinear mapping

$$Q : D(Q) \times D(Q) \to \mathbf{R}$$

such that

$$\begin{aligned} Q(u,v) &= Q(v,u) \quad \text{for all} \quad u,v \in D(Q)\ , \\ Q(u,u) &\geq 0 \quad \text{for all} \quad u \in D(Q)\ , \end{aligned}$$

where $D(Q)$ is a dense subspace of H, the **domain** of the form Q. We set

$$Q(u) = Q(u,u) \quad \text{for all} \quad u \in D(Q)\ .$$

The form Q is called **closed** if the space $D(Q)$ is complete for the norm

$$\|u\|_Q = (Q(u) + \|u\|_H^2)^{1/2}\ .$$

If Q is closed, then the operator A associated with Q is defined in the following way:

$$\begin{aligned} D(A) &= \{u \in D(Q) : \exists f \in H \quad \text{such that} \\ &\qquad Q(u,\varphi) = (f|\varphi)_H \quad \text{for all} \quad \varphi \in D(Q)\} \ , \\ Au &= f \ . \end{aligned}$$

The operator $-A$ is selfadjoint and generates a C_0-semigroup T on H satisfying $T(t) = T^*(t)$ and $\|T(t)\| \leq 1$ for all $t \geq 0$. We call T **the semigroup associated with the form** Q. Let us recall the following compactness criterion. The following are equivalent:

(i) $T(t)$ is compact for each $t > 0$;

(ii) the injection of $(D(Q), \| \ \|_Q)$ into H is compact;

(iii) the operator $(I + A)^{-1} \in \mathcal{L}(H)$ is compact.

We now suppose that $H = L^2(Y)$ where (Y, Σ, μ) is a σ-finite measure space. One says that $T = (T(t))_{t \geq 0}$ is a **symmetric Markov semigroup** if the following conditions are satisfied:

$$T(t) = T(t)^* \quad \text{for all} \quad t \geq 0 \ ; \tag{1.1}$$

$$T(t) \geq 0 \quad \text{for all} \quad t \geq 0 \tag{1.2}$$

$$\|T(t)f\|_\infty \leq \|f\|_\infty \quad \text{for all} \quad f \in L^2(Y) \cap L^\infty(Y) \text{ and all } t \geq 0 \ . \tag{1.3}$$

A **Dirichlet form** on $L^2(Y)$ is a closed positive form satisfying the following **two conditions of Beurling-Deny**

$$u \in D(Q) \quad \text{implies} \quad |u| \in D(Q) \text{ and } Q(|u|) \leq Q(u) \tag{1.4}$$

$$0 \leq u \in D(Q) \quad \text{implies} \quad u \wedge 1 \in D(Q) \text{ and } Q(u \wedge 1) \leq Q(u) \ . \tag{1.5}$$

Theorem 1.1 *([6, Theorem 1.3.3]). Let A be an operator on $L^2(Y)$. The following assertions are equivalent:*

(i) $-A$ generates a symmetric Markov semigroup;

(ii) A is associated with a Dirichlet form.

Next we recall a notion of ultracontractivity.

Theorem 1.2 *([6, Corollary 2.4.3]). Let Q be a Dirichlet form and $T = (T(t))_{t \geq 0}$ the associated semigroup. Let $\mu > 2$. The following assertions are equivalent:*

(i) $D(Q) \subset L^{2\mu/(\mu-2)}(Y)$;

(ii) there exists $c > 0$ such that

$$\|T(t)f\|_\infty \leq c t^{-\mu/4} \|f\|_2 \qquad (0 < t < 1)$$

for all $f \in L^2(Y)$.

If a $\mu > 2$ exists such that these equivalent conditions are satisfied, we call the semigroup T **ultracontractive**.

2 The semigroup on $L^2(\Omega) \oplus L^2(\Gamma)$

Let Ω be a bounded open subset of $\mathbf{R}^n$ with Lipschitz boundary $\Gamma = \partial\Omega$. We denote by

$$u \mapsto u_{|\Gamma}$$

the **trace** function, which is a bounded operator from the Sobolev space $H^1(\Omega)$ into $L^2(\Gamma, \sigma)$, where σ is the surface measure on Γ. To simplify the notation, we frequently write u instead of $u_{|\Gamma}$. Denote by $\Delta_{\max}$ the Laplacian in $L^2(\Omega)$ with maximal domain, i.e.,

$$\begin{aligned} D(\Delta_{\max}) &:= \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega)\} \\ \Delta_{\max} u &:= \Delta u \quad \text{(in the sense of distributions)}, \end{aligned}$$

and denote by $\nu(z)$ the exterior normal in $z \in \Gamma$. Let us introduce the notion of **weak normal derivative**.

Definition 2.1 *Let $u \in D(\Delta_{\max})$. We say that u has a* **weak normal derivative** *if there exists a function $b \in L^2(\Gamma)$ such that*

$$\int_\Omega \nabla u \nabla \varphi dx + \int_\Omega \Delta u \varphi dx = \int_\Gamma b \varphi d\sigma \tag{2.1}$$

for all $\varphi \in H^1(\Omega)$. In that case the function $b \in L^2(\Gamma)$ verifying (2.1) is unique and we denote it by $\frac{\partial u}{\partial \nu}$.

We now consider the space $H = L^2(\Omega) \oplus L^2(\Gamma)$. Note that H can be identified with a space $L^2(Y)$ for a suitable finite measure space (Y, Σ, μ) such that $L^\infty(Y)$ can be identified with $L^\infty(\Omega) \oplus L^\infty(\Gamma)$ with the norm

$$\|(u, b)\|_\infty := \max\{\|u\|_{L^\infty(\Omega)} \ , \ \|b\|_{L^\infty(\Gamma)}\}$$

for each $(u, b) \in L^\infty(\Omega) \oplus L^\infty(\Gamma)$. Let $\beta \in L^\infty(\Gamma)$ be such that $\beta(z) \geq 0$ for σ-a.a. $z \in \Gamma$ and define the operator A_β on H by

$$\begin{aligned} D(A_\beta) &:= \{(u, u_{|\Gamma}) : u \in D(\Delta_{\max}) \ , \ \frac{\partial u}{\partial \nu} \text{ exists in } L^2(\Gamma)\} \ , \\ A_\beta(u, u_{|\Gamma}) &:= (\Delta u, -\beta u_{|\Gamma} - \frac{\partial u}{\partial \nu}) \end{aligned}$$

Remark 2.2 It is possible to characterise the domain $D(A_\beta)$ in terms of fractional Sobolev spaces and traces. We have:

$$D(A_\beta) = \{u \in H^{3/2}(\Omega) : \ \Delta u \in L^2(\Omega)\}.$$

In fact, for every $u \in H^{3/2}(\Omega)$ with $\Delta u \in L^2(\Omega)$ a weak normal derivative exists, so one inclusion follows. Conversely, if u belongs to $D(A_\beta)$, setting $f = \Delta u$ and $b = \frac{\partial u}{\partial \nu}$, u is a variational solution of the boundary value problem $\Delta u = f$ in Ω,

$\frac{\partial u}{\partial \nu} = b$ on Γ. Moreover, there is a unique (up to constants) $v \in H^{3/2}(\Omega)$ solving the same problem, hence $u \in H^{3/2}(\Omega)$, as claimed.

If Γ is C^∞ these results are classical (see e.g. [19, Theorem 7.3, 7.4 p.186-7]), whereas if Γ is only Lipschitz continuous, the proof is much more delicate, and we refer to [17], [18].

Theorem 2.3 *The operator A_β generates a symmetric Markov semigroup on the space $L^2(\Omega) \oplus L^2(\Gamma)$.*

Proof. We define the positive form Q on H by

$$\begin{aligned} D(Q) &:= \{(u, u_{|\Gamma}) : u \in H^1(\Omega)\} \\ Q((u, u_{|\Gamma}), (v, v_{|\Gamma})) &:= \int_\Omega \nabla u \nabla v dx + \int_\Gamma uv\beta d\sigma \ . \end{aligned}$$

The proof is now given in several steps.

a) *$D(Q)$ is dense* in H. Let $b \in \mathcal{D}(\mathbf{R}^n)$ (i.e., b is a test function). Then there exists a sequence $(u_k)_{k\in\mathbf{N}}$ in $\mathcal{D}(\mathbf{R}^n)$ such that ${u_k}_{|\Gamma} = b$ and $u_k \to 0$ in $L^2(\Omega)$ as $k \to \infty$. Thus $(0, b) \in \overline{D(Q)}$. It follows that

$$\{0\} \oplus L^2(\Gamma) \subset \overline{D(Q)} \ .$$

Moreover,

$$(u, 0) = (u, u_{|\Gamma}) - (0, u_{|\Gamma}) \in \overline{D(Q)}$$

for all $u \in H^1(\Omega)$. Hence $L^2(\Omega) \oplus \{0\} \subset \overline{D(Q)}$.

b) *The form Q is closed.* Since the trace is a continuous operator from $H^1(\Omega)$ into $L^2(\Gamma)$, there exists a constant $c > 0$ such that

$$\|u_{|\Gamma}\|_{L^2(\Gamma)} \leq c\|u\|_{H^1(\Omega)}$$

for all $u \in H^1(\Omega)$. It follows that the form norm

$$\|(u, u_{|\Gamma})\|_Q = (Q(u, u_{|\Gamma}) + \|(u, u_{|\Gamma})\|_H^2)^{1/2}$$

is equivalent to the norm

$$\|(u, u_{|\Gamma})\| := \|u\|_{H^1(\Omega)} \ , \ u \in D(Q) \ .$$

Since $H^1(\Omega)$ is complete, also $D(Q)$ is complete.

c) *The first Beurling-Deny condition (1.4) is satisfied.* Let $u \in H^1(\Omega)$. Then $|u| \in H^1(\Omega)$ and $\nabla|u| = (\text{sign}u)\nabla u$ (see [14, § 7.6]). In particular, $\big|\nabla|u|\big|^2 = |\nabla u|^2$. Moreover, the trace of $|u|$ coincides with $|u_{|\Gamma}|$. Hence

$$Q(|u|, |u|_{|\Gamma}) = \int_\Omega |\nabla u|^2 dx + \int_\Gamma |u|^2\beta d\sigma = Q(u, u_{|\Gamma}) \ .$$

d) *The second Beurling-Deny condition (1.5) holds.* Let $0 \leq u \in H^1(\Omega)$. Then $u \wedge 1 \in H^1(\Omega)$ and $\nabla(u \wedge 1) = 1_{\{u<1\}}\nabla u$ (see [14, § 7.6]). Hence

$$(u, u_{|\Gamma}) \wedge (1_\Omega, 1_\Gamma) = (u \wedge 1_\Omega, (u \wedge 1_\Omega)_{|\Gamma}) \in D(Q)$$

and

$$\begin{aligned} Q((u,u_{|\Gamma})\wedge(1_\Omega,1_\Gamma)) &= \int_\Omega |\nabla(u\wedge 1_\Omega)|^2dx+\int_\Gamma (u\wedge 1_\Gamma)^2\beta d\sigma \\ &\le \int_\Omega |\nabla u|^2dx+\int_\Gamma |u|^2\beta d\sigma \\ &= Q(u,u_{|\Gamma})\ . \end{aligned}$$

Hence Q is a Dirichlet form.

e) $-A_\beta$ *is the operator associated with* Q. Denote by B the operator associated with Q. Let $(u,u_{|\Gamma})\in D(B)$, and let

$$B(u,u_{|\Gamma})=(f,b)\in L^2(\Omega)\oplus L^2(\Gamma)\ .$$

Then

$$\begin{aligned} \int_\Omega \nabla u\nabla\varphi dx+\int_\Gamma u\varphi\beta d\sigma &= Q((u,u_{|\Gamma}),(\varphi,\varphi_{|\Gamma})) \\ &= ((f,b)\mid(\varphi,\varphi_{|\Gamma}))_H \\ &= \int_\Omega f\varphi dx+\int_\Gamma b\varphi d\sigma\ , \end{aligned}$$

for all $\varphi\in H^1(\Omega)$. Choosing $\varphi\in\mathcal{D}(\Omega)$ we deduce that $f=-\Delta u$. Hence

$$\int_\Omega \nabla u\nabla\varphi dx+\int_\Omega \Delta u\varphi dx=\int_\Gamma (b-\beta u)\varphi d\sigma$$

for all $\varphi\in H^1(\Omega)$, i.e.,

$$\frac{\partial u}{\partial\nu}\text{ exists and }\frac{\partial u}{\partial\nu}=b-\beta u_{|\Gamma}\ .$$

Thus we have proved that

$$(u,u_{|\Gamma})\in D(A_\beta)\text{ and }A_\beta(u,u_{|\Gamma})=-B(u,u_{|\Gamma})\ .$$

In order to prove the converse, let $(u,u_{|\Gamma})\in D(A_\beta)$. Then

$$\begin{aligned} \int_\Omega \nabla u\nabla\varphi dx+\int_\Omega \Delta u\varphi dx &= \int_\Gamma \frac{\partial u}{\partial\nu}\varphi d\sigma \\ &= \int_\Gamma (b-\beta u_{|\Gamma})\varphi d\sigma \end{aligned}$$

where $b=\frac{\partial u}{\partial\nu}+\beta u_{|\Gamma}$ for all $\varphi\in H^1(\Omega)$. Hence

$$Q((u,u_{|\Gamma}),(\varphi,\varphi_{|\Gamma}))=-\int_\Omega \Delta u\varphi dx+\int_\Gamma b\varphi d\sigma$$

for all $\varphi\in H^1(\Omega)$. By the definition of the operator associated with the form Q we deduce that $(u,u_{|\Gamma})\in D(B)$ and

$$B(u,u_{|_\Gamma})=(-\Delta u,b)=-A_\beta(u,u_{|_\Gamma})\ .$$

□

Corollary 2.4 *Let $\lambda > 0$ and let $u \in D(\Delta_{\max})$ such that $\frac{\partial u}{\partial \nu}$ exists. Let*

$$\begin{aligned} f &= \lambda u - \Delta u \\ b &= \lambda u_{|\Gamma} + \beta u_{|\Gamma} + \frac{\partial u}{\partial \nu} \ . \end{aligned}$$

Then

$$\begin{aligned} \lambda \|(u, u_{|\Gamma})\|_{L^\infty(\Omega)\oplus L^\infty(\Gamma)} &\leq \|(f,b)\|_{L^\infty(\Omega)\oplus L^\infty(\Gamma)} \\ &= \max\{\|f\|_{L^\infty(\Omega)}, \|b\|_{L^\infty(\Gamma)}\} \ . \end{aligned}$$

Proof. It suffices to observe that

$$\|\lambda(\lambda - A_\beta)^{-1}\| \leq 1$$

where the norm is considered in the space of all linear operators on $L^\infty(\Omega) \oplus L^\infty(\Gamma)$. □

Remark 2.5 The C_0-semigroup given by the previous theorem extends to a positive contraction C_0-semigroup on $L^p(\Omega) \oplus L^p(\Gamma)$ for $1 \leq p < \infty$ which is holomorphic for $1 < p < \infty$. This follows directly from [5, Theorem 1.4.1 and 1.4.2] and can be also obtained as a special case of [10, Theorem 3.1].

Next we show that the C_0-semigroup $(T_\beta(t))_{t\geq 0}$ generated by A_β is ultracontractive.

Proposition 2.6 *Let $n \geq 3$. Then there exists a constant $c > 0$ such that*

$$\|T_\beta(t)(f,b)\|_\infty \leq c t^{-\frac{n-1}{2}} \|(f,b)\|_2 \quad (0 < t < 1)$$

for all $(f,b) \in L^2(\Omega) \oplus L^2(\Gamma)$. If $n \leq 2$, then for all $\mu > \frac{1}{2}$ there exists $c > 0$ such that

$$\|T_\beta(t)(f,b)\|_\infty \leq c t^{-\mu} \|(f,b)\|_2 \qquad (0 < t < 1)$$

for all $(f,b) \in L^2(\Omega) \oplus L^2(\Gamma)$.

Proof. By the embedding theorem [22, Chapter 2, Theorem 4.2], the trace operator $u \mapsto u_{|\Gamma}$ is continuous from $H^1(\Omega)$ to $L^q(\Gamma)$ where $q = \frac{2n-2}{n-2}$ for $n > 2$ and where $2 \leq q < \infty$ is arbitrary if $n = 2$. On the other hand, one has the following inclusions:

$$H^1(\Omega) \subset L^{\frac{2n}{n-2}}(\Omega) \quad \text{if } n > 2$$

and

$$H^1(\Omega) \subset L^q(\Omega) \text{ for } 2 \leq q < \infty \text{ arbitrary if } n = 2 \ .$$

Hence

$$D(Q) \subset L^q(\Omega) \oplus L^q(\Gamma) \text{ for } q = \frac{2n-2}{n-2} \text{ if } n > 2 \ .$$

Hence, letting $\mu = 2n-2$, one has $q = \frac{2\mu}{\mu-2}$ and the claim follows from Theorem 1.2 if $n > 2$. If $n \leq 2$, then $D(Q) \subset L^q(\Omega) \oplus L^q(\Gamma)$ for all $2 \leq q < \infty$, and the claim follows from Theorem 1.2 again. □

Corollary 2.7 *For all $t > 0$ the operator $T_\beta(t)$ is compact and the resolvent of A_β is compact.*

Proof. From Proposition 2.6 it follows that the operator $T_\beta(t)$ is Hilbert-Schmidt for all $t > 0$. □

Next we investigate how the semigroups depend on β. We denote by A_∞ the operator on $L^2(\Omega) \oplus L^2(\Gamma)$ with domain $D(A_\infty)$ given by

$$\begin{aligned} D(A_\infty) &:= \{(u,0) : u \in H_0^1(\Omega), \Delta u \in L^2(\Omega)\} \\ A_\infty(u,0) &:= (\Delta u, 0) \ . \end{aligned}$$

Then A_∞ generates a semigroup $T_\infty = (T_\infty(t))_{t\geq 0}$ on $L^2(\Omega) \oplus L^2(\Gamma)$ given by

$$T_\infty(t)(f,b) = (e^{t\Delta^D} f, b) \ ,$$

where Δ^D is the Dirichlet Laplacian on $L^2(\Omega)$.

Proposition 2.8 *Let $\beta_1, \beta_2 \in L^\infty(\Gamma)$ such that $0 \leq \beta_1 \leq \beta_2$. Then*

$$T_\infty(t) \leq T_{\beta_2}(t) \leq T_{\beta_1}(t) \leq T_0(t)$$

in the sense of positive semigroups, where $T_0 = (T_0(t))_{t\geq 0}$ denotes the semigroup T_β for $\beta = 0$.

Proof. Let Q_β be the form associated with $0 \leq \beta \in L^\infty(\Gamma)$. Then $D(Q_\beta)$ is independent of β. Moreover,

$$Q_0((u, u_{|\Gamma}), (v, v_{|\Gamma})) \leq Q_{\beta_1}((u, u_{|\Gamma}), (v, v_{|\Gamma})) \leq Q_{\beta_2}((u, u_{|\Gamma}), (v, v_{|\Gamma}))$$

if $u, v \geq 0$. From this, the second and third inequality in the statement follow from the domination criterion [23, Theorem 3.7] of Ouhabaz. Moreover, the form domain $D(Q_0) = \{(u,0) : u \in H_0^1(\Omega)\}$ is an ideal in $D(Q_{\beta_1})$ and the two forms Q_0 and Q_{β_1} coincide on $D(Q_0)$. Hence also the first inequality follows from Ouhabaz' criterion. □

Remark 2.9 (Semigroup on $H^1(\Omega)$) The operator B_β on $H^1(\Omega)$ given by $D(B_\beta) = \{u \in H^1(\Omega) : \Delta u \in H^1(\Omega), \frac{\partial u}{\partial \nu}$ exists in $L^2(\Gamma), (\Delta u)_{|\Gamma} + \beta u_{|\Gamma} + \frac{\partial u}{\partial \nu} = 0\}, B_\beta u = \Delta u$ generates a holomorphic C_0-semigroup on $H^1(\Omega)$. This follows directly from the proof of Theorem 2.3. In fact, the part $\tilde{B}_\beta$ of A_β in $D(Q)$ generates a holomorphic C_0-semigroup. This is just a property of forms which can easily be seen from the spectral theorem (cf. [2, § 7.1]). Now the mapping $u \in H^1(\Omega) \mapsto (u, u_{|\Gamma}) \in D(Q)$ is an isomorphism. With this identification, the operator $\tilde{B}_\beta$ induces the operator B_β on $H^1(\Omega)$. This result is valid on bounded open sets with Lipschitz boundary. For another approach on smooth domains allowing also degenerate elliptic operators we refer to [11].

3 The semigroup in the space $C(\bar{\Omega})$

In order to use Schauder estimates, we suppose in the sequel that Ω is a bounded open set in $\mathbf{R}^n$ of class $C^{2,\alpha}$ where $0 < \alpha < 1$.

We first consider the space $C(\bar{\Omega}) \oplus C(\Gamma)$ with the norm

$$\|(f,b)\|_\infty := \max\{\|f\|_{L^\infty(\Omega)}, \|b\|_{L^\infty(\Gamma)}\} .$$

Define the operator B_1 on the space $C(\bar{\Omega}) \oplus C(\Gamma)$ by

$$\begin{aligned} B_1(u, u_{|\Gamma}) &:= (\Delta u, -u_{|\Gamma} - \frac{\partial u}{\partial \nu}) \\ D(B_1) &:= \{(u, u_{|\Gamma}) : u \in C(\bar{\Omega}) \cap H^1(\Omega), \Delta u \in C(\bar{\Omega}), \frac{\partial u}{\partial \nu} \text{ exists in } C(\Gamma)\} \end{aligned}$$

Proposition 3.1 *The operator B_1 is m-dissipative and resolvent positive.*

Proof. At first we recall that the operator A_1 (i.e., A_β with $\beta \equiv 1$) is defined on $L^2(\Omega) \oplus L^2(\Gamma)$ by

$$A_1(u, u_{|\Gamma}) = (\Delta u, -u_{|\Gamma} - \frac{\partial u}{\partial \nu})$$

with domain

$$D(A_1) = \{(u, u_{|\Gamma}) : u \in D(\Delta_{\max}) \text{ such that } \frac{\partial u}{\partial \nu} \text{ exists}\} .$$

Observe that B_1 is the part of A_1 in $C(\bar{\Omega}) \oplus C(\Gamma)$; i.e.,

$$D(B_1) = \{w \in D(A_1) \cap (C(\bar{\Omega}) \oplus C(\Gamma)) : A_1 w \in C(\bar{\Omega}) \oplus C(\Gamma)\}$$

and

$$B_1 w = A_1 w, \; w \in D(B_1) .$$

Hence B_1 is dissipative by Corollary 2.4. Since A_1 is closed in $L^2(\Omega) \oplus L^2(\Gamma)$, also B_1 is closed in $C(\bar{\Omega}) \oplus C(\Gamma)$. Let $f \in C^\alpha(\bar{\Omega})$ and $b \in C^{1,\alpha}(\Gamma)$. By [14, Theorem 6.31] there exists $u \in C^{2,\alpha}(\bar{\Omega})$ such that

$$\Delta u = f \text{ and } -u_{|\Gamma} - \frac{\partial u}{\partial \nu} = b .$$

Hence $(u, u_{|\Gamma}) \in D(B_1)$ and $B_1(u, u_{|\Gamma}) = (f, b)$. By the Stone-Weierstrass Theorem, the space $C^\alpha(\bar{\Omega}) \oplus C^{1,\alpha}(\Gamma)$ is dense in $C(\bar{\Omega}) \oplus C(\Gamma)$, and then B_1 is m-dissipative. Since B_1 is the part of A_1 in the space $C(\bar{\Omega}) \oplus C(\Gamma)$, the resolvent $R(\lambda, B_1)$ of B_1 in $\lambda > 0$ is the restriction of $R(\lambda, A_1)$ to $C(\bar{\Omega}) \oplus C(\Gamma)$. Since the latter operator is positive, the same is true for $R(\lambda, B_1)$. □

Now we consider a perturbation of B_1. Let $0 \le \beta \in C(\Gamma)$ and let B_β the operator on the space $C(\bar{\Omega}) \oplus C(\Gamma)$ defined in the following way,

$$\begin{aligned} B_\beta(u, u_{|\Gamma}) &:= (\Delta u, -\frac{\partial u}{\partial \nu} - \beta u) \\ D(B_\beta) &= D(B_1) \ . \end{aligned}$$

Recall that B is called a Hille-Yosida operator if there exist $\omega \in \mathbf{R}, M \ge 0$ such that $(\omega, \infty) \subset \varrho(B)$ and

$$\|(\lambda - \omega)^{n+1} R(\lambda, B)^n\| \ \le M$$

for all $\lambda > \omega,\ n \in \mathbf{N},\ n \ge 1$ (see [2, § 3.5.]). We now show the following.

Proposition 3.2 *The operator B_β is a Hille-Yosida operator on $C(\bar{\Omega}) \oplus C(\Gamma)$ which is resolvent positive.*

Proof. Consider the bounded operator C on $L^2(\Omega) \oplus L^2(\Gamma)$ given by $C(f, b) = (0, (-\beta + 1)b)$ and its restriction C_0 to $C(\bar{\Omega}) \oplus C(\Gamma)$. Then by [2, Theorem 3.5.5] it follows that $A_1 + C$ and $B_1 + C_0$ are both Hille-Yosida operators. The semigroup $(e^{tC})_{t \ge 0}$ generated by C on $L^2(\Omega) \oplus L^2(\Gamma)$ is positive (in fact, $e^{tC}(f, b) = (f, e^{t(-\beta+1)}b))$. Hence $A_1 + C$ is resolvent positive. Consequently, also its part $B_1 + C_0$ in $C(\bar{\Omega}) \oplus C(\Gamma)$ is resolvent positive. □

Note that the operator B_β is not the generator of a C_0-semigroup since its domain is not dense. But its part in the closure of its domain generates a C_0-semigroup. This observation will finally lead to the principal result of the article.

Let $0 \le \beta \in C(\Gamma)$. Define the Laplacian with Wentzell-Robin boundary conditions on $C(\bar{\Omega})$ as the operator G_β given by

$$\begin{aligned} G_\beta u &:= \Delta u \\ D(G_\beta) &:= \{u \in C(\bar{\Omega}) \cap H^1(\Omega) : \Delta u \in C(\bar{\Omega}), \\ &\quad \frac{\partial u}{\partial \nu} \text{ exists in } C(\Gamma) \text{ and} \\ &\quad (\Delta u)_{|\Gamma} + \frac{\partial u}{\partial \nu} + \beta u_{|\Gamma} = 0\} \ . \end{aligned}$$

Theorem 3.3 *The operator G_β generates a compact, positive C_0-semigroup S_β on $C(\bar{\Omega})$.*

Proof. Consider the closed subspace F of $C(\bar{\Omega}) \oplus C(\Gamma)$ given by

$$F := \{(u, u_{|\Gamma}) : u \in C(\bar{\Omega})\} \ ,$$

which we will identify with $C(\bar{\Omega})$ in the sequel. Observe the following properties:
a) F **is the closure of** $D(B_\beta)$ **in** $C(\bar{\Omega}) \oplus C(\Gamma)$.
In fact, the domain $D(B_\beta)$ is contained in F and contains the set $\{(u, u_{|\Gamma}) : u \in$

$C^\infty(\bar\Omega)\}$ which is dense in F by the Stone-Weierstrass theorem.
b) By [2, Lemma 3.3.12] the part $\tilde G_\beta$ of B_β in F generates a C_0-semigroup $\tilde S_\beta$. This semigroup is positive since B_β is resolvent positive. Identifying F and $C(\bar\Omega)$ the operator $\tilde G_\beta$ becomes G_β. Thus G_β generates the C_0-semigroup S_β which can be identified with $\tilde S_\beta$.
c) **The semigroup S_β is compact**.

It is sufficient to prove that $\tilde S_\beta(t)$ is compact for $t > 0$. Recall that $\tilde S_\beta(t)$ is the restriction of $T_\beta(t)$ to F. This follows from the exponential formula $\tilde S_\beta(t) = \lim_{n\to\infty}(I - \frac{t}{n}\tilde G_\beta)^{-n}$ strongly. Recall that the operator $T_\beta(t)$ is compact. Since the semigroup T_β is ultracontractive, one has

$$T_\beta(t)(L^2(\Omega)\oplus L^2(\Gamma)) \subset L^\infty(\Omega)\oplus L^\infty(\Gamma)\ .$$

Factorising $T_\beta(2t)_{|L^\infty(\Omega)\oplus L^\infty(\Gamma)}$ as

$$L^\infty(\Omega)\oplus L^\infty(\Gamma) \hookrightarrow L^2(\Omega)\oplus L^2(\Gamma) \overset{T_\beta(t)}{\longrightarrow} L^2(\Omega)\oplus L^2(\Gamma) \overset{T_\beta(t)}{\longrightarrow} L^\infty(\Omega)\oplus L^\infty(\Gamma)$$

we deduce that $T_\beta(2t)_{|L^\infty(\Omega)\oplus L^\infty(\Gamma)}$ is compact. Hence also the restriction $S_\beta(2t)$ to F is compact. □

As a consequence of the previous results, we show that the semigroup is differentiable in $C(\bar\Omega)$.

Corollary 3.4 *The semigroup $(S_\beta(t))_{t\geq 0}$ is differentiable on $C(\bar\Omega)$.*

Proof. Let us show that for every $t > 0$ the operator $G_\beta S_\beta(t)$ is bounded on $C(\bar\Omega)$. First, recall that $(S_\beta(t))_{t\geq 0}$ concides in $C(\bar\Omega)$ with the semigroup $(T_\beta(t))_{t\geq 0}$ acting on $L^2(\Omega)\oplus L^2(\Gamma)$, which is holomorphic and ultracontractive. Therefore, we may write

$$G_\beta S_\beta(t) = G_\beta S_\beta(t/2)S_\beta(t/2) = S_\beta(t/2)G_\beta S_\beta(t/2) = T_\beta(t/2)A_\beta T_\beta(t/2).$$

But $A_\beta T_\beta(t/2)$ is a bounded operator from $L^2(\Omega)\oplus L^2(\Gamma)$ (hence, from $C(\bar\Omega)$) in $L^2(\Omega)\oplus L^2(\Gamma)$ and $T_\beta(t/2)$ is continuous from $L^2(\Omega)\oplus L^2(\Gamma)$ in $C(\bar\Omega)$ and the thesis follows. □

We next treat a monotonicity property. Denote by G_∞ the Dirichlet Laplacian on $C(\bar\Omega)$, i.e.,

$$\begin{aligned} D(G_\infty) &:= \{u \in C(\bar\Omega) : u_{|\Gamma} = 0,\ \Delta u \in C(\bar\Omega)\} \\ G_\infty u &:= \Delta u\ . \end{aligned}$$

Then G_∞ generates a positive holomorphic semigroup S_∞ on $C(\bar\Omega)$ such that $\|S_\infty(t)\| \leq 1$ for $t > 0$, which is **not** strongly continuous in 0 (see [2, Example 3.7.8, p. 156]).

Theorem 3.5 *Let $\beta_1, \beta_2 \in C(\Gamma)$ such that $0 \leq \beta_1 \leq \beta_2$. Then*

$$S_\infty(t) \leq S_{\beta_2}(t) \leq S_{\beta_1}(t) \leq S_0(t) \quad (t \geq 0)\ .$$

Proof. We identify $C(\bar{\Omega})$ with the subspace $\{(u, u_{|\Gamma}) : u \in C(\bar{\Omega})\}$ of $L^2(\Omega) \oplus L^2(\Gamma)$, so that the semigroups $S_\infty, S_{\beta_2}, S_{\beta_1}, S_0$ are restrictions of the semigroups $T_\infty, T_{\beta_2}, T_{\beta_1}$ and T_0 considered in Proposition 2.8. Thus the corresponding generators are obtained as parts of the corresponding generators on $L^2(\Omega) \oplus L^2(\Gamma)$ and the theorem is a consequence of Proposition 2.8. □

Finally, we consider the asymptotic behaviour of $(S_\beta(t))_{t\geq 0}$ as $t \to \infty$. If $\beta \geq 0$, then the operator G_β is dissipative and hence $\|S_\beta(t)\| \leq 1$ for $t \geq 0$.

If $\beta \equiv 0$, then $1_{\bar{\Omega}} \in D(G_0)$ and $G_0 1_{\bar{\Omega}} = 0$. Hence $S_\beta(t)1_{\bar{\Omega}} = 1_{\bar{\Omega}}$ for all $t \geq 0$ and the norm $\|S_\beta(t)\|$ does not converge to 0 as $t \to \infty$. But this case is exceptional. In fact, the following result holds:

Theorem 3.6 *Suppose that* Ω *is connected and* $0 \leq \beta \in C(\Gamma), \beta \not\equiv 0$. *Then there exist* $\varepsilon > 0, M \geq 0$ *such that*

$$\|S_\beta(t)\| \leq M e^{-\varepsilon t} \quad (t \geq 0) \ .$$

Proof. It suffices to consider the C_0-semigroup $\tilde{S}_\beta$ on $F := \{(u, u_{|\Gamma}) : u \in C(\bar{\Omega})\} \subset C(\bar{\Omega}) \oplus C(\Gamma)$, with generator $\tilde{G}_\beta$ (as in the proof of Theorem 3.3). Since $\tilde{S}_\beta(t)$ is compact, the resolvent of $\tilde{G}_\beta$ is compact (see [21, A-II Theorem 1.25]). We show that $\tilde{G}_\beta$ is injective. In order to do so, recall that $\tilde{G}_\beta$ is the part of A_β defined on $L^2(\Omega) \oplus L^2(\Gamma)$. Let $(u, u_{|\Gamma}) \in D(A_\beta)$ such that $A_\beta u = 0$, then

$$0 = Q(u, u_{|\Gamma}) = \int_\Omega |\nabla u|^2 dx + \int_\Gamma \beta |u|^2 d\sigma \ .$$

Hence, since $\nabla u = 0$ and since Ω is connected, u is constant on Ω, and since $\beta \not\equiv 0$, it follows that $u \equiv 0$. Thus $\tilde{G}_\beta$ is injective and hence invertible. Observing that $\tilde{S}_\beta$ is bounded, it follows that

$$\sigma(\tilde{G}_\beta) \cap i\mathbf{R} \subset \{0\} \ .$$

Moreover, the spectrum of $\tilde{G}_\beta$ is either finite or a sequence going to infinity. Thus, since $\tilde{S}_\beta$ is norm continuous, we deduce that the set

$$\{\lambda \in \sigma(\tilde{G}_\beta) : \mathrm{Re}\ \lambda \geq -1\}$$

is bounded (see [21, A-II Theorem 1.20]). This implies that the spectral bound $s(\tilde{G}_\beta)$ is negative. Applying [8, Theorem 1.10. p. 302] the result follows. □

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