Constrained BV functions on covering spaces for minimal networks and Plateau's type problems*

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Abstract

We link covering spaces with the theory of functions of bounded variation, in order to study minimal networks in the plane and Plateau's problem without fixing a priori the topology of solutions. We solve the minimization problem in the class of (possibly vector-valued) BV functions defined on a covering space of the complement of an $(n-2)$-dimensional compact embedded Lipschitz manifold S without boundary. This approach has several similarities with Brakke's "soap films" covering construction. The main novelty of our method stands in the presence of a suitable constraint on the fibers, which couples together the covering sheets. In the case of networks, the constraint is defined using a suitable subset of transpositions of m elements, m being the number of points of S. The model avoids all issues concerning the presence of the boundary S, which is automatically attained. The constraint is lifted in a natural way to Sobolev spaces, allowing also an approach based on Γ-convergence.

Introduction

In its earliest and simplest formulation, Plateau's problem consists in finding a surface Σ in the ambient space $\mathbb{R}^3$, spanning a fixed reference smooth loop S, and minimizing the area. As it is well known, several models have been proposed to solve the mathematical questions related to this problem (and to its generalizations in $\mathbb{R}^n$, for $n \geq 2$), depending on the definition of surface, boundary, and area: parametric and non parametric solutions, homology classes, integer rectifiable currents, varifolds, just to name a few. General references are for instance [?], [22], [24], [21], [9]; we refer the reader to [7] for a brief overview on the Plateau's problem. In connection with what we are going to discuss, we also mention the recent paper [8], where the authors, extending in a different setting some results of [12], look for a solution of Plateau's problem, minimizing the $(n-1)-$dimensional Hausdorff measure in the class of relatively closed subsets of $\mathbb{R}^n \setminus S$, with nonempty intersection with every loop having unoriented linking number with S equal to 1.

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In this paper we link the coverings with the theory of (possibly vector-valued) functions of bounded variation and Γ-convergence, in order to solve the problem of minimal networks in the plane, and to find an embedded solution to Plateau's problem, without fixing a priori the topology of solutions. This work has several similarities with the "soap films" covering space model, set up in [5] by Brakke as a new original approach to Plateau's problem in codimension one.

Our model mathematically reproduces the physical structure of an interface separating two (or more) phases. In this respect, for instance in case of two phases, it is useful to merge Plateau's problem in an n-dimensional ($n = 3$ being the physical case) manifold, which is a *covering space* of the open set

$$M := \Omega \setminus S,$$

where $\Omega \subset \mathbb{R}^n$ is usually a bounded connected Lipschitz open set containing the $(n-2)$-dimensional compact embedded Lipschitz manifold S withouth boundary. In the model of [5], mathematically reproducing the physical structure of soap films made of two layers with a thin liquid region between them, how to choose the covering is part of the model construction, and it can lead to different solutions. Then, one has to select some connected components of a *pair covering space* of M in order to pair the sheets and to set up the minimization problem in terms of a suitable notion of currents mass. Again, the choice of the pair covering space is part of the model construction.

In this work, we approach the problem *without making use of pair covering spaces*, which can be considered as a first simplification of the model. Typical situations that we shall consider are:

- $n = 2$, $S \subset \mathbb{R}^2$ a set of m-distinct points, and an m-sheeted covering space of M; the case $m = 3$ is already interesting, and related to the Steiner graphs (when $m \geq 3$, taking a two-sheeted covering space does not lead to any interesting conclusion). See Figures 1, 3 and 6.

- $n = 3$, $S \subset \mathbb{R}^3$ a link, and a two-sheeted covering space, see Figure 2. This leads to the Plateau's problem.

Our explicit construction of the covering, denoted by $(Y_{\mathbf{\Sigma}}, \pi_{\mathbf{\Sigma},M})$, requires a suitable pair of cuts $\mathbf{\Sigma} = (\Sigma, \Sigma')$, where Σ and Σ' are $(n-1)$-dimensional compact Lipschitz manifolds (not necessarily connected), having S as topological boundary (Definitions 1.2 and 1.3). The construction is made by "cut and paste", with the use of local parametrizations, which suggest the natural way to endow $Y_{\mathbf{\Sigma}}$ with the Euclidean metric. The metric aspects here play a role; as it will be clear from the discussion, we cannot confine ourselves to a purely topological construction of the covering (see Remark 1.4).

Let $m \geq 2$ be the number of sheets of $Y_{\mathbf{\Sigma}}$, and let $V \subset \mathbb{R}^{m-1}$ be the set of vertices of a regular simplex. Our idea is to minimize the total variation $|Du|(Y_{\mathbf{\Sigma}})$ among all BV functions $u\colon Y_{\mathbf{\Sigma}} \to V$, satisfying the following *constraint on the fibers*: for $j = 1, \dots, m$, denote by $v_j(u)$ the restriction of u to the j-th sheet of the covering (Definition 1.8); then we require that

$$v_j(u) = \tau^{j-1} \circ v_1(u), \qquad j = 1, \dots, m, \tag{0.1}$$

for a transposition τ of V of order m and independent of j. Roughly speaking, condition (0.1) means that u "behaves" the same way on each covering sheet, the only difference consisting in a fixed transposition of the elements of V having order m. For instance, if $y \in Y_{\mathbf{\Sigma}}$ is a jump point of u, then u has to jump at all points of the same fiber of y. Figure 4 shows an example where $m = 3$ and condition (0.1) is violated.

When $m = 2$ and $V = \{\pm 1\}$, condition (0.1) is equivalent to require

$$\sum_{\pi_{\mathbf{\Sigma},M}(y)=x} u(y) = 0, \quad \text{for a.e. } x \in M, \tag{0.2}$$

so that u takes opposite values on (the two) points of the same fiber. To have an idea of the geometric meaning of the total variation we are considering, it is useful to look at the elementary Example 1.10, which refers to the case $m = 3$. The usefulness of constraint (0.1) stands in the possibility of study the minimization problem handling with standard BV-functions defined on open subsets of $\mathbb{R}^n$. We also remark that (0.1) forces the boundary datum S to be attained (Corollary 2.17); this represents a difference with the approach of [5], where it may happen that the boundary S is not fully covered by a solution. See also Figure 5 for an explicit example (in dimension $n = 2$) where the two methods lead to different solutions.

What we call a *constrained covering solution with boundary* S will be (Definition 2.18) the projection *via* $\pi_{\mathbf{\Sigma},M}$ of the jump set of a minimizer. Existence of minimizers is proved in Theorem 2.15.

In Theorem 2.7 we prove that the constrained covering solutions are independent of $\mathbf{\Sigma}$.[1] In some sense, this is due to the fact that, working on the covering space, all information about the exact location of the cuts becomes irrelevant, since changing the cuts corresponds just to an isometry on the covering space. Once more, the constraint (0.1) plays a crucial role; for instance, it forces the minimum value to be strictly positive (Lemma 2.16). Perhaps, the most remarkable among its consequences is that all issues about the definition of "boundary" on S are avoided. It seemed to us not immediate to derive the constraint on the fibers from the approach of [5].

This paper is divided as follows: in Section 1 we define the family of admissible cuts, and the space $BV(Y_{\mathbf{\Sigma}}; V)$. For $u \in BV(Y_{\mathbf{\Sigma}}; V)$, $j = 1, \ldots, m$, and $j' = m+1, \ldots, 2m$, the functions $v_j(u)$, $v_{j'}(u)$ are defined in (1.10), and coincide with u read in the various charts of the atlas used to parametrize $Y_{\mathbf{\Sigma}}$. Then, for any admissible pair of cuts $\mathbf{\Sigma}$, the minimization problem is set up in Section 2, on the space

$$BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V) := \Big\{ u \in BV(Y_{\mathbf{\Sigma}}; V) \ : \ u \text{ satisfies } (0.1) \Big\}.$$

Regularity of constrained covering solutions is based on the well−established regularity theory for isoperimetric sets and minimizing clusters (see for instance [17] and references therein). Then, in Section 3, we lift the constraint on the fibers to the class of Sobolev functions on $Y_{\mathbf{\Sigma}}$, showing in Proposition 3.1 that our formulation naturally leads to a Γ-convergence result. In Section 4.1 we exploit the case $n = 2$, namely when S consists of $m \geq 2$ distinct points, and we show that a constrained covering solution coincides with the Steiner graph over S. In Section 4.2 we test the

[1] See [5, Proposition 12.1] for a similar result.

model in the case of the standard Plateau's problem in $\mathbb{R}^3$: in Theorem 4.3 we show that, at least when $2 < n < 8$, our model is equivalent to solving Plateau's problem using the theory of integral currents modulo 2 [10]. Finally, in the Appendix we present a standard abstract construction of a covering space of M, which we show to be isometric with $Y_{\mathbf{\Sigma}}$. The construction is performed avoiding the definition of admissible cuts.

We expect that our model could be generalized in a nontrivial way in various directions; in particular, to more general choices of S (for instance, taking as S the set of all 1-dimensional edges of a polyhedron). In this spirit, we briefly discuss in Section 4.3 the case when S is the one-skeleton of a tetrahedron ($n = 3$ and $m = 4$), and, by adapting an argument in [4], we give a regularity result[2] (Proposition 4.9) in the sense of Almgren's $(\mathbf{M}, 0, r)$-minimal sets [1], [26].

1 BV functions on coverings

Notation. Let $n \geq 2$. We denote by $\mathcal{H}^{n-1}$ the Euclidean $(n-1)-$dimensional Hausdorff measure in $\mathbb{R}^n$. We let $|\cdot|$ be the Euclidean norm. For any x, $x' \in \mathbb{R}^n$, we denote by $x \cdot x'$ the scalar product between x and x'. We also let $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$; for $r > 0$ and $x \in \mathbb{R}^n$, we set $B_r(x) := \{y \in \mathbb{R}^n : |y - x| < r\}$ and $B_r := B_r(0)$. For any $F \subseteq \mathbb{R}^n$, we denote by $\overline{F}$ the closure of F in $\mathbb{R}^n$.

Throughout this paper, $\Omega \subseteq \mathbb{R}^n$ denotes a nonempty connected open set. Unless otherwise specified, we let $S \subset \Omega$ be a boundaryless, compact, embedded, smooth submanifold of dimension $n - 2$, not necessarily connected nor oriented.

We define the base set as

$$M := \Omega \setminus S, \tag{1.1}$$

which is path connected.

Example 1.1. Typical choices will be:

- $n = 2$, and S a finite number m of distinct points;
- $n = 3$, and S a tame link (that is, a finite number of disjoint closed embedded smooth space curves).

In this paper we shall perform a different covering construction depending on the dimension n. Indeed, apart from the construction in Section 4.3, our covering space will consist of $m := m(n)$ sheets, where

$$m := \begin{cases} \text{cardinality of } S & \text{if } n = 2, \\ 2 & \text{if } n > 2. \end{cases} \tag{1.2}$$

The family of admissible cuts is defined distinguishing between the following two alternatives. We shall say that an $(n-1)$-dimensional submanifold $\Sigma \subset \Omega$ is Lipschitz provided that, locally around any of its points, Σ is the graph of a Lipschitz function defined on a suitable $(n-1)$-orthonormal frame.

[2] See also [5, Theorem 10.2] for a similar result.

Definition 1.2 (**Admissible cuts,** $n = 2$)**.** *Let* $n = 2$*, and* $S := \{p_1, \ldots, p_m\}$*. We denote by* $\mathrm{Cuts}(\Omega, S)$ *the set of all* $\Sigma := \cup_{i=1}^{m-1} \Sigma_i \subset \Omega$ *where:*

- *for* $i = 1, \ldots, m-1$*,* Σ_i *is a Lipschitz simple curve, starting at* p_i *and ending at* p_{i+1}*;*
- *if* $m > 2$*, then* $\Sigma_i \cap \Sigma_{i+1} = \{p_{i+1}\}$ *for* $i = 1, \ldots, m-2$*;*
- $\Sigma_i \cap \Sigma_l = \emptyset$ *for any* $i = 1, \ldots, m-2$*, and* $l = 1, \ldots, m-1$ *such that* $l \neq i, i+1$*.*

We also denote by $\mathbf{Cuts}(\Omega, S)$ *the set of all pairs* $\boldsymbol{\Sigma} := (\Sigma, \Sigma')$ *such that:*

(i) $\Sigma, \Sigma' \in \mathrm{Cuts}(\Omega, S)$*, and* $\Sigma \cap \Sigma' = S$*;*

(ii) for $m > 2$*, and for any* $i = 2, \ldots, m-1$*, let* C_i *be a sufficiently small disk centered at* p_i*, and denote by* x_i *(resp.* y_i*) the intersection of* C_i *with* Σ_{i-1} *(resp. with* Σ_{i+1}*). Then, there exists an arc of* C_i *connecting* x_i *and* y_i*, and not intersecting* $\Sigma' = \cup_{j=1}^{m-1} \Sigma'_j$*.*

Roughly speaking, condition (*ii*) in Definition 1.2 means that Σ lies from one side of Σ' locally around S.

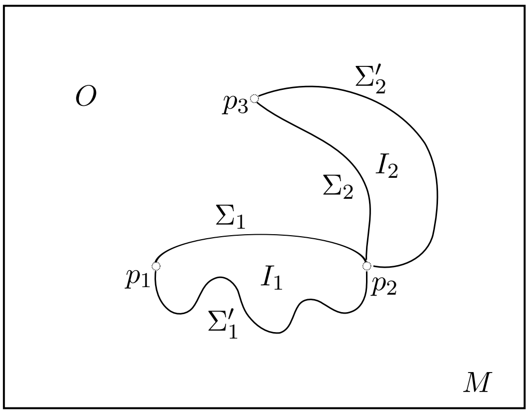

Figure 1: The base set $M = \Omega \backslash S$, when $n = 2$, $m = 3$, Ω is a rectangle, and $S = \{p_1, p_2, p_3\}$. In the figure, an example of admissible pair of cuts is shown.

Definition 1.3 (**Admissible cuts,** $n > 2$)**.** *Let* $n > 2$*. We denote by* $\mathrm{Cuts}(\Omega, S)$ *the set of all* $(n-1)$*-dimensional compact embedded Lipschitz submanifolds* $\Sigma \subset \Omega$ *having* S *as topological boundary.*

We also let $\mathbf{Cuts}(\Omega, S)$ *be the set of all pairs* $\boldsymbol{\Sigma} := (\Sigma, \Sigma')$ *such that* $\Sigma, \Sigma' \in \mathrm{Cuts}(\Omega, S)$*, and* $\Sigma \cap \Sigma' = S$*.*

Referring to Definitions 1.2-1.3, we call the elements of $\mathrm{Cuts}(\Omega, S)$ (resp. of $\mathbf{Cuts}(\Omega, S)$) *admissible cuts* (resp. *admissible pairs of cuts*). When $n > 2$, we shall always suppose that $\mathrm{Cuts}(\Omega, S)$ and $\mathbf{Cuts}(\Omega, S)$ are nonempty. A typical situation is when S is the topological boundary of some $(n-1)$–dimensional, compact, embedded, orientable, smooth submanifold $\Sigma \subset \Omega$.[3]

[3] Indeed, the orientability of Σ gives a unit normal vector field on $\Sigma \setminus S$ — hence, in particular,

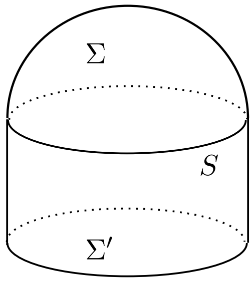

Figure 2: An example of admissible pair of cuts in the case $S \subset \mathbb{R}^3$ is a circle. In the figure, Σ is a closed half-sphere, while Σ' is a portion of cylinder over S, with the addiction of the lower base.

Remark 1.4. An m-sheeted covering of M can be constructed in a standard way [16, p.147] using a single orientable cut $\Sigma \in \mathrm{Cuts}(\Omega, S)$, by suitably identifying m copies of $\Omega \setminus \Sigma$. This construction is perhaps more intuitive than the one based on (1.5) and corresponds, essentially, to the case in which Σ and Σ' coincide. However, in order to rigorously define the covering, one needs to slightly separate the "faces" of Σ. Since our minimization problem (see (2.2) below) depends on the metric on the covering space, we find more convenient to use the construction via admissible pairs of cuts. However, it is worth noticing that, concretely, it will be enough to deal with only one of the two cuts of the pair $\mathbf{\Sigma}$.

1.1 "Cut and paste" construction of the covering

In this section we explicitly construct the covering $(Y_{\mathbf{\Sigma}}, \pi_{\mathbf{\Sigma},M})$ quoted in the introduction. As a consequence, we shall end up with local parametrizations which naturally bring the Euclidean metric on $Y_{\mathbf{\Sigma}}$.

Let $n \geq 2$ and m be as in (1.2). Let $\mathbf{\Sigma} = (\Sigma, \Sigma') \in \mathbf{Cuts}(\Omega, S)$. We consider m disjoint copies of the open sets

$$D := \Omega \setminus \Sigma, \quad D' := \Omega \setminus \Sigma', \tag{1.3}$$

which we denote respectively by

$$(D, j), \quad j = 1, \dots, m, \qquad (D', j'), \quad j' = m+1, \dots, 2m. \tag{1.4}$$

Points in the space

$$\mathcal{X} := \bigcup_{j=1}^{m} (D, j) \cup \bigcup_{j'=m+1}^{2m} (D', j')$$

are identified as follows. For $i = 1, \dots, m-1$, let I_i be the bounded open set enclosed by Σ_i and Σ_i'; set also $O := \Omega \setminus \cup_{i=1}^{m-1} \overline{I_i}$. Let $x, x' \in M$, $j \in \{1, \dots, m\}$,

a direction to follow in order to "enlarge" the cut, separating its two faces. The construction is standard (in the case $n = 3$, it is given for instance in [16, p.147]). Necessary and sufficient conditions for the existence of this $(n-1)$-dimensional orientable submanifold can be found in [27]. When $n = 3$, and S is a tame link, there exists [23, Theorem 4, p.120] an embedded orientable surface, called *Seifert surface*, whose boundary is S.

and $j' \in \{m+1, \dots, 2m\}$; then $(x,j) \sim (x',j')$ if and only if $x = x'$, and one of the following conditions holds:

$$\begin{cases} j \equiv j' \pmod{m}, & x = x' \in O, \\ j \equiv j' - i \pmod{m}, & x = x' \in I_i, \quad i = 1, \dots, m-1. \end{cases} \tag{1.5}$$

Of course, any point is also identified with itself. See Figures 1-3 for an example in the case $n = 2$, $m = 3$.

Then $\sim$ is an equivalence relation, and the quotient space[4]

$$Y_{\boldsymbol{\Sigma}} := \mathcal{X}/\sim$$

is endowed with the quotient topology given by the projection $\widetilde{\pi} : \mathcal{X} \to Y_{\boldsymbol{\Sigma}}$ induced by $\sim$. We set $\pi \colon (x,j) \in \mathcal{X} \mapsto x \in M$, and we denote by

$$\pi_{\boldsymbol{\Sigma},M} \colon Y_{\boldsymbol{\Sigma}} \to M \tag{1.6}$$

the projection $\pi_{\boldsymbol{\Sigma},M}(\widetilde{\pi}(x,j)) := x$, for any $(x,j) \in \mathcal{X}$. This latter map is well defined, since if $(x,j) \sim (x',j')$, then $\pi_{\boldsymbol{\Sigma},M}(\widetilde{\pi}(x,j)) = x = x' = \pi_{\boldsymbol{\Sigma},M}(\widetilde{\pi}(x',j'))$. Therefore, we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\widetilde{\pi}"] \arrow[dr, "\pi"'] & Y_{\boldsymbol{\Sigma}} \arrow[d, "\pi_{\boldsymbol{\Sigma},M}"] \\ & M \end{tikzcd} \tag{1.7}$$

Definition 1.5 (**Local parametrizations**). *We set*

$$\begin{aligned} &\Psi_j \colon D \to \widetilde{\pi}\big((D,j)\big), && \Psi_j := \widetilde{\pi} \circ \big(\pi_{|_{(D,j)}}\big)^{-1}, \quad j = 1, \dots, m, \\ &\Psi_{j'} \colon D' \to \widetilde{\pi}\big((D',j')\big), && \Psi_{j'} := \widetilde{\pi} \circ \big(\pi_{|_{(D',j')}}\big)^{-1}, \quad j' = m+1, \dots, 2m. \end{aligned} \tag{1.8}$$

The covering space[5] $Y_{\boldsymbol{\Sigma}}$ admits a natural structure of differentiable manifold, with $2m$ local parametrizations Ψ_j, $\Psi_{j'}$ given by (1.8).

Remark 1.6. For $j \in \{1, \dots, m\}$ and $j' \in \{m+1, \dots, 2m\}$, we have

$$\Psi_{j'}^{-1} \circ \Psi_j = \mathrm{id} = \Psi_j^{-1} \circ \Psi_{j'} \quad \text{on } D \cap D',$$

where id is the identity map on $D \cap D'$. The pair $(Y_{\boldsymbol{\Sigma}}, \pi_{\boldsymbol{\Sigma},M})$ is an m-sheeted covering of M. Notice that $\cup_{j=1}^m \Psi_j(D) = Y_{\boldsymbol{\Sigma}} \setminus \pi_{\boldsymbol{\Sigma},M}^{-1}(\Sigma \setminus S)$.

Remark 1.7 (**Non-zero thickness wires**). Our covering construction applies without modifications to the (simpler) case of a base domain $M := \Omega \setminus \overline{C}$, where $C \subset \Omega$ is a thin open neighborhood of S.

[4] $Y_{\boldsymbol{\Sigma}}$ depends on the choice of Ω; for notational simplicity we shall not indicate such a dependence.
[5] Since S has been removed, $Y_{\boldsymbol{\Sigma}}$ is not branched.

1.2 Total variation on the m-sheeted covering

The covering space $Y_{\mathbf{\Sigma}}$ is an n-dimensional connected orientable smooth non complete manifold; it is endowed with a natural volume measure μ, which is the push-forward $\mathcal{L}^N_{\#}$ of the n-dimensional Lebesgue measure $\mathcal{L}^n$ in M via the maps (1.8). More specifically, let $E \subseteq Y_{\mathbf{\Sigma}}$ be a Borel set. Then we can write E as the union of the following $2m$ disjoint Borel sets[6]

$$E \cap \widetilde{\pi}((D, j)),\ j = 1, \dots, m, \qquad E \cap \widetilde{\pi}((\Sigma \setminus S, j')),\ j' = m+1, \dots, 2m, \tag{1.9}$$

and we set

$$\mu(E) := \sum_{j=1}^{m} \Psi_{j\#}\mathcal{L}^n(E \cap \widetilde{\pi}((D, j))) = \sum_{j=1}^{m} \mathcal{L}^n(\pi_{\mathbf{\Sigma},M}(E \cap \widetilde{\pi}((D, j)))).$$

For $k \in \mathbb{N}$, $k \geq 1$, we set $L^1(Y_{\mathbf{\Sigma}}; \mathbb{R}^k) := L^1_\mu(Y_{\mathbf{\Sigma}}; \mathbb{R}^k)$ and $L^1_{\mathrm{loc}}(Y_{\mathbf{\Sigma}}; \mathbb{R}^k) := L^1_{\mu_{\mathrm{loc}}}(Y_{\mathbf{\Sigma}}; \mathbb{R}^k)$. The relevant case in this paper will be

$$k := m - 1,$$

where we recall that m is defined in (1.2).

Definition 1.8 (The functions $v_h(u)$). *Let $u : Y_{\mathbf{\Sigma}} \to \mathbb{R}^k$. For $j = 1, \dots, m$ and $j' = m+1, \dots, 2m$, we let $v_j(u) : D \to \mathbb{R}^k$, $v_{j'}(u) : D' \to \mathbb{R}^k$ be the maps defined by*

$$v_j(u) := u \circ \Psi_j, \qquad v_{j'}(u) := u \circ \Psi_{j'}. \tag{1.10}$$

Clearly, if $u \in L^1(Y_{\mathbf{\Sigma}}; \mathbb{R}^k)$ then $v_j(u) \in L^1(D; \mathbb{R}^k)$, $v_{j'}(u) \in L^1(D'; \mathbb{R}^k)$.
By construction (recall (1.5)), we have

$$v_j(u) = v_{j'}(u) \ \text{ in } O \quad \text{if } j \equiv j' \ (\mathrm{mod}\ m), \tag{1.11}$$

$$v_j(u) = v_{j'}(u) \ \text{ in } I_i \quad \text{if } j \equiv j' - i \ (\mathrm{mod}\ m), \quad i = 1, \dots, m-1. \tag{1.12}$$

Let Ω be bounded. Our aim is to define the total variation of a function $u \in L^1(Y_{\mathbf{\Sigma}}; \mathbb{R}^k)$. We say that u is in $BV_\mu(Y_{\mathbf{\Sigma}}; \mathbb{R}^k) =: BV(Y_{\mathbf{\Sigma}}; \mathbb{R}^k)$ if its distributional gradient[7] $Du \colon \eta \in (C^1_c(Y_{\mathbf{\Sigma}}))^k \mapsto -\int_{Y_{\mathbf{\Sigma}}} \sum_{l=1}^k u_l \nabla \eta_l \, d\mu \in \mathbb{R}^n$ is a bounded $(k \times n)$-matrix of Radon measures on $Y_{\mathbf{\Sigma}}$. Let us denote by $|Du|$ the *total variation* measure of Du [2]; we recall [2, Proposition 1.47] that, for any open subset $E \subseteq Y_{\mathbf{\Sigma}}$, we have

$$|Du|(E) = \sup \left\{ \sum_{l=1}^{k} \int_E u_l \ \mathrm{div}\eta_l \, d\mu \ : \ \eta \in \left(C^1_c(E; \mathbb{R}^n)\right)^k, \ ||\eta||_\infty \leq 1 \right\}, \tag{1.13}$$

which is $L^1(Y_{\mathbf{\Sigma}})$ − lower semicontinuous.

[6]Notice that Σ' does not appear in (1.9). Choosing D' in place of D amounts in considering Σ' in place of Σ and does not change the subsequent discussion.

[7]Let $\phi \in C^1_c(Y_{\mathbf{\Sigma}})$. For $i = 1, \dots, n$, let e_i be the i − th element of the canonical basis of $\mathbb{R}^n$. Then $\nabla_i \phi(y) := \lim_{h \to 0} h^{-1}\big(\phi(\Psi_1(\pi_{\mathbf{\Sigma},M}(y) + he_i)) - \phi(y)\big)$ is well − defined for every $y \in \widetilde{\pi}((D, 1))$. Similarly for other points in $Y_{\mathbf{\Sigma}}$. We set $\nabla\phi := (\nabla_1\phi, \dots, \nabla_n\phi)$. For $\Phi := (\phi_1, \dots, \phi_n) \in C^1_c(Y_{\mathbf{\Sigma}}; \mathbb{R}^n)$, we set $\mathrm{div}\Phi := \sum_{i=1}^n \nabla_i \phi_i$.

Remark 1.9 (Representation of the total variation, I). Let $u \in BV(Y_{\boldsymbol{\Sigma}};\mathbb{R}^k)$ and $E \subseteq Y_{\boldsymbol{\Sigma}}$ be a Borel set. Then

$$\begin{aligned}|Du|(E) =&\sum_{j=1}^{m} |Dv_j(u)|\Big(\pi_{\boldsymbol{\Sigma},M}\big(E\cap\widetilde{\pi}((D,j))\big)\Big)\\ &+\sum_{j'=m+1}^{2m}|Dv_{j'}(u)|\Big(\pi_{\boldsymbol{\Sigma},M}\big(E\cap\widetilde{\pi}((\Sigma\setminus S,j'))\big)\Big).\end{aligned} \tag{1.14}$$

In order to prove (1.14), let us first assume $E \subseteq \widetilde{\pi}((D,1))$ is open. Then, recalling (1.13), we have

$$\begin{aligned}|Du|(E) &= \sup\left\{\sum_{l=1}^{k}\int_{\Psi_1^{-1}(E)}(v_1(u))_l \ \mathrm{div}\eta_l\, d\mathcal{L}^n \ : \ \eta\in\left(C_c^1(\Psi_1^{-1}(E);\mathbb{R}^n)\right)^k,\ ||\eta||_\infty\le 1\right\}\\ &= |Dv_1(u)|(\Psi_1^{-1}(E)) = |Dv_1(u)|(\pi_{\boldsymbol{\Sigma},M}(E)),\end{aligned} \tag{1.15}$$

which gives (1.14). From (1.15) and [2, Proposition 1.43], we get (1.14) for every Borel set $E \subseteq Y_{\boldsymbol{\Sigma}}$ contained in a single chart. The general case follows by the splitting in (1.9).

Example 1.10. Let $n = 2$, $m = 3$, $S = \{p_1, p_2, p_3\}$, $\Sigma = \Sigma_1 \cup \Sigma_2$ and $\Sigma' = \Sigma_1' \cup \Sigma_2'$ be as in Figure 1. For $j = 1, 2, 3$, fix $\alpha_j, \beta_j \in \mathbb{R}^2$, and let $u \in BV(Y_{\boldsymbol{\Sigma}};\mathbb{R}^2)$ be such that, for every $j = 1, 2, 3$, $v_j(u)$ is equal to α_j inside a disk $B \subset M$ of radius $r > 0$ compactly contained in O (or in I_1, or in I_2) and β_j outside. Then, from (1.14), it follows

$$\begin{aligned}|Du|(Y_{\boldsymbol{\Sigma}}) =&\sum_{j=1}^{3}|Dv_j(u)|(B\cap D)+\sum_{j'=4}^{6}|Dv_{j'}(u)|(\Sigma\setminus S)\\ =&2\pi r\sum_{j=1}^{3}|\beta_j-\alpha_j| + \mathcal{H}^1(\Sigma)\sum_{\substack{j,l=1\\ j<l}}^{3}|\beta_l-\beta_j|.\end{aligned} \tag{1.16}$$

On the other hand, if B is centered at a point of $\Sigma \setminus S$, and $B \cap \Sigma' = \emptyset$, then

$$\begin{aligned}|Du|(Y_{\boldsymbol{\Sigma}}) =&2\pi r\sum_{j=1}^{3}|\beta_j-\alpha_j| + \mathcal{H}^1(\Sigma\cap B)\sum_{\substack{j,l=1\\ j<l}}^{3}|\alpha_l-\alpha_j|\\ &+\big(\mathcal{H}^1(\Sigma)-\mathcal{H}^1(\Sigma\cap B)\big)\sum_{\substack{j,l=1\\ j<l}}^{3}|\beta_l-\beta_j|.\end{aligned} \tag{1.17}$$

In particular, if $\alpha_1, \alpha_2, \alpha_3$ are the vertices of an equilateral triangle in $\mathbb{R}^2$ having side of length ℓ, and if $\beta_1 := \alpha_2$, $\beta_2 := \alpha_3$, $\beta_3 := \alpha_1$,[8] both (1.16) and (1.17) reduce to

$$3\ell\Big(2\pi r + \mathcal{H}^1(\Sigma)\Big).$$

[8]With this choice, and letting $V = \{\alpha_1, \alpha_2, \alpha_3\}$, u belongs to $BV_{\mathrm{constr}}(Y_{\boldsymbol{\Sigma}}; V)$, see Definition 2.1) in the next section.

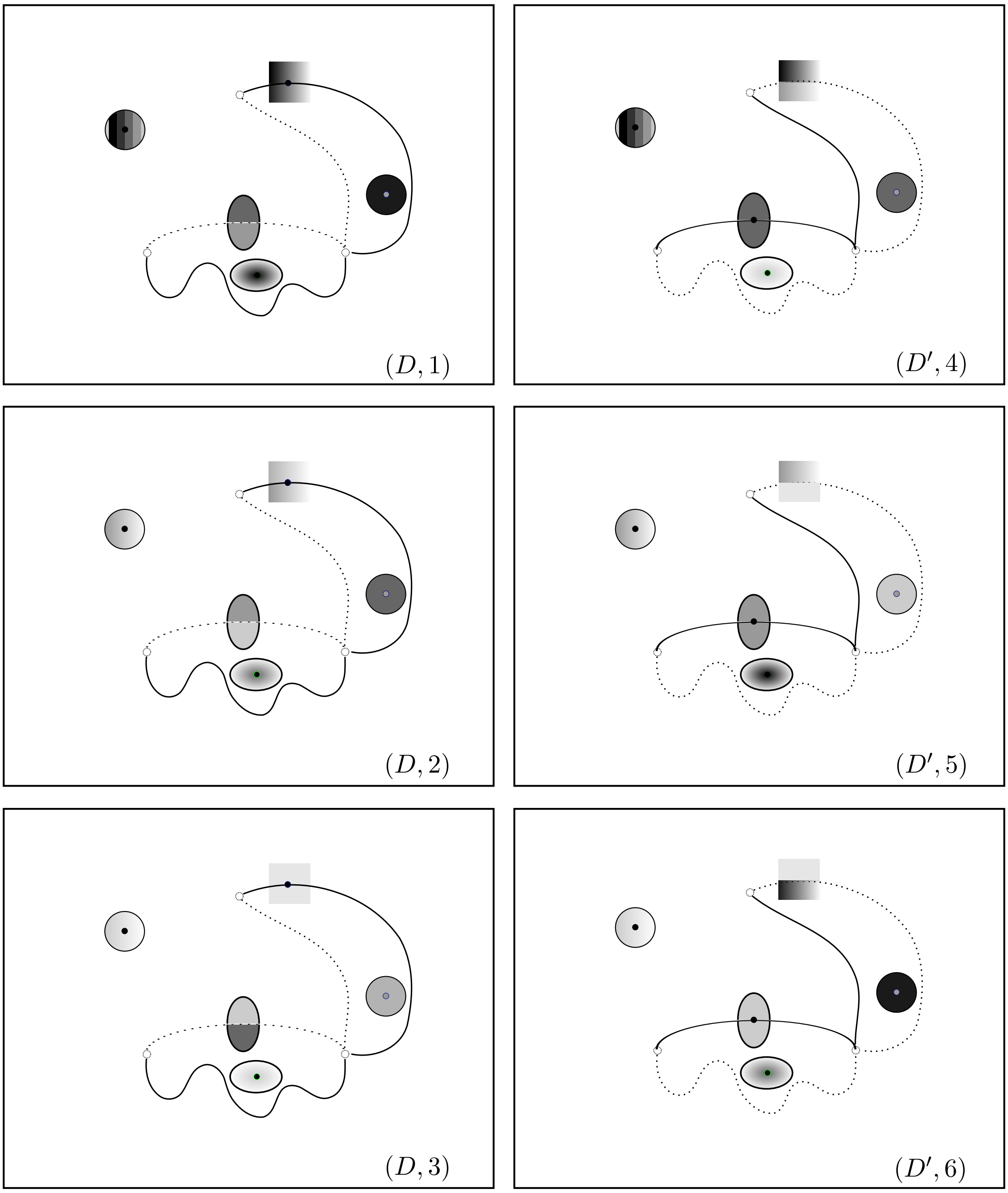

Figure 3: The triple covering space $Y_{\mathbf{\Sigma}}$, for M as in Figure 1. A dashed curve denotes that an admissible cut has been removed. In the picture, some examples of admissible neighbourhoods are shown. Identifications are meant by using the same grey level and shape. Note that a complete counterclockwise (small) turn around any point of S corresponds to move one sheet forward in $Y_{\mathbf{\Sigma}}$. Moreover, $m = 3$ turns around a point of S correspond to a single turn in $Y_{\mathbf{\Sigma}}$.

2 The constrained minimum problem

Let $\ell > 0$, and let $V := \{\alpha_1, \dots, \alpha_m\} \subset \mathbb{R}^{m-1}$ be such that

$$|\alpha_j - \alpha_l| = \ell, \qquad j, l = 1, \dots, m, \quad j \neq l.$$

We define

$$BV(Y_{\mathbf{\Sigma}}; V) := \Big\{ u \in BV(Y_{\mathbf{\Sigma}}; \mathbb{R}^{m-1}) \ : \ u(x) \in V \ \mu\text{−a.e. in } Y_{\mathbf{\Sigma}} \Big\}.$$

We denote by

$$\mathcal{T}(V)$$

the set of all maps $\tau : V \to V$ such that, for $h \in \{1, \dots, m-1\}$ coprime with m,

$$\tau(\alpha_j) = \alpha_l \ \text{ where } \ l \equiv j + h \ (\mathrm{mod}\ m), \ j \in \{1, \dots, m\}.$$

For $\tau \in \mathcal{T}(V)$, define $\tau^0 := \mathrm{id}$ in V, and $\tau^l := \tau \circ (\tau)^{l-1}$, for positive $l \in \mathbb{N}$. Notice that m coincides with the smallest positive integer κ such that $\tau^\kappa = \mathrm{id}$ (we call τ a transposition of V of order m).

Definition 2.1 (**Constrained BV functions on coverings**). *We denote by*

$$BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V)$$

the set of all $u \in BV(Y_{\mathbf{\Sigma}}; V)$ for which there exists $\tau \in \mathcal{T}(V)$ such that

$$v_j(u) = \tau^{j-1} \circ v_1(u), \qquad j = 1, \dots, m. \tag{2.1}$$

Remark 2.2. In view of (1.11) and (1.12), the constraint (2.1) is equivalent to require $v_{j'}(u) = \tau^{j'-1} \circ v_{m+1}(u)$, for $j' = m+1, \dots, 2m$.

To have an idea of the meaning of the constraint (2.1) in the case $m = 3$, the reader may refer to Figure 4.

Our constrained minimization problem, which in principle could depend on the choice of $\mathbf{\Sigma}$, can be now stated as follows:

$$\mathscr{A}^{\Omega}_{\mathrm{constr}}(S, \mathbf{\Sigma}) := \inf \Big\{ |Du|\big(Y_{\mathbf{\Sigma}}\big) \ : \ u \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V) \Big\}. \tag{2.2}$$

The independence of $\mathscr{A}^{\Omega}_{\mathrm{constr}}(S, \mathbf{\Sigma})$ of Σ will be shown in Corollary 2.8.

Remark 2.3. When $m = 2$, we fix the choice $\ell := 2$ and $V := \{\pm 1\}$, so that

$$BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; \{\pm 1\}) = \Big\{ u \in BV(Y_{\mathbf{\Sigma}}) \ : \ |u| = 1, \ v_1(u) = -v_2(u) \Big\}.$$

Clearly $u \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; \{\pm 1\})$ if and only if $u \in BV(Y_{\mathbf{\Sigma}}; \{\pm 1\})$ and

$$\sum_{\pi_{\mathbf{\Sigma}, M}(y) = x} u(y) = 0. \tag{2.3}$$

Notice that the sum in (2.3) contains only 2 terms.

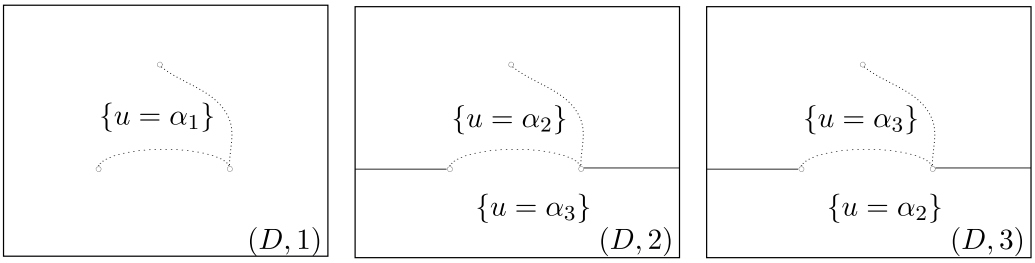

Figure 4: A function $u \in BV(Y_{\mathbf{\Sigma}}; V) \setminus BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V)$, where $Y_{\mathbf{\Sigma}}$ is the covering space in Figure 3. Notice that we need to specify the values of u just on the three charts drawn in the picture.

The functional in (2.2) attains the same value when evaluated at u and at $\tau \circ u$, for any $\tau \in \mathcal{T}(V)$. By virtue of the constraint, $\mathscr{A}^{\Omega}_{\mathrm{constr}}(S, \mathbf{\Sigma})$ will turn out to be strictly positive (see Theorem 2.15 below).

Remark 2.4 (Unbounded open sets). Let Ω be unbounded. Then, instead of (2.2), we shall consider the minimization problem

$$\mathscr{A}^{\Omega}_{\mathrm{constr}}(S, \mathbf{\Sigma}) := \inf \Big\{ |Du|(Y_{\mathbf{\Sigma}}) \ : \ u \in BV^{\mathrm{loc}}_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V) \Big\}, \tag{2.4}$$

where

$$\begin{aligned} BV^{\mathrm{loc}}_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V) := \Big\{ & u \in L^1_{\mathrm{loc}}(Y_{\mathbf{\Sigma}}; V) \ : \ |Du|(E) < \infty, \ E \subset Y_{\mathbf{\Sigma}} \text{ open rel. compact}, \\ & \exists\, \tau \in \mathcal{T}(V) \text{ s.t. } v_j(u) = \tau^{j-1} \circ v_1(u), \ j = 1, \dots, m \Big\}. \end{aligned}$$

We notice that the previous discussion (in particular, formula (1.14)) still holds true when Ω is unbounded.

The next observation shows a difference between our model and the model in [5], while Remark 2.6 seems to suggest a model closer to the one in [5].

Remark 2.5 (Monotonicity with respect to the base domain). Let Ω, $\Omega' \subseteq \mathbb{R}^n$ be connected open sets, such that $\Omega \subseteq \Omega'$. Then

$$\mathscr{A}^{\Omega}_{\mathrm{constr}}(S, \mathbf{\Sigma}) \leq \mathscr{A}^{\Omega'}_{\mathrm{constr}}(S, \mathbf{\Sigma}). \tag{2.5}$$

Indeed, let us assume that Ω' is bounded (the case in which Ω or Ω' are unbounded being similar). For $\mathbf{\Sigma} \in \mathbf{Cuts}(\Omega, S) \subseteq \mathbf{Cuts}(\Omega', S)$, let us denote by $Y'_{\mathbf{\Sigma}}$ the covering space of $M' := \Omega' \setminus S$. It is natural to see $Y_{\mathbf{\Sigma}}$ as a subset of $Y'_{\mathbf{\Sigma}}$, so that, for any $u \in BV_{\mathrm{constr}}(Y'_{\mathbf{\Sigma}}; V)$, we have $u_{|Y_{\mathbf{\Sigma}}} \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V)$. In particular, $|Du|(Y_{\mathbf{\Sigma}}) \leq |Du|(Y'_{\mathbf{\Sigma}})$, which gives (2.5).

Remark 2.6 (A Dirichlet-type formulation). By slightly modifying the space $BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V)$, it is possible to set up a minimization problem such that the minimum value decreases when the base domain becomes larger (the opposite of (2.5)). Let $\Omega, \Omega' \subset \mathbb{R}^n$ be connected open sets, such that $\Omega \subseteq \Omega'$ and $\Omega' \setminus \Omega \neq \emptyset$. Fix $\alpha \in V$, and let $\mathbf{\Sigma} \in \mathbf{Cuts}(\Omega, S) \subseteq \mathbf{Cuts}(\Omega', S)$. Let us consider the following Dirichlet-type problem:

$$\mathscr{B}^{\Omega}_{\mathrm{constr}}(S, \mathbf{\Sigma}, \Omega') := \inf\{|Du|(Y'_{\mathbf{\Sigma}}) : u \in BV_{\mathrm{constr}}(Y'_{\mathbf{\Sigma}}; V), \ v_1(u) = \alpha \text{ in } \Omega' \setminus \overline{\Omega}\}.$$

Then, the larger is Ω, the smaller is the value of $\mathscr{B}^{\Omega}_{\mathrm{constr}}(S, \mathbf{\Sigma}, \Omega')$.

2.1 Independence of the admissible pair of cuts

In this section we show that constrained−covering solutions are independent of admissible cuts. Our proof of Theorem 2.7 relies on general facts in coverings' theory, which we recall in the Appendix. Nevertheless, at least when $m = 2$, it is possibile to give a different proof which is independent of the abstract covering construction performed at the end of this paper.

For any $u \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V)$, let $J_u \subset Y_{\mathbf{\Sigma}}$ be the *set of approximate jump points*[9] of u in $Y_{\mathbf{\Sigma}}$.

Theorem 2.7. *Let Ω be bounded. Let $\mathbf{\Sigma} \in \mathbf{Cuts}(\Omega, S)$, and let $u \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V)$. Then, for any $\widehat{\mathbf{\Sigma}} \in \mathbf{Cuts}(\Omega, S)$, there exists $\widehat{u} \in BV_{\mathrm{constr}}(Y_{\widehat{\mathbf{\Sigma}}}; V)$ such that*

$$\pi_{\mathbf{\Sigma},M}(J_u) = \pi_{\widehat{\mathbf{\Sigma}},M}(J_{\widehat{u}}). \tag{2.6}$$

Proof. Let $f\colon Y_{\mathbf{\Sigma}} \to Y_{\widehat{\mathbf{\Sigma}}}$ be the homeomorphism defined in (A.5). We set $\widehat{u}\colon Y_{\widehat{\mathbf{\Sigma}}} \to V$ as

$$\widehat{u} := u \circ f^{-1}.$$

By definition of f, it follows that $u \in BV_{\mathrm{constr}}(Y_{\widehat{\mathbf{\Sigma}}}; V)$, and $J_{\widehat{u}} = f(J_u)$. Hence

$$\pi_{\widehat{\mathbf{\Sigma}},M}(J_{\widehat{u}}) = \pi_{\widehat{\mathbf{\Sigma}},M}(f(J_u)) = \pi_{\mathbf{\Sigma},M}(J_u),$$

where in the last equality we have made use of (A.5). □

Corollary 2.8 (Independence). *The value $\mathscr{A}^{\Omega}_{\mathrm{constr}}(S, \mathbf{\Sigma})$ in (2.2) is independent of $\mathbf{\Sigma} \in \mathbf{Cuts}(\Omega, S)$.*

Proof. We consider the case in which Ω is bounded, the unbounded case being similar. Let $\mathbf{\Sigma}, \widehat{\mathbf{\Sigma}} \in \mathbf{Cuts}(\Omega, S)$. Let $u_{\min} \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V)$ be such that $\mathscr{A}^{\Omega}_{\mathrm{constr}}(S, \mathbf{\Sigma}) = m\ell\, \mathcal{H}^{n-1}(\pi_{\mathbf{\Sigma},M}(J_{u_{\min}}))$. Let $\widehat{u} \in BV_{\mathrm{constr}}(Y_{\widehat{\mathbf{\Sigma}}}; V)$ be the function given by Theorem 2.7, applied with $u = u_{\min}$. Then, by (2.12) and (2.6), we have

$$\mathscr{A}^{\Omega}_{\mathrm{constr}}(S, \widehat{\mathbf{\Sigma}}) \le m\ell\mathcal{H}^{n-1}(\pi_{\widehat{\mathbf{\Sigma}},M}(J_{\widehat{u}})) = m\ell\, \mathcal{H}^{n-1}(\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})) = \mathscr{A}^{\Omega}_{\mathrm{constr}}(S, \mathbf{\Sigma}).$$

Arguing similarly for the converse inequality, we get $\mathscr{A}^{\Omega}_{\mathrm{constr}}(S, \widehat{\mathbf{\Sigma}}) = \mathscr{A}^{\Omega}_{\mathrm{constr}}(S, \mathbf{\Sigma})$. □

In accordance with Corollary 2.8, we set

$$\mathscr{A}^{\Omega}_{\mathrm{constr}}(S) := \mathscr{A}^{\Omega}_{\mathrm{constr}}(S, \mathbf{\Sigma}).$$

Corollary 2.9 (Upper bound). *We have*

$$\mathscr{A}^{\Omega}_{\mathrm{constr}}(S) \le m\ell \inf \Big\{ \mathcal{H}^{n-1}(\Sigma) \;:\; \Sigma \in \mathrm{Cuts}(\Omega, S) \Big\}. \tag{2.7}$$

[9] Here we follow [2, Definition 3.67, p.163]. Let $y \in \Psi_1(D, 1)$, $x := \pi_{\mathbf{\Sigma},M}(y)$, and let $r > 0$ be such that $B_r(x)$ is contained in D. Given a unit vector $\nu \in \mathbb{R}^n$, set $B_r(y) := \Psi_1(B_r(x))$, $B^+_{r,\nu}(y) := \{y' \in B_r(y) \;:\; (\pi_{\mathbf{\Sigma},M}(y') - x) \cdot \nu > 0\}$, $B^-_{r,\nu}(y) := \{y' \in B_r(y) \;:\; (\pi_{\mathbf{\Sigma},M}(y') - x) \cdot \nu < 0\}$. Now, we say that y is an approximate jump point of u if there exist a unit vector $\nu \in \mathbb{R}^n$, and two distinct $\alpha, \beta \in V$ satisfying $\lim_{r\to 0^+} r^{-n} \int_{B^+_{r,\nu}(y)} |u - \alpha|\, d\mu = 0 = \lim_{r\to 0^+} r^{-n} \int_{B^-_{r,\nu}(y)} |u - \beta|\, d\mu$. Similarly we proceed when y belongs to the other covering sheets.

Proof. Let $\tau \in \mathcal{T}(V)$. Let u be the τ-constrained lift of v (Definition 2.10 below), with v identically equal to some $\alpha \in V$. Then (2.9) holds, and (2.7) follows. $\square$

In Sections 4.1 and 4.2 we shall prove that, when $m = 2$, $n \le 7$, and $\Omega = \mathbb{R}^n$, (2.7) holds as an equality (see Corollary 4.2 and Theorem 4.6). Notice that, by the regularity of area minimizing currents modulo 2 [25, Theorem 6.2.1], the infimum on the right hand side of (2.7) is a minimum, provided $n \le 7$.

2.2 Existence of minimizers

Concerning functions defined on the base set, clearly $BV(M;V) = BV(\Omega;V)$. Moreover

$$BV(\Omega;V) = BV(D;V),$$

so that any $v \in BV(D;V)$ (or more generally any $v \in BV(\Omega \setminus C;V)$, with C a finite union of cuts) can be considered also as a BV function in Ω, whose total variation in general may increase by a contribution due to the two traces of v on Σ (more generally on C). In the following, we denote by

$$J_v \subset \Omega$$

the set of approximate jump points of v considered as a function in $BV(\Omega;V)$. The next definition will be of frequent use in the sequel.

Definition 2.10 (Constrained lift). *Let $v \in BV(D;V)$, and let $\tau \in \mathcal{T}(V)$. Then the function defined as*

$$u := \tau^{j-1} \circ v \circ \Psi_j^{-1} \ \textit{in}\ \Psi_j(D), \qquad j = 1, \dots, m, \tag{2.8}$$

is in $BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}};V)$, and $v_1(u) = v$. We call u the τ-constrained lift *of v.*

In particular, when v is identically equal to some $\alpha \in V$, we have

$$\pi_{\mathbf{\Sigma},M}(J_u) = \Sigma \setminus S, \tag{2.9}$$

for every $\tau \in \mathcal{T}(V)$.

Lemma 2.11 (Splitting of the projection of the jump). *Let $\mathbf{\Sigma} = (\Sigma, \Sigma') \in$ **Cuts**(Ω, S), and let $u \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}};V)$. Then*

$$\pi_{\mathbf{\Sigma},M}(J_u) = \Big(J_{v_1(u)} \setminus (\Sigma \setminus S)\Big) \cup \Big(J_{v_{m+1}(u)} \cap (\Sigma \setminus S)\Big). \tag{2.10}$$

Proof. Let us split J_u as the union of the following $2m$ disjoint sets:

$$J_u \cap \widetilde{\pi}((D,j)),\ j = 1, \dots, m, \qquad J_u \cap \widetilde{\pi}((\Sigma \setminus S, j')),\ j' = m+1, \dots, 2m. \tag{2.11}$$

By the constraint (2.1), for each $j = 2, \dots, m$ (resp. for each $j' = m+2, \dots, 2m$), to each point in $J_u \cap \widetilde{\pi}((D,j))$ (resp. in $J_u \cap \widetilde{\pi}((\Sigma \setminus S, j'))$) there corresponds a unique point in $J_u \cap \widetilde{\pi}((D,1))$ (resp. in $J_u \cap \widetilde{\pi}((\Sigma \setminus S, m+1))$), belonging to the same fiber, and *viceversa*. Hence

$$\pi_{\mathbf{\Sigma},M}(J_u) = \pi_{\mathbf{\Sigma},M}\Big(J_u \cap \widetilde{\pi}((D,1))\Big) \cup \pi_{\mathbf{\Sigma},M}\Big(J_u \cap \widetilde{\pi}((\Sigma \setminus S, m+1))\Big).$$

By definition of J_u, $J_{v_1(u)}$, $J_{v_{m+1}(u)}$, using also the local parametrizations Ψ_1, Ψ_{m+1}, it follows that $\pi_{\mathbf{\Sigma},M}\big(J_u \cap \widetilde{\pi}((D,1))\big) = J_{v_1(u)} \setminus (\Sigma \setminus S)$, and $\pi_{\mathbf{\Sigma},M}\big(J_u \cap \widetilde{\pi}((\Sigma \setminus S, m+1))\big) = J_{v_{m+1}(u)} \cap (\Sigma \setminus S)$, and (2.10) follows. $\square$

The next lemma seems to be consistent with [5, Lemma 10.1].

Lemma 2.12 (**Representation of the total variation on the covering, II**). *Let* $\mathbf{\Sigma} = (\Sigma, \Sigma') \in \mathbf{Cuts}(\Omega, S)$*, and let* $u \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V)$*. Then*

$$\begin{aligned} |Du|(Y_{\mathbf{\Sigma}}) =& m\ell \Big(\mathcal{H}^{n-1}(J_{v_1(u)} \setminus \Sigma) + \mathcal{H}^{n-1}(J_{v_{m+1}(u)} \cap \Sigma)\Big) \\ =& m\ell\, \mathcal{H}^{n-1}(\pi_{\mathbf{\Sigma},M}(J_u)). \end{aligned} \tag{2.12}$$

Proof. Recall the splitting in (1.14), with the choice $E := Y_{\mathbf{\Sigma}}$. By (1.11), we have

$$\begin{aligned} |Dv_j(u)|(D) = |Dv_1(u)|(D), \qquad & j = 1, \ldots, m, \\ |Dv'_j(u)|(\Sigma) = |Dv_{m+1}(u)|(\Sigma), \qquad & j' = m+1, \ldots, 2m. \end{aligned} \tag{2.13}$$

By [2, Theorem 3.84], we have

$$|Dv_1(u)|(D) = \ell\, \mathcal{H}^{n-1}(J_{v_1(u)} \setminus \Sigma), \quad |Dv_{m+1}(u)|(\Sigma) = \ell\, \mathcal{H}^{n-1}(J_{v_{m+1}(u)} \cap \Sigma). \tag{2.14}$$

Substituting (2.14) into (1.14), and recalling (2.13), we get the first equality in (2.12). The second equality is now a consequence of (2.10). $\square$

Remark 2.13. From formula (2.12), we see that $|Du|(Y_{\mathbf{\Sigma}})$ is indeed independent of the orientation of Σ.

Corollary 2.14 (**Compactness**). *Let* Ω *be bounded with Lipschitz boundary. Let* $(u_h)_{h\in\mathbb{N}} \subset BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V)$ *be such that* $\sup_{h\in\mathbb{N}} |Du_h|(Y_{\mathbf{\Sigma}}) < +\infty$*. Then there exist* $u \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V)$ *and a subsequence of* $(u_h)_{h\in\mathbb{N}}$ *converging to* u *in* $L^1(Y_{\mathbf{\Sigma}}; \mathbb{R}^{m-1})$*.*

Proof. Let $v_1(u_h) \in BV(D; V) = BV(\Omega; V)$ be as in (1.10). From (1.14) and (2.13) we have

$$\begin{aligned} \sup_{h\in\mathbb{N}} |Dv_1(u_h)|(\Omega) =& \sup_{h\in\mathbb{N}} \Big[|Dv_1(u_h)|(D) + |Dv_1(u_h)|(\Sigma)\Big] \\ \leq& \frac{1}{m} \sup_{h\in\mathbb{N}} |Du_h|(Y_{\mathbf{\Sigma}}) + \ell\mathcal{H}^{n-1}(\Sigma) < +\infty. \end{aligned}$$

Since Ω is a bounded Lipschitz domain, there exists $v \in BV(\Omega; V)$ such that, up to a not relabelled subsequence, $v_1(u_h) \to v$ in $L^1(\Omega; \mathbb{R}^k)$. The proof is completed, letting u be defined as in (2.8). $\square$

We are now in the position to show that problem (2.2) has a solution; a key result is represented by Lemma 2.16 below.

Theorem 2.15 (**Existence of minimizers**). *Let* Ω *be a bounded connected open set with Lipschitz boundary. Let* $\mathbf{\Sigma} \in \mathbf{Cuts}(\Omega, S)$*. Then* $\mathscr{A}^{\Omega}_{\mathrm{constr}}(S)$ *is a minimum, and* $\mathscr{A}^{\Omega}_{\mathrm{constr}}(S) > 0$*.*

Proof. By the lower semicontinuity of the total variation, also recalling Corollary 2.14, existence of minimizers for problem (2.2) follows by direct methods. Positivity of $\mathscr{A}^{\Omega}_{\mathrm{constr}}(S)$ follows from (2.16) below, with the choice $A := \Omega$. $\square$

The next lemma shows, in particular, that in the fibers over any open subset of Ω containing a loop around a point of S, the jump set of any function in $BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V)$ has strictly positive $\mathcal{H}^{n-1}-$measure. We stress that this is due just to the constraint (2.1).

Lemma 2.16 (**Non-constancy**). *Let $A \subseteq \Omega$ be a nonempty connected open set such that $\pi^{-1}_{\mathbf{\Sigma},M}(A \setminus S)$ does not consist of m connected components. Then, for every $u \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V)$,*

$$\mathcal{H}^{n-1}\big(A \cap \pi_{\mathbf{\Sigma},M}(J_u)\big) > 0. \tag{2.15}$$

Moreover, if A is bounded with Lipschitz boundary, then

$$\inf\big\{\mathcal{H}^{n-1}\big(A \cap \pi_{\mathbf{\Sigma},M}(J_u)\big) \;:\; u \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V)\big\} > 0. \tag{2.16}$$

Proof. In order to show (2.15), suppose by contradiction that there exists $u \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V)$ such that

$$\mathcal{H}^{n-1}\big(A \cap \pi_{\mathbf{\Sigma},M}(J_u)\big) = 0. \tag{2.17}$$

Applying (2.10) to (2.17), we get

$$0 = \mathcal{H}^{n-1}(A \cap (J_{v_1(u)} \setminus \Sigma)) + \mathcal{H}^{n-1}(A \cap J_{v_{m+1}(u)} \cap \Sigma). \tag{2.18}$$

Now, consider the (connected open) set $A^S := A \setminus S$. Applying (1.14) with the choice $E := \pi^{-1}_{\mathbf{\Sigma},M}(A^S)$, we get

$$\begin{aligned}
|Du|(\pi^{-1}_{\mathbf{\Sigma},M}(A^S)) =& m\,|Dv_1(u)|\,\Big(\pi_{\mathbf{\Sigma},M}(\pi^{-1}_{\mathbf{\Sigma},M}(A^S) \cap \widetilde{\pi}((D,1)))\Big)\\
&+ m\,|Dv_{m+1}(u)|\,\Big(\pi_{\mathbf{\Sigma},M}(\pi^{-1}_{\mathbf{\Sigma},M}(A^S) \cap \widetilde{\pi}(\Sigma \setminus S, m+1))\Big)\\
=& m\,|Dv_1(u)|\,\big(A^S \setminus \Sigma\big) + m\,|Dv_{m+1}(u)|\,\big(A^S \cap \Sigma\big)\\
=& m\ell\,\big(\mathcal{H}^{n-1}(A \cap (J_{v_1(u)} \setminus \Sigma)) + \mathcal{H}^{n-1}(A \cap J_{v_{m+1}(u)} \cap \Sigma)\big),
\end{aligned} \tag{2.19}$$

which, coupled with (2.18), implies $|Du|(\pi^{-1}_{\mathbf{\Sigma},M}(A^S)) = 0$. Then[10] u is constant on each connected component of $\pi^{-1}_{\mathbf{\Sigma},M}(A^S)$. By the assumption on A, there exists at least one connected component of $\pi^{-1}_{\mathbf{\Sigma},M}(A^S)$, not contained in a single covering sheet. This contradicts the validity of the constraint (2.1), proving (2.15).

Now, let us suppose, still by contradiction, that there exists a sequence $(u_h)_h \subset BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V)$ such that $\lim_{h\to+\infty} \mathcal{H}^{n-1}\big(A \cap \pi_{\mathbf{\Sigma},M}(J_{u_h})\big) = 0$. For $h \in \mathbb{N}$, set $\hat{u}_h := u_h|_{\pi^{-1}_{\mathbf{\Sigma},M}(A^S)}$. In particular, reasoning as above, $|D\hat{u}_h|(\pi^{-1}_{\mathbf{\Sigma},M}(A^S)) = m\ell\,\mathcal{H}^{n-1}(A \cap \pi_{\mathbf{\Sigma},M}(J_{u_h}))$. Let us apply Corollary 2.14, replacing Ω with A. Then, up to a not relabelled subsequence, there exists $\hat{u} \in BV_{\mathrm{constr}}(\pi^{-1}_{\mathbf{\Sigma},M}(A^S); V)$ such that $\hat{u}_h \to u$ in $L^1(\pi^{-1}_{\mathbf{\Sigma},M}(A^S); V)$, and by lower semicontinuity,

$$|D\hat{u}|(\pi^{-1}_{\mathbf{\Sigma},M}(A^S)) \le \liminf_{h\to+\infty} |D\hat{u}_h|(\pi^{-1}_{\mathbf{\Sigma},M}(A^S)) = m\ell \lim_{h\to+\infty} \mathcal{H}^{n-1}\big(A \cap \pi_{\mathbf{\Sigma},M}(J_{u_h})\big) = 0.$$

Hence $\hat{u}$ is constant on $\pi^{-1}_{\mathbf{\Sigma},M}(A^S)$, a contradiction with (2.15). □

As a further consequence of Lemma 2.16, the boundary datum S is covered by any constrained function in the covering space. In Theorem 4.3, using also (2.21) below, we shall prove that equality holds in (2.20) when $2 < n \le 7$ and u is a minimizer.

[10] See [2, Proposition 3.2]; this constancy result can be generalized to our setting, considering first the case in which a connected open set $E \subseteq Y_{\mathbf{\Sigma}}$ is contained in a single chart, and then reasoning for each connected component of $E \cap \widetilde{\pi}((D,1))$, $E \cap \widetilde{\pi}((D',m+1))$.

Corollary 2.17. *Let Ω be bounded (resp. unbounded), and let $u \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V)$ (resp. $u \in BV^{\mathrm{loc}}_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V)$). Then*

$$S \subseteq \overline{\pi_{\mathbf{\Sigma},M}(J_u)} \setminus \pi_{\mathbf{\Sigma},M}(J_u). \tag{2.20}$$

Proof. The relation $S \cap \pi_{\mathbf{\Sigma},M}(J_u) = \emptyset$ is trivial, recall also (2.10). Now, suppose by contradiction that there exists a point $p \in S \setminus \overline{\pi_{\mathbf{\Sigma},M}(J_u)}$. Take an open ball B centered at p, with $B \subset \Omega \setminus \overline{\pi_{\mathbf{\Sigma},M}(J_u)}$, and apply Lemma 2.16 with the choice $A := B$. Then, since $A \cap \pi_{\mathbf{\Sigma},M}(J_u) = \emptyset$, we end up with a contradiction with (2.15). □

In view of Lemma 2.12, we give the following definition.

Definition 2.18 (**Constrained − covering solutions**)**.** *Let Ω be bounded with Lipschitz boundary and let $u_{\min}$ be a minimizer of problem* (2.2)*. We call*

$$\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$$

a constrained − covering solution *(in Ω)* with boundary S.

A similar definition is given when Ω is unbounded, assuming existence of $u_{\min}$ minimizing (2.4).

Remark 2.19. No topological restrictions on $\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$ are required.

Recalling Remark 1.6, we observe that the proof of analytic regularity for the reduced boundary of minimizing clusters [17] applies in our setting. Indeed, since the classical arguments (such as monotonicity formula, excess decay, tilt lemma) are *local*, they can be symmetrically reproduced on the m sheets of the covering space, thus respecting the constraint on the fibers. In particular, the following results hold.

Theorem 2.20 (**Regularity,** $n = 2$)**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded connected open set with Lipschitz boundary (resp. an unbounded connected open set) , and let $u_{\min}$ be a minimizer of* (2.2) *(resp. of* (2.4)*). Then $J_{u_{\min}}$, and hence, $\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$, is the union of finitely many segments. Moreover, for each singular point x of $\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$ there exist exactly three segments of $\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$ having x as one of their endpoints, and meeting at x at $\frac{2\pi}{3}$-angles. Moreover,*

$$\overline{\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})} \setminus \pi_{\mathbf{\Sigma},M}(J_{u_{\min}}) \subseteq S \cup \partial\Omega. \tag{2.21}$$

Proof. We can confine ourselves to the proof of (2.21). Recalling Lemma 2.11, we have

$$\left(\overline{\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})} \setminus \pi_{\mathbf{\Sigma},M}(J_{u_{\min}})\right) \cap D = \left(\overline{J_{v_1(u_{\min})}} \setminus J_{v_1(u_{\min})}\right) \cap D. \tag{2.22}$$

By the regularity of local minimizing clusters, $J_{v_1(u_{\min})} \cap D$ coincides with the relative boundary in D of the set $\cup_{\alpha \in V}\{v_1(u_{\min}) = \alpha\}$. In particular, $J_{v_1(u_{\min})} \cap D$ is relatively closed in D, which by (2.22) implies

$$\left(\overline{\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})} \setminus \pi_{\mathbf{\Sigma},M}(J_{u_{\min}})\right) \cap D = \emptyset.$$

Similarly we argue on D', and (2.21) follows. □

The proof of the regularity result in the case $m = 2$ is analogous, so that we omit the details.

Theorem 2.21 (**Regularity,** $m = 2$)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set with Lipschitz boundary (resp. an unbounded connected open set), and let $u_{\min}$ be a minimizer of* (2.2) *(resp. of* (2.4)*). Then $J_{u_{\min}}$, and hence $\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$, is an analytic submanifold, possibly excepting for a set of Hausdorff dimension at most $n-8$. Moreover,* (2.21) *holds.*

3 Regularization

The interest in using V-valued BV functions in the context of covering spaces is substantiated by a $\Gamma-$convergence [6] result.

Let us first consider the case $m = 2$. Let Ω be bounded with Lipschitz boundary. The main idea is to lift the constraint (2.1) onto the Sobolev space $H^1(Y_{\mathbf{\Sigma}}) := \{u \in L^2_\mu(Y_{\mathbf{\Sigma}}) \ : Du \in L^2_\mu(Y_{\mathbf{\Sigma}};\mathbb{R}^n)\}$. Recalling Remark 2.3, we set

$$H^1_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}) := \Big\{u \in H^1(Y_{\mathbf{\Sigma}}) \ : \sum_{\pi_{\mathbf{\Sigma},M}(y)=x} u(y) = 0 \text{ for a.e. } x \in M\Big\}. \tag{3.1}$$

For $\epsilon \in (0,1)$, let us consider the functionals $F_\epsilon\colon L^1(Y_{\mathbf{\Sigma}}) \to [0,+\infty]$, defined as

$$F_\epsilon(u) := \int_{Y_{\mathbf{\Sigma}}} \left[\epsilon|\nabla u|^2 + \frac{1}{\epsilon}(1-u^2)^2\right] d\mu \qquad \text{if } u \in H^1_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}),$$

and extended to $+\infty$ in $L^1(Y_{\mathbf{\Sigma}}) \setminus H^1_{\mathrm{constr}}(Y_{\mathbf{\Sigma}})$.

Proposition 3.1 ($\Gamma-$**convergence,** $m = 2$)**.** *Assume $n \geq 2$ and $m = 2$. If $(u_{\epsilon_h})_h \subset L^1(Y_{\mathbf{\Sigma}})$ is such that $\sup_h F_{\epsilon_h}(u_{\epsilon_h}) < +\infty$, then there exist $u \in L^1(Y_{\mathbf{\Sigma}})$ and a subsequence of $(u_{\epsilon_h})_h$ converging to u in $L^1(Y_{\mathbf{\Sigma}})$. Moreover,*

$$\big(\Gamma(L^1(Y_{\mathbf{\Sigma}})) - \lim_{\epsilon\to 0^+} F_\epsilon\big)(u) = \begin{cases} \frac{c_0}{2}|Du|(Y_{\mathbf{\Sigma}}), & \text{if } u \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}};\{\pm 1\}), \\ +\infty, & \text{otherwise in } L^1(Y_{\mathbf{\Sigma}}), \end{cases}$$

where $c_0 := \xi(1) - \xi(-1)$, and $\xi(t) := 2\int_0^t |1-s^2|\,ds$.

Proof. The proof of the equicoerciveness statement is standard (see, *e.g.*, [18]). The $\Gamma - \liminf$ inequality follows using the lower semicontinuity of the total variation, and the fact that the constraint (2.3) is closed under almost everywhere convergence in $Y_{\mathbf{\Sigma}}$. The $\Gamma - \limsup$ construction follows by recalling that the local parametrizations of $Y_{\mathbf{\Sigma}}$ are the identity (Remark 1.6); in order to get the validity of the constraint in (3.1), it is sufficient to use the standard construction, since the optimal one-dimensional profile is odd (hence, the corresponding recovering sequence is in $H^1_{\mathrm{constr}}(Y_{\mathbf{\Sigma}})$). See [18] for the details. □

Now, let us conclude this section with the case $n = 2$ and $m = 3$. Let $V := \{\alpha_1, \alpha_2, \alpha_3\} \subset \mathbb{R}^2$ be the set of vertices of an equilateral triangle, centered at the origin. With a slight abuse of notation, it is natural to identify $\mathcal{T}(V)$ with $\{\frac{2\pi}{3}, \frac{4\pi}{3}\}$, see (3.3) below. The idea is now to lift the constraint (2.1) onto the Sobolev space $H^1(Y_{\mathbf{\Sigma}};\mathbb{R}^2) := \{u \in L^2_\mu(Y_{\mathbf{\Sigma}};\mathbb{R}^2) \ : \ Du \in L^2_\mu(Y_{\mathbf{\Sigma}};\mathbb{R}^2 \times \mathbb{R}^2)\}$, by asking that

$$\exists\, \theta \in \Big\{\frac{2\pi}{3}, \frac{4\pi}{3}\Big\} \quad \text{s.t.} \quad v_j(u) = e^{i(j-1)\theta} \circ v_1(u),\ j = 1, 2, 3, \tag{3.2}$$

and then setting

$$H^1_{\rm constr}(Y_{\mathbf{\Sigma}};\mathbb{R}^2) := \Big\{u \in H^1(Y_{\mathbf{\Sigma}};\mathbb{R}^2) \ : \ (3.2) \text{ holds}\Big\}. \tag{3.3}$$

where, for $\theta \in [0, 2\pi)$, $e^{i\theta} \colon \mathbb{R}^2 \to \mathbb{R}^2$ is the counterclockwise rotation of angle θ.

Let $W\colon \mathbb{R}^2 \to [0, +\infty)$ be a triple-well potential with superlinear growth at infinity, and such that $W^{-1}(0) = V$. We assume also that[11]

$$W(e^{i\theta} x) = W(x), \qquad x \in \mathbb{R}^2,\ \theta \in \Big\{\frac{2\pi}{3}, \frac{4\pi}{3}\Big\}. \tag{3.4}$$

For $\epsilon \in (0, 1)$, let us consider the functionals $G_\epsilon \colon L^1(Y_{\mathbf{\Sigma}};\mathbb{R}^2) \to [0, +\infty]$, defined as

$$G_\epsilon(u) := \int_{Y_{\mathbf{\Sigma}}} \Big[\epsilon|\nabla u|^2 + \frac{1}{\epsilon} W(u)\Big]\, d\mu \qquad \text{if } u \in H^1_{\rm constr}(Y_{\mathbf{\Sigma}};\mathbb{R}^2),$$

and extended to $+\infty$ in $L^1(Y_{\mathbf{\Sigma}};\mathbb{R}^2) \setminus H^1_{\rm constr}(Y_{\mathbf{\Sigma}};\mathbb{R}^2)$.

Proposition 3.2 ($\Gamma-$**convergence,** $m = 3$)**.** *Assume $n = 2$ and $m = 3$. If $(u_{\epsilon_h})_h \subset L^1(Y_{\mathbf{\Sigma}};\mathbb{R}^2)$ is such that $\sup_h G_{\epsilon_h}(u_{\epsilon_h}) < +\infty$, then there exist $u \in L^1(Y_{\mathbf{\Sigma}};\mathbb{R}^2)$ and a subsequence of $(u_{\epsilon_h})_h$ converging to u in $L^1(Y_{\mathbf{\Sigma}};\mathbb{R}^2)$. Moreover, there exists the $\Gamma(L^1(Y_{\mathbf{\Sigma}};\mathbb{R}^2))$-limit of $(G_\epsilon)_\epsilon$ as $\epsilon \to 0^+$, which is finite just on functions $u \in BV_{\rm constr}(Y_{\mathbf{\Sigma}};V)$, and it equals $|Du|(Y_{\mathbf{\Sigma}})$ up to a positive multiplicative constant depending only on W.*

Proof. Again, the proof of the equicoerciveness statement is standard (see, *e.g.*, [3]). Let $(u_\epsilon)_\epsilon \subset H^1_{\rm constr}(Y_{\mathbf{\Sigma}};\mathbb{R}^2)$ be such that $(G_\epsilon(u_\epsilon))_\epsilon$ is equibounded, and $u_\epsilon \to u$ in $L^1(Y_{\mathbf{\Sigma}};\mathbb{R}^2)$ for some $u \in L^1(Y_{\mathbf{\Sigma}};\mathbb{R}^2)$. Then $W(u) = 0$ a.e. in $Y_{\mathbf{\Sigma}}$, or equivalently $u(x) \in V$ a.e. in $Y_{\mathbf{\Sigma}}$. The fact that u satisfies (2.1) for some $\tau \in \mathcal{T}(V)$ follows at once by the constraint in (3.3). The $\Gamma - \liminf$ inequality is now a consequence of the lower semicontinuity of the total variation.

Let us sketch the proof of the $\Gamma - \limsup$ construction, which is a slight modification of the one provided in [3]. Without loss of generality, we can assume $u \in BV_{\rm constr}(Y_{\mathbf{\Sigma}};V)$, and $\pi_{\mathbf{\Sigma},M}(J_u)$ contained in the union of a finite number of segments. For small $\epsilon > 0$, consider an ϵ-tubular neighbourhood $T_\epsilon \subset \Omega$ of $\pi_{\mathbf{\Sigma},M}(J_u)$; let also $Z_\epsilon \subset T_\epsilon$ be the Lipschitz open set containing the triple junctions, such that $\pi_{\mathbf{\Sigma},M}(Z_\epsilon) = \bigcap_{j=1}^3 \{|d_j| < \epsilon\}$, where, for every $j = 1, 2, 3$, d_j denotes a signed distance from $\{u = \alpha_j\}$. Then, we construct a map $u_\epsilon \in H^1(Y_{\mathbf{\Sigma}};\mathbb{R}^2)$ so that:

- $u_\epsilon = u$ in $Y_{\mathbf{\Sigma}} \setminus \pi^{-1}_{\mathbf{\Sigma},M}(T_\epsilon)$,
- in $\pi^{-1}_{\mathbf{\Sigma},M}(T_\epsilon \setminus Z_\epsilon)$, u_ϵ realizes the transition between the two corresponding zeroes of W, along suitable optimal profiles which depend only on W (see [3]);

[11] For instance, one could consider the choice $W(x) := \prod_{j=1}^3 |x - \alpha_j|^2$.

- u_ϵ in $\pi^{-1}_{\mathbf{\Sigma},M}(Z_\epsilon)$ is defined by interpolating the trace of u_ϵ on $\partial\pi^{-1}_{\mathbf{\Sigma},M}(Z_\epsilon)$ with zero (the barycenter of V) along the segments starting at the triple junction.

Here we notice that, since $u \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V)$, and thanks to the simmetry assumption (3.4) on W, u_ϵ satisfies (3.2), and therefore $u_\epsilon \in H^1_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; \mathbb{R}^2)$. Moreover, the contribution to $G_\epsilon(u_\epsilon)$ on $\pi^{-1}_{\mathbf{\Sigma},M}(Z_\epsilon)$ is of order ϵ. Then the statement follows. □

Remark 3.3. Proposition 3.2 can be extended to the case $m \geq 3$, combining the standard tools in [3] (which actually hold for every $m \geq 2$).

4 Constrained covering solutions when $n = 2, 3$

4.1 Minimal networks in the plane

In this section we exploit the case $n = 2$, $m \geq 2$, and $S := \{p_1, \dots, p_m\} \subset \Omega$, with $p_j \neq p_l$ for any $j, l = 1, \dots, m$, $j \neq l$.

Theorem 4.1. *Assume that*

$$\mathrm{dist}(S, \partial\Omega) > \inf\{\mathcal{H}^1(\Sigma) \ : \ \Sigma \in \mathrm{Cuts}(\Omega, S)\}. \tag{4.1}$$

Then $\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$ is connected.

Proof. By contradiction, suppose that there exist two disjoint nonempty sets C_1, C_2, relatively closed in $\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$, and such that $C_1 \cup C_2 = \pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$. By Theorem 2.20, for each $j = 1, 2$, C_j consists of segments (possibly meeting at triple junctions); moreover, by virtue of (4.1), also recalling (2.12) and (2.7), we have $\overline{C_j} \cap \partial\Omega = \emptyset$. Set $S_j := \overline{C_j} \cap S$, for $j = 1, 2$. Note that

$$S_j \neq \emptyset, \qquad j = 1, 2. \tag{4.2}$$

Indeed, suppose by contradiction that (for example) $S_1 = \emptyset$; then, by (2.21), $\overline{C_1} \setminus C_1 \subset \partial\Omega$, and therefore there exists a connected open set $A \subset \Omega$ such that $\Omega \cap \partial A \subseteq C_1$, and $A \cap S = \emptyset$. Thanks to Theorem 2.7, it is not restrictive to assume also that $A \cap \Sigma = \emptyset$. Now, it is immediate to modify $v_1(u_{\min})$ inside A so that it does not jump anymore on $\Omega \cap \partial A$. Taking any constrained-lift of the modified function, minimality of $u_{\min}$ is contradicted, proving (4.2).

Let us choose now two tubular neighborhoods T, U of C_1, so that $T \subset\subset U$,

$$(U \setminus \overline{T}) \cap \pi_{\mathbf{\Sigma},M}(J_{u_{\min}}) = \emptyset, \tag{4.3}$$

and $T \cap C_2 = \emptyset$. In particular, there must be $j \in \{1, \dots, m\}$ such that Σ_j connects a point of S_1 with a point of S_2. Therefore, $\pi^{-1}_{\mathbf{\Sigma},M}(U \setminus \overline{T})$ does not consist of m distinct connected components, so that, applying Lemma 2.16 with the choice $A := U \setminus \overline{T}$, we get a contradiction with (4.3). □

Corollary 4.2. *Assume* (4.1). *Then $\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$ is a Steiner graph*[12] *connecting the points of S.*

[12] See, *e.g.*, [11].

Proof. Let $C \subset \Omega$ be a Steiner graph connecting the points of S. By Theorems 2.20-4.1, $\mathcal{H}^1(\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})) \leq \mathcal{H}^1(C)$. On the other hand, fix any $\Sigma \in \mathrm{Cuts}(\Omega, S)$ such that $\Sigma \cap C = S$. Then, define $v \in BV(\Omega; V)$ so that: for $j = 1, \dots, m-1$, $v := \alpha_j$ on the connected open set whose boundary contains Σ_j, and is contained in $\Sigma_j \cup C$; $v := \alpha_m$ elsewhere in Ω. Finally, consider the τ-contrained lift $u \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V)$ of v, where $\tau(j) := j - 1 \pmod m$. By construction, $\pi_{\mathbf{\Sigma},M}(J_u) = C$, and the statement follows. $\square$

In order to get Corollary 4.2, we cannot avoid condition (4.1), see Figure 5 for a counterexample when $m = 2$. This is another difference with respect to the model proposed in [5]: in our model the boundary of Ω is "wettable" in principle, and therefore, in order to avoid a minimizer to touch $\partial\Omega$, we need a condition of the form (4.1) (see also Remark 2.6).

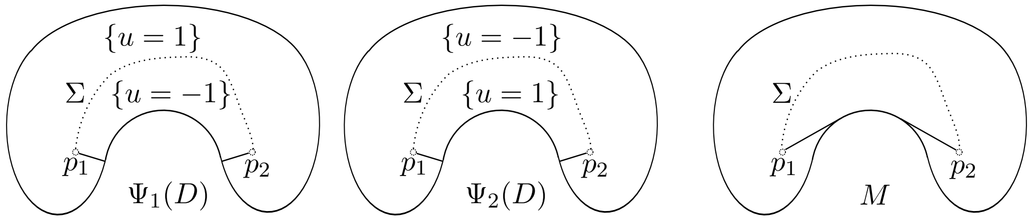

Figure 5: Let Ω be the "bean-shaped" domain in the picture, let $S := \{p_1, p_2\}$, and let $\Sigma \in \mathrm{Cuts}(\Omega, S)$ be the dashed curve. The two pictures on the left show the constrained covering solution, while the right picture shows the solution of [5].

4.2 Plateau's problem

In this section we exploit the case $n = 3$, hence $m = 2$, so that $Y_{\mathbf{\Sigma}}$ is a double-covering space of M.

Let $S \subset \mathbb{R}^3$ be a tame link. Let $\Omega \subset \mathbb{R}^3$ be bounded with Lipschitz boundary, and $\mathbf{\Sigma} \in \mathbf{Cuts}(\Omega, S)$. Let $u_{\min} \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; \{\pm 1\})$ be a minimizer of problem (2.2). By Theorem 2.21, $\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$ is an embedded analytic surface in M. We ask now whether $\overline{\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})} \setminus \pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$ coincides with S (compare with (2.21)). To this aim, we need an assumption, analogous to (4.1), in order to avoid components of $\overline{\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})}$ touching $\partial\Omega$; roughly speaking, we have to show that "long thin" hairs reaching the boundary of Ω cannot occur in a constrained double−covering solution.

Theorem 4.3 (**Attaining the boundary condition**)**.** *Let $2 \leq n \leq 7$. Let $\bar r > 0$ be such that $S \subset B_{\bar r}$. There exists $R > \bar r$ such that, if $\Omega \supset B_R$, then any minimizer $u_{\min} \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; \{\pm 1\})$ of problem* (2.2) *satisfies*

$$\overline{\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})} \setminus \pi_{\mathbf{\Sigma},M}(J_{u_{\min}}) = S. \tag{4.4}$$

Proof. Fix $\mathbf{\Sigma} = (\Sigma, \Sigma') \in \mathbf{Cuts}(B_{\bar r}, S) \subset \mathbf{Cuts}(\Omega, S)$. Set

$$\mathscr{A}(r) := \mathscr{A}^{B_r}_{\mathrm{constr}}(S), \qquad r \geq \bar r,$$

and let $u_r \in BV_{\mathrm{constr}}(Y^r_{\mathbf{\Sigma}}; \{\pm 1\})$ be a minimizer of problem (2.2) for $\Omega = B_r$; here, $Y^r_{\mathbf{\Sigma}}$ denotes the double covering space of the base set $B_r \setminus S$.

By (2.5), $\mathscr{A}(\cdot)$ is nondecreasing; in addition, it is bounded (see (2.7)). Set $\epsilon := 4\,\mathcal{H}^{n-1}(\Sigma) - \mathscr{A}(\bar r) \geq 0$, so that by (2.7),

$$\mathscr{A}(r) - \mathscr{A}(\bar r) \leq \epsilon, \qquad r \geq \bar r. \tag{4.5}$$

Write $\mathscr{A}(r) = 4\mathcal{H}^{n-1}(\pi_{\mathbf{\Sigma},M}(J_{u_r}) \cap B_{\bar r}) + 4t(r,\bar r)$, where $t(r,\bar r) := \mathcal{H}^{n-1}(\pi_{\mathbf{\Sigma},M}(J_{u_r}) \setminus B_{\bar r})$. Since $\pi_{\mathbf{\Sigma},M}(J_{u_r}) \cap B_{\bar r}$ is a competitor for the computation of $\mathscr{A}(\bar r)$, we have

$$\begin{aligned}\mathscr{A}(\bar r) \leq& 4\,\mathcal{H}^{n-1}(\pi_{\mathbf{\Sigma},M}(J_{u_r}) \cap B_{\bar r}) \\ \leq& 4\,\mathcal{H}^{n-1}(\pi_{\mathbf{\Sigma},M}(J_{u_r}) \cap B_{\bar r}) + 4\,t(r,\bar r) = \mathscr{A}(r).\end{aligned} \tag{4.6}$$

Coupling (4.5) and (4.6), we get

$$4t(r,\bar r) \leq \mathscr{A}(r) - \mathscr{A}(\bar r) \leq \epsilon, \qquad r \geq \bar r. \tag{4.7}$$

Suppose $\epsilon = 0$. Then, by (4.7), we have $\mathcal{H}^{n-1}(\pi_{\mathbf{\Sigma},M}(J_{u_r}) \setminus B_{\bar r}) = 0$, which, by the assumption $2 \leq n \leq 7$ and Theorem 2.21, implies that the constrained double–covering solution does not reach ∂B_r, for any $r > \bar r$. Then the statement follows, taking an arbitrary $R > \bar r$.
Suppose $\epsilon > 0$, and let $r > \bar r$ be such that $\left(\overline{\pi_{\mathbf{\Sigma},M}(J_{u_r})} \setminus B_{\bar r}\right) \cap \partial B_r \neq \emptyset$. By the assumption $2 \leq n \leq 7$ and Theorem 2.21, there exists $x \in (\pi_{\mathbf{\Sigma},M}(J_{u_r}) \setminus B_{\bar r}) \cap \partial B_{(r+\bar r)/2}$. Take $\delta \in (0, (r-\bar r)/2)$. By the lower density estimate for local minimizers of the perimeter functional (see for instance [17, Theorem 21.11]), we have

$$c_n \delta^{n-1} \leq 4\mathcal{H}^{n-1}(\pi_{\mathbf{\Sigma},M}(J_{u_r}) \cap B_\delta(x)) \leq 4t(r,\bar r) \leq \epsilon, \tag{4.8}$$

for some positive constant c_n depending only on n. Inequalities (4.8) hold for each $\delta \in (0, (r-\bar r)/2)$; this is possible only if $r \leq r_\epsilon := \bar r + 2(\epsilon/c_n)^{\frac{1}{n-1}}$. Hence, taking $R > r_\epsilon$, the assertion follows. $\square$

Now, we compare the constrained double–covering solutions with other classical notions of solutions to Plateau's problem.

Remark 4.4 (**Area-minimizing currents**). Let $n = 3$, and assume that Ω contains the closed convex envelope of S. Let $T_{\min}$ be a rectifiable two-current [10] solving Plateau's problem with boundary S in the sense of currents. By [19, Theorem 5.6], the support of $T_{\min}$ is contained in Ω; moreover, by [14], it is an embedded, orientable smooth surface $\Sigma_{\min} \subset \Omega$ up to the boundary S. In particular, $\Sigma_{\min} \in \mathrm{Cuts}(\Omega, S)$. Hence, by (2.7)

$$\mathscr{A}^{\Omega}_{\mathrm{constr}}(S) \leq 4\,\mathbf{M}(T_{\min}), \tag{4.9}$$

where $\mathbf{M}(T_{\min})$ is the mass of $T_{\min}$.

It is worth noticing that there is not an absolute positive constant $c \in (0, 4]$, satisfying

$$\mathscr{A}^{\Omega}_{\mathrm{constr}}(S) \geq c\,\mathbf{M}(T_{\min}) \tag{4.10}$$

for any S. As a counterexample, let $\widehat{B_1} := \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 < 1, x_3 = 0\}$, let $S := \partial \widehat{B_1}$, and, for $\epsilon > 0$, let $\Omega := (1+\epsilon)\widehat{B_1} \times (-2, 2)$. As admissible pair of cuts, we take as Σ the closure of $\widehat{B_1}$, and $\Sigma' := \{x \in \mathbb{R}^3 \ : \ x_1^2 + x_2^2 \leq 1, \ x_3 = -\sqrt{1 - x_1^2 - x_2^2}\}$. Now, let $v \in BV(\Omega; \{\pm 1\})$ be defined as $v(x_1, x_2, x_3) := 1$ if $x_3 > 0$, and -1 elsewhere. Finally, let $u \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; \{\pm 1\})$ be the constrained lift of v. Then, recalling (2.12), it is immediate to verify that

$$\mathscr{A}^{\Omega}_{\mathrm{constr}}(S) \leq \frac{|Du|(Y_{\mathbf{\Sigma}})}{4} = \pi\big((1+\epsilon)^2 - 1\big) \to 0 \quad \text{as } \epsilon \to 0^+.$$

At the same time, the minimal mass in the sense of currents is π (the area of $\widehat{B}_1$), independently of ϵ.

Another (not rigorous but more intuitive) example of the failure of inequality (4.10) can be obtained taking as S the boundary of a very thin Möbius band: in this case a surface similar to the Möbius band is expected to be the double covering solution with boundary S, while the support of the minimal current is expected to be approximately a double disk.

Remark 4.5 (Disk-type area-minimizers). Let $n = 3$ and suppose that S is connected. Recalling (4.9) and the results in [21], [9], we have

$$\mathscr{A}^{\Omega}_{\mathrm{constr}}(S) \leq 4 \min\{\mathrm{area}(X) \ : \ X \in H^1(\mathrm{D}; \mathbb{R}^3), \ X \text{ spans } S\}, \tag{4.11}$$

where $\mathrm{D} \subset \mathbb{R}^2$ is the unit disk, $\mathrm{area}(X) := \int_{\mathrm{D}} |\partial_{x_1} X \wedge \partial_{x_2} X| \, dx_1 \, dx_2$, and the meaning of "$X$ spans S" is given for instance in [9]. We observe that (4.11) can be obtained independently of (4.9), by reproducing the proof of Theorems 2.7 and 4.6.

Now, we show that, when $n < 8$, constrained double−covering solutions give an equivalent way to solve Plateau's problem in the sense of integral currents modulo 2 [10].

Theorem 4.6 (Area-minimizing integral currents mod 2). *Let $2 \leq n \leq 7$, and let Ω be as in Theorem 4.3. Let $u_{\min} \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; \{\pm 1\})$ be a minimizer of problem (2.2). Then $\overline{\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})}$ can be seen as an integral current modulo 2 with boundary S, and $\mathscr{A}^{\Omega}_{\mathrm{constr}}(S)$ coincides with $4\mathbf{M}_2(T_{2,\min})$, where $\mathbf{M}_2$ is the mass and $T_{2,\min}$ is a mass-minimizing integral current modulo 2 having boundary S.*

Proof. By Theorems 2.21 and 4.3, $\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$ is an embedded analytic hypersurface satisfying (4.4). In particular, $\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$ can be considered as the support of an integral current modulo 2 having S as boundary support. This gives

$$\mathscr{A}^{\Omega}_{\mathrm{constr}}(S) = 4\mathcal{H}^{n-1}(\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})) \geq 4\,\mathbf{M}_2(T_{2,\min}).$$

The converse inequality follows from the interior regularity of minimal integral currents modulo 2 [25, Theorem 6.2.1] and Corollary 2.9, since the area-minimizing current mod 2 with boundary S belongs to $\mathrm{Cuts}(\Omega, S)$. $\square$

Remark 4.7. Let $n \geq 2$. Recalling Theorem 2.21 and Lemma 2.16, we have

$$\begin{aligned}\mathscr{A}^{\Omega}_{\mathrm{constr}}(S) \geq 4 \inf \Big\{ \mathcal{H}^{n-1}(K) \ : \ K \subset M \text{ rel. closed, } K \cap \rho(\mathbb{S}^1) \neq \emptyset \\ \text{for every } S\text{-simple link } \rho \in C(\mathbb{S}^1; M) \Big\},\end{aligned} \tag{4.12}$$

where, according to [12, p.4], ρ is said an *S-simple link* if $\mathrm{link}(\rho; C) = 1$ for some connected component C of S, and $\mathrm{link}(\rho; C') = 0$ for all connected components C' of $S \setminus C$.[13] The right hand side of (4.12) has been recently investigated in [12] and [8], for more general choices of S.

We notice that, in general, we cannot expect the inequality in (4.12) to be an equality. A counterexample, with $n = 2$ and $m = 6$, is obtained taking S as the set of (six) vertices of two triangles, as in Figure 6. Then the right hand side of (4.12) is attained by the union of G_1 and G_2, the two Steiner graphs corresponding to the triangles. On the other hand, by Theorem 4.1, $\mathscr{A}^{\Omega}_{\mathrm{constr}}(S)$ is strictly larger than $\mathcal{H}^1(G_1) + \mathcal{H}^1(G_2)$.

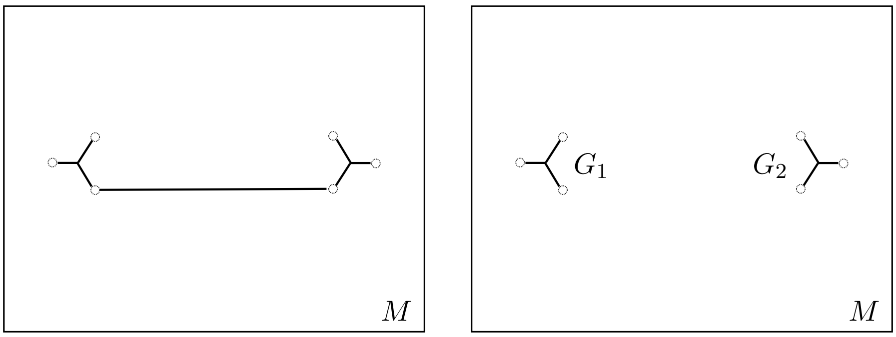

Figure 6: Let S be the set of vertices of two triangles, which are sufficiently far one from the other. In the left picture, the constrained covering solution is shown, in the case $\Omega = \mathbb{R}^2$. Notice that $\mathscr{A}^{\mathbb{R}^2}_{\mathrm{constr}}(S)$ is strictly larger than the length of the two Steiner graphs drawn in the right picture.

4.3 The tetrahedron

We end this section coming back to the m-sheeted covering construction given in Section 1, for a possible interesting extension in dimension $n = 3$. As for the case of minimal networks, Example 4.8 below shows that the covering construction has essentially to be chosen depending on the solution that one would like to obtain. In our present case, we aim to design a covering construction giving, possibly, the solution obtained by J. Taylor in [26].

[13] Here, $\mathrm{link}(\rho; C)$ denotes the linking number [15] between $\rho \in C(\mathbb{S}^1; M)$ and a boundaryless compact embedded Lipschitz $(n-1)$-dimensional submanifold $C \subset M$.

Example 4.8. Let $\mathrm{S} \subset \mathbb{R}^3$ be the one-skeleton of a regular tetrahedron T centered at 0 (here, Ω can be thought of as a large ball containing S). Referring to Figure 7, let us denote by F_j the (closed) facet of T opposite to the vertex p_j, for $j = 1, 2, 3$. We now aim to define a 4-sheeted, "cut and paste" covering of $\mathrm{M} := \Omega \setminus \mathrm{S}$ following the procedure described in Section 1.1. To this aim, we take as family $\mathrm{Cuts}(\Omega, \mathrm{S})$ of admissible cuts the collection of all $\Sigma = \cup_{j=1}^3 \Sigma_j \subset \Omega$ such that:

- for $j = 1, 2, 3$, Σ_j is a 2-dimensional compact embedded Lipschitz submanifold, having the edges of F_j as topological boundary;
- for $j, l = 1, 2, 3$, $j \neq l$, $\Sigma_j \cap \Sigma_l$ equals the intersection of the topological boundaries of F_j and F_l.

Clearly, the easiest example of an element of $\mathrm{Cuts}(\Omega, \mathrm{S})$ is given by $\cup_{j=1}^3 F_j$. Then, we select the family $\mathbf{Cuts}(\Omega, \mathrm{S})$ of admissible pairs of cuts as the collection of all pairs $\mathbf{\Sigma} := (\Sigma, \Sigma')$ such that $\Sigma, \Sigma' \in \mathrm{Cuts}(\Omega, \mathrm{S})$, and $\Sigma \cap \Sigma' = \mathrm{S}$; moreover, as in Definition 1.2-(ii), we require Σ to "lie on one side" of Σ' locally around S.

Fix now $\mathbf{\Sigma} = (\Sigma, \Sigma') \in \mathbf{Cuts}(\Omega, \mathrm{S})$. Then, the covering $(Y_{\mathbf{\Sigma}}, \pi_{\mathbf{\Sigma},\mathrm{M}})$ is obtained identifying four copies of the open sets $D := \Omega \setminus \Sigma$, $D' := \Omega \setminus \Sigma'$ as in (1.5) (with the choice $m = 4$). Namely, assuming for simplicity $\Sigma = \cup_{j=1}^3 F_j$, crossing the facet F_j coming from $\Omega \setminus T$ (resp. from T) corresponds to moving j-sheets forward (resp. backward) in the covering, for $j = 1, 2, 3$. Finally, the minimization problem can be set up as in Section 2; here, $V := \{\alpha_1, \dots, \alpha_4\} \subset \mathbb{R}^3$ is the set of vertices of a regular tetrahedron centered at 0 (not necessarily equal to T). Existence of minimizers in the class $BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V)$ follows by adapting the arguments in Theorem 2.15.[14] Concerning regularity of minimizers, and referring to [1] for the notion of $(1, \delta)$-restricted sets, we can state the following result.

Proposition 4.9. *Let $u_{\min} \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V)$ be a minimizer of problem* (2.2)*. Let $x \in \pi_{\mathbf{\Sigma},\mathrm{M}}(J_{u_{\min}})$, and let $r > 0$ be such that $B_r(x) \subseteq \Omega$. Then $\pi_{\mathbf{\Sigma},\mathrm{M}}(J_{u_{\min}}) \cap B_r(x)$ is $(1, \delta)$-restricted with respect to $\overline{\Omega} \setminus B_r(x)$, for any $\delta \in (0, r)$.*

Proof. Fix a perturbation $\varphi \in \mathrm{Lip}(\Omega; \Omega)$ of the identity, compactly supported in $B_r(x)$. Using the same construction as in [4, Theorem 2], we define a function $v^* \in BV(\Omega; V)$ such that

$$v^* = v_1(u_{\min}) \quad \text{outside } B_r(x), \qquad J_{v^*} \cap B_r(x) = \varphi(J_{v_1(u_{\min})} \cap B_r(x)). \tag{4.13}$$

Let $\tau \in \mathcal{T}(V)$ be such that $v_j(u_{\min}) = \tau^{j-1} \circ v_1(u_{\min})$, for $j = 2, 3, 4$. Then, we define $u^* \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V)$ as the τ-constrained lift of v^*. The statement now follows recalling [4, Corollary 1], (4.13), and using the minimality of $u_{\min}$. $\square$

Assume that there exists $r > 0$ such that $\mathrm{dist}(\pi_{\mathbf{\Sigma},\mathrm{M}}(J_{u_{\min}}), \partial\Omega) > r$. Then, as a consequence of Proposition 4.9, and by the general theory of Almgren's minimal sets [1], we get that $\pi_{\mathbf{\Sigma},\mathrm{M}}(J_{u_{\min}})$ is $(\mathbf{M}, 0, r)$-minimal. Figure 7 represents a minimizer $u_{\min} \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V)$ of problem (2.2), where $\Sigma := \cup_{j=1}^3 F_j$, and Σ' lies in $\Omega \setminus T$ (the cut Σ' is not drawn in the picture). More precisely, let $v \in BV(\Omega; V)$ be such that: $v := \alpha_1$ in $\Omega \setminus T$ and in the tetrahedron with vertices $0, p_1, p_2, p_3$; $v := \alpha_2$ in

[14] It is possible to check that, up to a homeomorphism, the covering construction is independent of the chosen admissible pair of cuts.

the tetrahedron with vertices $0, p_1, p_2, p_4$; $v := \alpha_3$ in the tetrahedron with vertices $0, p_1, p_3, p_4$; $v := \alpha_4$, in the tetrahedron with vertices $0, p_2, p_3, p_4$. Let also $\tau \in \mathcal{T}(V)$ be the transposition such that $\tau(j) := j + 1$, for $j = 1, 2, 3$. Then, $u_{\min}$ is defined as the τ-constrained lift of v, recall Definition 2.10. By construction, u does not jump on the fiber of the facets F_j's. Notice that $\pi_{\mathbf{\Sigma},\mathrm{M}}(J_{u_{\min}}) = \mathrm{int}(T) \cap C$, where C is the (infinite) cone over S, and it coincides with the solution provided by [26].

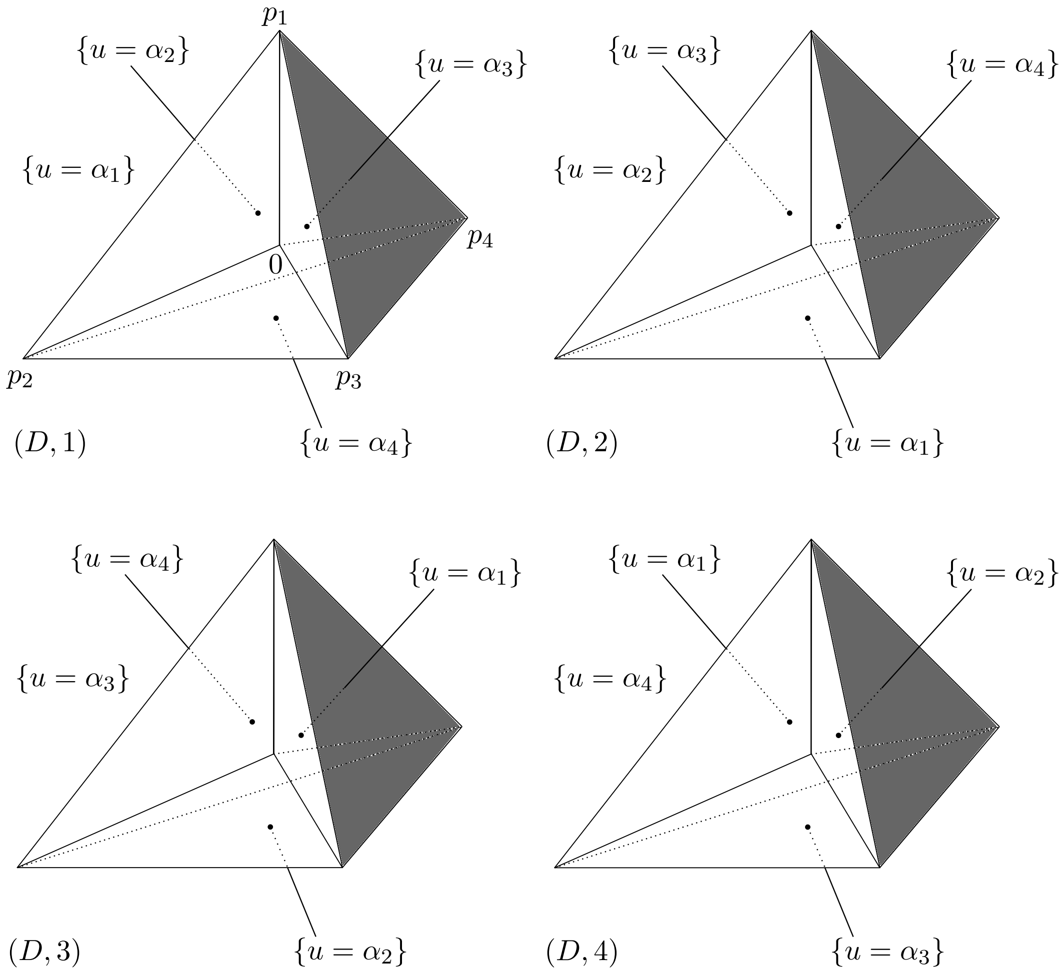

Figure 7: A minimizer $u \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; V)$, when S is the one-skeleton of a regular tetrahedron centered at 0. The picture refers to the choice $\Sigma := \cup_{j=1}^{3} F_j$. The copies of the facet F_2 have been coloured in grey to denote that they have been removed from the covering sheets drawn in the figure.

A Appendix: an abstract covering construction

In this appendix we give an alternative construction of the covering of M built up in Section 1. The construction is standard (see, *e.g.*, [13], [16]), and has the advantage to avoid all issues about the definition of admissible cuts. Setting up the minimization problem on the covering space M_H below could have an independent interest; we have preferred to use the "cut and paste" construction (and next proving independence of the cuts) in order to deal with a more "handy" formula (like (2.12)) for the total variation of a BV function defined on the covering space.

Let Ω, S, M, and m be as in Section 1. Fix $x_0 \in M$, and set $C_{x_0}([0,1];M) := \{\gamma \in C\big([0,1];M\big) \ : \ \gamma(0) = x_0\}$. For $\gamma \in C_{x_0}([0,1];M)$, let $[\gamma]$ be the class of paths in $C_{x_0}([0,1];M)$ which are homotopic to γ with fixed endpoints. We recall that the universal covering of M is the pair $(\widetilde{M}, \mathfrak{p})$, where $\widetilde{M} := \big\{[\gamma] \ : \ \gamma \in C_{x_0}([0,1];M)\big\}$ and $\mathfrak{p} \colon [\gamma] \in \widetilde{M} \mapsto \mathfrak{p}([\gamma]) := \gamma(1) \in M$. A basis for the topology of $\widetilde{M}$ is given by the family $\{[\gamma\lambda] : [\gamma] \in \widetilde{M},\, \gamma(1) \in B \text{ open ball},\, \lambda \in C([0,1];B),\, \lambda(0) = \gamma(1)\}$.

Let $\pi_1(M, x_0)$ be the first fundamental group of M with base point $x_0 \in M$, and let

$$H := \{[\rho] \in \pi_1(M, x_0) \ : \ \mathrm{link}(\rho; S) \equiv 0 \ (\mathrm{mod}\ m)\}.$$

Remark A.1. H is a (normal) subgroup of $\pi_1(M, x_0)$ of index m.

For $\gamma \in C_{x_0}([0,1];M)$, set $\bar{\gamma}(t) := \gamma(1-t)$ for all $t \in [0,1]$. Associated with H, we can consider the following equivalence relation $\sim_H$ on $\widetilde{M}$: for $[\gamma], [\lambda] \in \widetilde{M}$,

$$[\gamma] \sim_H [\lambda] \iff \gamma(1) = \lambda(1), \quad \mathrm{link}(\gamma\bar{\lambda}; S) \equiv 0 \ (\mathrm{mod}\ m).$$

We denote by $[\gamma]_H$ the equivalence class of $[\gamma] \in \widetilde{M}$ induced by $\sim_H$, and we set

$$M_H := \widetilde{M} / \sim_H .$$

Letting $\widetilde{\mathfrak{p}}_H \colon \widetilde{M} \to M_H$ be the projection induced by $\sim_H$, we endow M_H with the corresponding quotient topology. We set $\mathfrak{p}_{H,M} \colon [\gamma]_H \in M_H \mapsto \gamma(1) \in M$, so that we have the following commutative diagram

$$\begin{tikzcd} \widetilde{M} \arrow[r, "\widetilde{\mathfrak{p}}_H"] \arrow[dr, "\mathfrak{p}"'] & M_H \arrow[d, "\mathfrak{p}_{H,M}"] \\ & M \end{tikzcd} \tag{A.1}$$

and the pair $(M_H, \mathfrak{p}_{H,M})$ is a covering of M, see [13, Proposition 1.36].

Let (Y, π_Y) be a covering of M, and let $y_0 \in \pi_Y^{-1}(x_0)$. By $(\pi_Y)_* \colon \pi_1(Y, y_0) \to \pi_1(M, x_0)$ we denote the homomorphism defined as $(\pi_Y)_*([\varrho]) := [\pi_Y \circ \varrho]$. By [13, Proposition 1.36], we have

$$(\mathfrak{p}_{H,M})_*(\pi_1(M_H, [x_0]_H)) = H. \tag{A.2}$$

Proposition A.2. *Let $\mathbf{\Sigma} \in \mathrm{Cuts}(\Omega, S)$. Then $Y_{\mathbf{\Sigma}}$ and M_H are homeomorphic.*

Proof. Recall the notation in Section 1.1. By [13, p. 28], it is not restrictive to assume that $x_0 \in O$. Let $y_0 \in \pi_{\mathbf{\Sigma},M}^{-1}(x_0)$, and let $[\varrho] \in \pi_1(Y_{\mathbf{\Sigma}}, y_0)$. By [13, Proposition 1.36], since H and $(\pi_{\mathbf{\Sigma},M})_*\big(\pi_1(Y_{\mathbf{\Sigma}}, y_0)\big)$ have the same index, the statement follows if we are able to prove that $(\pi_{\mathbf{\Sigma},M})_*([\varrho]) \in H$, or equivalently that

$$\mathrm{link}(\pi_{\mathbf{\Sigma},M} \circ \varrho; S) \equiv 0 \ (\mathrm{mod}\ m), \tag{A.3}$$

where m is given in (1.2).

Let us first consider the case $n = 2$. Notice that

$$\mathrm{link}(\pi_{\mathbf{\Sigma},M} \circ \varrho; S) = \sum_{j=1}^{m} \mathrm{link}(\pi_{\mathbf{\Sigma},M} \circ \varrho; p_j), \tag{A.4}$$

and, for any $j = 1, \ldots, m$, $\mathrm{link}(\pi_{\mathbf{\Sigma},M} \circ \varrho; p_j)$ equals the number of times that $\pi_{\mathbf{\Sigma},M} \circ \varrho$ turns around p_j, a counterclockwise (resp. clockwise) turn around p_j being counted with positive (resp. negative) sign. By construction (see for instance Figure 3 when $m = 3$), any counterclockwise (resp. clockwise) turn of $\pi_{\mathbf{\Sigma},M} \circ \varrho$ around a point in S corresponds to moving one sheet forward (resp. backward) in $Y_{\mathbf{\Sigma}}$. Thus, the sum in the right hand side of (A.4) is equal to the number of sheets visited by the loop ϱ until it comes back to y_0. It is now clear that this number can be only a multiple of m, proving (A.3).

The case $n > 2$ is even simpler, since we have $m = 2$, and (A.3) follows noticing that $[\varrho]$ can change sheet in $Y_{\mathbf{\Sigma}}$ just an even number of times. $\square$

Let $\mathbf{\Sigma}$, $\widehat{\mathbf{\Sigma}} \in \mathbf{Cuts}(\Omega, S)$. By Proposition A.2, and by general results in coverings theory [13], there exists a homeomorphism $f \colon Y_{\mathbf{\Sigma}} \to Y_{\widehat{\mathbf{\Sigma}}}$ such that

$$\pi_{\mathbf{\Sigma},M} = \pi_{\widehat{\mathbf{\Sigma}},M} \circ f. \tag{A.5}$$

The map f is defined by path-lifting. More precisely, fix $x_0 \in M$, and let $y_0 \in Y_{\mathbf{\Sigma}}$, $\widehat{y}_0 \in Y_{\widehat{\mathbf{\Sigma}}}$ be such that $\pi_{\mathbf{\Sigma},M}(y_0) = x_0 = \pi_{\widehat{\mathbf{\Sigma}},M}(\widehat{y}_0)$. Let $y \in Y_{\mathbf{\Sigma}}$, and let $\gamma \in C([0,1]; Y_{\mathbf{\Sigma}})$ be such that $\gamma(0) = y_0$, $\gamma(1) = y$. Then, $f(y) \in Y_{\widehat{\mathbf{\Sigma}}}$ is defined as the ending point of the lift of $\pi_{\mathbf{\Sigma},M} \circ \gamma$ to $Y_{\widehat{\mathbf{\Sigma}}}$, starting at $\widehat{y}_0$.

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