On the Lagrangian structure of transport equations:  
the Vlasov-Poisson system

Luigi Ambrosio  * Maria Colombo † Alessio Figalli ‡

Abstract

The Vlasov-Poisson system is a classical model in physics used to describe the evolution of particles under their self-consistent electric or gravitational field. The existence of classical solutions is limited to dimensions $d \leq 3$ under strong assumptions on the initial data, while weak solutions are known to exist under milder conditions. However, in the setting of weak solutions it is unclear whether the Eulerian description provided by the equation physically corresponds to a Lagrangian evolution of the particles. In this paper we develop several general tools concerning the Lagrangian structure of transport equations with non-smooth vector fields and we apply these results: (1) to show that weak solutions of Vlasov-Poisson are Lagrangian; (2) to obtain global existence of weak solutions under minimal assumptions on the initial data.

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*Scuola Normale Superiore, Pisa. email: luigi.ambrosio@sns.it
†Scuola Normale Superiore, Pisa. email: maria.colombo@sns.it
‡University of Texas at Austin. email: figalli@math.utexas.edu

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5 The superposition principle under local integrability bounds on the velocity

1 Introduction

The $d$-dimensional Vlasov-Poisson system describes the evolution of a nonnegative distribution function $f : (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ according to Vlasov’s equation, under the action of a self-consistent force determined by the Poisson’s equation:

$$
\begin{align*}
\partial_t f_t + v \cdot \nabla_x f_t + E_t \cdot \nabla_v f_t &= 0 \quad \text{in } (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \\
\rho_t(x) &= \int_{\mathbb{R}^d} f_t(x, v) \, dv \quad \text{in } (0, \infty) \times \mathbb{R}^d \\
E_t(x) &= \sigma c_d \int_{\mathbb{R}^d} \rho_t(y) \frac{x - y}{|x - y|^d} \, dy \quad \text{in } (0, \infty) \times \mathbb{R}^d.
\end{align*}
$$

(1.1)

Here $f_t(x, v)$ stands for the density of particles having position $x$ and velocity $v$ at time $t$, $\rho_t(x)$ is the distribution of particles in the physical space, $E_t = -\sigma \nabla(\Delta^{-1} \rho_t)$ is the force field, $c_d > 0$ is a dimensional constant chosen in such a way that $c_d \operatorname{div} \left( \frac{x}{|x|^d} \right) = \delta_0$, and $\sigma \in \{\pm 1\}$. The case $\sigma = 1$ corresponds to the case of electrostatic forces between charged particles with the same sign (repulsion) while $\sigma = -1$ corresponds to the gravitational case (attraction).

This system appears in several physical models. For instance, when $\sigma = 1$ it describes in plasma physics the evolution of charged particles under their self-consistent electric field, while when $\sigma = -1$ the same system is used in astrophysics to describe the motion of galaxy clusters under the gravitational field. Many different models have been developed in connection with the Vlasov-Poisson equation: amongst others, we mention the relativistic version of (1.1) (where the velocity of particles is given by $v/\sqrt{1 + |v|^2}$) and the Vlasov-Maxwell system (which takes into account both the electric and magnetic fields of the Maxwell equations).

Regarding the existence of classical solutions, namely, solutions where all the relevant derivatives exist, the first contributions were given by Iordanskii [23] for the existence of solutions in dimension 1, by Ukai and Okabe [32] in dimension 2, and by Bardos and Degond [6] in dimension 3 for small data. For symmetric initial data, more existence results have been proven in [7, 33, 20, 31] (see also the presentation in [30] for an overview of the topic and the references quoted therein). Finally, in 1989 Pfaffelmoser [29] and Lions and Perthame [26] were able to prove global existence of classical solutions starting from general data. In [26] the authors consider an initial datum $f_0 \in L^1 \cap L^\infty(\mathbb{R}^6)$ with finite moments $|v|^m f_0(x, v) \in L^1(\mathbb{R}^6)$ for some $m > 3$, and, thanks to an a priori estimate on the propagation of moments, they show the existence of a distributional $f \in C((0, \infty); L^p(\mathbb{R}^6)) \cap L^\infty((0, \infty); L^\infty(\mathbb{R}^6))$ for every $1 \leq p < \infty$. Moreover, in [26] the problem of uniqueness is also addressed; under more restrictive assumptions on the initial datum, the authors show that there is uniqueness in the class of solutions with bounded space densities in $[0, \infty) \times \mathbb{R}^3$. Uniqueness is achieved by considering the Lagrangian flow associated to the vector field.
\( \mathbf{b}_t(x, v) := (v, E_t(x)) \), which is regular enough under a global bound on the space density (see also [27] for a different proof based on stability in the Wasserstein metric).

As one can see, the above results require strong assumptions on the initial data. However, it would be very desirable to get global existence of solutions under much weaker conditions. In the classical paper [5], Arsen’ev proved global existence of weak solutions under the assumption that the initial datum is bounded and has finite kinetic energy (see also [22]). This result has then been improved in [21], where the authors relaxed the boundedness assumption on an \( L^p \) bound for some suitable \( p > 1 \).

Notice that these higher integrability assumptions are needed even to give a meaning to the equation in the distributional sense: indeed, when \( f_t \) is merely \( L^1 \) the product \( E_tf_t \) does not belong to \( L^1_{\text{loc}} \) (when \( d = 3 \), for the term \( E_tf_t \) to belong to \( L^1_{\text{loc}} \) one needs to have \( f_t \in L^p \) with \( p \geq (12 + 2\sqrt{5})/11 \), see for instance [14]). To overcome this difficulty, in [14] the authors considered the concept of renormalized solutions and obtained global existence in the case \( \sigma = 1 \) under the assumption that the total energy is finite and \( f_0 \log(1 + f_0) \in L^1 \) (in the case \( \sigma = -1 \) they still need some \( L^p \) assumption on \( f \)). Also, under some suitable integrability assumptions on \( f_t \), they can show that the concepts of weak and renormalized solutions are equivalent.

It is important to observe that the Vlasov-Poisson system has a transport structure which allows one to prove that, when the solutions is sufficiently smooth, \( f_t \) is transported along the characteristics of the vector field \( \mathbf{b}_t(x, v) = (v, E_t(x)) \). However, when dealing with weak or renormalized solutions, it is not clear that such a vector field defines a flow on the phase-space and one loses the relation between the Eulerian and Lagrangian picture.

The goal of this paper is twofold: one the one hand we show that the Lagrangian picture is still valid even for weak/renormalized solutions, and secondly we obtain global existence of weak solutions under minimal assumptions on the initial data. Both results rely on a combination of the following tools:

(i) the local version of the DiPerna-Lions theory developed in [2];

(ii) the uniqueness of bounded compactly supported solutions to the continuity equation for a special class of vector fields obtained by convolving a singular kernel with a measure (this is based on the techniques developed in [11, 8], see Section 4.2);

(iii) the fact that the concept of Lagrangian solution is stronger than the one of renormalized solution (see Section 4.4);

(iv) a general superposition principle stating that every nonnegative solution of the continuity equation has a Lagrangian structure without any regularity or growth assumption on the vector field (see Section 5).

The above machinery is needed to prove a general result on the renormalization property for solutions of transport equations which is crucial in our proof. However, from a PDE viewpoint this renormalization property is all we shall need, so in order to keep the presentation as much as possible independent of this heavy machinery we shall organize the paper as follows: in the next section we state our results keeping the presentation on the Lagrangian structure of solutions at an informal level. Then in Sections 3.1 and 3.2 we prove our PDE results without introducing the tools mentioned above but simply using the consequences of them, and we postpone points (i)-(iv) above to Sections 4 and 5.
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2 Statement of the results

As already observed in the introduction, the Vlasov-Poisson system has a transport structure: indeed we can rewrite it as

$$\partial_t f_t + b_t \cdot \nabla_{x,v} f_t = 0,$$

(2.1)

where the vector field $b_t(x,v) = (v, E_t(x)) : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ is divergence-free, and is coupled to $f_t$ via the relation $E_t = \sigma c_d \rho_t \ast (x/|x|^d)$. Recalling that $c_d \text{div} \left( \frac{x}{|x|^d} \right) = \delta_0$, the vector field $E_t$ can also be found as $E_t = -\nabla_x V_t$ where the potential $V_t : (0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ solves

$$-\Delta V_t = \rho_t \quad \text{in } \mathbb{R}^d, \quad \lim_{|x| \to \infty} V_t(x) = 0.$$

(2.2)

Notice that, because the kernel $x/|x|^d$ is locally integrable, the electric field $E_t$ belongs to $L^1_{\text{loc}}(\mathbb{R}^d)$, therefore $b_t \in L^1_{\text{loc}}(\mathbb{R}^{2d})$.

Now, since $b_t$ is divergence-free, the above equation can be rewritten as

$$\partial_t f_t + \text{div}_{x,v}(b_t f_t) = 0,$$

and the equation can be reinterpreted in the distributional sense provided the product $b_t f_t$ belongs to $L^1_{\text{loc}}$. However, as mentioned before, this is not true if $f_t$ is merely $L^1$. To overcome this difficulty, one notices that if $f_t$ is a smooth solution of (2.1) then also $\beta(f_t)$ is a solution for all $C^1$ functions $\beta : \mathbb{R} \to \mathbb{R}$; indeed

$$\partial_t \beta(f_t) + b_t \cdot \nabla_{x,v} \beta(f_t) = \left[ \partial_t f_t + b_t \cdot \nabla_{x,v} f_t \right] \beta'(f_t) = 0,$$

or equivalently (since $\text{div}_{x,v}(b_t) = 0$)

$$\partial_t \beta(f_t) + \text{div}_{x,v}(b_t \beta(f_t)) = 0.$$

(2.3)

This motivates the introduction of the concept of renormalized solution [14]:

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This description is correct in dimension $d \geq 3$ since the fundamental solution of the Laplacian decays at infinity, while in dimension 2 $V_t$ is given by the convolution of $\rho_t$ with $-\frac{1}{d} \log |x|$. 

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Definition 2.1. A Borel function $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ is a renormalized solution of (2.1) (starting from $f_0$) if for every $\beta \in C^1 \cap L^\infty(\mathbb{R})$ we have that (2.3) holds in the sense of distributions, namely, for every $\phi \in C^\infty_c([0, T] \times \mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} \phi_0(x, v) \beta(f_0(x, v)) \, dx \, dv + \int_0^T \int_{\mathbb{R}^2} \left[ \partial_t \phi_t(x, v) + \nabla_{x,v} \phi_t(x, v) b_t(x, v) \right] \beta(f_t(x, v)) \, dx \, dv \, dt = 0. \quad (2.4)$$

Notice that because $\beta$ is bounded by assumption, $\beta(f_t) \in L^\infty$ so $b_t \beta(f_t) \in L^1_{\text{loc}}$ (recall that $b_t \in L^1_{\text{loc}}$) and (2.4) makes always sense.

This definition takes care of the integrability of the term $E_t f_t$ appearing in the equation. However a second problem comes when dealing with weak solutions: the vector field $b_t$ is not in general Lipschitz, so one cannot use the standard Cauchy-Lipschitz theory to construct a flow for such a vector field. In the seminal paper [17], DiPerna and Lions showed that, even for Sobolev vector fields, one can introduce a suitable notion of flow (this result has then been extended in several directions, see for instance [1, 13, 11]). However this theory requires the a priori assumption that the trajectories of the flow do not blow up in finite time, which is expressed in terms of the vector field by the following global hypothesis:

$$\frac{|b_t|(x, v)}{1 + |x| + |v|} \in L^1((0, T); L^1(\mathbb{R}^2)) + L^1((0, T); L^\infty(\mathbb{R}^d)). \quad (2.5)$$

We notice that for Vlasov-Poisson (or more in general for any Hamiltonian system where $b_t(x, v)$ is of the form $(v, -\nabla V_t(x))$) the above assumption is satisfied if and only if

$$E_t = -\nabla V_t \in L^1((0, T); L^\infty(\mathbb{R}^d; \mathbb{R}^d)).$$

Unfortunately this is a very restrictive assumption, as it requires both some integrability and moment (in $v$) conditions on $f_t$, so we cannot apply the classical DiPerna-Lions’ theory in this context.

In our recent paper [2] we developed a local version of the DiPerna-Lions’ theory under no global assumptions on the vector field, and this will be a crucial tool for us to give a Lagrangian description of solutions. More precisely, in Theorem 5.1 we shall first prove that every bounded nonnegative solution of a continuity equation can be always represented as a superposition of mass transported along integral curves of the vector field (notice that a priori these curves may split/intersect). Then, by a modification of the argument in [8], we shall prove that for any vector fields of the form $(v, \mu_t \ast x/|x|^d)$ with $\mu_t$ a time-dependent measure there is uniqueness of bounded compactly supported solutions of the continuity equation (see Theorem 4.5). Finally, combining these facts with the theory from [2] we can show that all bounded/renormalized solutions of Vlasov-Poisson are Lagrangian.

As mentioned before, to express the fact that solutions are Lagrangian we shall need to introduce the concept of Maximal Regular Flow. Roughly speaking, this is a (uniquely defined) incompressible flow on the phase-space composed of integral curves of $b_t$ that “transport” the density $f_t$ (notice that, since trajectories may blow-up in finite time, mass of
$f_t$ can disappear at infinity and/or come from infinity, but it has to follow the integral curves of $b_t$). However, since the definition is rather technical, in order to keep the presentation simpler we shall not introduce now the concept but postpone it to Section 4. This will leave the general reader with the intuitive concept of what is going on, and only the interested readers may decide to enter into the details of the definition and the proofs.

Our first main result shows that bounded or renormalized solutions of Vlasov-Poisson are Lagrangian. As shown in Theorem 4.11, the concept of Lagrangian solutions is stronger than the one of renormalized solutions, as all Lagrangian solutions of Vlasov-Poisson are renormalized. Here and in the sequel we shall use the notation $L^1_+$ to denote the space of nonnegative integrable functions. Also, by weakly continuous solutions we shall always mean that the map $t \mapsto \int_{\mathbb{R}^2d} f_t \varphi \, dx \, dv$ is continuous for any $\varphi \in C_c(\mathbb{R}^d)$.

**Theorem 2.2.** Let $T > 0$ and $f_t \in L^{\infty}((0,T); L^1_+((\mathbb{R}^2d)))$ be a weakly continuous, distributional solution of the Vlasov-Poisson equation (1.1). Assume that:
(i) either $f_t \in L^{\infty}((0,T); L^\infty(\mathbb{R}^2d)))$;
(ii) or $f_t$ is a renormalized solution.
Then $f_t$ is a Lagrangian solution transported by the Maximal Regular Flow associated to $b_t(x,v) = (v,E_t(x))$. In particular $f_t$ is renormalized.

Moreover, if $d \leq 4$ and $f_t \in L^{\infty}((0,T); L^q(\mathbb{R}^2d))$ with

$$ q = \begin{cases} 
\frac{23 + \sqrt{145}}{24} \approx 1.46 & \text{if } d = 2, \\
\frac{10 + \sqrt{37}}{7} \approx 2.30 & \text{if } d = 3, \\
13 + 3\sqrt{17} \approx 25.37 & \text{if } d = 4,
\end{cases} \quad (2.6)
$$

and the kinetic energy is integrable in time, that is

$$ \int_0^T \int_{\mathbb{R}^{2d}} |v|^2 f_t(x,v) \, dx \, dv \, dt < \infty, \quad (2.7) $$

then the flow is globally defined on $[0,T]$ (i.e., trajectories do not blow-up). In particular $f_t$ is the image of $f_0$ through an incompressible flow, hence, for all $\psi : [0,\infty) \to [0,\infty)$ Borel,

$$ [0,T] \ni t \mapsto \int_{\mathbb{R}^{2d}} \psi(f_t(x,v)) \, dx \, dv $$

is constant in time.

**Remark 2.3.** When $\sigma = 1$ the validity of (2.7) is guaranteed by the assumption

$$ \int_{\mathbb{R}^{2d}} |v|^2 f_0(x,v) \, dx \, dv + \int_{\mathbb{R}^d} |E_0(x)|^2 \, dx < \infty $$

(see also Corollary 2.6 below), while when $d = 3$ and $\sigma = -1$ one needs the additional hypothesis that $f_0 \in L^{9/7}(\mathbb{R}^d)$ (see [15, Equation (38)]). A similar result could also be given when $d = 2$ and $\sigma = -1$, but one would need to slightly change the form of the electric field (see Remark 2.7 below).
Our second result deals with existence of global Lagrangian solutions under minimal assumptions on the initial data. In this case the sign of $\sigma$ (i.e., whether the potential is attractive or repulsive) plays a crucial role, since in the repulsive case the total energy controls the kinetic part, while in the attractive case the loss of an a priori bound of the kinetic energy prevents us for showing such a result. However we can state a general existence theorem that holds both in the attractive and repulsive case, and then show that in the repulsive case gives us what we want.

The basic idea is the following: when proving existence of solutions by approximation it may happen that, in the approximating sequence, there are some particles that move at higher and higher speed while still remaining localized in a compact set in space (think of a family of particle rotating faster and faster along circles around the origin). Then, while in the limit these particles will disappear from the phase-space (having infinite velocity), the electric field generated by them will survive, since they are still in the physical space. Hence the electric field is not anymore generated by the marginal of the electric field generated by them, but by an “effective density” $\rho_{t_{\text{eff}}}(x)$ that is larger than $\rho_t(x)$.

So, our strategy will be first to prove global existence of Lagrangian (hence renormalized) solutions for a generalized Vlasov-Poisson system where the electric field is generated by $\rho_{t_{\text{eff}}}$ and then show that, in the particular case $\sigma = 1$, if the initial datum has finite total energy, then $\rho_{t_{\text{eff}}} = \rho_t$ and our solution solves the classical Vlasov-Poisson system.

We begin by introducing the concept of generalized solutions to Vlasov-Poisson. We use the notation $\mathcal{M}_+$ to denote the space of nonnegative measures with finite total mass.

**Definition 2.4** (Generalized solution of the Vlasov-Poisson equation). Given $\mathcal{F} \in L^1(\mathbb{R}^d)$, let $f_t \in L^\infty((0, \infty); L^1(\mathbb{R}^d))$ and $\rho_{t_{\text{eff}}} \in L^\infty((0, \infty); \mathcal{M}_+(\mathbb{R}^d))$. We say that the couple $(f_t, \rho_{t_{\text{eff}}})$, is a (global in time) generalized solution of the Vlasov-Poisson system starting from $\mathcal{F}$ if, setting

$$
\rho_t(x) := \int_{\mathbb{R}^d} f_t(x, v) \, dv, \quad E_{t_{\text{eff}}}(y) := \sigma c_d \int_{\mathbb{R}^d} \rho_{t_{\text{eff}}}(y) \frac{x - y}{|x - y|^d} \, dy, \quad b_t(x, v) := (v, E_{t_{\text{eff}}}(x)),
$$

$f_t$ is a renormalized solution of the continuity equation with vector field $b_t$ starting from $\mathcal{F}$,

$$
\rho_t \leq \rho_{t_{\text{eff}}} \quad \text{as measures for a.e. } t \in (0, \infty),
$$

and

$$
|\rho_{t_{\text{eff}}}(\mathbb{R}^d)| \leq |f_0|_{L^1(\mathbb{R}^{2d})} \quad \text{for a.e. } t \in (0, \infty).
$$

Notice that since $|\rho_t|_{L^1(\mathbb{R}^d)} = |f_t|_{L^1(\mathbb{R}^d)}$, it follows by (2.9) and (2.10) that whenever the mass of $f_t$ is conserved in time, that is $|f_t|_{L^1(\mathbb{R}^{2d})} = |f_0|_{L^1(\mathbb{R}^{2d})}$ for a.e. $t \in (0, \infty)$, then $\rho_{t_{\text{eff}}} = \rho_t$ and generalized solutions of the Vlasov-Poisson system are just standard distributional solutions.

We prove here that generalized solutions of the Vlasov-Poisson equation exist globally for any $L^1$ initial datum, both in the attractive and in the repulsive case.

**Theorem 2.5.** Let us consider $f_0 \in L^1_+(\mathbb{R}^{2d})$. Then there exists a generalized solution $(f_t, \rho_{t_{\text{eff}}})$ of the Vlasov-Poisson system starting from $f_0$. Moreover, $f_t$ is transported by the Maximal Regular Flow associated to $b_t(x, v) = (v, E_{t_{\text{eff}}}(x))$. 

As observed before, if $\rho_t^{\text{eff}} = \rho_t$ then $f_t$ is a renormalized solution of the Vlasov-Poisson system. When $\sigma = 1$ (i.e., in the repulsive case) the equality $\rho_t^{\text{eff}} = \rho_t$ is satisfied in many cases of interest, for instance whenever the initial energy is finite (namely $|v|^2 f_0 \in L^1(\mathbb{R}^{2d})$ and $E_0 \in L^2(\mathbb{R}^d)$, see Corollary 2.6 below), or in the case of infinite energy if other weaker conditions are satisfied as it happens in the context of [34] and [26] (see Remark 3.3).

The following result improves the result announced in [14], generalizing their statement to any dimension and with weaker conditions on the initial datum.

**Corollary 2.6.** Let $d \geq 3$, and let $f^0 \in L^1_+(\mathbb{R}^{2d})$ satisfy

$$\int_{\mathbb{R}^{2d}} |v|^2 f^0(x, v) \, dx \, dv + \int_{\mathbb{R}^d} H * \rho_0 \, dx < \infty,$$

where $\rho_0(x) := \int_{\mathbb{R}^d} f_0(x, v) \, dv$ and $H(x) := \frac{c_d}{d-2} |x|^{2-d}$. Assume that $\sigma = 1$. Then there exists a global Lagrangian (hence renormalized) solution of the Vlasov-Poisson system (1.1) with initial datum $f_0$.

**Remark 2.7.** In dimension $d = 2$, even with an initial datum $f_0 \in C^\infty_c(\mathbb{R}^d)$, the electric field $E_0$ cannot belong to $L^2$ (this is due to the fact that the kernel $x/|x|^d$ does not belong to $L^2$ at infinity) and therefore the initial potential energy, which coincides with $\|E_0\|_{L^2(\mathbb{R}^d)}^2$ is not finite. However, one can show that an analogous statement of Corollary 2.6 holds also for solutions of a slightly modified equation, which has the form

$$\begin{aligned}
\begin{cases}
\partial_t f_t + v \cdot \nabla x f_t + E_t \cdot \nabla v f_t &= 0 \\
\rho_t(x) &= \int_{\mathbb{R}^d} f_t(x, v) \, dv \\
E_t(x) &= \sigma c_d \int_{\mathbb{R}^d} \left( \rho_t(y) - \rho_b(y) \right) \frac{x - y}{|x - y|^d} \, dy
\end{cases}
\end{aligned}
$$

in $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$,

where $f_t$, $\rho_t$, $E_t$ play the same role as in the standard Vlasov-Poisson equation (1.1), and $\rho_b \in L^1_+(\mathbb{R}^d)$ represents a fixed background satisfying

$$\int_{\mathbb{R}^d} \rho_b(x) \, dx = \int_{\mathbb{R}^d} \rho_0(x) \, dx.$$

This allows for cancellations in the expression for the $L^2$ norm of $E_0$, which turns out to be finite if $\rho_b$ and $\rho_0$ are sufficiently nice.

**Remark 2.8.** In this paper we restricted ourselves to the Vlasov-Poisson equation but the argument and techniques introduced here generalize to other equations. For instance, a minor modification of our proofs allows one to obtain the same results in the context of the relativistic Vlasov-Poisson system.

The proofs of Theorems 2.2 and 2.5 and Corollary 2.6 are given in the next section.
3 Vlasov-Poisson: Lagrangian solutions and global existence

3.1 The flow associated to Vlasov-Poisson: proof of Theorem 2.2

Before proving the result, we recall a classical interpolation lemma (see for instance [14], where the lemma in the case α = 0 is stated).

**Lemma 3.1.** Let α ∈ [0, ∞), f ∈ L^1_{ loc}(R^d), and assume that f ∈ L^q(R^d) for some q ≥ 1 and that |v|^2 f ∈ L^1(R^d). Set \( p_α := \frac{d(q-1)+(α+2)q}{d(q-1)+α+2} \). Then \( \rho_α(x) := \int_{\mathbb{R}^d} \frac{f(x,v)}{(1+|v|)^α} dv \) belongs to \( L^{p_α}(\mathbb{R}^d) \) and there exists a constant \( C > 0 \), depending only on \( n, α \) and \( q \), such that

\[
\|\rho_α\|_{L^{p_α}(\mathbb{R}^d)} ≤ C\|\|v|^2 f\|_{L^1(\mathbb{R}^d)}^{θ_α} \|f\|_{L^q(\mathbb{R}^d)}^{1-θ_α}
\]

where \( θ_α ∈ [0, 1] \) is given by \( θ_α = \frac{d(q-1)}{d(q-1)+(2+α)q} \).

**Proof.** Assume for simplicity \( q < ∞ \). By Hölder inequality, for every \( x ∈ \mathbb{R}^d \) we estimate

\[
\rho_α(x) = \int_{\{v|v|<R\}} \frac{f(x,v)}{(1+|v|)^α} dv + \int_{\{v|v|≥R\}} \frac{f(x,v)}{(1+|v|)^α} dv
\]

\[
≤ R^{d(q-1)/q} \left( \int_{\mathbb{R}^d} f(x,v)^q dv \right)^{1/q} + \frac{1}{R^{2+α}} \int_{\mathbb{R}^d} |v|^2 f(x,v) dv.
\]

Minimizing the right-hand side in \( R \), for every \( x ∈ \mathbb{R}^d \) we deduce that

\[
\rho_α(x) ≤ \left( \int_{\mathbb{R}^d} f(x,v)^q dv \right)^{\frac{2+α}{(d(q-1)+(2+α)q)}} \left( \int_{\mathbb{R}^d} |v|^2 f(x,v) dv \right)^{\frac{d(q-1)}{(d(q-1)+(2+α)q)}}.
\]

Taking finally the \( L^{p_α} \)-norm of \( ρ_α \) and using Hölder inequality, we find the result. \( \square \)

We can now proceed with the proof of Theorem 2.2. Notice that the vector field \( b \) satisfies assumption (a) of Section 4.1 and is divergence-free. Also, by Theorem 4.5 it satisfies assumption (b). Therefore by Theorem 5.1 we deduce that \( f_t \) (resp. \( β(f_t) \) with \( β(s) = \arctan(s) \) if \( f_t \) is not bounded but is renormalized) is a Lagrangian solution. In particular Theorem 4.11 ensures that \( f_t \) is a renormalized solution.

Assume now that (2.7) holds and that \( f_t ∈ L^∞((0,T); L^q(\mathbb{R}^d)) \) with the choice of \( q \) given by (2.6). Let

\[
p = \frac{d(q-1)+2q}{d(q-1)+2} \quad \text{and} \quad r = \frac{d(q-1)+3q}{d(q-1)+3}.
\]

With this choice, the integrability exponent provided by Lemma 3.1 is precisely \( p \) if \( α = 0 \) and \( r \) if \( α = 1 \). In addition, by the choice of \( q \) in (2.6), we observe with some computations that for \( d = 2, 3, 4 \) the exponents \( p \) and \( r \) satisfy

\[
1 + \frac{1}{d} = \frac{1}{p} + \frac{1}{r}.
\]
In the rest of the proof we denote by $C$ any constant which depends only on $d$, on (2.7), and on the norm of $f_t$ in $L^q(\mathbb{R}^{2d})$.

Thanks to Lemma 3.1, applied with $\alpha = 0$ and $\alpha = 1$, we have that
\[
\|\rho_t\|_{L^p(\mathbb{R}^d)} \leq C\|v\|^2 f_t\|^{\theta_0}_{L^1(\mathbb{R}^{2d})} \|f_t\|^{1-\theta_0}_{L^q(\mathbb{R}^{2d})} \leq C
\]  
(3.1) and
\[
\|\eta_t\|_{L^r(\mathbb{R}^d)} \leq C\|v\|^2 f_t\|^{\theta_1}_{L^1(\mathbb{R}^{2d})} \|f_t\|^{1-\theta_1}_{L^q(\mathbb{R}^{2d})} \leq C,
\]
where $\eta_t := \int_{\mathbb{R}^d} f_t(x,v) dv \cdot \eta$. By (3.1), Sobolev inequality and Calderón-Zygmund estimates (see for instance [19, Corollary 9.10]) we deduce that, for a.e. $t \in (0,T)$,
\[
\|E_t\|_{L^{dp/(d-p)}(\mathbb{R}^d)} \leq C\|\nabla E_t\|_{L^p(\mathbb{R}^d)} \leq C\|\rho_t\|_{L^p(\mathbb{R}^d)} \leq C.
\]

Thus, noticing that our choice of $p$ and $r$ implies $\frac{r}{r-1} = \frac{dp}{d-p}$, thanks to Hölder’s inequality we deduce that, for every $t \in [0,T]$,
\[
\int_{\mathbb{R}^{2d}} |E_t(x)| f_t(x,v) \frac{1}{1+|v|} dx dv = \int_{\mathbb{R}^d} |E_t(x)| \eta_t(x) dx \leq \|E_t\|_{L^{r/(r-1)}(\mathbb{R}^d)} \|\eta_t\|_{L^{r'}(\mathbb{R}^d)} \leq C.
\]  
(3.2)

Integrating (2.7) and (3.2) with respect to time, we find that
\[
\int_0^T \int_{\mathbb{R}^{2d}} \frac{|b_t| f_t}{1 + (|x|^2 + |v|^2)^{1/2}} dx dv dt \leq \int_0^T \int_{\mathbb{R}^{2d}} f_t \frac{1}{1 + |v|} dx dv dt + \int_0^T \int_{\mathbb{R}^{2d}} |E_t| \eta_t \frac{f_t}{1 + |v|} dx dv dt < \infty.
\]

Hence, by the no blow-up criterion stated in Theorem 4.4, it follows that the Maximal Regular Flow $X(t,\cdot) : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ is globally defined on $[0,T]$, its trajectories $X(\cdot,x,v)$ belong to $AC([0,T];\mathbb{R}^{2d})$ for $f_0$-a.e. $(x,v) \in \mathbb{R}^{2d}$, and $f_t = X(t,\cdot) \# f_0 = f_0 \circ X(t,\cdot)^{-1}$. In particular, for all functions $\psi : [0,\infty) \to [0,\infty)$ Borel we have
\[
\int_{\mathbb{R}^{2d}} \psi(f_t) dx dv = \int_{\mathbb{R}^{2d}} \psi(f_0) \circ X(t,\cdot)^{-1} dx dv = \int_{\mathbb{R}^{2d}} \psi(f_0) dx dv,
\]
where the second equality follows by the incompressibility of the flow.

### 3.2 Global existence for Vlasov-Poisson

In this section we shall prove Theorem 2.5 and Corollary 2.6.

**Proof of Theorem 2.5.** To prove existence of global generalized Lagrangian solutions of Vlasov-Poisson we shall use an approximation procedure. Since the argument is rather long and involved, we divide the proof in five steps that we now describe briefly: In Step 1 we start from approximated solutions $f^n$, obtained by smoothing the initial datum and the kernel, and we decompose them along their level sets. Exploiting the incompressibility of the flow, these functions are still solutions of the continuity equation with the same vector
field and, when \( n \) varies, they are uniformly bounded. This allows us to take their limit as \( n \to \infty \) in Step 2, and show that the limit belongs to \( L^1 \). In Step 3 we introduce \( \rho^{\text{eff}} \) as the limit as \( n \to \infty \) of the approximated densities \( \rho^n \), and we motivate its properties. In Step 4 we show that the vector fields \( E^n \) converge to the vector field obtained by convolving \( \rho^{\text{eff}} \) with the Poisson kernel. Finally, in Step 5 we employ the stability results for the continuity equation and the results of Section 5 to take the limit in the approximated Vlasov-Poisson equation and show that the limiting solution is transported by the limiting incompressible flow. We now enter into the details of the proof.

**Step 1: approximating solutions.** Let \( K(x) := \sigma c_d x/|x|^d \) and let us consider approximating kernels \( K_n := K \ast \psi_n \), where \( \psi_n(x) = n^d \psi(nx) \) and \( \psi \in C_0^\infty(\mathbb{R}^d) \) is a standard convolution kernel in \( \mathbb{R}^d \). Let \( f_0^n \in C_0^\infty(\mathbb{R}^{2d}) \) be a sequence of functions such that

\[
f_0^n \to f_0 \quad \text{in } L^1(\mathbb{R}^{2d}).
\]

Let \( f_t^n \) be distributional solutions of the Vlasov system with initial datum \( f_0^n \) and kernel \( K_n \) (see [18] for this classical construction based on a fixed point argument in the Wasserstein metric, and [30]). For every \( t \in [0, \infty) \), let us denote by \( \rho_t^n = \int f_t^n \, dv \) and \( E_t^n = K_n \ast \rho_t^n \) the space distribution of particles and the field associated to \( f_t^n \). Notice that since \( K_n \) is smooth and decays at infinity, both \( E_t^n \) and \( \nabla E_t^n \) are bounded on \([0, \infty) \times \mathbb{R}^d \) (with a bound that depends on \( n \)). Hence \( b_t^n \) is a Lipschitz divergence-free vector field, and by standard theory for the transport equation we obtain that, for every \( t \in (0, \infty) \),

\[
f_t^n = f_t^n \circ X^n(t)^{-1}
\]

where \( X^n(t) : \mathbb{R}^d \to \mathbb{R}^d \) is the flow of the vector field \( b_t^n(x, v) = (v, E_t^n(x)) \), and

\[
\|\rho_t^n\|_{L^1(\mathbb{R}^d)} = \|f_t^n\|_{L^1(\mathbb{R}^{2d})} = \|f_0^n\|_{L^1(\mathbb{R}^{2d})}.
\]

Assuming without loss of generality that \( L^2(\{f_0 = k\}) = 0 \) for every \( k \in \mathbb{N} \) (otherwise we consider as level sets the values \( R + k \) in place of \( k \) for some \( R \in [0,1) \)), from (3.3) we deduce that

\[
f_0^{n,k} \to f_0^k := 1_{\{k \leq f_0^k < k+1\}} f_0 \quad \text{in } L^1(\mathbb{R}^{2d}).
\]

We then consider \( f_t^{n,k} := 1_{\{k \leq f_t^n < k+1\}} f_t^n \) for every \( k, n \in \mathbb{N} \), and by (3.4) we notice that, for every \( t \in (0, \infty) \),

\[
f_t^{n,k} = 1_{\{k \leq f_t^n < k+1\}} f_t^n \circ X^n(t)^{-1}
\]

is the image of \( f_0^{n,k} := 1_{\{k \leq f_0^n < k+1\}} f_0^n \) through the flow \( X^n(t) \), that \( f_t^{n,k} \) is a distributional solution of the continuity equation with vector field \( b_t^n(x, v) \), and that

\[
\|f_t^{n,k}\|_{L^1(\mathbb{R}^{2d})} = \|f_0^{n,k}\|_{L^1(\mathbb{R}^{2d})} \quad \text{for every } t \in (0, \infty).
\]

**Step 2: limit in the phase-space.** By construction the functions \( \{f_t^{n,k}\}_{n \in \mathbb{N}} \) are nonnegative and bounded by \( k+1 \) in \( L^\infty((0, \infty) \times \mathbb{R}^d) \), hence there exists \( f^k \in L^\infty((0, \infty) \times \mathbb{R}^d) \) nonnegative such that, up to subsequences,

\[
f^{n,k} \rightharpoonup f^k \quad \text{weakly* in } L^\infty((0, \infty) \times \mathbb{R}^d) \quad \text{for every } k \in \mathbb{N}.
\]
Moreover, for any $K$ compact subset of $\mathbb{R}^{2d}$ and any nonnegative function $\phi \in L^\infty(0, \infty)$ with compact support, using the test function $\phi(t)1_K(x, v)\text{sign}(f_k^t)(x, v)$ in the previous weak convergence, by Fatou’s Lemma, (3.8), and (3.6), we get

$$\int_0^\infty \phi(t)\|f_t^k\|_{L^1(K)} \, dt \leq \liminf_{n \to \infty} \int_0^\infty \phi(t)\|f_n^k\|_{L^1(K)} \, dt$$

$$\leq \liminf_{n \to \infty} \int_0^\infty \phi(t)\|f_n^k\|_{L^1(\mathbb{R}^{2d})} \, dt$$

$$= \liminf_{n \to \infty} \int_0^\infty \phi(t)\|f_0^k\|_{L^1(\mathbb{R}^{2d})} \, dt$$

$$= \int_0^\infty \phi(t)\|f_0^k\|_{L^1(\mathbb{R}^{2d})} \, dt.$$  \hfill (3.10)

Hence, taking the supremum among all compact subset $K \subset \mathbb{R}^{2d}$, this proves that

$$\|f_t^k\|_{L^1(\mathbb{R}^{2d})} \leq \|f_0^k\|_{L^1(\mathbb{R}^{2d})} \quad \text{for a.e. } t \in (0, \infty),$$

so, in particular, $f^k \in L^\infty((0, \infty); L^1(\mathbb{R}^{2d}))$.

Thanks to (3.11), we can define $f \in L^\infty((0, \infty); L^1(\mathbb{R}^{2d}))$ by

$$f := \sum_{k=0}^\infty f^k \quad \text{in } (0, \infty) \times \mathbb{R}^{2d},$$

where the global bound on the $L^1$-norm of $f_t$ comes from

$$\|f_t\|_{L^1(\mathbb{R}^{2d})} \leq \sum_{k=0}^\infty \|f_t^k\|_{L^1(\mathbb{R}^{2d})} \leq \sum_{k=0}^\infty \|f_0^k\|_{L^1(\mathbb{R}^{2d})} = \|f_0\|_{L^1(\mathbb{R}^{2d})}.$$  \hfill (3.13)

We now claim that, for every $T > 0$,

$$f^n \rightharpoonup f \quad \text{weakly in } L^1((0, T) \times \mathbb{R}^{2d}),$$

that is, for every $\varphi \in L^\infty((0, T) \times \mathbb{R}^{2d})$,

$$\lim_{n \to \infty} \int_0^T \int_{\mathbb{R}^{2d}} \varphi f^n \, dx \, dt = \int_0^T \int_{\mathbb{R}^{2d}} \varphi f \, dx \, dt.$$  \hfill (3.15)

Indeed, noticing that $f^n = \sum_{k=0}^\infty f_{n,k}$ and $f = \sum_{k=0}^\infty f^k$, by the triangle inequality we have that, for every $k_0 \geq 1$,

$$\left| \int_0^T \int_{\mathbb{R}^{2d}} \varphi(f^n - f) \, dx \, dv \, dt \right| = \left| \sum_{k=0}^\infty \int_0^T \int_{\mathbb{R}^{2d}} \varphi(f_{n,k} - f^k) \, dx \, dv \, dt \right|$$

$$\leq \left| \sum_{k=0}^{k_0-1} \int_0^T \int_{\mathbb{R}^{2d}} \varphi(f_{n,k} - f^k) \, dx \, dv \, dt \right|$$

$$+ \sum_{k=k_0}^\infty \int_0^T \int_{\mathbb{R}^{2d}} |\varphi||f_{n,k}| \, dx \, dv \, dt + \sum_{k=k_0}^\infty \int_0^T \int_{\mathbb{R}^{2d}} |\varphi||f^k| \, dx \, dv \, dt.$$
Now, let us consider any nonnegative function \( k \) and finally, letting \( n \to \infty \), we obtain that, for any \( f \),

\[
\lim \sup_{n \to \infty} \int T_0 \varphi (f^n - f) dxdvdt \leq \lim \sup_{n \to \infty} \sum_{k=0}^{k_0-1} \int \varphi (f^{n,k} - f^k) dxdvdt + 2T\|\varphi\|_\infty \|f_0\|_{L^1(\mathbb{R}^{2d})}.
\]

Notice that, thanks to (3.6) and (3.3), it follows that

\[
f_0^1\{f_0^0 \geq k_0\} \to f_0^1\{f_0 \geq k_0\} \quad \text{in } L^1(\mathbb{R}^{2d}),
\]

so by letting \( n \to \infty \) and using (3.9) we deduce that

\[
\lim \sup_{n \to \infty} \int T_0 \varphi (f^n - f) dxdvdt \leq \lim \sup_{n \to \infty} \sum_{k=0}^{k_0-1} \int \varphi (f^{n,k} - f^k) dxdvdt + 2T\|\varphi\|_\infty \|f_0^1\{f_0^0 \geq k_0\}\|_{L^1(\mathbb{R}^{2d})}.
\]

Finally, letting \( k_0 \to \infty \) we deduce (3.15), which proves the claim.

**Step 3: limit of physical densities.** Since by (3.5) the sequence \( \{\rho^n\}_{n \in \mathbb{N}} \) is bounded in \( L^\infty((0, \infty); \mathcal{M}_+ (\mathbb{R}^d)) \subset \left[L^1((0, \infty), C_0(\mathbb{R}^d))\right]^* \), there exists \( \rho^{\text{eff}} \in L^\infty((0, \infty); \mathcal{M}_+ (\mathbb{R}^d)) \) such that

\[
\rho^n \rightharpoonup \rho^{\text{eff}} \quad \text{weakly* in } L^\infty((0, \infty); \mathcal{M}_+ (\mathbb{R}^d)). \tag{3.16}
\]

Moreover, by the lower semicontinuity of the norm under weak* convergence, using (3.5) again we deduce that

\[
\text{ess sup}_{t \in (0, \infty)} |\rho_{t}^{\text{eff}}(\mathbb{R}^d)| \leq \lim_{n \to \infty} \left( \sup_{t \in (0, \infty)} \|\rho^n_t\|_{L^1(\mathbb{R}^d)} \right) = \lim_{n \to \infty} \|f_0^n\|_{L^1(\mathbb{R}^{2d})} = \|f_0\|_{L^1(\mathbb{R}^{2d})}. \tag{3.17}
\]

Now, let us consider any nonnegative function \( \varphi \in C_c((0, \infty) \times \mathbb{R}^d) \). By (3.16) and (3.14) we obtain that, for any \( R > 0 \),

\[
\int_0^\infty \int_{\mathbb{R}^d} \varphi_t(x) d\rho_t^{\text{eff}}(x) dt = \lim_{n \to \infty} \int_0^\infty \int_{\mathbb{R}^d} \rho^n_t(x) \varphi_t(x) dx dt
\]

\[
= \lim_{n \to \infty} \int_0^\infty \int_{\mathbb{R}^{2d}} f^n_t(x,v) \varphi_t(x) dv dx dt
\]

\[
\geq \lim_{n \to \infty} \int_0^\infty \int_{\mathbb{R}^d \times B_R} f^n_t(x,v) \varphi_t(x) dv dx dt
\]

\[
= \int_0^\infty \int_{\mathbb{R}^d \times B_R} f_t(x,v) \varphi_t(x) dv dx dt,
\]

13
Letting $\rho$ and that, for every ball $B$ that $R\phi$, by the arbitrariness of $\phi$ to identify the limit, we claim that for every $\rho \leq \rho_t^{\text{eff}}$ as measures for a.e. $t \in (0, \infty)$, (3.18)

as desired.

**Step 4: limit of vector fields.** Set $E_t^{\text{eff}} := K * \rho_t^{\text{eff}}$ and $b_t(x,v) := (v, E_t^{\text{eff}}(x))$. We claim that

$$b^n \rightharpoonup b$$ weakly in $L^1_{\text{loc}}((0, \infty) \times \mathbb{R}^d; \mathbb{R}^{2d})$ (3.19)

and that, for every ball $B_R \subset \mathbb{R}^d$,

$$[\rho^n_t * K_n](x+h) \to [\rho^n_t * K_n](x) \quad \text{as} \ |h| \to 0 \text{ in } L^1_{\text{loc}}((0, \infty); L^1(B_R)), \text{ uniformly in } n. \quad (3.20)$$

To show this we first prove that the sequence $\{b^n\}_{n \in \mathbb{N}}$ is bounded in $L^p_{\text{loc}}((0, \infty) \times \mathbb{R}^d; \mathbb{R}^{2d})$ for every $p \in [1, d/(d-1))$. Indeed, using Young’s inequality, for every $t \geq 0$, $n \in \mathbb{N}$, and $r > 0$,

$$\|\rho^n_t * K_n\|_{L^p(B_r)} = \|(\rho^n_t * \psi_n) * K\|_{L^p(B_r)}$$

$$\leq ||(\rho^n_t * \psi_n) * (K1_{B_1})||_{L^p(B_r)} + \|(\rho^n_t * \psi_n) * (K1_{\mathbb{R}^d \setminus B_1})||_{L^p(B_r)}$$

$$\leq ||(\rho^n_t * \psi_n) * (K1_{B_1})||_{L^p(\mathbb{R}^d)} + \mathcal{L}^d(B_r)^{1/p} ||(\rho^n_t * \psi_n) * (K1_{\mathbb{R}^d \setminus B_1})||_{L^\infty(\mathbb{R}^d)}$$

$$\leq \|\rho^n_t\|_{L^1(\mathbb{R}^d)} \|\psi_n\|_{L^1(\mathbb{R}^d)} \|K\|_{L^p(\mathbb{R}^d)} + \mathcal{L}^d(B_r)^{1/p} \|\rho^n_t\|_{L^1(\mathbb{R}^d)} \|\psi_n\|_{L^1(\mathbb{R}^d)} \|K\|_{L^\infty(\mathbb{R}^d \setminus B_1)}$$

hence, up to subsequences, the sequence $\{b^n\}_{n \in \mathbb{N}}$ converges locally weakly in $L^p$. In order to identify the limit, we claim that for every $\varphi \in C_c((0, \infty) \times \mathbb{R}^d)$

$$\lim_{n \to \infty} \int_0^\infty \int_{\mathbb{R}^d} \rho^n_t * K_n \varphi_t \, dx \, dt = \int_0^\infty \int_{\mathbb{R}^d} \rho_t^{\text{eff}} * K \varphi_t \, dx \, dt.$$ 

Indeed, by standard properties of convolution,

$$\left| \int_0^\infty \int_{\mathbb{R}^d} \rho^n_t * K_n \varphi_t \, dx \, dt - \int_0^\infty \int_{\mathbb{R}^d} \rho_t^{\text{eff}} * K \varphi_t \, dx \, dt \right|$$

$$= \left| \int_0^\infty \int_{\mathbb{R}^d} \rho^n_t \varphi_t * K_n \, dx \, dt - \int_0^\infty \int_{\mathbb{R}^d} \rho_t^{\text{eff}} \varphi_t * K \, dx \, dt \right|$$

$$\leq \left| \int_0^\infty \int_{\mathbb{R}^d} (\rho^n_t - \rho_t^{\text{eff}}) \varphi_t * K \, dx \, dt \right| + \left| \int_0^\infty \int_{\mathbb{R}^d} \rho_t^{\text{eff}} (\varphi_t * K - \varphi_t * K * \psi_n) \, dx \, dt \right|$$

$$\leq \left( \sup_{t \in (0, \infty)} \|\rho^n_t\|_{L^1(\mathbb{R}^d)} \right) \|\varphi_t * K - \varphi_t * K * \psi_n\|_{L^\infty((0, \infty) \times \mathbb{R}^d)}.$$
function, compactly supported in time and decaying at infinity in space. The second term, in turn, converges to 0 since the first factor is bounded by (3.17) and \( \varphi_t \ast K \ast \psi_n \) converges to \( \varphi_t \ast K \) uniformly in \((0, \infty) \times \mathbb{R}^d\).

This computation identifies the weak limit of \( \rho_t^n \ast K_n \) in \( L^1_{\text{loc}}([0, T] \times \mathbb{R}^d) \), showing that it coincides with \( \rho_t^{\text{eff}} \ast K \) and proving (3.19).

We now prove (3.20). First of all, since \( K \in W^{\alpha,p}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d) \) for every \( \alpha < 1 \) and \( p < n/(n-1+\alpha) \), using Young’s inequality we deduce that, for any \( t \in (0, \infty) \),

\[
\| \rho_t^n \ast K_n \|_{W^{\alpha,p}(B_R; \mathbb{R}^d)} = \| (\rho_t^n \ast \psi_n) \ast K \|_{W^{\alpha,p}(B_R; \mathbb{R}^d)} \leq C(R) \| \rho_t^n \ast \psi_n \|_{L^1(\mathbb{R}^d)}.
\]

Since \( \| \psi_n \|_{L^1(\mathbb{R}^d)} = 1 \), thanks to (3.5) we deduce that the last term is bounded independently of \( t \) and \( n \), that is, for every \( R > 0 \),

\[
\sup_{t \in (0, \infty)} \sup_{n \in \mathbb{N}} \| \rho_t^n \ast K_n \|_{W^{\alpha,p}(B_R; \mathbb{R}^d)} < \infty. \tag{3.21}
\]

Hence, by a classical embedding between fractional Sobolev spaces and Nikolsky spaces (see for instance [24, Lemma 2.3]) we find that, for \( |h| \leq R \),

\[
\int_{B_R} |\rho_t^n \ast K_n(x+h) - \rho_t^n \ast K_n(x)|^p \, dx \leq C(p, \alpha, R, \| \rho_t^n \ast K_n \|_{W^{\alpha,p}(B_{2R}; \mathbb{R}^d)}) |h|^\alpha p,
\]

from which (3.20) follows.

**Step 5: conclusion.** Thanks to (3.19) and (3.20), we can apply the stability result from [17, Theorem II.7] (which does not require any growth condition on the vector fields, see also [2, Proposition 6.5] for the stability of the associated flows) to deduce that, for every \( k \in \mathbb{N} \), \( f^k \) is a weakly continuous distributional solution of the continuity equation starting from \( f_0^k \). Since the continuity equation is linear, we deduce that also \( F^m := \sum_{k=1}^m f^k \) is a distributional solution for every \( m \in \mathbb{N} \).

Since \( F^m \) is bounded, Theorem 5.1 gives that \( F^m \) is a renormalized solution for every \( m \in \mathbb{N} \). Letting \( m \to \infty \), since \( F^m \to f \) strongly in \( L^1_{\text{loc}}((0, \infty) \times \mathbb{R}^d) \), also \( f \) is a renormalized solution of the continuity equation starting from \( f_0 \) with vector field \( b \). Together with (3.18), (3.13), and (3.17) this proves that \( (f_t, \rho_t^{\text{eff}}) \) is a generalized solution of the Vlasov-Poisson equation starting from \( f_0 \) according to Definition 2.4.

Finally, the fact that \( f \) is transported by the Maximal Regular Flow associated to \( b_t \) simply follows by the fact that each density \( f^k \) is transported by Maximal Regular Flow associated to \( b_t \) (thanks to Theorem 2.2) and that \( f = \sum_{k=0}^\infty f^k \) is an absolutely convergent series (see (3.13)).

The proof of Corollary 2.6 is an easy adaptation of the proof of Theorem 2.5, obtained by approximating the initial datum with a sequence of smooth data with bounded energy. In turn, this bound ensures that the approximating sequence of phase-space distributions is tight in the \( v \) variable uniformly in time, allowing us to show that \( \rho_t^{\text{eff}} = \rho_t \) for a.e. \( t \in (0, \infty) \). The approximation of the initial datum with a smooth sequence having uniformly bounded energy is a technical task that we describe in the next lemma.

---

This can be seen by a direct computation, using the definition of fractional Sobolev spaces.
Lemma 3.2. Let $d \geq 3$, let $\psi$ be a standard convolution kernel, and set $\psi_k(x) := k^d \psi(kx)$ for every $k \geq 1$. Let $f_0 \in L^1(\mathbb{R}^d)$ be an initial datum of finite energy, namely
\[
\int_{\mathbb{R}^d} |v|^2 f_0(x, v) \, dv + \int_{\mathbb{R}^d} [H \ast \rho_0](x) \rho_0(x) \, dx < \infty,
\]
where $\rho_0(x) := \int_{\mathbb{R}^d} f_0(x, v) \, dv$ and $H(x) := c_d(d - 2)^{-1} |x|^{2-d}$ for every $x \in \mathbb{R}^d$. Then there exist a sequence of functions $\{f_n^0\}_{n \in \mathbb{N}} \subset C_c^\infty(\mathbb{R}^d)$ and a sequence $\{k_n\}_{n \in \mathbb{N}}$ such that $k_n \to \infty$ and, setting $\rho_0^n(x) = f_0^n(x, v) \, dv$,
\[
\lim_{n \to \infty} \left( \int_{\mathbb{R}^d} |v|^2 f_0^n \, dv + \int_{\mathbb{R}^d} H \ast \psi_{k_n} \rho_0^n \, dx \right) = \int_{\mathbb{R}^d} |v|^2 f_0 \, dv + \int_{\mathbb{R}^d} H \ast \rho_0 \rho_0 \, dx. \tag{3.22}
\]

**Proof.** We split the approximation procedure in three steps. We use the notation $L_c^\infty$ to denote the space of bounded functions with compact support.

**Step 1: approximation of the initial datum when $f_0 \in L_c^\infty(\mathbb{R}^d)$.** Assuming that $f_0 \in L_c^\infty(\mathbb{R}^d)$, we claim that there exists $\{f_n^0\}_{n \in \mathbb{N}} \subset C_c^\infty(\mathbb{R}^d)$ such that
\[
\lim_{n \to \infty} \left( \int_{\mathbb{R}^d} |v|^2 f_0^n \, dv + \int_{\mathbb{R}^d} H \ast \rho_0^n \rho_0 \, dx \right) = \int_{\mathbb{R}^d} |v|^2 f_0 \, dv + \int_{\mathbb{R}^d} H \ast \rho_0 \rho_0 \, dx. \tag{3.23}
\]

To this end, consider smooth functions $f_0^n$ which converge to $f_0$ pointwise, whose $L^\infty$ norms are bounded by $\|f_0\|_{L^\infty(\mathbb{R}^d)}$, and whose supports are all contained in the same ball. By construction the densities $\rho_0^n$ are bounded as well and their supports are also contained in a fixed ball; moreover, the functions $H \ast \rho_0^n$ are bounded and converge to $H \ast \rho_0$ locally in every $L^p$. These observations show the validity of (3.23), by dominated convergence.

**Step 2: approximation of the initial datum when $f_0 \in L^1(\mathbb{R}^d)$.** Assuming that $f_0 \in L^1(\mathbb{R}^d)$, we claim that there exists a sequence of functions $\{f_n^0\}_{n \in \mathbb{N}} \subset C_c^\infty(\mathbb{R}^d)$ such that (3.23) holds.

Indeed, by Step 1 it is enough to approximate $f_0$ with a sequence in $L_c^\infty(\mathbb{R}^d)$ and converging energies. To this aim, for every $n \in \mathbb{N}$ we define the truncations of $f_0$ given by
\[
f_0^n(x, v) := \min\{n, 1_B_n(x, v) f_0(x, v)\} \quad (x, v) \in \mathbb{R}^d.
\]

Since $H \geq 0$ the integrands in the left-hand side of (3.23) converge monotonically, hence the integrals converge by monotone convergence.

**Step 3: approximation of the kernel.** We conclude the proof of the lemma. In order to approximate the kernel, we notice that, given the sequence of functions $f_0^n \in C_c^\infty(\mathbb{R}^d)$ provided by Steps 1-2, for $n \in \mathbb{N}$ fixed we have
\[
\lim_{k \to \infty} \int_{\mathbb{R}^d} H \ast \psi_k \ast \rho_0^n \rho_0 \, dx = \int_{\mathbb{R}^d} H \ast \rho_0^n \rho_0 \, dx.
\]

Hence, choosing $k_n$ sufficiently large so that
\[
\left| \int_{\mathbb{R}^d} H \ast \psi_{k_n} \ast \rho_0^n \rho_0 \, dx - \int_{\mathbb{R}^d} H \ast \rho_0^n \rho_0 \, dx \right| \leq \frac{1}{n},
\]
we conclude the proof of the approximation lemma. \qed
Proof of Corollary 2.6. Given \( f_0 \) of finite energy, let \( \{f^n_0\}_{n \in \mathbb{N}} \subset C_c^\infty(\mathbb{R}^d) \) and \( \{k_n\}_{n \in \mathbb{N}} \) be as in Lemma 3.2. Also let \( K := c_4 x / |x|^d \) and \( K_n := K * \psi_{k_n} \). Applying verbatim the arguments in Steps 1-3 in the proof of Theorem 2.5 we get a sequence \( f_n \) of smooth solutions with kernels \( K_n \) such that

\[
f^n \rightharpoonup f \quad \text{weakly in } L^1([0, T] \times \mathbb{R}^d) \quad \text{for any } T > 0,
\]

and

\[
\rho^n \rightharpoonup \rho^{\text{eff}} \quad \text{weakly* in } L^\infty((0, T); \mathscr{M}_+(\mathbb{R}^d)),
\]

where \( \rho^n_t(x) := \int_{\mathbb{R}^d} f_t(x, v) \, dv \).

In addition, the conservation of the energy along classical solutions gives that, for every \( n \in \mathbb{N} \) and \( t \in [0, \infty) \)

\[
\int_{\mathbb{R}^d} |v|^2 f^n_t \, dx \, dv + \int_{\mathbb{R}^d} H \ast \psi_{k_n} \ast \rho^n_t \rho^n_t \, dx = \int_{\mathbb{R}^d} |v|^2 f^n_t \, dx \, dv + \int_{\mathbb{R}^d} H \ast \psi_{k_n} \ast \rho^n_0 \rho^n_0 \, dx \leq C,
\]

Hence, since \( H \geq 0 \) we deduce that

\[
\sup_{n \in \mathbb{N}} \sup_{t \in [0, \infty)} \int_{\mathbb{R}^d} |v|^2 f^n_t \, dx \, dv \leq C,
\]

and by lower semicontinuity of the kinetic energy we deduce that, for every \( T > 0 \),

\[
\int_0^T \int_{\mathbb{R}^d} |v|^2 f_t \, dx \, dv \, dt \leq \liminf_{n \to \infty} \int_0^T \int_{\mathbb{R}^d} |v|^2 f^n_t \, dx \, dv \, dt \leq CT.
\]

We now want to exploit (3.24) and (3.25) to show that \( \rho^{\text{eff}} = \rho \), where \( \rho_t(x) := \int_{\mathbb{R}^d} f_t(x, v) \, dv \in L^\infty((0, T); L^1(\mathbb{R}^d)) \). For this, we want to show that for any \( \varphi \in C_c((0, \infty) \times \mathbb{R}^d) \)

\[
\lim_{n \to \infty} \int_0^\infty \int_{\mathbb{R}^d} \varphi \rho^n_t \, dx \, dt = \int_0^\infty \int_{\mathbb{R}^d} \varphi \rho_t \, dx \, dt.
\]

To prove this, for every \( k \in \mathbb{N} \) we consider a continuous nonnegative function \( \zeta_k : \mathbb{R}^d \to [0, 1] \) which equals 1 inside \( B_k \) and 0 outside \( B_{k+1} \), and observe that

\[
\begin{align*}
\int_0^\infty \int_{\mathbb{R}^d} \varphi (\rho^n_t - \rho_t) \, dx \, dt &= \int_0^\infty \int_{\mathbb{R}^d} \varphi_t(x) f^n_t(x, v)(1 - \zeta_k(v)) \, dx \, dv \\
&\quad + \int_0^\infty \int_{\mathbb{R}^d} \varphi_t(x)(f^n_t(x, v) - f(x, v)) \zeta_k(v) \, dx \, dv \\
&\quad + \int_0^\infty \int_{\mathbb{R}^d} \varphi_t(x)f_t(x, v)(\zeta_k(v) - 1) \, dx \, dv.
\end{align*}
\]

The second term in the right-hand side converges to 0 by the weak convergence of \( f^n \) to \( f \) in \( L^1 \), while the other two terms are estimated by the finiteness of energy (3.24) and (3.25) as

\[
\left| \int_0^\infty \int_{\mathbb{R}^d} \varphi f^n_t(x, v)(1 - \zeta_k(v)) \, dx \, dv \, dt \right| \leq \frac{\| \varphi \|_\infty}{k^2} \int_0^T \int_{\mathbb{R}^d} f^n_t(x, v)|v|^2 \, dx \, dv \, dt \leq \frac{CT\| \varphi \|_\infty}{k^2},
\]
and similarly
\[ \left| \int_0^\infty \int_{\mathbb{R}^d} \varphi f_t(x, v)(1 - \zeta_k(v)) \, dx \, dv \, dt \right| \leq \frac{C T \| \varphi \|_\infty}{k^2}. \]
Letting \( k \to \infty \), this proves (3.26). Thanks to this fact, the conclusion of the proof proceeds exactly as in Steps 4 and 5 in the proof of Theorem 2.5 with \( \rho^{\text{eff}}_t = \rho_t \).

**Remark 3.3.** The construction in Theorem 2.5 provides distributional solutions of the Vlasov-Poisson system if further assumptions are assumed on the initial datum such as finiteness of the total energy, as shown in Corollary 2.6. Still, there are examples of infinite energy data such that the generalized solution built in Theorem 2.5 is in fact distributional. For instance, in [28] Perthame considers an initial datum \( f_0 \in L^1 \cap L^\infty(\mathbb{R}^6) \) with \( (1 + |x|^2) f_0 \in L^1(\mathbb{R}^6) \) and infinite energy, and he shows the existence of a solution \( f \in L^\infty([0, \infty); L^1 \cap L^\infty(\mathbb{R}^6)) \) of the Vlasov-Poisson system such that the quantities

\[ t^{1/2} \| E_t \|_{L^2}, \quad t^{3/5} \| \rho_t \|_{L^{5/3}}, \quad \int_{\mathbb{R}^6} \frac{|x - vt|^2}{t} f_t(x, v) \, dx \, dv \]  
(3.27)

are bounded for all \( t \in (0, \infty) \).

It can be easily seen that, under Perthame’s assumptions, the construction in the proof of Theorem 2.5 provides a solution of the Vlasov-Poisson equation as the one built in [28]. In particular, thanks to the a priori estimate (3.27) on the approximating sequence, it is easy to see that \( \rho^{\text{eff}} = \rho \), therefore providing a Lagrangian (and therefore renormalized and distributional) solution of Vlasov-Poisson.

Similarly, under the assumptions of [34], a similar argument shows that the generalized solutions built in Theorem 2.5 solve the classical Vlasov-Poisson system.

## 4 Maximal Regular Flows of the state space and renormalized solutions

The aim of this and next section is to develop the abstract theory of Maximal Regular Flows and Lagrangian/renormalized solutions that are behind the results presented in the previous sections. We warn the reader that from now on, since the theory is completely general, we shall often consider flows of vector field in \( \mathbb{R}^d \) and denote by \( x \) a point in \( \mathbb{R}^d \). Then, for the applications to kinetic equations in the phase-space \( \mathbb{R}^{2d} \), one should apply these results replacing \( d \) with \( 2d \) and \( x \) with \( (x, v) \).

### 4.1 Preliminaries on Maximal Regular Flows

In this section we recall the basic results in [2], where a local version of the theory of DiPerna and Lions [17] and Ambrosio [1] was developed. First we recall the definition of a local (in space and time) version of the Regular Lagrangian Flow introduced by Ambrosio [1]. Here and in the sequel, \( B(\mathbb{R}^d) \) denotes the collection of Borel sets in \( \mathbb{R}^d \), and \( AC([\tau_1, \tau_2]; \mathbb{R}^d) \) is the space of absolutely continuous curves on \( [\tau_1, \tau_2] \) with values in \( \mathbb{R}^d \).
Before stating the result, we recall these assumptions. For the following two properties hold:

(i) for a.e. $x \in B$, $X(\cdot,x) \in AC([\tau_1,\tau_2];\mathbb{R}^d)$ and solves the ODE $\dot{x}(t) = b(t,x(t))$ a.e. in $(\tau_1,\tau_2)$, with the initial condition $X(\tau_1,x) = x$;

(ii) there exists a constant $C = C(X)$ satisfying $X(t,\cdot)_{\#}(\mathcal{L}^d \setminus B) \leq C\mathcal{L}^d$ for all $t \in [\tau_1,\tau_2]$.

Let $T \in (0,\infty)$ and let $b : (0,T) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a Borel vector field. The main object of our analysis is the Maximal Regular Flow, which takes into account the possibility of blow-up before time $T$ (or not up to time 0, when an initial condition $s \in (0,T)$ is under consideration).

Definition 4.2 (Maximal Regular Flow). For every $s \in (0,T)$ we say that a Borel map $X(\cdot,s,\cdot)$ is a Maximal Regular Flow starting at time $s$ if there exist two Borel maps $T_{s,X}^-, T_{s,X}^+ : \mathbb{R}^d \rightarrow (s,T], T_{s,X}^-, T_{s,X}^+ : [s,T) \rightarrow [0,s)$ such that $X(\cdot,x)$ is defined in $(T_{s,X}^-(x), T_{s,X}^+(x))$ and the following two properties hold:

(i) for a.e. $x \in \mathbb{R}^d$, $X(\cdot,x) \in AC_{loc}((T_{s,X}^-(x), T_{s,X}^+(x);\mathbb{R}^d)$ and solves the ODE $\dot{x}(t) = b(t,x(t))$ a.e. in $(T_{s,X}^-(x), T_{s,X}^+(x))$, with the initial condition $X(s,s,x) = x$;

(ii) there exists a constant $C = C(s,X) \in C(s)$ such that

$$\forall t \in [0,T].$$

(iii) for a.e. $x \in \mathbb{R}^d$, either $T_{s,X}^+(x) = T$ (resp. $T_{s,X}^-(x) = 0$) and $X(\cdot,s,x)$ can be continuously extended up to $t = T$ (resp. $t = 0$) so that $X(\cdot,s,x) \in C([s,T];\mathbb{R}^d)$ (resp. $X(\cdot,s,x) \in C([0,s];\mathbb{R}^d)$), or

$$\lim_{t \uparrow T_{s,X}^+(x)} |X(t,s,x)| = \infty \quad \text{(resp. } \lim_{t \downarrow T_{s,X}^-(x)} |X(t,s,x)| = \infty).$$

In particular, $T_{s,X}^+(x) < T$ (resp. $T_{s,X}^-(x) > 0$) implies (4.2).

The definition of Maximal Regular Flow can be extended up to the extreme times $s = 0$, $s = T$, setting $T_{0,X}^- \equiv 0$ and $T_{T,X}^+ \equiv T$.

A Maximal Regular Flow has been built in [2] under general local assumptions on $b$. Before stating the result, we recall these assumptions. For $T \in (0,\infty)$ we are given a Borel vector field $b : (0,T) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfying:

(a) $\int_0^T \int_{B_R} |b(t,x)| \, dx \, dt < \infty$ for any $R > 0$;

(b) for any nonnegative $\bar{\rho} \in L_+^\infty(\mathbb{R}^d)$ with compact support and any closed interval $I = [a,b] \subset [0,T]$, the continuity equation

$$\frac{d}{dt} \rho_t + \text{div}(b(t,x) \rho_t) = 0 \quad \text{in } (a,b) \times \mathbb{R}^d \quad (4.3)$$
has at most one solution in the class of all weakly* nonnegative continuous functions \( I \ni t \mapsto \rho_t \) with \( \rho_0 = \bar{\rho} \) and \( \cup_{t \in I} \text{supp} \rho_t \subseteq \mathbb{R}^d \).

Since the vector fields that arise in the applications we have in mind are divergence-free, we assume throughout the paper that our velocity field \( b \) satisfies

\[
\text{div } b = 0 \quad \text{in } (0, T) \times \mathbb{R}^d \quad \text{in the sense of distributions.} \quad (4.4)
\]

Equivalently, \( \text{div } b_x = 0 \) in the sense of distributions for a.e. \( t \in (0, T) \).

The existence and uniqueness of the Maximal Regular Flow after time \( s \), as well as the semigroup property, were proved in [2, Theorems 5.7, 6.1, 7.1] assuming a one sided bound (specifically a lower bound) on the divergence. In this context, uniqueness should be understood as follows: if \( X \) and \( Y \) are Maximal Regular Flows, for all \( s \in [0, T] \) one has

\[
\begin{cases}
T_{s,X}^+(x) = T_{s,Y}^+(x) \text{ for a.e. } x \in \mathbb{R}^d \\
X(\cdot, s, x) = Y(\cdot, s, x) \text{ in } (T_{s,X}^-(x), T_{s,X}^+(x)) \text{ for a.e. } x \in \mathbb{R}^d.
\end{cases} \quad (4.5)
\]

Under our assumptions on the divergence, by simply reversing the time variable the Maximal Regular Flow can be built both forward and backward in time, so we state the result directly in the time-reversible case.

**Theorem 4.3** (Existence, uniqueness, and semigroup property). Let \( b : (0, T) \times \mathbb{R}^d \rightarrow \mathbb{R}^d \) be a Borel vector field which satisfies (a) and (b). Then the Maximal Regular Flow starting from any \( s \in [0, T] \) is unique according to (4.5), and existence is ensured under the additional assumption (4.4). In addition, still assuming (4.4), for all \( s \in [0, T] \) the following properties hold:

(i) the compressibility constant \( C(s, X) \) in Definition 4.2 equals 1 and for every \( t \in [0, T] \)

\[
X(t, s, \cdot)\#(\mathcal{L}^d \Delta \{ T_{s,X}^- < t < T_{s,X}^+ \}) = \mathcal{L}^d \Delta \{ X(t, s, \cdot)(\{ T_{s,X}^- < t < T_{s,X}^+ \}) \}; \quad (4.6)
\]

(ii) if \( \tau_1 \in [0, s], \tau_2 \in [s, T], \) and \( Y \) is a Regular Flow in \( [\tau_1, \tau_2] \times B \), then \( T_{s,X}^+ > \tau_2, \)

\[
T_{s,X}^- < \tau_1 \quad \text{a.e. in } B; \quad \text{moreover}
\]

\[
X(\cdot, s, x) = Y(\cdot, X(\tau_1, s, x)) \text{ in } [\tau_1, \tau_2], \text{ for a.e. } x \in B; \quad (4.7)
\]

(iii) the Maximal Regular Flow satisfies the semigroup property, namely for all \( s, s' \in [0, T] \)

\[
T_{s',X}^\pm(X(s', s, x)) = T_{s,X}^\pm(x), \quad \text{for } \mathcal{L}^d\text{-a.e. } x \in \{ T_{s,X}^+ > s' > T_{s,X}^- \}, \quad (4.8)
\]

and, for a.e. \( x \in \{ T_{s,X}^+ > s' > T_{s,X}^- \}, \)

\[
X(t, s', X(s', s, x)) = X(t, s, x) \quad \forall t \in (T_{s,X}^-(x), T_{s,X}^+(x)). \quad (4.9)
\]

The following criterion, taken from [2, Theorem 7.6], provides a simple condition for global existence of the maximal flow. We state the result in the “forward” case, i.e. for \( X(\cdot, 0, \cdot) \), because this is the case explicitly considered in [2], the statement can be immediately adapted to cover the other cases when \( s \in (0, T] \).
Theorem 4.4 (No blow-up criterion). Let \( b : (0, T) \times \mathbb{R}^d \to \mathbb{R}^d \) be a divergence-free Borel vector field which satisfies (a) and (b). Assume that \( \rho_t \in L^\infty((0, T); L^\infty(\mathbb{R}^d)) \) is a weakly continuous solution of the continuity equation satisfying the integrability condition

\[
\int_0^T \int_{\mathbb{R}^d} \frac{\|b(t, x)\|}{1 + |x|} \rho_t(x) \, dx \, dt < \infty. \tag{4.10}
\]

Then \( T^+_{0, X}(x) = T \) and \( X(\cdot, 0, x) \in AC([0, T]; \mathbb{R}^d) \) for \( \rho_0, L^d \text{-a.e. } x \in \mathbb{R}^d \).

4.2 Uniqueness for the continuity equation and singular integrals

In this section we deal with uniqueness of solutions to the continuity equation when the gradient of the vector field is given by the singular integral of a time dependent family of measures. The theorem is a minor variant of a result by Bohun, Bouchut, and Crippa [8] (see also [11], where the uniqueness is proved for vector fields whose gradient is the singular integral of an \( L^1 \) function). We give the proof of the theorem under the precise assumptions that we need later on, since [8] deals with globally defined regular flows (hence the authors need to assume global growth conditions on the vector field) whereas here we present a local version of such result.

Theorem 4.5. Let \( b : (0, T) \times \mathbb{R}^{2d} \to \mathbb{R}^{2d} \) be given by \( b_t(x, v) = (b_{1t}(v), b_{2t}(x)) \), where

\[
b_{1t} \in L^\infty((0, T); W^{1,\infty}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d)), \quad b_{2t} = K * \rho_t
\]

with \( \rho \in L^\infty((0, T); \mathcal{M}_+(\mathbb{R}^d)) \) and \( K(x) = x/|x|^d \).

Then \( b \) satisfies (b) of Section 4.1, namely the uniqueness of bounded, compactly supported, nonnegative distributional solutions of the continuity equation.

Proof. To simplify the notation we give the proof in the case of autonomous vector fields, but the same computations work for the general statement.

It is enough to show that, given \( B_R \subset \mathbb{R}^d \) and \( \eta \in \mathcal{P}(C([0, T]; B_R \times B_R)) \) concentrated on integral curves of \( b \) and such that \( \eta_0 \# \eta \leq C_0 L^d \) for all \( t \in [0, T] \), the disintegration \( \eta_x \) of \( \eta \) with respect to the map \( \eta_0 \) is a Dirac delta for \( \eta_0 \# \eta \text{-a.e. } x \). Indeed, any two nonnegative, bounded, compactly supported, distributional solutions with the same initial datum \( \tilde{\rho} \) can be represented through the superposition principle (see [4, Theorem 8.2.1] and the more recent versions in [3, Theorem 12], [2], and Theorem 5.1 below) by \( \eta_1, \eta_2 \in \mathcal{P}(C([0, T]; B_R \times B_R)) \).

Hence, setting \( \eta = (\eta_1 + \eta_2)/2 \), if we can prove that \( \eta_x \) is a Dirac delta for \( \tilde{\rho} \text{-a.e. } x \) we deduce that \( (\eta_1)_x = (\eta_2)_x = \eta_x \) for \( \tilde{\rho} \text{-a.e. } x \), thus \( \eta_1 = \eta_2 \).

To show that \( \eta_x \) is a Dirac delta for \( \eta_0 \# \eta \text{-a.e. } x \), let us consider the function

\[
\Phi_{\delta, \zeta}(t) := \iint \log \left( 1 + \frac{|\gamma^1(t) - \eta^1(t)|}{\zeta \delta} + \frac{|\gamma^2(t) - \eta^2(t)|}{\delta} \right) d\eta_x(\gamma) d\eta_x(\eta) \, d\tilde{\rho}(x),
\]

where \( \delta, \zeta \in (0, 1) \) are small parameters to be chosen later, \( t \in [0, T], \tilde{\rho} := (\eta_0)_\# \eta \), and we use the notation \( \gamma(t) = (\gamma^1(t), \gamma^2(t)) \in \mathbb{R}^d \times \mathbb{R}^d \). It is clear that \( \Phi_{\delta, \zeta}(0) = 0 \).
Let us define the probability measure \( \mu \in \mathcal{P}(\mathbb{R}^d \times C([0,T];\mathbb{R}^d)^2) \) by \( d\mu(x,\eta,\gamma) := d\eta_x(\eta)d\eta_x(\gamma)d\rho(x) \), and assume by contradiction that \( \eta_x \) is not a Dirac delta for \( \rho \)-a.e. \( x \).

This means that there exists a constant \( a > 0 \) such that

\[
\iint\int \left( \int_0^T \min\{ |\gamma(t) - \eta(t)|, 1 \} \, dt \right) d\mu(x,\eta,\gamma) \geq a.
\]

By Fubini’s Theorem this implies that there exists a time \( t_0 \in (0,T) \) such that

\[
\iint\int \min\{ |\gamma(t_0) - \eta(t_0)|, 1 \} \, d\mu(x,\eta,\gamma) \geq \frac{a}{T}.
\]

Since the integrand is bounded by 1 and the measure \( \mu \) has mass 1, this means that the set

\[
A := \left\{ (x,\eta,\gamma) : \min\{ |\gamma(t_0) - \eta(t_0)|, 1 \} \geq \frac{a}{2T} \right\}
\]

has \( \mu \)-measure at least \( a/(2T) \). Then, assuming without loss of generality that \( a \leq 2T \), this implies that \( |\gamma(t_0) - \eta(t_0)| \geq a/(2T) \) for all \( (x,\eta,\gamma) \in A \), hence

\[
\Phi_{\delta,\zeta}(t_0) \geq \iint\int_A \log \left( 1 + \frac{|\gamma^1(t_0) - \eta^1(t_0)|}{\zeta \delta} + \frac{|\gamma^2(t_0) - \eta^2(t_0)|}{\delta} \right) d\mu(x,\eta,\gamma) \geq \frac{a}{2T} \log \left( 1 + \frac{a}{2\delta T} \right). \tag{4.11}
\]

We now want to show that this is impossible.

Computing the time derivative of \( \Phi_{\delta,\zeta} \) we see that

\[
\frac{d\Phi_{\delta,\zeta}}{dt}(t) \leq \int_{\mathbb{R}^d} \iint\int \left( \frac{\|b_1(\gamma^2(t)) - b_1(\eta^2(t))\|}{\zeta \delta + |\gamma^2(t) - \eta^2(t)|} + \frac{\|b_2(\gamma^1(t)) - b_2(\eta^1(t))\|}{\zeta \delta + |\gamma^1(t) - \eta^1(t)|} \right) d\mu(x,\eta,\gamma). \tag{4.12}
\]

By our assumption on \( b_1 \), the first summand is easily estimated using the Lipschitz regularity of \( b_1 \) in \( B_R \):

\[
\int_{\mathbb{R}^d} \iint |b_1(\gamma^2(t)) - b_1(\eta^2(t))| \frac{1}{\zeta (\delta + |\gamma^2(s) - \eta^2(s)|)} d\mu(x,\eta,\gamma) \leq \frac{\|\nabla b_1\|_{L^\infty(B_R)}}{\zeta}. \tag{4.13}
\]

To estimate the second integral we show that for some constant \( C \), which depends only on \( d, \|\rho\|_{(\mathbb{R}^d)} \) and \( R \), one has

\[
\iint\int \frac{\zeta |K \ast \rho(t)(\gamma^1(t)) - K \ast \rho(t)(\eta^1(t))|}{\zeta \delta + |\gamma^1(t) - \eta^1(t)|} d\mu(x,\eta,\gamma) \leq C \zeta \left( 1 + \log \left( \frac{C}{\zeta \delta} \right) \right). \tag{4.14}
\]

To this end, we first recall the definition of weak \( L^p \) norm of a \( \mu \)-measurable function \( f : X \to \mathbb{R} \) in a measure space \((X,\mu)\):

\[
\|f\|_{L^p(X,\mu)} := \sup\{ \lambda : \mu(\{|f| > \lambda\})^{1/p} : \lambda > 0 \}.
\]
Similarly, the second term in the right hand side can be estimated using (4.15) and the weak-$L^1$ estimate

$$|||\hat{M}(DK*\rho)|||_{M^1(B_R)} \leq C|\rho|(\mathbb{R}^d)$$

(4.16)

holds with a constant $C$ which depends only on $d$ and $R$. Applying (4.15), we see that

$$\iint \int \frac{|K*\rho(\gamma^1(t)) - K*\rho(\eta^1(t))|}{\zeta \delta + |\gamma^1(t) - \eta^1(t)|} d\mu \leq \int g_t(x, \eta, \gamma) d\mu,$$

(4.17)

where

$$g_t(x, \eta, \gamma) := \min \left\{ C\hat{M}(DK*\rho)(\gamma^1(t)) + C\hat{M}(DK*\rho)(\eta^1(t)), \frac{|K*\rho(\gamma^1(t))| + |K*\rho(\eta^1(t))|}{\zeta \delta} \right\}.$$

Let us fix $p := \frac{d}{d-1/2} \in \left(1, \frac{d}{d-1}\right)$, so that $|K| \in L^p_{\text{loc}}(\mathbb{R}^d)$. The last term in (4.17) can be estimated thanks to the following interpolation inequality (see [11, Lemma 2.2])

$$||g_t||_{L^1(\mu)} \leq \frac{p}{p-1} \left(||g_t||_{M^1(\mu)} \left(1 + \log\left(\frac{||g_t||_{M^p(\mu)}}{||g_t||_{M^1(\mu)}}\right)\right)\right).$$

Then, the first term in the right-hand side above can be estimated using our assumption $(e_t)_{\#} \eta \leq C_0 \mathcal{L}^d$ and (4.16):

$$||g_t||_{M^1(\mu)} \leq 2||\hat{M}(DK*\rho)(\eta^1(t))||_{M^1(\mu)}$$

$$= 2||\hat{M}(DK*\rho)(\eta^1(t))||_{M^1(\eta)}$$

$$= 2||\hat{M}(DK*\rho)(x)||_{M^1(B_R \times B_{\eta^1(t)} \times \mu \# \eta)}$$

$$\leq 2C_0||\hat{M}(DK*\rho)(x)||_{M^1(B_R \times B_{\eta^1(t)} \times \mathcal{L}^d)}$$

$$\leq 2C_0 \mathcal{L}^d(B_R) |||\hat{M}(DK*\rho)(x)|||_{M^1(B_R, \mathcal{L}^d)}$$

$$\leq 2C_0C. \mathcal{L}^d(B_R)||\rho||(\mathbb{R}^d).$$

Similarly, the second term in the right hand side can be estimated using $(e_t)_{\#} \eta \leq C_0 \mathcal{L}^d$ and Young’s inequality:

$$||g_t||_{M^p(\mu)} \leq 2(\zeta \delta)^{-1}||K*\rho(\eta^1(t))||_{L^p(\mu)} = 2(\zeta \delta)^{-1}||K*\rho(\eta^1(t))||_{L^p(\eta)}$$

$$\leq 2C_0(\zeta \delta)^{-1}||K*\rho(x)||_{L^p(B_R \times B_R)} \leq 2C_0(\zeta \delta)^{-1} \mathcal{L}^d(B_R)||K*\rho||_{L^p(B_R)}$$

$$\leq 2C(\zeta \delta)^{-1} \mathcal{L}^d(B_R)||K||_{L^p(B_R)}||\rho||(\mathbb{R}^d)$$

$$\leq C(\zeta \delta)^{-1},$$
where \( C \) depends on \( d, R, \) and \( |\rho|(\mathbb{R}^d) \). Combining these last estimates with (4.17), we obtain (4.14).

Then, using (4.12), (4.13), and (4.14), we deduce that

\[
\frac{d\Phi_{\delta,\zeta}(t)}{dt} \leq \frac{C}{\zeta} + \zeta + \zeta \log \left( \frac{C}{\zeta \delta} \right)
\]

for some constant \( C \) depending only on \( d, R, |\rho| (\mathbb{R}^d) \). Integrating with respect to time in \([0, t_0]\), we find that

\[
\Phi_{\delta,\zeta}(t_0) \leq C t_0 \left( \frac{1}{\zeta} + \zeta + \zeta \log \left( \frac{C}{\zeta} \right) + \zeta \log \left( \frac{1}{\delta} \right) \right).
\]

Choosing first \( \zeta > 0 \) small enough in order to have \( C t_0 \zeta < a/(2T) \) and then letting \( \delta \to 0 \), we find a contradiction with (4.11), which concludes the proof. 

\[
\square
\]

### 4.3 Generalized flows and Maximal Regular Flows

We denote by \( \mathbb{R}^d = \mathbb{R}^d \cup \{\infty\} \) the one-point compactification of \( \mathbb{R}^d \) and we recall the definition of generalized flow and of regular generalized flow in our context, as introduced in [2, Definition 5.3].

**Definition 4.6 (Generalized flow).** Let \( b : (0, T) \times \mathbb{R}^d \to \mathbb{R}^d \) be a Borel vector field. The measure \( \eta \in \mathcal{M}_+(C([0, T]; \mathbb{R}^d)) \) is said to be a generalized flow of \( b \) if \( \eta \) is concentrated on the set

\[
\Gamma := \{ \eta \in C([0, T]; \mathbb{R}^d) : \eta \in AC_{\text{loc}}(\{\eta \neq \infty\}; \mathbb{R}^d) \text{ and } \dot{\eta}(t) = b(t, \eta(t)) \text{ for a.e. } t \in \{\eta \neq \infty\} \}. \tag{4.18}
\]

In connection with this definition, let us provide a sketch of proof of the fact that the set \( \Gamma \) in (4.18) is Borel in \( C([0, T]; \mathbb{R}^d) \).

First of all one notices that for all intervals \([a, b] \subset [0, T]\) the set \( \{ \eta : \eta([a, b]) \subset \mathbb{R}^d \} \) is Borel. Then, considering the absolute continuity of a curve \( \eta \) in the integral form

\[
|\eta(t) - \eta(s)| \leq \int_s^t |b(r, \eta(r))| \, dr \quad \forall s, t \in [a, b], \ s \leq t
\]

it is sufficient to verify (arguing componentwise and splitting in positive and negative part) that for any nonnegative Borel function \( c \) and for any \( s, t \in [0, T] \) with \( s \leq t \) fixed, the function

\[
\eta \mapsto \int_s^t c(r, \eta(r)) \, dr
\]

is Borel in \( \{ \eta : \eta([a, b]) \subset \mathbb{R}^d \} \). This follows by a monotone class argument, since the property is obviously true for continuous functions and it is stable under equibounded and monotone convergence. Finally, as soon as the absolute continuity property is secured, also the verification of the Borel regularity of the class

\[
\Gamma \cap \{ \eta : \eta([a, b]) \subset \mathbb{R}^d \} = \{ \eta \in C([0, T]); \mathbb{R}^d) : \eta \in AC([a, b]; \mathbb{R}^d), \ \dot{\eta}(t) = b(t, \eta(t)) \text{ a.e. in } (a, b) \}
\]

can be achieved following similar lines. Finally, by letting the endpoints \( a, b \) vary in a countable dense set we obtain that \( \Gamma \) is Borel.
We say that a generalized flow $\eta$ is regular if there exists $L_0 \geq 0$ satisfying

$$
(e_t)_{\#}\eta \subseteq \mathbb{R}^d \leq L_0 \mathcal{L}^d \quad \forall t \in [0, T].
$$

(4.19)

In the case of a smooth, bounded vector field, a particular class of generalized flows is the one generated by transporting the initial measure along the integral lines of the flow:

$$
\eta = \int_{\mathbb{R}^d} \delta_{\mathbf{X}(\cdot, x)} \, d([e_0]_{\#}\eta)(x).
$$

In the next definition we propose a generalization of this construction involving Maximal Regular Flows.

**Definition 4.7** (Measures transported by the Maximal Regular Flow). Let $\mathbf{b} : (0, T) \times \mathbb{R}^d \to \mathbb{R}^d$ be a Borel vector field having a Maximal Regular Flow $\mathbf{X}$ and let $\eta \in \mathcal{M}_+(C([0, T]; \mathbb{R}^d))$ with $(e_t)_{\#}\eta \ll \mathcal{L}^d$ for all $t \in [0, T]$. We say that $\eta$ is transported by $\mathbf{X}$ if, for all $s \in [0, T]$, $\eta$ is concentrated on

$$
\{ \eta \in C([0, T]; \mathbb{R}^d) : \eta(s) = \infty \text{ or } \eta(\cdot) = \mathbf{X}(\cdot, s, \eta(s)) \text{ in } (T^-_{s, \mathbf{X}}(\eta(s)), T^+_{s, \mathbf{X}}(\eta(s))) \}. \quad (4.20)
$$

The absolute continuity assumption $(e_t)_{\#}\eta \ll \mathcal{L}^d$ on the marginals of $\eta$ is needed to ensure that this notion is invariant with respect to the uniqueness property in (4.5). In other words, if $\mathbf{X}$ and $\mathbf{Y}$ are related as in (4.5), then $\eta$ is transported by $\mathbf{X}$ if and only if $\eta$ is transported by $\mathbf{Y}$.

It is easily seen that if $\eta$ is transported by a Maximal Regular Flow, then $\eta$ is a generalized flow according to Definition 4.6, but in connection with the proof of the renormalization property we are more interested to the converse statement. As shown in the next theorem, this holds for regular generalized flows and for divergence-free vector fields satisfying (a)-(b) of Section 4.1.

**Theorem 4.8** (Regular generalized flows are transported by $\mathbf{X}$). Let $\mathbf{b} : (0, T) \times \mathbb{R}^d \to \mathbb{R}^d$ be a divergence-free vector field which satisfies (a)-(b) of Section 4.1 and let $\eta \in \mathcal{M}_+(C([0, T]; \mathbb{R}^d))$ be a regular generalized flow according to Definition 4.6. Consider $s \in [0, T]$ and a Borel family $\{\eta^s_x\} \subset \mathcal{P}(C([0, T]; \mathbb{R}^d))$, $x \in \mathbb{R}^d$, of conditional probability measures representing $\eta$ with respect to the marginal $(e_s)_{\#}\eta$, i.e., $\int \eta^s_x \, d(e_s)_{\#}\eta(x) = \eta$. Then for $(e_s)_{\#}\eta$-almost every $x \in \mathbb{R}^d$ we have that $\eta^s_x$ is concentrated on the set

$$
\Gamma_s := \left\{ \eta \in C([0, T]; \mathbb{R}^d) : \eta(s) = x, \eta(\cdot) = \mathbf{X}(\cdot, s, \eta(s)) \text{ in } (T^-_{s, \mathbf{X}}(\eta(s)), T^+_{s, \mathbf{X}}(\eta(s))) \right\}. \quad (4.21)
$$

In particular $\eta$ is transported by $\mathbf{X}$.

**Proof.** First of all we notice that the set $\Gamma_s$ in (4.21) is Borel. Indeed, the maps $\eta \mapsto T^-_{s, \mathbf{X}}(\eta(s))$ are Borel because $T^-_{s, \mathbf{X}}$ are Borel in $\mathbb{R}^d$, and the map $\eta \mapsto \mathbf{X}(t, s, \eta(s))$ is Borel as well for any $t \in [0, T]$. Therefore, choosing a countable dense set of times $t \in [0, T]$ the Borel regularity of $\Gamma_s$ is achieved.
The fact that $\eta^s_x$ is concentrated on the set $\{\eta : \eta(s) = x\}$ is immediate from the definition of $\eta^s_x$. We now show that for $(e_s)_#\eta$-almost every $x \in \mathbb{R}^d$ the measure $\eta^s_x$ is concentrated on the set

$$\{\eta \in C([0,T];\mathbb{R}^d) : \eta(\cdot) = X(\cdot, s, x) \text{ in } [s, T^+_s(x)]\}. \quad (4.22)$$

Notice that applying the same result after reversing the time variable, this proves the concentration on the set $\Gamma_s$ in (4.21).

For $r \in (s, T]$ we denote by $\Sigma^{s,r} : C([0,T];\mathbb{R}^d) \to C([s,r];\mathbb{R}^d)$ the map induced by restriction to $[s, r]$, namely $\Sigma^{s,r}(\eta) := \eta|[s,r]$.

For every $R > 0$, $r \in (s, T]$, let us consider $\eta^{R,r} := \Sigma^{s,r}(\eta \mathbb{L}\{\eta : \exists t \in [s, r] \text{ s.t. } \eta(t) \in B_R\})$.

By construction $\eta^{R,r}$ is a regular generalized flow relative to $b$ with compact support, hence our regularity assumption on $b$ allows us to apply [2, Theorem 3.4] to deduce that

$$\eta^{R,r} = \int \delta_X(\cdot, x) d[(e_s)_#\eta^{R,r}](x), \quad (4.23)$$

where $Y(\cdot, x)$ is an integral curve of $b$ in $[s, r]$ for $(e_s)_#\eta$-a.e. $x \in \mathbb{R}^d$. Let us denote by $\rho_{R,r}$ the density of $(e_s)_#\eta^{R,r}$ with respect to $\mathcal{L}^d$, which is bounded by $L_0$ thanks to (4.19).

For every $\delta > 0$ we have that

$$Y(t, \cdot)_#(\mathcal{L}^d \mathbb{L}\{\rho_{R,r} > \delta\}) = (e_t)_# \int_{\{\rho_{R,r} > \delta\}} \delta_Y(\cdot, x) d\mathcal{L}^d(x)$$

$$\leq \frac{1}{\delta} (e_t)_# \int_{\{\rho_{R,r} > \delta\}} \delta_Y(\cdot, x) d[(e_s)_#\eta^{R,r}](x) \quad (4.24)$$

$$\leq \frac{1}{\delta} (e_t)_# \eta^{R,r} \leq \frac{1}{\delta} (e_t)_# \eta \mathbb{L}\mathcal{L}^d \leq \frac{L_0}{\delta} \mathcal{L}^d,$$

hence $Y(\cdot, x)$ is a Regular Flow of $b$ in $[s, r] \times \{\rho_{R,r} > \delta\}$ according to Definition 4.1. By Theorem 4.3(ii) we deduce that $Y(\cdot, x) = X(\cdot, s, x)$ for a.e. $x \in \{\rho_{R,s} > \delta\}$ and therefore, letting $\delta \to 0$,

$$Y(\cdot, x) = X(\cdot, s, x) \quad \text{ in } [s, r] \text{ for } (e_s)_#\eta^{R,s}_\# \text{-a.e. } x \in \mathbb{R}^d. \quad (4.25)$$

Letting $R \to \infty$ we have that $\eta^{R,r} \to \sigma^r$ increasing, where

$$\sigma^r := \Sigma^{s,r}(\eta \mathbb{L}\{\eta : \eta(t) \neq \infty \text{ for every } t \in [s, r]\}).$$

By (4.23) and (4.25) we deduce that for every $r \in (s, T]$

$$\sigma^r = \int \delta_X(\cdot, s, x) d[(e_s)_#\sigma^r](x). \quad (4.26)$$
Arguing by contradiction, let us assume that there exists a Borel set \( E \subset \mathbb{R}^d \) such that 
\( (e_s)_\# \eta(E) > 0 \) and \( \eta^s \) is not concentrated on the set (4.22) for every \( x \in E \), namely
\[
\eta^s \left( \{ \eta \in C([0,T];\mathbb{R}^d) : \eta \neq X(\cdot, s, x) \text{ as a curve in } [s, T^{+}_{s,x}(x)] \} \right) > 0.
\]

Since this can be rewritten as
\[
\eta^s \left( \bigcup_{r \in \mathbb{Q} \cap (s, T^{+}_{s,x}(x))} \{ \eta \in C([0,T];\mathbb{R}^d) : \eta \neq X(\cdot, s, x) \text{ in } [s, r], \eta([s, r]) \subset \mathbb{R}^d \} \right) = 0,
\]
for every \( x \in E \) there exists \( r_x \in \mathbb{Q} \cap (s, T^{+}_{s,x}(x)) \) such that
\[
\eta^s \left( \{ \eta \in C([0,T];\mathbb{R}^d) : \eta \neq X(\cdot, s, x) \text{ as a curve in } [s, r_x], \eta([s, r_x]) \subset \mathbb{R}^d \} \right) > 0.
\]

In other words, for every \( x \in E \) there exists a rational number \( r_x \) such that
\[
\Sigma^{s,r_x} \left( \eta^s \mathbb{L}\{ \eta : \eta(t) \neq \infty \text{ for every } t \in [s, r_x] \} \right)
\]
is nonzero and not multiple of \( \delta_{X(\cdot, s, x)} \).

Therefore, there exist a Borel set \( E' \subset E \) of positive \( (e_s)_\# \eta \)-measure and \( r \in (s, T] \cap \mathbb{Q} \) such that for every \( x \in E' \)
\[
\Sigma^{s,r} \left( \eta^s \mathbb{L}\{ \eta : \eta(t) \neq \infty \text{ for every } t \in [s, r] \} \right)
\]
is nonzero and not multiple of \( \delta_{X(\cdot, s, x)} \).

By (4.26) and \( (e_s)_\# \sigma^r \leq (e_s)_\# \eta \) we have that
\[
\int \delta_{X(\cdot, s, x)} d(e_s)_\# \eta(x) \geq \sigma^r = \int \Sigma^{s,r} \left( \eta^s \mathbb{L}\{ \eta : \eta(t) \neq \infty \text{ for every } t \in [s, r] \} \right) d(e_s)_\# \eta(x).
\]

This yields \( \delta_{X(\cdot, s, x)} \geq \Sigma^{s,r} \left( \eta^s \mathbb{L}\{ \eta : \eta(t) \neq \infty \text{ for every } t \in [s, r] \} \right) \) for \( (e_s)_\# \eta \)-a.e. \( x \), and therefore a contradiction with the existence of \( E' \). This proves that \( \eta^s \) is concentrated on the set defined in (4.22), as desired.

Finally, in order to prove that \( \eta \) is transported by \( X \) we apply the definition of disintegration and the fact that for \( (e_s)_\# \eta \)-a.e. \( x \in \mathbb{R}^d \) the measure \( \eta^s_x \) is concentrated on the set \( \Gamma_x \) in (4.21) to obtain that
\[
\eta(\Gamma) = \int \eta^s_x(\Gamma) d(e_s)_\# \eta(x) = 1,
\]
where \( \Gamma \) is the set in (4.20).

\[\Box\]

### 4.4 Regular generalized flows and renormalized solutions

We now recall the well-known concept of renormalized solution to a continuity equation. This was already introduced in Section 2 in the context of the Vlasov-Poisson system, but we prefer to reintroduce it here in its general formulation for the convenience of the reader. To fix the ideas we consider the interval \((0,T)\) and 0 as initial time, but the definition can be immediately adapted to general intervals, forward and backward in time.
Definition 4.9 (Renormalized solutions). Let $b \in L^{1}_{\text{loc}}((0,T) \times \mathbb{R}^{d};\mathbb{R}^{d})$ be a Borel and divergence-free vector field. A Borel function $\rho : (0,T) \times \mathbb{R}^{d} \to \mathbb{R}$ is a renormalized solution of the continuity equation relative to $b$ if
\[
\partial_{t}\beta(\rho) + \nabla \cdot (b\beta(\rho)) = 0 \quad \text{in } (0,T) \times \mathbb{R}^{d} \quad \forall \beta \in C^{1} \cap L^{\infty}(\mathbb{R}) \tag{4.27}
\]
in the sense of distributions. Analogously, we say that $\rho$ is a renormalized solutions starting from a Borel function $\rho_{0} : \mathbb{R}^{d} \to \mathbb{R}$ if
\[
\int_{\mathbb{R}^{d}} \phi_{0}(x)\beta(\rho_{0}(x)) \, dx + \int_{0}^{T} \int_{\mathbb{R}^{d}} [\partial_{t}\phi_{t}(x) + \nabla \phi_{t}(x) \cdot b_{t}(x)]\beta(\rho_{t}(x)) \, dx \, dt = 0 \tag{4.28}
\]
for all $\phi \in C_{c}^{\infty}([0,T) \times \mathbb{R}^{d})$ and all $\beta \in C^{1} \cap L^{\infty}(\mathbb{R})$.

Remark 4.10 (Equivalent formulations). The definition is equivalent to test equation (4.27) with compactly supported functions in the space variable (see for instance [4, Section 8.1]); in other words, (4.28) holds if and only if for every $\varphi \in C_{c}^{\infty}(\mathbb{R}^{d})$ the function $\int_{\mathbb{R}^{d}} \varphi(x)\beta(\rho_{t}(x)) \, dx$ coincides a.e. with an absolutely continuous function $t \mapsto A(t)$ such that $A(0) = \int_{\mathbb{R}^{d}} \varphi(x)\beta(\rho_{0}(x)) \, dx$ and
\[
\frac{d}{dt} A(t) = \int_{\mathbb{R}^{d}} \nabla \varphi(x) \cdot b_{t}(x)\beta(\rho_{t}(x)) \, dx \quad \text{for a.e. } t \in (0,T). \tag{4.29}
\]

Moreover, by an easy approximation argument, the same holds for every Lipschitz, compactly supported $\varphi : \mathbb{R}^{d} \to \mathbb{R}$. This way, possibly splitting $\varphi$ in positive and negative parts, only nonnegative test functions need to be considered. Analogously, by writing every $\beta \in C^{1}(\mathbb{R}^{d})$ as the sum of a $C^{1}$ nondecreasing function and of a $C^{1}$ nonincreasing function, we can use the linearity of the equation with respect to $\beta(\rho_{t})$ to reduce to the case of $\beta \in C^{1} \cap L^{\infty}(\mathbb{R})$ nondecreasing.

In the next theorem we show first that, flowing an initial datum $\rho_{0} \in L^{1}(\mathbb{R}^{d})$ through the maximal flow, we obtain a renormalized solution of the continuity equation. This is, in turn, a key tool to prove the second part of the lemma, namely that any regular generalized flow induces, with its marginals, renormalized solutions. The proof of these facts heavily relies on the incompressibility of the flow and therefore on the assumption that the vector field is divergence-free. A generalization of this lemma to the case of vector fields with bounded divergence is possible, but rather technical and long. Hence, since it is not needed for the application to the Vlasov-Poisson system, we shall not present it here.

To fix the ideas, in part (i) of the theorem below we consider only 0 as initial time. An analogous statement can be given for any other initial time $s \in [0,T]$, considering intervals $[0,s]$ or $[s,T]$, with no additional assumption on $b$.

Theorem 4.11. Let $b : (0,T) \times \mathbb{R}^{d} \to \mathbb{R}^{d}$ be a divergence-free vector field which satisfies (a)-(b) of Section 4.1. Let $X(t,s,x)$ be the maximal flow of $b$.

(i) If $\rho_{0} \in L^{1}(\mathbb{R}^{d})$, we define $\rho_{t} \in L^{1}(\mathbb{R}^{d})$ by
\[
\rho_{t} := X(t,0,\cdot)(\rho_{0} \mathbb{1}_{(T_{0}^{+},T)_{X} > t}) \quad t \in [0,T).
\]
Then \( \rho_t \) is a renormalized solution of the continuity equation starting from \( \rho_0 \). In addition the map \( t \mapsto \rho_t \) is strongly continuous on \([0,T]\) w.r.t. the \( L^1_{\text{loc}} \) convergence, and even strongly \( L^1 \) continuous on \([0,T]\) from the right.

(ii) If \( \eta \in \mathcal{M}_+(C([0,T];\mathbb{R}^d)) \) is transported by \( X \), and \((e_t)_{\#} \eta \subset \mathcal{L}^d \) for every \( t \in [0,T] \), then the density \( \rho_t \) of \( (e_t)_{\#} \eta \subset \mathcal{L}^d \) with respect to \( \mathcal{L}^d \) is a renormalized solution of the continuity equation.

**Proof.** In the proof of (i) we set for simplicity \( X(t,x) = X(t,0,x) \) and \( T^+_0X = T_X \). We first notice that by the incompressibility of the flow (4.6) and by the definition of \( \rho_t \), for every \( t \in [0,T] \) and \( \varphi \in C_c(\mathbb{R}^d) \) one has

\[
\int_{\{T_X > t\}} \varphi(X(t,x))\rho_t(X(t,x)) \, dx = \int_{X(t,\cdot)\{t(T_X > t)\}} \varphi \rho_t \, dx = \int_{\{T_X > t\}} \varphi(X(t,x))\rho_0 \, dx.
\]

Hence, for any \( t \in [0,T] \) it holds

\[
\rho_t(X(t,x)) = \rho_0(x) \quad \text{for a.e. } x \in \{T_X > t\}. \tag{4.30}
\]

Let \( \beta \in C^1 \cap L^\infty(\mathbb{R}) \). By the incompressibility of the flow (4.6) and by (4.30) we have that

\[
\int_{\mathbb{R}^d} \varphi \beta(\rho_t) \, dx = \int_{X(t,\cdot)\{T_X > t\}} \varphi \beta(\rho_t) \, dx = \int_{\{T_X > t\}} \varphi(X(t,\cdot))\beta(\rho_0) \, dx \tag{4.31}
\]

for any \( \varphi \in C_c(\mathbb{R}^d) \). In addition, the blow-up property (4.2) ensures that \( t \mapsto \varphi(X(t,x)) \) can be continuously extended to be identically 0 on the time interval \([T_X(x),T)\) (in the case of blow-up before time \( T \)); in addition, for the same reason, if \( \varphi \in C^1_c(\mathbb{R}^d) \) the extended map is absolutely continuous in \([0,T]\) and

\[
\frac{d}{dt} \varphi(X(t,x)) = \chi([0,T_X(x))] \, t) \nabla \varphi(X(t,x)) \cdot b_t(X(t,x)) \quad \text{for a.e. } t \in (0,T). \tag{4.32}
\]

Therefore, using (4.31) and integrating (4.32), for all \( \varphi \in C^1_c(\mathbb{R}^d) \) we find that

\[
\frac{d}{dt} \int_{\mathbb{R}^d} \varphi \beta(\rho_t) \, dx = \int_{\{T_X > t\}} \nabla \varphi(X(t,\cdot)) \cdot b_t(X(t,\cdot)) \beta(\rho_0) \, dx = \int_{\mathbb{R}^d} \nabla \varphi \cdot b_t \beta(\rho_t) \, dx,
\]

for a.e. \( t \in (0,T) \), which proves the renormalization property.

We notice that, as a consequence of the possibility of continuously extending the map \( t \mapsto \varphi(X(t,x)) \) after \( T_X(x) \) for \( \varphi \in C_c(\mathbb{R}^d) \), the map \([0,T) \ni t \mapsto \rho_t \) is weakly continuous in duality with \( C_c(\mathbb{R}^d) \). Let us prove now the strong continuity of \( t \mapsto \rho_t \). We start with the proof for \( t = 0 \). Fix \( \epsilon > 0 \), let \( \psi \in C_c(\mathbb{R}^d) \) with \( \|\psi - \rho_0\|_1 < \epsilon \), and notice that the positivity a.e. in \( \mathbb{R}^d \) of \( T_X \) gives

\[
\int_{\mathbb{R}^d} |\rho_t(x) - \psi(x)| \, dx \leq \int_{X(t,\cdot)\{T_X > t\}} |\rho_t(x) - \psi(x)| \, dx + \int_{X(t,\cdot)\{0 < T_X \leq t\}} |\psi(x)| \, dx
\]
and that the second summand in the right hand side is infinitesimal. Changing variables and using (4.30) together the incompressibility of the flow, it follows that
\[
\int_{X(t,\cdot) \setminus \{T_X > t\}} |\rho_t(x) - \psi(x)| \, dx = \int_{\{T_X > t\}} |\rho_0(x) - \psi(X(t, x))| \, dx,
\]
therefore
\[
\limsup_{t \downarrow 0} \int_{\mathbb{R}^d} |\rho_t - \psi| \, dx \leq \limsup_{t \downarrow 0} \int_{\{T_X > t\}} |\rho_0(x) - \psi(X(t, x))| \, dx \leq \int_{\mathbb{R}^d} |\rho_0 - \psi| \, dx.
\]
This proves that \(\limsup_t \|\rho_t - \rho_0\|_1 \leq 2\epsilon\) and, by the arbitrariness of \(\epsilon\), the desired strong continuity for \(t = 0\).

We now notice that the same argument together with the semigroup property of Theorem 4.3(iii) shows that the map \(t \mapsto \rho_t\) is strongly continuous from the right in \(L^1\). In addition, reversing the time variable and using again the semigroup property, we deduce the identity \(\rho_t(x) = \rho_s(X(t, s, x))1_{\{T_X > t\}}(X(0, s, x))\), therefore
\[
\lim_{s \uparrow t} \int_{\mathbb{R}^d} |\rho_t(x) - \rho_s(x)1_{\{T_X > t\}}(X(0, s, x))| \, dx = 0 \quad \forall t \in (0, T).
\]
Hence, in order to prove that the map \(t \mapsto \rho_t\) is strongly continuous in \(L^1_{\text{loc}}\), we are left to show that for every \(R > 0\) and \(t \in (0, T)\) one has
\[
\lim_{s \uparrow t} \int_{B_R} |\rho_t(x) - \rho_s(x)1_{\{T_X > t\}}(X(0, s, x))| \, dx = 0.
\]
(4.33)

For this, we observe that by (4.30) and the incompressibility of the flow, we have that
\[
\int_{B_R} |\rho_s(x) - \rho_s(x)1_{\{T_X > t\}}(X(0, s, x))| \, dx = \int_{B_R} |\rho_s(x)1_{\{T_X \leq t\}}(X(0, s, x))| \, dx
\]
\[
= \int_{\mathbb{R}^d} |\rho_0(y)1_{\{T_X \leq t\}}(y)1_{B_R}(X(s, 0, y))| \, dy,
\]
(4.34)

Since trajectories go to infinity when the time approaches \(T_X\) (see (4.2)), it follows that
\[
1_{\{T_X \leq t\}}(y)1_{B_R}(X(s, 0, y)) \to 0 \quad \text{for a.e. } y \text{ as } s \uparrow t,
\]
so (4.33) follows by dominated convergence. This concludes the proof of (i).

To prove (ii), we notice that by Remark 4.10 it is enough to prove that, given a bounded nondecreasing \(\beta \in C^1(\mathbb{R})\) and a nonnegative \(\varphi \in C_c^\infty(\mathbb{R}^d)\), the function \(t \mapsto \int_{\mathbb{R}^d} \varphi\beta(\rho_t) \, dx\) is absolutely continuous in \([0, T]\) and
\[
\frac{d}{dt} \int_{\mathbb{R}^d} \varphi\beta(\rho_t) \, dx = \int_{\mathbb{R}^d} \nabla \varphi \cdot \mathbf{b}_t \beta(\rho_t) \, dx \quad \text{for a.e. } t \in (0, T).
\]
(4.35)
To show that the map is absolutely continuous, let us consider \( s, t \in [0, T] \) and let \( \tilde{\rho}_r^t \) be the evolution of \( \rho_t \) through the flow \( X(\cdot, t, x) \), namely
\[
\tilde{\rho}_r^t := X(r, t, \cdot) \# (\rho_t \mathbb{1}_{\{T_{r,X}^+ > r > T_{r,X}^-\}}) \quad \text{for every } r \in [0, T].
\] (4.36)
Since, by Theorem 4.8, \( \eta \) is transported by \( X \), we can prove that
\[
\tilde{\rho}_r^t \leq \rho_r
\] for every \( r \in [0, T] \). (4.37)
Indeed, with the notation of the statement of Theorem 4.8, since \( \delta X(r, t, x) = (e_r)^I \# \eta_t^I \) for \( \rho_t \)-a.e. \( x \in \{T_{r,X}^+ > r > T_{r,X}^-\} \), for every \( r \in [0, T] \) one has
\[
\tilde{\rho}_r^t \mathbb{L}^d = \int_{\{T_{r,X}^+ > r > T_{r,X}^-\}} \delta X(r, t, x) \rho_t(x) \, dx \leq \int_{\mathbb{R}^d} (e_r)^I \# \eta_t^I \rho_t(x) \, dx
\]
\[
= (e_r)^I \int_{\mathbb{R}^d} \eta_t^I \rho_t(x) \, dx = (e_r)^I \# \eta = \rho_r \mathbb{L}^d.
\]
Combining (4.37), the equality \( \tilde{\rho}_r^t = \rho_t \), the monotonicity of \( \beta \), and statement (i), we deduce that
\[
\int_{\mathbb{R}^d} [\beta(\rho_t) - \beta(\rho_s)] \varphi \, dx \leq \int_{\mathbb{R}^d} [\beta(\tilde{\rho}_r^t) - \beta(\tilde{\rho}_r^s)] \varphi \, dx = \int_s^t \int_{\mathbb{R}^d} \beta(\tilde{\rho}_r^t) \nabla \varphi \cdot b_t \, dx \, dr
\] (4.38)
and similarly
\[
\int_{\mathbb{R}^d} [\beta(\rho_t) - \beta(\rho_s)] \varphi \, dx \geq \int_{\mathbb{R}^d} [\beta(\tilde{\rho}_r^t) - \beta(\tilde{\rho}_r^s)] \varphi \, dx = \int_s^t \int_{\mathbb{R}^d} \beta(\tilde{\rho}_r^t) \nabla \varphi \cdot b_r \, dx \, dr.
\] (4.39)
We deduce that
\[
\left| \int_{\mathbb{R}^d} [\beta(\rho_t) - \beta(\rho_s)] \varphi \, dx \right| \leq \|\beta\|_\infty \int_{\mathbb{R}^d} \int_s^t |\nabla \varphi| |b_r| \, dr \, dx,
\]
which shows that the function \( t \mapsto \int_{\mathbb{R}^d} \varphi(\rho_t) \, dx \) is absolutely continuous in \([0, T]\).

In order to prove (4.35) it is sufficient to notice that (4.38) and the strong continuity of \( r \mapsto \tilde{\rho}_r^t \) at \( r = t \) (ensured by statement (i)) give
\[
\int_{\mathbb{R}^d} [\beta(\rho_t) - \beta(\rho_s)] \varphi \, dx \leq (t - s) \int_{\mathbb{R}^d} \beta(\rho_t) \nabla \varphi \cdot b_t \, dx + o(t - s),
\]
hence (4.35) holds at any differentiability point of \( t \mapsto \int_{\mathbb{R}^d} \varphi(\rho_t) \, dx \). \( \square \)

5 The superposition principle under local integrability bounds on the velocity

In order to represent the solution to the continuity equation by means of a generalized flow, we would like to apply the superposition principle (see [3, Theorem 12] or [2, Theorem 2.1]). However, the lack of global bounds makes this approach very difficult to implement. An analogous of the classical superposition principle is the content of the following theorem.
Theorem 5.1 (Extended superposition principle). Let $b \in L^1_{\text{loc}}([0, T] \times \mathbb{R}^d; \mathbb{R}^d)$ be a Borel vector field. Let $\rho_t \in L^\infty((0, T); L^1_\text{loc}(\mathbb{R}^d))$ be a distributional solution of the continuity equation, weakly continuous in duality with $C_c(\mathbb{R}^d)$. Assume that:

(i) either $|b_t| \rho_t \in L^1_{\text{loc}}([0, T] \times \mathbb{R}^d)$;
(ii) or $\text{div} \ b_t = 0$ and $\rho_t$ is a renormalized solution.

Then there exists $\eta \in \mathcal{M}_+(C([0, T]; \mathbb{R}^d))$ with

\[ |\eta|(C([0, T]; \mathbb{R}^d)) \leq \sup_{t \in [0, T]} \|\rho_t\|_{L^1(\mathbb{R}^d)}, \]

which is concentrated on the set $\Gamma$ defined in (4.18) and satisfies

\[ (e_t)_\# \eta \in \mathbb{R}^d = \rho_t \mathcal{L}^d \quad \text{for every } t \in [0, T]. \]

In addition, if $b$ is divergence-free and satisfies (a)-(b) of Section 4.1, then $\eta$ is transported by the Maximal Regular Flow of $X$.

Remark 5.2. If, in addition, we assume that

\[ \int_0^T \int_{\mathbb{R}^d} \frac{|b_t(x)|}{1 + |x|} \rho_t(x) \, dx \, dt < \infty, \quad (5.1) \]

then $\rho_t$ is transported by the Maximal Flow, namely $T_{0,x}^+(x) = T$, $X(\cdot, 0, x) \in AC([0, T]; \mathbb{R}^d)$ for a.e. $x \in \{\rho_0 > 0\}$ and $\rho_t \mathcal{L}^d = X(t, \cdot) \# \rho_0 \mathcal{L}^d$.

Indeed, by Theorem 4.4 and (5.1) we know that the Maximal Regular Flow is well defined in $[0, T]$ for a.e. $x \in \mathbb{R}^d$. Since $\eta$ is transported by $X$, for $\eta$-a.e. $\eta$ we know that $\eta = X(\cdot, 0, \eta(0))$ in $[0, T]$. This implies that for a.e. $x \in \{\rho_0 > 0\}$ the measure $\eta_x$, obtained through disintegration of $\eta$ with respect to $\mathbf{e}_0$, coincides with $\delta_{X(\cdot, 0, x)}$, therefore

\[ (e_t)_\# \eta = \int_{\mathbb{R}^d} (e_t)_\# \eta_x \rho_0(x) \, dx = \int_{\mathbb{R}^d} (e_t)_\# \delta_{X(\cdot, 0, x)} \rho_0(x) \, dx = X(\cdot, 0, x) \# \rho_0 \mathcal{L}^d, \]

as desired.

Let us first briefly explain the idea behind the proof of the theorem above. To overcome the lack of global bounds on $b$ we introduce a kind of “damped” stereographic projection, with damping depending on the growth of $|b|$ at $\infty$, and we look at the flow of $b$ on the $d$-dimensional sphere $S^d$ in such a way that the north pole $N$ of the sphere corresponds to the points at infinity of $\mathbb{R}^d$. Then we apply the superposition principle in these new variables and eventually, reading this limit in the original variables, we obtain a representation of the solution as a generalized flow. Let us observe that it is crucial for us that the map sending $\mathbb{R}^d$ onto $S^d$ is chosen a function of $b$: indeed, as we shall see, by shrinking enough distances at infinity we can ensure that the vector field read on the sphere becomes globally integrable.

We denote by $N$ be the north pole of the $d$-dimensional sphere $S^d$, thought of as a subset of $\mathbb{R}^{d+1}$. For our constructions, we will use a smooth diffeomorphism which maps $\mathbb{R}^d$ onto $S^d \setminus \{N\}$ and whose derivative has a prescribed decay at $\infty$. 

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Lemma 5.3. Let $D : [0, \infty) \to (0, 1]$ be a nonincreasing function. Then there exist $r_0 > 0$ and a smooth diffeomorphism $\psi : \mathbb{R}^d \to S^d \setminus \{N\} \subset \mathbb{R}^{d+1}$ such that

\[
\psi(x) \to N \text{ as } |x| \to \infty,
\]

\[
|\nabla \psi(x)| \leq D(0) \quad \forall x \in \mathbb{R}^d,
\]

\[
|\nabla \psi(x)| \leq D(|x|) \quad \forall x \in \mathbb{R}^d \setminus B_{r_0}.
\]

Proof. We split the construction in two parts: first we perform a 1-dimensional construction, and then we use this construction to build the desired diffeomorphism.

Step 1: 1-dimensional construction. Let $D_0 : [0, \infty) \to (0, 1]$ be a nonincreasing function. We claim that there exists a smooth diffeomorphism $\psi_0 : [0, \infty) \to [0, \pi)$ such that

\[
\lim_{r \to \infty} \psi_0(r) = \pi, \quad \lim_{r \to 0} \psi_0'(r) = 0,
\]

\[
\psi_0(r) = c_0 D_0(0) r \quad \forall r \in [0, \pi/D_0(0)), \quad \text{for some } c_0 \in (0, 1),
\]

\[
|\psi_0'(r)| \leq D_0(0) \quad \forall r \in [0, \infty),
\]

\[
|\psi_0'(r)| \leq D_0(r) \quad \forall r \in [2\pi/D_0(0), \infty).
\]

Indeed, define the nonincreasing $L^1$ function $D_1 : [0, \infty) \to (0, \infty)$ as

\[
D_1(r) := \begin{cases}
D_0(0) & \forall r \in [0, 1 + \pi/D_0(0)] \\
\min\{D_0(r), r^{-2}\} & \forall r \in (1 + \pi/D_0(0), \infty).
\end{cases}
\]

We then consider an asymmetric convolution kernel, namely a nonnegative function $\sigma \in C^\infty_c((0, 1))$ with $\int \sigma = 1$, and consider the convolution of $D_1(r)$ with $\sigma(-r)$:

\[
\psi_1(r) := \int_0^r \sigma(r') D_1(r + r') \, dr' \quad \forall r \in [0, \infty).
\]

Notice that $\psi_1$ is smooth on $(0, \infty)$, positive, nonincreasing, and $\psi_1 \leq D_1$ in $[0, \infty)$ (in particular $\psi_1 \in L^1(0, \infty)$). Moreover we have that $\psi_1 \equiv D_0(0)$ in $[0, \pi/D_0(0)]$, hence $\|\psi_1\|_{L^1(0, \infty)} \geq \pi$ and $c_0 := \pi \|\psi_1\|_{L^1(0, \infty)}^{-1} \in (0, 1)$. Finally, we define $\psi_0$ as

\[
\psi_0(r) := c_0 \int_0^r \psi_1(s) \, ds \quad \forall r \in [0, \infty).
\]

Since $|\psi_0'(r)| = c_0|\psi_1(r)| \leq D_1(r)$, taking into account that $\pi/D_0(0) > 1$ it easy to check that all the desired properties are satisfied.

Step 2: “radial” diffeomorphism in any dimension. Let $D_0 : [0, \infty) \to (0, 1]$ be chosen later and consider $\psi_0, c_0$ as in Step 1. We define $\psi : \mathbb{R}^d \to S^d \setminus \{N\} \subset \mathbb{R}^{d+1}$ which maps every half-line starting at the origin to an arc of sphere between the south pole and the north pole:

\[
\psi(x) := \sin(\psi_0(|x|)) \left( \frac{x}{|x|}, 0 \right) - \cos(\psi_0(|x|)) (0, \ldots, 0, 1).
\]
Thanks to (5.6) and to the fact that the functions $x \mapsto |x|^2$, $t \mapsto \sin(\sqrt{t})/\sqrt{t}$, and $t \mapsto \cos(\sqrt{t})$ are all of class $C^\infty$, we obtain that $\psi \in C^\infty(\mathbb{R}^d; \mathbb{R}^{d+1})$. We also notice that its inverse $\phi : \mathbb{S}^d \setminus \{N\} \to \mathbb{R}^d$ can be explicitly computed:

$$\phi(x_1, \ldots, x_{d+1}) = \psi^{-1}_0(\arccos(-x_{d+1}))/\|(x_1, \ldots, x_d)\| \\
= \psi^{-1}_0(\arcsin(\|(x_1, \ldots, x_d)\|))/\|(x_1, \ldots, x_d)\|.$$  

Writing $r = |x|$ and denoting by $I_d$ the identity matrix on the first $d$ components, we compute the gradient of $\psi$:

$$\nabla \psi(x) = \frac{\cos(\psi_0(r))\psi'_0(r)r - \sin(\psi_0(r))(x, 0) \otimes (x, 0)}{r^3} + \frac{\sin(\psi_0(r))}{r} I_d \\
- \frac{\sin(\psi_0(r))\psi'_0(r)(x, 0) \otimes (0, \ldots, 0, 1)}{r}.$$  

It is immediate to check that $|\nabla \psi(x)| \neq 0$ for all $x \in \mathbb{R}^d$, so it follows by the Inverse Function Theorem that $\phi$ is smooth as well. Also, we can estimate

$$|\nabla \psi(x)| \leq 2|\psi'_0(r)| + 2 \frac{\sin(\psi_0(r))}{r}. \tag{5.9}$$

Using now (5.7) and (5.8), the first term in the right hand side above can be estimated with $2D_0(0)$ for every $x \in \mathbb{R}^d$, and with $2D_0(r)$ for every $x \in \mathbb{R}^d$ such that $r \geq 2\pi/D_0(0)$. As regards the second term, for $r \in [0, \pi/D_0(0)]$ we have that

$$\frac{\sin(\psi_0(r))}{r} = \frac{\sin(c_0D_0(0)r)}{r} \leq c_0D_0(0), \tag{5.10}$$

while for $r \in [\pi/D_0(0), \infty)$ we estimate the numerator with 1 to get

$$\frac{\sin(\psi_0(r))}{r} \leq \frac{D_0(0)}{\pi}. \tag{5.11}$$

Therefore, since $c_0 < 1$, by (5.9), (5.10), and (5.11) we get

$$|\nabla \psi(x)| \leq 4D_0(0) \quad \forall \ x \in \mathbb{R}^d. \tag{5.12}$$

Now, for $r \in [2\pi/D_0(0), \infty)$, thanks to (5.5) and (5.8) we can estimate

$$\frac{\sin(\psi_0(r))}{r} = \frac{1}{r} \int_r^\infty -\cos(\psi_0(s))\psi'_0(s) \, ds \leq \frac{1}{r} \int_r^\infty |\psi'_0(s)| \, ds \leq \frac{1}{r} \int_r^\infty D_0(s) \, ds, \tag{5.13}$$

thus by (5.8), (5.9), and (5.13) we obtain

$$|\nabla \psi(x)| \leq 2D_0(r) + \frac{2}{r} \int_r^\infty D_0(s) \, ds \quad \forall \ x \in \mathbb{R}^d \setminus B_{2\pi/D_0(0)}. \tag{5.14}$$
In this way we obtain a smooth diffeomorphism \( \psi \) such that \( \psi \) holds, \( r \)

Step 1: construction of a diffeomorphism between the Maximal Regular Flow. \( \rho \)

In Step 1, based on Lemma 5.3, we construct a diffeomorphism between \( \mathbb{R}^d \) in this case. This is done in two steps:

- In Step 2 we associate a solution of the continuity equation on the sphere to the solution with the property that the vector field \( b \), read on the sphere, becomes globally integrable.

- In Step 3 how to handle the case when \( \rho_t \) is a renormalized solution.

Once the theorem has been proved for \( |b_t| \rho_t \in L^1_{\text{loc}}([0,T] \times \mathbb{R}^d) \), we show in Step 3 how to prove the result in this case. This is done in two steps:


\[
|\nabla \psi(x)| \leq \frac{D(r)}{2} + \frac{1}{r} \int_r^{\infty} \frac{D(s)}{s^2} ds \leq \frac{D(r)}{2} + \frac{\int_0^{\infty} \frac{r^2}{s^2} ds}{r^2} \leq D(r) \quad \forall x \in \mathbb{R}^d \setminus B_{r_0},
\]

proving (5.4) and concluding the proof.

\[ \square \]

**Proof of Theorem 5.1.** We first assume that \( |b_t| \rho_t \in L^1_{\text{loc}}([0,T] \times \mathbb{R}^d) \) and we prove the result in this case. This is done in two steps:

- In Step 1, based on Lemma 5.3, we construct a diffeomorphism between \( \mathbb{R}^d \) and \( \mathbb{S}^d \setminus \{N\} \) with the property that the vector field \( b \), read on the sphere, becomes globally integrable.

- In Step 3 how to handle the case when \( \rho_t \) is a renormalized solution.

Finally, in Step 4 we exploit the results of Section 4 to show that \( \rho_t \) is transported by the Maximal Regular Flow.

**Step 1: construction of a diffeomorphism between \( \mathbb{R}^d \) and \( \mathbb{S}^d \).** We build a diffeomorphism \( \psi \in C^\infty(\mathbb{R}^d; \mathbb{S}^d \setminus \{N\}) \) such that

\[
\lim_{x \to \infty} \psi(x) = N,
\]

which maps \( \mathbb{R}^d \) onto \( \mathbb{S}^d \setminus \{N\} \) such that \( (5.15) \) holds, \( |\nabla \psi(x)| \leq 1 \) on \( \mathbb{R}^d \), and

\[
|\nabla \psi(x)| \leq \frac{1}{2^n C_n} \quad \forall x \in B_{2^n} \setminus B_{2^{n-1}}, \; n \geq n_0,
\]

for some \( n_0 > 0 \). Thanks to these facts we deduce that

\[
\begin{align*}
&\int_0^T \int_{\mathbb{R}^d} |\nabla \psi(x)||b_t(x)| \rho_t(x) \, dx \, dt \\
\leq &\int_0^T \int_{B_{2^{n_0}}} |b_t(x)| \rho_t(x) \, dx \, dt + \sum_{i=n_0+1}^{\infty} \int_0^T \int_{B_{2^i} \setminus B_{2^{i-1}}} |\nabla \psi(x)||b_t(x)| \rho_t(x) \, dx \, dt \\
\leq &\int_0^T \int_{B_{2^{n_0}}} |b_t(x)| \rho_t(x) \, dx \, dt + \sum_{i=n_0+1}^{\infty} \frac{1}{2^i} < \infty,
\end{align*}
\]

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which proves (5.16).

**Step 2: superposition principle on the sphere.** We build $\eta \in \mathcal{M}_+([0,T];\mathbb{R}^d)$ such that $|\eta|(C([0,T];\mathbb{R}^d)) \leq \sup_{t \in [0,T]} \|\rho_t\|_{L^1(\mathbb{R}^d)}$, $\eta$ is concentrated on curves $\eta$ which are locally absolutely continuous integral curves of $b$ in $\{\eta \neq \infty\}$, and whose marginal at time $t$ in $\mathbb{R}^d$ is $\rho_t \mathcal{L}^d$.

Without loss of generality, possibly dividing every $\rho_t$ by $\sup_{t \in [0,T]} \|\rho_t\|_{L^1(\mathbb{R}^d)}$, we can assume that $\sup_{t \in [0,T]} \|\rho_t\|_{L^1(\mathbb{R}^d)}$, we can assume that $\sup_{t \in [0,T]} \|\rho_t\|_{L^1(\mathbb{R}^d)}$.

Define $m_t := \|\rho_t\|_{L^1(\mathbb{R}^d)} \leq 1$,

$$
c(t, y) := \begin{cases} 
\nabla \psi(\phi(y))b(t, \phi(y)) & \text{if } y \in S^d \setminus \{N\} \\
0 & \text{if } y = N
\end{cases}
$$

and

$$
\mu_t := \psi_#(\rho_t \mathcal{L}^d) + (1 - m_t)\delta_N \in \mathcal{P}(S^d), \quad t \in [0,T].
$$

Since $c(t, N) = 0$ we can neglect the mass at $N = \psi(\infty)$ to get

$$
\int_0^T \int_{S^d} |c| \, d\mu_t \, dt = \int_0^T \int_{S^d \setminus \{N\}} |\nabla \psi|(\phi(y))|b|(t, \phi(y)) \, d\mu_t(y) \, dt
$$

$$
= \int_0^T \int_\mathbb{R}^d |\nabla \psi|(x)|b|(t, x) \rho_t(x) \, dx \, dt < \infty,
$$

where in the last inequality we used (5.16).

We now show that the probability measure $\mu_t$ is a solution to the continuity equation on $S^d \subset \mathbb{R}^{d+1}$ with vector field $c_t$. To this end we first notice that, by the weak continuity in duality with $C_c(\mathbb{R}^d)$ of $\rho_t$ and by the fact that all the measures $\mu_t$ have unit mass, we deduce that $\mu_t$ is weakly continuous in time. Indeed, any limit point of $\mu_s$ as $s \to t$ is uniquely determined on $S^d \setminus \{N\}$, and then the mass normalization gives that it is completely determined. We want to prove that the function $t \mapsto \int_{S^d} \varphi \, d\mu_t$ is absolutely continuous and satisfies

$$
\frac{d}{dt} \int_{S^d} \varphi \, d\mu_t = \int_{S^d} c_t \cdot \nabla \varphi \, d\mu_t \quad \text{a.e. on } (0,T)
$$

(5.20) for every $\varphi \in C^\infty(\mathbb{R}^{d+1})$. We remark that, since $\rho_t$ is a solution to the continuity equation in $\mathbb{R}^d$ with vector field $b_t$, changing variables with the diffeomorphism $\psi$ we obtain that (5.20) holds for every $\varphi \in C^\infty_c(\mathbb{R}^{d+1} \setminus \{N\})$, hence we are left to check that (5.20) holds also when $\varphi$ is not necessarily 0 in a neighborhood of the north pole.

Let us consider $\varphi \in C^\infty_c(\mathbb{R}^{d+1})$. By $\mu_t(N) = 1 - m_t = 1 - \mu_t(S^d \setminus \{N\})$, for every $t \in [0,T]$ we have that

$$
\int_{S^d} \varphi \, d\mu_t = \int_{S^d \setminus \{N\}} \varphi \, d\mu_t + \varphi(N)\mu_t(N) = \varphi(N) + \int_{S^d} (\varphi - \varphi(N)) \, d\mu_t.
$$

(5.21)

For every $\varepsilon > 0$ let us consider a function $\chi_\varepsilon : \mathbb{R}^{d+1} \to \mathbb{R}$ which is 0 in $B_\varepsilon(N)$, 1 outside $B_{2\varepsilon}(N)$, and whose gradient is bounded by $2/\varepsilon$. Since $\rho_t$ is a solution to the continuity
equation in $\mathbb{R}^d$ and since $\chi_{\varepsilon}(\varphi - \varphi(N))$ is a smooth, compactly supported function in $C^\infty_c(\mathbb{R}^{d+1} \setminus \{N\})$ we deduce that

$$
\frac{d}{dt} \int_{\mathbb{S}^d} \chi_{\varepsilon}(\varphi - \varphi(N)) \, d\mu_t = \int_{\mathbb{S}^d \setminus \{N\}} c_t \cdot \nabla [\chi_{\varepsilon}(\varphi - \varphi(N))] \, d\mu_t
$$

$$
= \int_{\mathbb{S}^d \setminus \{N\}} (\varphi - \varphi(N)) c_t \cdot \nabla \chi_{\varepsilon} \, d\mu_t + \int_{\mathbb{S}^d \setminus \{N\}} \chi_{\varepsilon} c_t \cdot \nabla \varphi \, d\mu_t.
$$

(5.22)

To estimate the first term in the right-hand side of (5.22) we use that $|\varphi - \varphi(N)| \leq \varepsilon \|\nabla \varphi\|_\infty$ in $B_\varepsilon(N)$ and that $|\nabla \chi_{\varepsilon}| \leq 2/\varepsilon$ to get that

$$
\left| \int_{\mathbb{S}^d \setminus \{N\}} c_t \cdot \nabla \chi_{\varepsilon}(\varphi - \varphi(N)) \, d\mu_t \right| \leq 2\|\nabla \varphi\|_\infty \int_{B_{2\varepsilon}(N) \setminus B_\varepsilon(N)} |c_t| \, d\mu_t,
$$

and notice the latter goes to 0 in $L^1(0, T)$ as $\varepsilon \to 0$ since $|c|$ is integrable with respect to $\mu_t dt$ in space-time thanks to (5.20). Since the second term in the right-hand side of (5.22) converges in $L^1(0, T)$ to $\int_{\mathbb{S}^d \setminus \{N\}} c_t \cdot \nabla \varphi \, d\mu_t$, taking the limit as $\varepsilon \to 0$ in (5.22) we obtain that $t \mapsto \int_{\mathbb{S}^d} (\varphi - \varphi(N)) \, d\mu_t$ is absolutely continuous in $[0, T]$ and that for a.e. $t \in (0, T)$ one has

$$
\frac{d}{dt} \int_{\mathbb{S}^d} (\varphi - \varphi(N)) \, d\mu_t = \int_{\mathbb{S}^d} c_t \cdot \nabla \varphi \, d\mu_t.
$$

Using the identity (5.21), this formula can be rewritten in the form (5.20), as desired.

Since $\mu_t$ is a weakly continuous solution of the continuity equation and the integrability condition (5.20) holds, we can apply the superposition principle (see [3, Theorem 12] or [2, Theorem 2.1]) to deduce the existence of a measure $\sigma \in \mathcal{P}(C([0, T]; \mathbb{S}^d))$ which is concentrated on integral curves of $c$ and such that $(e_t)_{#} \sigma = \mu_t$ for all $t \in [0, T]$.

We then consider $\phi : \mathbb{S}^d \to \hat{\mathbb{R}}^d$ to be the inverse of $\psi$ extended to $N$ as $\phi(N) = \infty$, and define $\Phi : C([0, T]; \mathbb{S}^d) \to C([0, T]; \hat{\mathbb{R}}^d)$ as $\Phi(\eta) := \phi \circ \eta$. Then the measure

$$
\eta := \Phi_{#} \sigma \in \mathcal{P}(C([0, T]; \hat{\mathbb{R}}^d))
$$

is concentrated on locally absolutely continuous integral curves of $b$ in the sense stated in (4.18), and

$$
(e_t)_{#} \eta \subseteq \mathbb{R}^d = \phi_{#} (e_t)_{#} \sigma \subseteq \mathbb{R}^d = \phi_{#} \mu_t \subseteq \mathbb{R}^d = \rho_t \mathcal{L}^d.
$$

Step 3: the case of renormalized solutions. We now show how to prove the result when $\text{div } b_t = 0$ and $\rho_t$ is a renormalized solution. Notice that in this case we have no local integrability information on $|b_t|\rho_t$, so the argument above does not apply. However, exploiting the fact that $\rho_t$ is renormalized we can easily reduce to that case.

More precisely, we begin by observing that, by a simple approximation argument, the renormalization property (see Definition 4.9) is still true when $\beta$ is a bounded Lipschitz function. Thanks to this observation we consider, for $k \geq 0$, the functions

$$
\beta_k(s) := \begin{cases} 
0 & \text{if } s \leq k, \\
 s - k & \text{if } k \leq s \leq k + 1, \\
 1 & \text{if } s \geq k + 1.
\end{cases}
$$

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Since $\rho_t$ is renormalized, $\beta_k(\rho_t)$ is a bounded distributional solution of the continuity equation, hence by Steps 1-2 above there exists a measure $\eta_k \in \mathcal{M}_+(C([0,T];\mathbb{R}^d))$ with

$$|\eta_k|(C([0,T];\mathbb{R}^d)) \leq \sup_{t\in[0,T]} \|\beta_k(\rho_t)\|_{L^1(\mathbb{R}^d)},$$

which is concentrated on the set defined in (4.18) and satisfies

$$(e_t)_{#}\eta_k \subset \mathbb{R}^d = \beta_k(\rho_t) \mathcal{L}^d \quad \text{for every } t \in [0,T].$$

Since $\sum_{k \geq 0} \beta_k(s) = s$, we immediately deduce that the measure $\eta := \sum_{k \geq 0} \eta_k$ satisfies all the desired properties.

**Step 4: representation via the Maximal Regular Flow.** If we assume in addition that $b$ is divergence-free and satisfies (a)-(b) of Section 4.1, then by Theorem 4.8 $\eta$ is transported by the Maximal Regular Flow.

\[\square\]

**References**


