A STAMPACCHIA-TYPE INEQUALITY FOR A FOURTH-ORDER ELLIPTIC OPERATOR ON KÄHLER MANIFOLDS AND APPLICATIONS

Luca Lussardi

Dipartimento di Matematica, Politecnico di Torino
c.so Duca degli Abruzzi 24, 10129 Torino (Italy)
email: luca.lussardi@polito.it

Abstract. In this paper we will prove an integral inequality of Stampacchia-type for a fourth-order elliptic operator on complete and connected Kähler manifolds. Our inequality implies a Hodge-Kodaira orthogonal decomposition for the Sobolev-type space $\mathrm{W}^{p,q}(X)$. In particular we will able to prove, under suitable topological conditions on the manifold X, the existence of an isomorphism between the Aeppli groups $\Lambda^{p,q}(X)$ and the groups $\mathrm{H}^{p,q}(X)$ of all global harmonic forms of bidegree (p,q).

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1. Introduction

Let X be a complex manifold, and let $p,q \geq 1$ integers. The Aeppli groups, even called $\partial\bar{\partial}$-cohomology groups, defined for the first time by Aeppli in [1] and studied, principally, by Bigolin in [10] and in [11], were introduced in order to study cycles of algebraic manifolds (see [8]). More recently the Aeppli groups are under consideration in order to investigate integral transformations (see [17]), properties of balanced manifolds (see [2],[5],[6]) and properties of 1-convex manifolds (see [3],[7],[4]). The Aeppli groups were originally defined in [1] by

$$\Lambda^{p,q}(X) = \frac{\mathrm{Ker}\{\mathrm{A}^{p,q}(X) \stackrel{d}{\to} \mathrm{A}^{p+1,q}(X) \oplus \mathrm{A}^{p,q+1}(X)\}}{\partial\bar{\partial}\mathrm{A}^{p-1,q-1}(X)}$$

$$\mathrm{V}^{p,q}(X) = \frac{\mathrm{Ker}\{\mathrm{A}^{p,q}(X) \stackrel{\partial\bar{\partial}}{\to} \mathrm{A}^{p+1,q+1}(X)\}}{\partial\mathrm{A}^{p-1,q}(X) + \bar{\partial}\mathrm{A}^{p,q-1}(X)}$$

where $\mathrm{A}^{p,q}(X)$ denotes the space af all (p,q)-differential forms with coefficients in $C^\infty(X)$ and with complex values. If X is a Stein manifolds then Aeppli, in [1], proves that the Aeppli groups are isomorphic to the complex De Rham cohomology: more precisely $\Lambda^{p,q}(X)$ and $\mathrm{V}^{p,q}(X)$ are isomorphic, respectively, to the spaces $\mathrm{H}^{p+q}(X)$ and $\mathrm{H}^{p+q+1}(X)$, where $\mathrm{H}^r(X)$ denotes the space of all global harmonic r-forms. The result of Aeppli gives a characterization of the De Rham cohomology for Stein manifolds. If the manifold X is Kähler and compact then Bigolin, in [10], proves, as a consequence of a orthogonal decomposition for the space of all $\partial\bar{\partial}$-closed forms, that both $V^{p,q}(X)$ and $\Lambda^{p,q}(X)$ are isomorphic to $\mathrm{H}^{p,q}(X)$, where $\mathrm{H}^{p,q}(X)$ denotes the space of all forms in $\mathrm{H}^{p+q}(X)$ of bidegree (p,q); moreover in the same paper some results proved by Aeppli in [1] for Stein manifolds are recovered. If we remove the compactness assumption on the manifold X then, at the moment, it is unknown the relation between Aeppli groups and $\mathrm{H}^{p,q}(X)$. In this paper we study the non-compact case. We will able to prove, under a technical topological condition on X (see assumption (5.1)), that the Aeppli groups $\Lambda^{p,q}(X)$ are isomorphic to $\mathrm{H}^{p,q}(X)$ whenever X is a connected and complete Kähler manifold. The main tool for the proof of our result is a suitable Hodge-Kodaira orthogonal decomposition. More precisely denoting by $\mathrm{D}^{p,q}(X)$ the space of all forms in $\mathrm{A}^{p,q}(X)$ with compact support in X, we can consider, on $\mathrm{D}^{p,q}(X)$, the standard complex scalar product $(\cdot,\cdot)_X$ of L^2-type and the complex scalar product

$$(u,v)_{1,X} := (u,v)_X + (\bar\partial u, \bar\partial v)_X + (\vartheta u, \vartheta v)_X$$

where ∂ and $\bar\partial$ are the classical complex differential operators and $\bar\vartheta$ and ϑ are their adjoints, respectively. Then if we denote by $\mathrm{W}^{p,q}(X)$ the completion of $\mathrm{D}^{p,q}(X)$ with respect to the scalar product $(\cdot,\cdot)_{1,X}$, in §4 we will able to prove that on Kähler manifolds the following Hodge-Kodaira decomposition holds:

$$\mathrm{W}^{p,q}(X) = [\partial\bar\partial \mathrm{D}^{p-1,q-1}(X)]_1 \oplus_\perp [\vartheta \mathrm{D}^{p,q+1}(X) + \bar\vartheta \mathrm{D}^{p+1,q}(X)]_1 \oplus_\perp \operatorname{Ker}\Box \cap \mathrm{W}^{p,q}(X) \tag{1.1}$$

where the square brackets with subscript 1 stands for the closure in $\mathrm{W}^{p,q}(X)$ and $\oplus_\perp$ says that the direct sum is orthogonal in the sense of the scalar product $(\cdot,\cdot)_{1,X}$. The proof of (1.1), in the absence of compactness, requires an integral inequality of "Stampacchia-type" for a suitable elliptic operator, and such a inequality is the crucial point. Let us briefly recall the history of Stampacchia-type inequalities.

Let X be a complete and connected hermitian manifold. The classical Stampacchia inequality is an integral inequality which involves the complex Laplace operator $\Box$; Andreotti and Vesentini proved it in [9] in order to obtain applications to the study of vanishing theorems by means of an extension of a Kodaira theorem ([13]). More precisely if $\mathrm{L}^{p,q}(X)$ denotes the completion of $\mathrm{D}^{p,q}(X)$ with respect to the scalar product $(\cdot,\cdot)_X$ and if B_r denotes the ball of radius r and centered in a fixed point $0 \in X$, then for any $r, R, \sigma > 0$, with $r < R$, it holds

$$(\bar\partial u, \bar\partial u)_{B_r} + (\vartheta u, \vartheta u)_{B_r} \le \left(\frac{1}{\sigma} + \frac{c}{(R-r)^2}\right)(u,u)_{B_R} + \sigma(\Box u, \Box u)_{B_R} \tag{1.2}$$

for all $u \in \mathrm{A}^{p,q}(X)$, where $c > 0$ is a constant which depends only by the complex dimension of X. In particular it descends the following characterization of the square-summable harmonic forms on X:

$$\operatorname{Ker}\Box \cap \mathrm{L}^{p,q}(X) = \{u \in \mathrm{A}^{p,q}(X) \cap \mathrm{L}^{p,q}(X) : \bar\partial u = \vartheta u = 0\}. \tag{1.3}$$

A real version of inequality (1.2) was proved by Vesentini, with the same technique, in [18]: if M denotes a complete and connected riemannian manifold then for any $r, R, \sigma > 0$, with $r < R$, it holds

$$(du, du)_{B_r} + (\delta u, \delta u)_{B_r} \le \left(\frac{1}{\sigma} + \frac{c}{(R-r)^2}\right)(u,u)_{B_R} + \sigma(\Delta u, \Delta u)_{B_R} \tag{1.4}$$

for any $u \in \mathrm{A}^p(M)$, where $c > 0$ is a constant which depends only by the dimension of M. The Stampacchia-type inequality (1.4) implies that

$$\operatorname{Ker}\Delta \cap \mathrm{L}^p(M) = \{u \in \mathrm{A}^p(M) \cap \mathrm{L}^p(M) : du = \delta u = 0\}$$

from which it follows the Hodge-Kodaira decomposition of $\mathrm{L}^p(M)$:

$$\mathrm{L}^p(M) = [d\mathrm{D}^{p-1}(M)]_{\mathrm{L}^p(M)} \oplus_\perp [\delta \mathrm{D}^{p+1}(M)]_{\mathrm{L}^p(M)} \oplus_\perp \operatorname{Ker}\Delta \cap \mathrm{L}^p(M). \tag{1.5}$$

In this this paper we will prove a Stampacchia-type inequality like (1.2) for the fourth-order elliptic operator $\mathcal{D}$ given by

$$\mathcal{D} = \partial\bar\partial\vartheta\bar\vartheta + \vartheta\bar\vartheta\partial\bar\partial + \bar\vartheta\bar\partial\vartheta\partial + \vartheta\partial\bar\vartheta\bar\partial + \bar\vartheta\partial + \vartheta\bar\partial$$

which was first considered by Kodaira and Spencer in [15] for the study of the stability of Kähler manifolds under small deformations (see moreover the important book of Morrow and Kodaira [16], Ch. 4, § 4). Such a operator $\mathcal{D}$ was also considered by Bigolin [10] in the compact case. More precisely in § 3, following the same technique of Andreotti and Vesentini, we will prove that there exist four positive constants c_1, c_2, c_3, c_4, eventually depending only by the complex dimension of X, such that for any $r, R, \sigma > 0$, with $r < R$, it holds

$$(\Box u, \Box u)_{B_R} + (\vartheta\bar\vartheta u, \vartheta\bar\vartheta u)_{B_r} + (\partial\bar\partial u, \partial\bar\partial u)_{B_r} + (\vartheta\partial u, \vartheta\partial u)_{B_r} + (\bar\vartheta\bar\partial u, \bar\vartheta\bar\partial u)_{B_r} + \tag{1.6}$$

$$+ (\partial u, \partial u)_{B_r} + (\bar\partial u, \bar\partial u)_{B_r} \le \left(\frac{c_1}{(R-r)^2} + \frac{c_2}{(R-r)^4} + \frac{1}{\sigma}\right)(u,u)_{B_R} +$$

$$+\frac{c_3}{(R-r)^2}((\bar\partial u,\bar\partial u)_{B_R}+(\vartheta u,\vartheta u)_{B_R})+c_4\sigma(\mathcal{D}u,\mathcal{D}u)_{B_R}$$

for any $u\in \mathrm{A}^{p,q}(X)$. By means of inequality (1.6) we will able to prove the decomposition (1.1) and then, in the last section, we will apply such a decomposition in order to study a relation between the Aeppli cohomology and classical De Rham cohomology.

2. Riemannian and hermitian manifolds

2.1. **Riemannian manifolds.** For a thorough treatment of the argument we refer the reader to [12]. Let M be a n-dimensional orientable complete riemannian manifold. Let $g_{\alpha\beta}$ be the metric tensor on M and let $g^{\alpha\beta}$ be the inverse of $g_{\alpha\beta}$; we also denote by $g=\det g_{\alpha\beta}$. For any positive integer p, with $p\le n$, we will denote by $\mathrm{K}^p(M)$ the space of all currents on M of degree p; the subspace $\mathrm{A}^p(M)$ will denote the space of all p-differential forms with C^∞-coefficients and real values. In this setting it is well defined the volume form $e_{\alpha_1\dots\alpha_n}dx^1\wedge\cdots\wedge dx^n$. Given $u\in\mathrm{A}^p(M)$ the adjoint of u is the form given, in local coordinates, by $*u_{\beta_1\dots\beta_{n-p}}=e_{\alpha_1\dots\alpha_p\beta_1\dots\beta_{n-p}}u^{\alpha_1\dots\alpha_p}$. The operator $*\colon \mathrm{A}^p(M)\to\mathrm{A}^{n-p}(M)$ can be extended to a unique operator $*\colon\mathrm{K}^p(M)\to\mathrm{K}^{n-p}(M)$. On the subspace $\mathrm{D}^p(M)$ given by all forms in $\mathrm{A}^p(M)$ which have compact support in M the operator $*$ permits us to define the real scalar product given by

$$(u,v)_M:=\int_M u\wedge *v.$$

We will denote by $\mathrm{L}^p(M)$ the completion of the space $\mathrm{D}^p(M)$ with respect to the scalar product $(\cdot,\cdot)_M$. It turns out that $\mathrm{L}^p(M)$ is an Hilbert space. Let $d\colon\mathrm{K}^p(M)\to\mathrm{K}^{p+1}(M)$ be the exterior differential and let $\delta\colon\mathrm{K}^p(M)\to\mathrm{K}^{p-1}(M)$ its formal adjoint, i.e. $\delta=(-1)^{np+n+1}*d*$; it is well known that $d^2=\delta^2=0$. The laplacian of a current $T\in\mathrm{K}^p(M)$ is given by $\Delta T=d\delta T+\delta dT$; the currents belong to $\operatorname{Ker}\Delta$ are called harmonic currents, and the forms belong to $\operatorname{Ker}\Delta$ are called harmonic forms. By ellipticity it turns out that if $T\in\operatorname{Ker}\Delta$ then actually $T\in\mathrm{A}^p(M)$.

2.2. **Hermitian manifolds.** For a thorough treatment of the subject we refer the reader to [16] and [19]. Let X be a complete hermitian manifold of complex dimension n, let $g_{\alpha\beta}$ be the hermitian metric on X, and let $g^{\alpha\beta}$ be its inverse; as in the real case we denote by $g=\det g_{\alpha\beta}$. For any positive integers p,q, with $p,q\le n$, we will denote by $\mathrm{K}^{p,q}(X)$ the space of all currents on X of bidegree (p,q); the subspace $\mathrm{A}^{p,q}(X)$ will denote the space of all (p,q)-differential forms with C^∞-coefficients and complex values. Associated to an hermitian metric we have the fundamental real form $\omega=ig_{\alpha\beta}dz^\alpha d\bar z^\beta$; X is a Kähler manifold if $d\omega=0$. Let, in local coordinates, $e_{\alpha_1\dots\alpha_n\beta_1\dots\beta_n}dz^{\alpha_1}\wedge\cdots\wedge dz^{\alpha_n}\wedge d\bar z^{\beta_1}\wedge\cdots\wedge d\bar z^{\beta_n}$ be the volume form on X. Given $u\in\mathrm{A}^{p,q}(X)$ the adjoint of u is the form given, in local coordinates, by $*u_{\mu_1\dots\mu_{n-q}\nu_1\dots\nu_{n-p}}=e_{\mu_1\dots\mu_{n-q}\alpha_1\dots\alpha_q\nu_1\dots\nu_{n-p}\beta_1\dots\beta_p}u^{\alpha_1\dots\alpha_q\beta_1\dots\beta_p}$. The operator $*\colon\mathrm{A}^{p,q}(X)\to\mathrm{A}^{n-q,n-p}(X)$ can be extended to a unique operator $*\colon\mathrm{K}^{p,q}(X)\to\mathrm{K}^{n-q,n-p}(X)$. As in the riemannian case on the subspace $\mathrm{D}^{p,q}(X)$ given by all forms in $\mathrm{A}^{p,q}(X)$ which have compact support in X the operator $*$ permits us to define a complex scalar product given by

$$(u,v)_X:=\int_X u\wedge\overline{*v}.$$

We will denote by $\mathrm{L}^{p,q}(X)$ the completion of $\mathrm{D}^{p,q}(X)$ with respect to the scalar product $(\cdot,\cdot)_X$. It turns out that $\mathrm{L}^{p,q}(X)$ is an Hilbert space. Let $\partial\colon\mathrm{K}^{p,q}(X)\to\mathrm{K}^{p+1,q}(X)$ and $\bar\partial\colon\mathrm{K}^{p,q}(X)\to\mathrm{K}^{p,q+1}(X)$ be the classical complex differential operators. It is well known that $\partial^2=\bar\partial^2=0$ and $d=\partial+\bar\partial$. The operators $\vartheta\colon\mathrm{K}^{p,q}(X)\to\mathrm{K}^{p,q-1}(X)$ and $\bar\vartheta\colon\mathrm{K}^{p,q}(X)\to\mathrm{K}^{p-1,q}(X)$ can be defined by setting $\vartheta=-*\partial*$ and $\bar\vartheta=-*\bar\partial*$, and we get $\vartheta^2=\bar\vartheta^2=0$. Let us now recall the following useful formulas: If at least one form between u and v belong to $\mathrm{D}^{p,q}(X)$ then

$$(\bar\partial u,v)_X=(u,\vartheta v)_X\quad\text{and}\quad(\partial u,v)_X=(u,\bar\vartheta v)_X;\tag{2.1}$$

moreover it holds $\partial\bar\partial=-\bar\partial\partial$ and $\vartheta\bar\vartheta=-\bar\vartheta\vartheta$. If X is a Kähler manifold then it is well known that

$$\partial\vartheta+\vartheta\partial=0,\quad\bar\partial\bar\vartheta+\bar\vartheta\bar\partial=0,\quad\bar\partial\vartheta+\vartheta\bar\partial=\partial\bar\vartheta+\bar\vartheta\partial.\tag{2.2}$$

We recall that the complex laplacian $\Box\colon \mathrm{K}^{p,q}(X) \to \mathrm{K}^{p,q}(X)$ is defined by $\Box = \bar\partial\vartheta + \vartheta\bar\partial$; the currents belong to $\operatorname{Ker}\Box$ are called harmonic currents, and the forms belong to $\operatorname{Ker}\Box$ are called harmonic forms. On Kähler manifolds by (2.2) it descends

$$\Box = \overline{\Box} := \partial\bar\vartheta + \bar\vartheta\partial.$$

By ellipticity it turns out that if $T \in \operatorname{Ker}\Box$ then actually $T \in \mathrm{A}^{p,q}(X)$. On Kähler manifolds it holds $\Box = \frac{1}{2}\Delta$. Finally we will denote by $\mathrm{W}^{p,q}(X)$ the Sobolev-type space given by the completion of $\mathrm{D}^{p,q}(X)$ with respect to the scalar product

$$(u,u)_{1,X} := (u,u)_X + (\bar\partial u, \bar\partial u)_X + (\vartheta u, \vartheta u)_X.$$

It turns out that $\mathrm{W}^{p,q}(X)$ is an Hilbert space.

3. A Stampacchia-type inequality for the operator $\mathcal{D}$

In the rest of the paper X will denote a complete and connected Kähler manifold of complex dimension n. Let $p, q \le n$ be positive integers. Consider the fourth-order operator $\mathcal{D}\colon \mathrm{K}^{p,q}(X) \to \mathrm{K}^{p,q}(X)$ given by

$$\mathcal{D} = \vartheta\bar\partial\vartheta\bar\partial + \bar\partial\vartheta\bar\partial\vartheta + \bar\vartheta\partial + \vartheta\bar\partial \tag{3.1}$$

Remark 3.1. *An easy application of formulas* (2.2) *shows that*

$$\Box^2 = \vartheta\bar\partial\vartheta\bar\partial + \bar\partial\vartheta\bar\partial\vartheta \tag{3.2}$$

and

$$\vartheta\bar\partial\vartheta\bar\partial + \bar\partial\vartheta\bar\partial\vartheta = \partial\bar\partial\vartheta\bar\vartheta + \vartheta\bar\vartheta\partial\bar\partial + \bar\vartheta\bar\partial\vartheta\partial + \vartheta\partial\bar\vartheta\bar\partial. \tag{3.3}$$

In [15] Kodaira and Spencer show that $\mathcal{D}$ (they used, for the principal part of $\mathcal{D}$, the form given by the right-hand side of (3.3)) is an elliptic operator, since its principal part is given by

$$\sum_{\alpha\beta\gamma\delta} g^{\beta\alpha} g^{\delta\gamma} \frac{\partial^4}{\partial z^\alpha \partial \bar z^\beta \partial z^\gamma \partial \bar z^\delta}$$

in any local coordinates system. For any $u \in \mathrm{A}^{p,q}(X)$ let

$$(u,u)_{2,X} := (\vartheta\bar\vartheta u, \bar\vartheta\bar\vartheta)_X + (\partial\bar\partial u, \partial\bar\partial u)_X + (\vartheta\partial u, \vartheta\partial u)_X + (\bar\vartheta\bar\partial u, \bar\vartheta\bar\partial u)_X.$$

Let $0 \in X$ be a fixed point; for any $r > 0$ we will denote by B_r the ball centered in 0 with radius r. For the sake of simplicity we will use the notation $(\cdot,\cdot)_r$ and $(\cdot,\cdot)_{2,r}$ respectively for the quantities $(\cdot,\cdot)_{B_r}$ and $(\cdot,\cdot)_{2,B_r}$. Notice that the completeness of X ensures that the generic ball B_r is relatively compact in X, by Hopf-Rinow theorem; in particular all quantities $(u,u)_r$ and $(u,u)_{2,r}$ are finite. The fundamental result of this section is an integral inequality of Stampacchia-type for the operator $\mathcal{D}$.

Theorem 3.2. (Stampacchia-type inequality) *For every $R, r, \sigma > 0$ with $r < R$ it holds*

$$(\Box u, \Box u)_r + (u,u)_{2,r} + (\partial u, \partial u)_r + (\bar\partial u, \bar\partial u)_r \le \tag{3.4}$$

$$\le \left(\frac{c_1}{(R-r)^4} + \frac{c_2}{(R-r)^2} + \frac{1}{\sigma}\right)(u,u)_R + \frac{c_3}{(R-r)^2}((\bar\partial u, \bar\partial u)_R + (\vartheta u, \vartheta u)_R) + c_4\sigma(\mathcal{D}u, \mathcal{D}u)_R$$

for any $u \in \mathrm{A}^{p,q}(X)$, with c_1, c_2, c_3, c_4 positive constants eventually depending only by the complex dimension n.

Proof. Using the same argument of Lemma 6 in [9] we can construct a function $\varphi\colon X \to [0,1]$ with $\varphi = 1$ on B_r, $\varphi = 0$ on $X \setminus B_R$ such that there exist two positive constants M_1 and M_2, depending only by n, with

$$(L\varphi \wedge u, L\varphi \wedge u)_R \le \frac{M_1}{(R-r)^2}(u,u)_R, \quad (N\varphi \wedge u, N\varphi \wedge u)_R \le \frac{M_2}{(R-r)^4}(u,u)_R \tag{3.5}$$

for any $u \in \mathrm{A}^{p,q}(X)$, whenever $L \in \{\partial, \bar\partial, \vartheta, \bar\vartheta\}$ and $N \in \{\partial\bar\partial, \vartheta\bar\vartheta, \vartheta\partial, \bar\vartheta\bar\partial\}$. Let $u \in \mathrm{A}^{p,q}(X)$; then $\varphi^m u$ has support in B_R for any positive integer m. Now we divide the proof in two steps; first we collect some useful estimates for the first and the second order terms that appears in $\mathcal{D}$, and then we will prove (3.4).

Step 1. Let us consider the first order terms. We have

$$\partial(\varphi^4 u) = 4\varphi^3\partial\varphi\wedge u + \varphi^4\partial u$$

and then

$$(\partial u, \partial(\varphi^4 u))_R = (\varphi^2\partial u, 4\varphi\partial\varphi\wedge u + \varphi^2\partial u)_R = 4(\varphi^2\partial u, \varphi\partial\varphi\wedge u)_R + (\varphi^2\partial u, \varphi^2\partial u)_R.$$

Taking into account formulas (2.1) we deduce that

$$(\varphi^2\partial u, \varphi^2\partial u)_R = (\bar\vartheta\partial u, \varphi^4 u)_R - 4(\varphi^2\partial u, \varphi\partial\varphi\wedge u)_R. \tag{3.6}$$

By applying the same argument we get

$$(\varphi^2\bar\partial u, \varphi^2\bar\partial u)_R = (\vartheta\bar\partial u, \varphi^4 u)_R - 4(\varphi^2\bar\partial u, \varphi\bar\partial\varphi\wedge u)_R. \tag{3.7}$$

Let us now consider the second order terms. We easily have

$$\begin{aligned}\vartheta\bar\partial(\varphi^4 u) &= \vartheta(2\varphi^3\bar\partial\varphi\wedge u + \varphi^2\bar\partial(\varphi^2 u)) = \\ &= 6\varphi^2\vartheta\varphi\wedge\bar\partial\varphi\wedge u + 2\varphi^3\vartheta\bar\partial\varphi\wedge u + (-1)^{p+q}2\varphi^3\bar\partial\varphi\wedge\vartheta u + 2\varphi\vartheta\varphi\wedge\bar\partial(\varphi^2 u) + \varphi^2\vartheta\bar\partial(\varphi^2 u) = \\ &= 10\varphi^2\vartheta\varphi\wedge\bar\partial\varphi\wedge u + 2\varphi^3\vartheta\bar\partial\varphi\wedge u + (-1)^{p+q}2\varphi^3\bar\partial\varphi\wedge\vartheta u + 2\varphi^3\vartheta\varphi\wedge\bar\partial u + \varphi^2\vartheta\bar\partial(\varphi^2 u)\end{aligned}$$

from which we obtain

$$\begin{aligned}&(\vartheta\bar\partial u, \vartheta\bar\partial(\varphi^4 u))_R = \\ &= (\varphi^2\vartheta\bar\partial u, 10\vartheta\varphi\wedge\bar\partial\varphi\wedge u + 2\varphi\vartheta\bar\partial\varphi\wedge u + (-1)^{p+q}2\varphi\bar\partial\varphi\wedge\vartheta u + 2\varphi\vartheta\varphi\wedge\bar\partial u + \vartheta\bar\partial(\varphi^2 u))_R = \\ &= (\vartheta\bar\partial(\varphi^2 u) - 2\vartheta\varphi\wedge\bar\partial\varphi\wedge u - 2\varphi\vartheta\bar\partial\varphi\wedge u - (-1)^{p+q}2\varphi\bar\partial\varphi\wedge\vartheta u - 2\varphi\vartheta\varphi\wedge\bar\partial u, 10\vartheta\varphi\wedge\bar\partial\varphi\wedge u + \\ &\quad + 2\varphi\vartheta\bar\partial\varphi\wedge u + (-1)^{p+q}2\varphi\bar\partial\varphi\wedge\vartheta u + 2\varphi\vartheta\varphi\wedge\bar\partial u + \vartheta\bar\partial(\varphi^2 u))_R.\end{aligned}$$

Then taking into account (2.1) we get

$$\begin{aligned}(\vartheta\bar\partial(\varphi^2 u), \vartheta\bar\partial(\varphi^2 u))_R &= (\vartheta\bar\partial\vartheta\bar\partial u, \varphi^4 u)_R - 10(\vartheta\bar\partial(\varphi^2 u), \vartheta\varphi\wedge\bar\partial\varphi\wedge u)_R + \\ &+ 20(\vartheta\varphi\wedge\bar\partial\varphi\wedge u, \vartheta\varphi\wedge\bar\partial\varphi\wedge u)_R + 20(\varphi\vartheta\bar\partial\varphi\wedge u, \vartheta\varphi\wedge\bar\partial\varphi\wedge u)_R + \\ &+ 20(-1)^{p+q}(\varphi\bar\partial\varphi\wedge\vartheta u, \vartheta\varphi\wedge\bar\partial\varphi\wedge u)_R + 20(\varphi\vartheta\varphi\wedge\bar\partial u, \vartheta\varphi\wedge\bar\partial\varphi\wedge u)_R + \\ &- 2(\vartheta\bar\partial(\varphi^2 u), \varphi\vartheta\bar\partial\varphi\wedge u)_R + 4(\vartheta\varphi\wedge\bar\partial\varphi\wedge u, \varphi\vartheta\bar\partial\varphi\wedge u)_R + 4(\varphi\vartheta\bar\partial\varphi\wedge u, \varphi\vartheta\bar\partial\varphi\wedge u)_R + \\ &+ 2(-1)^{p+q}(\varphi\bar\partial\varphi\wedge\vartheta u, \varphi\vartheta\bar\partial\varphi\wedge u)_R + 4(\varphi\vartheta\varphi\wedge\bar\partial u, \varphi\vartheta\bar\partial\varphi\wedge u)_R - 2(-1)^{p+q}(\vartheta\bar\partial(\varphi^2 u), \varphi\bar\partial\varphi\wedge\vartheta u)_R + \\ &+ 4(-1)^{p+q}(\vartheta\varphi\wedge\bar\partial\varphi\wedge u, \varphi\bar\partial\varphi\wedge\vartheta u)_R + 4(-1)^{p+q}(\varphi\vartheta\bar\partial\varphi\wedge u, \varphi\bar\partial\varphi\wedge\vartheta u)_R + \\ &+ 4(\varphi\bar\partial\varphi\wedge\vartheta u, \varphi\bar\partial\varphi\wedge\vartheta u)_R + 4(-1)^{p+q}(\varphi\vartheta\varphi\wedge\bar\partial u, \varphi\bar\partial\varphi\wedge\vartheta u)_R - 2(\vartheta\bar\partial(\varphi^2 u), \varphi\vartheta\varphi\wedge\bar\partial u)_R + \\ &+ 4(\vartheta\varphi\wedge\bar\partial\varphi\wedge u, \varphi\vartheta\varphi\wedge\bar\partial u)_R + 4(\varphi\vartheta\bar\partial\varphi\wedge u, \varphi\vartheta\varphi\wedge\bar\partial u)_R + 4(-1)^{p+q}(\varphi\bar\partial\varphi\wedge\vartheta u, \varphi\vartheta\varphi\wedge\bar\partial u)_R + \\ &+ 4(\varphi\vartheta\varphi\wedge\bar\partial u, \varphi\vartheta\varphi\wedge\bar\partial u)_R + 2(\vartheta\varphi\wedge\bar\partial\varphi\wedge u, \vartheta\bar\partial(\varphi^2 u))_R + 2(\varphi\vartheta\bar\partial\varphi\wedge u, \vartheta\bar\partial(\varphi^2 u))_R + \\ &+ 2(-1)^{p+q}(\varphi\bar\partial\varphi\wedge\vartheta u, \vartheta\bar\partial(\varphi^2 u))_R + 2(\varphi\vartheta\varphi\wedge\bar\partial u, \vartheta\bar\partial(\varphi^2 u))_R.\end{aligned} \tag{3.8}$$

After the same computation we can obtain a similar identity for the term $(\bar\partial\vartheta(\varphi^2 u), \bar\partial\vartheta(\varphi^2 u))_R$.

Step 2. Now we will prove (3.4). By taking the sum of (3.6),(3.7),(3.8) and the similar identity for the term $(\bar\partial\vartheta(\varphi^2 u), \bar\partial\vartheta(\varphi^2 u))_R$, taking into account the very definition of $\mathcal{D}$, Young inequality and (3.5) we easily obtain

$$\begin{aligned}&(\vartheta\bar\partial(\varphi^2 u), \vartheta\bar\partial(\varphi^2 u))_R + (\bar\partial\vartheta(\varphi^2 u), \bar\partial\vartheta(\varphi^2 u))_R + (\varphi^2\partial u, \varphi^2\partial u)_R + (\varphi^2\bar\partial u, \varphi^2\bar\partial u)_R \le \\ &\le |(\mathcal{D}u, \varphi^4 u)_R| + \frac{1}{2}[(\vartheta\bar\partial(\varphi^2 u), \vartheta\bar\partial(\varphi^2 u)_R + (\bar\partial\vartheta(\varphi^2 u), \bar\partial\vartheta(\varphi^2 u))_R + (\varphi^2\partial u, \varphi^2\partial u)_R + (\varphi^2\bar\partial u, \varphi^2\bar\partial u)_R] + \\ &+ \left(\frac{\alpha}{(R-r)^2} + \frac{\beta}{(R-r)^4}\right)(u,u)_R + \frac{\gamma}{(R-r)^2}((\bar\partial u, \bar\partial u)_R + (\vartheta u, \vartheta u)_R)\end{aligned} \tag{3.9}$$

for some positive constants α, β, γ depending only on the complex dimension n. Then

$$\begin{aligned}&(\vartheta\bar\partial(\varphi^2 u), \vartheta\bar\partial(\varphi^2 u))_R + (\bar\partial\vartheta(\varphi^2 u), \bar\partial\vartheta(\varphi^2 u))_R + (\varphi^2\partial u, \varphi^2\partial u)_R + (\varphi^2\bar\partial u, \varphi^2\bar\partial u)_R \le \\ &\le 2|(\mathcal{D}u, \varphi^4 u)_R| + \left(\frac{2\alpha}{(R-r)^2} + \frac{2\beta}{(R-r)^4}\right)(u,u)_R + \frac{2\gamma}{(R-r)^2}((\bar\partial u, \bar\partial u)_R + (\vartheta u, \vartheta u)_R).\end{aligned}$$

Now observe that

$$(\vartheta\bar\partial(\varphi^2 u),\vartheta\bar\partial(\varphi^2 u))_R+(\bar\partial\vartheta(\varphi^2 u),\bar\partial\vartheta(\varphi^2 u))_R=(\Box(\varphi^2 u),\Box(\varphi^2 u))_R$$

and, at the same time, applying (3.3),

$$(\vartheta\bar\partial(\varphi^2 u),\vartheta\bar\partial(\varphi^2 u))_R+(\bar\partial\vartheta(\varphi^2 u),\bar\partial\vartheta(\varphi^2 u))_R=(\varphi^2 u,\varphi^2 u)_{2,R}.$$

Thus, since $\varphi=1$ on B_r, we deduce that

$$(\Box u,\Box u)_r+(u,u)_{2,r}+(\partial u,\partial u)_r+(\bar\partial u,\bar\partial u)_r\le\left(\frac{4\alpha}{(R-r)^2}+\frac{4\beta}{(R-r)^4}\right)(u,u)_R+$$

$$+\frac{4\gamma}{(R-r)^2}((\bar\partial u,\bar\partial u)_R+(\vartheta u,\vartheta u)_R)+4|(\mathcal{D}u,\varphi^4 u)_R|.$$

Finally, applying again Young inequality, we obtain, for any $\eta>0$,

$$(\Box u,\Box u)_r+(u,u)_{2,r}+(\partial u,\partial u)_r+(\bar\partial u,\bar\partial u)_r\le\left(\frac{4\alpha}{(R-r)^2}+\frac{4\beta}{(R-r)^4}+\frac{4}{\eta}\right)(u,u)_R+$$

$$+\frac{4\gamma}{(R-r)^2}((\bar\partial u,\bar\partial u)_R+(\vartheta u,\vartheta u)_R)+4\eta|(\mathcal{D}u,\mathcal{D}u)_R|$$

which is, up to constants, inequality (3.4). □

4. A Hodge-Kodaira decomposition for the space $\mathrm{W}^{p,q}(X)$

This section is devoted to the proof of a Hodge-Kodaira orthogonal decompostion for the space $\mathrm{W}^{p,q}(X)$.

Proposition 4.1. *It holds*

(4.1) $\quad \operatorname{Ker}\Box\cap\mathrm{W}^{p,q}(X)=\operatorname{Ker}\mathcal{D}\cap\mathrm{W}^{p,q}(X)=\{u\in\mathrm{A}^{p,q}(X)\cap\mathrm{W}^{p,q}(X):\vartheta\bar\vartheta u=\partial u=\bar\partial u=0\}.$

Proof. Let $u\in\mathrm{W}^{p,q}(X)$ with $\mathcal{D}u=0$. Then inequality (3.4) implies that

$$(\vartheta\bar\vartheta u,\vartheta\bar\vartheta u)_r+(\partial u,\partial u)_r+(\bar\partial u,\bar\partial u)_r\le\left(\frac{c_1}{(R-r)^4}+\frac{c_2}{(R-r)^2}+\frac{1}{\sigma}\right)(u,u)_X+$$

$$+\frac{c_3}{(R-r)^2}((\bar\partial u,\bar\partial u)_X+(\vartheta u,\vartheta u)_X)$$

for any $R,r,\sigma>0$ with $r<R$. Observe that since X is connected we get

$$(\vartheta\bar\vartheta u,\vartheta\bar\vartheta u)_r+(\partial u,\partial u)_r+(\bar\partial u,\bar\partial u)_r\to(\vartheta\bar\vartheta u,\vartheta\bar\vartheta u)_X+(\partial u,\partial u)_X+(\bar\partial u,\bar\partial u)_X$$

as $r\to+\infty$. Choosing $r=R/2$ and by taking the lim sup as $R,\sigma\to+\infty$ we deduce that $(\vartheta\bar\vartheta u,\vartheta\bar\vartheta u)_X=(\partial u,\partial u)_X=(\bar\partial u,\bar\partial u)_X=0$. Then $\vartheta\bar\vartheta u=\partial u=\bar\partial u=0$. Conversely if $u\in\mathrm{A}^{p,q}(X)$ and if $\vartheta\bar\vartheta u=\partial u=\bar\partial u=0$ then recalling (3.3) we immediately have $\mathcal{D}u=0$. Then

$$\operatorname{Ker}\mathcal{D}\cap\mathrm{W}^{p,q}(X)=\{u\in\mathrm{A}^{p,q}(X)\cap\mathrm{W}^{p,q}(X):\vartheta\bar\vartheta u=\partial u=\bar\partial u=0\}.$$

Now if $u\in\mathrm{A}^{p,q}(X)\cap\mathrm{W}^{p,q}(X)$ and $\Box u=0$ then applying (1.2) and the same for $\overline{\Box}$ we get $\partial u=\bar\partial u=\vartheta u=\bar\vartheta u=0$, and thus $\mathcal{D}u=0$. Conversely if $u\in\mathrm{A}^{p,q}(X)\cap\mathrm{W}^{p,q}(X)$ and $\mathcal{D}u=0$ then by (3.4) we have

$$(\Box u,\Box u)_r\le\left(\frac{c_1}{(R-r)^4}+\frac{c_2}{(R-r)^2}+\frac{1}{\sigma}\right)(u,u)_X+\frac{c_3}{(R-r)^2}((\bar\partial u,\bar\partial u)_X+(\vartheta u,\vartheta u)_X).$$

Reasoning as before we conclude. □

Lemma 4.2. *If at least one form between u and v belong to $\mathrm{D}^{p,q}(X)$ then*

(4.2) $\quad (\partial u,v)_{1,X}=(u,\bar\vartheta v)_{1,X}$ *and* $(\bar\partial u,v)_{1,X}=(u,\vartheta v)_{1,X}.$

Proof. By direct computation we have, since (2.1) and (2.2) hold,

$$(\partial u, v)_{1,X} = (\partial u, v)_X + (\bar\partial\partial u, \bar\partial v)_X + (\vartheta\partial u, \vartheta v)_X = (u, \bar\vartheta v)_X + (\vartheta\bar\partial\partial u, v)_X + (\bar\partial\vartheta\partial u, v)_X =$$
$$= (u, \bar\vartheta v)_X + ((\vartheta\bar\partial + \bar\partial\vartheta)\partial u, v)_X = (u, \bar\vartheta v)_X + ((\partial\bar\vartheta + \bar\vartheta\partial)\partial u, v)_X =$$
$$= (u, \bar\vartheta v)_X + (\partial\bar\vartheta\partial u, v)_X = (u, \bar\vartheta v)_X + (\bar\vartheta\partial u, \bar\vartheta v)_X$$

and

$$(u, \bar\vartheta v)_{1,X} = (u, \bar\vartheta v)_X + (\bar\partial u, \bar\partial\bar\vartheta v)_X + (\vartheta u, \vartheta\bar\vartheta v)_X =$$
$$= (u, \bar\vartheta v)_X + (\vartheta\bar\partial u, \bar\vartheta v)_X + (\bar\partial\vartheta u, \bar\vartheta v)_X = (u, \bar\vartheta v)_X + (\vartheta\bar\partial u + \bar\partial u, \bar\vartheta v)_X =$$
$$= (u, \bar\vartheta v)_X + (\partial\bar\vartheta u + \bar\vartheta\partial u, \bar\vartheta v)_X = (u, \bar\vartheta v)_X + (\bar\vartheta\partial u, \bar\vartheta v)_X.$$

The other one is similar. □

Theorem 4.3. (Hodge-Kodaira decomposition) *The following Hodge-Kodaira orthogonal decomposition holds:*

$$\mathrm{W}^{p,q}(X) = [\partial\bar\partial\mathrm{D}^{p-1,q-1}(X)]_1 \oplus_\perp [\vartheta\mathrm{D}^{p,q+1}(X) + \bar\vartheta\mathrm{D}^{p+1,q}(X)]_1 \oplus_\perp \mathrm{Ker}\,\Box \cap \mathrm{W}^{p,q}(X). \tag{4.3}$$

Proof. Taking into account (4.1) it is sufficient to show that

$$\mathrm{W}^{p,q}(X) = [\partial\bar\partial\mathrm{D}^{p-1,q-1}(X)]_1 \oplus_\perp [\vartheta\mathrm{D}^{p,q+1}(X) + \bar\vartheta\mathrm{D}^{p+1,q}(X)]_1 \oplus_\perp$$
$$\oplus_\perp \{u \in \mathrm{A}^{p,q}(X) \cap \mathrm{W}^{p,q}(X) : \vartheta\bar\vartheta u = \partial u = \bar\partial u = 0\}.$$

Step 1. First we prove that the subspaces

$$[\partial\bar\partial\mathrm{D}^{p-1,q-1}(X)]_1 \quad \text{and} \quad [\vartheta\mathrm{D}^{p,q+1}(X) + \bar\vartheta\mathrm{D}^{p+1,q}(X)]_1$$

are orthogonal in the space $\mathrm{W}^{p,q}(X)$. Let $u = \partial\bar\partial\tilde u$ for some $\tilde u \in \mathrm{D}^{p-1,q-1}(X)$ and let $v = \vartheta\tilde v_1 + \bar\vartheta\tilde v_2$ for some $\tilde v_1 \in \mathrm{D}^{p,q+1}(X)$ and $\tilde v_2 \in \mathrm{D}^{p+1,q}(X)$. Then taking into account (4.2) we get

$$(u, v)_{1,X} = (\partial\bar\partial\tilde u, \vartheta\tilde v_1)_{1,X} + (\partial\bar\partial\tilde u, \bar\vartheta\tilde v_2)_{1,X} = -(\bar\partial\partial\tilde u, \vartheta\tilde v_1)_{1,X} + (\partial\bar\partial\tilde u, \bar\vartheta\tilde v_2)_{1,X} =$$
$$= -(\partial\tilde u, \vartheta^2\tilde v_1)_{1,X} + (\bar\partial\tilde u, \bar\vartheta^2\tilde v_2)_{1,X} = 0.$$

Passing to the closures in $\mathrm{W}^{p,q}(X)$ we conclude.

Step 2. Taking into account *Step 1* and applying the projection theorem in an Hilbert space we obtain the orthogonal decomposition

$$\mathrm{W}^{p,q}(X) = [\partial\bar\partial\mathrm{D}^{p-1,q-1}(X)]_1 \oplus_\perp [\vartheta\mathrm{D}^{p,q+1}(X) + \bar\vartheta\mathrm{D}^{p+1,q}(X)]_1 \oplus_\perp$$
$$\oplus_\perp [\partial\bar\partial\mathrm{D}^{p-1,q-1}(X)]_1^\perp \cap [\vartheta\mathrm{D}^{p,q+1}(X) + \bar\vartheta\mathrm{D}^{p+1,q}(X)]_1^\perp.$$

Using the same argument as before we easily get

$$\{u \in \mathrm{A}^{p,q}(X) \cap \mathrm{W}^{p,q}(X) : \vartheta\bar\vartheta u = \partial u = \bar\partial u = 0\} \subseteq$$
$$\subseteq [\partial\bar\partial\mathrm{D}^{p-1,q-1}(X)]_1^\perp \cap [\vartheta\mathrm{D}^{p,q+1}(X) + \bar\vartheta\mathrm{D}^{p+1,q}(X)]_1^\perp.$$

Now if $u \in [\partial\bar\partial\mathrm{D}^{p-1,q-1}(X)]_1^\perp \cap [\vartheta\mathrm{D}^{p,q+1}(X) + \bar\vartheta\mathrm{D}^{p+1,q}(X)]_1^\perp$ then for each $v \in \mathrm{D}^{p-1,q-1}(X)$, $w \in \mathrm{D}^{p,q+1}(X)$ and $z \in \mathrm{D}^{p+1,q}(X)$ we have

$$(\partial\bar\partial v, u)_{1,X} = (\vartheta w + \bar\vartheta z, u)_{1,X} = 0. \tag{4.4}$$

Let $\bar u \in \mathrm{D}^{p,q}(X)$; then considering (3.3) we have

$$(\mathcal{D}\bar u, u)_{1,X} = (\partial\bar\partial\omega_1, u)_{1,X} + (\vartheta\omega_2 + \bar\vartheta\omega_3, u)_{1,X}$$

where

$$\omega_1 = \vartheta\bar\vartheta\bar u \in \mathrm{D}^{p-1,q-1}(X), \quad \omega_2 = \bar\partial\vartheta\partial\bar u \in \mathrm{D}^{p,q+1}(X), \quad \omega_3 = \bar\vartheta\partial\bar\partial\bar u + \partial\bar\vartheta\bar\partial\bar u \in \mathrm{D}^{p+1,q}(X)$$

and thus from (4.4) we deduce that $(\mathcal{D}\bar u, u)_X = 0$. Then u is a weak solution of the equation $\mathcal{D} = 0$; since $\mathcal{D}$ is an elliptic operator we get $u \in \mathrm{A}^{p,q}(X)$. By (4.2) we finally obtain

$$(v, \vartheta\bar\vartheta u)_{1,X} = (w, \bar\partial u)_{1,X} = (z, \partial u)_{1,X} = 0$$

for all $v \in \mathrm{D}^{p-1,q-1}(X)$, $w \in \mathrm{D}^{p,q+1}(X)$ and $z \in \mathrm{D}^{p+1,q}(X)$. Therefore $\vartheta\bar\vartheta u = \partial u = \bar\partial u = 0$, and this concludes the proof of (4.3). □

5. Applications to the study of Aeppli groups

Let $p, q \geq 1$ integers. As recalled in the Introduction, the Aeppli groups $\Lambda^{p,q}$ were originally defined by

$$\Lambda^{p,q} = \frac{\mathrm{Ker}\{\mathrm{A}^{p,q}(X) \xrightarrow{d} \mathrm{A}^{p+1,q+1}(X)\}}{\partial\bar{\partial}\mathrm{A}^{p-1,q-1}(X)}.$$

Bigolin, in [11], proves, using certain resolutions of the sheaf of germs of $\partial\bar{\partial}$-closed functions, that there exists an algebraic isomorphism between $\Lambda^{p,q}$ and

$$\tilde{\Lambda}^{p,q} := \frac{\mathrm{Ker}\{\mathrm{K}^{p,q}(X) \xrightarrow{d} \mathrm{K}^{p+1,q+1}(X)\}}{\partial\bar{\partial}\mathrm{K}^{p-1,q-1}(X)}.$$

First we prove the following lemma.

Lemma 5.1. *The natural map*

$$\frac{[\mathrm{Ker}\{\mathrm{D}^{p,q}(X) \xrightarrow{d} \mathrm{D}^{p+1,q}(X) \oplus \mathrm{D}^{p,q+1}(X)\}]_1}{[\partial\bar{\partial}\mathrm{D}^{p-1,q-1}(X)]_1} \to \frac{[\mathrm{Ker}\{\mathrm{D}^{p+q}(X) \xrightarrow{d} \mathrm{D}^{p+q+1}(X)\}]_{\mathrm{L}^{p+q}}}{[d\mathrm{D}^{p+q-1}(X)]_{\mathrm{L}^{p+q}}}.$$

is injective.

Proof. Using the same argument of the proof of theorem 4.3 it is possible to show that

$$[\mathrm{Ker}\{\mathrm{D}^{p,q}(X) \xrightarrow{d} \mathrm{D}^{p+1,q}(X) \oplus \mathrm{D}^{p,q+1}(X)\}]_1 = [\vartheta\mathrm{D}^{p,q+1}(X) + \bar{\vartheta}\mathrm{D}^{p+1,q}(X)]_1^{\perp}.$$

Taking into account the Hodge-Kodaira decomposition (4.3) we deduce that

$$\frac{[\mathrm{Ker}\{\mathrm{D}^{p,q}(X) \xrightarrow{d} \mathrm{D}^{p+1,q}(X) \oplus \mathrm{D}^{p,q+1}(X)\}]_1}{[\partial\bar{\partial}\mathrm{D}^{p-1,q-1}(X)]_1} \quad \text{and} \quad \mathrm{Ker}\,\Box \cap \mathrm{W}^{p,q}(X)$$

are isomorphic. Now $\mathrm{Ker}\,\Box \cap \mathrm{W}^{p,q}(X) \subseteq \mathrm{Ker}\,\Delta \cap \mathrm{L}^{p+q}(X)$. Since

$$[\mathrm{Ker}\{\mathrm{D}^{p+q}(X) \xrightarrow{d} \mathrm{D}^{p+q+1}(X)\}]_{\mathrm{L}^{p+q}(X)} = [\delta\mathrm{D}^{p+1}(X)]^{\perp}_{\mathrm{L}^{p}(X)}$$

then, by the classical Hodge-Kodaira decomposition (1.5),

$$\mathrm{Ker}\,\Delta \cap \mathrm{L}^{p+q}(X) \quad \text{and} \quad \frac{[\mathrm{Ker}\{\mathrm{D}^{p+q}(X) \xrightarrow{d} \mathrm{D}^{p+q+1}(X)\}]_{\mathrm{L}^{p+q}(X)}}{[d\mathrm{D}^{p+q-1}(X)]_{\mathrm{L}^{p+q}(X)}}$$

are isomorphic, and this concludes the proof. □

In order to prove the main theorem of this section, i.e. a characterization of the Aeppli groups $\Lambda^{p,q}(X)$, we have to assume a technical topological condition on the manifold X. More precisely we will assume that

$$\partial\bar{\partial}\mathrm{K}^{p-1,q-1}(X) \quad \text{is weakly closed in } \mathrm{K}^{p,q}(X) \tag{5.1}$$

where the weak topology on $\mathrm{K}^{p,q}(X)$ is the usual weak topology of distributions (recall that $\mathrm{K}^{p,q}(X)$ is the dual space of $\mathrm{D}^{p,q}(X)$). It is well known that compact manifolds and Stein manifolds are examples of manifolds satisfying condition (5.1), so that our result extends the results contained in [1] and [10]. Moreover we point out that condition (5.1) is a necessary condition in order to prove only the next theorem: all the rest of the paper holds independently from this assumption; in particular the Stampacchia-type inequality (3.4) and the Hodge-Kodaira decomposition (4.3) hold for any connected and complete Kähler manifolds.

Theorem 5.2. *Let us assume* (5.1)*. Then the Aeppli group* $\Lambda^{p,q}(X)$ *is isomorphic to the group* $\mathrm{H}^{p,q}(X)$*, where we recall that* $\mathrm{H}^{p,q}(X)$ *denotes the space of all global harmonic* $(p+q)$*-forms of bidegree* (p,q)*.*

Proof. Since

$$\mathrm{H}^{p+q}(X) \simeq \frac{\operatorname{Ker}\{\mathrm{K}^{p+q}(X) \xrightarrow{d} \mathrm{K}^{p+q+1}(X)\}}{d\mathrm{K}^{p+q-1}(X)}$$

and since the image of the natural map

$$i\colon \frac{\operatorname{Ker}\{\mathrm{K}^{p,q}(X) \xrightarrow{d} \mathrm{K}^{p+1,q+1}(X)\}}{\partial\bar{\partial}\mathrm{K}^{p-1,q-1}(X)} \to \frac{\operatorname{Ker}\{\mathrm{K}^{p+q}(X) \xrightarrow{d} \mathrm{K}^{p+q+1}(X)\}}{d\mathrm{K}^{p+q-1}(X)}$$

is exactly $\mathrm{H}^{p,q}(X)$, then it is sufficient to show that i is injective. Let $T \in \mathrm{K}^{p,q}(X)$ with $T = dS$ for some $S \in \mathrm{K}^{p+q-1}(X)$. Then we have to show that there exists $R \in \mathrm{K}^{p-1,q-1}(X)$ such that $T = \partial\bar{\partial}R$. Since $\mathrm{D}^{p+q-1}(X)$ is dense in $\mathrm{K}^{p+q-1}(X)$ then there exists a sequence $(S_h)_{h\in\mathbb{N}} \subseteq \mathrm{D}^{p+q-1}(X)$ with $S_h \to S$ as $h \to +\infty$. Then $dS_h \to T$ and we can suppose, without loss of generality, that $dS_h \in \mathrm{D}^{p,q}(X)$. Let $T_h = dS_h$. Taking into account lemma 5.1 we get

$$T_h \in [\partial\bar{\partial}\mathrm{D}^{p-1,q-1}(X)]_1$$

so that

$$T_h = \lim_{k\to+\infty} T_h^k$$

with $T_h^k = \partial\bar{\partial}U_h^k$ for some $U_h^k \in \mathrm{D}^{p-1,q-1}(X)$. Since we are assuming $\partial\bar{\partial}\mathrm{K}^{p-1,q-1}(X)$ weakly closed in $\mathrm{K}^{p,q}(X)$ then

$$T_h = \partial\bar{\partial}R_h$$

for some $R_h \in \mathrm{K}^{p-1,q-1}(X)$, and then $T = \partial\bar{\partial}R$ for some $R \in \mathrm{K}^{p-1,q-1}(X)$, which ends the proof. □

Remark 5.3. *One can repeat all the considerations on the operator*

$$\mathcal{D}^* = \vartheta\bar{\partial}\vartheta\bar{\partial} + \bar{\partial}\vartheta\bar{\partial}\vartheta + \bar{\partial}\vartheta + \partial\bar{\vartheta}$$

and in particular we get the Hodge-Kodaira orthogonal decomposition

$$\mathrm{W}^{p,q}(X) = [\vartheta\bar{\vartheta}\mathrm{D}^{p+1,q+1}(X)]_1 \oplus_\perp [\partial\mathrm{D}^{p-1,q}(X) + \bar{\partial}\mathrm{D}^{p,q-1}(X)]_1 \oplus_\perp \operatorname{Ker}\square \cap \mathrm{W}^{p,q}(X)$$

which permits to study the Aeppli groups $\mathrm{V}^{p,q}(X)$ *reasoning as in lemma 5.1 and in theorem 5.2.*

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