THE SECTOR OF ANALYTICITY OF NONSYMMETRIC SUBMARKOVIAN SEMIGROUPS GENERATED BY ELLIPTIC OPERATORS

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Abstract. We estimate the sector of analyticity of not necessarily symmetric submarkovian semigroups generated by second order elliptic operators in divergence form or by Ornstein-Uhlenbeck operators. A lower bound for the angle θ_p of the sector of analyticity in L^p_μ is given by the formula

$$\cot\theta_p = \frac{\sqrt{(p-2)^2 + p^2(\cot\theta_2)^2}}{2\sqrt{p-1}}.$$

If the semigroup is symmetric then we recover known results. In general, this lower bound is optimal.

Let $\Omega \subset \mathbb{R}^N$ be open, and let $S \in L^\infty(\Omega;\mathbb{R}^{N\times N})$ be uniformly elliptic. Assume that S is in addition uniformly sectorial, i.e., there exists a constant $c_2 \geq 0$ such that

$$|\mathrm{Im}\,\langle S(x)\xi,\xi\rangle| \leq c_2 \mathrm{Re}\,\langle S(x)\xi,\xi\rangle \text{ for all } x\in\Omega,\ \xi\in\mathbb{C}^N, \tag{1}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual hermitian product in $\mathbb{C}^N$.

Let m be positive function in Ω such that $m, m^{-1} \in L^\infty_{loc}(\Omega)$ and define the Borel measure $d\mu = m\ d\lambda$, where λ is the Lebesgue measure on Ω. Let us introduce the spaces $L^p_\mu = L^p(\Omega; d\mu)$ and

$$W^{1,p}_\mu = \{u \in W^{1,p}_{loc}(\Omega) : u,\ \nabla u \in L^p_\mu\}.$$

We shall write H^1_μ instead of $W^{1,2}_\mu$, as usual. The operator A_2 on L^2_μ defined by

$$\begin{aligned} D(A_2) &:= \{u \in L^2_\mu : \exists v \in L^2_\mu \text{ s.t. } \forall\varphi\in H^1_\mu : \int_\Omega \langle S(x)\nabla u,\nabla v\rangle\, d\mu = \langle v,\varphi\rangle_{L^2_\mu}\}, \\ A_2 u &:= v, \end{aligned}$$

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which is associated with the form (a, H^1_μ)

$$a(u,v) := \int_\Omega \langle S(x)\nabla u, \nabla v\rangle \, d\mu, \quad u,\, v \in H^1_\mu,$$

is the negative generator of a submarkovian semigroup $(e^{-tA_2})_{t\geq 0}$, i.e., e^{-tA_2} is a positive contraction which is also L^∞-contractive. Let us stress that we are not assuming that A_2 is self-adjoint.

By (1), the form a and the operator A_2 are sectorial, and the semigroup $(e^{-tA_2})_{t\geq 0}$ extends to an analytic contraction semigroup on the sector $\Sigma_{\theta_2} := \{z \in \mathbb{C} : |\arg z| < \theta_2\}$, where the angle θ_2 is determined by the constant c_2 in (1) through the equation $\cot\theta_2 = c_2$.

Moreover, by L^∞ contractivity, the semigroup $(e^{-tA_2})_{t\geq 0}$ extrapolates on all L^p_μ, $2 \leq p \leq \infty$. Since A_2^* is the operator associated with the matrix S^*, it is also the negative generator of a submarkovian semigroup and therefore, by duality, the semigroup $(e^{-tA_2})_{t\geq 0}$ extrapolates on all L^p_μ, $1 < p \leq \infty$. The negative generator on L^p_μ will be denoted by A_p. We prove the following theorem.

Theorem 1. *For every $1 < p < \infty$, the semigroup $(e^{-tA_p})_{t\geq 0}$ on L^p_μ extends to an analytic semigroup of contractions on the sector Σ_{θ_p}, where*

$$\cot\theta_p = \frac{\sqrt{(p-2)^2 + p^2 c_2^2}}{2\sqrt{p-1}}.$$

Example 2. Theorem 1 applies to semigroups generated by second order elliptic operators in divergence form with bounded measurable real coefficients:

$$Au = -\operatorname{div} S(x)\nabla u \quad \text{on } \Omega.$$

In this example one takes $\mu = \lambda$, where λ is the Lebesgue measure on Ω, so that $L^2_\mu = L^2$ and $H^1_\mu = H^1$ are the usual Lebesgue and Sobolev spaces on Ω. The form (a, H^1) is given by

$$a(u,v) = \int_\Omega \langle S(x)\nabla u, \nabla v\rangle \, d\lambda, \quad u,\, v \in H^1.$$

Example 3. Theorem 1 applies to semigroups generated by *Ornstein-Uhlenbeck operators* of the form

$$Au = -\Delta u - Bx\,\nabla u \quad \text{on } \mathbb{R}^N,$$

where B is a real matrix having only eigenvalues with negative real part. Theorem 1 applies if μ is the invariant measure for the Ornstein-Uhlenbeck semigroup:

$$d\mu(x) = \frac{1}{\sqrt{(4\pi)^N \det Q_\infty}} \, e^{-\frac{1}{4}\langle Q_\infty^{-1}x, x\rangle} \, d\lambda(x),$$

where

$$Q_\infty := \int_0^\infty e^{sB} e^{sB^*}\, ds.$$

In this example, the form (a, H^1_μ) is given by

$$a(u,v) = -2 \int_{\mathbb{R}^N} \langle Q_\infty B^* \nabla u, \nabla v\rangle \, d\mu, \quad u,\, v \in H^1_\mu.$$

Example 4. Theorem 1 also applies to semigroups generated by *Ornstein-Uhlenbeck operators* of the form

$$Au = -\mathrm{div}\, S\nabla u - S^* \nabla\varphi\, \nabla u \quad \text{on } \Omega,$$

where $\varphi \in C^1(\Omega)$ and $S \in \mathbb{R}^{N\times N}$. In this example

$$d\mu(x) = e^{-\varphi(x)}\, d\lambda(x),$$

and (a, H^1_μ) is given by

$$a(u,v) = \int_{\mathbb{R}^N} \langle S\nabla u, \nabla v\rangle \, d\mu, \quad u,\, v \in H^1_\mu.$$

Remark 5. The angle of analyticity θ_p from Theorem 1 is in general better than the angle of analyticity which one would obtain by the Stein interpolation theorem, see [11]. That angle would be $\theta_2(1 - |\frac{2}{p} - 1|)$.

Remark 6. In the case when the $S(x)$ are symmetric, so that A_2 is self-adjoint, Theorem 1 has been proved in [1, 3, 8]. For general symmetric submarkovian semigroups, see [5, 12].

Remark 7. The proof of Theorem 1 will show that instead of the form domain H^1_μ one may also choose $H^1_{\mu,0}$ (the closure of $C_0^\infty(\Omega)$ in H^1_μ), which corresponds to Dirichlet boundary conditions in the case of nondegenerate m. One may take also as form domain H^1_μ but change the form to

$$a(u,v) := \int_\Omega \langle S(x)\nabla u, \nabla v\rangle \, d\mu + \int_{\partial\Omega} \beta(x) u\bar{v}, \quad u,\, v \in H^1_\mu,$$

which then corresponds to Robin boundary conditions.

Remark 8. Clearly, it can happen that the semigroup $(e^{-tA_p})_{t\geq 0}$ extends analytically to a larger sector than the sector described in Theorem 1. This can happen even if $p = 2$ and the constant c_2 from (1) is optimal; for nonsymmetric but *constant* S one has $c_2 > 0$ but if $\mu = \lambda$ is the Lebesgue measure on Ω then the operator A_2 is self-adjoint. In this case, it is known that $(e^{tA_p})_{t\geq 0}$ extends analytically to the sector Σ_{θ_p} where

$$\cot\theta_p = \frac{|p-2|}{2\sqrt{p-1}},$$

which is our θ_p for $c_2 = 0$.

However, in general the angle θ_p from Theorem 1 is optimal as the following proposition shows. It is an immediate consequence of [2, Theorem 2].

Proposition 9. *Consider the Ornstein-Uhlenbeck operator on the space* L^p_μ *from Example 3. Then for every* $p \in (1,\infty)$ *the angle of analyticity* θ_p *from Theorem 1 is optimal. More precisely: whenever* $(e^{-tA_p})_{t\geq 0}$ *extends to an analytic semigroup on a sector* Σ_θ *(the extended semigroup need* a priori *not be a contraction semigroup), then* $\theta \leq \theta_p$.

Proof of Theorem 1. Fix $p \in (2,\infty)$. By the Lumer-Phillips theorem, the semigroup $(e^{-tA_p})_{t\geq 0}$ on L^p_μ extends to an analytic semigroup of contractions on the sector Σ_{θ_p} if and only if $-e^{i\varphi}A_p$ is dissipative for every $\varphi \in (-\theta_p, \theta_p)$, i.e. if and only if for every $u \in D(A_p)$

$$\left|\operatorname{Im}\int_\Omega A_p u\, u^*\, d\mu\right| \leq \cot\theta_p \operatorname{Re}\int_\Omega A_p u\, u^*\, d\mu, \tag{2}$$

where

$$u^* := \bar{u}\,|u|^{p-2} \quad (\in L^{p'}_\mu,\, p' = \frac{p}{p-1}).$$

Note that for every $u \in D =: D(A_2) \cap D(A_p) \cap L^\infty$ one has $u \in H^1_\mu \cap L^\infty$ and therefore also $u^* \in H^1_\mu \cap L^\infty$. Hence, for every $u \in D$ one has

$$\int_\Omega A_p u\, u^*\, d\mu = \int_\Omega A_2 u\, u^*\, d\mu = a(u,u^*) = \int_\Omega S(x)\nabla u\, \nabla u^*\, d\mu;$$

here, $\xi\,\eta = \sum_{i=1}^N \xi_i \eta_i$ for $\xi,\, \eta \in \mathbb{C}^N$. Since D is a core for A_p (note that D is dense in L^p_μ and invariant under the semigroup), inequality (2) holds for every $u \in D(A_p)$ if and only if for every $u \in D$ one has

$$\left|\operatorname{Im}\int_\Omega S(x)\nabla u\, \nabla u^*\, d\mu\right| \leq \cot\theta_p \operatorname{Re}\int_\Omega S(x)\nabla u\, \nabla u^*\, d\mu. \tag{3}$$

Fix $x \in \Omega$ and $u \in D$. Set $S := S(x)$, and let

$$S_1 := (S + S^*)/2 \text{ and } S_2 := (S - S^*)/2$$

be the symmetric and the antisymmetric part of S, respectively.

Write $u = v + iw$, where v and w are real-valued. If u^* is defined as above, then

$$\nabla u^* = \nabla\bar{u}|u|^{p-2} + (p-2)\bar{u}(v\nabla v + w\nabla w)|u|^{p-4}.$$

Writing $|u|^{p-2} = |u|^{p-4}(v^2 + w^2)$, we thus obtain

$$\begin{aligned} S\nabla u\, \nabla u^* \;=\; & |u|^{p-4}(v^2+w^2)\big(S(\nabla v + i\nabla w)\,(\nabla v - i\nabla w)\big) \\ & -|u|^{p-4}(v - iw)\big(S(\nabla v + i\nabla w)\,(v\nabla v + w\nabla w)\big) \\ & +(p-1)|u|^{p-4}(v-iw)\big(S(\nabla v + i\nabla w)\,(v\nabla v + w\nabla w)\big). \end{aligned}$$

By simplifying, we obtain

$$\begin{aligned} S\nabla u\,\nabla u^* &= |u|^{p-4}\big[w^2 S\nabla v\,\nabla v + v^2 S\nabla w\,\nabla w \\ &\quad + i(w^2 S\nabla w\,\nabla v - v^2 S\nabla v\,\nabla w) \\ &\quad - vwS\nabla v\,\nabla w - vwS\nabla w\,\nabla v \\ &\quad + i(vwS\nabla v\,\nabla v - vwS\nabla w\,\nabla w) \\ &\quad + (p-1)\big(v^2 S\nabla v\,\nabla v + vwS\nabla v\,\nabla w\big) \\ &\quad + i(p-1)\big(v^2 S\nabla w\,\nabla v + vwS\nabla w\,\nabla w\big) \\ &\quad + (p-1)(vwS\nabla w\,\nabla v + w^2 S\nabla w\,\nabla w) \\ &\quad - i(p-1)(vwS\nabla v\,\nabla v + w^2 S\nabla v\,\nabla w)\big] \\ &= |u|^{p-4}\left[S(v\nabla w - w\nabla v)\,(v\nabla w - w\nabla v)\right. \\ &\quad + (p-1)S(v\nabla v + w\nabla w)\,(v\nabla v + w\nabla w) \\ &\quad + i(p-1)S(v\nabla w - w\nabla v)\,(v\nabla v + w\nabla w) \\ &\quad \left. - iS(v\nabla v + w\nabla w)\,(v\nabla w - w\nabla v)\right]. \end{aligned}$$

Observe that

$$S\xi\,\eta = S^*\eta\,\xi \text{ for every } \xi,\, \eta \in \mathbb{R}^N,$$

and that

$$\begin{aligned} \text{Re }(\bar{u}\nabla u) &= v\nabla v + w\nabla w \quad \text{ and} \\ \text{Im }(\bar{u}\nabla u) &= v\nabla w - w\nabla v. \end{aligned}$$

Hence

$$\begin{aligned} S\nabla u\,\nabla u^* &= |u|^{p-4}\left[\langle S\,\text{Im }(\bar{u}\nabla u), \text{Im }(\bar{u}\nabla u)\rangle\right. \\ &\quad + (p-1)\langle S\,\text{Re }(\bar{u}\nabla u), \text{Re }(\bar{u}\nabla u)\rangle \\ &\quad \left. + i\,\big\langle\big((p-1)S - S^*\big)\,\text{Im }(\bar{u}\nabla u), \text{Re }(\bar{u}\nabla u)\big\rangle\right]. \end{aligned}$$

Since

$$\langle S\xi, \xi\rangle = \langle S_1\xi, \xi\rangle \text{ for every } \xi \in \mathbb{R}^N,$$

and

$$(p-1)S - S^* = (p-2)S_1 - pS_2,$$

we finally obtain

$$\begin{aligned} S\nabla u\,\nabla u^* &= |u|^{p-4}\left[\langle S_1\,\text{Im}\,(\bar{u}\nabla u), \text{Im}\,(\bar{u}\nabla u)\rangle\right. \\ &\quad + (p-1)\langle S_1\,\text{Re}\,(\bar{u}\nabla u), \text{Re}\,(\bar{u}\nabla u)\rangle \\ &\quad \left. + i\,\big\langle\big((p-2)S_1 - pS_2\big)\,\text{Im}\,(\bar{u}\nabla u), \text{Re}\,(\bar{u}\nabla u)\big\rangle\right]. \end{aligned}$$

Since S_1 is elliptic, there exist $S_1^{\frac12}$ and $S_1^{-\frac12}$. By the Cauchy-Schwarz inequality

$$\begin{aligned}|\mathrm{Im}\, S\nabla u\, \nabla u^*| &= \\ &= |u|^{p-4} \left|\langle\big((p-2)I - pS_1^{-\frac12} S_2 S_1^{-\frac12}\big)(S_1^{\frac12}\mathrm{Im}\,(\bar u\nabla u)), (S_1^{\frac12}\mathrm{Re}\,(\bar u\nabla u))\rangle\right| \\ &\qquad \le |u|^{p-4}\, \|(p-2)I - pS_1^{-\frac12} S_2 S_1^{-\frac12}\|\, \|S_1^{\frac12}\mathrm{Im}\,(\bar u\nabla u)\|\, \|S_1^{\frac12}\mathrm{Re}\,(\bar u\nabla u)\|.\end{aligned}$$

On the other hand,

$$\mathrm{Re}\, S\nabla u\, \nabla u^* = |u|^{p-4}\big(\|S_1^{\frac12}\mathrm{Im}\,(\bar u\nabla u)\|^2 + (p-1)\|S_1^{\frac12}\mathrm{Re}\,(\bar u\nabla u)\|^2\big).$$

Since the matrix $S_1^{-\frac12} S_2 S_1^{-\frac12}$ is skew-adjoint, the norm of the normal matrix $(p-2)I - pS_1^{-\frac12} S_2 S_1^{-\frac12}$ is equal to its spectral radius. The latter can be easily computed by using Pythagoras' theorem and one obtains

$$\|(p-2)I - pS_1^{-\frac12} S_2 S_1^{-\frac12}\| = \sqrt{(p-2)^2 + p^2\|S_1^{-\frac12} S_2 S_1^{-\frac12}\|^2}.$$

By assumption (1),

$$\|S_1^{-\frac12} S_2 S_1^{-\frac12}\| \le c_2,$$

so that

$$\|(p-2)I - pS_1^{-\frac12} S_2 S_1^{-\frac12}\| \le \sqrt{(p-2)^2 + p^2 c_2^2} =: \kappa.$$

It is easy to verify that for

$$\gamma := \frac{\sqrt{(p-2)^2 + p^2 c_2^2}}{2\sqrt{p-1}}$$

one has

$$\kappa\, ab \le \gamma\,(a^2 + (p-1)b^2) \text{ for every } a,\, b \ge 0.$$

Hence, we have proved that for every $x \in \Omega$ and every $u \in W^{1,p}_\mu$,

$$|\mathrm{Im}\, S(x)\nabla u\, \nabla u^*| \le \gamma\, \mathrm{Re}\, S(x)\nabla u\, \nabla u^*.$$

Integrating this inequality over Ω, we obtain (3).

Now let $p \in (1,2)$, let p' be the conjugate exponent and A_p^* be the adjoint operator. Notice that A_2^* is obtained as A_2, starting from S^* instead of S, and that the constant c_2 in (1) is the same for S and S^*. Moreover, A_p^* is obtained by extrapolation from A_2^* with the same procedure as $A_{p'}$, hence we may apply the first part of the proof, obtaining for A_p^* an analyticity angle $\theta_{p'} = \theta_p$. By duality, the angles of analyticity for A_p and A_p^* coincide, and the claim is proved. □

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