

CONVERGENCE OF INTERACTION-DRIVEN EVOLUTIONS OF DISLOCATIONS WITH WASSERSTEIN DISSIPATION AND SLIP-PLANE CONFINEMENT

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ABSTRACT. We consider systems of n parallel edge dislocations in a single slip system, represented by points in a two-dimensional domain; the elastic medium is modelled as a continuum. We formulate the energy of this system in terms of the empirical measure of the dislocations, and prove several convergence results in the limit $n \rightarrow \infty$.

The main aim of the paper is to study the convergence of the evolution of the empirical measure as $n \rightarrow \infty$. We consider rate-independent, quasi-static evolutions, in which the motion of the dislocations is restricted to the same slip plane. This leads to a formulation of the quasi-static evolution problem in terms of a modified Wasserstein distance, which is only finite when the transport plan is slip-plane-confined.

Since the focus is on interaction between dislocations, we renormalize the elastic energy to remove the potentially large self- or core energy. We prove Gamma-convergence of this renormalized energy, and we construct joint recovery sequences for which both the energies and the modified distances converge. With this augmented Gamma-convergence we prove the convergence of the quasi-static evolutions as $n \rightarrow \infty$.

1. INTRODUCTION

It is well known that plastic, or permanent, deformation in metals is caused by the concerted movement of many curve-like defects in the crystal lattice, called *dislocations*. What is not yet known is how to use this insight to predict behaviour at continuum scales. It would be natural to take a sequence of systems with increasing numbers of dislocations, and derive an effective description in terms of dislocation *densities*. In various cases formal and rigorous convergence results have been proved of the elastic energies generated by the dislocations (see the discussion of the literature below), and this provides a good starting point.

However, macroscopic plasticity heavily depends on *dynamic* properties of the dislocations. The most basic aspect of the motion of dislocation curves of *edge* type, that we study in this paper, is a strong confinement to *slip planes*: each curve can move only in the plane spanned by its *Burgers vector* and the tangent to the curve. Other essential dynamic phenomena are *creation* and *annihilation* of dislocations, and their behaviour at *obstacles*; especially in small systems, these latter phenomena are believed to be the main determining factors in the plastic behaviour of metals [DNG05, CC10].

Although models exist that describe the motion, creation, annihilation, and obstacle behaviour of dislocations at the level of densities (see e.g. [GB99, GCZ03, YGG04, YG05a, YG05b, BGGZ06, GKG07, GGI10]), these are phenomenological in nature, and the connection between these models and more microscopic descriptions of dislocation motion is tenuous. Many different models exist, and at this moment no good method is available to compare these or choose between them. For a better understanding of the complex motion of dislocations it is therefore essential to understand the upscaling of discrete-dislocation models to descriptions at the level of densities.

A mathematical theory of creation, annihilation, and obstacles can only be formulated in the context of dislocation motion that is confined to the slip plane. In this paper we therefore make a first step in the direction of the dynamics of dislocations, by proving a rigorous upscaling of a system of moving edge dislocations in two dimensions *with slip-plane confinement*. Although this confinement may seem a minor restriction, it actually makes the proof of the upscaling far more complex.

1.1. Setup. We restrict ourselves to straight and parallel *edge* dislocations in plane strain, and we assume that only one slip system is active, with Burgers vector $b = e_1 \in \mathbb{R}^2$, and that all dislocations have the same sign. Since we will take the many-dislocation limit, we describe the positions of the dislocations by a measure on a bounded open set $\Omega \subset \mathbb{R}^2$ (the cross section) of the form

$$\mu = \frac{1}{n} \sum_{i=1}^n \delta_{z_i}, \quad \{z_i\}_{i=1}^n \subset \Omega.$$

The dislocations interact with each other through the elasticity of the medium, which we assume to be a homogeneous, isotropic, linearly elastic continuum; its properties are characterized by the fourth-order stress-strain tensor \mathbb{C} , which satisfies $\mathbb{C}F = \lambda \operatorname{tr}(\operatorname{sym} F)\operatorname{Id} + 2\mu \operatorname{sym} F$ with Lamé constants $\lambda + \mu > 0$ and $\mu > 0$.

Since the elastic energy density is not integrable close to a dislocation, we employ the *core-region approach* of removing disks of radius $\varepsilon_n \rightarrow 0$ around the dislocations, leading to an effective domain $\Omega_n(\mu) := \Omega \setminus \bigcup_{j=1}^n \overline{B}_{\varepsilon_n}(z_j)$. In addition we assume that the dislocations are separated from each other by a distance $r_n \rightarrow 0$ and that they can only live in a closed rectangle \mathcal{R} in Ω whose sides are parallel to the coordinate axes and which has a distance $\ell > 0$ from the boundary $\partial\Omega$. We assume that ε_n and r_n satisfy

$$\varepsilon_n \rightarrow 0, \quad r_n \rightarrow 0, \quad \varepsilon_n/r_n^3 \rightarrow 0, \quad r_n n \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (1.1)$$

See Section 1.7.1 for comments on this choice.

At finite n , admissible measures μ belong to the set

$$X_n := \left\{ \mu = \frac{1}{n} \sum_{i=1}^n \delta_{z_i} \text{ such that } \{z_i\}_{i=1}^n \subset \mathcal{R} \text{ and } |z_i - z_j| \geq r_n \text{ for } i \neq j \right\}. \quad (1.2)$$

The elastic energy of a measure $\mu \in X_n$, defined for convenience on the set $\mathcal{P}(\Omega)$ of all probability measures, is

$$F_n(\mu) := \begin{cases} \inf_{\beta \in \mathcal{A}_n(\mu)} E_n(\mu, \beta) & \text{if } \mu \in X_n, \\ +\infty & \text{if } \mu \in \mathcal{P}(\Omega) \setminus X_n, \end{cases} \quad (1.3)$$

where

$$E_n(\mu, \beta) := \frac{1}{2} \int_{\Omega_n(\mu)} \mathbb{C}\beta : \beta \, dx. \quad (1.4)$$

The tensor-valued field β has the interpretation of the elastic part of the strain. The admissibility class $\mathcal{A}_n(\mu)$ characterizes the incompatibility conditions on β generated by the dislocations:

$$\mathcal{A}_n(\mu) := \left\{ \beta \in L^2(\Omega; \mathbb{R}^{2 \times 2}) : \beta = 0 \text{ in } \Omega \setminus \Omega_n(\mu), \quad \operatorname{Curl} \beta = 0 \text{ in } \Omega_n(\mu), \right. \\ \left. \int_{\partial B_{\varepsilon_n}(z_i)} \beta \tau \, d\mathcal{H}^1 = \frac{b}{n} \text{ for every } i = 1, \dots, n \right\}, \quad (1.5)$$

where τ is the tangent to $\partial B_{\varepsilon_n}(z_i)$ and the integrand $\beta \tau$ is understood in the sense of traces (see [CL05] for details). The minimization problem (1.3) is a reformulation of the standard elasticity problem, in terms of the elastic strain β , with stress-free boundary conditions at $\partial\Omega_n(\mu)$.

Note that since the dislocation density $\mu \in X_n$ is rescaled to have mass one, the incompatibility of the strain β at every dislocation is of order $1/n$. We also observe that in alternative to the integral incompatibility condition in (1.5) one could require the more familiar condition on the circulation of the strain:

$$\operatorname{Curl} \beta = \frac{1}{n} \frac{b}{2\pi\varepsilon_n} \sum_{i=1}^n \mathcal{H}^1 \llcorner \partial B_{\varepsilon_n}(z_i) \quad \text{in } \Omega.$$

For more background on dislocations in general, see e.g. [Krö81, HL82, Cal07]; a more detailed derivation of this model can be found in [GN95, CL05, GLP10].

1.2. Γ -convergence of the renormalized energy. The discrete evolutionary system at finite n is defined by the energy functional F_n and a dissipation distance that we introduce below. We first study the behaviour of F_n as $n \rightarrow \infty$.

Garroni, Ponsiglione, and co-workers [GLP10, DLGP12] show that as $n \rightarrow \infty$ and $\varepsilon_n \rightarrow 0$ the energy F_n has contributions of order 1 and of order $n^{-1}|\log \varepsilon_n|$. The contributions of order 1 stem from the interaction between pairs of distant dislocations, of which there are n^2 , each with strength of order n^{-2} by the scaling in (1.5). The contributions of order $n^{-1}|\log \varepsilon_n|$ arise from the energy of order $n^{-2}|\log \varepsilon_n|$ contained in a neighbourhood of each of the n dislocations. Depending on the relative size of n and $|\log \varepsilon_n|$, one or the other will dominate.

In this paper we consider an evolution that conserves the total number of dislocations, and therefore the self-energy of order $n^{-1}|\log \varepsilon_n|$ is of no relevance, even when it is large. To extract the interaction energy, we renormalize F_n by defining

$$\mathcal{F}_n(\mu) := \begin{cases} F_n(\mu) - \frac{1}{2n^2} \sum_{i=1}^n \int_{\Omega_n(\mu)} \mathbb{C}K_{z_i}^n : K_{z_i}^n dx & \text{if } \mu \in X_n, \mu = \frac{1}{n} \sum_{i=1}^n \delta_{z_i}, \\ +\infty & \text{if } \mu \in \mathcal{P}(\Omega) \setminus X_n. \end{cases} \quad (1.6)$$

Here K_z^n is a small correction of the canonical strain field K_z generated by a single dislocation at z in \mathbb{R}^2 , as defined in Section 2.1. The second term has the same scaling $n^{-1}|\log \varepsilon_n|$ as the self-energy, and approximately cancels the self-energy in the first term. The aim of this renormalization is to extract that part of the energy that characterizes the interaction, and our first main result (Theorem 3.3) makes clear in which sense this is indeed the case:

Theorem 1.1. *Under conditions (1.1) on ε_n and r_n , the functionals \mathcal{F}_n Γ -converge in the space of probability measures $\mathcal{P}(\Omega)$ endowed with the narrow topology to the limit functional*

$$\mathcal{F}(\mu) := \frac{1}{2} \iint_{\Omega \times \Omega} V(y, z) d\mu(y) d\mu(z) + \inf_v \left\{ \frac{1}{2} \int_{\Omega} \mathbb{C} \nabla v : \nabla v dx + \int_{\Omega} \int_{\partial \Omega} \mathbb{C} K(x; y) \nu(x) \cdot v(x) d\mathcal{H}^1(x) d\mu(y) \right\}, \quad (1.7)$$

where

$$V(y, z) := \int_{\Omega} \mathbb{C} K_y(x) : K_z(x) dx. \quad (1.8)$$

The first term in the limit \mathcal{F} is a two-point interaction functional, with interaction potential $V(y, z)$. The potential in (1.8) is related to the formula for the interaction between segments of dislocations in infinite space that is widely used in the engineering literature (see e.g. [HL82, (5–16)]). The distinction between this formula and V , and the second term in (1.7), is generated by the boundedness of Ω .

1.3. Quasi-static evolution of dislocations. In the second part of this paper we use the convergence result above to pass to the limit in a *rate-independent* or *quasi-static* evolution that is driven by \mathcal{F}_n . Considering rate-independent evolution is a specific choice, and we motivate this first.

At the atomic scale, dislocation motion is a complex, thermally driven random rearrangement process, displaying such features as splitting into half-dislocations, joining of dislocations of different Burgers vectors, dislocation-core spreading, and many others. In addition, the presence of interstitial atoms, vacancies, stacking faults, grain boundaries, and dislocations belonging to different slip systems all interfere with the motion of a given dislocation line.

In this paper we consider dislocations as points in two dimensions, corresponding to straight and parallel lines in three dimensions (often called the Discrete Dislocation Dynamics (DDD) paradigm [GN95, CGN99, Zbi12]). For the evolution of DDD systems a choice has to be made that expresses the lateral velocity of a dislocation line in terms of the local stress. Atomistic, molecular-dynamics simulations suggest various motion laws, ranging from linear (e.g. [CCBY02]) to the rate-independent one of the type we adopt below [MT09]. The various types of obstacles that a dislocation may encounter further modify the mobility (see e.g. [TS76, OP82]).

These observations imply that there is no universal law for the mobility of edge dislocations in two dimensions. Since we are interested in the mathematical side of these models, and especially the mathematical formulation and treatment of slip-plane confinement, we choose a simple mobility law for this paper: a rate-independent law. This rate-independent evolution is observed in a variety of situations where the system can be compared to a point particle in a periodic potential. The wells of this potential correspond to energetically favourable positions of the system, which translates in the case of dislocations to energetically favourable positions of the defect. The spacing of the wells can be either the Burgers vector, in the case of dislocations in pure crystals, or the typical spacing of obstacles.¹

This locally periodic potential is tilted by a global driving force that varies on larger scales and arises from the bulk elasticity. Motion in this tilted periodic potential is assumed to arise from thermal fluctuations, leading to a jump process between wells with a rate that is determined by the rate of escape from a well. Orowan [Oro40] first formulated this concept of dislocation motion and the Arrhenius rate expression that follows from it. In the one-dimensional case, with a periodic potential with wells of depth e and spacing ℓ , and a global force \mathfrak{f} , the rates of jumping right and left are

$$r_+ = \alpha e^{-\beta(e-\mathfrak{f}\ell)} \quad r_- = \alpha e^{-\beta(e+\mathfrak{f}\ell)},$$

where $\alpha > 0$ is a fixed constant and $\beta = 1/kT$ is inverse temperature. The expected velocity therefore is equal to $2\alpha \exp(-\beta e) \sinh(\beta \mathfrak{f}\ell)$.

‘Rate-independence’ appears in this expression for the rate of motion when we take the low-temperature limit $\beta \rightarrow \infty$:

$$2\alpha e^{-\beta e} \sinh(\beta \mathfrak{f}\ell) \xrightarrow{\beta \rightarrow \infty} m\left(\frac{\mathfrak{f}\ell}{e}\right), \quad m(s) := \begin{cases} \emptyset & \text{if } s < -1 \\ (-\infty, 0] & \text{if } s = -1 \\ \{0\} & \text{if } -1 < s < 1 \\ [0, \infty) & \text{if } s = 1 \\ \emptyset & \text{if } s > 1. \end{cases} \quad (1.9)$$

Here the convergence is in the sense of graphs, and the limit m is a graph, i.e., a set-valued function. In words: at low temperatures, motion only takes place when $\mathfrak{f} = \pm e/\ell$, and for those two values of \mathfrak{f} any velocity is possible; for smaller forces \mathfrak{f} the motion is arrested, and larger forces should never appear (see Figure 1). What constitutes low temperatures can be understood in terms of an energy comparison: low temperatures are those in which the thermal energy $kT = \beta^{-1}$ is small with respect to the activation energy e .

Inspired by these arguments we choose a rate-independent evolution for the discrete system. In order to force a non-trivial evolution, we add a smoothly time-varying external load of the form

$$\int_{\Omega} f(x, t) d\mu(x). \quad (1.10)$$

The function f can be interpreted as a time-varying potential for dislocations, in the sense that the derivative $\partial_{x_1} f(x, t)$ acts as an additional horizontal force on a dislocation at x . For instance, applying an appropriate time-varying boundary traction, one can achieve a spatially uniform shear stress $\sigma(t)$ in the bulk, which can be represented with the potential $f(x, t) = \sigma(t)x_1$. The total energy for this system is then

$$\tilde{\mathcal{F}}_n(\mu, t) = \mathcal{F}_n(\mu) - \int_{\Omega} f(x, t) d\mu(x). \quad (1.11)$$

We assume that each of the dislocations z_i evolves by a rate-independent law of the type (1.9). The driving force \mathfrak{f} for each of the dislocations is the derivative of the energy with respect to the position z_i , $-\nabla_{z_i} \tilde{\mathcal{F}}_n(\frac{1}{n} \sum_{j=1}^n \delta_{z_j}, t)$. Since our edge dislocations are restricted to movement in the

¹Note added in proof: In the meantime Patrick van Meurs has proved a related evolutionary convergence result in the case of a linear driving law [Meu15].

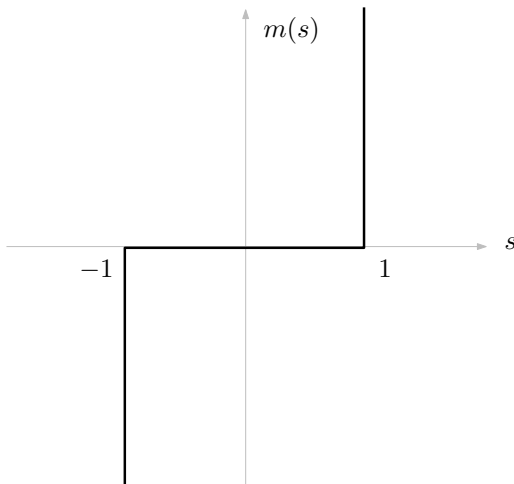


FIGURE 1. The rate-independent limit.

direction of the Burgers vector $b = e_1$, the motion is one-dimensional, and only the horizontal or first component of this force leads to motion:

$$\dot{z}_i(t) \in e_1 m\left(-e_1 \cdot \nabla_{z_i} \tilde{\mathcal{F}}_n\left(\frac{1}{n} \sum_{j=1}^n \delta_{z_j(t)}, t\right)\right). \quad (1.12)$$

Note how the ‘equation’ above is an inclusion, since the right-hand side is multivalued. In (1.12) we have normalized the constant ℓ/e to 1 for simplicity.

The flow rule (1.12) can be interpreted as follows: for each i separately, and at each t , *either* the velocity $\dot{z}_i(t)$ is zero, *or* the force $-e_1 \cdot \nabla_{z_i} \tilde{\mathcal{F}}_n$ equals ± 1 ; in the latter case, \dot{z}_i is parallel to e_1 and points in the same direction as the force.

1.4. Measure-valued formulation. Instead of considering solutions of (1.12) in the classical sense, we focus on solutions in the *energetic* or *quasi-static* sense [Mie05], given by Definition 1.2 below. For this we recast the problem as a rate-independent system in the space of probability measures $\mathcal{P}(\Omega)$, driven by $\tilde{\mathcal{F}}_n$, with a Wasserstein-type dissipation, which we now introduce.

To motivate the definition, consider a smooth curve $t \mapsto \mu(t) = \frac{1}{n} \sum_{i=1}^n \delta_{z_i(t)}$. Since dislocations move parallel to $b = e_1$, $\dot{z}_i(t) \cdot e_2 = 0$; therefore

$$\partial_t \int_{\Omega} \varphi(x_2) d\mu(t)(x) = \frac{1}{n} \partial_t \sum_{i=1}^n \varphi(z_i(t) \cdot e_2) = 0 \quad \text{for any } \varphi \in C^1(\mathbb{R}).$$

Along the curve the integral on the left-hand side is preserved, implying that if μ and ν can be transported into each other, then $\int \varphi(x_2) d(\mu - \nu)(x) = 0$; or equivalently, $(\pi_2)_\# \mu = (\pi_2)_\# \nu$, where $\pi_2 : \Omega \rightarrow \mathbb{R}$ is the coordinate mapping $\pi_2(x) := x_2$. This leads us to define the distance function

$$d(\mu, \nu) := \begin{cases} \inf_{\gamma \in \Gamma(\mu, \nu)} \iint_{\Omega \times \Omega} |x - y| d\gamma(x, y) & \text{if } (\pi_2)_\# \mu = (\pi_2)_\# \nu, \\ +\infty & \text{otherwise,} \end{cases} \quad (1.13)$$

where $\Gamma(\mu, \nu)$ is a restricted set of couplings of μ and ν ,

$$\Gamma(\mu, \nu) := \left\{ \gamma \in \mathcal{P}(\Omega \times \Omega) : \gamma(A \times \Omega) = \mu(A), \gamma(\Omega \times A) = \nu(A) \text{ for all Borel sets } A \subset \Omega, \right. \\ \left. \text{and } \pi_2(x) = \pi_2(y) \text{ for } \gamma\text{-a.e. } (x, y) \in \Omega \times \Omega \right\}. \quad (1.14)$$

This is the usual 1-Wasserstein or Monge-Kantorovich transport distance on $\mathcal{P}(\Omega)$ [Vil03], except for the additional restriction that $\pi_2(x) = \pi_2(y)$ for γ -a.e. (x, y) ; this restriction forces the transport to move parallel to $b = e_1$. We describe some properties of this distance in Lemma 4.1.

Now we have defined the driving functional $\tilde{\mathcal{F}}_n$ and the dissipation distance d , a solution of the quasi-static evolution is defined as follows.

Definition 1.2. A solution of the quasi-static evolution associated with $\tilde{\mathcal{F}}_n$ is a curve $t \mapsto \mu(t)$ from $[0, T]$ into X_n that satisfies the two following conditions:

- *global stability*: for every $t \in [0, T]$ we have

$$\tilde{\mathcal{F}}_n(\mu(t), t) \leq d(\nu, \mu(t)) + \tilde{\mathcal{F}}_n(\nu, t)$$

for every $\nu \in X_n$;

- *energy balance*: for every $t \in [0, T]$

$$\tilde{\mathcal{F}}_n(\mu(t), t) + \mathcal{D}(\mu, [0, t]) = \tilde{\mathcal{F}}_n(\mu(0), 0) - \int_0^t \int_{\Omega} \dot{f}(s) d\mu(s) ds,$$

where $\mathcal{D}(\mu, [0, t])$ is the total dissipation of μ on $[0, t]$,

$$\mathcal{D}(\mu, [0, t]) := \sup \left\{ \sum_{i=1}^M d(\mu(t_i), \mu(t_{i-1})) : 0 = t_0 < \dots < t_M = t, M \in \mathbb{N} \right\}. \quad (1.15)$$

In Section 4 we prove the existence of a solution for this evolution at fixed $n \in \mathbb{N}$ (Theorem 4.4).

1.5. Convergence of the quasi-static evolutions. Now we have defined the energy of the discrete system and its evolution in time, we turn to the convergence of this evolutionary system as $n \rightarrow \infty$.

The renormalized Γ -convergence result in terms of measures (Theorem 1.1) is the first ingredient in passing to the limit, as $n \rightarrow \infty$, in the quasi-static evolution that we formulated above. The second ingredient is a stronger recovery result, the existence of a *joint* or *mutual* recovery sequence [MRS08], for which both \mathcal{F}_n and d converge. Specifically, if $\mu_n \rightarrow \mu$ narrowly, and if ν is given, we need to construct ν_n such that

$$\mathcal{F}_n(\nu_n) + d(\mu_n, \nu_n) \rightarrow \mathcal{F}(\mu) + d(\mu, \nu).$$

This is a stronger property than Theorem 1.1 in a non-trivial way. To start with, the topology of d is stronger than the narrow topology, implying that the recovery sequence generated by Theorem 1.1 does not necessarily converge in d . The situation is worse, however: since $d(\mu, \nu)$ is only finite on measures with equal vertical marginals ($(\pi_2)_{\#}\mu = (\pi_2)_{\#}\nu$), the finiteness of $d(\mu_n, \nu_n)$ requires ν_n to be an exact horizontal transport of μ_n , i.e., the slip planes of μ_n and ν_n have to coincide, and the numbers of dislocations on each of the slip planes have to be the same for the two. Note that the recovery sequence ν_n necessarily depends on the whole sequence μ_n .

Because of this tight binding between μ_n and ν_n , we prove the existence of such a joint recovery sequence under a restriction on μ_n : we require the minimal slip-plane spacing in μ_n to be bounded from below, and the numbers of dislocations on each slip plane to be bounded from above, both by powers of n (see Definition 5.1 and Theorem 5.5). In the context of convergence in the evolutions this can relatively easily be achieved by imposing the same restrictions on the approximations of the initial datum μ^0 , since the evolution preserves these restrictions. The set of measures that can be approximated this way includes all measures with bounded two-dimensional Lebesgue density, but also some concentrated measures; see Lemma 5.2 for details.

Armed with this joint recovery sequence and setting

$$\tilde{\mathcal{F}}(\mu, t) := \mathcal{F}(\mu) - \int_{\Omega} f(t, x) \mu(t, x) dx,$$

in Section 5 we prove the following theorem (Theorem 5.8):

Theorem 1.3. Consider $\mu^0 \in \mathcal{P}(\Omega)$ such that there exists an approximating sequence μ_n^0 according to Theorem 5.5, with $\mu_n^0 \rightarrow \mu^0$ narrowly and $\mathcal{F}_n(\mu_n^0) \rightarrow \mathcal{F}(\mu^0)$. Assume that $f \in W^{1,1}(0, T; C(\Omega))$ is given. For every $n \in \mathbb{N}$ let $t \mapsto \mu_n(t)$ be a quasi-static evolution on $[0, T]$ associated with $\tilde{\mathcal{F}}_n$ and with initial value μ_n^0 , according to Definition 1.2. Then

- (1) **Compactness:** *There exists a subsequence μ_n (without change in notation) and a limit curve $\mu : [0, T] \rightarrow \mathcal{P}(\Omega)$ such that $\mu_n(t) \rightarrow \mu(t)$ narrowly for all $t \in [0, T]$;*
- (2) **Convergence:** *The curve μ is a quasi-static evolution associated with $\tilde{\mathcal{F}}$ and with initial value μ^0 (see (5.35) and (5.36)).*

1.6. Strong formulations. In Section 6 we show that, as we claimed above, the quasi-static formulation of Definition 1.2 is equivalent to the strong formulation of (1.12) when the solution is smooth. In that section, we also give a formal argument suggesting that the quasi-static evolution satisfied by the limiting measures μ , as given by Theorem 1.3, has a similar structure, which we now describe.

We consider evolving measures $t \mapsto \mu(t)$ that are smooth in space and time, and which have the property that there exists a smooth scalar field $\phi = \phi(t, x)$ such that

$$\partial_t \mu + \partial_{x_1}(\phi \mu) = 0 \quad \text{in the sense of distributions.}$$

The field ϕ has the interpretation of the horizontal velocity of the dislocations. Such a curve of measures solves the quasi-static evolution problem associated with $\tilde{\mathcal{F}}$, i.e., conditions (5.35) and (5.36), if and only if

$$-\phi(t, x) \partial_{x_1} \frac{\delta \tilde{\mathcal{F}}}{\delta \mu}(\mu(t), t) = |\phi(t, x)| \quad \text{for } \mu(t)\text{-a.e. } x.$$

Here $\delta \tilde{\mathcal{F}}/\delta \mu$ is the variational derivative of $\tilde{\mathcal{F}}$, and its x_1 -derivative at a point x has the interpretation of the x_1 -component of the force acting on a dislocation at x . Similarly to the discrete case, this implies that:

In $\mu(t)$ -a.e. x , either $\phi(x, t) = 0$, or the total force $-\partial_{x_1} \delta \tilde{\mathcal{F}}/\delta \mu(\mu(t), t)(x)$ equals ± 1 ; in the latter case ϕ has the same sign as the force.

We briefly mention that the evolution systems of this paper, both at finite n and in the limit $n \rightarrow \infty$, have formal counterparts as solutions of the so-called Energy-Dissipation Inequality, see e.g. [MRS09, MRS12]. However, the approach followed in these works generates a different class of solutions than Definition 1.2, since solutions of the Energy-Dissipation Inequality only jump when they lose their *local* minimality, while solutions of Definition 1.2 jump as soon as they lose *global* minimality.

1.7. Discussion. The research of this paper was driven by several aims. To start with, we wanted to make a first step in connecting dynamic discrete-dislocation models with their upscaled dislocation-density counterparts in a mathematically rigorous way; the slip-plane confinement is an important aspect of this. Secondly, we wanted to formulate a clear Γ -convergence statement for edge dislocations *in terms of dislocation densities* (or measures), which is more convenient for dislocation dynamics than the results of [GLP10], which are formulated in terms of the matrix field β . Finally, we wanted to bring the mathematical and the engineering literature on dislocations closer together; one striking difference is the preference in engineering to formulate problems in terms of the interaction energy V , while this interaction energy is nearly absent from the mathematical literature. Hence the explicit characterization of the Γ -limit in terms of V (Theorem 1.1).

These aims suggested a number of differences with the current literature. For instance, we chose to renormalize the elastic energy by subtracting the self-energy, which amounts to disregarding that energy contribution; in the context of the evolution of preserved dislocation numbers this makes sense, and leads to a simpler Gamma-convergence result (one single scaling regime instead of three regimes, as in [GLP10]). We also restrict ourselves to the single-Burgers-vector case, since the treatment of cancellation of dislocations of opposite sign should arise from a separate modelling of the creation and annihilation of dislocation pairs.

The main methodological contributions of this paper lie in the details of the construction of the recovery sequences in the two Gamma-convergence results (Theorems 1.1 and 5.5). Even without the slip-plane restrictions (Theorem 1.1), avoiding the singularity in the interaction potential requires careful placing of the dislocations. In the case of slip-plane confinement, the restrictions are much more severe, and subtle management of the different scales is required.

We now comment more in detail on some issues.

1.7.1. Conditions on the separation of dislocations. For Theorem 1.1 the sequence μ_n of empirical measures is required to satisfy two separation requirements: the defects should be separated from each other by at least $r_n \rightarrow 0$ and from the boundary by at least the fixed distance $\ell > 0$.

The parameter r_n represents the radius of the hard-core region. We require r_n to decay to 0 slower than the atomic distance parameter ε_n . More specifically, condition (1.1) arises from the difference between K^n and K , which in turn arises from the difference between imposing the circulation condition via an integrated boundary condition as in (2.3) or a Dirac delta as in (2.1). The interaction energy difference between the two is bounded by $O(\sqrt{\varepsilon_n/r_n^3})$, and condition (1.1) makes this difference small. Without this separation we believe the limiting energy to be the same, but we have no proof. In particular, the condition $r_n n \rightarrow 0$ implies that, if the dislocations are all on the same plane, the intersection of the hard-core region with the plane has length converging to zero, as $n \rightarrow \infty$.

The separation of the dislocations from the boundary $\partial\Omega$ is related to the choice of boundary conditions. With the no-stress boundary conditions that we impose at $\partial\Omega$, dislocations can reduce their energy by moving to the boundary (in essence they ‘vanish’ at the boundary). Taking this possibility into account would require a modification of the renormalization term in (1.6), and we leave this to a future publication. Note that similar conditions are present in other work; for instance, the authors of [ADLGP14] prove the liminf inequality only for the case that no dislocations are ‘lost’ in the limit. Although one could in principle use different, Dirichlet-type boundary conditions to avoid this problem, it is not clear how to do this in our setup. The energy we consider is indeed expressed in terms of a stress, and there is no natural deformation associated to it; we could obtain a deformation via a Hodge-type decomposition of the stress, but this procedure seems ad-hoc and not physically sound.

The condition that dislocations can only live in a rectangle $\mathcal{R} \subset \Omega$ is a simplification that proves to be very useful in the approximation results in Section 5 and in the construction of the recovery sequence in Theorem 5.5. A crucial point in our constructions is the possibility to spread out the dislocations horizontally and our assumption on \mathcal{R} guarantees that the modified positions are still in the domain.

In addition, as pointed out before, for the joint recovery sequence in Theorem 5.5 we require the admissible measures to satisfy an upper bound on the maximum number of dislocations per slip plane and a lower bound on the minimum distance between slip planes, both in terms of powers of the number n of dislocations (see Definition 5.1). This condition allows concentration on a slip plane or on a vertical line, but does not allow dislocations to be too close both horizontally and vertically at the same time.

1.7.2. Related Work. The asymptotic behaviour of the quadratic dislocation energy for edge dislocations was already studied by Garroni, Leoni and Ponsiglione in [GLP10]. One of the main difference with our work is that we consider the reduced energy (1.3) instead of (1.4), and use as main variable the dislocation density rather than the strain. Moreover, while in [GLP10] the authors focus on the self-energy term of (1.4) in the case of edge dislocations with multiple Burgers vectors, we instead focus on the next term in the expansion, namely the interaction energy, and simplify by restricting to one Burgers vector. A similar analysis as in [GLP10] has been done in [DLGP12] without the well-separation assumption that we make (see (1.1)).

The focus on the interaction term of the energy for edge dislocations was already present in the work of Cermelli and Leoni [CL05]. In [CL05] the authors define the renormalised energy starting from a quadratic dislocation energy, and focus on the interaction term of the energy. They however consider the case of a finite number of dislocations with different Burgers vectors, and do not phrase their result in terms of Γ -convergence, but instead they keep n fixed, and therefore express their renormalised energy in terms of the positions of dislocations, rather than densities.

For screw dislocations the interaction energy was derived from discrete models in [ADLGP14]. Moreover in [ADLGP14] the authors also considered the time-dependent case. More precisely they

proved the convergence of the discrete gradient flow for the discrete dislocation energy with flat dissipation to the gradient flow of the renormalised energy.

As for the slip-plane-confined motion, the only related work in the mathematical domain that we know of is [BFLM14], where the authors consider screw dislocations that may move along a finite set of directions. Such a system presents different mathematical difficulties, since each dislocation can, in theory, reach each point in the plane, in contrast to the single-slip-plane confinement of this paper.

Finally, there is an intriguing question that arises from the comparison with current continuum-scale modelling of plasticity (as in e.g. [GA05, AGLG05]). The limiting energy of Theorem 1.1 is non-local, with an interaction kernel that has no intrinsic length scale. However, ‘defect energies’ in the continuum-level modelling are usually assumed to be local (see e.g. [AGLG05, Eq. (8.8)] or [GA05, Eq. (6.16)]). It is unclear to us how these two descriptions can be reconciled.

1.7.3. Extensions and open questions. Various extensions of the present work would be relatively straightforward. The isotropy of \mathbb{C} is only assumed for convenience, and without this property we expect similar results to hold. The type of loading in the quasi-static evolution (the form of (1.10)) is also chosen for convenience; we expect that other types of loading can be treated with minor changes, although we expect that the elasticity problem should then be formulated in terms of displacements u rather than elastic strains β .

The current restriction on the set of admissible initial data (see Definition 5.1) is not fully satisfactory. It would be very interesting, and useful, to understand which class of quasi-static evolutions can be approximated using discrete systems.

In this work we restricted our attention to the single slip case, namely to the case of parallel slip planes, and with no loss of generality we considered $b = e_1$. We moreover assumed that all the dislocations are *positive* dislocations. This further assumption excludes the case of annihilation and the presence of *dipoles*. Natural extensions would be to allow for both positive and negative dislocations, and to consider the multiple Burgers vector case. It would also be very interesting to include creation and annihilation in such a model, allowing the total variation of the density of dislocations to change in time.

Another possible direction of investigation would be to consider other significant dissipations, e.g., the flat dissipation considered in [ADLGP14] (but still keeping our slip-confinement condition); another example of this would be considering gradient-flow (quadratic-dissipation) time evolution rather than quasi-static evolution.

One of the most interesting directions of extension is towards the three-dimensional case, where dislocations are three-dimensional curves, and entanglement between dislocations has a crucial role in evolution. For static Gamma-convergence very first results have recently been proved by Conti, Garroni, and Massaccesi [CGM13], but the mathematical understanding of the three-dimensional situation is still very much in its infancy.

1.8. Notation. We recall some definitions. We say that a sequence μ_n in $\mathcal{P}(\Omega)$ converges *narrowly* to $\mu \in \mathcal{P}(\Omega)$, and we write $\mu_n \rightharpoonup \mu$, if

$$\int_{\Omega} f d\mu_n \rightarrow \int_{\Omega} f d\mu, \quad (1.16)$$

as $n \rightarrow \infty$, for every continuous and bounded function $f : \Omega \rightarrow \mathbb{R}$. If $(\mu_n) \subset \mathcal{P}(\Omega)$ is such that $\text{supp } \mu_n \subset \Omega'$ for every n , where Ω' is an open set compactly contained in Ω , and (1.16) holds for every $f \in C_c^\infty(\Omega)$, then $\mu \in \mathcal{P}(\Omega)$ and $\mu_n \rightharpoonup \mu$ narrowly.

Below we list some symbols and abbreviations that are going to be used throughout the paper. Since the problem is in the setting of planar elasticity, all functions and vectors below are defined on a two-dimensional domain.

a^\perp	vector orthogonal to $a \in \mathbb{R}^2$ obtained by a counterclockwise rotation by $\pi/2$	
$[r]$	integer part of $r \in \mathbb{R}$	
skew F	skew-symmetric part of a matrix F , skew $F = (F - F^T)/2$	
sym F	symmetric part of a matrix F , sym $F = (F + F^T)/2$	
Eu	symmetric gradient, $Eu = \text{sym } \nabla u$	
$\mathcal{L}^1, \mathcal{L}^2$	one- and two-dimensional Lebesgue measure	
\mathcal{H}^1	one-dimensional Hausdorff measure	
π_i	projection onto the coordinate e_i	
$\mathcal{P}(\Omega)$	non-negative Borel measures on Ω of mass 1	
$ \mu (A)$	total variation of $\mu \in \mathcal{P}(\Omega)$ in $A \subseteq \Omega$	
$f\# \mu$	push forward of μ by f	
β_μ	elastic strain associated with μ	Sec. 2.1
K_z^n, K_z	singular strains associated with μ	Sec. 2.1
u_μ, v_μ	displacements associated with μ	Sec. 2.1
$I_{n,\mu}, I_\mu$	auxiliary functionals associated with μ	Sec. 2.1
d	slip-plane-confined Wasserstein distance	(1.13)
d_1	Monge-Kantorovich distance with cost $c(x, y) = x - y $	[Vil03]
$\Gamma(\mu, \nu)$	set of couplings of μ and ν with only horizontal transport	(1.14)
$\Gamma_1(\mu, \nu)$	set of couplings of μ and ν	[Vil03]
\mathcal{F}_n	renormalized energy	(2.13)
$\tilde{\mathcal{F}}_n$	total energy	(1.11)
X_n	set of admissible discrete dislocation measures	(1.2)
$Y_n(\gamma, c)$	set of admissible recovery measures	(5.2)
$\mathcal{P}_{\gamma,c}^\infty(\Omega)$	set of admissible limit measures	Lemma 5.2

2. DEFINITIONS AND PRELIMINARIES

In the following $\Omega \subset \mathbb{R}^2$ is a simply connected bounded domain with Lipschitz boundary. We moreover assume that it contains a closed rectangle \mathcal{R} whose sides are parallel to the coordinate axes and with $\text{dist}(\mathcal{R}, \partial\Omega) = \ell > 0$.

Before turning to the main theorems of this paper, starting from Section 3, we introduce five auxiliary functions (Section 2.1), and use these to rewrite the renormalized energy (Section 2.2).

2.1. Auxiliary functions. In the arguments of this paper, five auxiliary functions play an important role: K , K^n , β_μ , u_μ , and v_μ . We now introduce these. For the notation we follow [CL05], and we set the Burgers vector $b = e_1$.

The matrix-valued function K is defined as

$$K(x; z) := \frac{1}{2\pi|x-z|^2} e_1 \otimes (x-z)^\perp + \nabla v(x-z) \quad \text{for every } x \neq z,$$

where

$$v(x) := -\frac{\mu \log|x|}{2\pi(\lambda+2\mu)} e_2 - \frac{\lambda+\mu}{4\pi(\lambda+2\mu)|x|^2} [(e_1 \cdot x^\perp)x + (e_1 \cdot x)x^\perp]$$

and λ, μ are the Lamé constants of the stress-strain tensor \mathbb{C} . The function $K(\cdot; z)$ is the strain field in \mathbb{R}^2 generated by a single dislocation at z , with Burgers vector e_1 , and is a distributional solution of

$$\begin{cases} \text{div } \mathbb{C}K(\cdot; z) = 0 & \text{in } \mathbb{R}^2, \\ \text{Curl } K(\cdot; z) = e_1 \delta_z & \text{in } \mathbb{R}^2. \end{cases} \quad (2.1)$$

For brevity we often write K_z for $K(\cdot; z)$.

The matrix-valued function K^n is a perturbation of K , defined by

$$K^n(x; z) := K(x; z) + \varepsilon_n^2 \nabla w(x-z), \quad (2.2)$$

with

$$w(x) := \frac{(\lambda+\mu)}{2\pi(\lambda+2\mu)|x|^4} [(e_1 \cdot x^\perp)x + (e_1 \cdot x)x^\perp].$$

As before, we will often write $K_z^n(\cdot)$ and $w_z(\cdot)$ instead of $K^n(\cdot; z)$ and $w(\cdot - z)$. The function K^n is also the strain generated by a dislocation at z , but in the context of the exterior domain $\mathbb{R}^2 \setminus B_{\varepsilon_n}(z)$; the mismatch associated with the dislocation is enforced by a combination of a stress-free condition and a circulation condition on the boundary:

$$\begin{cases} \operatorname{div} \mathbb{C}K^n(\cdot; z) = 0 & \text{in } \mathbb{R}^2 \setminus B_{\varepsilon_n}(z), \\ \mathbb{C}K^n(\cdot; z)\nu = 0 & \text{on } \partial B_{\varepsilon_n}(z), \\ \int_{\partial B_{\varepsilon_n}(z)} K^n(x; z) \tau(x) d\mathcal{H}^1(x) = e_1. \end{cases} \quad (2.3)$$

The functions K and K^n have the same far-field behaviour, and K^n converges pointwise to K as $\varepsilon_n \rightarrow 0$.

For given $\mu \in X_n$, **the matrix-valued function** β_μ is the unique (up to skew-symmetric matrices) minimizer of the energy E_n in (1.4) subject to the circulation constraint (1.5); it is therefore the strain field generated by the dislocations that μ represents. It satisfies the equations

$$\begin{cases} \operatorname{div} \mathbb{C}\beta_\mu = 0 & \text{in } \Omega_n(\mu), \\ \mathbb{C}\beta_\mu\nu = 0 & \text{on } \partial\Omega_n(\mu). \end{cases} \quad (2.4)$$

For given $\mu \in X_n$, **the vector-valued function** u_μ is a displacement generated by the dislocations, as follows: since the left-hand side of

$$\beta_\mu(x) - \frac{1}{n} \sum_{i=1}^n K^n(x, z_i) = \nabla u_\mu(x) \quad \text{for } x \in \Omega_n(\mu) \quad (2.5)$$

is curl-free in $\Omega_n(\mu)$, with zero-circulation boundary conditions on each of the $\partial B_{\varepsilon_n}(z_i)$, it is the gradient of a function u_μ . It can be interpreted as a corrector displacement field, which cancels the non-stress-free boundary values of $\frac{1}{n} \sum_i K_{z_i}^n$.

We choose a fixed ball B contained in Ω such that $\operatorname{dist}(x, \partial\Omega) < \ell/2$ for every $x \in B$ (and thus, contained in $\Omega_n(\mu)$). We will require that

$$\int_B u_\mu(x) dx = 0, \quad \int_B \operatorname{skew} \nabla u_\mu(x) dx = 0. \quad (2.6)$$

The function u_μ can be alternatively characterised in terms of a minimisation problem for the functional $I_{n,\mu}$ defined as

$$I_{n,\mu}(u) := \frac{1}{2} \int_{\Omega_n(\mu)} \mathbb{C}\nabla u(x) : \nabla u(x) dx + \frac{1}{n} \sum_{i=1}^n \int_{\partial\Omega_n(\mu)} \mathbb{C}K^n(x; z_i)\nu(x) \cdot u(x) d\mathcal{H}^1(x). \quad (2.7)$$

This will be proved in Lemma 2.1 at the end of this section.

The vector-valued function v_μ is very similar to u_μ : for given $\mu \in \mathcal{P}(\Omega)$, it is the unique minimizer of

$$I_\mu(v) := \frac{1}{2} \int_\Omega \mathbb{C}\nabla v(x) : \nabla v(x) dx + \int_\Omega \int_{\partial\Omega} \mathbb{C}K(x; y)\nu(x) \cdot v(x) d\mathcal{H}^1(x) d\mu(y) \quad (2.8)$$

on the class

$$\left\{ v \in H^1(\Omega; \mathbb{R}^2) : \int_B v dx = 0, \int_B \operatorname{skew} \nabla v dx = 0 \right\}. \quad (2.9)$$

Note the similarity between $I_{n,\mu}$, which defines u_μ , and I_μ : if we take $\mu = \frac{1}{n} \sum_i \delta_{z_i}$, then $I_{n,\mu}$ and I_μ are very similar, differing only in the distinction between K^n and K and in the domain of integration. Indeed we will see below (proof of Theorem 3.3) that if $\mu_n \rightharpoonup \mu$, then u_{μ_n} converges to v_μ and $I_{n,\mu_n}(u_{\mu_n}) \rightarrow I_\mu(v_\mu)$.

2.2. Rewriting the energy. Recall that the energies F_n and \mathcal{F}_n are defined in (1.3) and (1.6) in terms of the minimiser β_μ of the energy E_n in (1.4) subject to the circulation constraint (1.5).

Using (2.5) we rewrite $F_n(\mu)$ for $\mu \in X_n$ as

$$\begin{aligned} F_n(\mu) &= \frac{1}{2} \int_{\Omega_n(\mu)} \mathbb{C} \beta_\mu : \beta_\mu \, dx \\ &= \frac{1}{2n^2} \sum_{i=1}^n \int_{\Omega_n(\mu)} \mathbb{C} K^n(x; z_i) : K^n(x; z_i) \, dx + \frac{1}{2n^2} \sum_{i=1}^n \sum_{j \neq i} \int_{\Omega_n(\mu)} \mathbb{C} K^n(x; z_i) : K^n(x; z_j) \, dx \\ &\quad + \frac{1}{n} \sum_{i=1}^n \int_{\Omega_n(\mu)} \mathbb{C} K^n(x; z_i) : \nabla u_\mu(x) \, dx + \frac{1}{2} \int_{\Omega_n(\mu)} \mathbb{C} \nabla u_\mu(x) : \nabla u_\mu(x) \, dx. \end{aligned} \quad (2.10)$$

Note that the first term in the above expression represents the *self-energy* contributions of the dislocations, while the second term represents the *mesoscopic* pairwise interaction energy between two dislocations located at z_i and z_j . Concerning the last two terms, integrating by parts and applying (2.3) yield

$$\begin{aligned} &\frac{1}{n} \sum_{i=1}^n \int_{\Omega_n(\mu)} \mathbb{C} K^n(x; z_i) : \nabla u_\mu(x) \, dx \\ &= \frac{1}{n} \sum_{i=1}^n \int_{\partial\Omega} \mathbb{C} K^n(x; z_i) \nu \cdot u_\mu(x) \, d\mathcal{H}^1(x) - \frac{1}{n} \sum_{i=1}^n \sum_{j \neq i} \int_{\partial B_{\varepsilon_n}(z_j)} \mathbb{C} K^n(x; z_i) \nu \cdot u_\mu(x) \, d\mathcal{H}^1(x). \end{aligned}$$

Similarly, using (2.3)–(2.5) we obtain

$$\begin{aligned} &\int_{\Omega_n(\mu)} \mathbb{C} \nabla u_\mu(x) : \nabla u_\mu(x) \, dx \\ &= -\frac{1}{n} \sum_{i=1}^n \int_{\partial\Omega} \mathbb{C} K^n(x; z_i) \nu \cdot u_\mu(x) \, d\mathcal{H}^1(x) + \frac{1}{n} \sum_{i=1}^n \sum_{j \neq i} \int_{\partial B_{\varepsilon_n}(z_j)} \mathbb{C} K^n(x; z_i) \nu \cdot u_\mu(x) \, d\mathcal{H}^1(x). \end{aligned} \quad (2.11)$$

By these computations we have that for $\mu \in X_n$

$$\begin{aligned} \mathcal{F}_n(\mu) &= \frac{1}{2n^2} \sum_{i=1}^n \sum_{j \neq i} \int_{\Omega_n(\mu)} \mathbb{C} K_{z_i}^n : K_{z_j}^n \\ &\quad + \frac{1}{2n} \sum_{i=1}^n \int_{\partial\Omega} \mathbb{C} K_{z_i}^n \nu \cdot u_\mu \, d\mathcal{H}^1 - \frac{1}{2n} \sum_{i=1}^n \sum_{j \neq i} \int_{\partial B_{\varepsilon_n}(z_j)} \mathbb{C} K_{z_i}^n \nu \cdot u_\mu \, d\mathcal{H}^1 \end{aligned} \quad (2.12)$$

$$= \frac{1}{2n^2} \sum_{i=1}^n \sum_{j \neq i} \int_{\Omega_n(\mu)} \mathbb{C} K_{z_i}^n : K_{z_j}^n + \frac{1}{2n} \sum_{i=1}^n \int_{\partial\Omega_n(\mu)} \mathbb{C} K_{z_i}^n \nu \cdot u_\mu \, d\mathcal{H}^1, \quad (2.13)$$

where in the last step we used the boundary condition (2.3) to add the missing term in the double sum.

The two terms above will result in the two terms in the limiting energy \mathcal{F} (see (1.7)). To recognize the second term, note that by the same calculation (2.11), the second term is equal to $I_{n,\mu}(u_\mu)$, which will converge in the limit to $I_\mu(v_\mu)$.

The energy decomposition above proves useful for the characterisation of u_μ in the following lemma.

Lemma 2.1. *Assume (1.1) and let $\mu = \frac{1}{n} \sum_{i=1}^n \delta_{z_i} \in X_n$ for some $n \in \mathbb{N}$. Then there exists a unique $u_\mu \in H^1(\Omega_n(\mu); \mathbb{R}^2)$ that satisfies (2.5) and (2.6). Moreover, there exist a constant $C > 0$, independent of n , μ , and u_μ , and an extension $\tilde{u}_\mu \in H^1(\Omega; \mathbb{R}^2)$ of u_μ such that*

$$\|\tilde{u}_\mu\|_{H^1(\Omega)} \leq C \|Eu_\mu\|_{L^2(\Omega_n(\mu))} \leq C. \quad (2.14)$$

In addition, the function u_μ minimizes the functional $I_{n,\mu}$ defined in (2.7) on $H^1(\Omega_n(\mu); \mathbb{R}^2)$.

Proof. As mentioned before, the curl-free property of the left-hand side of (2.5) implies that there exists some $u_\mu \in H^1(\Omega_n(\mu); \mathbb{R}^2)$ satisfying (2.5). Since the minimizer of $E_n(\mu, \cdot)$ in $\mathcal{A}_n(\mu)$ is unique up to addition of skew-symmetric matrices, condition (2.6) guarantees the uniqueness of u_μ .

We now prove (2.14). For every $u \in H^1(\Omega_n(\mu); \mathbb{R}^2)$ the function

$$\beta(x) := \chi_{\Omega_n(\mu)}(x) \left(\frac{1}{n} \sum_{i=1}^n K^n(x; z_i) + \nabla u(x) \right)$$

belongs to $\mathcal{A}_n(\mu)$. Thus, by (1.3) we have

$$F_n(\mu) \leq E_n(\mu, \beta) = \frac{1}{2} \int_{\Omega_n(\mu)} \mathbb{C} \beta : \beta \, dx.$$

Developing the quadratic form on the right-hand side and taking into account the decomposition (2.10) and the first equation in (2.3), one easily deduces that the function u_μ minimizes the functional $I_{n,\mu}$ on $H^1(\Omega_n(\mu); \mathbb{R}^2)$. In particular, considering $u \equiv 0$ as a competitor and using the coercivity of \mathbb{C} on symmetric matrices, we deduce

$$\begin{aligned} & C_1 \|Eu_\mu\|_{L^2(\Omega_n(\mu))}^2 \\ & \leq \frac{1}{n} \sum_{i=1}^n \|\mathbb{C}K^n(\cdot; z_i)\|_{L^2(\partial\Omega)} \|u_\mu\|_{L^2(\partial\Omega)} + \frac{1}{n} \sum_{i=1}^n \sum_{j \neq i} \|\mathbb{C}K^n(\cdot; z_i)\|_{L^2(\partial B_{\varepsilon_n}(z_j))} \|u_\mu\|_{L^2(\partial B_{\varepsilon_n}(z_j))}. \end{aligned} \quad (2.15)$$

From the definition (2.2) of K^n , the definition (1.2) of the admissible class, and assumption (1.1) it follows that

$$\sup_{x \in \partial\Omega} |\mathbb{C}K^n(x; z_i)| \leq C \sup_{x \in \partial\Omega} \left(\frac{1}{|x - z_i|} + \varepsilon_n^2 \frac{1}{|x - z_i|^3} \right) \leq C \left(\frac{1}{\ell} + \frac{\varepsilon_n^2}{\ell^3} \right) \leq C,$$

and, analogously, for $i \neq j$,

$$\sup_{x \in \partial B_{\varepsilon_n}(z_j)} |\mathbb{C}K^n(x; z_i)| \leq C \sup_{x \in \partial B_{\varepsilon_n}(z_j)} \left(\frac{1}{|x - z_i|} + \varepsilon_n^2 \frac{1}{|x - z_i|^3} \right) \leq C \left(\frac{1}{r_n} + \frac{\varepsilon_n^2}{r_n^3} \right) \leq \frac{C}{r_n}. \quad (2.16)$$

Combining the two inequalities above with (2.15), we have

$$\|Eu_\mu\|_{L^2(\Omega_n(\mu))}^2 \leq C \|u_\mu\|_{L^2(\partial\Omega)} + C \frac{\sqrt{\varepsilon_n}}{r_n} \sum_{j=1}^n \|u_\mu\|_{L^2(\partial B_{\varepsilon_n}(z_j))}. \quad (2.17)$$

To control the norms of u_μ on the right-hand side, we proceed as follows. By Theorem 7.1 in the Appendix we can extend u_μ to a function $\tilde{u}_\mu \in H^1(\Omega; \mathbb{R}^2)$ such that

$$\|E\tilde{u}_\mu\|_{L^2(\Omega)} \leq C \|Eu_\mu\|_{L^2(\Omega_n(\mu))},$$

where C is a constant independent of n , μ , and u_μ . Moreover, since $\tilde{u}_\mu = u_\mu$ on the ball B where (2.6) is satisfied, by the Korn and Poincaré inequalities we have

$$\|\tilde{u}_\mu\|_{H^1(\Omega)} \leq C \|E\tilde{u}_\mu\|_{L^2(\Omega)} \leq C \|Eu_\mu\|_{L^2(\Omega_n(\mu))}, \quad (2.18)$$

and by the continuity of the trace operator on $\partial\Omega$

$$\|u_\mu\|_{L^2(\partial\Omega)} = \|\tilde{u}_\mu\|_{L^2(\partial\Omega)} \leq C \|\tilde{u}_\mu\|_{H^1(\Omega)} \leq C \|Eu_\mu\|_{L^2(\Omega_n(\mu))}. \quad (2.19)$$

To estimate the norm of u_μ on the boundaries $\partial B_{\varepsilon_n}(z_j)$ we use polar coordinates centered at the point z_j on the annulus $B_{r_n}(z_j) \setminus B_{\varepsilon_n}(z_j)$. For every $\theta \in [0, 2\pi]$ and every $\varepsilon_n \leq s \leq r \leq r_n$ we may write

$$u_\mu(s, \theta) = u_\mu(r, \theta) - \int_s^r \frac{\partial u_\mu}{\partial \rho}(\rho, \theta) \, d\rho.$$

Thus,

$$|u_\mu(s, \theta)|^2 \leq 2|u_\mu(r, \theta)|^2 + 2r_n \int_s^{r_n} \left| \frac{\partial u_\mu}{\partial \rho}(\rho, \theta) \right|^2 \, d\rho.$$

Integrating over $[0, 2\pi]$ yields

$$\int_0^{2\pi} \varepsilon_n |u_\mu(s, \theta)|^2 d\theta \leq 2 \int_0^{2\pi} r |u_\mu(r, \theta)|^2 d\theta + 2r_n \int_{B_{r_n}(z_j) \setminus B_{\varepsilon_n}(z_j)} |\nabla u_\mu|^2 dx.$$

We now integrate over $r \in [\varepsilon_n, r_n]$ and divide by $r_n - \varepsilon_n$:

$$\int_{\partial B_{\varepsilon_n}(z_j)} |u_\mu|^2 d\mathcal{H}^1 \leq \frac{2}{r_n - \varepsilon_n} \int_{B_{r_n}(z_j) \setminus B_{\varepsilon_n}(z_j)} |u_\mu|^2 dx + 2r_n \int_{B_{r_n}(z_j) \setminus B_{\varepsilon_n}(z_j)} |\nabla u_\mu|^2 dx.$$

Hence, by (2.18) we conclude that

$$\sum_{j=1}^n \|u_\mu\|_{L^2(\partial B_{\varepsilon_n}(z_j))} \leq \frac{C}{\sqrt{r_n}} \|u_\mu\|_{H^1(\Omega_n(\mu))} \leq \frac{C}{\sqrt{r_n}} \|\tilde{u}_\mu\|_{H^1(\Omega)} \leq \frac{C}{\sqrt{r_n}} \|Eu_\mu\|_{L^2(\Omega_n(\mu))}. \quad (2.20)$$

Combining (2.17), (2.19), and (2.20), and taking into account (1.1), we obtain

$$\|Eu_\mu\|_{L^2(\Omega_n(\mu))}^2 \leq C \|Eu_\mu\|_{L^2(\Omega_n(\mu))} + \frac{C\sqrt{\varepsilon_n}}{r_n^{3/2}} \|Eu_\mu\|_{L^2(\Omega_n(\mu))} \leq C \|Eu_\mu\|_{L^2(\Omega_n(\mu))}.$$

Therefore, using also (2.18), we deduce (2.14). \square

3. Γ -CONVERGENCE OF THE RENORMALIZED ENERGY

In this section we prove Theorem 1.1, the Gamma-convergence of the renormalised energy defined in (1.6). Before stating and proving the main theorem we state some useful properties of the interaction potential V appearing in the definition of the limit energy.

Lemma 3.1. *The function $V : \Omega \times \Omega \rightarrow \mathbb{R} \cup \{+\infty\}$ defined as*

$$V(y, z) := \begin{cases} \int_{\Omega} \mathbb{C}K(x; y) : K(x; z) dx & \text{if } y \neq z, \\ +\infty & \text{if } y = z, \end{cases} \quad (3.1)$$

is well defined, symmetric, and continuous. Moreover, we have the following estimates:

- there exist two constants $C > 0$ and $L > 0$ such that for every $y, z \in \Omega$ with $y \neq z$

$$|V(y, z)| \leq C \left(1 - \log \frac{|y - z|}{L}\right); \quad (3.2)$$

- for every open set Ω' compactly contained in Ω there exist $C(\Omega') > 0$ and $\bar{R} > 0$ such that for every $y, z \in \Omega'$ with $0 < |y - z| \leq \bar{R}$

$$C(\Omega')(1 - \log |y - z|) \leq V(y, z). \quad (3.3)$$

Proof. From the definition of K it follows that there exists a constant $C > 0$ such that

$$|K(x; y)| \leq \frac{C}{|x - y|} \quad (3.4)$$

for every $x \neq y$. This implies that the integral in the definition of $V(y, z)$ for $y \neq z$ is finite. Clearly V is symmetric, that is, $V(y, z) = V(z, y)$.

We now prove (3.2). Let $y, z \in \Omega$ with $y \neq z$. From the definition of V and from (3.4) we have

$$|V(y, z)| \leq C \int_{\Omega} \frac{dx}{|x - y||x - z|} \leq C \int_{B_L(z)} \frac{dx}{|x - y||x - z|}, \quad (3.5)$$

because, Ω being bounded, $\Omega \subset B_L(z)$ for some finite $L > 0$ and every $z \in \Omega$. With no loss of generality we can assume that $L \geq 2|y - z|$. For convenience we set $\tilde{x} := x - z$ and $\tilde{y} := y - z$; then for the last integral in (3.5) we have

$$\int_{B_L(z)} \frac{dx}{|x - y||x - z|} = \int_{B_L(0)} \frac{d\tilde{x}}{|\tilde{x}||\tilde{x} - \tilde{y}|} = \int_{B_{2|\tilde{y}|}(0)} \frac{d\tilde{x}}{|\tilde{x}||\tilde{x} - \tilde{y}|} + \int_{B_L(0) \setminus B_{2|\tilde{y}|}(0)} \frac{d\tilde{x}}{|\tilde{x}||\tilde{x} - \tilde{y}|}. \quad (3.6)$$

If $\tilde{x} \in B_L(0) \setminus B_{2|\tilde{y}|}(0)$, then $|\tilde{x} - \tilde{y}| \geq \|\tilde{x}\| - \|\tilde{y}\| = |\tilde{x}| - |\tilde{y}|$; hence

$$\begin{aligned} \int_{B_L(0) \setminus B_{2|\tilde{y}|}(0)} \frac{d\tilde{x}}{|\tilde{x}||\tilde{x} - \tilde{y}|} &\leq \int_{B_L(0) \setminus B_{2|\tilde{y}|}(0)} \frac{d\tilde{x}}{|\tilde{x}|(|\tilde{x}| - |\tilde{y}|)} = 2\pi \int_{2|\tilde{y}|}^L \frac{dr}{r - |\tilde{y}|} \\ &= 2\pi \log \left(\frac{L}{|\tilde{y}|} - 1 \right) \leq -2\pi \log \left(\frac{|\tilde{y}|}{L} \right). \end{aligned} \quad (3.7)$$

We now split the integral on $B_{2|\tilde{y}|}(0)$ in (3.6) into three terms:

$$\int_{B_{2|\tilde{y}|}(0)} \frac{d\tilde{x}}{|\tilde{x}||\tilde{x} - \tilde{y}|} = \left\{ \int_{B_{|\tilde{y}|/2}(0)} + \int_{B_{|\tilde{y}|/2}(\tilde{y})} + \int_{B_{2|\tilde{y}|}(0) \setminus (B_{|\tilde{y}|/2}(0) \cup B_{|\tilde{y}|/2}(\tilde{y}))} \right\} \frac{d\tilde{x}}{|\tilde{x}||\tilde{x} - \tilde{y}|}. \quad (3.8)$$

For the first integral in the right-hand side of (3.8) we have

$$\int_{B_{|\tilde{y}|/2}(0)} \frac{d\tilde{x}}{|\tilde{x}||\tilde{x} - \tilde{y}|} \leq \frac{2}{|\tilde{y}|} \int_{B_{|\tilde{y}|/2}(0)} \frac{d\tilde{x}}{|\tilde{x}|} = \frac{4\pi}{|\tilde{y}|} \int_0^{|\tilde{y}|/2} dr = 2\pi,$$

since $|\tilde{x} - \tilde{y}| \geq \frac{|\tilde{y}|}{2}$ for $\tilde{x} \in B_{|\tilde{y}|/2}(0)$. Analogously,

$$\int_{B_{|\tilde{y}|/2}(\tilde{y})} \frac{d\tilde{x}}{|\tilde{x}||\tilde{x} - \tilde{y}|} \leq 2\pi.$$

Moreover, for the last integral in the right-hand side of (3.8) we have

$$\int_{B_{2|\tilde{y}|}(0) \setminus (B_{|\tilde{y}|/2}(0) \cup B_{|\tilde{y}|/2}(\tilde{y}))} \frac{d\tilde{x}}{|\tilde{x}||\tilde{x} - \tilde{y}|} \leq \frac{4}{|\tilde{y}|^2} \int_{B_{2|\tilde{y}|}(0)} d\tilde{x} = 16\pi.$$

In conclusion, from (3.8), we deduce

$$\int_{B_{2|\tilde{y}|}(0)} \frac{d\tilde{x}}{|\tilde{x}||\tilde{x} - \tilde{y}|} \leq 20\pi.$$

Finally, putting together (3.5)–(3.8), and substituting back $\tilde{y} = y - z$ we obtain

$$|V(y, z)| \leq C \left(-2\pi \log \left(\frac{|y - z|}{L} \right) + 20\pi \right),$$

which implies (3.2).

We now prove (3.3). Let Ω' be an open set contained in Ω with $\delta := \text{dist}(\Omega', \partial\Omega) > 0$ and let $y, z \in \Omega'$ with $y \neq z$. Here we use an argument by [CL05]. Let $\gamma_{y,z}$ be a line segment parallel to $z - y$ connecting z to $\partial\Omega$, so that

$$\gamma_{y,z} = \left\{ x \in \Omega : x = z + s \frac{z - y}{|z - y|}, s \in [0, \bar{s}_{y,z}] \right\}.$$

Let also $m_{y,z}$ be the unit normal vector to $\gamma_{y,z}$. Since $\Omega \setminus \gamma_{y,z}$ is simply connected and $\text{Curl } K(\cdot; z) = 0$ in $\Omega \setminus \gamma_{y,z}$, there exists a function $v_{y,z}$ in $\Omega \setminus \gamma_{y,z}$ such that $K(\cdot; z) = \nabla v_{y,z}$ in $\Omega \setminus \gamma_{y,z}$ and the jump of $v_{y,z}$ across $\gamma_{y,z}$ satisfies $[v_{y,z}] = -e_1$. We can also assume that $v_{y,z}$ has zero mean value on $\Omega \setminus \gamma_{y,z}$. By the divergence theorem we obtain

$$\begin{aligned} V(y, z) &= \int_{\Omega \setminus \gamma_{y,z}} \mathbb{C}K(x; y) : K(x; z) dx \\ &= \int_{\Omega \setminus \gamma_{y,z}} \mathbb{C}K(x; y) : \nabla v_{y,z}(x) dx \\ &= \int_{\partial\Omega} \mathbb{C}K(x; y) \nu(x) \cdot v_{y,z}(x) d\mathcal{H}^1(x) - \int_{\gamma_{y,z}} \mathbb{C}K(x; y) m_{y,z}(x) \cdot [v_{y,z}(x)] d\mathcal{H}^1(x). \end{aligned} \quad (3.9)$$

We first observe that the first integral on the right-hand side is uniformly bounded with respect to y and z . Indeed, by (3.4)

$$|\mathbb{C}K(x; y) \nu(x)| \leq \frac{C}{|x - y|} \leq \frac{C}{\delta} \quad (3.10)$$

for every $x \in \partial\Omega$ and every $y \in \Omega'$. Moreover, we have

$$\begin{aligned} \int_{\Omega \setminus \gamma_{y,z}} |K(x; z)| dx &\leq C \int_{\Omega} \frac{1}{|x-z|} dx \\ &= C \int_{B_{\delta/2}(z)} \frac{1}{|x-z|} dx + C \int_{\Omega \setminus B_{\delta/2}(z)} \frac{1}{|x-z|} dx \\ &\leq \pi C \delta + \frac{2C}{\delta}. \end{aligned}$$

Therefore, the measures $Dv_{y,z} = K(\cdot; z)\mathcal{L}^2 - e_1 \otimes m_{y,z}\mathcal{H}^1 \llcorner \gamma_{y,z}$ are uniformly bounded in Ω . By Poincaré-Wirtinger inequality we deduce that the functions $v_{y,z}$ are uniformly bounded with respect to y and z in the BV norm, hence their traces are bounded in $L^1(\partial\Omega; \mathbb{R}^2)$. In conclusion, by the previous estimate and by (3.10) follows that the first term in (3.9) is bounded.

We focus on the second integral on the right-hand side of (3.9). We note that for $x = z + s \frac{z-y}{|z-y|} \in \gamma_{y,z}$ we have

$$-\mathbb{C}K(x; y)m_{y,z}(x) \cdot [v_{y,z}(x)] = \mathbb{C}K(x; y)m_{y,z}(x) \cdot e_1 = \frac{\mu(\lambda + \mu)}{\pi(\lambda + 2\mu)} \frac{1}{|z-y| + s},$$

therefore

$$-\int_{\gamma_{y,z}} \mathbb{C}K(x; y)m_{y,z}(x) \cdot [v_{y,z}(x)] d\mathcal{H}^1(x) = \frac{\mu(\lambda + \mu)}{\pi(\lambda + 2\mu)} \left(-\log|y-z| + \log(|y-z| + \bar{s}_{y,z}) \right).$$

Since $\bar{s}_{y,z} \geq \text{dist}(z, \partial\Omega) \geq \delta$, we have

$$\log(|y-z| + \bar{s}_{y,z}) \geq \log \delta.$$

Combining (3.9) with the previous estimates, we conclude that

$$V(y, z) \geq -C - \frac{\mu(\lambda + \mu)}{\pi(\lambda + 2\mu)} \log|y-z| \geq C'(1 - \log|y-z|),$$

where the last inequality is satisfied for $|y-z| < \bar{R}$, with \bar{R} small enough. This shows (3.3).

Finally, we prove the continuity of V . Let $y, z \in \Omega$ with $y \neq z$ and let $(y_n), (z_n)$ be two sequences in Ω converging to y and z , respectively. For n large enough we clearly have $y_n \neq z_n$. It is easy to see that

$$\mathbb{C}K(x; y_n) : K(x, z_n) \rightarrow \mathbb{C}K(x; y) : K(x; z) \quad \text{for a.e. } x \in \Omega.$$

Moreover, the sequence is dominated. Indeed, for $0 < \delta < \frac{1}{4}|y-z|$ we have

$$\chi_{\Omega \setminus (B_\delta(y_n) \cup B_\delta(z_n))} |\mathbb{C}K(x; y_n) : K(x, z_n)| \leq \frac{C}{\delta^2},$$

while

$$\begin{aligned} \chi_{B_\delta(y_n) \cup B_\delta(z_n)} |\mathbb{C}K(x; y_n) : K(x, z_n)| &\leq \chi_{B_\delta(y_n) \cup B_\delta(z_n)} \frac{C}{|x-y_n||x-z_n|} \\ &\leq \frac{4C}{|y-z|} \left(\frac{1}{|x-y_n|} + \frac{1}{|x-z_n|} \right) \end{aligned}$$

and the function on the right-hand side is dominated since it is strongly converging in L^1 (by the continuity property in L^1). Thus, by the dominated convergence theorem we have

$$\begin{aligned} \lim_{(y_n, z_n) \rightarrow (y, z)} V(y_n, z_n) &= \lim_{(y_n, z_n) \rightarrow (y, z)} \int_{\Omega} \mathbb{C}K(x; y_n) : K(x; z_n) dx \\ &= \int_{\Omega} \mathbb{C}K(x; y) : K(x; z) dx = V(y, z). \end{aligned}$$

Let now $y = z \in \Omega$ and let $(y_n), (z_n)$ be two sequences converging to y . Let Ω' be an open set compactly contained in Ω such that $y \in \Omega'$. Without loss of generality we can assume that $y_n \neq z_n$ and $y_n, z_n \in \Omega'$ for every n . Thus, we can use (3.3) for n large enough and deduce that

$$\lim_{(y_n, z_n) \rightarrow (y, y)} V(y_n, z_n) = +\infty = V(y, y).$$

This concludes the proof of the continuity of V . \square

Remark 3.2. Note that from (3.2) and (3.3) follows that for every open set Ω' compactly contained in Ω there exists a constant $C_1(\Omega') > 0$ such that for every $y, z \in \Omega'$

$$V(y, z) \geq -C_1(\Omega'). \quad (3.11)$$

We are now ready to prove the Γ -convergence of the renormalised energy, which constitutes the main result of this section.

Theorem 3.3. *Assume (1.1). Then the functionals \mathcal{F}_n Γ -converge with respect to the narrow convergence of measures, as $n \rightarrow \infty$, to the functional $\mathcal{F} : \mathcal{P}(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}$ defined as*

$$\mathcal{F}(\mu) := \frac{1}{2} \iint_{\Omega \times \Omega} V(y, z) d\mu(y) d\mu(z) + \min_v I_\mu(v),$$

if $\text{supp } \mu \subset \mathcal{R}$, while $\mathcal{F}(\mu) := +\infty$ otherwise. In the formula above V is the function defined in (3.1) and I_μ is the functional defined in (2.8),

$$I_\mu(v) := \frac{1}{2} \int_{\Omega} \mathbb{C} \nabla v : \nabla v dx + \int_{\Omega} \int_{\partial \Omega} \mathbb{C} K(x; y) \nu(x) \cdot v(x) d\mathcal{H}^1(x) d\mu(y)$$

on the class

$$\left\{ v \in H^1(\Omega; \mathbb{R}^2) : \int_B v dx = 0, \int_B \text{skew } \nabla v dx = 0 \right\}. \quad (3.12)$$

Remark 3.4. Since $\text{supp } \mu \subset \mathcal{R}$ and V is bounded from below on open sets compactly contained in Ω by Remark 3.2, the interaction energy term in \mathcal{F} is always well defined, possibly equal to $+\infty$.

Proof of Theorem 3.3. Preliminary estimates. Let (μ_n) be such that $\mu_n \in X_n$ for every $n \in \mathbb{N}$, and let $(z_i^n)_{i=1, \dots, n} \subset \mathcal{R}$ be such that

$$\mu_n = \frac{1}{n} \sum_{i=1}^n \delta_{z_i^n}.$$

For brevity we set $\beta_n := \beta_{\mu_n}$, $u_n := u_{\mu_n}$ and simply denote $\Omega_n(\mu_n)$ by Ω_n . Finally, let

$$\mu_n \boxtimes \mu_n := \frac{1}{n^2} \sum_{i=1}^n \sum_{j \neq i}^n \delta_{(z_i^n, z_j^n)}.$$

As a preliminary step, we shall prove that

$$\frac{1}{2n^2} \sum_{i=1}^n \sum_{j \neq i}^n \int_{\Omega_n} \mathbb{C} K^n(x; z_i^n) : K^n(x; z_j^n) dx = \frac{1}{2} \iint_{\Omega \times \Omega} V(y, z) d(\mu_n \boxtimes \mu_n)(y, z) + o(1) \quad (3.13)$$

and

$$\frac{1}{n} \sum_{i=1}^n \int_{\partial \Omega_n} \mathbb{C} K^n(x; z_i^n) \nu(x) \cdot u_n(x) d\mathcal{H}^1(x) = \frac{1}{n} \sum_{i=1}^n \int_{\partial \Omega} \mathbb{C} K(x; z_i^n) \nu(x) \cdot u_n(x) d\mathcal{H}^1(x) + o(1), \quad (3.14)$$

where $o(1)$ denotes an infinitesimal quantity, as $n \rightarrow \infty$. Moreover, if we assume in addition that $\mu_n \rightarrow \mu$ narrowly, as $n \rightarrow \infty$, then

$$\lim_{n \rightarrow \infty} \frac{1}{2n} \sum_{i=1}^n \int_{\partial \Omega} \mathbb{C} K(x; z_i^n) \nu(x) \cdot u_n(x) d\mathcal{H}^1(x) = I_\mu(v_\mu) = \min I_\mu. \quad (3.15)$$

The proof of (3.13)–(3.15) is split into several steps.

Step 1. Proof of (3.13). We first show that, as $n \rightarrow \infty$,

$$\frac{1}{2n^2} \sum_{i=1}^n \sum_{j \neq i}^n \int_{\Omega_n} \mathbb{C} K^n(x; z_i^n) : K^n(x; z_j^n) dx = \frac{1}{2n^2} \sum_{i=1}^n \sum_{j \neq i}^n \int_{\Omega} \mathbb{C} K(x; z_i^n) : K(x; z_j^n) dx + o(1). \quad (3.16)$$

Here we follow an argument in [CL05, Proof of Theorem 5.1]. We recall that by (2.2) we have $K^n(x; z_i^n) = K(x; z_i^n) + \varepsilon_n^2 \nabla w_{n,i}$, where we set $w_{n,i}(x) := w(x - z_i^n)$. Therefore,

$$\begin{aligned} \int_{\Omega_n} \mathbb{C}K^n(x; z_i^n) : K^n(x; z_j^n) dx &= \int_{\Omega_n} \mathbb{C}K(x; z_i^n) : K(x; z_j^n) dx \\ &+ \varepsilon_n^2 \int_{\Omega_n} \mathbb{C}K(x; z_i^n) : \nabla w_{n,j}(x) dx + \varepsilon_n^2 \int_{\Omega_n} \mathbb{C}K(x; z_j^n) : \nabla w_{n,i}(x) dx \\ &+ \varepsilon_n^4 \int_{\Omega_n} \mathbb{C}\nabla w_{n,i}(x) : \nabla w_{n,j}(x) dx. \end{aligned}$$

Applying the divergence theorem the last three integrals can be written as

$$\begin{aligned} &\varepsilon_n^2 \int_{\partial\Omega} (\mathbb{C}K(x; z_i^n)\nu \cdot w_{n,j}(x) + \mathbb{C}K(x; z_j^n)\nu \cdot w_{n,i}(x) + \varepsilon_n^2 \mathbb{C}\nabla w_{n,i}(x)\nu \cdot w_{n,j}(x)) d\mathcal{H}^1(x) \\ &- \varepsilon_n^2 \sum_{k=1}^n \int_{\partial B_{\varepsilon_n}(z_k^n)} (\mathbb{C}K(x; z_i^n)\nu \cdot w_{n,j}(x) + \mathbb{C}K(x; z_j^n)\nu \cdot w_{n,i}(x) + \varepsilon_n^2 \mathbb{C}\nabla w_{n,i}(x)\nu \cdot w_{n,j}(x)) d\mathcal{H}^1(x). \end{aligned}$$

Since $|K(x; x_0)| \leq C|x - x_0|^{-1}$ and $|w(x)| \leq C|x|^{-2}$, while $|\nabla w(x)| \leq C|x|^{-3}$, we easily deduce that the first term in the formula above is of order ε_n^2 . As for the second term, by (1.1) we have

$$\begin{aligned} \varepsilon_n^2 \left| \sum_{k \neq i,j} \int_{\partial B_{\varepsilon_n}(z_k^n)} (\mathbb{C}K(x; z_i^n)\nu \cdot w_{n,j}(x) + \mathbb{C}K(x; z_j^n)\nu \cdot w_{n,i}(x) + \varepsilon_n^2 \mathbb{C}\nabla w_{n,i}(x)\nu \cdot w_{n,j}(x)) d\mathcal{H}^1(x) \right| \\ \leq C\varepsilon_n^3 n \left(\frac{1}{r_n^3} + \frac{\varepsilon_n^2}{r_n^5} \right) \leq C \frac{\varepsilon_n^3}{r_n^4} \left(1 + \frac{\varepsilon_n^2}{r_n^2} \right), \end{aligned}$$

while

$$\begin{aligned} \varepsilon_n^2 \left| \sum_{k=i,j} \int_{\partial B_{\varepsilon_n}(z_k^n)} (\mathbb{C}K(x; z_i^n)\nu \cdot w_{n,j}(x) + \mathbb{C}K(x; z_j^n)\nu \cdot w_{n,i}(x) + \varepsilon_n^2 \mathbb{C}\nabla w_{n,i}(x)\nu \cdot w_{n,j}(x)) d\mathcal{H}^1(x) \right| \\ \leq C\varepsilon_n^3 \left(\frac{1}{\varepsilon_n^2 r_n} + \frac{1}{\varepsilon_n r_n^2} + \frac{1}{r_n^3} \right) = C \frac{\varepsilon_n}{r_n} \left(1 + \frac{\varepsilon_n}{r_n} + \frac{\varepsilon_n^2}{r_n^2} \right). \end{aligned}$$

Combining the previous estimates, we obtain

$$\frac{1}{2n^2} \sum_{i=1}^n \sum_{j \neq i} \int_{\Omega_n} \mathbb{C}K^n(x; z_i^n) : K^n(x; z_j^n) dx = \frac{1}{2n^2} \sum_{i=1}^n \sum_{j \neq i} \int_{\Omega_n} \mathbb{C}K(x; z_i^n) : K(x; z_j^n) dx + o(1).$$

To conclude the proof of (3.16), it remains to show that

$$\lim_{n \rightarrow \infty} \frac{1}{2n^2} \sum_{i=1}^n \sum_{j \neq i} \int_{\Omega \setminus \Omega_n} \mathbb{C}K(x; z_i^n) : K(x; z_j^n) dx = 0. \quad (3.17)$$

Using again the estimate $|K(x; x_0)| \leq C|x - x_0|^{-1}$ and (1.1), we infer

$$\begin{aligned} &\frac{1}{2n^2} \left| \sum_{i=1}^n \sum_{j \neq i} \int_{\Omega \setminus \Omega_n} \mathbb{C}K(x; z_i^n) : K(x; z_j^n) dx \right| \\ &= \frac{1}{2n^2} \left| \sum_{i=1}^n \sum_{j \neq i} \sum_{k=1}^n \int_{B_{\varepsilon_n}(z_k^n)} \mathbb{C}K(x; z_i^n) : K(x; z_j^n) dx \right| \\ &\leq C \left(n \frac{\varepsilon_n^2}{r_n^2} + \frac{\varepsilon_n}{r_n} \right) = C \frac{\varepsilon_n}{r_n} \left(1 + \frac{\varepsilon_n}{r_n^2} \right), \end{aligned}$$

which proves (3.17) and, in turn, (3.16).

Identity (3.13) follows immediately from (3.16), since by the definitions of V and of $\mu_n \boxtimes \mu_n$ we may write

$$\begin{aligned} \frac{1}{2n^2} \sum_{i=1}^n \sum_{j \neq i} \int_{\Omega} \mathbb{C}K(x; z_i^n) : K(x; z_j^n) dx &= \frac{1}{2} \iint_{\Omega \times \Omega} \left(\int_{\Omega} \mathbb{C}K(x; y) : K(x; z) dx \right) d(\mu_n \boxtimes \mu_n)(y, z) \\ &= \frac{1}{2} \iint_{\Omega \times \Omega} V(y, z) d(\mu_n \boxtimes \mu_n)(y, z). \end{aligned}$$

Step 2. Proof of (3.14). By Lemma 2.1 for every $n \in \mathbb{N}$ there exists an extension $\tilde{u}_n \in H^1(\Omega; \mathbb{R}^2)$ of u_n such that

$$\|\tilde{u}_n\|_{H^1(\Omega)} \leq C \|Eu_n\|_{L^2(\Omega_n)} \leq C, \quad (3.18)$$

with a constant C independent of n . This implies that, up to subsequences, \tilde{u}_n converges weakly in $H^1(\Omega; \mathbb{R}^2)$ to some function $v \in H^1(\Omega; \mathbb{R}^2)$. Since u_n (and thus \tilde{u}_n) satisfies the conditions (2.6) for every n , the function v belongs to the class (3.12). We also recall that by (2.20) and (3.18) we have

$$\sum_{j=1}^n \|u_n\|_{L^2(\partial B_{\varepsilon_n}(z_j^n))} \leq \frac{C}{\sqrt{r_n}} \|Eu_n\|_{L^2(\Omega_n)} \leq \frac{C}{\sqrt{r_n}}. \quad (3.19)$$

Using the boundary condition in (2.3), we have

$$\begin{aligned} &\frac{1}{n} \sum_{i=1}^n \int_{\partial\Omega_n} \mathbb{C}K^n(x; z_i^n) \nu \cdot u_n(x) d\mathcal{H}^1(x) \\ &= \frac{1}{n} \sum_{i=1}^n \int_{\partial\Omega} \mathbb{C}K^n(x; z_i^n) \nu \cdot u_n(x) d\mathcal{H}^1(x) - \frac{1}{n} \sum_{i=1}^n \sum_{j \neq i} \int_{\partial B_{\varepsilon_n}(z_j^n)} \mathbb{C}K^n(x; z_i^n) \nu \cdot u_n(x) d\mathcal{H}^1(x). \end{aligned} \quad (3.20)$$

Arguing as in the proof of (3.16), it is easy to see that

$$\frac{1}{n} \sum_{i=1}^n \int_{\partial\Omega} \mathbb{C}K^n(x; z_i^n) \nu \cdot u_n(x) d\mathcal{H}^1(x) = \frac{1}{n} \sum_{i=1}^n \int_{\partial\Omega} \mathbb{C}K(x; z_i^n) \nu \cdot u_n(x) d\mathcal{H}^1(x) + o(1). \quad (3.21)$$

Moreover, as a consequence of (2.16) and (3.19), we obtain

$$\frac{1}{n} \left| \sum_{i=1}^n \sum_{j \neq i} \int_{\partial B_{\varepsilon_n}(z_j^n)} \mathbb{C}K^n(x; z_i^n) \nu \cdot u_n(x) d\mathcal{H}^1(x) \right| \leq C \frac{\sqrt{\varepsilon_n}}{r_n^{3/2}} \rightarrow 0. \quad (3.22)$$

Combining (3.20)–(3.22), we deduce (3.14).

Step 3. Convergence of the boundary energy terms. Assume now that $\mu_n \rightarrow \mu$ narrowly, as $n \rightarrow \infty$. We prove that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \int_{\partial\Omega} \mathbb{C}K(x; z_i^n) \nu \cdot u_n(x) d\mathcal{H}^1(x) = \int_{\Omega} \int_{\partial\Omega} \mathbb{C}K(x; y) \nu(x) \cdot v(x) d\mathcal{H}^1(x) d\mu(y). \quad (3.23)$$

Since $\tilde{u}_n = u_n$ on $\partial\Omega$, we have

$$\frac{1}{n} \sum_{i=1}^n \int_{\partial\Omega} \mathbb{C}K(x; z_i^n) \nu \cdot u_n(x) d\mathcal{H}^1(x) = \int_{\Omega} \int_{\partial\Omega} \mathbb{C}K(x; y) \nu(x) \cdot \tilde{u}_n(x) d\mathcal{H}^1(x) d\mu_n(y). \quad (3.24)$$

Let now Ω' be the open set of all points in Ω with distance from $\partial\Omega$ larger than $\ell/2$. By the compactness of the trace operator the weak convergence of \tilde{u}_n to v in $H^1(\Omega; \mathbb{R}^2)$ guarantees strong convergence of the traces in $L^2(\partial\Omega; \mathbb{R}^2)$. This, in turn, implies

$$\begin{aligned} \sup_{y \in \Omega'} \left| \int_{\partial\Omega} \mathbb{C}K(x; y) \nu(x) \cdot (\tilde{u}_n(x) - v(x)) d\mathcal{H}^1(x) \right| &\leq \|\tilde{u}_n - v\|_{L^2(\partial\Omega)} \sup_{y \in \Omega'} \|K(\cdot; y)\|_{L^2(\partial\Omega)} \\ &\leq \frac{C}{\ell} \|\tilde{u}_n - v\|_{L^2(\partial\Omega)} \rightarrow 0. \end{aligned}$$

Since $\text{supp } \mu_n \subset \Omega'$ for every $n \in \mathbb{N}$, the estimate above, together with the narrow convergence of μ_n to μ , allows us to pass to the limit in (3.24) and obtain (3.23).

Step 4. Characterization of v . We now prove that $v = v_\mu$, that is, v is the unique minimizer of the functional I_μ on the class (3.12).

We have that $|\Omega \setminus \Omega_n| \rightarrow 0$ since (1.1) implies that $nr_n^2 \rightarrow 0$ as $n \rightarrow \infty$. Therefore, by lower semicontinuity of the elastic energy with respect to weak convergence in L^2 we have

$$\liminf_{n \rightarrow \infty} \frac{1}{2} \int_{\Omega_n} \mathbb{C} \nabla u_n : \nabla u_n \, dx = \liminf_{n \rightarrow \infty} \frac{1}{2} \int_{\Omega_n} \mathbb{C} \nabla \tilde{u}_n : \nabla \tilde{u}_n \, dx \geq \frac{1}{2} \int_{\Omega} \mathbb{C} \nabla v : \nabla v \, dx.$$

Combining the inequality above with (3.14) and (3.23) yields

$$\liminf_{n \rightarrow \infty} I_{n, \mu_n}(u_n) \geq I_\mu(v).$$

On the other hand, since u_n minimizes I_{n, μ_n} on $H^1(\Omega_n; \mathbb{R}^2)$ by Lemma 2.1, we have

$$I_{n, \mu_n}(u_n) \leq I_{n, \mu_n}(u)$$

for every $u \in H^1(\Omega; \mathbb{R}^2)$. Arguing as in (3.20)–(3.22), one can show that the functionals I_{n, μ_n} converge pointwise to I_μ on $H^1(\Omega; \mathbb{R}^2)$. Thus, we deduce that

$$I_\mu(v) = \liminf_{n \rightarrow \infty} I_{n, \mu_n}(u_n) \leq \lim_{n \rightarrow \infty} I_{n, \mu_n}(u) = I_\mu(u).$$

We conclude that v is the minimizer of I_μ on the class (3.12), that is, $v = v_\mu$, and $I_{n, \mu_n}(u_n) \rightarrow I_\mu(v_\mu)$. Together with (3.23), this implies (3.15), since

$$I_\mu(v_\mu) = \frac{1}{2} \int_{\Omega} \int_{\partial\Omega} \mathbb{C} K(x; y) \nu(x) \cdot v_\mu(x) \, d\mathcal{H}^1(x) \, d\mu(y),$$

where we used integration by parts and the Euler-Lagrange equations satisfied by v_μ .

Liminf inequality. Let now (μ_n) be a sequence of measures such that $\mu_n \in X_n$ for every $n \in \mathbb{N}$ and $\mu_n \rightarrow \mu$ narrowly, as $n \rightarrow \infty$. We shall prove that

$$\liminf_{n \rightarrow \infty} \mathcal{F}_n(\mu_n) \geq \mathcal{F}(\mu). \quad (3.25)$$

Clearly μ satisfies $\text{supp } \mu \subset \mathcal{R}$. We consider the decomposition of \mathcal{F}_n given in (2.13). Equation (3.13) guarantees that

$$\liminf_{n \rightarrow \infty} \frac{1}{2n^2} \sum_{i=1}^n \sum_{j \neq i} \int_{\Omega_n} \mathbb{C} K^n(x; z_i^n) : K^n(x; z_j^n) \, dx = \liminf_{n \rightarrow \infty} \frac{1}{2} \iint_{\Omega \times \Omega} V(y, z) \, d(\mu_n \boxtimes \mu_n)(y, z). \quad (3.26)$$

For $M > 0$ let $V_M := V \wedge M$. We observe that $\text{supp } \mu_n, \text{supp } \mu \subset \Omega'$ for every n , where Ω' is the open set of all points in Ω with distance from $\partial\Omega$ larger than $\ell/2$; thus, by Lemma 3.1 and Remark 3.2 the function V_M is continuous and bounded on $\Omega' \times \Omega'$. Using the narrow convergence of $\mu_n \boxtimes \mu_n$ to $\mu \otimes \mu$, we infer

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{1}{2} \iint_{\Omega \times \Omega} V(y, z) \, d(\mu_n \boxtimes \mu_n)(y, z) &\geq \liminf_{n \rightarrow \infty} \frac{1}{2} \iint_{\Omega \times \Omega} V_M(y, z) \, d(\mu_n \boxtimes \mu_n)(y, z) \\ &\geq \frac{1}{2} \iint_{\Omega \times \Omega} V_M(y, z) \, d\mu(y) \, d\mu(z). \end{aligned} \quad (3.27)$$

Since M is arbitrary, (3.26) and (3.27) yield

$$\liminf_{n \rightarrow \infty} \frac{1}{2n^2} \sum_{i=1}^n \sum_{j \neq i} \int_{\Omega_n} \mathbb{C} K^n(x; z_i^n) : K^n(x; z_j^n) \, dx \geq \frac{1}{2} \iint_{\Omega \times \Omega} V(y, z) \, d\mu(y) \, d\mu(z).$$

Combining this inequality with (3.14) and (3.15), we obtain (3.25).

Limsup inequality. Let $\mu \in \mathcal{P}(\Omega)$ be such that $\mathcal{F}(\mu) < +\infty$. In particular, we have that $\text{supp } \mu \subset \mathcal{R}$. Since the energy estimates (3.13)–(3.15) are valid for any admissible sequence

converging to μ , it is sufficient to construct a sequence μ_n such that $\mu_n \in X_n$, with $\mu_n \rightharpoonup \mu$ narrowly and

$$\limsup_{n \rightarrow \infty} \frac{1}{2} \iint_{\Omega \times \Omega} V(y, z) d(\mu_n \boxtimes \mu_n)(y, z) \leq \frac{1}{2} \iint_{\Omega \times \Omega} V(y, z) d\mu(y) d\mu(z).$$

Also in this case the proof will be split into several steps.

Step 1. Approximation of the measure μ . Define the family of squares

$$\tilde{Q}^h := \{[0, 2h]^2 + 2h(m, \ell) : (m, \ell) \in \mathbb{Z}^2\},$$

and let $\{\tilde{Q}_k^h\}_{k=1, \dots, N_h}$ be the squares in \tilde{Q}^h intersecting the support of μ . For $k = 1, \dots, N_h$ let Q_k^h be defined as

$$Q_k^h := \{x \in \tilde{Q}_k^h : x + \lambda_1 e_1 + \lambda_2 e_2 \in \tilde{Q}_k^h, 0 \leq \lambda_i < h\}.$$

We define the approximating measures μ^h as

$$\mu^h := \sum_{k=1}^{N_h} \frac{\mu(\tilde{Q}_k^h)}{h^2} \chi_{Q_k^h}$$

for every h . Without loss of generality we can assume that $\text{supp } \mu^h \subset \mathcal{R}$ for every h . Clearly $\mu^h \rightharpoonup \mu$ narrowly, as $h \rightarrow 0$.

We claim that

$$\limsup_{h \rightarrow 0} \iint_{\Omega \times \Omega} V(y, z) d\mu^h(y) d\mu^h(z) \leq \iint_{\Omega \times \Omega} V(y, z) d\mu(y) d\mu(z). \quad (3.28)$$

For $M > 0$ we consider the truncated function $V_M := V \wedge M$ and we write $V = V_M + (V - V_M)$.

Let Ω' be the open set of all points in Ω with distance from $\partial\Omega$ larger than $\ell/2$. Since $\text{supp } \mu^h, \text{supp } \mu \subset \Omega'$ for every h and V_M is continuous and bounded on $\Omega' \times \Omega'$, narrow convergence yields

$$\begin{aligned} \lim_{h \rightarrow 0} \iint_{\Omega \times \Omega} V_M(y, z) d\mu^h(z) d\mu^h(y) &= \iint_{\Omega \times \Omega} V_M(y, z) d\mu(z) d\mu(y) \\ &\leq \iint_{\Omega \times \Omega} V(y, z) d\mu(z) d\mu(y). \end{aligned}$$

Therefore, (3.28) is proved if we show that

$$\lim_{M \rightarrow \infty} \limsup_{h \rightarrow 0} \iint_{\Omega \times \Omega} (V(y, z) - V_M(y, z)) d\mu^h(z) d\mu^h(y) = 0. \quad (3.29)$$

We observe that the bound (3.2) on V implies that

$$|V(y, z)| > M \implies |y - z| < R_M, \quad (3.30)$$

for some $R_M > 0$ such that $R_M \rightarrow 0$, as $M \rightarrow \infty$. Therefore,

$$\iint_{\Omega \times \Omega} (V(y, z) - V_M(y, z)) d\mu^h(z) d\mu^h(y) \leq \int_{\Omega} \left(\int_{B_{R_M}(y)} V(y, z) d\mu^h(z) \right) d\mu^h(y). \quad (3.31)$$

By estimate (3.3) of Lemma 3.1 there exist $C(\Omega') > 0$ and $\bar{R} > 0$ such that

$$C(\Omega')(1 - \log |y - z|) \leq V(y, z) \quad (3.32)$$

for every $y, z \in \Omega'$ with $0 < |y - z| \leq \bar{R}$. Since $R_M \rightarrow 0$, as $M \rightarrow \infty$, we can assume that $R_M \ll \bar{R}$. For every $k = 1, \dots, N_h$ and every $p = 1, \dots, P_k^h$, with $P_k^h = \lfloor R_M/(2h) \rfloor + 1$, we consider the set of indices

$$I_{k,p}^h := \{1 \leq j \leq N_h : Q_j^h = Q_k^h + 2h(m, \ell), |m| \vee |\ell| = p\}.$$

$$\begin{aligned}
& \int_{\Omega} \left(\int_{B_{R_M}(y)} V(y, z) d\mu^h(z) \right) d\mu^h(y) \\
&= \sum_{k=1}^{N_h} \sum_{p=1}^{P_k^h} \sum_{j \in I_{k,p}^h} \frac{\mu^h(\tilde{Q}_k^h) \mu^h(\tilde{Q}_j^h)}{h^4} \int_{Q_k^h} \left(\int_{B_{R_M}(y) \cap Q_j^h} V(y, z) dz \right) dy \\
&\quad + \sum_{k=1}^{N_h} \frac{(\mu^h(\tilde{Q}_k^h))^2}{h^4} \int_{Q_k^h} \left(\int_{B_{R_M}(y) \cap Q_k^h} V(y, z) dz \right) dy.
\end{aligned}$$

For $k = 1, \dots, N_h$ and $j \in I_{k,p}^h$ we have that $|y - z| \geq (2p - 1)h$ for every $y \in Q_k^h$, $z \in Q_j^h$; thus, estimate (3.2) implies that

$$V(y, z) \leq C \left(1 - \log \frac{(2p - 1)h}{L} \right) \quad \text{for every } y \in Q_k^h, z \in Q_j^h, j \in I_{k,p}^h.$$

On the other hand, since $|y - z| \leq 2\sqrt{2}(p + 1)h$ for every $y \in \tilde{Q}_k^h$, $z \in \tilde{Q}_j^h$ with $j \in I_{k,p}^h$, estimate (3.32) yields

$$C(\Omega')(1 - \log(2\sqrt{2}(p + 1)h)) \leq V(y, z) \quad \text{for every } y \in \tilde{Q}_k^h, z \in \tilde{Q}_j^h, j \in I_{k,p}^h.$$

Therefore, we obtain

$$\begin{aligned}
& \sum_{j \in I_{k,p}^h} \frac{\mu(\tilde{Q}_k^h) \mu(\tilde{Q}_j^h)}{h^4} \int_{Q_k^h} \left(\int_{B_{R_M}(y) \cap Q_j^h} V(y, z) dz \right) dy \\
&\leq C \sum_{j \in I_{k,p}^h} \mu(\tilde{Q}_k^h) \mu(\tilde{Q}_j^h) \left(1 - \log \frac{(2p - 1)h}{L} \right) \\
&= C \sum_{j \in I_{k,p}^h} \mu(\tilde{Q}_k^h) \mu(\tilde{Q}_j^h) \left(1 - \log(2\sqrt{2}(p + 1)h) + \log \frac{2\sqrt{2}(p + 1)L}{2p - 1} \right) \\
&\leq C \sum_{j \in I_{k,p}^h} \int_{\tilde{Q}_k^h} \left(\int_{\tilde{Q}_j^h} V(y, z) d\mu(z) \right) d\mu(y) + C \sum_{j \in I_{k,p}^h} \mu(\tilde{Q}_k^h) \mu(\tilde{Q}_j^h).
\end{aligned}$$

Similarly, by applying (3.2) and (3.32) and using polar coordinates we have

$$\begin{aligned}
\frac{(\mu(\tilde{Q}_k^h))^2}{h^4} \int_{Q_k^h} \left(\int_{B_{R_M}(y) \cap Q_k^h} V(y, z) dz \right) dy &\leq C \frac{(\mu(\tilde{Q}_k^h))^2}{h^4} \int_{Q_k^h} \left(\int_{B_{\sqrt{2}h}(y) \cap Q_k^h} V(y, z) dz \right) dy \\
&\leq C (\mu(\tilde{Q}_k^h))^2 (1 - \log(\sqrt{2}h)) \\
&\leq C \int_{\tilde{Q}_k^h} \left(\int_{\tilde{Q}_k^h} V(y, z) d\mu(z) \right) d\mu(y) + C (\mu(\tilde{Q}_k^h))^2.
\end{aligned}$$

Combining together the previous estimates, we conclude that

$$\begin{aligned}
& \int_{\Omega} \left(\int_{B_{R_M}(y)} V(y, z) d\mu^h(z) \right) d\mu^h(y) \\
&\leq C \int_{\Omega} \left(\int_{B_{R_M}(y)} V(y, z) d\mu(z) \right) d\mu(y) + C \int_{\Omega} \mu(B_{R_M}(y)) d\mu(y),
\end{aligned}$$

where the right-hand side tends to zero, as $R_M \rightarrow 0$, since $\mathcal{F}(\mu) < +\infty$. Owing to (3.31), this proves (3.29), hence the claim (3.28). In virtue of (3.28) and of the metrizable of the narrow convergence on $\mathcal{P}(\Omega)$, it is sufficient to construct a recovery sequence for the approximating measures μ^h .

Step 2. Construction of the recovery sequence. Let us fix $h > 0$. For every n we denote with μ_n^h an approximation of the measure μ^h defined in the previous step, of the following form:

$$\mu_n^h := \sum_{k=1}^{N_h} c_{k,n} \chi_{Q_k^h},$$

where the constants $c_{k,n}$ are chosen in such a way that $\sqrt{n\mu_n^h(Q_k^h)} \in \mathbb{N}_0$ for every k and $\mu_n^h(Q_k^h) \rightarrow \mu^h(Q_k^h)$, as $n \rightarrow \infty$.

At this point we construct the recovery sequence $\mu_n = \frac{1}{n} \sum_{i=1}^n \delta_{z_i^n}$ by choosing the points $z_i^n \in \Omega$ in the following way:

- (1) to each square Q_k^h , allocate $n\mu_n^h(Q_k^h)$ dislocations;
- (2) in a given square Q_k^h , choose $\sqrt{n\mu_n^h(Q_k^h)}$ equispaced horizontal coordinates and $\sqrt{n\mu_n^h(Q_k^h)}$ equispaced vertical coordinates. Then the dislocations will be located at the nodes of this square grid, and their distance within the same square Q_k^h satisfies the bound

$$|z_i^n - z_j^n| > \frac{h}{\sqrt{n\mu_n^h(Q_k^h)}} \sim \frac{C(h)}{\sqrt{n}} \gg r_n,$$

since, by assumption (1.1) $nr_n \rightarrow 0$, as $n \rightarrow \infty$. Such a sequence is therefore admissible.

By construction we have that $\mu_n \llcorner Q_k^h \rightarrow \mu^h \llcorner Q_k^h$ narrowly, as $n \rightarrow \infty$, for every k . This implies that $\mu_n \rightarrow \mu^h$ narrowly as $n \rightarrow \infty$.

Step 3. Convergence of the energy. It remains to prove that

$$\limsup_{n \rightarrow \infty} \frac{1}{2} \iint_{\Omega \times \Omega} V(y, z) d(\mu_n \boxtimes \mu_n)(y, z) \leq \frac{1}{2} \iint_{\Omega \times \Omega} V(y, z) d\mu^h(y) d\mu^h(z). \quad (3.33)$$

For $k = 1, \dots, N_h$ and $n \in \mathbb{N}$ we introduce the set

$$J_k^n := \{1 \leq i \leq n : z_i^n \in Q_k^h\}.$$

We then calculate

$$\begin{aligned} \frac{1}{2} \iint_{\Omega \times \Omega} V(y, z) d(\mu_n \boxtimes \mu_n)(y, z) &= \frac{1}{2n^2} \sum_{\substack{i,j=1 \\ i \neq j}}^n V(z_i^n, z_j^n) \\ &= \frac{1}{2n^2} \sum_{\substack{k,\ell=1 \\ k \neq \ell}}^{N_h} \sum_{\substack{i \in J_k^n \\ j \in J_\ell^n}} V(z_i^n, z_j^n) + \frac{1}{2n^2} \sum_{k=1}^{N_h} \sum_{\substack{i,j \in J_k^n \\ i \neq j}} V(z_i^n, z_j^n). \end{aligned} \quad (3.34)$$

For the first term in the right-hand side of (3.34) we have that

$$\lim_{n \rightarrow \infty} \frac{1}{2n^2} \sum_{\substack{k,\ell=1 \\ k \neq \ell}}^{N_h} \sum_{\substack{i \in J_k^n \\ j \in J_\ell^n}} V(z_i^n, z_j^n) = \frac{1}{2} \sum_{\substack{k,\ell=1 \\ k \neq \ell}}^{N_h} \iint_{Q_k^h \times Q_\ell^h} V(y, z) d\mu^h(y) d\mu^h(z), \quad (3.35)$$

since V is continuous and bounded on $Q_k^h \times Q_\ell^h$ for $k \neq \ell$. For the second term in (3.34) we consider, for fixed $M > 0$, the decomposition $V = V_M + (V - V_M)$, where $V_M := V \wedge M$. Since the truncated function V_M is continuous and bounded on an open set containing $\mathcal{R} \times \mathcal{R}$, we have

$$\lim_{n \rightarrow \infty} \frac{1}{2n^2} \sum_{k=1}^{N_h} \sum_{\substack{i,j \in J_k^n \\ i \neq j}} V_M(z_i^n, z_j^n) = \frac{1}{2} \sum_{k=1}^{N_h} \iint_{Q_k^h \times Q_k^h} V_M(y, z) d\mu^h(y) d\mu^h(z). \quad (3.36)$$

Combining together (3.34)–(3.36), since $V_M \leq V$, the claim (3.33) follows if we prove that

$$\lim_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{2n^2} \sum_{k=1}^{N_h} \sum_{\substack{i,j \in J_k^n \\ i \neq j}} (V - V_M)(z_i^n, z_j^n) = 0. \quad (3.37)$$

From (3.30), for large enough M (so that, in particular, $R_M < h$) we have that, for every $k = 1, \dots, N_h$,

$$\frac{1}{2n^2} \sum_{\substack{i,j \in J_k^n \\ i \neq j}} (V - V_M)(z_i^n, z_j^n) \leq \frac{1}{2n^2} \sum_{\substack{i,j \in J_k^n \\ |z_i^n - z_j^n| < R_M \\ i \neq j}} V(z_i^n, z_j^n). \quad (3.38)$$

Now we denote with L_k^n the size of the square grid in Q_k^h formed by the dislocations locations, i.e.,

$$L_k^n := \frac{h}{\sqrt{n\mu_n^h(Q_k^h)}}.$$

Then we can rewrite the last term in (3.38) as

$$\frac{1}{2n^2} \sum_{\substack{i,j \in J_k^n \\ |z_i^n - z_j^n| < R_M \\ i \neq j}} V(z_i^n, z_j^n) = \frac{1}{2n^2} \sum_{i \in J_k^n} \sum_{p=2}^{P_k^n} \sum_{j \in J_k^n(i,p)} V(z_i^n, z_j^n), \quad (3.39)$$

where P_k^n is the integer part of R_M/L_k^n and, for $p = 2, \dots, P_k^n$,

$$J_k^n(i,p) := \{j \in J_k^n : (p-1)L_k^n \leq |z_j^n - z_i^n| < pL_k^n\}.$$

The cardinality of the set $J_k^n(i,p)$ can be estimated as

$$\#(J_k^n(i,p)) \leq C(p^2 - (p-1)^2) = C(2p+1);$$

moreover, for every $j \in J_k^n(i,p)$ we have, in virtue of (3.2),

$$|V(z_i^n, z_j^n)| \leq C \left(1 - \log \frac{(p-1)L_k^n}{L}\right).$$

Hence, we deduce

$$\begin{aligned} \left| \frac{1}{2n^2} \sum_{i \in J_k^n} \sum_{p=2}^{P_k^n} \sum_{j \in J_k^n(i,p)} V(z_i^n, z_j^n) \right| &\leq \frac{C}{2n^2} \sum_{i \in J_k^n} \sum_{p=2}^{P_k^n} (2p+1) \left(1 - \log \frac{(p-1)L_k^n}{L}\right) \\ &\leq \frac{C \#(J_k^n)}{n^2} \sum_{p=2}^{P_k^n} \left(2p+1 - 2(p-1) \log \frac{(p-1)L_k^n}{L} - 3 \log \frac{L_k^n}{L}\right). \end{aligned} \quad (3.40)$$

It is easy to show that, for every $p = 2, \dots, P_k^n$,

$$-2(p-1) \log \frac{(p-1)L_k^n}{L} \leq \frac{C}{L_k^n};$$

therefore, the sum in the right-hand side of (3.40) can be estimated as follows:

$$\sum_{p=2}^{P_k^n} \left(2p+1 + \frac{C}{L_k^n} - 3 \log \frac{L_k^n}{L}\right) \leq CP_k^n \left(1 + \frac{1}{L_k^n}\right) + CP_k^n(P_k^n + 1).$$

Combining this estimate with (3.40) and the definition of P_k^n and L_k^n , we deduce

$$\begin{aligned} \left| \frac{1}{2n^2} \sum_{i \in J_k^n} \sum_{p=2}^{P_k^n} \sum_{j \in J_k^n(i,p)} V(z_i^n, z_j^n) \right| &\leq C \frac{\mu_n^h(Q_k^h)}{n} \left(\frac{R_M}{L_k^n} + \frac{R_M}{(L_k^n)^2} + \frac{R_M^2}{(L_k^n)^2} \right) \\ &\leq CR_M \frac{(\mu_n^h(Q_k^h))^{3/2}}{h\sqrt{n}} + CR_M \frac{(\mu_n^h(Q_k^h))^2}{h^2} + CR_M^2 \frac{(\mu_n^h(Q_k^h))^2}{h^2}. \end{aligned}$$

From the last estimate it follows that

$$\limsup_{n \rightarrow \infty} \left| \frac{1}{2n^2} \sum_{i \in J_k^n} \sum_{p=2}^{P_k^n} \sum_{j \in J_k^n(i,p)} V(z_i^n, z_j^n) \right| \leq CR_M \frac{(\mu^h(Q_k^h))^2}{h^2} + CR_M^2 \frac{(\mu^h(Q_k^h))^2}{h^2}. \quad (3.41)$$

Since $R_M \rightarrow 0$, as $M \rightarrow \infty$, (3.38), (3.39), and (3.41) prove the claim (3.37). This concludes the proof of (3.33) and of the theorem. \square

Remark 3.5. We observe that there exists a positive constant C such that

$$\mathcal{F}(\mu) \geq -C \quad (3.42)$$

for every $\mu \in \mathcal{P}(\Omega)$. Indeed, from (3.11) follows directly that the interaction energy term is uniformly bounded from below. Moreover, considering $v = 0$ as test function for I_μ in the class (3.12), we obtain

$$\frac{1}{2} \int_{\Omega} \mathbb{C} \nabla v_\mu : \nabla v_\mu \, dx \leq - \int_{\Omega} \int_{\partial\Omega} \mathbb{C} K(x; y) \nu(x) \cdot v(x) \, d\mathcal{H}^1(x) \, d\mu(y),$$

hence

$$\int_{\Omega} |E v_\mu|^2 \, dx \leq \frac{C}{\ell} \|v_\mu\|_{L^2(\partial\Omega)}.$$

By Korn's inequality and the definition of the class (3.12) we deduce that the functions v_μ are uniformly bounded in $H^1(\Omega; \mathbb{R}^2)$ with respect to μ , hence their traces are uniformly bounded in $L^2(\partial\Omega; \mathbb{R}^2)$ with respect to μ . Finally, this implies that the minimum values $I_\mu(v_\mu)$ are uniformly bounded from below, as well.

We conclude this section with a characterization of the class where the Γ -limit functional \mathcal{F} is finite.

Proposition 3.6. *Let $\mu \in \mathcal{P}(\Omega)$. Then $\mathcal{F}(\mu) < +\infty$ if and only if $\mu \in H^{-1}(\Omega)$.*

Proof. For every $\mu \in \mathcal{P}(\Omega)$ we set

$$\beta_\mu^*(x) := \nabla v_\mu(x) + \int_{\Omega} K(x; y) \, d\mu(y), \quad (3.43)$$

where v_μ is the function associated to μ according to Section 2.1, that is, the unique minimizer of I_μ on the class (2.9). By Fubini Theorem we have that $\beta_\mu^* \in L^1(\Omega; \mathbb{R}^{2 \times 2})$. Moreover, β_μ^* is the unique (up to skew-symmetric matrices) solution of the system

$$\begin{cases} \operatorname{div} \mathbb{C} \beta_\mu^* = 0 & \text{in } \Omega, \\ \mathbb{C} \beta_\mu^* \nu = 0 & \text{on } \partial\Omega, \\ \operatorname{Curl} \beta_\mu^* = \mu e_1 & \text{in } \Omega. \end{cases} \quad (3.44)$$

Let now $\mu \in \mathcal{P}(\Omega) \cap H^{-1}(\Omega)$. Then the problem

$$\min \left\{ \frac{1}{2} \int_{\Omega} \mathbb{C} \beta(x) : \beta(x) \, dx : \beta \in L^2(\Omega; \mathbb{R}^{2 \times 2}), \operatorname{Curl} \beta = \mu e_1 \right\}$$

has a solution $\tilde{\beta}$ (indeed, the class where the minimum is taken is not empty: for instance, we can solve $\Delta u = \mu e_1$, $u \in H_0^1(\Omega; \mathbb{R}^2)$, and take as β a $\pi/2$ rotation of ∇u). Since $\tilde{\beta}$ is a solution to (3.44), we deduce that $\beta_\mu^* \in L^2(\Omega; \mathbb{R}^{2 \times 2})$. Now, by Fubini Theorem we may write

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} \mathbb{C} \beta_\mu^* : \beta_\mu^* \, dx \\ &= \frac{1}{2} \int_{\Omega} \mathbb{C} \nabla v_\mu : \nabla v_\mu \, dx + \int_{\Omega} \int_{\Omega} \mathbb{C} K(x; y) : \nabla v_\mu \, dx \, d\mu(y) + \frac{1}{2} \int_{\Omega \times \Omega} V(y; z) \, d\mu(y) \, d\mu(z) \\ &= \mathcal{F}(\mu), \end{aligned}$$

hence we conclude that $\mathcal{F}(\mu) < +\infty$.

Conversely, let $\mu \in \mathcal{P}(\Omega)$ be such that $\mathcal{F}(\mu) < +\infty$. By Step 1 in the proof of the limsup inequality of Theorem 3.3 there exists a sequence $(\mu^h) \subset L^\infty(\Omega)$ such that $\mu^h \rightharpoonup \mu$ narrowly and

$$\limsup_{h \rightarrow 0} \mathcal{F}(\mu^h) \leq \mathcal{F}(\mu) < +\infty. \quad (3.45)$$

Let β_h^* be defined as in (3.43) with μ replaced by μ^h . Since $(\mu^h) \subset L^\infty(\Omega)$ (hence, in particular, in $H^{-1}(\Omega)$), the previous argument guarantees that $\beta_h^* \in L^2(\Omega; \mathbb{R}^{2 \times 2})$ and by Fubini Theorem

$$\frac{1}{2} \int_{\Omega} \mathbb{C} \beta_h^* : \beta_h^* dx = \mathcal{F}(\mu_h) \quad (3.46)$$

for every h . Combining (3.45) and (3.46), we obtain that the sequence $(\text{sym } \beta_h^*)$ is uniformly bounded in $L^2(\Omega; \mathbb{R}^{2 \times 2})$.

On the other hand, since $\mu^h \rightharpoonup \mu$, we have that $\nabla v_{\mu^h} \rightharpoonup \nabla v_{\mu}$ weakly in $L^2(\Omega; \mathbb{R}^{2 \times 2})$ and that

$$\int_{\Omega} K(\cdot; y) d\mu^h(y) \rightharpoonup \int_{\Omega} K(\cdot; y) d\mu(y)$$

in the sense of distributions. Thus, $\beta_h^* \rightharpoonup \beta_{\mu}^*$ in the sense of distributions. We conclude that $\text{sym } \beta_{\mu}^* \in L^2(\Omega; \mathbb{R}^{2 \times 2})$. By the generalized Korn inequality [GLP10, Theorem 11] we deduce that $\beta_{\mu}^* \in L^2(\Omega; \mathbb{R}^{2 \times 2})$, hence $\mu \in H^{-1}(\Omega)$. \square

4. QUASI-STATIC EVOLUTION OF DISLOCATION ENERGIES

In this section we prove the existence of a quasi-static evolution for the energy-dissipation system given by the renormalised dislocation energy \mathcal{F}_n defined in (1.6), and the modified, slip-plane-confined, Wasserstein distance d in (1.13) as dissipation distance.

4.1. Wasserstein-type dissipation. The dissipation term is defined in terms of a Wasserstein-type distance d (1.13)–(1.14), which is finite only on pairs of measures with the same vertical marginals. This implies that during the evolution only the horizontal positions of the dislocations will be allowed to vary, i.e., each dislocation will be only allowed to move within its slip plane.

The next lemma summarises a number of properties of the distance d and its relation with the standard Wasserstein distance d_1 and with narrow convergence. Recall that d_1 is defined by

$$d_1(\mu, \nu) := \inf_{\gamma \in \Gamma_1(\mu, \nu)} \iint_{\Omega \times \Omega} |x - y| d\gamma(x, y), \quad (4.1)$$

where $\Gamma_1(\mu, \nu)$ is the following set of couplings of μ and ν :

$$\Gamma_1(\mu, \nu) := \{ \gamma \in \mathcal{P}(\Omega \times \Omega) : \gamma(A \times \Omega) = \mu(A), \gamma(\Omega \times A) = \nu(A) \text{ for all Borel sets } A \subset \Omega \}. \quad (4.2)$$

Since Ω is bounded, the distance d_1 generates the topology of narrow convergence.

Lemma 4.1. *The following properties are satisfied.*

- (1) *Convergence in d implies narrow convergence.*
- (2) *The d_1 -distance of measures $\mu, \nu \in \mathcal{P}(\Omega)$ is bounded from below by the d_1 -distance of their horizontal marginals, namely*

$$d_1((\pi_1)_\# \mu, (\pi_1)_\# \nu) \leq d_1(\mu, \nu) \leq d(\mu, \nu).$$

- (3) *The metric d is lower semicontinuous with respect to narrow convergence: if $\mu_k \rightharpoonup \mu$ and $\nu_k \rightharpoonup \nu$, then $d(\mu, \nu) \leq \liminf_{k \rightarrow \infty} d(\mu_k, \nu_k)$.*
- (4) *The dissipation \mathcal{D} , introduced in (1.15), is lower semicontinuous with respect to pointwise narrow convergence: if $\mu_k, \mu : [0, T] \rightarrow \mathcal{P}(\Omega)$ are such that $\mu_k(t) \rightharpoonup \mu(t)$ for all $t \in [0, T]$, then $\mathcal{D}(\mu, [0, T]) \leq \liminf_{k \rightarrow \infty} \mathcal{D}(\mu_k, [0, T])$.*
- (5) *We have the dual characterization*

$$d(\mu, \nu) = \sup \left\{ \int_{\Omega} \phi (d\mu - d\nu) : \phi \in C_c^\infty(\mathbb{R}^2), \right. \\ \left. |\phi(x_1, x_2) - \phi(x'_1, x_2)| \leq |x_1 - x'_1| \text{ for all } x_1, x'_1, x_2 \right\}. \quad (4.3)$$

- (6) *For $\varepsilon > 0$ let $d_{1,\varepsilon}$ be the Monge-Kantorovich transport distance with cost function $c_\varepsilon(x, y) = |x_1 - y_1| + \varepsilon^{-1}|x_2 - y_2|$; then*

$$d(\mu, \nu) = \lim_{\varepsilon \rightarrow 0^+} d_{1,\varepsilon}(\mu, \nu). \quad (4.4)$$

(7) For any measurable curve $t \mapsto \mu(t, \cdot)$, define

$$\mathcal{D}(\mu, [0, T]) := \inf \left\{ \int_0^T \int_{\Omega} |\phi(t, x)| d\mu(t) dt : \phi : (0, T) \times \mathbb{R}^2 \rightarrow \mathbb{R} \text{ Borel measurable,} \right. \\ \left. \partial_t \mu + \partial_{x_1}(\phi \mu) = 0 \text{ as distributions on } (0, T) \times \mathbb{R}^2 \right\}, \quad (4.5)$$

with the convention that $\inf \emptyset = +\infty$. If $\mathcal{D}(\mu, [0, T]) < \infty$, then $\mathcal{D}(\mu, [0, T]) = \mathcal{D}(\mu, [0, T])$.

Remark 4.2. The condition that $\mathcal{D} < \infty$ in Part 7 is necessary: for instance, for two points $x, \tilde{x} \in \Omega$ with $x_2 = \tilde{x}_2$, the curve $\mu(t) = t\delta_x + (1-t)\delta_{\tilde{x}}$ has $\mathcal{D}(\mu, [0, 1]) = |x_1 - \tilde{x}_1|$. However, $\mathcal{D}(\mu, [0, 1]) = +\infty$, since the ‘jumping’ of the mass from x to \tilde{x} can not be represented by the continuity equation $\partial_t \mu + \partial_{x_1}(\phi \mu) = 0$.

Proof of Lemma 4.1. By definition $d_1(\mu, \nu) \leq d(\mu, \nu)$, which implies Part 1. For Part 2 we observe that

$$\gamma \in \Gamma_1(\mu, \nu) \Rightarrow (\pi_1 \times \pi_1)_{\#} \gamma \in \Gamma_1((\pi_1)_{\#} \mu, (\pi_1)_{\#} \nu).$$

Thus, using the fact that $\pi_1(x) = x_1$, we have

$$\begin{aligned} d_1((\pi_1)_{\#} \mu, (\pi_1)_{\#} \nu) &= \inf_{\hat{\gamma} \in \Gamma_1((\pi_1)_{\#} \mu, (\pi_1)_{\#} \nu)} \iint_{\mathbb{R} \times \mathbb{R}} |x_1 - y_1| d\hat{\gamma}(x_1, y_1) \\ &\leq \inf_{\gamma \in \Gamma_1(\mu, \nu)} \iint_{\mathbb{R} \times \mathbb{R}} |x_1 - y_1| d((\pi_1 \times \pi_1)_{\#} \gamma)(x_1, y_1) \\ &= \inf_{\gamma \in \Gamma_1(\mu, \nu)} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} |\pi_1(x) - \pi_1(y)| d\gamma(x, y) \\ &\leq \inf_{\gamma \in \Gamma_1(\mu, \nu)} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} |x - y| d\gamma(x, y) = d_1(\mu, \nu). \end{aligned}$$

We next prove Part 6. Note that $d_{1,\varepsilon}(\mu, \nu) \leq d(\mu, \nu)$. To prove $d(\mu, \nu) = \lim_{\varepsilon \rightarrow 0^+} d_{1,\varepsilon}(\mu, \nu)$, first assume that $d(\mu, \nu) < \infty$ and let $\gamma_{1,\varepsilon}$ be an optimal transport plan for $d_{1,\varepsilon}(\mu, \nu)$; since μ and ν are fixed, $\gamma_{1,\varepsilon}$ is tight, and as $\varepsilon \rightarrow 0^+$ we can assume that $\gamma_{1,\varepsilon} \rightarrow \gamma_{1,0}$ in $\mathcal{P}(\Omega \times \Omega)$. The inequality

$$\iint_{\Omega \times \Omega} \left(|x_1 - y_1| + \frac{1}{\varepsilon} |x_2 - y_2| \right) d\gamma_{1,\varepsilon}(x, y) = d_{1,\varepsilon}(\mu, \nu) \leq d(\mu, \nu)$$

then implies that the limit satisfies $\gamma_{1,0} \in \Gamma(\mu, \nu)$. Therefore $\gamma_{1,0}$ is admissible for $d(\mu, \nu)$, and

$$\liminf_{\varepsilon \rightarrow 0^+} d_{1,\varepsilon}(\mu, \nu) \geq \iint_{\Omega \times \Omega} |x_1 - y_1| d\gamma_{1,0}(x, y) \geq d(\mu, \nu).$$

This, together with the inequality $d_{1,\varepsilon}(\mu, \nu) \leq d(\mu, \nu)$, proves (4.4). On the other hand, if $d(\mu, \nu) = \infty$, then this argument implies that $d_{1,\varepsilon}(\mu, \nu)$ can not remain bounded as $\varepsilon \rightarrow 0^+$, which again implies (4.4).

From (4.4) directly follows Part 3, since $d_{1,\varepsilon}$ is lower semicontinuous with respect to narrow convergence for each $\varepsilon > 0$ and $d_{1,\varepsilon}(\mu, \nu) \leq d(\mu, \nu)$. Part 4 follows from Part 3 and the definition (1.15).

We next prove Part 5. For $\alpha > 0$ define the norm $|(x_1, x_2)|_{\alpha} = |x_1| + \alpha^{-1}|x_2|$, so that $d_{1,\varepsilon}$ is the transport distance with respect to the norm $|\cdot|_{\varepsilon}$. Fix $\eta > 0$, and choose $\varepsilon > 0$ such that $d(\mu, \nu) \leq d_{1,\varepsilon}(\mu, \nu) + \eta$; using the dual characterization for the Monge-Kantorovich transport distance (e.g. [Vil03, Theorem 1.14]), there exists a function ϕ with $|\phi(x) - \phi(y)| \leq |x - y|_{\varepsilon}$, and such that $d_{1,\varepsilon}(\mu, \nu) = \int_{\Omega} \phi(d\mu - d\nu)$. At the cost of another η we can also assume that $\phi \in C_c^{\infty}(\mathbb{R}^2)$, so that

$$d(\mu, \nu) \leq 2\eta + \int_{\Omega} \phi(d\mu - d\nu).$$

Since $|\phi(x_1, x_2) - \phi(x'_1, x_2)| \leq |(x_1, x_2) - (x'_1, x_2)|_{\varepsilon} = |x_1 - x'_1|$, ϕ is admissible in the supremum in (4.3). Since η is arbitrary, it follows that $d(\mu, \nu)$ is bounded from above by the right-hand side in (4.3). For the opposite inequality, let $\gamma \in \Gamma(\mu, \nu)$. By applying the disintegration theorem to μ and ν , we can write $\mu(x_1, x_2) = \zeta(x_2)\tilde{\mu}_{x_2}(x_1)$ and $\nu(y_1, y_2) = \zeta(y_2)\tilde{\nu}_{y_2}(y_1)$, where $\zeta := (\pi_2)_{\#} \mu =$

$(\pi_2)_\# \nu$ is the common vertical marginal of μ and ν . Since γ only transports from points x to y with $x_2 = y_2$, we have $\gamma(x_1, x_2, y_1, y_2) = \zeta(x_2) \delta_{x_2}(y_2) \tilde{\gamma}_{x_2}(x_1, y_1)$, where the marginal transport plan $\tilde{\gamma}_{x_2}$ is an element of $\Gamma_1(\tilde{\mu}_{x_2}, \tilde{\nu}_{x_2})$ for ζ -a.e. x_2 . We can then write

$$\begin{aligned} \iint_{\Omega \times \Omega} |x - y| d\gamma(x, y) &= \int_{\mathbb{R}} \int_{\mathbb{R}^2} |x_1 - y_1| d\tilde{\gamma}_{x_2}(x_1, y_1) d\zeta(x_2) \\ &\geq \int_{\mathbb{R}} d_1(\tilde{\mu}_{x_2}, \tilde{\nu}_{x_2}) d\zeta(x_2) \\ &\geq \int_{\mathbb{R}} \int_{\mathbb{R}} \phi(x_1, x_2) (d\tilde{\mu}_{x_2} - d\tilde{\nu}_{x_2}) d\zeta(x_2) \end{aligned}$$

for any ϕ 1-Lipschitz in x_1 , from which follows the assertion.

For Part 7, we first show the inequality $\mathcal{D} \leq \mathcal{D}$. Since we assume $\mathcal{D}(\mu, [0, T]) < \infty$, we may choose an admissible ϕ . For any $\xi \in C_c^\infty(\mathbb{R}^2)$ with $|\partial_{x_1} \xi| \leq 1$, write

$$f(t) := \int_{\mathbb{R}^2} \xi(x) d\mu(t) \quad \text{and} \quad g(t) := \int_{\mathbb{R}^2} \phi(x, t) \partial_{x_1} \xi(x) d\mu(t),$$

and note that $g \in L^1(0, T)$ since $\mathcal{D}(\mu, [0, T]) < \infty$. From the continuity equation it follows that $\partial_t f = g$ in the sense of distributions on $(0, T)$, which implies that f is absolutely continuous. From the fundamental theorem of calculus and the bound $|\partial_{x_1} \xi| \leq 1$ we then deduce

$$\int_{\Omega} \xi(x) (d\mu(\sigma) - d\mu(s)) = \int_s^\sigma \int_{\Omega} \phi(t, x) \partial_{x_1} \xi(x) d\mu(t) dt \leq \int_s^\sigma \int_{\Omega} |\phi(t, x)| d\mu(t) dt.$$

In combination with the dual characterization (part 5) this implies the inequality $\mathcal{D} \leq \mathcal{D}$.

To show the opposite inequality $\mathcal{D} \leq \mathcal{D}$, by the same argument as above we have, for any partition $\{t_i\}$ of $[0, T]$, and $\xi_i \in C_c^\infty(\mathbb{R}^2)$ with $|\partial_{x_1} \xi_i| \leq 1$,

$$\sum_i \int_{t_i}^{t_{i+1}} \int_{\Omega} \phi(t, x) \partial_{x_1} \xi_i(x) d\mu(t) dt = \sum_i \int_{\Omega} \xi_i(x) (d\mu(t_{i+1}) - d\mu(t_i)) \leq \mathcal{D}(\mu, [0, T]). \quad (4.6)$$

By approximation it follows that

$$\int_0^T \int_{\mathbb{R}^2} \phi(t, x) \partial_{x_1} \zeta(x) d\mu(t) dt \leq \mathcal{D}(\mu, [0, T]) \quad \text{for all } \zeta \in C_c^\infty([0, T] \times \mathbb{R}^2), |\partial_{x_1} \zeta| \leq 1,$$

and by defining $\partial_{x_1} \zeta =: \psi$, up to a modification outside of Ω to obtain compact support, we find

$$\int_0^T \int_{\mathbb{R}^2} \phi(t, x) \psi(t, x) d\mu(t) dt \leq \mathcal{D}(\mu, [0, T]) \quad \text{for all } \psi \in C_c^\infty([0, T] \times \mathbb{R}^2), |\psi| \leq 1.$$

The inequality $\mathcal{D} \leq \mathcal{D}$ then follows from the dual characterization of the total variation. \square

4.2. Loading term. Let $[0, T]$ be the interval of time in which we observe our process. The evolution of the system is driven by a forcing term of the form

$$\int_{\Omega} f(t, x) d\mu(x),$$

where $f \in W^{1,1}(0, T; C(\Omega))$, which leads to the time-dependent total energy $\tilde{\mathcal{F}}_n(\mu, t)$ defined as

$$\tilde{\mathcal{F}}_n(\mu, t) := \mathcal{F}_n(\mu) - \int_{\Omega} f(t, x) d\mu(x).$$

The function f in the force term can be interpreted as the potential of a force, in the sense that $\partial_{x_1} f$ can be seen as a shear stress imposed on the system in the horizontal direction. This is clear in the case $f(t, x) = \sigma(t)x_1$, where σ is a time dependent but spatially uniform shear stress acting on the body.

4.3. Minimisation problem for the semi-discrete energy. In order to prove existence of solutions for the continuous-time quasi-static problem (Definition 1.2) we follow the common strategy of time discretization, and we first study the minimisation problem that defines a single time step. Given the current value μ^0 of the dislocation density and the updated value of the force potential f (for this discussion the time t is fixed), the updated value of the dislocation density is obtained by solving the minimisation problem

$$\min_{\mu \in X_n} \left\{ \mathcal{F}_n(\mu) + d(\mu, \mu^0) - \int_{\Omega} f d\mu \right\}. \quad (4.7)$$

Lemma 4.3 (Continuity of \mathcal{F}_n and existence of solutions of the incremental problem). *Let $n \in \mathbb{N}$ and let $\mu^k, \mu \in X_n$ be such that $\mu^k \rightharpoonup \mu$ narrowly. Then*

$$\lim_{k \rightarrow \infty} \mathcal{F}_n(\mu^k) = \mathcal{F}_n(\mu).$$

For $\mu^0 \in X_n$ and $f \in C(\Omega)$ the minimisation problem (4.7) has a solution.

Proof. Since $\mu^k \rightharpoonup \mu$, we can choose a numbering of the n dislocation points z_i^k of μ^k and z_i of μ such that $z_i^k \rightarrow z_i$. By Lemma 2.1, the extended \tilde{u}_{μ^k} are uniformly bounded in $H^1(\Omega; \mathbb{R}^2)$, and a subsequence (not relabeled) converges weakly to some $u \in H^1(\Omega; \mathbb{R}^2)$, as $k \rightarrow \infty$. The trace of \tilde{u}_{μ^k} converges strongly in $L^2(\partial\Omega; \mathbb{R}^2)$, as does the trace of \tilde{u}_{μ^k} on the moving boundaries $\partial B_{\varepsilon_n}(z_i^k)$ (indeed, this follows from the convergence $z_i^k \rightarrow z_i$, and the boundedness in $H^1(B_{\varepsilon_n}(0); \mathbb{R}^2)$ of translates $x \mapsto \tilde{u}_{\mu^k}(x - z_i^k)$). From the inequalities

$$I_{n,\mu}(u) \leq \liminf_{k \rightarrow \infty} I_{n,\mu^k}(u_{\mu^k}) \leq \liminf_{k \rightarrow \infty} I_{n,\mu^k}(v) = I_{n,\mu}(v) \quad \text{for all } v \in H^1(\Omega; \mathbb{R}^2),$$

it follows that $u = u_{\mu}$ on $\Omega_n(\mu)$ and that the whole sequence converges.

To prove the continuity of \mathcal{F}_n , consider the decomposition in (2.13) for $\mathcal{F}_n(\mu^k)$. The first term converges as $k \rightarrow \infty$ since $K^n(\cdot; z_i^k)$ is uniformly bounded on $\Omega_n(\mu^k)$. For the second term in (2.13) we use the strong convergence of u_{μ^k} on the boundaries $\partial\Omega(\mu^k)$.

Finally, to prove the existence of a minimizer in (4.7), let $(\mu^k) \subset X_n$ be a minimising sequence, and assume without loss of generality that $\mu^k \rightharpoonup \mu$ narrowly and $u_{\mu^k} \rightharpoonup u_{\mu}$ in $H^1(\Omega; \mathbb{R}^2)$. Moreover, because of the separation condition in X_n , $\mu \in X_n$. The forcing term clearly converges; by the lower semicontinuity of d (Lemma 4.1) and the continuity of \mathcal{F}_n ,

$$\liminf_{k \rightarrow \infty} \left\{ \mathcal{F}_n(\mu^k) + d(\mu^k, \mu^0) - \int_{\Omega} f d\mu^k \right\} \geq \mathcal{F}_n(\mu) + d(\mu, \mu^0) - \int_{\Omega} f d\mu,$$

which proves the minimality of the limit measure μ and concludes the proof of the lemma. \square

4.4. Quasi-static evolution. In this section we prove the existence of a quasi-static evolution in the sense of Definition 1.2 for a time-dependent force potential $f \in W^{1,1}(0, T; C(\Omega))$.

Theorem 4.4. *Let $n \in \mathbb{N}$ and let $\mu_0 \in X_n$ satisfy the stability condition*

$$\mathcal{F}_n(\mu_0) - \int_{\Omega} f(0) d\mu_0 \leq \mathcal{F}_n(\nu) + d(\nu, \mu_0) - \int_{\Omega} f(0) d\nu,$$

for every $\nu \in X_n$. Then there exists a quasi-static evolution $t \mapsto \mu(t)$ from $[0, T]$ into $\mathcal{P}(\Omega)$ such that $\mu(0) = \mu_0$. In other words, the following two conditions hold:

(qs1) $_n$ *global stability: for every $t \in [0, T]$ we have $\mu(t) \in X_n$ and*

$$\mathcal{F}_n(\mu(t)) - \int_{\Omega} f(t) d\mu(t) \leq \mathcal{F}_n(\nu) + d(\nu, \mu(t)) - \int_{\Omega} f(t) d\nu, \quad (4.8)$$

for every $\nu \in X_n$;

(qs2) $_n$ *energy balance: the map $s \mapsto \int_{\Omega} \dot{f}(s) d\mu(s)$ is integrable on $[0, T]$ and for every $t \in [0, T]$*

$$\mathcal{F}_n(\mu(t)) + \mathcal{D}(\mu, [0, t]) - \int_{\Omega} f(t) d\mu(t) = \mathcal{F}_n(\mu(0)) - \int_{\Omega} f(0) d\mu(0) - \int_0^t \int_{\Omega} \dot{f}(s) d\mu(s) ds. \quad (4.9)$$

Remark 4.5. Note that if $t \mapsto \mu(t)$ is a quasi-static evolution, then

$$(\pi_2)_\# \mu(t) = (\pi_2)_\# \mu(0) \quad \text{for every } t \in [0, T]. \quad (4.10)$$

Indeed, owing to the regularity assumptions on f , condition (qs2) $_n$ implies that $\mathcal{D}(\mu, [0, T]) < +\infty$, which in turn gives (4.10) by the definition of the dissipation distance d .

Proof of Theorem 4.4. The theorem is a direct application of [MM05, Theorem 4.5]. We sketch the proof for the reader's convenience.

For every $k \in \mathbb{N}$ let $(t_k^i)_{i=0, \dots, k}$ be a partition of the interval $[0, T]$ such that

$$0 = t_k^0 < t_k^1 < \dots < t_k^k = T \quad \text{and} \quad \max_{i=1, \dots, k} |t_k^i - t_k^{i-1}| \rightarrow 0, \quad \text{as } k \rightarrow \infty.$$

For every $k \in \mathbb{N}$ we set $\mu_k^0 := \mu_0$ and for $i \geq 1$ we define μ_k^i as a solution of the following minimisation problem (well-defined by Lemma 4.3):

$$\min_{\nu \in X_n} \left\{ \mathcal{F}_n(\nu) + d(\nu, \mu_k^{i-1}) - \int_{\Omega} f(t_k^i) d\nu \right\}.$$

Since μ_k^{i-1} is a competitor for this problem with finite energy, the minimum is finite. This implies that $d(\mu_k^i, \mu_k^{i-1}) < +\infty$, so that $(\pi_2)_\# \mu_k^i = (\pi_2)_\# \mu_k^{i-1}$ for every i and k . In other words, $(\pi_2)_\# \mu_k^i = (\pi_2)_\# \mu_0$ for every i and k . By the minimality of μ_k^i and the triangle inequality we infer that

$$\mathcal{F}_n(\mu_k^i) - \int_{\Omega} f(t_k^i) d\mu_k^i \leq \mathcal{F}_n(\nu) + d(\nu, \mu_k^i) - \int_{\Omega} f(t_k^i) d\nu \quad (4.11)$$

for every $\nu \in X_n$. Moreover, using again the minimality, the following discrete energy inequality can be proved: for all $1 \leq j \leq k$

$$\mathcal{F}_n(\mu_k^j) + \sum_{i=1}^j d(\mu_k^i, \mu_k^{i-1}) \leq \mathcal{F}_n(\mu_0) + \sum_{i=1}^j \int_{\Omega} f(t_k^i) d(\mu_k^i - \mu_k^{i-1}). \quad (4.12)$$

Now we let $\mu_k(t)$ be the piecewise constant right-continuous interpolation of $(\mu_k^i)_{i=1, \dots, k}$. In particular, we have $(\pi_2)_\# \mu_k(t) = (\pi_2)_\# \mu_0$ for every $t \in [0, T]$ and every k .

By the definition (1.6) it is clear that the functional \mathcal{F}_n is bounded from below by a fixed constant (since F_n is nonnegative, while the self-energy is bounded by $C(n\varepsilon_n)^{-1}$). Using this fact and (4.12) we deduce that $\mathcal{D}(\mu_k, [0, T]) \leq C$ for every k . Therefore, by Theorem 7.2 in the Appendix (or by applying the standard Helly's Theorem directly to the sequences (in k) of maps $t \mapsto z_j^k(t)$, $j = 1, \dots, n$, where $\mu_k(t) = \frac{1}{n} \sum_{j=1}^n \delta_{z_j^k(t)}$) there exist a subsequence (not relabelled) and a function $t \mapsto \mu(t)$ from $[0, T]$ into $\mathcal{P}(\Omega)$, with $\mathcal{D}(\mu, [0, T]) < +\infty$, such that $\mu_k(t) \rightharpoonup \mu(t)$ narrowly, as $k \rightarrow +\infty$, for every $t \in [0, T]$. It is also easy to check that $\mu(t) \in X_n$ for every $t \in [0, T]$ and $(\pi_2)_\# \mu(t) = (\pi_2)_\# \mu_0$ for every $t \in [0, T]$.

Condition (qs1) $_n$ follows now by passing to the limit in (4.11). This is possible since \mathcal{F}_n and the force term are continuous with respect to narrow convergence, and convergence in d is equivalent to convergence of the dislocation points for sequences of measures in X_n .

To prove condition (qs2) $_n$, we can pass to the limit in (4.12) and deduce an energy inequality. The converse inequality can be proved using the global stability (qs1) $_n$ and an approximation of $t \mapsto \mu(t)$ in terms of piecewise constant maps. This is possible since the condition $\mathcal{D}(\mu, [0, T]) < +\infty$ is equivalent to saying that the maps $t \mapsto z_j(t)$ are BV functions from $[0, T]$ into \mathbb{R}^2 for every $j = 1, \dots, n$, where $\mu(t) = \frac{1}{n} \sum_{j=1}^n \delta_{z_j(t)}$. Therefore, in particular, $t \mapsto z_j(t)$ is continuous on $[0, T]$, except possibly on a countable set of times. \square

5. CONVERGENCE OF THE EVOLUTIONS

We now prove the convergence of the evolutions associated to \mathcal{F}_n to an evolution associated to the Γ -limit functional \mathcal{F} , i.e., Theorem 1.3 in the introduction (as Theorem 5.8 below). We will consider evolutions for \mathcal{F}_n with initial values μ_n^0 belonging to a subset $Y_n(\gamma, c)$ of the admissible class X_n , defined as follows.

Definition 5.1. Let $n \in \mathbb{N}$ and let $\mu = \frac{1}{n} \sum_{i=1}^n \delta_{z_i}$, for some $z_i \in \Omega$. Let S_μ be the set of slip planes of μ , and for $s \in S_\mu$ let $m_{\mu,s}$ be the number of dislocations on the slip plane s , i.e.,

$$S_\mu := \text{supp}[(\pi_2)_\# \mu], \quad m_{\mu,s} := \#\{j : \pi_2(z_j) = s\}. \quad (5.1)$$

For $-\frac{1}{2} < \gamma \leq \frac{1}{2}$ and $c > 0$ we introduce the following class of measures:

$$Y_n(\gamma, c) := \left\{ \mu \in X_n : \min_{s, s' \in S_\mu, s \neq s'} |s - s'| \geq cn^{-\frac{1}{2} + \gamma}, \max_{s \in S_\mu} m_{\mu,s} \leq \frac{1}{c} n^{\frac{1}{2} + \gamma} \right\}. \quad (5.2)$$

In words, measures in $Y_n(\gamma, c)$ have slip-plane spacing larger than $cn^{-\frac{1}{2} + \gamma}$ and no more than $c^{-1}n^{\frac{1}{2} + \gamma}$ dislocations per slip plane. Note that the two properties required in the definition of the class $Y_n(\gamma, c)$ depend only on the vertical marginal of a measure; therefore if $\mu \in Y_n(\gamma, c)$ for some γ and c and $\nu \in X_n$ is such that $(\pi_2)_\# \nu = (\pi_2)_\# \mu$, then $\nu \in Y_n(\gamma, c)$.

Although the class $Y_n(\gamma, c)$ imposes restrictions on the structure of admissible measures, we can nevertheless recover a large class of measures in the limit, as shown in the following lemma.

Lemma 5.2. Assume (1.1) and let $-\frac{1}{2} < \gamma \leq \frac{1}{2}$ and $c > 0$. Let $\mathcal{P}_{\gamma,c}^\infty(\Omega)$ be the class of measures $\mu \in \mathcal{P}(\Omega)$ such that there exists a sequence $\mu_n \rightarrow \mu$ narrowly, as $n \rightarrow \infty$, with $\mu_n \in Y_n(\gamma, c)$ for every n .

- If $-\frac{1}{2} < \gamma < \frac{1}{2}$, then

$$\mathcal{P}_{\gamma,c}^\infty(\Omega) = \left\{ \mu \in \mathcal{P}(\Omega) : \text{supp } \mu \subset \mathcal{R}, (\pi_2)_\# \mu \leq \frac{1}{c^2} \mathcal{L}^1 \right\}. \quad (5.3)$$

- If $\gamma = \frac{1}{2}$, then $\mathcal{P}_{\gamma,c}^\infty(\Omega)$ is the set of all measures $\mu \in \mathcal{P}(\Omega)$ such that $\text{supp } \mu \subset \mathcal{R}$, $S_\mu := \text{supp}[(\pi_2)_\# \mu]$ is finite,

$$\min \{|s - s'| : s, s' \in S_\mu, s \neq s'\} \geq c, \quad (5.4)$$

and

$$\mu(\Omega \cap (\mathbb{R} \times \{s\})) \leq \frac{1}{c} \quad \text{for every } s \in S_\mu. \quad (5.5)$$

Remark 5.3. We note that for $-\frac{1}{2} < \gamma < \frac{1}{2}$ the class $\mathcal{P}_{\gamma,c}^\infty(\Omega)$ is nonempty if and only if $c^2 \leq \mathcal{L}^1(\pi_2(\Omega))$. This follows from the second condition in (5.3) and the fact that μ is a probability measure. Analogously, for $\gamma = \frac{1}{2}$ the class $\mathcal{P}_{\gamma,c}^\infty(\Omega)$ is nonempty if and only if $c^2 \leq \max\{c, \mathcal{L}^1(\pi_2(\Omega))\}$. This follows from (5.5) and the fact that μ is a probability measure, taking into account that $\#S_\mu \leq \max\{1, \mathcal{L}^1(\pi_2(\Omega))/c\}$ by (5.4).

Proof of Lemma 5.2. In what follows all the measures that appear are extended to zero outside Ω . Let $\mu \in \mathcal{P}_{\gamma,c}^\infty(\Omega)$ and let $\mu_n \rightarrow \mu$ narrowly, as $n \rightarrow \infty$, with $\mu_n \in Y_n(\gamma, c)$ for every n . Clearly $\text{supp } \mu \subset \mathcal{R}$. Let I_r be an open interval of length $r > 0$.

If $-\frac{1}{2} < \gamma < \frac{1}{2}$, then the maximum number of dislocations of μ_n contained in the horizontal strip $\mathbb{R} \times I_r$ is rn/c^2 . Therefore,

$$(\pi_2)_\# \mu(I_r) = \mu(\mathbb{R} \times I_r) \leq \liminf_{n \rightarrow \infty} \mu_n(\mathbb{R} \times I_r) \leq \frac{1}{c^2} \mathcal{L}^1(I_r).$$

This inequality implies that $(\pi_2)_\# \mu \leq \frac{1}{c^2} \mathcal{L}^1$ on every open set of \mathbb{R} and thus, by approximation on every measurable set.

If $\gamma = \frac{1}{2}$, then the maximum number of dislocations of μ_n in $\mathbb{R} \times I_r$ is $\max\{n/c, rn/c^2\}$. This implies (5.5), since $S_\mu := \text{supp}[(\pi_2)_\# \mu]$ clearly has finite cardinality and satisfies (5.4).

Let now $-\frac{1}{2} < \gamma < \frac{1}{2}$ and let $\mu \in \mathcal{P}(\Omega)$ be such that $\text{supp } \mu \subset \mathcal{R}$ and $(\pi_2)_\# \mu \leq \frac{1}{c^2} \mathcal{L}^1$. In particular, for every interval I_r of length $r > 0$ we have

$$\mu(\mathbb{R} \times I_r) \leq \frac{r}{c^2}. \quad (5.6)$$

As a first step we assume that $\text{supp } \mu \subset \text{int } \mathcal{R}$, where $\text{int } \mathcal{R}$ denotes the interior of \mathcal{R} .

For every $n \in \mathbb{N}$ we consider a covering $\{Q_i^n\}$ of Ω made of a grid of squares of side h_n with $h_n \rightarrow 0$ chosen such that

$$n^{\frac{1}{2} - \gamma} h_n \rightarrow \infty, \quad n^{\frac{1}{2} + \gamma} h_n \rightarrow \infty, \quad \text{as } n \rightarrow \infty. \quad (5.7)$$

We construct the approximating measures μ_n as follows. Let us consider the straight lines $\ell_j^n := \mathbb{R} \times \{cjn^{-\frac{1}{2}+\gamma}\}$, $j \in \mathbb{Z}$. For every i and j such that $Q_i^n \cap \ell_j^n \neq \emptyset$ we allocate m_i^n equidistant dislocations on $Q_i^n \cap \ell_j^n$ at a distance r_n from $\partial Q_i^n \cap \ell_j^n$, where

$$m_i^n := \left\lfloor \frac{\mu(Q_i^n)}{h_n} cn^{\frac{1}{2}+\gamma} \right\rfloor.$$

Note that $h_n/r_n \rightarrow +\infty$, as $n \rightarrow \infty$, by (1.1) and (5.7). Moreover, by (5.6) the distance between two dislocations in Q_i^n and on the same line is larger than

$$\frac{h_n - 2r_n}{m_i^n} \geq \frac{1}{2c} \frac{h_n^2}{\mu(Q_i^n)n^{\frac{1}{2}+\gamma}} \geq \frac{c}{2} \frac{h_n}{n^{\frac{1}{2}+\gamma}} \gg r_n,$$

where the last inequality is satisfied in view of (1.1) and (5.7).

Let I_j^n be the set of indices i for which Q_i^n intersects the line ℓ_j^n . Then the total number of dislocations on ℓ_j^n can be estimated by

$$\sum_{i \in I_j^n} m_i^n \leq \sum_{i \in I_j^n} \frac{\mu(Q_i^n)}{h_n} cn^{\frac{1}{2}+\gamma} \leq \frac{\mu(S_j^n)}{h_n} cn^{\frac{1}{2}+\gamma} \leq \frac{1}{c} n^{\frac{1}{2}+\gamma},$$

where $S_j^n := \cup_{i \in I_j^n} Q_i^n$ is a strip of height h_n and the last inequality follows from (5.6). Since $\text{supp } \mu_n \subset \mathcal{R}$ for n large enough, we conclude that $\mu_n \in Y_n(\gamma, c)$ (if $\mu_n(\Omega) < 1$, we can add the dislocations needed in order for μ_n to be a probability measure in the activated slip planes, at distance larger than r_n from other dislocations).

We now check that $\mu_n \rightarrow \mu$ narrowly. Note that the number of lines ℓ_j^n intersecting a square Q_i^n is $\lfloor h_n n^{\frac{1}{2}-\gamma}/c \rfloor$. Therefore, the total number of dislocations on each square Q_i^n is

$$p_i^n := \left\lfloor \frac{h_n n^{\frac{1}{2}-\gamma}}{c} \right\rfloor m_i^n.$$

Denoting by $\{z_j^{i,n}\}_{j=1, \dots, p_i^n}$ the dislocations of the measures μ_n in Q_i^n , we may write

$$\mu_n = \frac{1}{n} \sum_i \sum_{j=1}^{p_i^n} \delta_{z_j^{i,n}}.$$

Let now $\phi \in C(\Omega)$. We have

$$\begin{aligned} \int_{\Omega} \phi d\mu_n &= \frac{1}{n} \sum_i \sum_{j=1}^{p_i^n} \phi(z_j^{i,n}) \\ &= \frac{1}{n} \sum_i p_i^n \phi(z_1^{i,n}) + \frac{1}{n} \sum_i \sum_{j=1}^{p_i^n} (\phi(z_j^{i,n}) - \phi(z_1^{i,n})), \end{aligned}$$

where the last term tends to 0, as $n \rightarrow \infty$, by the uniform continuity of ϕ and the fact that $|z_j^{i,n} - z_1^{i,n}| \leq \sqrt{2}h_n \rightarrow 0$. Moreover,

$$\frac{1}{n} \sum_i p_i^n \phi(z_1^{i,n}) = \frac{1}{n} \sum_i (p_i^n - n\mu(Q_i^n))\phi(z_1^{i,n}) + \sum_i \int_{Q_i^n} (\phi(z_1^{i,n}) - \phi) d\mu + \int_{\Omega} \phi d\mu,$$

where again by the absolute continuity of ϕ the second term on the right-hand side tends to 0, as $n \rightarrow \infty$. To conclude, it is enough to observe that

$$\begin{aligned} \frac{1}{n} \sum_i \left| (p_i^n - n\mu(Q_i^n))\phi(z_1^{i,n}) \right| &\leq \frac{1}{n} \|\phi\|_{\infty} \sum_i \left(\frac{h_n n^{\frac{1}{2}-\gamma}}{c} + \frac{\mu(Q_i^n)}{h_n} cn^{\frac{1}{2}+\gamma} - 1 \right) \\ &\leq C \left(\frac{1}{n^{\frac{1}{2}-\gamma} h_n} + \frac{1}{n^{\frac{1}{2}+\gamma} h_n} + \frac{1}{nh_n^2} \right) \end{aligned}$$

which goes to 0 by (5.7). In the last inequality we have used the fact that the number of squares Q_i^n covering Ω is of order $1/h_n^2$.

In the general case, one can proceed as follows.

For every $k \in \mathbb{N}$ let μ^k be a push-forward of μ by a $1/k$ -retraction of its support; clearly $\text{supp } \mu^k \subset \text{int } \mathcal{R}$. Then, we can construct an approximation (μ_n^k) of μ^k as above. In particular, for every $k \in \mathbb{N}$ there exists $N^k \in \mathbb{N}$ such that $d(\mu_n^k, \mu^k) < 1/k$ for every $n \geq N^k$ and we can assume that N_k is strictly increasing with respect to k . The required approximation μ_n of μ is finally given by $\mu_n := \mu_n^k$ for $N^k \leq n < N^{k+1}$.

Assume now $\gamma = \frac{1}{2}$ and let $\mu \in \mathcal{P}(\Omega)$ be such that $\text{supp } \mu \subset \mathcal{R}$, $S_\mu := \text{supp } [(\pi_2)_\# \mu]$ has finite cardinality, and (5.4)–(5.5) hold. In this case, to construct the approximating sequence, we locate the dislocations on the slip planes of μ (which by assumption are finite in number and spaced at least c from each other) by following a one-dimensional procedure on every slip plane. More precisely, for every $s \in S_\mu$ we subdivide the straight line $\mathbb{R} \times \{s\}$ into a family of segments $\{I_i^{s,n}\}$ of length h_n , where $h_n \rightarrow 0$ is chosen so that $nh_n \rightarrow \infty$ and $h_n/(nr_n) \rightarrow \infty$. In every segment $I_i^{s,n}$ we allocate $m_i^{s,n}$ equispaced dislocations at a distance r_n from $\partial I_i^{s,n}$, where

$$m_i^{s,n} := \lfloor \mu(I_i^{s,n})n \rfloor.$$

Note that the distance between two dislocations in $I_i^{s,n}$ is larger than

$$\frac{h_n - 2r_n}{m_i^{s,n}} \geq \frac{1}{2} \frac{h_n}{n\mu(I_i^{s,n})} \geq C \frac{h_n}{n} \gg r_n,$$

where the last two inequalities follow from (5.5) and from our choice of h_n . Moreover, again by (5.5) the number of dislocations on every slip plane is less than n/c . Therefore, the sequence μ_n supported on the dislocations positions defined above is in $Y_n(\gamma, c)$ and similarly as before we can show that it converges narrowly to μ . \square

Remark 5.4. We note that the case $\gamma = -\frac{1}{2}$ is irrelevant for our analysis since the corresponding class of limit measures is contained in the class $\mathcal{P}_{\gamma,c}^\infty(\Omega)$ with $-\frac{1}{2} < \gamma < \frac{1}{2}$, as it can be easily deduced from the proof of Lemma 5.2.

Let now $f \in W^{1,1}(0, T; C(\Omega))$ and let (μ_n^0) be a sequence of initial data with $\mu_n^0 \in Y_n(\gamma, c)$ for every $n \in \mathbb{N}$. By Theorem 4.4 for every n there exists a quasi-static evolution $t \mapsto \mu_n(t)$ with initial value μ_n^0 and potential f , according to Definition 1.2. Note that $(\pi_2)_\# \mu_n(t) = (\pi_2)_\# \mu_n^0$ for every $t \in [0, T]$ and every n by Remark 4.5. Therefore, $\mu_n(t) \in Y_n(\gamma, c)$ for every $t \in [0, T]$ and every $n \in \mathbb{N}$. Now, in order to pass to the limit in $(\text{qs1})_n$ and $(\text{qs2})_n$ we need to construct a so-called joint recovery sequence for \mathcal{F}_n relative to the distance d (as defined in [MRS08]). This construction is done in the next theorem and will also be used to guarantee the existence of admissible sequences of initial data (see Remark 5.9).

Theorem 5.5. *Assume (1.1) and fix $-\frac{1}{2} < \gamma \leq \frac{1}{2}$ and $c > 0$. Let $\mu \in \mathcal{P}_{\gamma,c}^\infty(\Omega)$ be such that $\mathcal{F}(\mu) < +\infty$ and let $\mu_n \rightarrow \mu$ narrowly, as $n \rightarrow \infty$, with $\mu_n \in Y_n(\gamma, c)$ for every n . Then for every $\nu \in \mathcal{P}(\Omega)$ with $(\pi_2)_\# \mu = (\pi_2)_\# \nu$ and $\mathcal{F}(\nu) < +\infty$, there exists a sequence (ν_n) such that*

- (i) $\nu_n \in Y_n(\gamma, c)$ and $(\pi_2)_\# \mu_n = (\pi_2)_\# \nu_n$ for every n ;
- (ii) $\nu_n \rightarrow \nu$ narrowly, as $n \rightarrow \infty$;
- (iii) $\limsup_{n \rightarrow \infty} d(\mu_n, \nu_n) \leq d(\mu, \nu)$;
- (iv) $\limsup_{n \rightarrow \infty} \mathcal{F}_n(\nu_n) \leq \mathcal{F}(\nu)$.

The latter two conditions in Theorem 5.5 motivate the name *joint recovery sequence* for the sequence ν_n . The construction of (ν_n) relies on some technical steps which are summarised in the following Lemmas 5.6 and 5.7.

Lemma 5.6. *Assume (1.1). For $n \in \mathbb{N}$ let $\{z_i^n : i = 1, \dots, n\}$ be a family of points in Ω with $\{z_i^n\}_i \subset \mathcal{R}$, and let $\mu_n := \frac{1}{n} \sum_{i=1}^n \delta_{z_i^n}$ denote the corresponding probability measure. Assume that $\mu_n \rightarrow \mu$ narrowly, as $n \rightarrow \infty$. Then for every $\nu \in \mathcal{P}(\Omega)$ with $\text{supp } \nu \subset \mathcal{R}$ and $(\pi_2)_\# \mu = (\pi_2)_\# \nu$,*

there exists a sequence $(\hat{\nu}_n) \subset \mathcal{P}(\Omega)$, $\hat{\nu}_n = \frac{1}{n} \sum_{i=1}^n \delta_{z_i^n}$, such that $\text{supp } \hat{\nu}_n \subset \mathcal{R}$, $(\pi_2)_\# \hat{\nu}_n = (\pi_2)_\# \mu_n$ for every n , $\hat{\nu}_n \rightarrow \nu$ narrowly, as $n \rightarrow \infty$, and

$$\limsup_{n \rightarrow \infty} d(\mu_n, \hat{\nu}_n) \leq d(\mu, \nu).$$

Proof. Let $\nu \in \mathcal{P}(\Omega)$ be such that $\text{supp } \nu \subset \mathcal{R}$ and $(\pi_2)_\# \mu = (\pi_2)_\# \nu$. Fix $\delta > 0$. For every $m \in \mathbb{Z}$ we set $S_m := \Omega \cap (\mathbb{R} \times [m\delta, (m+1)\delta])$ and we define $\mu^m := \mu \llcorner S_m$ and $\nu^m := \nu \llcorner S_m$. From the assumption $(\pi_2)_\# \mu = (\pi_2)_\# \nu$ it follows immediately that $(\pi_2)_\# \mu^m = (\pi_2)_\# \nu^m$ and $\mu^m(\Omega) = \nu^m(\Omega)$. Moreover, from the definition of the distance d we have

$$d(\mu, \nu) = \sum_{m \in \mathbb{Z}} d(\mu^m, \nu^m). \quad (5.8)$$

We can construct a family $\{I_n^m\}_{m \in \mathbb{Z}}$ of disjoint subsets of $\{1, \dots, n\}$ with the following properties:

$$\mu_n^m := \frac{1}{n} \sum_{i \in I_n^m} \delta_{z_n^i} \rightarrow \mu^m \quad \text{as } n \rightarrow \infty, \quad \text{for every } m \in \mathbb{Z}, \quad (5.9)$$

$$\text{dist}(z_n^i, S_m) \leq \delta \quad \text{for every } i \in I_n^m, m \in \mathbb{Z}, n \in \mathbb{N}, \quad (5.10)$$

$$\frac{1}{n} \#(I_n^*) \rightarrow 0, \quad (5.11)$$

where $I_n^* := \{1, \dots, n\} \setminus \cup_{m \in \mathbb{Z}} I_n^m$.

For simplicity of notation we denote the Cartesian coordinates of z_n^i by (ξ_n^i, ζ_n^i) . For every $n \in \mathbb{N}$, $m \in \mathbb{Z}$, and $i \in I_n^m$ let $\tilde{\xi}_n^{m,i} \in \mathbb{R}$ be chosen so that

$$\lambda_n^m := \frac{1}{n} \sum_{i \in I_n^m} \delta_{\tilde{\xi}_n^{m,i}} \rightarrow (\pi_1)_\# \nu^m, \quad (5.12)$$

as $n \rightarrow \infty$. Let T_n^m be the non-decreasing map from $\{\xi_n^i\}_{i \in I_n^m}$ to $\{\tilde{\xi}_n^{m,i}\}_{i \in I_n^m}$. We define

$$\hat{\nu}_n^m := \frac{1}{n} \sum_{i \in I_n^m} \delta_{(T_n^m(\xi_n^i), \zeta_n^i)} \quad \text{for } m \in \mathbb{Z}.$$

Up to modifying slightly the choice of $\tilde{\xi}_n^{m,i}$, if needed, we can assume that $\text{supp } \hat{\nu}_n^m \subset \mathcal{R}$. We note that (5.12) and the definition of $\hat{\nu}_n^m$ imply

$$(\pi_1)_\# \hat{\nu}_n^m = \lambda_n^m \rightarrow (\pi_1)_\# \nu^m, \quad (5.13)$$

as $n \rightarrow \infty$. Clearly $(\pi_2)_\# \mu_n^m = (\pi_2)_\# \hat{\nu}_n^m$. Moreover, using the fact that T_n^m is non-decreasing, we have

$$d(\mu_n^m, \hat{\nu}_n^m) = \frac{1}{n} \sum_{i \in I_n^m} |\xi_n^i - T_n^m(\xi_n^i)| = d_1((\pi_1)_\# \mu_n^m, (\pi_1)_\# \hat{\nu}_n^m),$$

so that by (5.9) and (5.13) we obtain

$$\lim_{n \rightarrow \infty} d(\mu_n^m, \hat{\nu}_n^m) = d_1((\pi_1)_\# \mu^m, (\pi_1)_\# \nu^m) \quad (5.14)$$

for every $m \in \mathbb{Z}$.

Now we define

$$\hat{\nu}_n := \sum_{m \in \mathbb{Z}} \hat{\nu}_n^m + \frac{1}{n} \sum_{i \in I_n^*} \delta_{z_n^i}.$$

Clearly $\text{supp } \hat{\nu}_n \subset \mathcal{R}$ since both $\hat{\nu}_n^m$ and μ_n satisfy the same condition. We will now prove that the sequence $(\hat{\nu}_n)$ satisfies

- (1) $(\pi_2)_\# \hat{\nu}_n = (\pi_2)_\# \mu_n$ for every n ;
- (2) $\limsup_{n \rightarrow \infty} d_1(\hat{\nu}_n, \nu) \leq 3\delta$;
- (3) $\limsup_{n \rightarrow \infty} d(\mu_n, \hat{\nu}_n) \leq d(\mu, \nu)$.

Property (1) holds by construction. Moreover, by Part 2 of Lemma 4.1, (5.8), and (5.14), we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} d(\mu_n, \hat{\nu}_n) &\leq \sum_{m \in \mathbb{Z}} \limsup_{n \rightarrow \infty} d(\mu_n^m, \hat{\nu}_n^m) = \sum_{m \in \mathbb{Z}} d_1((\pi_1)_\# \mu^m, (\pi_1)_\# \nu^m) \\ &\leq \sum_{m \in \mathbb{Z}} d(\mu^m, \nu^m) = d(\mu, \nu), \end{aligned}$$

which proves condition (3).

To show (2), we introduce the auxiliary measures

$$\bar{\nu}_n^m := \begin{cases} \frac{\hat{\nu}_n^m(\Omega)}{\nu^m(\Omega)} \nu^m & \text{if } \nu^m(\Omega) > 0 \\ \hat{\nu}_n^m & \text{if } \nu^m(\Omega) = 0 \end{cases}$$

for every $n \in \mathbb{N}$, $m \in \mathbb{Z}$, and

$$\bar{\nu}_n := \sum_{m \in \mathbb{Z}} \bar{\nu}_n^m + \frac{1}{n} \sum_{i \in I_n^*} \delta_{z_i}$$

for every $n \in \mathbb{N}$. Since $\hat{\nu}_n^m(\Omega) = \mu_n^m(\Omega) \rightarrow \mu^m(\Omega)$, as $n \rightarrow \infty$, and (5.11) holds, it is immediate to see that $\bar{\nu}_n \rightarrow \nu$, as $n \rightarrow \infty$, and thus, $d_1(\bar{\nu}_n, \nu) \rightarrow 0$, as $n \rightarrow \infty$. In particular, we have

$$(\pi_1)_\# \bar{\nu}_n^m \rightarrow (\pi_1)_\# \nu^m, \quad (5.15)$$

as $n \rightarrow \infty$.

Since $\lim_{n \rightarrow \infty} d_1(\bar{\nu}_n, \nu) = 0$, to prove condition (2) it is enough to show that

$$\limsup_{n \rightarrow \infty} d_1(\hat{\nu}_n, \bar{\nu}_n) \leq 3\delta. \quad (5.16)$$

By (5.10) the support of $\hat{\nu}_n^m$ and $\bar{\nu}_n^m$ is contained in $\mathbb{R} \times [(m-1)\delta, (m+2)\delta]$ for every $n \in \mathbb{N}$ and $m \in \mathbb{Z}$. Therefore, for $m \in \mathbb{Z}$ such that $\nu^m(\Omega) > 0$, we have, by (4.1) and (4.2):

$$\begin{aligned} d_1(\hat{\nu}_n^m, \bar{\nu}_n^m) &\leq \inf \left\{ \iint_{\Omega \times \Omega} (|x_1 - y_1| + 3\delta) d\gamma(x, y) : \gamma \in \Gamma_1(\hat{\nu}_n^m, \bar{\nu}_n^m) \right\} \\ &= \inf \left\{ \iint_{\pi_1(\Omega) \times \pi_1(\Omega)} |x_1 - y_1| d\gamma(x_1, y_1) : \gamma \in \Gamma((\pi_1)_\# \hat{\nu}_n^m, (\pi_1)_\# \bar{\nu}_n^m) \right\} + 3\delta \\ &= d_1((\pi_1)_\# \hat{\nu}_n^m, (\pi_1)_\# \bar{\nu}_n^m) + 3\delta \hat{\nu}_n^m(\Omega). \end{aligned}$$

Combining this inequality with (5.13) and (5.15), we deduce that

$$\limsup_{n \rightarrow \infty} d_1(\hat{\nu}_n^m, \bar{\nu}_n^m) \leq 3\delta \hat{\nu}_n^m(\Omega)$$

for every $m \in \mathbb{Z}$ such that $\nu^m(\Omega) > 0$. Since $\bar{\nu}_n^m = \hat{\nu}_n^m$ for every $m \in \mathbb{Z}$ such that $\nu^m(\Omega) = 0$, we conclude that

$$\limsup_{n \rightarrow \infty} d_1(\hat{\nu}_n, \bar{\nu}_n) \leq \sum_{m \in \mathbb{Z}} \limsup_{n \rightarrow \infty} d_1(\hat{\nu}_n^m, \bar{\nu}_n^m) \leq 3\delta \sum_{m \in \mathbb{Z}} \nu^m(\Omega) = 3\delta,$$

which proves the claim (5.16), hence condition (2).

To conclude the proof of the lemma it is enough to consider a sequence $\delta_k \searrow 0$ and apply a diagonal argument. \square

Lemma 5.7. *Let $\hat{z}_1, \dots, \hat{z}_n \in \Omega$, $n \in \mathbb{N}$, be such that $\{\hat{z}_i\}_i \subset \mathcal{R}$, and let $\hat{\nu} := \frac{1}{n} \sum_{i=1}^n \delta_{\hat{z}_i}$ denote the corresponding probability measure. Then there exists $\eta_0 > 0$, depending only on \mathcal{R} , such that for every $\eta \in (0, \eta_0)$ there exists a modification ν of $\hat{\nu}$, $\nu = \frac{1}{n} \sum_{i=1}^n \delta_{z_i} \in \mathcal{P}(\Omega)$, satisfying the following properties:*

- (a) $(\pi_2)_\# \nu = (\pi_2)_\# \hat{\nu}$;
- (b) $d(\hat{\nu}, \nu) \leq \eta$;
- (c) $\text{supp } \nu \subset \mathcal{R}$;
- (d) $|\pi_1(z_i - z_j)| \geq \frac{\eta}{m}$ for every $z_i, z_j \in \text{supp } \nu$ such that $\pi_2(z_i) = \pi_2(z_j)$, $i \neq j$,

where m is the maximum number of dislocations per slip plane of $\hat{\nu}$, that is,

$$m := \max \# \{i : \pi_2(\hat{z}_i) = s, s \in \text{supp}[(\pi_2)_\# \hat{\nu}]\}.$$

Proof. Let $S := \text{supp}[(\pi_2)_\# \hat{\nu}]$ and for $s \in S$ let m_s be the number of dislocations on the s -th slip plane, as in (5.1). By definition $m = \max_{s \in S} m_s$. We can rewrite the measure $\hat{\nu}$ by splitting its vertical and horizontal marginals as

$$\hat{\nu}(x, y) = \sum_{s \in S} \frac{m_s}{n} \delta_s(y) \hat{\nu}_s(x),$$

where $\hat{\nu}_s \in \mathcal{P}(\mathbb{R})$ is the normalised dislocation density on the s -th slip plane. More explicitly, we write

$$\hat{\nu}_s = \frac{1}{m_s} \sum_{i=1}^{m_s} \delta_{\hat{x}_i^s} \quad \text{with } \{\hat{x}_i^s : i = 1 \dots, m_s\} := \{\pi_1(\hat{z}_i) : \hat{z}_i \in \text{supp } \hat{\nu}, \pi_2(\hat{z}_i) = s\},$$

and we assume without loss of generality that the points $\{\hat{x}_i^s\}$ of the support of $\hat{\nu}_s$ are ordered increasingly.

To construct the required modification ν of $\hat{\nu}$, since we want to preserve the vertical marginals, we will only modify the *horizontal* marginals $\hat{\nu}_s$, for every $s \in S$. More precisely, for every $s \in S$ we consider a grid of size η/m_s and define G_s as

$$G_s := \left\{ x \in \frac{\eta}{m_s} \mathbb{Z} : (x, s) \in \mathcal{R} \right\}.$$

Now let $\nu_s := \frac{1}{m_s} \sum_{i=1}^{m_s} \delta_{x_i^s}$ be a minimiser of the following problem:

$$\min \left\{ d_1(\hat{\nu}_s, \mu) : \mu = \frac{1}{m_s} \sum_{i=1}^{m_s} \delta_{x_i}, x_i \in G_s \text{ for every } i, x_i \neq x_j \text{ if } i \neq j \right\},$$

where the points $\{x_i^s\}$ are ordered increasingly. Note that to guarantee that the above minimum is taken over a class that is nonempty, η has to be smaller than the width of \mathcal{R} (that we denote by η_0). By construction we have that

$$|x_i^s - x_j^s| \geq \frac{\eta}{m_s} \quad \text{if } i \neq j \quad \text{and} \quad (x_i^s, s) \in \mathcal{R} \quad \text{for every } i. \quad (5.17)$$

Moreover, from the minimality it follows that

$$d_1(\hat{\nu}_s, \nu_s) \leq \eta. \quad (5.18)$$

We now define the measure $\nu \in \mathcal{P}(\Omega)$ as

$$\nu(x, y) := \sum_{s \in S} \frac{m_s}{n} \delta_s(y) \nu_s(x).$$

From (5.17) it follows immediately that ν satisfies (a), (c), and (d). As for property (b), we have

$$d(\hat{\nu}, \nu) = \sum_{s \in S} \frac{m_s}{n} d_1(\hat{\nu}_s, \nu_s) \leq \eta,$$

where the last inequality follows from (5.18). \square

We are now ready for the proof of Theorem 5.5. Roughly speaking, given $\mu \in \mathcal{P}_{\gamma, c}^\infty(\Omega)$, a competitor ν with $(\pi_2)_\# \mu = (\pi_2)_\# \nu$, and $\mu_n \in Y_n(\gamma, c)$ for every n with $\mu_n \rightarrow \mu$ narrowly, Lemmas 5.6 and 5.7 will provide a joint recovery sequence ν_n by *moving* the horizontal coordinates of the points in the support of μ_n so that the modified sequence converges to the given measure ν .

Proof of Theorem 5.5. Let $\nu \in \mathcal{P}(\Omega)$ be such that $(\pi_2)_\# \mu = (\pi_2)_\# \nu$ and $\mathcal{F}(\nu) < +\infty$. We consider the cases $\gamma \in (-\frac{1}{2}, \frac{1}{2})$ and $\gamma = \frac{1}{2}$ separately.

Let $\gamma \in (-\frac{1}{2}, \frac{1}{2})$. We apply Lemma 5.6 to the sequence (μ_n) and to the measure ν and construct a sequence $(\hat{\nu}_n)$. Then we apply Lemma 5.7 to each measure $\hat{\nu}_n$ with $\eta = n^{-\frac{1}{2} + \gamma}$. In this way we

obtain a new sequence (ν_n) such that $\text{supp } \nu_n \subset \mathcal{R}$, $(\pi_2)_\# \nu_n = (\pi_2)_\# \mu_n$ for every n , ν_n converges to ν narrowly (in view of Part 1 of Lemma 4.1 and (b) of Lemma 5.7), and

$$\limsup_{n \rightarrow \infty} d(\mu_n, \nu_n) \leq d(\mu, \nu).$$

We shall now prove that (ν_n) is the required joint recovery sequence. Taking into account that $(\pi_2)_\# \nu_n = (\pi_2)_\# \mu_n$, by property (d) of Lemma 5.7 and by our choice of η we have that for every $z_i^n, z_j^n \in \text{supp } \nu_n$, with $i \neq j$,

$$|z_i^n - z_j^n| \geq cn^{-1} \wedge cn^{-\frac{1}{2} + \gamma} \geq cn^{-1} \stackrel{(1.1)}{\gg} r_n. \quad (5.19)$$

Therefore, $\nu_n \in X_n$. Since $\mu_n \in Y_n(\gamma, c)$ and $(\pi_2)_\# \nu_n = (\pi_2)_\# \mu_n$, we conclude that $\nu_n \in Y_n(\gamma, c)$. In what follows we set $c = 1$ for notational simplicity.

To complete the proof of the theorem we show that

$$\limsup_{n \rightarrow \infty} \mathcal{F}_n(\nu_n) \leq \mathcal{F}(\nu). \quad (5.20)$$

Applying the preliminary estimates (3.13)–(3.15) in the proof of Theorem 3.3 and the decomposition (2.13), inequality (5.20) is proved if we show that

$$\limsup_{n \rightarrow \infty} \frac{1}{2} \iint_{\Omega \times \Omega} V(y, z) d(\nu_n \boxtimes \nu_n)(y, z) \leq \frac{1}{2} \iint_{\Omega \times \Omega} V(y, z) d\nu(y) d\nu(z). \quad (5.21)$$

To prove (5.21) we proceed similarly to the proof of the limsup inequality of Theorem 3.3. Let $M > 0$ be fixed, and set $V_M := V \wedge M$. By definition of ν_n we have

$$\begin{aligned} \frac{1}{2} \iint_{\Omega \times \Omega} V(y, z) d(\nu_n \boxtimes \nu_n)(y, z) &= \frac{1}{2n^2} \sum_{\substack{i, j=1 \\ i \neq j}}^n V_M(z_i^n, z_j^n) + \frac{1}{2n^2} \sum_{\substack{i, j=1 \\ i \neq j}}^n (V - V_M)(z_i^n, z_j^n) \\ &\leq \frac{1}{2n^2} \sum_{\substack{i, j=1 \\ i \neq j}}^n V_M(z_i^n, z_j^n) + \frac{1}{2n^2} \sum_{i=1}^n \sum_{\substack{j \neq i \\ |z_i^n - z_j^n| < R_M}} V(z_i^n, z_j^n), \end{aligned} \quad (5.22)$$

where $R_M \rightarrow 0$ as $M \rightarrow \infty$. Since the truncated function V_M is continuous and bounded on an open set containing $\mathcal{R} \times \mathcal{R}$, for the first term in the right-hand side of (5.22) we have that

$$\lim_{n \rightarrow \infty} \frac{1}{2n^2} \sum_{\substack{i, j=1 \\ i \neq j}}^n V_M(z_i^n, z_j^n) = \frac{1}{2} \iint_{\Omega \times \Omega} V_M(y, z) d\nu(y) d\nu(z).$$

Since $V_M \leq V$, claim (5.21) follows if we prove that for every $i = 1, \dots, n$

$$\lim_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{n^2} \sum_{i=1}^n \sum_{\substack{j \neq i \\ |z_i^n - z_j^n| < R_M}} |V(z_i^n, z_j^n)| = 0. \quad (5.23)$$

Let $Q_{R_M}(z)$ denote the open square centred at z with side length $2R_M$. Clearly $B_{R_M}(z_i^n) \subset Q_{R_M}(z_i^n)$ for every i ; therefore, if we prove

$$\lim_{M \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{n^2} \sum_{i=1}^n \sum_{\substack{j \neq i \\ z_j^n \in Q_{R_M}(z_i^n)}} |V(z_i^n, z_j^n)| = 0, \quad (5.24)$$

then (5.23) will follow. First of all, by (3.2), we have the bound

$$\sum_{i=1}^n \sum_{\substack{j \neq i \\ z_j^n \in Q_{R_M}(z_i^n)}} |V(z_i^n, z_j^n)| \leq C \sum_{i=1}^n \sum_{\substack{j \neq i \\ z_j^n \in Q_{R_M}(z_i^n)}} \left(1 - \log \frac{|z_i^n - z_j^n|}{L}\right). \quad (5.25)$$

Since the right-hand side of (5.25) is a decreasing function of the distances between the dislocation locations, the energy of any dislocation arrangement can be estimated from above in terms of the energy of the most densely packed configuration. More precisely, taking into account that

$\nu_n \in Y_n(\gamma, 1)$ and (5.19) holds, an upper bound in (5.25) is generated by the arrangement where the slip planes are all at the minimum distance $n^{-\frac{1}{2}+\gamma}$ from each other, and are packed with dislocations located at the minimum distance $1/n$ from one another. Without loss of generality we therefore limit our analysis to this special arrangement, and we denote by \tilde{z}_j^n the corresponding dislocation positions. Since replacing $\{z_i^n\}$ with $\{\tilde{z}_i^n\}$ increases the right-hand side of (5.25), we have

$$\frac{1}{n^2} \sum_{i=1}^n \sum_{\substack{j \neq i \\ z_j^n \in Q_{R_M}(z_i^n)}} |V(z_i^n, z_j^n)| \leq \frac{C}{n^2} \sum_{i=1}^n \sum_{\substack{j \neq i \\ \tilde{z}_j^n \in Q_{R_M}(\tilde{z}_i^n)}} \left(1 - \log \frac{|\tilde{z}_i^n - \tilde{z}_j^n|}{L}\right). \quad (5.26)$$

Let $i = 1, \dots, n$ be fixed. To estimate the right-hand side of (5.26), we first consider the region $A_i^n \subset Q_{R_M}(\tilde{z}_i^n)$ defined as the following union of concentric square annuli:

$$A_i^n := \bigcup_{p=2}^{P_1^n} Q_{\frac{p}{n}}(\tilde{z}_i^n) \setminus Q_{\frac{p-1}{n}}(\tilde{z}_i^n), \quad P_1^n := \lfloor n^{\frac{1}{2}+\gamma} \rfloor.$$

We note that each annulus in the set A_i^n intersects only the slip plane containing \tilde{z}_i^n , since the largest annulus of the family has outer side length $2P_1^n/n$, which by definition is smaller than $2n^{-\frac{1}{2}+\gamma}$. Since, by assumption, two active consecutive slip planes are at a distance $n^{-\frac{1}{2}+\gamma}$, every annulus in the set A_i^n only contains dislocations in the same slip plane as \tilde{z}_i^n . Moreover, there are at most two dislocations in each annulus since, by assumption, the distance between dislocations is $1/n$, which is exactly the width of each annulus.

By these arguments, for $p = 2, \dots, P_1^n$ we have

$$|\tilde{z}_i^n - \tilde{z}_j^n| = \frac{p-1}{n}, \quad \forall \tilde{z}_j^n \in Q_{\frac{p}{n}}(\tilde{z}_i^n) \setminus Q_{\frac{p-1}{n}}(\tilde{z}_i^n). \quad (5.27)$$

Therefore we can estimate the energy contribution in the sets A_i^n by using (5.27), and we obtain

$$\begin{aligned} \frac{1}{n^2} \sum_{i=1}^n \sum_{z_j^n \in A_i^n} \left(1 - \log \frac{|\tilde{z}_i^n - \tilde{z}_j^n|}{L}\right) &\leq \frac{2}{n} \sum_{p=2}^{P_1^n} \left(1 - \log \frac{p-1}{nL}\right) \\ &\leq \frac{2}{n} \left(P_1^n - \log(P_1^n!) + P_1^n \log(nL)\right). \end{aligned} \quad (5.28)$$

Since $P_1^n \rightarrow \infty$, as $n \rightarrow \infty$, we use the Stirling approximation

$$\log(P!) = P \log P - P + O(\log P) \quad \text{as } P \rightarrow \infty.$$

For the last term in (5.28) we have

$$\frac{2}{n} \left(P_1^n - \log(P_1^n!) + P_1^n \log(nL)\right) = \frac{2P_1^n}{n} \left(2 - \log \frac{P_1^n}{nL}\right) + O\left(\frac{\log P_1^n}{n}\right). \quad (5.29)$$

Since $P_1^n \simeq n^{\frac{1}{2}+\gamma}$, $P_1^n/n \simeq n^{-\frac{1}{2}+\gamma} \rightarrow 0$ as $n \rightarrow \infty$. Therefore, the right-hand side of (5.29) tends to zero, and by (5.28) we deduce that the energy contribution in A_i^n is negligible as $n \rightarrow \infty$.

It remains now to estimate the energy contribution in the complement of the sets A_i^n , i.e., in

$$B_i^n := Q_{R_M}(\tilde{z}_i^n) \setminus A_i^n.$$

Note that B_i^n contains only dislocations located on different slip planes than \tilde{z}_i^n . This is because the horizontal length of the region containing dislocations is less than $n^{-\frac{1}{2}+\gamma}$, while the outer side of the largest annulus in A_i^n is $2\lfloor n^{\frac{1}{2}+\gamma} \rfloor/n$, which is larger than $n^{-\frac{1}{2}+\gamma}$ for n large enough.

We now write the region B_i^n as the union of horizontal strips of height $n^{-\frac{1}{2}+\gamma}$, as

$$B_i^n = \bigcup_{p=1}^{P_2^n} S_p^{i,n}, \quad S_p^{i,n} := \left\{x \in B_i^n : pn^{-\frac{1}{2}+\gamma} \leq |\pi_2(x - \tilde{z}_i^n)| < (p+1)n^{-\frac{1}{2}+\gamma}\right\},$$

where $P_2^n := \lfloor R_M n^{\frac{1}{2}-\gamma} \rfloor$. Then we have

$$\frac{1}{n^2} \sum_{i=1}^n \sum_{z_j^n \in B_i^n} \left(1 - \log \frac{|\tilde{z}_i^n - \tilde{z}_j^n|}{L}\right) \leq \frac{1}{n^2} \sum_{i=1}^n \sum_{p=1}^{P_2^n} \sum_{\tilde{z}_j^n \in S_p^{i,n}} \left(1 - \log \frac{|\tilde{z}_i^n - \tilde{z}_j^n|}{L}\right).$$

Since the maximum number of dislocations per slip plane is $n^{\frac{1}{2}+\gamma}$,

$$\frac{1}{n^2} \sum_{i=1}^n \sum_{p=1}^{P_2^n} \sum_{\tilde{z}_j^n \in S_p^{i,n}} \left(1 - \log \frac{|\tilde{z}_i^n - \tilde{z}_j^n|}{L}\right) \leq \frac{1}{n} \sum_{p=1}^{P_2^n} n^{\frac{1}{2}+\gamma} \left(1 - \log \frac{pn^{-\frac{1}{2}+\gamma}}{L}\right). \quad (5.30)$$

Now, the right-hand side of (5.30) can be estimated as follows:

$$\frac{1}{n} \sum_{p=1}^{P_2^n} n^{\frac{1}{2}+\gamma} \left(1 - \log \frac{pn^{-\frac{1}{2}+\gamma}}{L}\right) \leq C \int_0^{R_M/L} (1 - \log x) dx \leq CR_M(1 - \log R_M);$$

thus, it tends to zero, as $M \rightarrow \infty$. This concludes the estimate of the energy contribution in the sets B_i^n . In view of (5.26), claim (5.24) is proved.

We now consider the case $\gamma = \frac{1}{2}$. For simplicity we assume that there is only one active slip plane and $c = 1$; moreover, we assume that $\text{supp}[(\pi_2)_\# \mu_n] = \text{supp}[(\pi_2)_\# \mu] = \text{supp}[(\pi_2)_\# \nu] = \{s\}$. In this case the problem becomes one-dimensional and we can construct the recovery sequence by hand, using the same ideas as in the proof of Theorem 3.3. More precisely, following the steps of the limsup inequality in Theorem 3.3 we first approximate the measure ν by means of a sequence (ν^h) supported on a subset of $h\mathcal{I} \times \{s\}$, where

$$\mathcal{I} := \bigcup_{m \in \mathbb{Z}} I_m, \quad I_m := [2m, 2m+1].$$

Up to a further approximation, we can assume that $n\nu^h(hI_m) \in \mathbb{N}_0$ for every h, n , and m . This can be done without increasing the interaction energy. Moreover, since $\text{supp}(\pi_2)_\# \nu^h = \{s\}$ and $\nu^h \rightarrow \nu$ narrowly, as $h \rightarrow 0$, we have $d(\nu^h, \nu) \rightarrow 0$, as $h \rightarrow 0$. In other words, it is enough to construct a joint recovery sequence for the measures ν^h .

Let $h > 0$ be fixed. The recovery sequence ν_n^h is obtained by arranging $n\nu^h(hI_m)$ equispaced dislocations $\{z_i^n\}$ in every interval hI_m . We observe that $\nu_n^h \in Y_n(\frac{1}{2}, 1)$ and $(\pi_2)_\# \nu_n^h = (\pi_2)_\# \mu_n$ for every n ; in particular, for every $i \neq j$ we have

$$|z_i^n - z_j^n| \geq \frac{C(h)}{n} \geq r_n. \quad (5.31)$$

Moreover, by construction, $\nu_n^h \rightarrow \nu^h$ narrowly. Since $\text{supp}(\pi_2)_\# \nu_n^h = \{s\}$, this implies that $d(\mu_n, \nu_n^h) \rightarrow d(\mu, \nu^h)$.

To prove that ν_n^h is a recovery sequence it remains to show (5.20) and, as for the case $\gamma \neq \frac{1}{2}$, this reduces to proving (5.23). By (3.2) we have that

$$\begin{aligned} \frac{1}{n^2} \sum_{i=1}^n \sum_{\substack{j \neq i \\ |z_i^n - z_j^n| \leq R_M}} |V(z_i^n, z_j^n)| &\leq \frac{C}{n^2} \sum_{i=1}^n \sum_{\substack{j \neq i \\ |z_i^n - z_j^n| \leq R_M}} \left(1 - \log \frac{|z_i^n - z_j^n|}{L}\right) \\ &\leq \frac{C}{n} \sum_{p=1}^{\lfloor nR_M/C(h) \rfloor} \left(1 - \log \frac{C(h)p}{nL}\right), \end{aligned} \quad (5.32)$$

where the last inequality follows by estimating the energy of the distribution of dislocations of ν_n^h with the most densely packed configuration, from (5.31). By taking the limit as $n \rightarrow \infty$ in (5.32) we have

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{i=1}^n \sum_{\substack{j \neq i \\ |z_i^n - z_j^n| \leq R_M}} |V(z_i^n, z_j^n)| \leq C \int_0^{R_M/L} (1 - \log x) dx,$$

and the right-hand side converges to zero, as $M \rightarrow \infty$. Therefore (5.23) is proved also for $\gamma = \frac{1}{2}$.

This concludes the proof of the theorem. \square

5.1. Convergence of the evolutions. In this section we prove the convergence of the quasi-static evolution for the renormalized energy defined as in (4.8) and (4.9).

Theorem 5.8. *Assume (1.1). Let $f \in W^{1,1}(0, T; C(\Omega))$ and let $-\frac{1}{2} < \gamma \leq \frac{1}{2}$ and $c > 0$. Let $\mu^0 \in \mathcal{P}(\Omega)$ with $\mathcal{F}(\mu^0) < +\infty$ and let (μ_n^0) be such that $\mu_n^0 \rightharpoonup \mu^0$ narrowly, and*

$$\mathcal{F}_n(\mu_n^0) \rightarrow \mathcal{F}(\mu^0), \quad (5.33)$$

as $n \rightarrow \infty$. For every $n \in \mathbb{N}$ assume also that $\mu_n^0 \in Y_n(\gamma, c)$ and μ_n^0 satisfies the stability condition:

$$\mathcal{F}_n(\mu_n^0) - \int_{\Omega} f(0) d\mu_n^0 \leq \mathcal{F}_n(\nu) + d(\nu, \mu_n^0) - \int_{\Omega} f(0) d\nu \quad (5.34)$$

for every $\nu \in X_n$. For every $n \in \mathbb{N}$ let $t \mapsto \mu_n(t)$ be a quasi-static evolution on $[0, T]$ with initial value μ_n^0 and potential force f , according to Theorem 4.4. Then

- (1) **Compactness:** *There exists a subsequence μ_n (without change in notation) and a limit curve $\mu : [0, T] \rightarrow \mathcal{P}_{\gamma, c}^{\infty}(\Omega)$ such that $\mu_n(t) \rightharpoonup \mu(t)$ for all $t \in [0, T]$;*
- (2) **Convergence:** *The curve μ is a quasi-static evolution with initial value μ^0 and force f for the limit energy \mathcal{F} and the dissipation \mathcal{D} . More precisely, the following two conditions are satisfied:*

(qs1) $_{\infty}$ *global stability: for every $t \in [0, T]$ and for every $\nu \in \mathcal{P}(\Omega)$,*

$$\mathcal{F}(\mu(t)) - \int_{\Omega} f(t) d\mu(t) \leq \mathcal{F}(\nu) + d(\nu, \mu(t)) - \int_{\Omega} f(t) d\nu; \quad (5.35)$$

(qs2) $_{\infty}$ *energy balance: the map $t \mapsto \int_{\Omega} \dot{f}(t) d\mu(t)$ is integrable on $[0, T]$ and for every $t \in [0, T]$*

$$\mathcal{F}(\mu(t)) + \mathcal{D}(\mu, [0, t]) - \int_{\Omega} f(t) d\mu(t) = \mathcal{F}(\mu^0) - \int_{\Omega} f(0) d\mu(0) - \int_0^t \int_{\Omega} \dot{f}(s) d\mu(s) ds. \quad (5.36)$$

Remark 5.9. The existence of an admissible sequence of initial data satisfying all the assumptions of Theorem 5.8 is guaranteed by Theorems 3.3 and 5.5. It can be constructed as follows. Let $\hat{\mu}^0 \in \mathcal{P}_{\gamma, c}^{\infty}(\Omega)$ be such that $\mathcal{F}(\hat{\mu}^0) < +\infty$. By definition of $\mathcal{P}_{\gamma, c}^{\infty}(\Omega)$ there exist $\hat{\mu}_n^0 \rightharpoonup \hat{\mu}^0$ narrowly, as $n \rightarrow \infty$, such that $\hat{\mu}_n^0 \in Y_n(\gamma, c)$ for every n . By applying Theorem 5.5 to $\mu = \nu = \hat{\mu}^0$ and $\mu_n = \hat{\mu}_n^0$ we can replace the sequence $(\hat{\mu}_n^0)$ with a new sequence on which the renormalized energies \mathcal{F}_n converge to $\mathcal{F}(\hat{\mu}^0)$. In other words, up to replacing $(\hat{\mu}_n^0)$ with this new sequence, we can assume that $\mathcal{F}_n(\hat{\mu}_n^0) \leq C$ for every n . By Lemma 4.3 the minimum problems

$$\min_{\mu \in X_n} \left\{ \mathcal{F}_n(\mu) + d(\mu, \hat{\mu}_n^0) - \int_{\Omega} f(0) d\mu \right\}$$

have a solution μ_n^0 with finite energy. In particular, $d(\mu_n^0, \hat{\mu}_n^0) < \infty$ for every n , which implies $(\pi_2)_{\#} \mu_n^0 = (\pi_2)_{\#} \hat{\mu}_n^0$ for every n . Thus, $\mu_n^0 \in Y_n(\gamma, c)$ for every n . By the triangle inequality μ_n^0 satisfies the stability condition (5.34). Moreover, up to subsequences, there exists $\mu^0 \in \mathcal{P}(\Omega)$ such that $\mu_n^0 \rightharpoonup \mu^0$ narrowly, as $n \rightarrow \infty$. From the minimality we also deduce that

$$\mathcal{F}_n(\mu_n^0) + d(\mu_n^0, \hat{\mu}_n^0) - \int_{\Omega} f(0) d\mu_n^0 \leq \mathcal{F}_n(\hat{\mu}_n^0) - \int_{\Omega} f(0) d\hat{\mu}_n^0,$$

hence $\mathcal{F}_n(\mu_n^0) \leq C$ for every n and from the liminf inequality of Theorem 3.3 we deduce that $\mathcal{F}(\mu^0) < +\infty$. It remains to show (5.33). By Theorem 5.5 there exists a sequence (ν_n^0) such that $\nu_n^0 \in X_n$ for every n , $\nu_n^0 \rightharpoonup \mu^0$ narrowly, as $n \rightarrow \infty$, and

$$\lim_{n \rightarrow \infty} d(\mu_n^0, \nu_n^0) = 0, \quad \limsup_{n \rightarrow \infty} \mathcal{F}_n(\nu_n^0) \leq \mathcal{F}(\mu^0). \quad (5.37)$$

From the stability condition (5.34) we have

$$\mathcal{F}_n(\mu_n^0) - \int_{\Omega} f(0) d\mu_n^0 \leq \mathcal{F}_n(\nu_n^0) + d(\nu_n^0, \mu_n^0) - \int_{\Omega} f(0) d\nu_n^0.$$

Owing to the liminf inequality of Theorem 3.3 and to (5.37), we can pass to the limit in the above expression and get (5.33).

Remark 5.10. Note that if $t \mapsto \mu(t)$ is a quasi-static evolution, then

$$(\pi_2)_\# \mu(t) = (\pi_2)_\# \mu(0) \quad \text{for every } t \in [0, T]. \quad (5.38)$$

Indeed, owing to the regularity assumptions on f and to (3.42), condition (qs2) $_\infty$ implies that $\mathcal{D}(\mu, [0, T]) < +\infty$, which in turn gives (5.38) by the definition of the dissipation distance d .

Proof of Theorem 5.8. Compactness. Using the regularity assumption on f and assumption (5.33) on the initial data, we infer from the energy balance (qs2) $_n$

$$\mathcal{F}_n(\mu_n(t)) + \mathcal{D}(\mu_n, [0, t]) \leq C \quad (5.39)$$

for every n and every $t \in [0, T]$. On the other hand, the energy estimates (3.13) and (3.14) in the proof of Theorem 3.3 yield

$$\begin{aligned} \mathcal{F}_n(\mu_n(t)) &= \frac{1}{2} \iint_{\Omega \times \Omega} V(y, z) d(\mu_n(t) \boxtimes \mu_n(t))(y, z) \\ &\quad + \frac{1}{2n} \sum_{i=1}^n \int_{\partial\Omega} \mathbb{C}K^n(x; z_i^n(t)) \nu \cdot u_{\mu_n}(t) d\mathcal{H}^1(x) + o(1), \end{aligned}$$

where $o(1)$ is a quantity tending to 0, as $n \rightarrow \infty$, uniformly in t . By Lemma 2.1 and Remark 3.2 we deduce that

$$\mathcal{F}_n(\mu_n(t)) \geq -C$$

for every n and every t . Combining this inequality with (5.39), we obtain

$$\mathcal{D}(\mu_n, [0, T]) \leq C. \quad (5.40)$$

In particular, this implies that $d(\mu_n(t), \mu_n^0) < \infty$ for every $t \in [0, T]$ and every n and, in turn, that $(\pi_2)_\# \mu_n(t) = (\pi_2)_\# \mu_n^0$ for every $t \in [0, T]$ and every n . Since $\mu_n^0 \in Y_n(\gamma, c)$ by assumption, we conclude that $\mu_n(t) \in Y_n(\gamma, c)$ for every $t \in [0, T]$ and every n .

By Theorem 7.2 and (5.40) we can guarantee that there exists a map $t \mapsto \mu(t)$ from $[0, T]$ into $\mathcal{P}(\Omega)$ such that, up to a subsequence,

$$\mu_n(t) \rightharpoonup \mu(t) \quad \text{narrowly,}$$

as $n \rightarrow \infty$, for every $t \in [0, T]$. By definition of $\mathcal{P}_{\gamma, c}^\infty(\Omega)$ we clearly have that $\mu(t) \in \mathcal{P}_{\gamma, c}^\infty(\Omega)$ for every $t \in [0, T]$. Moreover, by (5.39) and Theorem 3.3 we have

$$\mathcal{F}(\mu(t)) \leq \liminf_{n \rightarrow \infty} \mathcal{F}_n(\mu_n(t)) \leq C \quad (5.41)$$

for every $t \in [0, T]$; in other words, $\mathcal{F}(\mu(t)) < \infty$ for every $t \in [0, T]$.

Convergence. We now prove condition (qs1) $_\infty$. Let us fix $t \in [0, T]$ and let $\nu \in \mathcal{P}(\Omega)$. Clearly, it is enough to prove (5.35) for $\nu \in \mathcal{P}(\Omega)$ such that $\mathcal{F}(\nu) < \infty$ and $d(\nu, \mu(t)) < \infty$. This last condition implies, in particular, that $(\pi_2)_\# \nu = (\pi_2)_\# \mu(t)$. By Theorem 5.5 there exists a sequence (ν_n) such that $\nu_n \in Y_n(\gamma, c)$ for every n , $\nu_n \rightharpoonup \nu$ narrowly, as $n \rightarrow \infty$, and

$$\limsup_{n \rightarrow \infty} d(\nu_n, \mu_n(t)) \leq d(\nu, \mu(t)) \quad \text{and} \quad \limsup_{n \rightarrow \infty} \mathcal{F}_n(\nu_n) \leq \mathcal{F}(\nu). \quad (5.42)$$

Since $\nu_n \in X_n$ for every n , the minimality condition (qs1) $_n$ implies

$$\mathcal{F}_n(\mu_n(t)) - \int_{\Omega} f(t) d\mu_n(t) \leq d(\nu_n, \mu_n(t)) + \mathcal{F}_n(\nu_n) - \int_{\Omega} f(t) d\nu_n.$$

Combining (5.41) and (5.42), we can pass to the limit in this inequality and obtain (5.35).

By (5.41), the lower semicontinuity of the dissipation, and (5.33), we can now pass to the limit in the energy balance (qs2) $_n$ and prove an energy inequality. The converse energy inequality can be deduced from the global stability. \square

6. STRONG FORMULATION OF THE EVOLUTION PROBLEMS

In this section we derive the strong formulation of the quasi-static evolution for the discrete renormalised energy (1.6) and for the limit energy (1.7), under the assumption that they are sufficiently regular for the various arguments to be valid.

6.1. Discrete energy. Our starting point is the formulation (4.8)–(4.9) of the quasi-static evolution for the renormalised energy. For simplicity of notation, since $\mu(t) = \frac{1}{n} \sum_{i=1}^n \delta_{z_i(t)}$ we write

$$\mathcal{F}_n(\mu(t)) = \frac{1}{n} \int_{\Omega} W_n(x, z_1(t), \dots, z_n(t)) dx.$$

We note that the force term reduces to

$$\int_{\Omega} f(t) d\mu(t) = \frac{1}{n} \sum_{i=1}^n f(t, z_i(t)).$$

We assume that $\text{supp } \mu(t) \subset \text{int } \mathcal{R}$ and $|z_i(t) - z_j(t)| > r_n$, $i \neq j$, for every $t \in [0, T]$. Moreover, we suppose that $\mu(t)$ varies smoothly with respect to time, so that

$$\mathcal{D}(\mu, [0, t]) = \frac{1}{n} \sum_{i=1}^n \int_0^t |\dot{z}_i(s) \cdot e_1| ds. \quad (6.1)$$

Note also that by (4.10) we have

$$\dot{z}_i(t) \cdot e_2 = 0 \quad (6.2)$$

for every $i = 1, \dots, n$. We also assume that the forcing term is regular enough, for instance $f \in W^{1,1}(0, T; C^1(\Omega))$, and W_n is continuously differentiable with respect to the points in the support of μ (this latter condition depends on the regularity of the functions $z \mapsto K_z$).

In (qs1)_n we consider the measure $\frac{1}{n} \sum_{i=1}^n \delta_{z_i(t) + \eta_i e_1}$ as a competitor of $\mu(t)$, with $|\eta_i|$ small enough, and we obtain

$$\begin{aligned} & \frac{1}{n} \int_{\Omega} W_n(x, z_1(t), \dots, z_n(t)) dx - \frac{1}{n} \sum_{i=1}^n f(t, z_i(t)) \\ & \leq \frac{1}{n} \int_{\Omega} W_n(x, z_1(t) + \eta_1 e_1, \dots, z_n(t) + \eta_n e_1) dx + \frac{1}{n} \sum_{i=1}^n |\eta_i| - \frac{1}{n} \sum_{i=1}^n f(t, z_i(t) + \eta_i e_1). \end{aligned}$$

Choosing $\eta_i > 0$ and $\eta_j = 0$ for $i \neq j$, we can divide by η_i and pass to the limit as $\eta_i \rightarrow 0$, and obtain

$$- \int_{\Omega} \nabla_{z_i} W_n(x, z_1(t), \dots, z_n(t)) \cdot e_1 dx + \partial_{x_1} f(t, z_i(t)) \leq 1. \quad (6.3)$$

Repeating the same argument for $\eta_i < 0$ we have

$$- \int_{\Omega} \nabla_{z_i} W_n(x, z_1(t), \dots, z_n(t)) \cdot e_1 dx + \partial_{x_1} f(t, z_i(t)) \geq -1.$$

Therefore, we deduce

$$\left| - \int_{\Omega} \nabla_{z_i} W_n(x, z_1(t), \dots, z_n(t)) \cdot e_1 dx + \partial_{x_1} f(t, z_i(t)) \right| \leq 1 \quad (6.4)$$

for every $i = 1, \dots, n$.

Differentiating (qs2)_n with respect to time, we obtain

$$\sum_{i=1}^n \dot{z}_i(t) \cdot \left(- \int_{\Omega} \nabla_{z_i} W_n(x, z_1(t), \dots, z_n(t)) dx + \nabla_x f(t, z_i(t)) \right) = \sum_{i=1}^n |\dot{z}_i(t) \cdot e_1|.$$

In view of (6.4) and (6.2), this is equivalent to

$$\left(- \int_{\Omega} \nabla_{z_i} W_n(x, z_1(t), \dots, z_n(t)) \cdot e_1 dx + \partial_{x_1} f(t, z_i(t)) \right) (\dot{z}_i(t) \cdot e_1) = |\dot{z}_i(t) \cdot e_1| \quad (6.5)$$

for every $i = 1, \dots, n$.

The flow rule (6.5) is rate-independent. Moreover, together with (6.4), it implies that when the inequality in (6.4) is strict for some i , then it must be $\dot{z}_i \cdot e_1 = 0$. When (6.4) holds with the equality for some i , then $\dot{z}_i \cdot e_1$ may be different from zero and satisfies

$$\dot{z}_i(t) \cdot e_1 = \lambda \left(- \int_{\Omega} \nabla_{z_i} W_n(x, z_1(t), \dots, z_n(t)) \cdot e_1 dx + \partial_{x_1} f(t, z_i(t)) \right)$$

for some $\lambda \geq 0$.

6.2. Limiting energy. The limiting quasi-static evolution is in this case given by (5.35)–(5.36), where we set

$$\tilde{\mathcal{F}}(\mu, t) := \mathcal{F}(\mu) - \int_{\Omega} f(t) d\mu.$$

We now follow the same arguments as in Section 6.1, but for the energy $\tilde{\mathcal{F}}$. We assume for simplicity that $\text{supp } \mu(t) \subset \text{int } \mathcal{R}$ for every $t \in [0, T]$. We choose $\varphi \in C_c^\infty(\Omega)$, and for $\eta > 0$ small enough we define perturbations μ^η of $\mu(t)$ by (writing $x = (x_1, x_2)$) for an element of Ω)

$$\mu^\eta = T_\#^\eta \mu(t), \quad T^\eta(x) := (x_1 + \eta\varphi(x), x_2).$$

Note that if η is small enough, then T^η is a smooth one-to-one map from Ω to Ω . Since the map $x \mapsto T^\eta(x)$ is an admissible transport map, we have

$$d(\mu^\eta, \mu(t)) \leq \int_{\Omega} |x_1 - (x_1 + \eta\varphi(x))| d\mu(t)(x) = \eta \int_{\Omega} |\varphi(x)| d\mu(t)(x).$$

To estimate the effect of this perturbation on $\tilde{\mathcal{F}}$, we note that $\partial_\eta \mu^\eta = -\partial_{x_1}(\varphi\mu^\eta)$ at $\eta = 0$, and calculate

$$\lim_{\eta \rightarrow 0} \frac{1}{\eta} [\tilde{\mathcal{F}}(\mu^\eta, t) - \tilde{\mathcal{F}}(\mu(t), t)] \Big|_{\eta=0} = \int_{\Omega} \frac{\delta \tilde{\mathcal{F}}}{\delta \mu}(\mu(t), t) \partial_\eta \mu^\eta \Big|_{\eta=0} = \int_{\Omega} \varphi \partial_{x_1} \frac{\delta \tilde{\mathcal{F}}}{\delta \mu}(\mu(t), t) d\mu(t), \quad (6.6)$$

where

$$\frac{\delta \tilde{\mathcal{F}}}{\delta \mu}(\mu, t)(x) = \int_{\Omega} V(x, y) d\mu(y) + \frac{1}{2} \int_{\partial \Omega} \mathbb{C}K(x; y) \nu(y) \cdot v_\mu(y) d\mathcal{H}^1(y) - f(x, t). \quad (6.7)$$

The second term follows from (1.7) using the general principle that if $h(s) = \inf_v g(v, s)$, with unique minimizer v_s depending smoothly on s , then $h'(s) = (\partial_s g)(v_s, s)$.

Now using μ^η in $(\text{qs1})_\infty$, dividing by η and taking the limit $\eta \rightarrow 0$, we find

$$- \int_{\Omega} \varphi \partial_{x_1} \frac{\delta \tilde{\mathcal{F}}}{\delta \mu}(\mu(t), t) d\mu(t) \leq \int_{\Omega} |\varphi| d\mu(t),$$

which is the equivalent of (6.3). Since φ is arbitrary, we similarly deduce that

$$\left| \partial_{x_1} \frac{\delta \tilde{\mathcal{F}}}{\delta \mu}(\mu(t), t)(x) \right| \leq 1 \quad \text{for } \mu(t)\text{-a.e. } x \in \Omega. \quad (6.8)$$

Continuing the argument from the discrete case, we assume that μ is smooth in space and time, and we write $d\mu(t) = \mu(t, x) dx$. By (4.5) there exists a Borel measurable ϕ satisfying the equation

$$\partial_t \mu(t, x) + \partial_{x_1}(\phi(t, x)\mu(t, x)) = 0 \quad \text{for all } t \text{ and for } \mu(t)\text{-a.e. } x,$$

such that

$$\mathcal{D}(\mu, [0, t]) = \int_0^t \int_{\Omega} |\phi(s, x)| \mu(s, x) dx ds.$$

Differentiating $(\text{qs2})_\infty$ in time and calculating as in (6.6) we find

$$\int_{\Omega} \phi(t, x) \partial_{x_1} \frac{\delta \tilde{\mathcal{F}}}{\delta \mu}(\mu(t), t) \mu(t, x) dx + \int_{\Omega} |\phi(t, x)| \mu(t, x) dx = 0.$$

Again, in combination with (6.8), this implies

$$-\phi(t, x) \partial_{x_1} \frac{\delta \tilde{\mathcal{F}}}{\delta \mu}(\mu(t), t) = |\phi(t, x)| \quad \text{for } \mu(t)\text{-a.e. } x,$$

which is the counterpart of (6.5). At $\mu(t)$ -a.e. x , therefore, *either* $\phi(x, t) = 0$, *or* the total force satisfies $-\partial_{x_i} \delta \tilde{\mathcal{F}} / \delta \mu(\mu(t), t)(x) = \pm 1$; in the latter case, ϕ points in the direction of the force.

7. APPENDIX

In this appendix we state and prove an extension result (Theorem 7.1), and a Helly-type theorem.

Theorem 7.1. *Let Ω be an open set in \mathbb{R}^2 . Let δ and let z_1, \dots, z_n , $n \in \mathbb{N}$, be such that $|z_j - z_k| \geq 4\delta$ for every $j \neq k$ and $\text{dist}(z_i, \partial\Omega) \geq 2\delta$ for every i . Then there exists a positive constant C , independent of δ , of the z_i 's and of n , with the following property: for every $u \in H^1(\Omega \setminus \cup_{i=1}^n \overline{B}_\delta(z_i); \mathbb{R}^2)$ there exists an extension $\tilde{u} \in H^1(\Omega; \mathbb{R}^2)$ of u such that*

$$\|E\tilde{u}\|_{L^2(\Omega)} \leq C \|Eu\|_{L^2(\Omega \setminus \cup_{i=1}^n \overline{B}_\delta(z_i))}. \quad (7.1)$$

Proof. Let $u \in H^1(\Omega \setminus \cup_{i=1}^n \overline{B}_\delta(z_i); \mathbb{R}^2)$. By [OSY92, Lemma 4.1] for every $i = 1, \dots, n$ there exists a function $\tilde{u}^i \in H^1(B_{2\delta}(z_i))$ such that $\tilde{u}^i = u$ on $B_{2\delta}(z_i) \setminus \overline{B}_\delta(z_i)$ and

$$\|E\tilde{u}^i\|_{L^2(B_{2\delta}(z_i))} \leq C_1 \|Eu\|_{L^2(B_{2\delta}(z_i) \setminus \overline{B}_\delta(z_i))}, \quad (7.2)$$

where the constant C_1 is independent of u . By a scaling argument it is easy to see that C_1 is also independent of δ and of z_i .

It is now enough to define

$$\tilde{u} := \begin{cases} \tilde{u}^i & \text{in } B_\delta(z_i), \\ u & \text{otherwise.} \end{cases}$$

Clearly \tilde{u} is an extension of u and $\tilde{u} \in H^1(\Omega; \mathbb{R}^2)$. Moreover, (7.2) yields

$$\begin{aligned} \|E\tilde{u}\|_{L^2(\Omega)}^2 &= \|Eu\|_{L^2(\Omega \setminus \cup_{i=1}^n \overline{B}_\delta(z_i))}^2 + \sum_{i=1}^n \|E\tilde{u}^i\|_{L^2(B_\delta(z_i))}^2 \\ &\leq \|Eu\|_{L^2(\Omega \setminus \cup_{i=1}^n \overline{B}_\delta(z_i))}^2 + C_1^2 \sum_{i=1}^n \|Eu\|_{L^2(B_{2\delta}(z_i) \setminus \overline{B}_\delta(z_i))}^2 \\ &\leq (1 + C_1^2) \|Eu\|_{L^2(\Omega \setminus \cup_{i=1}^n \overline{B}_\delta(z_i))}^2, \end{aligned}$$

which proves (7.1). \square

Theorem 7.2. *Let $t \mapsto \mu_k(t)$ be a sequence of maps from $[0, T]$ into the space $\mathcal{P}(\Omega)$. We assume that there exist an open set $\Omega' \subset \subset \Omega$ and $C > 0$ such that*

$$\text{supp } \mu_k(t) \subset \Omega' \quad (7.3)$$

for every $t \in [0, T]$ and every k , and

$$\mathcal{D}(\mu_k, [0, T]) \leq C \quad (7.4)$$

for every k . Then there exists a subsequence (not relabelled) and a map $t \mapsto \mu(t)$ from $[0, T]$ into $\mathcal{P}(\Omega)$, with $\mathcal{D}(\mu, [0, T]) \leq C$, such that

$$\mu_k(t) \rightharpoonup \mu(t) \quad \text{narrowly} \quad (7.5)$$

for every $t \in [0, T]$.

Proof. Let Q be a countable and dense subset of $[0, T]$ containing 0. Using a diagonal argument, assumption (7.3) implies the existence of a subsequence $k_j \rightarrow \infty$ and a map $t \mapsto \mu(t)$ from Q into $\mathcal{P}(\Omega)$ such that

$$\mu_{k_j}(t) \rightharpoonup \mu(t) \quad \text{narrowly}$$

for every $t \in Q$. We now define

$$D_{k_j}(t) := \mathcal{D}(\mu_{k_j}, [0, t]).$$

The functions D_{k_j} are non-decreasing with respect to time; moreover, $D_{k_j}(T) \leq C$ and $D_{k_j}(0) = 0$ for every j . By Helly's Theorem we deduce that, up to subsequences,

$$\lim_{j \rightarrow \infty} D_{k_j}(t) = D(t) \quad \forall t \in [0, T],$$

and the limit function $D(t)$ is also non decreasing in t and bounded by the same constant C . We now define the set

$$\Theta := \{0\} \cup \left\{ t \in (0, T] : \lim_{s \rightarrow t^-} D(s) = D(t) \right\}.$$

Note that, by definition, for every $0 \leq s \leq t \leq T$ we have

$$d(\mu_{k_j}(t), \mu_{k_j}(s)) \leq D_{k_j}(t) - D_{k_j}(s). \quad (7.6)$$

Let $t \in \Theta$. Then, for every $\varepsilon > 0$ there exists $s \in Q$ such that $s \leq t$ and $D(t) - D(s) < \varepsilon$. Then, by (7.6), for large enough j (note that $j = j(t, s)$)

$$d(\mu_{k_j}(t), \mu_{k_j}(s)) < \varepsilon.$$

By Part 2 of Lemma 4.1 this implies that

$$d_1(\mu_{k_j}(t), \mu_{k_j}(s)) < \varepsilon.$$

Then, for every $i, j \in \mathbb{N}$ we have the bound

$$d_1(\mu_{k_i}(t), \mu_{k_j}(t)) < 2\varepsilon + d_1(\mu_{k_i}(s), \mu_{k_j}(s)),$$

entailing that $(\mu_{k_j}(t))$ is a Cauchy sequence in d_1 . Therefore, $\mu_{k_j}(t)$ converges narrowly to some limit measure $\mu(t)$ for $t \in \Theta$. Since the complement of Θ in $[0, T]$ is at most countable, up to extracting a further subsequence we have by a diagonal argument that $\mu_{k_j}(t)$ converges narrowly to a measure $\mu(t)$ for every $t \in [0, T]$. \square

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