A NEW DIFFERENTIATION, SHAPE OF THE UNIT BALL AND PERIMETER MEASURE

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Abstract. We present a new blow-up method that allows for establishing the first general formula to compute the perimeter measure with respect to the spherical Hausdorff measure in noncommutative nilpotent groups. Our techniques are new also in the classical Euclidean framework. When the distance is sub-Riemannian, we are lead to an unexpected relationship between the validity of a suitable area formula with respect to a distance and the profile of its corresponding unit ball.

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1. Introduction

In the last decade, the study of sub-Riemannian Geometry, in short SR Geometry, has known a strong impulse in different areas, from PDE and Control Theory to Differential Geometry and Geometric Measure Theory. In particular, a number of Riemannian problems may have a sub-Riemannian interpretation in a large framework and this often leads to foundational questions or to new viewpoints.

The challenging project of developing Geometric Measure Theory on SR manifolds has shown that both of these aspects can happen. With this aim in mind, finding a theory of area in SR Geometry represents the starting point of a demanding program.
Historically, since the seminal works by Carathéodory [12] and Hausdorff [30], many theories grew to study $k$-dimensional Lebesgue area, smoothness conditions for area and coarea formulae, extensions to Finsler spaces, etc. We mention only a few relevant references [1], [2], [3], [9], [10], [13], [15], [16], [17], [18], [19], [20], [21], [27], [29], [32], [38], [40], [42], [43], [46], to give a very small glimpse of the much wider literature in both old and new research lines.

In this paper we provide a new approach to the surface area in SR Geometry. In the modern view, the Riemannian surface area can be easily computed by Euclidean tools, since smooth subsets have Lipschitz parametrizations, the Rademacher theorem holds and standard change of variables formulae perfectly fit with the density given by the Riemannian metric.

The previous techniques fail completely in the SR case and this is due to two important obstructions. First of all, we need not have Lipschitz parametrizations, even for smooth subsets of a sub-Riemannian manifold. This forces the use of abstract differentiation theorems for measures, but the second obstacle comes up exactly at this stage. In fact, the Besicovitch covering theorem, shortly B.C.T., in general may fail to hold both in sub-Riemannian and homogeneous Heisenberg groups, [34], [45]. Thus, in our framework we do not have any general theorem to differentiate an arbitrary Radon measure.

We will show how to overcome these difficulties, restricting our attention to the theory of finite perimeter sets. However, the general idea of our method is clearly expected to have similar applications for higher codimensional subsets, although in this case the corresponding theory of currents is still very far from being understood.

For finite perimeter sets, it is convenient to start from the relationship between perimeter measure and Hausdorff measure, that holds for every finite perimeter set $E$ in an Ahlfors $Q$-regular metric space, equipped with a Poincaré inequality. Under these conditions, Ambrosio established the formula

$$|\partial E| = \theta \mathcal{H}^{Q-1} \cup \partial^* E,$$

where $|\partial E|$ denotes the perimeter measure defined by a relaxation procedure, $\partial^* E$ is the essential boundary and $\theta$ is measurable and bounded below by a positive geometric constant, see [4] for more details.

Our ambient space is a homogeneous stratified group, namely a stratified group equipped with a homogeneous distance, see Section 2. Since homogeneous distances are left invariant and 1-homogeneous with respect to dilations one can easily verify that a stratified group $\mathbb{G}$ is automatically Ahlfors $Q$-regular, where $Q$ denotes its Hausdorff dimension. Moreover, it satisfies a Poincaré inequality and the previous perimeter measure has an equivalent variational formulation, [26], [41], then (1) joined with [4] give the formula

$$|\partial_H E| = \beta \mathcal{S}_0^{Q-1} \cup \mathcal{F}_H E,$$
where $\beta$ is measurable, $E \subset G$ is an h-finite perimeter set, $\mathcal{F}_H E$ is the reduced boundary and $|\partial_H E|$ is the variational perimeter measure on groups, see Section 4 for precise definitions. The symbol $S_0^{Q-1}$ denotes the $(Q - 1)$-dimensional spherical Hausdorff measure with $c_{Q-1} = 2^{1-Q}$, see Definition 3.1.

A general notion of perimeter measure on SR manifolds has been recently introduced in [7], where the divergence operator is only defined by the volume measure. This leads to the observation that in stratified groups the notion of perimeter measure can be equivalently introduced by the standard divergence, see (28).

Finding a geometric expression for $\beta$ is the crucial question, since it allows for computing the perimeter measure. From the classical De Giorgi’s theory, [15], in the Euclidean case, $G = \mathbb{R}^n$, the classical area formula and the rectifiability of the reduced boundary give $\beta = \omega_{n-1}$, that is the volume of the unit ball in $\mathbb{R}^{n-1}$. Further extensions to the case of Finsler spaces have been also established, [9]. When the ambient space is a noncommutative stratified group, we have a drastic change of the problem. The new fundamental difficulty consists in the very low regularity of the reduced boundary $\mathcal{F}_H E$, that in general is not rectifiable in the classical sense of 3.2.14 of [20], so that all of the known methods fail.

At present there are no results to find $\beta$ when the spherical Hausdorff measure is replaced by the Hausdorff measure. However, some integral representations for Borel measures with respect to the spherical Hausdorff measure can be written, with an explicit, but not manageable, formula for the density.

**Theorem 1.1 ([37]).** Let $X$ be a diametrically regular metric space, let $\alpha > 0$ and let $\mu$ be a Borel regular measure over $X$ such that there exists a countable open covering of $X$ whose elements have $\mu$ finite measure. If $B \subset A \subset X$ are Borel sets and $S_{\mu, \alpha}$ covers $A$ finely, then $\theta^\alpha(\mu, \cdot)$ is Borel on $A$. In addition, if $\mathcal{S}^\alpha(A) < +\infty$ and $\mu \mathcal{L} A$ is absolutely continuous with respect to $\mathcal{S}^\alpha \mathcal{L} A$, then we have

$$
\mu(B) = \int_B \theta^\alpha(\mu, x) \, d\mathcal{S}^\alpha(x).
$$

The general hypotheses of Theorem 1.1 are obviously satisfied in all stratified groups, since B.C.T. is not required. The spherical Federer density $\theta^\alpha(\mu, \cdot)$ has been recently introduced in [37]. The crucial aspect is its general formula

$$
\theta^\alpha(\mu, x) = \inf_{\varepsilon > 0} \sup \left\{ \frac{\mu(B)}{c_{\alpha, \diam(B)^\alpha}} : x \in B \in \mathcal{F}_b, \diam B < \varepsilon \right\},
$$

where $\mathcal{F}_b$ denotes the family of all closed balls in $X$ and $c_{\alpha} > 0$ plays the role of the geometric constant to be fixed in relation to the geometry of the metric ball. Then finding $\beta$ in (2) exactly corresponds to find a more explicit formula for $\theta^{Q-1}(\partial_H E, \cdot)$, when a specific homogeneous distance $d$ is fixed. This computation is more delicate than the one with centered densities in the sense of 2.10.19 [20], and it requires a new blow-up method, along with new regularity conditions on the homogeneous distance.
We introduce this regularity in Definition 2.2, using the terminology “\((n − 1)\)-vertical regularity”, since we require a negligibility condition for the intersection of \((n − 1)\)-dimensional vertical subspaces with the boundary of the metric unit ball.

Our blow-up method is applied to \(G\)-regular hypersurfaces, see Definition 2.1, that play the role of \(C^1\) smooth hypersurfaces, although they might be very far from being rectifiable in the classical sense, [33]. On the other hand, these hypersurfaces can be used to introduce an adapted notion of rectifiability, called \(G\)-rectifiability. These notions have been introduced and studied in a series of papers by Franchi, Serapioni and Serra Cassano, [24], [25], [26], see Section 4.

The special graph structure of \(G\)-regular hypersurfaces allows us to overcome both the possible Euclidean unrectifiability of \(FHE\) and the absence of B.C.T. These facts are summarized in the first result of this work.

**Theorem 1.2** (Upper blow-up). Let \(\Sigma\) be a parametrized \(G\)-regular hypersurface and let \(x \in \Sigma\) and let \(\sigma_\Sigma\) be its associated perimeter measure. If the homogeneous distance \(d\) is \((n − 1)\)-vertically regular, then we have

\[
\theta^{Q-1}(\sigma_\Sigma, x) = \beta(d, \nu_\Sigma(x)).
\]

The number \(\beta(d, \nu_\Sigma(x))\) is the maximal area of the intersection between a vertical hyperplane orthogonal to \(\nu_\Sigma(x)\) and the metric unit ball, see Definition 2.2. The computation of \(\theta^{Q-1}(\sigma_\Sigma, x)\) essentially requires a “non-centered blow-up limit” at \(x\), that to the author’s knowledge seems to appear for the first time in the theory of finite perimeter sets. This corresponds to compute the minimal upper bound for

\[
\frac{|\partial_H E| (B(y, r))}{r^{Q-1}}
\]

as \(y\) varies in \(B(x, r)\) and \(r \to 0^+\). This limit superior yields a blow-up with varying centers in such a way that the quotient of (6) is maximized in the limit process. We consider the maximizing sequence of “blow-up centers” \((y_k)\) and an infinitesimal sequence \((t_k)\) with \(y_k \in B(x, t_k)\) that in the limit of (6) give \(\theta^{Q-1}(\sigma_\Sigma, x)\). Then we exploit both the weak regularity of \(G\)-regular hypersurfaces and their special graph structure to establish the first of the following inequalities

\[
\theta^{Q-1}(\sigma_\Sigma, x) \leq H^{n-1}(N_x \cap B(z, 1)) \leq \beta(d, \nu_\Sigma(x)),
\]

where, up to considering a subsequence, we have \(\delta_{1/t_k}(x^{-1}y_k) \to z \in B(0, 1)\) and \(N_x\) is the intrinsic blow-up of \(\Sigma\) at \(x\), that is a vertical subgroup of \(G\). The Hausdorff measure \(H^{n-1}\) is computed with respect to the underlying Euclidean distance. The last inequality of (7) comes from the definition of \(\beta(d, \nu_\Sigma(x))\) and it leads to one inequality of (5). For the opposite inequality, we select a special curve of blow-up centers \(\gamma(t) = x\delta_t z_0 \in B(x, t)\), where \(z_0\) is the point of maximal intersection, hence

\[
H^{n-1}(B(z_0, 1) \cap N_x) = \beta(d, \nu_\Sigma(x)).
\]
The blow-up along this curve joined with the \((n - 1)\)-vertical regularity of the distance lead eventually to the opposite inequality, proving (5).

The main application of the upper blow-up is the general representation of the perimeter measure \(|\partial H E|\), whenever \(F H E\) is \(G\)-rectifiable.

**Theorem 1.3** (Area formula for the perimeter measure). Let \(G\) be a stratified group and let \(E \subset G\) be an \(h\)-finite perimeter set such that \(F H E\) is \(G\)-rectifiable. If \(d\) is an \((n - 1)\)-vertically regular distance, then we have

\[
|\partial H E| = \beta(d, \nu E) S^{n - 1}_0 \cap F H E.
\]

The first aspect related to the proof of Theorem 1.3 is the important role played by the \(G\)-rectifiability of \(F H E\), although this notion is completely independent of the classical Euclidean rectifiability. In our hypothesis, we have to assume that \(F H E\) is \(G\)-rectifiable, since this fact is still an important open question for an arbitrary stratified group \(G\). Only for two step groups and some special classes of higher step groups the \(G\)-rectifiability of all reduced boundaries is known, [26], [39]. In fact, for general groups we do not known whether the rescaled sets

\[
E_{x,r} = \delta_1/r(x^{-1}E)
\]

converge to a halfspace for \(|\partial H E|\)-a.e. \(x \in G\). To prove Theorem 1.3 we circumvent this problem by joining different results. The asymptotically doubling property, [4], allows us to differentiate an integral version of the perimeter measure for \(G\)-regular hypersurfaces, [25]. As a consequence, the \(|\partial H E|\)-a.e. existence of the limit giving the density allows us to consider only special sequences of radii \((r_k)\) in (9), hence the \(|\partial H E|\)-a.e. existence of vertical halfspaces as suitable tangents, [5], leads us to the computation of this density. A similar use of the existence of halfspaces among the tangent measures first appeared in [6], to prove the locality of the perimeter measure in general stratified groups.

The proof of (8) also shows an unexpected relationship with the shape of the metric unit ball associated with the homogeneous distance. This is another new phenomenon entailed by our approach. The verification of the \((n - 1)\)-regularity of the SR distance is strictly related to the regularity of the profile of the SR ball and this is in turn related to the regularity of SR geodesics. For instance, if there are no singular SR geodesics, then the profile of the SR unit ball has Lipschitz regularity, [44], hence the SR distance is clearly \((n - 1)\)-vertically regular and (8) holds for this distance. This regularity is an interesting open question for general Carnot groups.

In Theorem 5.2 we prove that whenever the unit ball \(B(0, 1)\) is convex, then the homogeneous distance \(d\) is \((n - 1)\)-vertically regular and there holds

\[
\beta(d, v) = \mathcal{H}^{n - 1}(N(v) \cap B(0, 1)).
\]

This provides a more manageable formula for the density. The key to prove this result is an interesting concavity-type property for the area of the intersections of parallel
hyperplanes with a convex set, that was proved by Busemann, [11], see Theorem 5.1. Let us point out that SR balls in general are not convex sets. This happens in the Heisenberg group, see for instance [8], but it is a more general phenomenon.

We also observe that homogeneous distances where $\beta(d, \cdot)$ is a constant function are clearly more convenient. In fact, in this case we can set $\omega_{G, Q-1} = \beta(d, \cdot)$, where $\omega_{G, Q-1}$ plays the role of $\omega_{n-1}$ in the Euclidean space, and we may define the spherical Hausdorff measure $S_{G}^{Q-1}$ by setting $c_{Q-1} = \omega_{G, Q-1}/2^{Q-1}$ in Definition 3.1. We identify these special distances as $V_1$-vertically symmetric distances, see Definition 6.1, that precisely make $\beta(d, \cdot)$ a constant function, see Theorem 6.1. This theorem joined with the previous definitions leads us to a more manageable area formula.

**Theorem 1.4** (Area formula for symmetric distances). Let $G$ be a stratified group equipped with an $(n-1)$-vertically regular distance $d$, which is $V_1$-vertically symmetric, and let $E \subset G$ be an $h$-finite perimeter set. If $\mathcal{F}_H E$ is $G$-rectifiable, then

$$|\partial_H E| = S_{G}^{Q-1} \cap \mathcal{F}_H E. \quad (11)$$

In each stratified group, it is possible to find a homogeneous distance whose unit ball $B(0, 1)$ coincides with an Euclidean ball of sufficiently small radius, see Theorem 2 of [31], hence this distance satisfies the hypotheses of Theorem 1.4 and we can define the associated measure $S_{G}^{Q-1}$. These hypotheses are also satisfied by the Korányi-Cygan distance in all H-type groups, that is also $V_1$-vertically symmetric, see Section 6. Formula (11) clearly includes the case where $G$ is the standard Euclidean space, where our method provides a new approach also with respect to the classical one.

A formula for $\beta$ in (2) appeared in [24], under the assumption that the homogeneous distance is $d_\infty$, see Example 5.4, and the ambient space is an Heisenberg group. It is easy to observe that $d_\infty$ is $V_1$-vertically symmetric and its unit ball is a convex set, hence in the Heisenberg group (11) yields the known formula. For a general homogeneous distance our approach provides the new arguments to establish the relationship between the geometry of the distance and the corresponding area formula. For instance, in two step groups (8) corrects the corresponding area formula of [26].

In those cases where the SR distance is $(n-1)$-vertically regular, the complicated shape of the metric ball does not make easy to establish whether the maximal area $\beta(d_c, v)$ satisfies (10). There are no special reasons to expect this equality to hold in general. In fact, for intersections of the SR ball by vertical lines, the corresponding version of (10) fails to hold, [37].

We may say that whenever explicit formulae for the profile of the SR ball are known, then (8) holds. For instance, this is the case of the Heisenberg group, where one can check that the SR distance is both $V_1$-vertically symmetric and 2-vertically regular. Incidentally, this example shows how the SR distance can provide an example of homogeneous distance that is $V_1$-vertically symmetric, 2-vertically regular, but whose metric unit ball is not convex.
2. Notation, terminology and basic facts

A stratified group can be seen as a graded linear space \( G = V_1 \oplus \cdots \oplus V_i \) with graded Lie algebra \( G = V_1 \oplus \cdots \oplus V_i \), satisfying the properties \( [V_1, V_j] = V_{j+1} \) for all integers \( j \geq 0 \), where \( V_i = \{0\} \) for all \( i > i \) and \( V_i \neq \{0\} \). This terminology is due to Folland, [22]. Stratified groups equipped with a sub-Riemannian distance are also well known as Carnot groups, according to the terminology introduced in a 1989 paper by Pansu.

A homogeneous distance \( d \) on \( G \) is a continuous and left invariant distance with \( d(\delta_r x, \delta_r y) = r d(x, y) \) for all \( x, y \in G \) and \( r > 0 \). We define the open and closed balls

\[
B(y, r) = \{ z \in G : d(z, y) < r \} \quad \text{and} \quad B(y, r) = \{ z \in G : d(z, y) \leq r \}.
\]

The corresponding homogeneous norm is denoted by \( \|x\| = d(x, 0) \) for all \( x \in G \). When we wish to stress that the stratified group is equipped with a homogeneous distance we may use the terminology homogeneous stratified group, although often the homogeneous distance will be understood. In fact, throughout a homogeneous distance will be fixed and \( \Omega \) will denote an open subset of a stratified group \( G \). Notice that any homogeneous distance is bi-Lipschitz equivalent to the SR distance.

In our terminology, a \( C^1_h \) smooth function \( f : \Omega \to \mathbb{R} \) on an open set \( \Omega \) of a stratified group \( G \) has the property that for all \( x \in \Omega \) and \( X \in V_1 \) the horizontal derivative

\[
Xf(x) = \lim_{t \to 0} \frac{f(\Phi_t^X(x)) - f(x)}{t}
\]

exists and it is continuous in \( \Omega \), where \( \Phi_t^X \) denotes the flow of \( X \). We denote by \( C^1_h(\Omega) \) the linear space of all \( C^1_h \) smooth functions on \( \Omega \).

We also define \( d_h f(x) : H_x^1 G \to \mathbb{R} \) as follows

\[
d_h f(x)(w) = Wf(x),
\]

where \( w \in H_x^1 G \) and \( W \in V_1 \) is the unique left invariant vector field such that \( W(x) = w \). For every \( j = 1, \ldots, i \), we have also defined

\[
H_x^j G = \{ X(x) \in T_x G \mid X \in V_j \}.
\]

The class of \( C^1_h \) smooth functions has an associated implicit function theorem.

**Theorem 2.1** (Implicit function theorem). Let \( x \in \Omega \), \( X \in V_1 \) and \( f \in C^1_h(\Omega) \) with \( Xf(x) \neq 0 \). Let \( N_1 \subset V_1 \) be the kernel of \( d_h f(x) \), let \( N = N_1 \oplus V_2 \oplus \cdots \oplus V_i \) and let \( H = \mathbb{R} v_X \), where \( v_X = \exp X \). Then we have an open set \( V \subset N \) with \( 0 \in V \), a continuous function \( \varphi : V \to H \) and an open neighbourhood \( U \subset G \) of \( x \), such that

\[
f^{-1}(f(x)) \cap U = \{ xn \varphi(n) \mid n \in V \}.
\]

This result is an immediate consequence of the Euclidean implicit function theorem, once the proper system of coordinates is fixed. It shows that regular level sets of \( C^1_h \) smooth functions are locally graphs with respect to the group operation, [25]. A more general implicit function theorem for mappings between two stratified groups \( G \) and
M can be also obtained. The new algebraic and topological difficulties of this case are partially overcome using the topological degree and assuming special algebraic factorizations of the source space. Level sets of these mappings define the general class of \((G, M)\)-regular sets of \(G\), see [35] and [36] for more information. For the purposes of this work, the next definition refers to the case \(M = \mathbb{R}\). In this special case, these sets have first appeared in [25] and called \(G\)-regular hypersurfaces.

**Definition 2.1.** We say that a subset \(\Sigma \subset G\) is a parametrized \(G\)-regular hypersurface if there exists \(f \in C^1_h(\Omega)\) such that \(d_h f\) is everywhere nonvanishing and \(\Sigma = f^{-1}(0)\). A \(G\)-regular hypersurface \(S \subset G\) has a countable open covering such that the intersection of \(S\) with each element of the covering is a parametrized \(G\)-regular hypersurface.

A graded basis \((e_1, \ldots, e_n)\) of \(G\) is defined by assuming that the families of vectors
\[
(e_{m_j-1+1}, e_{m_j-1+2}, \ldots, e_{m_j})
\]
are bases of the subspaces \(V_j\) and \(m_j = \sum_{i=1}^j \dim V_i\) for every \(j = 1, \ldots, \iota\), where \(m_0 = 0\). We also set \(m = m_1\).

In the sequel, a graded basis is fixed and the corresponding Lebesgue measure \(\mathcal{L}^n\) is automatically defined on \(G\). Its left invariance makes it proportional to the Haar measure \(\mu\) of \(G\). We fix an auxiliary scalar product on \(G\), whose restriction to \(V_1\) defines the left invariant sub-Riemannian metric \(g\) on each horizontal fiber \(H_x G\). In particular, this scalar product is chosen to make the previous graded basis orthonormal. We denote by \(|\cdot|\) the associated norm. The choice of this scalar product for instance appears in Definition 2.2, where we use the Euclidean \(n - 1\) dimensional Hausdorff measure \(\mathcal{H}^{n-1}\).

For each parametrized \(G\)-regular hypersurface \(\Sigma \subset G\) defined by \(f : \Omega \to \mathbb{R}\), we may define the intrinsic measure of \(\Sigma\) through the perimeter measure
\[
\sigma_\Sigma(O) = \sup \left\{ \int_{E_f} \text{div} X \, d\mu \mid X \in C^1_c(O, H G), |X| \leq 1 \right\}
\]
for any open set \(O \subset \Omega\), where \(E_f = \{x \in \Omega : f(x) < 0\}\). The symbol \(\text{div}\) denotes the divergence operator with respect to the Haar measure \(\mu\), that in our coordinate system fives the standard divergence operator. The horizontal normal for a parametrized \(G\)-regular hypersurface \(\Sigma\) with defining function \(f\) is defined for each \(y \in \Sigma\) as follows
\[
\nu_\Sigma(y) = \frac{\nabla_H f(y)}{|\nabla_H f(y)|},
\]
where \((X_1, \ldots, X_m)\) is an orthonormal basis of \(V_1\) and \(\nabla_H f = (X_1 f, \ldots, X_m f)\). For our purposes we do not claim an orientation, hence a sign, for the horizontal normal. The following definition requires a little regularity of the unit ball.
Definition 2.2 (Vertical subgroups and regular distances). For each \( \nu \in V_1 \setminus \{0\} \), we define its corresponding vertical subgroup \( N(\nu) = \nu^\perp \oplus V_2 \oplus \cdots \oplus V_t \), where \( \nu^\perp \) is the subspace of \( V_1 \) that is orthogonal to \( \nu \). Then we define \( \beta(d, \nu) = \max_{z \in B(0,1)} H^{n-1}(B(z,1) \cap N(\nu)) \).

A homogeneous distance \( d \) is \((n-1)\)-vertically regular if for each \( \nu \in V_1 \setminus \{0\} \) there exists an element \( z(\nu) \in B(0,1) \) such that
\[
(15) \quad H^{n-1}(N(\nu) \cap \partial B(z(\nu),1)) = 0 \quad \text{and} \quad \beta(d, \nu) = H^{n-1}(B(z(\nu),1) \cap N(\nu)).
\]

3. Upper blow-up of the perimeter measure

In this section we prove the one of the central results of this work, namely, a new type of blow-up theorem for the perimeter measure. We recall here the standard notion of spherical Hausdorff measure.

Definition 3.1. Let \( \mathcal{F}_b \subset \mathcal{P}(G) \) a class of closed balls and let \( \zeta_{b,\alpha} : S \to [0, +\infty] \) represent the size function on metric balls \( \zeta_{b,\alpha}(B) = c_{\alpha} \operatorname{diam}(B)^\alpha \), where \( \alpha > 0 \) and \( c_\alpha \) is a geometric constant to be fixed. If \( \delta > 0 \) and \( R \subset X \), then we define
\[
\phi_\delta(R) = c_\alpha \inf \left\{ \sum_{j=0}^{\infty} \operatorname{diam}(B_j)^\alpha : B_j \in \mathcal{F}_b, \ \operatorname{diam}(B_j) \leq \delta \text{ for all } j \in \mathbb{N}, R \subset \bigcup_{j \in \mathbb{N}} B_j \right\}.
\]
The \( \alpha \)-dimensional spherical Hausdorff measure is defined by
\[
S^\alpha(E) = \sup_{\delta > 0} \phi_\delta(E),
\]
for every \( E \subset G \). The spherical Hausdorff measure with no geometric constant is denoted by \( S^\alpha_0 \) and it corresponds to the special case where \( c_\alpha = 2^{-\alpha} \).

Proof of Theorem 1.2. We consider \( f \) as the defining function of \( \Sigma \) and select \( X_1 \in \mathcal{V} \) such that \( X_1(x) \) has unit length and it is orthogonal to the kernel of \( d_Hf(x) \). Let \( X_2, \ldots, X_m \in \mathcal{V}_1 \) be such that \( (X_2(x), \ldots, X_m(x)) \) is an orthonormal basis of this kernel. By Theorem 2.1, we have an open neighbourhood \( V \subset N \) of the origin, with \( N = \ker d_Hf(x) \oplus V_2 \oplus \cdots \oplus V_t \), a continuous function \( \varphi : V \to \mathbb{R} \), the vector \( v_{X_1} = \exp X_1 \in G \) and an open neighbourhood \( U \subset G \) of \( x \), such that
\[
(16) \quad \Sigma \cap U = \left\{ xn(\varphi(n)e_{X_1}) \mid n \in V \right\}.
\]
Up to changing the sign of \( f \) and possibly shrinking \( U \), we assume that \( X_1f > 0 \) bounded away from zero on \( U \). To define the intrinsic measure of \( \Sigma \), we define the open set
\[
E_f = \{ y \in U \mid f(y) < 0 \} = \{ xn(te_{X_1}) \in U \mid n \in V, \ t < \varphi(n) \}.
\]
From [25], this set has finite perimeter and defining the graph mapping \( \Phi : V \to \Sigma \) by \( \Phi(n) = xn(\varphi(n)e_{X_1}) \) for every \( n \in V \), we also have the formula

\[
\sigma_{\Sigma}(B(y,t)) = |\partial E|_{H}(B(y,t)) = \int_{\Phi^{-1}(B(y,t))} \frac{\sqrt{\sum_{j=1}^{m} X_j f(\Phi(n))^2}}{X_1 f(\Phi(n))} \, dH^{n-1}(n)
\]

for \( t > 0 \) small and \( y \in U \). We make the change of variables \( n = \Lambda_{t}\eta \), where \( \Lambda_{t} : N \to N \) and

\[
\Lambda_{t} = \sum_{j=2}^{m} t \eta_j e_j + \sum_{i=2}^{m} \sum_{j=m_{i-1}+1}^{m_i} t^i \eta_j e_j.
\]

We notice that \( \delta_t |_{N} = \Lambda_{t} \) and the Jacobian of \( \Lambda_{t} \) is \( t^{Q-1} \), where \( Q \) is the Hausdorff dimension of \( G \). By a change of variables, we get

\[
\sigma_{\Sigma}(B(y,t)) = t^{Q-1} \int_{\Lambda_{1/t}(\Phi^{-1}(B(y,t)))} \frac{\sqrt{\sum_{j=1}^{m} X_j f(\Lambda_{t}\eta)^2}}{X_1 f(\Lambda_{t}\eta)} \, dH^{n-1}(\eta).
\]

The general definition of spherical Federer’s density, [37], in our setting gives

\[
\theta^{Q-1}(\sigma_{\Sigma}, x) = \inf_{r > 0} \sup_{0 < t < r} \frac{\sigma_{\Sigma}(B(y,t))}{t^{Q-1}}.
\]

In view of (17), to find \( \theta^{Q-1}(\sigma_{\Sigma}, x) \) we first observe that the sets

\[
\Lambda_{1/t}(\Phi^{-1}(B(y,t))) = \{ \eta \in \Lambda_{1/t}V \mid (\delta_{1/t}(y^{-1}x)) \eta \left( \frac{\varphi(\delta_{t}\eta)}{t}e_{X_1} \right) \in B(0, 1) \}
\]

are uniformly bounded uniformly with respect to \( t \). Then for \( t > 0 \) sufficiently small we have a compact set \( K_0 \) such that

\[
\Lambda_{1/t}(\Phi^{-1}(B(y,t))) \subset K_0.
\]

The first consequence of this inclusion is that \( \theta^{Q-1}(\sigma_{\Sigma}, x) < +\infty \), hence there exist a sequence \( \{t_k\} \subset (0, +\infty) \) converging to zero and a sequence of elements \( y_k \in B(x, t_k) \) such that

\[
\theta^{Q-1}(\sigma_{\Sigma}, x) = \lim_{k \to \infty} \int_{\Lambda_{1/t_k}(\Phi^{-1}(B(y_k,t_k)))} \frac{\sqrt{\sum_{j=1}^{m} X_j f(\Lambda_{t_k}\eta)^2}}{X_1 f(\Lambda_{t_k}\eta)} \, dH^{n-1}(\eta).
\]

Possibly extracting a subsequence, there exists \( z \in B(0, 1) \) such that

\[
\delta_{1/t_k}(y_k^{-1}x) \to z^{-1} \in B(0, 1).
\]

Setting \( S_z = N \cap B(z, 1) \), we wish to show that for each \( w \in N \setminus S_z \) there holds

\[
\lim_{k \to \infty} \mathbf{1}_{A_k}(w) = 0,
\]
where $A_k = \Lambda_{1/t_k}(\Phi^{-1}(B(y_k, t_k)))$. For this, we have to prove that

\begin{equation}
\lim_{t \to 0^+} \frac{\varphi(\delta_t w)}{t} = 0.
\end{equation}

Since $\varphi$ is only continuous, this makes the proof of this limit more delicate. We define

\[ A(w) = \left\{ t \in \mathbb{R} \mid t > 0, \varphi(\delta_t w) \neq 0 \right\}. \]

If $A(w)$ does not contain zero, the limit (23) becomes obvious. If $0 \in A(w)$, then we choose an arbitrary infinitesimal sequence $\{\tau_k\} \subset A(w)$. Using the stratified mean value inequality in (1.41) of [23], we notice that

\begin{equation}
\lim_{k \to \infty} f(x\delta_{\tau_k} w) \frac{\varphi(\delta_{\tau_k} w)}{\tau_k} = 0.
\end{equation}

Since $\varphi(\delta_{\tau_k} w) \neq 0$, we can multiply and divide by $\varphi(\delta_{\tau_k} w)$, getting

\begin{equation}
0 = \lim_{k \to \infty} f(x\delta_{\tau_k} w) \frac{\varphi(\delta_{\tau_k} w)}{\tau_k} = \left( -X_1 f(x) \right) \lim_{k \to \infty} \frac{\varphi(\delta_{\tau_k} w)}{\tau_k}.
\end{equation}

This proves (23), hence (22) follows. We consider the following integral as the sum

\[
\int_{A_k} \sqrt{\sum_{j=1}^m X_j f(\Phi(\Lambda t_k \eta))^2} d\mathcal{H}^{n-1}(\eta) = I_k + J_k,
\]

where, introducing the density function

\[ \alpha(t, \eta) = \left( X_1 f(\Phi(\Lambda t \eta)) \right)^{-1} \sqrt{\sum_{j=1}^m X_j f(\Phi(\Lambda t \eta))^2}, \]

we have set

\[ I_k = \int_{A_k \cap S_z} \alpha(t_k, \eta) d\mathcal{H}^{n-1}(\eta) \quad \text{and} \quad J_k = \int_{A_k \setminus S_z} \alpha(t_k, \eta) d\mathcal{H}^{n-1}(\eta). \]

In principle, when $w \in N \cap \partial B(z, 1)$ we do not have information on the limit of $1_{A_k \cap S_z}(w)$ as $k \to \infty$. Taking into account (19), this depend on the geometry of $x^{-1}\Sigma \cap B(0, 1)$. However, in this step we wish to prove only one inequality. Then we consider the following inequality

\begin{equation}
I_k \leq \int_{S_z} \alpha(t_k, \eta) d\mathcal{H}^{n-1}(\eta).
\end{equation}

Due to (20), we have

\[ J_k \leq \int_{K_0 \setminus S_z} 1_{A_k}(\eta) \alpha(t_k, \eta) d\mathcal{H}^{n-1}(\eta), \]
the integrand goes to zero as $k \to \infty$ due to (22) and it is uniformly bounded, then we can apply Lebesgue’s convergence theorem, proving that $J_k \to 0$. Again Lebesgue’s theorem gives

$$\lim_{k \to \infty} \int_{S_z} \alpha(t, \eta) \, d\mathcal{H}^{n-1}(\eta) = \mathcal{H}^{n-1}(S_z).$$

In fact, $X_j f(0) = 0$ for $j = 2, \ldots, m$, hence $\alpha(t, \eta) \to 1$ as $k \to \infty$. This gives

$$\theta^{Q-1}(\sigma, x) = \lim_{k \to \infty} \int_{A_k} \frac{\sqrt{\sum_{j=1}^m X_j f(\Lambda_{t, \eta})^2}}{X_1 f(\Lambda_{t, \eta})} \, d\mathcal{H}^{n-1}(\eta) \leq \mathcal{H}^{n-1}(S_z) \leq \mathcal{H}^{n-1}(S_{z_0}),$$

where the vertical regularity of $d$ allows us to choose $z_0 \in \mathbb{B}(0, 1)$ such that

$$(26) \quad \mathcal{H}^{n-1}(S_{z_0}) = \beta(d, \nu_{\Sigma}(x)) \quad \text{and} \quad \mathcal{H}^{n-1}(N \cap \partial \mathbb{B}(z_0, 1)) = 0,$$

where $N = N(\nu_{\Sigma}(x))$. To prove the opposite inequality, we select $y^0_t = x \delta_t z_0 \in \mathbb{B}(x, t)$ and observe that

$$\sup_{0 < t < r} \frac{\mathcal{H}(y^0_t, t)}{t^{Q-1}} \leq \sup_{0 < t < r} \frac{\mathcal{H}(y, t)}{t^{Q-1}},$$

for every $r > 0$, therefore the definition of spherical Federer density (18) yields

$$\limsup_{t \to 0^+} \frac{\mathcal{H}(y^0_t, t)}{t^{Q-1}} \leq \theta^{Q-1}(\sigma, x).$$

Taking into account (19), we get

$$(27) \quad A^0_t = \Lambda_{1/t} \left( \Phi^{-1}(\mathbb{B}(y^0_t, t)) \right) = \left\{ \eta \in \Lambda_{1/t} V \mid \eta \left( \frac{\varphi(\delta_t \eta)}{t} e_{X_1} \right) \in \mathbb{B}(z_0, 1) \right\},$$

that implies

$$\int_{N \cap B(z_0, 1)} 1_{A^0_t}(w) \alpha(t, \eta) \, d\mathcal{H}^{n-1}(\eta) \leq \int_{A^0_t} \alpha(t, \eta) \, d\mathcal{H}^{n-1}(\eta) = \frac{\mathcal{H}(y^0_t, t)}{t^{Q-1}}.$$

Since $\alpha(t, \eta)$ is well defined and bounded on $[0, \varepsilon] \times (N \cap \mathbb{B}(z_0, 1))$, for $\varepsilon$ sufficiently small, and taking into account that

$$\lim_{t \to 0^+} 1_{A^0_t}(w) = 1$$

for all $w \in N \cap B(z_0, 1)$, Lebesgue’s convergence yields

$$\mathcal{H}^{n-1}(N \cap B(z_0, 1)) = \limsup_{t \to 0^+} \int_{N \cap B(z_0, 1)} 1_{A^0_t}(w) \alpha(t, \eta) \, d\mathcal{H}^{n-1}(\eta) \leq \theta^{Q-1}(\sigma, x).$$

The equalities of (26) give the opposite inequality, hence concluding the proof. □
4. **Area formulae for the perimeter measure**

This section is devoted to establish the general relationship between perimeter measure and spherical Hausdorff measure for the class of \((n-1)\)-vertically regular distances. Joining Theorem 1.2 with Theorem 1.1, we obtain the following result.

**Theorem 4.1 (Area formula).** Let \(\Sigma\) be a parametrized \(G\)-regular hypersurface and let \(\sigma_\Sigma\) be its associated perimeter measure \((13)\). If \(d\) is an \((n-1)\)-vertically regular distance, then

\[
\sigma_\Sigma = \beta(d, \nu_\Sigma) S_0^{Q-1} \cup \Sigma.
\]

This theorem also yields an area formula for the perimeter measure. To present this result we introduce a few more definitions. A subset \(S \subset G\) is \(G\)-rectifiable if there exists a countable family \(\{\Sigma_j \mid j \in \mathbb{N}\}\) of parametrized \(G\)-regular hypersurfaces \(\Sigma_j\) such that

\[
S_0^{Q-1}(S \setminus \bigcup \Sigma_j) = 0.
\]

A measurable set \(E \subset G\) has \(h\)-finite perimeter if

\[
\sup \left\{ \int_E \text{div } X \, d\mu \bigg| X \in C_c^1(G, H G), |X| \leq 1 \right\} < +\infty.
\]

This allows for defining a finite Radon measure \(|\partial H E|\) on \(G\), see for instance [7]. The **reduced boundary** \(F_{HE}\) is the set of points \(x \in G\) such that there exists \(\nu_E(x) \in H_0 G\) with \(|\nu_E(x)| = 1\) and

\[
\lim_{r \to 0^+} \frac{1}{|\partial H E|(B(x, r))} \int_{B(x, r)} \nu_E(y) \, d|\partial H E|(y) = \nu_E(x),
\]

where \(\nu_E\) is the generalized inward normal of \(E\), see [26] for more information.

**Proof of Theorem 1.3.** Since the perimeter measure is asymptotically doubling, the perimeter measure \(|\partial H E|\) on \(G\) is concentrated on the reduced boundary \(F_{HE}\), see [4]. Moreover, the \(G\)-rectifiability of \(F_{HE}\) implies the existence of a countable family \(\{\Sigma_j \mid j \in \mathbb{N}\}\) of parametrized \(G\)-regular hypersurfaces \(\Sigma_j\) such that

\[
S_0^{Q-1}(F_{HE} \setminus \bigcup \Sigma_j) = 0.
\]

On each \(\Sigma_j\) we define the Radon measure

\[
\mu_j = \beta(d, \nu_{\Sigma_j}) S_0^{Q-1} \cup \Sigma_j,
\]

hence the area formula of Theorem 4.1 gives \(\mu_j = \sigma_{\Sigma_j}\). The argument of the upper blow-up theorem clearly simplifies in the case of the following centered blow-up, giving

\[
\lim_{r \to 0^+} \frac{\sigma_{\Sigma_j}(B(y, r))}{r^{Q-1}} = \mathcal{H}^{n-1}(N(\nu_{\Sigma_j}(y)) \cap B(0, 1))
\]
for all $y \in \Sigma_j$. From Theorem 1.2 of [5], there exists $R_E \subset \mathcal{F}_H E$ such that

$$|\partial_H E|(\mathcal{F}_H E \setminus R_E) = 0$$

and for every $x \in R_E$ the following vertical halfspace

$$Z(\nu_E(x)) = \{ v \in \mathbb{G} \mid v = v_1 + v_0, v_1 \in V_1, v_0 \in V_2 \oplus \cdots \oplus V_\iota, \langle v_1, \nu_E(x) \rangle < 0 \}$$

belongs to $\text{Tan}(E, x)$, see Definition 6.3 of [5]. Since $|\partial_H E|$ is asymptotically doubling, [4], the perimeter measure $|\partial_H E|$ can differentiate $\mu_j$, hence we can define

$$(30) \quad \sigma_E(x) = \lim_{r \to 0^+} \frac{\mu_j(B(x, r))}{|\partial_H E|(B(x, r))}$$

for each $x \in R^1_E \subset R_E \cap \Sigma_j$, where $|\partial_H E|((R_E \cap \Sigma_j) \setminus R^1_E) = 0$. If $x \in R^1_E \subset R_E \cap \Sigma_j$, then $Z(\nu_E(x)) \in \text{Tan}(E, x)$ and there exists an infinitesimal positive sequence $(r_k)$ of radii such that

$$(31) \quad |\partial_H E_{x,r_k}|(B(0,1)) \longrightarrow |\partial_H Z(\nu_E(x))|(B(0,1)) \quad \text{as} \quad k \to \infty.$$ 

The boundary of $Z(\nu_E(x))$ is $N(\nu_E(x))$, that is a $\mathbb{G}$-regular hypersurface, hence

$$(32) \quad |Z(\nu_E(x))|(B(0,1)) = \mathcal{H}^{n-1}(N(\nu_E(x)) \cap B(0,1)).$$

Joining (29), (31) and (32), we obtain that $\sigma_E(x) = 1$, hence (30) immediately leads us to the conclusion. \hfill \square

5. Vertical sections of convex homogeneous balls

In this section we study those homogeneous distances with convex unit ball. The following classical result of convex geometry will play a key role.

**Theorem 5.1** ([11]). Let $H$ be an $n$-dimensional Hilbert space with $n \geq 2$ and let $C$ be a compact convex set with nonempty interior that contains the origin. Let $v \in H \setminus \{0\}$ and let $N$ denote the orthogonal space to $v$. Then the function

$$\psi(t) = \left[\mathcal{H}^{n-1}(C \cap (tv + N))\right]^{1/(n-1)}$$

is concave on the interval $\{ t \in \mathbb{R} : C \cap (tv + N) \neq \emptyset \}$.

Thus, we are in the position to establish the result of this section.

**Theorem 5.2.** If $d$ is a homogeneous distance such that the corresponding unit ball $B(0,1)$ is convex and $v \in V_1 \setminus \{0\}$, then $d$ is $(n - 1)$-vertically regular and there holds

$$(33) \quad \beta(d, v) = \mathcal{H}^{n-1}(N(v) \cap B(0,1)).$$
Proof. Let us set \( N = N(v) \), where \( N(v) \) is the vertical subgroup, see Definition 2.2. The Euclidean Jacobian of the translation \( \tau : N \rightarrow zN \) is one for all \( z \in G \), hence

\[
\mathcal{H}^{n-1}(N \cap B(z, 1)) = \mathcal{H}^{n-1}(B(0, 1) \cap z^{-1}N).
\]

To study the previous function with respect to \( z \), we introduce

\[
a(z) = \mathcal{H}^{n-1}(B(0, 1) \cap zN).
\]

Defining \( H = \mathbb{R}v \), we have two canonical projections \( \pi_1 : G \rightarrow H \) and \( \pi_2 : G \rightarrow N \) such that \( y = \pi_1(y)\pi_2(y) \) for all \( y \in G \), see Proposition 7.6 of [36]. Since \( H \) is a horizontal subspace, one can also check that \( \pi_1 : G \rightarrow H \) is precisely the linear projection onto \( H \) with respect to the direct sum of linear spaces \( H \oplus N = G \). As a consequence,

\[
a(z) = \mathcal{H}^{n-1}(B(0, 1) \cap zN) = \mathcal{H}^{n-1}(B(0, 1) \cap \pi_1(z)N).
\]

Furthermore, \( \pi_1(z)N = \pi_1(z) + N \) and \( B(0, 1)^{-1} = -B(0, 1) \), therefore

\[
a(z) = \mathcal{H}^{n-1}(B(0, 1) \cap (\pi_1(z) + N)) = \mathcal{H}^{n-1}(B(0, 1) \cap (-\pi_1(z) + N))
\]

and the property \(-\pi_1(z) = \pi_1(z^{-1})\) yields

\[
a(z) = \mathcal{H}^{n-1}(B(0, 1) \cap (\pi_1(z^{-1}) + N)) = \mathcal{H}^{n-1}(B(0, 1) \cap (\pi_1(z^{-1})N)).
\]

Thus, by (34) we get \( a(z) = \mathcal{H}^{n-1}(B(0, 1) \cap (z^{-1}N)) = a(z^{-1}) = a(-z) \), hence \( a \) is an even function. For every \( t \in \mathbb{R} \) we may define the function

\[
b(t) = \left[ \mathcal{H}^{n-1}(B(0, 1) \cap (tv + N)) \right]^{1/(n-1)}
\]

By Theorem 5.1, the function \( b(t) = \sqrt[n]{a(tv)} \) is concave and even on the compact interval

\[
I = \{ t \in \mathbb{R} : B(0, 1) \cap (tv + N) \neq \emptyset \},
\]

hence we get

\[
\beta(d, v) = \max_{z \in \partial B(0, 1)} \mathcal{H}^{n-1}(B(z, 1) \cap N(v)) = \mathcal{H}^{n-1}(N(v) \cap \partial B(0, 1)).
\]

Being \( N(v) \cap \partial B(0, 1) \) locally parametrized by Lipschitz mappings on an \((n - 2)\) dimensional open set, then we obviously have \( \mathcal{H}^{n-1}(N(v) \cap \partial B(0, 1)) = 0 \), concluding the proof.

In any general homogeneous group we can always find a homogeneous distance with convex unit ball, [31]. The next examples provide other distances with this property.

Example 5.3. Let \( N = V_1 \oplus V_2 \) be an H-type group. We have an explicit formula for a homogeneous distance \( d(x, y) = \|x^{-1}y\| \) such that

\[
\|x\| = \sqrt{|x_1|^4 + 16|x_2|^2}
\]
where \( x, y \in N \), \( x = x_1 + x_2 \) and \( x_i \in V_i \) for \( i = 1, 2 \), see [14]. The unit ball with respect to this distance is clearly a convex set.

**Example 5.4.** Let \( G = V_1 \oplus V_2 \oplus \cdots \oplus V_i \) be any stratified group. From the Baker-Campbell-Hausdorff formula it is easy to see the existence of constants \( \varepsilon_j > 0 \), with \( j = 1, \ldots, \iota \) and \( \varepsilon_1 = 1 \), such that setting

\[
\|x\| = \max\{\varepsilon_j |x_j|^{1/j}\}
\]

with \( x_i \in V_i \) for all \( i = 1, \ldots, \iota \), we have actually defined the homogeneous distance \( d_\infty(x, y) = \|x^{-1}y\| \), as it was observed in [26]. The unit ball \( B(0, 1) \) with respect to \( d_\infty \) is clearly a convex set.

### 6. Vertically Symmetric Distances

The next definition introduces those distances whose symmetries allows for having a precise geometric constant in the definition of the spherical Hausdorff measure \( S^d_G \), as discussed in the introduction. Theorem 6.1 below will prove this fact.

**Definition 6.1.** Let \( G \) be a stratified group of topological dimension \( n \), with direct decomposition \( G = V_1 \oplus W \) and \( W = V_2 \oplus \cdots \oplus V_i \). We equip \( G \) by a scalar product that makes \( V_1 \) and \( W \) orthogonal. We consider a family \( F_1 \subset O(V_1) \) that acts transitively on \( V_1 \), where \( O(V_1) \) denotes the group of isometries of \( V_1 \). We set

\[
F_1 = \{T \in GL_n(G) : T_w = \text{Id}_W, T|_{V_1} \in F_1\}.
\]

We denote by \( p_1 : G \rightarrow V_1 \) the orthogonal projection onto \( V_1 \) and set

\[
B(0, 1) = \{y \in G : d(y, 0) \leq 1\},
\]

where \( d \) is a homogeneous distance of \( G \). We say that \( d \) is \( V_1 \)-vertically symmetric if

1. \( p_1(B(0, 1)) = B(0, 1) \cap V_2 = \{h \in V_1 : |h| \leq r_0\} \) for some \( r_0 > 0 \),
2. \( T(B(0, 1)) = B(0, 1) \) for all \( T \in F_1 \),

It is not difficult to observe that the distances of Example 5.3 and Example 5.4 are both \( V_1 \)-vertically symmetric.

**Theorem 6.1.** If a homogeneous distance \( d \) is \( V_1 \)-vertically symmetric, then \( \beta(d, \cdot) \) is a constant function.

**Proof.** Let \( z \in B(0, 1) \) and choose \( \nu_1, \nu_2 \in V_1 \). Since \( F_1 \) is transitive on \( V_1 \) there exists \( T \in F_1 \) such that \( T(\nu_1) = \nu_2 \). We set \( H = \mathbb{R}\nu_1 \) and see \( G \) as the inner semidirect product between \( H \) and \( N(\nu_1) \). In fact, we have the two canonical projections

\[
\pi_1 : G \rightarrow H \quad \text{and} \quad \pi_2 : G \rightarrow N(\nu_1)
\]

such that \( y = \pi_1(y)\pi_2(y) \) for all \( y \in G \). We set \( \pi_1(z^{-1}) = h \) and \( \pi_2(z^{-1}) = n \), therefore

\[
\mathcal{H}^{n-1}(B(z, 1) \cap N(\nu_1)) = \mathcal{H}^{n-1}(B(0, 1) \cap (h + N(\nu_1)))
\]
By property (2) of Definition 6.1, it follows that 
\[ \mathcal{H}^{n-1}(B(z,1) \cap N(\nu_1)) = \mathcal{H}^{n-1}(B(0,1) \cap (Th + T(N(\nu_1)))) . \]
Since \( \pi_1 \) is the linear projection onto \( H \) with respect to the decomposition \( H \oplus N(\nu_1) \) and \( H \perp N(\nu_1) \), we can write \( z^{-1} \) as the following sum of orthogonal vectors 
\[ w + h' + h, \]
where \( w \in N(\nu_1) \) and \( h, h' \in V_1 \). By property (1) of Definition 6.1, we get 
\[ p_1(z^{-1}) = h' + h \in B(0,1) \cap V_1 = \{ y \in V_1 : |y| \leq r_0 \}. \]
Since \( h \) and \( h' \) are orthogonal, we get \( h \in \{ y \in V_1 : |y| \leq r_0 \} = B(0,1) \cap V_1 \), hence 
\[ T(h) \in B(0,1) \cap V_1. \]
Since \( T \) is orthogonal, \( T(N(\nu_1)) = N(\nu_2) \) and we obtain 
\[ \mathcal{H}^{n-1}(B(z,1) \cap N(\nu_1)) = \mathcal{H}^{n-1}(B(0,1) \cap (Th)''N(\nu_2))). \]
It follows that 
\[ \mathcal{H}^{n-1}(B(z,1) \cap N(\nu_1)) = \mathcal{H}^{n-1}(B(T(h)^{-1},1) \cap N(\nu_2)) \leq \beta(d,\nu_2). \]
The arbitrary choice of \( z \in B(0,1) \) yields \( \beta(d,\nu_1) \leq \beta(d,\nu_2) \). Exchanging the role of \( \nu_1 \) for that of \( \nu_2 \), we conclude the proof. \( \square \)

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