

SHARP DIMENSION FREE QUANTITATIVE ESTIMATES FOR THE GAUSSIAN ISOPERIMETRIC INEQUALITY

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Abstract. We provide a full quantitative version of the Gaussian isoperimetric inequality. Our estimate is independent of the dimension, sharp on the decay rate with respect to the asymmetry and with optimal dependence on the mass.

1. INTRODUCTION

The isoperimetric inequality in Gauss space states that among all sets with a given Gaussian measure the half-space has the smallest Gaussian perimeter. This result was first proved by Borell [7] and independently by Sudakov-Tsirelson [25]. Since then many alternative proofs have been proposed, e.g. [3, 4, 12], but the issue of completely characterizing the extremals was settled only more recently by Carlen-Kerce [9], establishing that half-spaces are the unique solutions to the Gaussian isoperimetric problem.

The natural issue of proving a quantitative version of the isoperimetric inequality turns out to be a much more delicate task. An estimate in terms of the Fraenkel asymmetry, i.e., the Gaussian measure of the symmetric difference between a given set and a half-space, was recently established by Cianchi-Fusco-Maggi-Pratelli [10]. This result provides the sharp decay rate with respect to the Fraenkel asymmetry but with a non-explicit, dimensionally dependent constant. As for the analogous result in the groundbreaking paper in the Euclidean space [15], the proof is purely geometric and is based on a reflection argument in order to reduce the problem to sets which are $(n-1)$ -symmetric. This will cause the constant to blow up at least exponentially with respect to the dimension. However, the fact that in Gauss space most geometric and functional inequalities are independent of the dimension suggests that the quantitative isoperimetric inequality should also be dimension free. This would also be important for possible applications, see [20, 21, 22] and the references therein. Indeed, after [10], Mossel-Neeman [21, 22] and Eldan [13] have provided quantitative estimates which are dimension free but have a sub-optimal decay rate with respect to the Fraenkel asymmetry. It is therefore a natural open problem whether the sharp form of the quantitative isoperimetric inequality holds without dimensional dependence. This was explicitly conjectured in [22] (see also the discussion in [13]).

In this paper we prove this conjecture. Our result is valid not only for the Fraenkel asymmetry but for a stronger one introduced in [13] which measures the difference of the barycenter of a given set from the barycenter of a half-space. Our quantitative isoperimetric inequality is completely explicit, and it is rather striking that it also has the optimal dependence on the mass. The main result is given in terms of the strong asymmetry since in our opinion this is a more natural way to measure the stability of the Gaussian isoperimetric inequality. We will also see that the strong asymmetry appears naturally when one considers an asymmetry which we call the *excess* of the set. This is the Gaussian counterpart of the oscillation asymmetry in the Euclidean setting introduced by Fusco and the third author in [16] (see also [5, 6]).

Subsequent to [15], different proofs in the Euclidean case have been given in [14] (by the optimal transport) and in [1, 11] (using the regularity theory of minimal surfaces and the selection

principle). Both of these strategies are rather flexible and have been adopted to prove many other geometric inequalities in a sharp quantitative form. Nevertheless, they do not seem easily implementable for our purpose. Indeed, it is not known if the Gaussian isoperimetric inequality itself can be retrieved from optimal transport (see [27]). On the other hand, the approach via selection principle is by contradiction. Therefore, if it may be adapted to the Gaussian setting, it cannot be used as it is to provide explicit information about the constant in the quantitative isoperimetric inequality. Finally, the proof in [13] is based on stochastic calculus and provides sharp estimates for the Gaussian noise stability inequality. As a corollary this gives a quantitative estimate for the Gaussian isoperimetric inequality which instead is not sharp. In order to prove the sharp quantitative estimate we introduce a technique which is based on a direct analysis of the first and the second variation conditions of solutions to a suitable minimum problem. It is suprising that this enables us to obtain the sharp result with a very short proof. We will outline the proof at the end of the introduction.

In order to describe the problem more precisely we introduce our setting. Given a Borel set $E \subset \mathbb{R}^n$, $\gamma(E)$ denotes its *Gaussian measure*, defined as

$$\gamma(E) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_E e^{-\frac{|x|^2}{2}} dx.$$

If E is an open set with Lipschitz boundary, $P_\gamma(E)$ denotes its *Gaussian perimeter*, defined as

$$P_\gamma(E) := \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{\partial E} e^{-\frac{|x|^2}{2}} d\mathcal{H}^{n-1}(x). \quad (1)$$

Moreover, given $\omega \in \mathbb{S}^{n-1}$ and $s \in \mathbb{R}$, $H_{\omega,s}$ denotes the half-space of the form

$$H_{\omega,s} := \{x \in \mathbb{R}^n : x \cdot \omega < s\}.$$

We define also the function $\phi : \mathbb{R} \rightarrow (0, 1)$ as

$$\phi(s) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^s e^{-\frac{t^2}{2}} dt.$$

Then we have $\gamma(H_{\omega,s}) = \phi(s)$ and $P_\gamma(H_{\omega,s}) = e^{-s^2/2}$. The isoperimetric inequality states that, given an open set E with Lipschitz boundary and mass $\gamma(E) = \phi(s)$, one has

$$P_\gamma(E) \geq e^{-s^2/2}, \quad (2)$$

and the equality holds if and only if $E = H_{\omega,s}$ for some $\omega \in \mathbb{S}^{n-1}$.

A natural question is the stability of the inequality (2). Let us denote by $D(E)$ the Gaussian isoperimetric deficit (i.e., the gap between the two side of the isoperimetric inequality),

$$D(E) := P_\gamma(E) - e^{-s^2/2},$$

and by $\alpha(E)$ the Fraenkel (or the standard) asymmetry,

$$\alpha(E) := \min_{\omega \in \mathbb{S}^{n-1}} \gamma(E \Delta H_{\omega,s}),$$

where Δ stands for the symmetric difference between sets. As we mentioned, it is proved in [10] that for every set $E \subset \mathbb{R}^n$ with $\gamma(E) = \phi(s)$ the isoperimetric deficit controls the square of the Fraenkel asymmetry, i.e.,

$$\alpha(E)^2 \leq c(n, s) D(E), \quad (3)$$

and the exponent 2 on the left-hand side is sharp. On the other hand, in [22] a similar estimate is proved (for $s = 0$), with a sub-optimal exponent but with a constant independent of the dimension. The following natural conjecture is stated explicitly in [22].

Conjecture. *The inequality (3) holds for a constant $c(s)$ depending only on the mass s .*

In [13] the author introduces a new asymmetry which is equivalent to

$$\beta(E) := \min_{\omega \in \mathbb{S}^{n-1}} |b(E) - b(H_{\omega,s})|, \quad (4)$$

where $b(E) := \int_E x d\gamma$ is the (non-renormalized) barycenter of the set E , and s is chosen such that $\gamma(E) = \phi(s)$. We call this *strong asymmetry* since it controls the standard one as (see Proposition 4)

$$\beta(E) \geq \frac{e^{\frac{s^2}{2}}}{4} \alpha(E)^2. \quad (5)$$

In [13, Corollary 4] it is proved that

$$\beta(E) |\log \beta(E)|^{-1} \leq c(s) D(E)$$

for an unexplicit constant $c(s)$ depending only on s . Together with (5), this proves the conjecture up to a logarithmic factor.

In this paper we prove the conjecture. In fact, we prove an even stronger result, since we provide the optimal quantitative estimate in terms of the strong asymmetry. Our main result then reads as follows.

Main Theorem. *There exists an absolute constant c such that for every $s \in \mathbb{R}$ and for every set $E \subset \mathbb{R}^n$ with $\gamma(E) = \phi(s)$ the following estimate holds*

$$\beta(E) \leq c(1 + s^2) D(E). \quad (6)$$

In Remark 1 we show that the dependence on the mass is optimal. This can be seen by comparing a one-dimensional interval $(-\infty, s)$ with a union of two intervals $(-\infty, -a) \cup (a, \infty)$ with the same Gaussian length. The numerical constant c is explicit, although not optimal. Since the decay rate with respect to the Fraenkel asymmetry in (3) is sharp this implies that the linear dependence on $\beta(E)$ in (6) is optimal.

We may state the result of the main theorem in a more geometrical way. Define for a given (sufficiently regular) set E its *excess* as

$$\mathcal{E}(E) := \min_{\omega \in \mathbb{S}^{n-1}} \left\{ \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{\partial E} |\nu_E - \omega|^2 e^{-\frac{|x|^2}{2}} d\mathcal{H}^{n-1}(x) \right\},$$

where ν_E is the exterior normal of E . In Corollary 1 at the end of section 4 we show that for every set E it holds

$$\mathcal{E}(E) = 2D(E) + 2\sqrt{2\pi}\beta(E).$$

Therefore by the main theorem we conclude that the deficit controls also the excess of the set. Roughly speaking this means that the closer the perimeter of E is to the perimeter of half-space, the flatter its boundary has to be. This is the Gaussian counterpart of the result in [16] for the Euclidean case, and it highlights the importance of the strong asymmetry.

As we already mentioned, the proof of the main theorem is based on a direct variational method. The idea is to write the inequality (6) as a minimization problem

$$\min \left\{ P_\gamma(E) + \frac{\varepsilon}{2} |b(E)|^2 : \gamma(E) = \phi(s) \right\}$$

and deduce directly from the first and the second variation conditions that when $\varepsilon > 0$ is small enough the only solutions are half-spaces. It is not difficult to see that this is equivalent to the statement of the main theorem. We note that for technical reasons we replace the volume constraint by a volume penalization. In section 3 we study the regularity of the solutions to the above problem and derive the Euler equation (i.e. the first variation is zero) and the quadratic form associated with the second variation. The latter is nothing but the second derivative of

the above functional in a direction of a given test function. In section 4 we give the proof of the main theorem. The key point of the proof is step 3 where, with a careful choice of test functions in the second variation condition, we are able to conclude directly that when ε is sufficiently small every minimizer is a union of parallel stripes. Since this is true in every dimension and the choice of ε does not depend on n , this argument reduces the problem to the one-dimensional case. At the end we would like to mention recent works [19, 23] where the authors use the second variation condition to study isoperimetric inequalities in Gauss space.

2. NOTATION AND PRELIMINARIES

In this section we briefly introduce our basic notation and recall some elementary results from geometric measure theory. For an introduction to the theory of sets of finite perimeter we refer to [2] and [18].

We denote by $\{e^{(1)}, \dots, e^{(n)}\}$ the canonical base of \mathbb{R}^n . For generic point $x \in \mathbb{R}^n$ we denote its j -component by $x_j := \langle x, e^{(j)} \rangle$ and use the notation $x = (x', x_n)$ when we want to specify the last component. The family of the Borelian sets in \mathbb{R}^n is denoted by \mathcal{B} . We denote the Hausdorff measure with Gaussian weight by \mathcal{H}_γ^{n-1} , i.e., for every set $A \in \mathcal{B}$ we define

$$\mathcal{H}_\gamma^{n-1}(A) := \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_A e^{-\frac{|x|^2}{2}} d\mathcal{H}^{n-1}(x).$$

A set $E \in \mathcal{B}$ has *locally finite perimeter* if for every ball $B_R \subset \mathbb{R}^n$ it holds

$$\sup \left\{ \int_E \operatorname{div} \varphi \, dx : \varphi \in C_0^\infty(B_R; \mathbb{R}^n), \sup |\varphi| \leq 1 \right\} < \infty.$$

If E is a set of locally finite perimeter then its *reduced boundary* is denoted by $\partial^* E$ and $\nu_E(x)$ is the *exterior normal* at $x \in \partial^* E$. When no confusion arises we shall simply write ν and use the notation $\nu_j = \langle \nu, e^{(j)} \rangle$. If E has locally finite perimeter then its perimeter in $A \in \mathcal{B}$ is

$$P(E; A) := \mathcal{H}^{n-1}(\partial^* E \cap A).$$

Moreover, by the divergence theorem we have

$$\int_E \operatorname{div} X \, dx = \int_{\partial^* E} \langle X, \nu_E \rangle d\mathcal{H}^{n-1}$$

for every Lipschitz continuous vector field $X : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with compact support.

In (2) the Gaussian isoperimetric problem was stated for sets with Lipschitz boundary, but this can be extended to more general and more natural class of sets. Indeed, if $E \in \mathcal{B}$ is a set of locally finite perimeter with $\mathcal{H}_\gamma^{n-1}(\partial^* E) < \infty$ then it has *finite Gaussian perimeter* and we denote its Gaussian perimeter by

$$P_\gamma(E) := \mathcal{H}_\gamma^{n-1}(\partial^* E).$$

Otherwise we set $P_\gamma(E) := \infty$. It follows from the divergence theorem that

$$P_\gamma(E) = \sqrt{2\pi} \sup \left\{ \int_E (\operatorname{div} \varphi - \langle \varphi, x \rangle) d\gamma(x) : \varphi \in C_0^\infty(B_R; \mathbb{R}^n), \sup |\varphi| \leq 1 \right\}$$

for every $E \in \mathcal{B}$. We will assume that every set has finite Gaussian perimeter if not otherwise mentioned. If E is an open set with Lipschitz boundary then trivially $\partial^* E = \partial E$. Therefore the above notion of Gaussian perimeter provides an extension of (1).

If $M \subset \mathbb{R}^n$ is a smooth, oriented manifold with or without boundary, then D_τ and div_τ denotes the tangential differential and the tangential divergence respectively. We recall that if

$X : M \rightarrow \mathbb{R}^n$ is a Lipschitz continuous vector field with compact support then the divergence theorem on manifold states

$$\int_M \operatorname{div}_\tau X \, d\mathcal{H}^{n-1} = \int_M \mathcal{H} \langle X, \nu_M \rangle \, d\mathcal{H}^{n-1}$$

where \mathcal{H} is the sum of the principle curvatures.

3. MINIMUM PROBLEM

We begin by analysing a specific minimum problem, which will be instrumental in the proof of our quantitative estimate. Given $\varepsilon > 0$ and $\Lambda > 0$, we consider the functional $\mathcal{F} : \mathcal{B} \rightarrow \mathbb{R}^+$ defined by

$$\mathcal{F}(E) = P_\gamma(E) + \frac{\varepsilon}{2}|b(E)|^2 + \Lambda|\gamma(E) - \phi(s)|. \quad (7)$$

The last term is a volume penalization that forces (for Λ large enough) the minimizers of \mathcal{F} to have Gaussian measure $\phi(s)$. For simplicity we will indicate by b_s the norm of $b(H_{\omega,s})$, since it does not depend on ω . We have $b(H_{\omega,s}) = -b_s\omega$ and $b_s = e^{-\frac{s^2}{2}}/\sqrt{2\pi}$. It is important to observe that the half-spaces maximize the norm of the barycenter:

$$b_s \geq |b(E)| \quad (8)$$

for every set E such that $\gamma(E) = \phi(s)$. Indeed, if $b(E) \neq 0$, by taking $\omega = -b(E)/|b(E)|$, we have

$$\begin{aligned} |b(E)| - b_s &= \langle b(E) + b_s\omega, -\omega \rangle = - \int_E \langle x, \omega \rangle d\gamma + \int_{H_{\omega,s}} \langle x, \omega \rangle d\gamma \\ &= \int_{E \setminus H_{\omega,s}} (\langle x, -\omega \rangle + s) d\gamma + \int_{H_{\omega,s} \setminus E} (\langle x, \omega \rangle - s) d\gamma \leq 0, \end{aligned}$$

because the integrands in the last term are both negative. This enlightens the fact that in minimizing \mathcal{F} the two terms $P_\gamma(E)$ and $|b(E)|$ are in competition. Minimizing $P_\gamma(E)$ means to push the set E at infinity in one direction, so that it becomes closer to a half-space. On the other hand, minimizing $|b(E)|$ means to balance the mass of E with respect to the origin. We will see, and this is the main point of our analysis, that for ε small enough the perimeter term overcomes the barycenter, so that the only minima of \mathcal{F} are the half-spaces $H_{\omega,s}$.

In this section we study the existence of minimizers of the functional \mathcal{F} , and their regularity. We calculate also the Euler equation and the second variation of \mathcal{F} . All these results are nowadays standard, but for the readers convenience we prefer to give here the outlines of the proofs. Specific properties of the minimizers will be analysed in the next section, along the proof of our main theorem.

Proposition 1. *The functional \mathcal{F} has a minimizer.*

Proof. Consider a sequence E_h in \mathcal{B} such that

$$\lim_{h \rightarrow \infty} \mathcal{F}(E_h) = \inf\{\mathcal{F}(F) : F \in \mathcal{B}\}.$$

Since for any bounded open set $A \subset \mathbb{R}^n$ one has that $\sup_h P(E_h; A)$ is finite, the compactness theorem for BV functions (see [2, Theorem 3.23]) ensures the existence of a Borel set $E \subset \mathbb{R}^n$ such that, up to a subsequence, $\chi_{E_h} \rightarrow \chi_E$ strongly in $L^1_{\text{loc}}(\mathbb{R}^n)$. Given $R > 0$, let r_h and r be such that

$$\phi(r_h) = \gamma(E_h \setminus B_R) \quad \text{and} \quad \phi(r) = \gamma(\mathbb{R}^n \setminus B_R).$$

From inequality (8) we get

$$\left| \int_{E_h \setminus B_R} x \, d\gamma \right| \leq \frac{e^{-\frac{r_h^2}{2}}}{\sqrt{2\pi}} \leq \frac{e^{-\frac{r^2}{2}}}{\sqrt{2\pi}}.$$

A similar estimate holds also for the set $F \setminus B_R$. Therefore, since

$$\left| \int_{E_h} x \, d\gamma - \int_E x \, d\gamma \right| \leq \left| \int_{\mathbb{R}^n} (\chi_{E_h} - \chi_E) \chi_{B_R} x \, d\gamma \right| + \frac{2e^{-\frac{r^2}{2}}}{\sqrt{2\pi}},$$

we have that $b(E) = \lim_{h \rightarrow \infty} b(E_h)$. By the lower semicontinuity of the Gaussian perimeter we have also $P_\gamma(E) \leq \liminf_{h \rightarrow \infty} P_\gamma(E_h)$, so that $\mathcal{F}(E) \leq \mathcal{F}(F)$ for every set $F \in \mathcal{B}$. \square

The regularity of minimizer of \mathcal{F} follows from the regularity theory of almost minimizers of the perimeter [26]. From the regularity point of view the advantage of having the strong asymmetry in the functional (7) instead of the standard one is that the minimizers are smooth outside the singular set. The fact that one may gain regularity by replacing the standard asymmetry by a stronger one was observed in [8] in a different context.

Proposition 2. *Let E be a minimizer of \mathcal{F} . Then the reduced boundary $\partial^* E$ is a relatively open, smooth hypersurface and satisfies the Euler equation*

$$\mathcal{H} - \langle x, \nu \rangle + \varepsilon \langle b, x \rangle = \lambda \quad \text{on } \partial^* E, \quad (9)$$

where $b = b(E)$ is the barycenter of E . Here λ is the Lagrange multiplier which can be estimated by

$$|\lambda| \leq \Lambda.$$

The singular part of the boundary $\partial E \setminus \partial^* E$ is empty when $n < 8$, while for $n \geq 8$ its Hausdorff dimension can be estimated by $\dim_{\mathcal{H}}(\partial E \setminus \partial^* E) \leq n - 8$.

Proof. First of all we note that ∂E is the topological boundary of the correct representative of the set (see [18, Proposition 12.19]).

Let us fix $x_0 \in \partial E$ and $r \in (0, 1)$. From the minimality we deduce that for every set $F \subset \mathbb{R}^n$ with locally finite perimeter such that $F \triangle E \subset B_{2r}(x_0)$ it holds

$$P_\gamma(E) \leq P_\gamma(F) + C\gamma(F \triangle E) \quad (10)$$

for some constant C depending on $|x_0|$. If we choose $F = E \setminus B_r(x_0)$ we get from (10) that

$$P(E; B_r(x_0)) \leq C_0 r^{n-1} \quad (11)$$

for some constant $C_0 = C_0(|x_0|)$. Note that for every $x \in B_r(x_0)$ and $r \in (0, 1)$ it holds

$$\left| e^{-\frac{|x|^2}{2}} - e^{-\frac{|x_0|^2}{2}} \right| \leq Cr$$

for some constant C . Therefore (10) and (11) imply that for all sets F with $F \triangle E \subset B_r(x_0)$ and $r \leq 1$ it holds

$$P(E; B_r(x_0)) \leq P(F; B_r(x_0)) + Cr^n \quad (12)$$

for some constant C depending on $|x_0|$. It follows from [26, Theorem 1.9] (see also [18]) that $\partial^* E$ is relatively open (in ∂E), $C^{1,\sigma}$ hypersurface for every $\sigma < 1/2$ and the singular set $\partial E \setminus \partial^* E$ is empty when $n < 8$ and $\dim_{\mathcal{H}}(\partial E \setminus \partial^* E) \leq n - 8$ when $n \geq 8$.

To prove the Euler equation we fix $x_0 \in \partial^* E$. Since $\partial^* E$ is relatively open we find an open set $U \subset \mathbb{R}^n$ such that $\partial E \cap U \subset \partial^* E$. We show that (9) holds on $\partial E \cap U$ in a weak sense, i.e., for every $X \in C_0^1(U; \mathbb{R}^n)$ with $\int_{\partial^* E} \langle X, \nu \rangle d\mathcal{H}_\gamma^{n-1} = 0$ we have

$$\int_{\partial^* E} \operatorname{div}_\tau X - \langle X, x \rangle d\mathcal{H}_\gamma^{n-1} + \varepsilon \int_{\partial^* E} \langle b, x \rangle \langle X, \nu \rangle d\mathcal{H}_\gamma^{n-1} = 0.$$

To this aim let $\Phi : U \times (-\delta, \delta) \rightarrow U$ be the flow associated with X , i.e.,

$$\frac{\partial}{\partial t} \Phi(x, t) = X(\Phi(x, t)), \quad \Phi(x, 0) = x$$

and define $E_t = \Phi(E, t)$. Then it holds (see [18])

$$\frac{\partial}{\partial t} \Big|_{t=0} \gamma(E_t) = \int_{\partial^* E} \langle X, \nu \rangle d\mathcal{H}_\gamma^{n-1} = 0.$$

Therefore $\frac{\partial}{\partial t} \Big|_{t=0} |\gamma(E_t) - \phi(s)| = 0$. The weak form of the Euler equation then follows from the minimality of E and

$$\frac{\partial}{\partial t} \Big|_{t=0} P_\gamma(E_t) = \int_{\partial^* E} \operatorname{div}_\tau X - \langle X, x \rangle d\mathcal{H}_\gamma^{n-1}$$

and (see [18])

$$\frac{\partial}{\partial t} \Big|_{t=0} |b(E_t)|^2 = 2 \int_{\partial^* E} \langle b, x \rangle \langle X, \nu \rangle d\mathcal{H}_\gamma^{n-1}.$$

Since the reduced boundary $\partial^* E$ is $C^{1,\sigma}$ manifold and since it satisfies the Euler equation (9) in a weak sense, from classical Schauder estimates we deduce that $\partial^* E$ is in fact C^∞ manifold.

Finally, in order to bound the Lagrange multiplier λ , we fix $x_0 \in \partial^* E$ and an open set $U \subset \mathbb{R}^n$ such that $\partial E \cap U \subset \partial^* E$. Let $X \in C_0^1(U; \mathbb{R}^n)$ be any vector field and assume that is $\Phi(x, t)$ the associated flow and denote $E_t = \Phi(E, t)$. Then by the above calculations we have

$$\begin{aligned} \frac{\partial}{\partial t} \Big|_{t=0} \left(P_\gamma(E_t) + \frac{\varepsilon}{2} |b(E_t)|^2 \right) &= \int_{\partial^* E} \operatorname{div}_\tau X - \langle X, x \rangle + \varepsilon \langle b, x \rangle \langle X, \nu \rangle d\mathcal{H}_\gamma^{n-1} \\ &= \int_{\partial^* E} (\mathcal{H} - \langle x, \nu \rangle + \varepsilon \langle b, x \rangle) \langle X, \nu \rangle d\mathcal{H}_\gamma^{n-1} \\ &= \lambda \int_{\partial^* E} \langle X, \nu \rangle d\mathcal{H}_\gamma^{n-1} \end{aligned}$$

and

$$\limsup_{t \rightarrow 0} \frac{|\gamma(E_t) - \phi(s)| - |\gamma(E) - \phi(s)|}{t} \leq \left| \frac{\partial}{\partial t} \Big|_{t=0} \gamma(E_t) \right| = \left| \int_{\partial^* E} \langle X, \nu \rangle d\mathcal{H}_\gamma^{n-1} \right|.$$

Therefore by the minimality of E we have

$$\lambda \int_{\partial^* E} \langle X, \nu \rangle d\mathcal{H}_\gamma^{n-1} + \Lambda \left| \int_{\partial^* E} \langle X, \nu \rangle d\mathcal{H}_\gamma^{n-1} \right| \geq 0$$

for every $X \in C_0^1(U; \mathbb{R}^n)$. This proves the claim. \square

Next we derive the second order condition for minimizers of the functional \mathcal{F} , i.e., the quadratic form associated with the second variation is non-negative. Let us briefly explain what we mean by this. Let $\varphi : \partial^* E \rightarrow \mathbb{R}$ be a smooth function with compact support such that it has zero average, i.e., $\int_{\partial^* E} \varphi d\mathcal{H}_\gamma^{n-1} = 0$. We choose a specific vector field $X : \mathbb{R}^n \rightarrow \mathbb{R}^n$ (see (13)), such that $X := \varphi \nu_E$ on $\partial^* E$. We denote the associated flow by $\Phi(x, t)$ and define $E_t := \Phi(E, t)$. The second variation of the functional \mathcal{F} at E in direction φ is then defined to be the value

$$\frac{d^2}{dt^2} \Big|_{t=0} \mathcal{F}(E_t).$$

It turns out that the choice of the vector field X ensures that the second derivative exists and it follows from the minimality of E that this value is non-negative. Moreover the second variation at E defines a quadratic form over all functions $\varphi \in C_0^\infty(\partial^* E)$ with zero average.

The calculations of the second variation are standard (see [1, 18, 19, 24] for similar calculations) but since they are technically challenging we include them for the convenience of the

reader. We note that since ∂E is not necessarily smooth we may only perturb the regular part of the boundary. We denote $u \in C_0^\infty(\partial^* E)$ when $u : \partial^* E \rightarrow \mathbb{R}$ is a smooth function with compact support.

Proposition 3. *Let E be a minimizer of \mathcal{F} . The quadratic form associated with the second variation is non-negative*

$$J[\varphi] := \int_{\partial^* E} (|D_\tau \varphi|^2 - |B_E|^2 \varphi^2 - \varphi^2 + \varepsilon \langle b, \nu \rangle \varphi^2) d\mathcal{H}_\gamma^{n-1} + \varepsilon \left| \int_{\partial^* E} x \varphi d\mathcal{H}_\gamma^{n-1} \right|^2 \geq 0$$

for every $\varphi \in C_0^\infty(\partial^* E)$ which satisfies

$$\int_{\partial^* E} \varphi d\mathcal{H}_\gamma^{n-1} = 0.$$

Here $b = b(E)$ and $\nu = \nu_E$, while $|B_E|^2$ is the sum of the squares of the curvatures.

Proof. Assume that $\varphi \in C_0^\infty(\partial^* E)$ satisfies $\int_{\partial^* E} \varphi d\mathcal{H}_\gamma^{n-1} = 0$. Let $d_E : \mathbb{R}^n \rightarrow \mathbb{R}$ be the signed distance function of E

$$d_E(x) = \begin{cases} \text{dist}(x, \partial E) & \text{for } x \in \mathbb{R}^n \setminus E \\ -\text{dist}(x, \partial E) & \text{for } x \in E. \end{cases}$$

It follows from Proposition 2 that there is an open set $U \subset \mathbb{R}^n$ such that d_E is smooth in U and the support of φ is in $\partial^* E \cap U$. We extend φ to U , and call the extension simply by φ , such that $\varphi \in C_0^\infty(U)$ and

$$\partial_\nu \varphi = (\langle x, \nu \rangle - \mathcal{H})\varphi \quad \text{on } \partial^* E. \quad (13)$$

Finally we define the vector field $X : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $X := \varphi \nabla d_E$ in U and $X := 0$ in $\mathbb{R}^n \setminus U$. Note that X is smooth and $X = \varphi \nu$ on $\partial^* E$.

Let $\Phi : \mathbb{R}^n \times (-\delta, \delta) \rightarrow \mathbb{R}^n$ be the flow associated with X , i.e.,

$$\frac{\partial}{\partial t} \Phi(x, t) = X(\Phi(x, t)), \quad \Phi(x, 0) = x$$

and define $E_t = \Phi(E, t)$. Let us denote the Jacobian of $\Phi(\cdot, t)$ by $J\Phi(x, t)$ and the tangential Jacobian on $\partial^* E$ by $J_\tau \Phi(x, t)$. We recall the formulas (see [24])

$$\begin{aligned} \frac{\partial}{\partial t} \Big|_{t=0} J\Phi(x, t) &= \text{div} X \\ \frac{\partial^2}{\partial t^2} \Big|_{t=0} J\Phi(x, t) &= \text{div}((\text{div} X)X) \end{aligned} \quad (14)$$

and

$$\begin{aligned} \frac{\partial}{\partial t} \Big|_{t=0} J_\tau \Phi(\cdot, t) &= \text{div}_\tau X \\ \frac{\partial^2}{\partial t^2} \Big|_{t=0} J_\tau \Phi(\cdot, t) &= |(D_\tau X)^T \nu|^2 + (\text{div}_\tau X)^2 + \text{div}_\tau Z - \text{Tr}(D_\tau X)^2 \end{aligned} \quad (15)$$

where $Z := DXX = \frac{\partial^2 \Phi(x, t)}{\partial t^2} \Big|_{t=0}$ is the acceleration field. Recall also that by definition $\Phi(x, 0) = x$ and $\frac{\partial}{\partial t} \Big|_{t=0} \Phi(x, t) = X$.

We begin by differentiating the Gaussian volume. We use (14) to calculate

$$\frac{\partial}{\partial t} \Big|_{t=0} \gamma(E_t) = \int_{\partial^* E} \varphi d\mathcal{H}_\gamma^{n-1} = 0$$

and

$$\frac{\partial^2}{\partial t^2} \Big|_{t=0} \gamma(E_t) = \int_E \text{div} \left(\text{div} \left(X e^{-\frac{|x|^2}{2}} \right) X \right) dx = \int_{\partial^* E} \varphi \partial_\nu \varphi + (\mathcal{H} - \langle x, \nu \rangle) \varphi^2 d\mathcal{H}_\gamma^{n-1} = 0$$

where we have used (13). Hence, $\gamma(E_t) = \gamma(E) + o(t^2)$ and

$$\frac{\partial^2}{\partial t^2} \Big|_{t=0} |\gamma(E_t) - \phi(s)| = 0.$$

Since $t \mapsto P_\gamma(E_t)$ and $t \mapsto |b(E_t)|^2$ are smooth with respect to t we have by the minimality of E that

$$0 \leq \frac{\partial^2}{\partial t^2} \Big|_{t=0} \mathcal{F}(E_t) = \frac{\partial^2}{\partial t^2} \Big|_{t=0} P_\gamma(E_t) + \frac{\varepsilon}{2} \frac{\partial^2}{\partial t^2} \Big|_{t=0} |b(E_t)|^2. \quad (16)$$

Thus we need to differentiate the perimeter and the barycenter.

To differentiate the perimeter we write

$$P_\gamma(E_t) = \int_{\partial^* E} e^{-\frac{|\Phi(x,t)|^2}{2}} J_\tau \Phi(x, t) d\mathcal{H}^{n-1}.$$

We differentiate this twice and use (15) to get

$$\begin{aligned} \frac{\partial^2}{\partial t^2} \Big|_{t=0} P_\gamma(E_t) &= \int_{\partial^* E} (|(D_\tau X)^T \nu|^2 + (\operatorname{div}_\tau X)^2 + \operatorname{div}_\tau Z - \operatorname{Tr}(D_\tau X)^2) d\mathcal{H}_\gamma^{n-1} \\ &\quad + \int_{\partial^* E} (-2\operatorname{div}_\tau X \langle X, x \rangle - \langle Z, x \rangle - |X|^2 + \langle X, x \rangle^2) d\mathcal{H}_\gamma^{n-1} \\ &= \int_{\partial^* E} |D_\tau \varphi|^2 - |B_E|^2 \varphi^2 - \varphi^2 d\mathcal{H}_\gamma^{n-1} \\ &\quad + \int_{\partial^* E} (\mathcal{H} - \langle x, \nu \rangle)(\varphi \partial_\nu \varphi + (\mathcal{H} - \langle x, \nu \rangle) \varphi^2) d\mathcal{H}_\gamma^{n-1}. \end{aligned} \quad (17)$$

Let us denote $b_t = b(E_t)$, $b = b(E)$, $b' = \frac{\partial}{\partial t} \Big|_{t=0} b_t$ and $b'' = \frac{\partial^2}{\partial t^2} \Big|_{t=0} b_t$. Then

$$\frac{\partial^2}{\partial t^2} \Big|_{t=0} |b_t|^2 = 2\langle b, b'' \rangle + 2|b'|^2.$$

To differentiate the barycenter we write

$$b_t = \int_E \Phi(x, t) e^{-\frac{|\Phi(x,t)|^2}{2}} J\Phi(x, t) dx.$$

We use (14) and get after differentiating once that

$$b' = \int_{\partial^* E} x \varphi d\mathcal{H}_\gamma^{n-1} \quad (18)$$

and after differentiating twice that

$$\begin{aligned} b'' &= \int_E (x \operatorname{div}((\operatorname{div} X)X) + 2X(\operatorname{div} X) - 2x \langle X, x \rangle (\operatorname{div} X) - 2X \langle X, x \rangle) e^{-\frac{|x|^2}{2}} dx \\ &\quad + \int_E (DX X + x \langle X, x \rangle^2 - x \langle DX X, x \rangle - x |X|^2) e^{-\frac{|x|^2}{2}} dx \\ &= \int_E \left(DX X e^{-\frac{|x|^2}{2}} + 2X \operatorname{div}(X e^{-\frac{|x|^2}{2}}) + x \operatorname{div} \left(\operatorname{div}(X e^{-\frac{|x|^2}{2}}) X \right) \right) dx. \end{aligned}$$

Thus we obtain by the divergence theorem that

$$\begin{aligned} \langle b, b'' \rangle &= \int_E \operatorname{div} \left(\langle X, b \rangle X e^{-\frac{|x|^2}{2}} \right) + \operatorname{div} \left(\langle x, b \rangle \left(\operatorname{div}(X e^{-\frac{|x|^2}{2}}) X \right) \right) dx \\ &= \int_{\partial^* E} \langle X, b \rangle \langle X, \nu \rangle d\mathcal{H}_\gamma^{n-1} + \int_{\partial^* E} \langle b, x \rangle \langle X, \nu \rangle \left(\operatorname{div}(X e^{-\frac{|x|^2}{2}}) \right) d\mathcal{H}^{n-1} \\ &= \int_{\partial^* E} \langle b, \nu \rangle \varphi^2 d\mathcal{H}_\gamma^{n-1} + \int_{\partial^* E} \langle b, x \rangle (\varphi \partial_\nu \varphi + (\mathcal{H} - \langle x, \nu \rangle) \varphi^2) d\mathcal{H}_\gamma^{n-1}. \end{aligned} \quad (19)$$

Therefore (16), (17), (18) and (19) imply

$$\begin{aligned}
0 \leq \frac{\partial^2}{\partial t^2} \Big|_{t=0} \mathcal{F}(E_t) &= \frac{\partial^2}{\partial t^2} \Big|_{t=0} P_\gamma(E_t) + \varepsilon (\langle b, b'' \rangle + |b'|^2) \\
&= \int_{\partial^* E} (|D_\tau \varphi|^2 - |B_E|^2 \varphi^2 - \varphi^2 + \varepsilon \langle b, \nu \rangle \varphi^2) d\mathcal{H}_\gamma^{n-1} + \varepsilon \left| \int_{\partial^* E} x \varphi d\mathcal{H}_\gamma^{n-1} \right|^2 \\
&\quad + \int_{\partial^* E} (\mathcal{H} - \langle x, \nu \rangle + \varepsilon \langle b, x \rangle) (\varphi \partial_\nu \varphi + (\mathcal{H} - \langle x, \nu \rangle) \varphi^2) d\mathcal{H}_\gamma^{n-1}.
\end{aligned} \tag{20}$$

We use the Euler equation (9) and (13) to conclude that

$$\begin{aligned}
&\int_{\partial^* E} (\mathcal{H} - \langle x, \nu \rangle + \varepsilon \langle b, x \rangle) (\varphi \partial_\nu \varphi + (\mathcal{H} - \langle x, \nu \rangle) \varphi^2) d\mathcal{H}_\gamma^{n-1} \\
&= \lambda \int_{\partial^* E} \varphi \partial_\nu \varphi + (\mathcal{H} - \langle x, \nu \rangle) \varphi^2 d\mathcal{H}_\gamma^{n-1} = 0.
\end{aligned}$$

Hence, the claim follows from (20). \square

We would like to extend the quadratic form in Proposition 3 to more general functions than $\varphi \in C_0^\infty(\partial^* E)$. To this aim we define the function space $H_\gamma^1(\partial^* E)$ as the closure of $C_0^\infty(\partial^* E)$ with respect to the norm $\|u\|_{H_\gamma^1(\partial^* E)} = \|u\|_{L^2(\partial^* E)} + \|D_\tau u\|_{L^2(\partial^* E)}$. A priori this definition seems rather restrictive since it is not clear if even constant functions belong to $H_\gamma^1(\partial^* E)$. However, the information on the singular set $\dim_{\mathcal{H}}(\partial E \setminus \partial^* E) \leq n - 8$ from Proposition 2 ensures that the singular set has capacity zero and it is therefore negligible. It follows that every smooth function $u : \partial^* E \rightarrow \mathbb{R}$ which has finite H_γ^1 -norm is in $H_\gamma^1(\partial^* E)$. In particular, if $u : \mathbb{R}^n \rightarrow \mathbb{R}$ is a smooth function such that the $H_\gamma^1(\partial^* E)$ norm of its restriction on $\partial^* E$ is bounded, then the restriction is in $H_\gamma^1(\partial^* E)$.

Lemma 1. *Let E be a minimizer of \mathcal{F} . If $u : \partial^* E \rightarrow \mathbb{R}$ is smooth function with $\|u\|_{H_\gamma^1(\partial^* E)} < \infty$, then $u \in H_\gamma^1(\partial^* E)$.*

Proof. By truncation we may assume that u is bounded and by a standard mollification argument it is enough to find Lipschitz continuous functions u_k with a compact support on $\partial^* E$ such that $\lim_{k \rightarrow \infty} \|u - u_k\|_{H_\gamma^1(\partial^* E)} = 0$. We will show that there exist Lipschitz continuous functions $\zeta_k : \partial^* E \rightarrow \mathbb{R}$ with compact support such that $0 \leq \zeta_k \leq 1$, $\zeta_k \rightarrow 1$ in $H_\gamma^1(\partial^* E)$ and $\zeta_k(x) \rightarrow 1$ pointwise on $\partial^* E$. We may then choose $u_k = u \zeta_k$ and the claim follows.

Let us fix $k \in \mathbb{N}$. First of all let us choose a large radius R_k such that the Gaussian perimeter of E outside the ball B_{R_k} is small, i.e., $P_\gamma(E; \mathbb{R}^n \setminus B_{R_k}) \leq 1/k$. We choose a cut-off function $\eta_k \in C_0^\infty(B_{2R_k})$ such that $|D\eta_k(x)| \leq 1$ for every $x \in \mathbb{R}^n$ and $\zeta \equiv 1$ in B_{R_k} .

Denote the singular set by $\Sigma := \partial E \setminus \partial^* E$. Proposition 2 implies that Σ is a closed set with $\mathcal{H}^{n-3}(\Sigma) = 0$. Therefore we may cover $\Sigma \cap \overline{B_{2R_k}}$ with balls $B_{r_i} := B_{r_i}(x_i)$, $i = 1, \dots, N_k$, with radii $r_i \leq 1/2$ such that

$$\sum_{i=1}^{N_k} r_i^{n-3} \leq \frac{1}{C_0} \frac{1}{k} \tag{21}$$

where $C_0 = C_0(2R_k)$ is the constant from the estimate (11) for the radius $2R_k$. For every ball B_{2r_i} we define a cut-off function $\psi_i \in C_0^\infty(B_{2r_i})$ such that $\psi_i \equiv 1$ in B_{r_i} , $0 \leq \psi_i \leq 1$ and $|D\psi_i| \leq \frac{2}{r_i}$. Define

$$\theta_k(x) := \max_i \psi_i(x), \quad x \in \mathbb{R}^n.$$

Then $\theta_k(x) = 1$ for $x \in \cup_i B_{r_i}$, $\theta_k(x) = 0$ for $x \notin \cup_i B_{2r_i}$ and it is Lipschitz continuous. We may estimate its weak tangential gradient on $\partial^* E$ by

$$|D_\tau \theta_k(x)| \leq \max_i |D_\tau \psi_i(x)| \leq \left(\sum_{i=1}^{N_k} |D\psi_i(x)|^2 \right)^{1/2}$$

for \mathcal{H}^{n-1} -almost every $x \in \partial^* E$. Since $\Sigma \cap B_{2R_k} \subset \cup_i B_{r_i}$ the function

$$\zeta_k = (1 - \theta_k)\eta_k$$

has compact support on $\partial^* E$. Note that by (11) it holds $P(E; B_{2r_i}) \leq C_0 r_i^{n-1}$. Hence we have that

$$\begin{aligned} \|D_\tau \zeta_k\|_{L_\gamma^2(\partial^* E)}^2 &\leq 2 \int_{\partial^* E} (|D_\tau \eta_k|^2 + |D_\tau \theta_k|^2) d\gamma \\ &\leq 2P_\gamma(E; \mathbb{R}^n \setminus B_{R_k}) + 2 \sum_{i=1}^{N_k} \int_{\partial^* E \cap B_{2r_i}} |D\psi_i|^2 d\mathcal{H}^{n-1} \\ &\leq \frac{2}{k} + 8 \sum_{i=1}^{N_k} r_i^{-2} P(E; B_{2r_i}) \\ &\leq \frac{2}{k} + 8C_0 \sum_{i=1}^{N_k} r_i^{n-3} \leq \frac{10}{k}. \end{aligned}$$

Similarly we conclude that $\|\zeta_k - 1\|_{L_\gamma^2(\partial^* E)}^2 \rightarrow 0$ as $k \rightarrow \infty$. \square

4. QUANTITATIVE ESTIMATES

In this section we focus on the proof of our main result, as well as on some of its direct consequences.

Proof of the main theorem. Since $\beta(E) = \beta(\mathbb{R}^n \setminus E)$, we may restrict ourselves to the case $s \leq 0$. We begin by writing the strong asymmetry $\beta(E)$ in an alternative way. As we have already observed at the beginning of the previous section, the half-spaces maximize the norm of the barycenter. In particular, the strong asymmetry is nothing else than the gap between this maximum and the norm of $b(E)$. When $b(E) \neq 0$, the minimum in (4) is attained by $\omega = -b(E)/|b(E)|$ and with this choice of ω we have

$$\beta(E) = |b(E) + b_s \omega| = |(-|b(E)| + b_s) \omega| = b_s - |b(E)|.$$

If we show that for some ε and Λ the only minimizers of the functional \mathcal{F} are the half-spaces $H_{\omega, s}$, $\omega \in \mathbb{S}^{n-1}$, then we are done, since this implies that for every set $E \subset \mathbb{R}^n$ with $\gamma(E) = \phi(s)$ it holds

$$\begin{aligned} D(E) &\geq \frac{\varepsilon}{2} (b_s^2 - |b(E)|^2) = \frac{\varepsilon}{2} (b_s + |b(E)|) \beta(E) \\ &\geq \frac{\varepsilon}{2\sqrt{2\pi}} e^{-\frac{s^2}{2}} \beta(E). \end{aligned}$$

We will prove that this is indeed the case when we choose ε and Λ as

$$\varepsilon = \frac{e^{\frac{s^2}{2}}}{40\pi^2(1+s^2)} \quad \text{and} \quad \Lambda = \frac{\sqrt{2}e^{-\frac{s^2}{2}}}{\phi(s)}. \quad (22)$$

With this choice we have (6) with the constant

$$c = 80\pi^2\sqrt{2\pi}.$$

Assume now that E is a minimizer of \mathcal{F} and, without loss of generality, that its barycenter is in the direction of $-e^{(n)}$, i.e., $b(E) = -|b|e^{(n)}$. Along the proof we will use the shorter notation $H_s = H_{e^{(n)},s}$. We are going to show that $E = H_s$. We divide the proof into four steps.

Step 1. As a first step we prove an upper bound for the quantity $\int_{\partial^* E} \langle x, \omega \rangle^2 d\gamma$, i.e., for every $\omega \in \mathbb{S}^{n-1}$ it holds

$$\int_{\partial^* E} \langle x, \omega \rangle^2 d\mathcal{H}_\gamma^{n-1} \leq 20\pi^2(1+s^2)e^{-\frac{s^2}{2}}.$$

The proof is similar to the classical Caccioppoli inequality in the theory of elliptic equations.

We begin with few observations. Using H_s as a competitor, the minimality of E implies

$$P_\gamma(E) \leq \mathcal{F}(H_s) = P_\gamma(H_s) + \frac{\varepsilon}{2}|b(H_s)|^2 \leq \frac{10}{9}e^{-\frac{s^2}{2}}. \quad (23)$$

Let r be such that $\phi(r) = \gamma(E)$. Since H_r maximizes the length of the barycenter we have by the Gaussian isoperimetric inequality that

$$|b| \leq |b(H_r)| = \frac{1}{\sqrt{2\pi}}P_\gamma(H_r) \leq \frac{1}{\sqrt{2\pi}}P_\gamma(E) \leq \frac{10}{9\sqrt{2\pi}}e^{-\frac{s^2}{2}}.$$

From our choice of ε in (22) it follows that

$$\varepsilon|b| \leq \frac{1}{4}. \quad (24)$$

By second order analysis it is easy to check that the function

$$g(s) := e^{-\frac{s^2}{2}} + (\sqrt{2\pi}s - \pi)\phi(s)$$

is non-positive in $(-\infty, 0]$. Indeed, g' is non-positive and $\lim_{s \rightarrow -\infty} g(s) = 0$. Therefore,

$$\Lambda^2 + 1 = 2\frac{e^{-s^2}}{\phi(s)^2} + 1 \leq 2(\pi - \sqrt{2\pi}s)^2 + 1 \leq \frac{9}{2}\pi^2(1+s^2). \quad (25)$$

Since $\partial^* E$ is smooth we deduce from the Euler equation (9) that for every Lipschitz continuous vector field $X : \partial^* E \rightarrow \mathbb{R}^n$ with compact support it holds

$$\int_{\partial^* E} (\operatorname{div}_\tau X - \langle X, x \rangle) d\mathcal{H}_\gamma^{n-1} - \varepsilon|b| \int_{\partial^* E} x_n \langle X, \nu \rangle d\mathcal{H}_\gamma^{n-1} = \lambda \int_{\partial^* E} \langle X, \nu \rangle d\mathcal{H}_\gamma^{n-1}. \quad (26)$$

To obtain (26) simply multiply the Euler equation (9) by $\langle X, \nu \rangle$ and use the divergence theorem on manifolds.

Let $\zeta_k : \partial^* E \rightarrow \mathbb{R}$ be the sequence of Lipschitz continuous functions from the proof of Lemma 1 which have compact support, $0 \leq \zeta_k \leq 1$ and $\zeta_k \rightarrow 1$ in $H_\gamma^1(\partial^* E)$. Let us fix $\omega \in \mathbb{S}^{n-1}$ and choose $X = -\zeta_k^2 x_\omega \omega$ in (26), where $x_\omega = \langle x, \omega \rangle$. We use (24), (26) and Young's inequality to get

$$\begin{aligned} & \int_{\partial^* E} (x_\omega^2 - (1 - \langle \nu, \omega \rangle^2)) \zeta_k^2 d\gamma - \frac{1}{8} \int_{\partial^* E} (x_\omega^2 + x_n^2) \zeta_k^2 d\gamma \\ & \leq |\lambda| \int_{\partial^* E} |x_\omega| \zeta_k^2 d\gamma + 2 \int_{\partial^* E} \zeta_k |x_\omega| |D_\tau \zeta_k| d\gamma \\ & \leq \lambda^2 P_\gamma(E) + \frac{1}{2} \int_{\partial^* E} x_\omega^2 \zeta_k^2 d\gamma + 4 \int_{\partial^* E} |D_\tau \zeta_k|^2 d\gamma. \end{aligned}$$

This yields

$$\frac{3}{8} \int_{\partial^* E} x_\omega^2 \zeta_k^2 d\gamma - \frac{1}{8} \int_{\partial^* E} x_n^2 \zeta_k^2 d\gamma \leq (\lambda^2 + 1)P_\gamma(E) + 4 \int_{\partial^* E} |D_\tau \zeta_k|^2 d\gamma.$$

Maximizing over $\omega \in \mathbb{S}^{n-1}$ gives

$$\max_{\omega \in \mathbb{S}^{n-1}} \left(\frac{1}{4} \int_{\partial^* E} x_\omega^2 \zeta_k^2 d\gamma \right) \leq (\lambda^2 + 1)P_\gamma(E) + 4 \int_{\partial^* E} |D_\tau \zeta_k|^2 d\gamma.$$

By letting $k \rightarrow \infty$, from the bound $|\lambda| \leq \Lambda$ proved in Proposition 2, and from (23) and (25) we deduce

$$\max_{\omega \in \mathbb{S}^{n-1}} \int_{\partial^* E} \langle x, \omega \rangle^2 d\mathcal{H}_\gamma^{n-1} \leq 4(\Lambda^2 + 1)P_\gamma(E) \leq 20\pi^2(1 + s^2)e^{\frac{-s^2}{2}}.$$

Step 2. In this step we use the previous step and Proposition 3 to conclude that for every $\varphi \in H_\gamma^1(\partial^* E)$ with $\int_{\partial^* E} \varphi d\mathcal{H}_\gamma^{n-1} = 0$ it holds

$$\int_{\partial^* E} \left(|D_\tau \varphi|^2 - |B_E|^2 \varphi^2 - \frac{1}{2} \varphi^2 - \varepsilon |b| \nu_n \varphi^2 \right) d\mathcal{H}_\gamma^{n-1} \geq 0. \quad (27)$$

Recall that $H_\gamma^1(\partial^* E)$ is the closure of $C_0^\infty(\partial^* E)$ with respect to H_γ^1 -norm.

Let $\varphi \in H_\gamma^1(\partial^* E)$ with $\int_{\partial^* E} \varphi d\mathcal{H}_\gamma^{n-1} = 0$. Then there exists $\varphi_k \in C_0^\infty(\partial^* E)$ such that $\varphi_k \rightarrow \varphi$ in $H_\gamma^1(\partial^* E)$. In particular, $\lim_{k \rightarrow \infty} \int_{\partial^* E} \varphi_k d\mathcal{H}_\gamma^{n-1} = 0$. Therefore by slightly changing the functions φ_k we may assume that they satisfy $\int_{\partial^* E} \varphi_k d\mathcal{H}_\gamma^{n-1} = 0$ and still converge to φ in $H_\gamma^1(\partial^* E)$. Let $\omega_k \in \mathbb{S}^{n-1}$ be vectors such that

$$\left| \int_{\partial^* E} \varphi_k x d\mathcal{H}_\gamma^{n-1} \right| = \left\langle \int_{\partial^* E} \varphi_k x d\mathcal{H}_\gamma^{n-1}, \omega_k \right\rangle = \int_{\partial^* E} \langle x, \omega_k \rangle \varphi_k d\mathcal{H}_\gamma^{n-1}.$$

We use Proposition 3 and step 1 to conclude

$$\begin{aligned} \int_{\partial^* E} (|D_\tau \varphi_k|^2 - |B_E|^2 \varphi_k^2 - \varphi_k^2 - \varepsilon |b| \nu_n \varphi_k^2) d\mathcal{H}_\gamma^{n-1} &\geq -\varepsilon \left(\int_{\partial^* E} \langle x, \omega_k \rangle^2 d\mathcal{H}_\gamma^{n-1} \right) \left(\int_{\partial^* E} \varphi_k^2 d\mathcal{H}_\gamma^{n-1} \right) \\ &\geq -\varepsilon 20\pi^2(1 + s^2)e^{\frac{-s^2}{2}} \left(\int_{\partial^* E} \varphi_k^2 d\mathcal{H}_\gamma^{n-1} \right). \end{aligned}$$

From our choice of ε in (22) we conclude that (27) holds for every φ_k . Since $\varphi_k \rightarrow \varphi$ in $H_\gamma^1(\partial^* E)$, (27) follows by letting $k \rightarrow \infty$ and by noticing that Fatou's lemma implies

$$\liminf_{k \rightarrow \infty} \int_{\partial^* E} |B_E|^2 \varphi_k^2 d\mathcal{H}_\gamma^{n-1} \geq \int_{\partial^* E} |B_E|^2 \varphi^2 d\mathcal{H}_\gamma^{n-1}.$$

Before the next step we remark that by (27) we have

$$\int_{\partial^* E} |B_E|^2 \varphi^2 d\mathcal{H}_\gamma^{n-1} \leq C \|\varphi\|_{H_\gamma^1(\partial^* E)}^2$$

for every $\varphi \in H_\gamma^1(\partial^* E)$ with zero average. Recalling Lemma 1 it is not difficult to see that this implies

$$\int_{\partial^* E} |B_E|^2 d\gamma < \infty. \quad (28)$$

We leave the proof of this estimate to the reader.

Step 3. In this step we will prove that our minimizer E is the half-space

$$E = H_t = \{x \in \mathbb{R}^n : x_n < t\} \quad \text{for some } t \in \mathbb{R}. \quad (29)$$

This step is the core of the proof.

Let $j \in \{1, \dots, n-1\}$. The divergence theorem yields

$$0 = \langle b(E), e^{(j)} \rangle = \int_E x_j d\gamma = - \int_E \operatorname{div}(e^{(j)} e^{-\frac{|x|^2}{2}}) dx = - \int_{\partial^* E} \nu_j d\gamma.$$

In other words, the function ν_j has zero average. Moreover (28) implies

$$\int_{\partial^* E} |D_\tau \nu_j|^2 d\mathcal{H}_\gamma^{n-1} \leq \int_{\partial^* E} |B_E|^2 d\mathcal{H}_\gamma^{n-1} < \infty.$$

From Lemma 1 we deduce that $\nu_j \in H_\gamma^1(\partial^* E)$ and we may thus use (27) to conclude

$$\int_{\partial^* E} |D_\tau \nu_j|^2 - |B_E|^2 \nu_j^2 - \frac{1}{2} \nu_j^2 - \varepsilon |b| \nu_n \nu_j^2 d\mathcal{H}_\gamma^{n-1} \geq 0. \quad (30)$$

Next we introduce the notation δ_i for the tangential derivative in $e^{(i)}$ -direction, i.e., $\delta_i u = \partial_{x_i} u - \langle \nabla u, \nu \rangle \nu_i$. Moreover we denote by Δ_τ the tangential Laplacian, i.e., $\Delta_\tau u = \sum_{i=1}^n \delta_i \delta_i u$. We recall the well known equation (see e.g. [17, Lemma 10.7])

$$\Delta_\tau \nu_j = -|B_E|^2 \nu_j + \delta_j \mathcal{H} \quad \text{on } \partial^* E.$$

We differentiate the Euler equation (9) and with the above equation we deduce that

$$\Delta_\tau \nu_j - \langle D_\tau \nu_j, x \rangle = -|B_E|^2 \nu_j - \varepsilon |b| \nu_n \nu_j \quad \text{on } \partial^* E$$

by using the fact that $\delta_j \nu_i = \delta_i \nu_j$ and $j \neq n$. Let $\zeta_k : \partial^* E \rightarrow \mathbb{R}$ be as in step 1. We multiply the previous equation by $\zeta_k \nu_j$, integrate over $\partial^* E$ and use the divergence theorem on manifold to conclude

$$\begin{aligned} \int_{\partial^* E} \zeta_k (|B_E|^2 \nu_j^2 + \varepsilon |b| \nu_n \nu_j^2) d\mathcal{H}_\gamma^{n-1} &= - \int_{\partial^* E} \zeta_k \nu_j (\Delta_\tau \nu_j - \langle D_\tau \nu_j, x \rangle) d\mathcal{H}_\gamma^{n-1} \\ &= - \int_{\partial^* E} \zeta_k \nu_j \operatorname{div}_\tau \left(D_\tau \nu_j e^{-\frac{|x|^2}{2}} \right) d\mathcal{H}_\gamma^{n-1} \\ &= - \int_{\partial^* E} \operatorname{div}_\tau \left(\zeta_k \nu_j D_\tau \nu_j e^{-\frac{|x|^2}{2}} \right) d\mathcal{H}_\gamma^{n-1} + \int_{\partial^* E} \langle D_\tau (\zeta_k \nu_j), D_\tau \nu_j \rangle d\mathcal{H}_\gamma^{n-1} \\ &= \int_{\partial^* E} \zeta_k |D_\tau \nu_j|^2 d\mathcal{H}_\gamma^{n-1} + \int_{\partial^* E} \nu_j \langle D_\tau \zeta_k, D_\tau \nu_j \rangle d\mathcal{H}_\gamma^{n-1}. \end{aligned}$$

Since $\|D_\tau \zeta_k\|_{L^2(\partial^* E)} \rightarrow 0$ as $k \rightarrow \infty$ we deduce from the previous equation that

$$\int_{\partial^* E} |B_E|^2 \nu_j^2 + \varepsilon |b| \nu_n \nu_j^2 d\mathcal{H}_\gamma^{n-1} = \int_{\partial^* E} |D_\tau \nu_j|^2 d\mathcal{H}_\gamma^{n-1}.$$

Thus we get from (30) that

$$-\frac{1}{2} \int_{\partial^* E} \nu_j^2 d\mathcal{H}_\gamma^{n-1} \geq 0.$$

This implies $\nu_j \equiv 0$ on $\partial^* E$. Since E has locally finite perimeter in \mathbb{R}^n , De Giorgi's structure theorem (see [18, Proposition II.4.9]) yields

$$D\chi_E = -\nu \mathcal{H}^{n-1} \llcorner \partial^* E.$$

Therefore, the distributional partial derivatives $D_j \chi_E$, $j = 1, \dots, n-1$, are all zero and necessarily $E = \mathbb{R}^{n-1} \times F$ for some set F of locally finite perimeter in \mathbb{R} . In particular, the topological boundary of E is smooth and $\partial^* E = \partial E$.

We will show that the boundary of E is connected, which will imply that E is a half-space. To this aim we use the argument from [24]. We argue by contradiction and assume that there are

two disjoint closed sets $\Gamma_1, \Gamma_2 \subset \partial E$ such that $\partial E = \Gamma_1 \cup \Gamma_2$. Let $a_1 < 0 < a_2$ be two numbers such that the function $\varphi : \partial E \rightarrow \mathbb{R}$

$$\varphi := \begin{cases} a_1, & \text{on } \Gamma_1 \\ a_2, & \text{on } \Gamma_2 \end{cases}$$

has zero average. Then clearly $\varphi \in H_\gamma^1(\partial E)$ and therefore (27) implies

$$\int_{\partial E} |B_E|^2 \varphi^2 + \frac{1}{2} \varphi^2 + \varepsilon |b| \nu_n \varphi^2 d\mathcal{H}_\gamma^{n-1} \leq 0.$$

From (24) we deduce

$$\int_{\partial E} |B_E|^2 \varphi^2 + \frac{1}{4} \varphi^2 d\mathcal{H}_\gamma^{n-1} \leq 0$$

which is obviously impossible. Hence, ∂E is connected.

Step 4. We need yet to show that E has the correct volume, i.e., $\gamma(E) = \phi(s)$. Since we have proved (29) we need only to show that the function $f : \mathbb{R} \rightarrow (0, \infty)$

$$f(t) := \mathcal{F}(H_t) = e^{-\frac{t^2}{2}} + \frac{\varepsilon}{4\pi} e^{-t^2} + \Lambda |\phi(t) - \phi(s)|$$

attains its minimum at $t = s \leq 0$.

Note that for every $t < 0$ it holds $f(t) < f(|t|)$. Moreover the function f is clearly increasing on $(s, 0)$. Hence, we only need to show that $f(s) < f(t)$ for every $t < s$. In $(-\infty, s)$ we have

$$f'(t) = -te^{-\frac{t^2}{2}} - \frac{\varepsilon}{2\pi} te^{-t^2} - \frac{\Lambda}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}.$$

In particular, f increases, reaches its maximum and decreases to $f(s)$. From our choices of Λ and ε in (22) we have

$$\lim_{t \rightarrow -\infty} f(t) = \Lambda \phi(s) \geq \sqrt{2} e^{-\frac{s^2}{2}} > f(s).$$

Thus the function f attains its minimum at $t = s$ which implies

$$\gamma(E) = \phi(s).$$

This concludes the proof. \square

Remark 1. We remark that the dependence on the mass in (6) is optimal. This can be verified by considering one-dimensional set which is a union of two intervals $E_s = (-\infty, a(s)) \cup (-a(s), \infty)$ where $s < 0$, and $a(s) < s$ is a number such that

$$\frac{2}{\sqrt{2\pi}} \int_{-\infty}^{a(s)} e^{-\frac{t^2}{2}} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^s e^{-\frac{t^2}{2}} dt, \quad (31)$$

i.e., $\gamma(E_s) = \phi(s)$. Then $b(E_s) = 0$ and $\beta(E_s) = \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}}$. The sharp mass dependence follows from

$$\liminf_{s \rightarrow -\infty} \frac{D(E_s)}{s^{-2} \beta(E_s)} = \frac{1}{\sqrt{2\pi}} \liminf_{s \rightarrow -\infty} \frac{2e^{-\frac{a(s)^2}{2}} - e^{-\frac{s^2}{2}}}{s^{-2} e^{-\frac{s^2}{2}}} \leq \frac{2}{\sqrt{2\pi}}. \quad (32)$$

For the readers convenience we will give the calculations below.

To show (32) we write $a(s) = s - \varepsilon(s)$. From (31) it follows that $\varepsilon(s) \rightarrow 0$ as $s \rightarrow -\infty$. We claim that

$$\liminf_{s \rightarrow -\infty} \frac{\varepsilon'(s)}{s^{-2}} \leq 1.$$

Indeed, if this were not true then we would have $\varepsilon(s) \geq \frac{1}{|s|}$ when $|s|$ is large. Then it follows from (31) that

$$\frac{1}{2} \leq \lim_{s \rightarrow -\infty} \frac{\int_{-\infty}^{s+1/s} e^{-\frac{t^2}{2}} dt}{\int_{-\infty}^s e^{-\frac{t^2}{2}} dt} = \lim_{s \rightarrow -\infty} \frac{(1 - \frac{1}{s^2}) e^{-\frac{(s+1/s)^2}{2}}}{e^{-\frac{s^2}{2}}} = \frac{1}{e}$$

which is a contradiction. By differentiating (31) with respect to s and substituting in the left-hand side of (32) we obtain

$$\liminf_{s \rightarrow -\infty} \frac{2e^{-\frac{(s-\varepsilon(s))^2}{2}} - e^{-\frac{s^2}{2}}}{s^{-2} e^{-\frac{s^2}{2}}} = \liminf_{s \rightarrow -\infty} \frac{2\varepsilon'(s) e^{-\frac{(s-\varepsilon(s))^2}{2}}}{s^{-2} e^{-\frac{s^2}{2}}} \leq 2.$$

We proceed by proving that the strong asymmetry controls the square of the standard one. Let us introduce a variant of the Fraenkel asymmetry. Given a Borel set E with $\gamma(E) = \phi(s)$ we define

$$\hat{\alpha}(E) := \begin{cases} 2\phi(-|s|) & \text{if } b(E) = 0, \\ \gamma(E \triangle H_{\omega,s}) & \text{if } b(E) \neq 0, \end{cases}$$

where $\omega = -b(E)/|b(E)|$. Since $\alpha(E) \leq 2\phi(-|s|)$, it is straightforward that $\hat{\alpha}(E) \geq \alpha(E)$. Compared to the asymmetry α , the asymmetry $\hat{\alpha}$ has the advantage that the half-space is chosen to be in the direction of the barycenter. The following estimate can be found in [13] but without explicit constant. We give a prove where we obtain the optimal dependence on the mass.

Proposition 4. *Let $E \subset \mathbb{R}^n$ be a set with $\gamma(E) = \phi(s)$. Then*

$$\beta(E) \geq \frac{e^{\frac{s^2}{2}}}{4} \hat{\alpha}(E)^2. \quad (33)$$

Proof. Since $\hat{\alpha}(E) = \hat{\alpha}(\mathbb{R}^n \setminus E)$ we may restrict ourselves to the case $s \leq 0$. By first order analysis it is easy to check that the function

$$f(s) := e^{-\frac{s^2}{2}} - \sqrt{\frac{2}{\pi}} \int_{-\infty}^s e^{-\frac{x_n^2}{2}} dx_n$$

is non-negative in $(-\infty, 0]$ or, equivalently, that $e^{-\frac{s^2}{2}} \geq 2\phi(s)$. Therefore, if $b(E) = 0$ we immediately have

$$\beta(E) = b_s = \frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}} \geq \frac{e^{\frac{s^2}{2}}}{\sqrt{2\pi}} \hat{\alpha}(E)^2.$$

Assume now that $b(E) \neq 0$ and, without loss of generality, that $e^{(n)} = -b(E)/|b(E)|$. For simplicity we write $H = H_{e^{(n)},s}$. Let a_1 and a_2 be positive numbers such that

$$\gamma(E \setminus H) = \frac{1}{\sqrt{2\pi}} \int_{s-a_1}^s e^{-\frac{x_n^2}{2}} dx_n = \frac{1}{\sqrt{2\pi}} \int_s^{s+a_2} e^{-\frac{x_n^2}{2}} dx_n.$$

Consider the sets $E^+ := E \setminus H$, $E^- := E \cap H$, $F^+ := \mathbb{R}^{n-1} \times [s, s+a_2]$, $F^- := \mathbb{R}^{n-1} \times (-\infty, s-a_1]$, and $F := F^+ \cup F^-$. By construction $\gamma(F) = \phi(s)$, $\gamma(F^+) = \gamma(E^+)$, and $\gamma(F^-) = \gamma(E^-)$. We

have

$$\begin{aligned}\beta(E) - \beta(F) &= \int_E x_n d\gamma - \int_F x_n d\gamma \\ &= \int_{E^+ \setminus F^+} (x_n - s - a_2) d\gamma + \int_{F^+ \setminus E^+} (-x_n + s + a_2) d\gamma \\ &\quad + \int_{E^- \setminus F^-} (x_n - s + a_1) d\gamma + \int_{F^- \setminus E^-} (-x_n + s - a_1) d\gamma \geq 0,\end{aligned}$$

because the integrands in the last term are all positive.

Since $\gamma(E \setminus H) = \gamma(H \setminus E)$ it is sufficient to show that $\beta(F) \geq e^{\frac{s^2}{2}} \gamma(E \setminus H)^2$. By first order analysis it is easy to check that for a fixed $s \leq 0$ the function

$$g(t) := \int_{s-t}^s (-x_n + s) e^{-\frac{x_n^2}{2}} dx_n - \frac{e^{\frac{s^2}{2}}}{2} \left(\int_{s-t}^s e^{-\frac{x_n^2}{2}} dx_n \right)^2$$

is non-negative in $[0, \infty)$. Indeed, g' is non-negative and $g(0) = 0$. By rearranging terms as above we deduce

$$\begin{aligned}\beta(F) &= \int_F x_n d\gamma - \int_H x_n d\gamma \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^{n-1}} e^{-\frac{|x'|^2}{2}} dx' \left(\int_{s-a_1}^s (-x_n + s) e^{-\frac{x_n^2}{2}} dx_n + \int_s^{s+a_2} (x_n - s) e^{-\frac{x_n^2}{2}} dx_n \right) \\ &\geq \frac{1}{\sqrt{2\pi}} \int_{s-a_1}^s (-x_n + s) e^{-\frac{x_n^2}{2}} dx_n \geq \frac{e^{\frac{s^2}{2}}}{2\sqrt{2\pi}} \left(\int_{s-a_1}^s e^{-\frac{x_n^2}{2}} dx_n \right)^2 \\ &= \sqrt{\frac{\pi}{2}} e^{\frac{s^2}{2}} \gamma(E \setminus H)^2.\end{aligned}$$

□

Remark 2. *The reduction to the set F in the previous proof gives in particular that the dependence on the mass in (33) is optimal. We note that even though the dependence on the mass in the main theorem (5) and in (33) are optimal, we do not know if these together provide the optimal mass dependence for (3).*

Given a set E of finite Gaussian perimeter, the *excess* of E is defined as

$$\mathcal{E}(E) := \min_{\omega \in \mathbb{S}^{n-1}} \left\{ \int_{\partial^* E} |\nu_E - \omega|^2 d\mathcal{H}_\gamma^{n-1}(x) \right\}. \quad (34)$$

We conclude by proving that the isoperimetric deficit controls the excess of the set.

Corollary 1. *There exists an absolute constant c such that for every $s \in \mathbb{R}$ and for every set of finite Gaussian perimeter $E \subset \mathbb{R}^n$ with $\gamma(E) = \phi(s)$ the following estimate holds*

$$\mathcal{E}(E) \leq c(1 + s^2)D(E). \quad (35)$$

Moreover, if $b(E) \neq 0$, the minimum in (34) is attained by $\omega = -b(E)/|b(E)|$.

Proof. By the divergence theorem

$$\begin{aligned}\langle b(E), \omega \rangle &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_E \langle x, \omega \rangle e^{-\frac{|x|^2}{2}} dx \\ &= -\frac{1}{(2\pi)^{\frac{n}{2}}} \int_E \operatorname{div} \left(e^{-\frac{|x|^2}{2}} \omega \right) dx = -\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\partial^* E} \langle \omega, \nu_E \rangle e^{-\frac{|x|^2}{2}} d\mathcal{H}^{n-1} \\ &= \frac{1}{2\sqrt{2\pi}} \int_{\partial^* E} |\omega - \nu_E|^2 d\mathcal{H}_\gamma^{n-1} - \frac{1}{\sqrt{2\pi}} \int_{\partial^* E} d\mathcal{H}_\gamma^{n-1}.\end{aligned}$$

By minimizing over $\omega \in \mathbb{S}^{n-1}$ we get

$$\mathcal{E}(E) = 2P_\gamma(E) - 2\sqrt{2\pi}|b(E)| = 2D(E) + 2\sqrt{2\pi}\beta(E).$$

Finally, thanks to the estimate (6), we obtain (35). \square

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