Asymptotic analysis of periodically-perforated nonlinear media at and close to the critical exponent

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Abstract

We give a general Γ-convergence result for vector-valued non-linear energies defined on perforated domains for integrands with p-growth in the critical case $p = n$. We characterize the limit extra term by a formula of homogenization type. We also prove that for p close to n there are three regimes, two with a non trivial size of the perforation (exponential and mixed polynomial-exponential), and one where the Γ-limit is always trivial.

1 Introduction

An interesting and much studied class of problems are variational problems defined on varying domains. The prototype of these domains are *perforated domains*; i.e., those obtained from a fixed Ω by removing some periodic set, the simplest of which is a periodic array of closed sets:

$$\Omega_\delta = \Omega \setminus \bigcup_{i\in\mathbb{Z}^n} (\delta i + \varepsilon K), \tag{1}$$

with $\varepsilon = \varepsilon(\delta)$. On the set K, we suppose that it is a bounded closed set with non empty interior. When we consider Dirichlet boundary conditions on the boundary of Ω_δ (or on the boundary of Ω_δ interior to Ω) the asymptotic behaviour of such problems is obtained by studying the Γ-convergence of the functionals

$$F_\delta(u) = \begin{cases} \displaystyle\int_\Omega f(Du)\,dx & \text{if } u \in W_0^{1,p}(\Omega;\mathbb{R}^m) \text{ and } u = 0 \text{ on } \Omega\setminus\Omega_\delta \\ +\infty & \text{otherwise,} \end{cases} \tag{2}$$

where f is an energy density growing as $|Du|^p$. Taking $f(Du) = |Du|^p$ above, we encounter some by-now 'classical' results, first observed by Marchenko and Khruslov [7], and subsequently recast in a variational setting by Cioranescu and Murat [6], in which case for a *non-trivial scaling* of the perforation the Γ-limit contains an additional 'strange term' in place of the internal boundary conditions. To obtain this form of the Γ-limit different choices of ε must be made according to the space dimension n, that in this case are:

$$\text{(polynomial scaling)} \qquad \varepsilon = R\delta^{n/n-p} \qquad \text{if } p < n \text{ (with } R > 0) \tag{3}$$

$$\text{(exponential scaling)} \qquad \varepsilon = \exp(-a/\delta^{n/n-1}) \qquad \text{if } p = n \text{ (with } a > 0). \tag{4}$$

A complete analysis by Γ-convergence for energies with a general (quasiconvex) integrand f with p-growth, and depending on vector-valued functions has been performed by Ansini and Braides in the case leading to the polynomial scaling ($p < n$) [2]. In this paper we treat the case $p = n$, which is the one leading to the exponential scaling, by first giving general convergence result for this critical case, and then exploring the case when p is varying and close to n.

2 Asymptotic behaviour at the critical scaling

In the case $p = n$ we have the following general convergence result.

Theorem 2.1 (Asymptotic behaviour at the critical exponent) *Let* $f : \mathbb{M}^{m\times n} \to [0, \infty)$ *be a quasiconvex function with* $f(0) = 0$*; we suppose that there exist* $c_1, c_2, k > 0$ *such that*

$$c_1|A|^n \le f(A) \le c_2|A|^n, \qquad |f(A) - f(B)| \le k|A - B|||A|^{n-1} + |B|^{n-1}|$$

for all $A, B \in \mathbb{M}^{m\times n}$*. Let* δ_j *be a positive infinitesimal sequence and let* $a > 0$*. Then, upon passing to a subsequence of* (δ_j) *(not relabelled) and having set* $T_j = \exp(a\delta_j^{-n/n-1})$*, the limit*

$$\varphi(z) \;=\; \sup_{s>0} \lim_{j\to\infty} \frac{(\log T_j)^{n-1}}{a^{n-1}} \min\Big\{\int_{B_{s\delta_j T_j}} \frac{f(T_j Du)}{T_j^n}\,dx : u \in z + W_0^{1,n}(B_{s\delta_j T_j};\mathbb{R}^m),\ u = 0 \text{ on } K\Big\} \tag{5}$$

exists for all z *and the functionals* F_{δ_j} *defined in* (2) Γ*-converge (with respect to the strong convergence of* $L^n(\Omega;\mathbb{R}^m)$*) to the functional* $F_0 : L^n(\Omega;\mathbb{R}^m) \to [0 + \infty)$ *defined by*

$$F_0(u) = \begin{cases} \displaystyle\int_\Omega f(Du)\,dx + \int_\Omega \varphi(u)\,dx & \text{if } u \in W_0^{1,n}(\Omega;\mathbb{R}^m) \\ +\infty & \text{otherwise.} \end{cases} \tag{6}$$

Furthermore, if f *is positively homogeneous of degree* n *the function* φ *and the functional* F_0 *are independent of the subsequence, so that the whole family* (F_δ) Γ*-converges. Moreover the simplified formula*

$$\varphi(z) = \lim_{T\to\infty} \frac{(\log T)^{n-1}}{a^{n-1}} \min\Big\{\int_{B_{T_j}} f(Du)\,dx : u \in z + W_0^{1,n}(B_{T_j};\mathbb{R}^m),\ u = 0 \text{ on } B_1\Big\} \tag{7}$$

holds, independent of K.

The proof of this results relies on a general argument by Ansini and Braides [2], which reduces the computation of the 'extra term' along a sequence $u_\delta \to u$ to an estimate close to the perforation; *i.e.*, on balls $B_{\rho\delta}(\delta i)$ for some small $\rho > 0$ (*a posteriori* independent of ρ). It is easily seen that the limit is not trivial only when $\varepsilon << \delta$ so that $K \subset B_{\rho\delta/\varepsilon}$ for ε small enough. If u is continuous and f is p-homogeneous this estimate reads

$$\begin{aligned} &\int_{B_{\rho\delta}(\delta i)} f(Du_\delta)\,dx \\ \geq\ & \varepsilon^{n-p}|u(\delta i)|^p \min\Big\{\int_{B_{\rho\delta/\varepsilon}(0)} f(Dv)\,dy : v = 0 \text{ on } K,\ v = 1 \text{ on } \partial B_{\rho\delta/\varepsilon}(0)\Big\}. \end{aligned} \tag{8}$$

When $p < n$ the minimum problem in (8) is estimated in [2] by the *p-capacity* of the set K (with respect to $\mathbb{R}^n$). Summing up in i, we obtain a Riemann sum provided that $\varepsilon^{n-p} = M\delta^n + o(\delta)$, which gives the scaling $\varepsilon = R\delta^{\frac{n}{n-p}}$. In the case $p = n$ the same argument gives a trivial lower bound since the corresponding limit computation of the n-capacity of K (with respect to $\mathbb{R}^n$) gives

$$\inf\Big\{\int_{\mathbb{R}^n} |Dv|^n\,dy : v = 0 \text{ on } K,\ 1 - v \in W^{1,n}(\mathbb{R}^n)\Big\} = 0,$$

from which we deduce that the limit of the right-hand side of (8) is 0. We have therefore to depart from the proof in [2] by a more difficult analysis of the behaviour of the energies defined by the minimum problems in (8). This can be done explicitly if K is a ball, and gives (4) as a result. Note that in this case the radius of K does not affect the result; we can therefore extend the result to arbitrary K with non-empty interior by comparison with the case of balls containing K or contained in K, respectively, and conclude that the form of the limit is indeed independent of the shape of K. Further technical arguments are needed when f is not positively homogeneous; a detailed proof can be found in [9].

Remark 1 Using the terminology introduced in [5] our result can be summarized by saying that the functionals F_δ in (2) are equivalent to G_δ defined as

$$G_\delta(u) = \int_\Omega f(Du)\,dx + \frac{|\log\varepsilon|^{n-1}}{\delta^n}\int_\Omega \varphi(u)\,dx, \qquad u \in W_0^{1,n}(\Omega;\mathbb{R}^m), \tag{9}$$

for $\delta \to 0$, meaning that both families have the same Γ-limits on all Γ-converging sequences with δ_j and ε_j tending to 0. Our arguments show that the exponential regime derives from the scaling invariance of the problems in *(8)*, which eliminates the pre-factor ε^{n-p}, and from the logarithmic behaviour of minimizers. We have shown that the usual 'capacitary' formula for the limit integrand φ in the case $p < n$ is substituted by an interesting 'homogenization' formula. This highlights that in this critical case the energy does not concentrate at the same scale as the perforation radius, in a fashion similar to optimal sequences for Ginzburg-Landau functionals [3, 8, 1].

3 Asymptotic behaviour close to the critical scaling

Theorem 2.1, together with the companion analysis for $p < n$, shows a passage from a polynomial to an exponential decay of the relevant perforations at the critical scaling. To overcome the discontinuity in the description of the asymptotic analysis of energies (2) at $p = n$ we consider their dependence also on varying p. Since we are interested in a scale analysis, it is sufficient to consider the (scalar) case $f(Du) = |Du|^p$ and $K = \overline{B}_1$. We set

$$F^p_\delta(u) = \begin{cases} \displaystyle\int_\Omega |Du|^p\,dx & \text{if } u \in W^{1,p}_0(\Omega) \text{ and } u = 0 \text{ on } \Omega \setminus \Omega_\delta \\ +\infty & \text{otherwise.} \end{cases} \tag{10}$$

By letting at the same time $\delta \to 0$ and $p \to n$ we can highlight three different behaviours of the perforation scaling. If $p - n >> \delta^{n/n-1}$ then the functionals behave as in the case $p > n$ where every perforation gives a trivial limit since it enforces the constraint $u = 0$ on limits of sequences bounded in energy. In the other two regimes there exists a scaling giving a non-trivial limit. If $|p - n| = O(\delta^{n/n-1})$ then the critical perforation scale is exponential as for $p = n$, while in the remaining case $n - p >> \delta^{n/n-1}$ it is an interpolation between the exponential and the polynomial scaling. The precise form of the Γ-limit in dependence of the perforation is described by the following theorem, in which we also explicitly link the radii of the perforation to the coefficient κ of the additional term in the limit.

Theorem 3.1 (Asymptotic behaviour close to the critical exponent) *The Γ-limit of the energies F^p_δ defined in (10) as $\delta \to 0$ and $p = p(\delta) \to n$ exists and is described explicitly in the following three regimes. In the first two there exists a choice of the perforation $\varepsilon = \varepsilon(\delta)$ such that the limit is*

$$F_0(u) = \begin{cases} \displaystyle\int_\Omega |Du|^n\,dx + \kappa \int_\Omega |u|^n\,dx & \text{if } u \in W^{1,n}_0(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \tag{11}$$

The link between ε and κ is expressed by

(i) (interpolation between the polynomial and the exponential regime) *if $p < n$ and $n - p >> \delta^{n/n-1}$ then*

$$\varepsilon = R^{\frac{1}{n-p}}\delta^{\frac{n}{n-p}}(n-p)^{-\frac{(n-1)}{(n-p)}}, \qquad \kappa = R\frac{\omega_{n-1}}{(n-1)^{(n-1)}}, \qquad \text{with } R > 0;$$

(ii) (exponential regime) *if $n - p = \gamma\,\delta^{n/n-1} + o(\delta^{n/n-1})$ for $\gamma \in \mathbb{R}$ then*

$$\varepsilon = \exp(-a/\delta^{n/n-1}), \qquad \kappa = \frac{\omega_{n-1}}{(n-1)^{(n-1)}}e^{-a\gamma}\Big(\frac{1-e^{-a\gamma/(n-1)}}{\gamma}\Big)^{1-n}, \ \text{with } a > 0, \ \text{if } \gamma \neq 0$$

$$\kappa = \frac{\omega_{n-1}}{a^{n-1}}, \ \text{with } a > 0, \ \text{if } \gamma = 0.$$

Note that in this regime $n-p$ can also be negative;

(iii) (rigid regime) *if $p>n$ and $p-n>>\delta^{n/n-1}$ then the limit is finite (and null) only on the constant function zero (this can be seen as a degenerate case on* (11) *when we take take $\kappa=+\infty$).*

The proof of this result relies on adapting the arguments in [2] to the case of varying exponent p, to carefully estimate the minimum problems in (8), and understand their interplay with the (unknown) prefactor ε^{n-p}. A detailed proof is contained in [10].

Remark 2 Our arguments show that the functionals F^p_δ are equivalent to G^p_δ given by

$$G^p_\delta(u)=\int_\Omega |Du|^p\,dx+C_p\frac{\varepsilon^{n-p}}{\delta^n}\Big(\frac{1-\varepsilon^{n-p/p-1}}{n-p}\Big)^{1-p}\int_\Omega |u|^p\,dx,\qquad u\in W^{1,p}_0(\Omega;\mathbb{R}^m)\quad (12)$$

(C_p explicitly computable) as $p\to n$. This description can be extended to all $p>1$ upon noticing that the scaling in regime (i) reduces to the usual polynomial perforation scaling for $p<n$ fixed. It is worth noting that this description is uniform in p, in the sense that the Γ-limits of Γ-converging subsequences of the two families of energies are the same also if we let p vary as δ and ε tend to 0. A general analysis of this kind of uniform equivalence between functionals can be found in [5].

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