Global Properties of Invariant Measures*

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Abstract

We study global regularity properties of invariant measures associated with second order differential operators in $\mathbf{R}^N$. Under suitable conditions, we prove global boundedness of the density, Sobolev regularity, a Harnack inequality and pointwise upper and lower bounds.

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1 Introduction.

In this paper we study global regularity properties of invariant measures associated with second-order elliptic partial differential operators in $\mathbf{R}^N$

$$A = \sum_{i,j=1}^{N} D_i(a_{ij} D_j) + \sum_{i=1}^{N} F_i D_i = A_0 + F \cdot D. \tag{1.1}$$

We assume that there exists a Borel probability measure μ on $\mathbf{R}^N$ such that

$$\int_{\mathbf{R}^N} A\phi \, d\mu = 0 \tag{1.2}$$

for every $\phi \in C_c^\infty(\mathbf{R}^N)$. If the operator A, endowed with a certain domain $D(A)$, generates a semigroup $(T(t))_{t\geq 0}$ in a suitable function space X, then (1.2) holds for every $\phi \in D(A)$ if and only if

$$\int_{\mathbf{R}^N} T(t) f \, d\mu = \int_{\mathbf{R}^N} f \, d\mu \tag{1.3}$$

for every $f \in X$ and $t \geq 0$ and this means that the measure μ is an invariant distribution for the Markov process described by $(A, D(A))$. For this reason a probability measure μ satisfying (1.2) is called *invariant*, even though no semigroup explicitly appears. We refer the reader to [9, Chapter 4] for a general background on invariant measures of Markov processes and to [15], see also [8], for the investigation of the problem of existence of a semigroup satisfying (1.3).

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Many local regularity properties are known for invariant measures, even under very weak conditions on the coefficients, see e.g. [5]. On the other hand, to our knowledge the only available results dealing with global regularity are [6], [3], which have been the starting point of our investigation, and the very recent [13] where $W^{2,p}(\mathbf{R}^N)$ regularity of the invariant measure is established assuming that the diffusion coefficients a_{ij} belong to $C^1_b(\mathbf{R}^N)$ and that the drift F is slightly less than globally Lipschitz continuous.

In order to describe the main results of this paper, let us state precisely our assumptions on the coefficients of A which will be kept in the whole paper without further mentioning.

(H0) $a_{ij} = a_{ji}, F_i : \mathbf{R}^N \to \mathbf{R}$, with $a_{ij} \in W^{1,p}_{loc}(\mathbf{R}^N)$, $F_i \in L^p_{loc}(\mu)$ for some $p > N$ and

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$$

for every $x, \xi \in \mathbf{R}^N$ and a suitable $\lambda > 0$.

(H1) For every $i, j = 1, \ldots, N$, $(1 + |x|^2)^{-1}a_{ij} \in L^1(\mu)$ and $(1 + |x|)^{-1}D_i a_{ij} \in L^1(\mu)$.

(H2) $F \in L^1(\mu)$.

Notice that neither the matrix (a_{ij}) nor the drift $F = (F_1, \ldots, F_N)$ are assumed to be bounded in $\mathbf{R}^N$. Note also that (H1) is always satisfied if the a_{ij} grow at most quadratically and their gradients at most linearly at infinity. As regards the local regularity of the coefficients, we recall that (H0) guarantees that μ is given by a density $\rho \in W^{1,p}_{loc}(\mathbf{R}^N)$, see [5, Corollary 2.10]; in particular, ρ is a continuous function. If $F \in L^p_{loc}(\mathbf{R}^N)$, i.e. it is locally integrable with respect to the Lebesgue measure and not with respect to μ (which implies $F \in L^p_{loc}((\mu))$, then ρ is positive, see [5, Corollary 2.11]. The comments following [5, Corollary 2.10] motivate why, in some situations, is also more natural to require the integrability of F with respect to μ and not to the Lebesgue measure.

The plan of the paper is the following.

In Section 2 we recall some known facts on local regularity of μ and show how the integrability of certain unbounded functions with respect to μ can be obtained via Lyapunov functions techniques. Moreover, these results allow us to give some growth conditions on the coefficients a_{ij}, F in order that the integrability properties with respect to μ contained in (H1), (H2) hold true, see Remark 2.6.

In Section 3 we show global boundedness of the density ρ, a first global regularity result which will be crucial in the developments of the subsequent sections. In Theorem 3.1 we prove that $\sqrt{\rho}$ belongs to $W^{1,2}(\mathbf{R}^N)$, provided that F belongs to $L^2(\mu)$, thus extending a result from [3], where, in addition, $a_{ij} \in C^1_b(\mathbf{R}^N)$ was assumed. Observe, however, that the condition $F \in L^p_{loc}(\mu)$ is not needed in [3]. Then we show that, if $F \in L^k(\mu)$ for some $k > N$ or $F, \operatorname{div} F \in L^k(\mu)$ for $k > N/2$, $k \geq 2$, the density ρ is bounded in $\mathbf{R}^N$. The proof relies upon Moser's iteration technique, whose starting point is Theorem 3.1. The cases $2 \leq k < N$ are also examined. The local regularity of ρ which follows from (H0) is crucial to perform the needed integration by parts. In fact, in our approach, global regularity is deduced from local regularity and this, in turn, holds since the diffusion matrix (a_{ij}) is locally uniformly elliptic. However, the assumption $F \in L^p_{loc}(\mu)$ for some $p > N$, though a weak one, looks too strong when the global integrability condition $F \in L^k(\mu)$ is required only for $k \leq N$ and it is possible that further investigation will remove it in these cases. Results in this direction have been obtained in [3] using an approximation procedure that leads directly to global regularity.

In Section 4 we prove Sobolev regularity assuming that $a_{ij} \in C_b^1(\mathbf{R}^N)$. Moreover, we also consider the case $F \in L^k(\mu)$ with $1 \le k < 2$, excluded in the previous section. We prove both $W^{1,p}$ and $W^{2,p}$ regularity; in the second case, however, we need also assumptions on the divergence of the drift F. We point out that the results on global boundedness and $W^{1,p}$ regularity are precise as regards the exponents involved: in fact, they reduce to the Sobolev embeddings when $A = \Delta - D\Phi \cdot D$, so that $\rho = e^{-\Phi}$, see Remarks 3.11, 4.5. On the other hand, those concerning $W^{2,p}$ regularity are not optimal. This depends upon the fact that we can prove that $\sqrt{\rho}$ belongs to $W^{1,2}(\mathbf{R}^N)$ when $F \in L^2(\mu)$, whereas the conjecture $\rho^{1/k} \in W^{1,k}(\mathbf{R}^N)$ when $F \in L^k(\mu)$, needed to improve our conditions, remains open.

In Section 5 we prove a Harnack-type inequality for ρ finding explicit bounds on its logarithmic derivative. These bounds are used later to obtain sufficient conditions under which $D\rho/\rho$ belongs to $L^p(\mu)$ for $1 \le p < \infty$. We point out that, in contrast with the case $p = 2$ which was already known, see [3], the general case is obtained requiring more regularity on the coefficients and using a different approach.

In Section 6 we prove both upper and lower bounds on ρ assuming that certain exponentials are integrable with respect to μ. Basically we show that if $\exp\{\delta|x|^\beta\}$ belongs to $L^1(\mu)$ for some $\delta, \beta > 0$, then $\rho(x) \le c_1 \exp\{-c_2|x|^\beta\}$ for related constants $c_1, c_2 > 0$. Explicit conditions for the integrability of the above exponentials are given in Section 2. Lower bounds for ρ are deduced from the Harnack inequality of Section 5 assuming growth conditions of polynomial type on the coefficients. Combining upper and lower bounds, the precise decay of ρ is given for a class of operators.

Notation $C_b^k(\mathbf{R}^N)$ is the space of all k times continuously differentiable functions in $\mathbf{R}^N$, bounded together their derivatives up to the order k, $C_0(\mathbf{R}^N)$ is the space of continuous functions on $\mathbf{R}^N$ vanishing as $|x| \to \infty$ and $C_c^\infty(\mathbf{R}^N)$ is the space of test functions. For $1 \le p \le \infty$, $k \in \mathbf{N}$, $W^{k,p}(\mathbf{R}^N)$ denotes the classical Sobolev space of all L^p-functions having weak derivatives in L^p up to the order k. Its norm is denoted by $\|\cdot\|_{k,p}$ and by $\|\cdot\|_p$ when $k = 0$. All integrals where the underlying measure is not explicitly indicated are understood with respect to the Lebesgue measure dx. Accordingly, we write $L^p(\mathbf{R}^N)$ when the Lebesgue measure is understood. The L^p-space with respect to a measure μ is denoted by $L^p(\mu)$.

If $G : \mathbf{R}^N \to \mathbf{R}^m$ is a C^1-function, then $|DG|^2 = \sum_{i,j} |D_i G_j|^2$ and $|D^2 G|^2 = \sum_{i,j,h} |D_{ij} G_h|^2$.

We define

$$\lambda = \inf_{x \in \mathbf{R}^N} \lambda(x) \qquad \Lambda = \sup_{x \in \mathbf{R}^N} \Lambda(x), \tag{1.4}$$

where $\lambda(x)$ and $\Lambda(x)$ are the minimum and the maximum eigenvalue of the matrix $(a_{ij}(x))$, respectively. Observe that λ is the same as in (H0) and it is supposed to be positive. On the other hand, we do not assume that Λ is finite.

We write $a(\xi, \eta)$ for $\sum_{i,j} a_{ij}(\cdot)\xi_i \eta_j$, $\xi, \eta \in \mathbf{R}^N$.

2 Existence, uniqueness and integrability properties

In this section we briefly recall some results on invariant measures.

First, μ is absolutely continuous with respect to Lebesgue measure: we write $d\mu = \rho dx$, and state a result concerning the local regularity of ρ which allows us to perform some integrations by parts. We refer to [5, Corollaries 2.10, 2.11] for the proof (see also [9, Chapter 4] for the absolute continuity of μ and the positivity of its density ρ).

Theorem 2.1 *Assume that μ is an invariant measure for A. Then $d\mu = \rho dx$ with $\rho \in W^{1,p}_{loc}(\mathbf{R}^N)$, where $p > N$ is the summability exponent in (H0). Moreover, if $F \in L^p_{loc}(\mathbf{R}^N)$ then (the continuous representative of) ρ is positive.*

Throughout the paper we always identify ρ with its continuous representative.

As regards existence and uniqueness of invariant measures we quote the following improvement of Hasminskii's criterion proved in [7], see also [8, Corollary 3.3] for the uniqueness part. A function V as in the following theorem is often named a *Lyapunov function*.

Theorem 2.2 *Assume $F \in L^p_{loc}(\mathbf{R}^N)$ and that there exists a C^2-function $V : \mathbf{R}^N \to \mathbf{R}$ such that $V(x) \to \infty$, $AV(x) \to -\infty$ as $|x| \to \infty$. Then A has a unique invariant measure μ.*

It is a consequence of the proof of [7, Theorem 1.1] that AV belongs to $L^1(\mu)$. Since this fact will be useful later and for reader's convenience we extract from [7, Lemma 1.1] a short proof.

Proposition 2.3 *Assume that there exists a C^2-function $V : \mathbf{R}^N \to \mathbf{R}$ such that $V(x) \to \infty$ as $|x| \to \infty$ and $AV(x) \le 0$ for large $|x|$. Then AV belongs to $L^1(\mu)$.*

Proof. A simple approximation argument shows that (1.2) is satisfied for every $\phi \in C^2(\mathbf{R}^N)$ with compact support. Therefore, since $A1 = 0$, it holds for every $\phi \in C^2(\mathbf{R}^N)$ constant outside of a large ball. For every n, we consider $\psi_n \in C^\infty(\mathbf{R})$ such that $\psi_n(t) = t$ for $t \le n$, ψ_n is constant in $[n+1, \infty[$, $\psi_n' \ge 0$, $\psi_n'' \le 0$. Then (1.2) holds for $\psi_n \circ V$. Let B be a ball such that $AV(x) \le 0$ if $x \notin B$. Then

$$A(\psi_n \circ V) = (\psi_n' \circ V)AV + (\psi_n'' \circ V) \sum_{i,j=1}^{N} a_{ij} D_i V D_j V \le 0$$

outside B. Then, for large n

$$\int_{\mathbf{R}^N \setminus B} |A(\psi_n \circ V)|\, d\mu = -\int_{\mathbf{R}^N \setminus B} A(\psi_n \circ V)\, d\mu = \int_B AV\, d\mu \le C$$

and the statement follows letting $n \to \infty$ and using Fatou's lemma. □

The integrability of certain exponential functions will be important in Section 6 to derive upper bounds on ρ. A sufficient condition to this aim is given in the following proposition.

Proposition 2.4 *Let $\Lambda(x)$ be the maximum eigenvalue of $(a_{ij}(x))$. Assume that*

$$\limsup_{|x| \to \infty} \left(c\Lambda(x) + |x|^{1-\beta} G(x) \cdot \frac{x}{|x|} \right) < 0 \tag{2.1}$$

for some $c > 0$, $\beta > 0$, where $G = (g_1, \ldots, g_N)$ and $g_i = F_i + \sum_j D_j a_{ji}$. Then $V(x) = \exp\{\delta |x|^\beta\}$ for $|x| \ge 1$ is a Lyapunov function for $\delta < \beta^{-1} c$. Moreover, $\exp\{\delta |x|^\beta\}$ is integrable with respect to μ, for $\delta < \beta^{-1} c$.

Proof. Let $V(x) = \exp\{\delta |x|^\beta\}$ for $|x| \ge 1$. We obtain, by a straightforward computation,

$$\begin{aligned} AV(x) &= \delta\beta |x|^{\beta-1} e^{\delta |x|^\beta} \left(\frac{\sum_i a_{ii}(x)}{|x|} + \frac{\beta - 2}{|x|^3} \sum_{i,j=1}^{N} a_{ij}(x) x_i x_j \right. \\ &\qquad \left. + \delta\beta |x|^{\beta-3} \sum_{i,j=1}^{N} a_{ij}(x) x_i x_j + G \cdot \frac{x}{|x|} \right). \end{aligned}$$

Since the quadratic form $|\sum_{i,j} a_{ij}(x)x_i x_j|$ can be estimated by $\Lambda(x)|x|^2$, the first statement can be checked by elementary arguments. As regards the second, observe that $|AV|$ is integrable with respect to μ, by Proposition 2.3, and that either $|AV|$ is bigger than V (when $\beta \geq 1$) or $|AV|$ is bigger than $V^{1-\varepsilon}$ for every $\varepsilon > 0$ (when $0 < \beta < 1$) for large $|x|$. □

Similar computations prove the following result which will be useful in Section 6. Observe that, since $\beta > 1$ and $a_{ij} \in C_b^1(\mathbf{R}^N)$, it is no longer necessary to introduce the function G of the above proposition.

Corollary 2.5 *Assume that $a_{ij} \in C_b^1(\mathbf{R}^N)$ and that*

$$\limsup_{|x|\to\infty} |x|^{1-\beta} F(x) \cdot \frac{x}{|x|} = -c, \tag{2.2}$$

$0 < c \leq \infty$, for some $\beta > 1$. Then $V(x) = \exp\{\delta|x|^\beta\}$ for $|x| \geq 1$ is a Lyapunov function for $\delta < (\beta\Lambda)^{-1}c$. Moreover, $\exp\{\delta|x|^\beta\}$ is integrable with respect to μ for $\delta < (\beta\Lambda)^{-1}c$.

Remark 2.6 Proposition 2.4 and Corollary 2.5 can be used to check assumptions (H1) and (H2). In fact, under the hypotheses of those statements, if the functions $(1+|x|^2)^{-1}|a_{ij}|$, $(1+|x|)^{-1}|D_i a_{ij}|$ and $|F|$ grow at infinity not faster than $\exp\{|x|^\gamma\}$ for some $\gamma < \beta$ then (H1) and (H2) are satisfied.

Remark 2.7 Equation (2.2) is a radial assumption on F and, if $0 < c < \infty$, it says that the inward radial component of F has a prescribed polynomial behaviour. Of course, changing $x/|x|$ to $(x - x_0)/|x - x_0|$ leads to a new condition that, though not equivalent to (2.2), yields similar conclusions.

Remark 2.8 Assume that μ is the invariant measure of a Feller semigroup $(T(t))_{t\geq 0}$. The integrability of the exponential functions $\exp\{\delta|x|^2\}$, hence the validity of (2.2) with $\beta = 2$, is strongly connected with hypercontractivity and supercontractivity properties of the semigroup in L^p-spaces with respect to μ, see [14]. We also remark that if $\beta > 2$ is allowed in (2.2), then $T(t))_{t\geq 0}$ is ultracontractive, see [14, Corollary 2.5] and compact in $C_b(\mathbf{R}^N)$, see [12, Corollary 3.11].

3 Global boundedness

First we state and prove a global regularity result which generalises to our setting [3, Thereom 1.1] and [6, Theorem 3.1]. We do not assume that the diffusion matrix (a_{ij}) is bounded and globally Lipschitz continuous. However we suppose that $F \in L^p_{loc}(\mu)$ and this is not needed in [6], [3]. In the sequel, we use the convention that $|D\rho|^2/\rho^s = 0$ on the set $\{\rho = 0\}$, for any $s > 0$.

Theorem 3.1 *Assume that $d\mu = \rho\, dx$ is an invariant measure for A and that $F \in L^2(\mu)$. Then $\sqrt{\rho} \in W^{1,2}(\mathbf{R}^N)$. Moreover*

$$\int_{\mathbf{R}^N} \frac{|D\rho|^2}{\rho} \leq \frac{1}{\lambda^2} \int_{\mathbf{R}^N} |F|^2 d\mu. \tag{3.1}$$

PROOF. From Theorem 2.1 we know that $\rho \in W^{1,p}_{loc}(\mathbf{R}^N)$ where $p > N$ is the exponent in (H0). The invariance of μ then implies

$$\int_{\mathbf{R}^N} a(D\rho, D\phi) = \int_{\mathbf{R}^N} F \cdot D\phi\, \rho \tag{3.2}$$

for every $\phi \in C_c^\infty(\mathbf{R}^N)$. Since ρ is continuous, then $F\rho \in L^2_{loc}(\mathbf{R}^N)$ and, by density, equality (3.2) holds if ϕ belongs to $W^{1,2}(\mathbf{R}^N)$ and has a compact support. Let us take $\eta \in C_c^\infty(\mathbf{R}^N)$ such that $\eta(x) = 1$ for $|x| \le 1$ and $\eta(x) = 0$ for $|x| \ge 2$, $\eta_n(x) = \eta(x/n)$ and observe that for every ε, k such that $0 < \varepsilon < k$, the function $\log((\rho \vee \varepsilon) \wedge k)$ belongs to $W^{1,p}_{loc}(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$, $p > N \ge 2$. Plugging $\phi = \eta_n^2 \log((\rho \vee \varepsilon) \wedge k)$ in (3.2), since

$$D \log((\rho \vee \varepsilon) \wedge k) = \frac{D\rho}{\rho} \chi_{\{\varepsilon < \rho < k\}}$$

we obtain

$$\begin{aligned} \int_{\mathbf{R}^N} \eta_n^2 \chi_{\{\varepsilon<\rho<k\}} \frac{a(D\rho, D\rho)}{\rho} &= -2 \int_{\mathbf{R}^N} \eta_n \log((\rho \vee \varepsilon) \wedge k) a(D\rho, D\eta_n) \\ &\quad + \int_{\mathbf{R}^N} \eta_n^2 F \cdot D\rho \chi_{\{\varepsilon<\rho<k\}} \\ &\quad + 2 \int_{\mathbf{R}^N} \eta_n \rho \log((\rho \vee \varepsilon) \wedge k) F \cdot D\eta_n. \end{aligned}$$

The above equality yields

$$\begin{aligned} \int_{\mathbf{R}^N} \eta_n^2 \chi_{\{\varepsilon<\rho<k\}} \frac{a(D\rho, D\rho)}{\rho} &\le \left(\int_{\mathbf{R}^N} \eta_n^2 \chi_{\{\varepsilon<\rho<k\}} \frac{|D\rho|^2}{\rho} \right)^{1/2} \left(\int_{\mathbf{R}^N} |F|^2 \rho \right)^{1/2} \\ &\quad + \frac{C(\varepsilon, k)}{n} \int_{\mathbf{R}^N} |F| \rho + I_n, \end{aligned} \tag{3.3}$$

where

$$I_n = -2 \int_{\mathbf{R}^N} \eta_n \log((\rho \vee \varepsilon) \wedge k) a(D\rho, D\eta_n).$$

Integrating by parts we obtain

$$\begin{aligned} I_n &= 2 \int_{\{n \le |x| \le 2n\}} \rho \Bigg(\log((\rho \vee \varepsilon) \wedge k) a(D\eta_n, D\eta_n) + \eta_n \log((\rho \vee \varepsilon) \wedge k) \sum_{i,j=1}^N a_{ij} D_{ij} \eta_n \\ &\quad + \eta_n \log((\rho \vee \varepsilon) \wedge k) \sum_{i,j=1}^N D_i a_{ij} D_j \eta_n + \eta_n \chi_{\{\varepsilon<\rho<k\}} \frac{a(D\rho, D\eta_n)}{\rho} \Bigg). \end{aligned}$$

Since $|D\eta_n| \le C(1 + |x|)^{-1}$, $|D^2\eta_n| \le C(1 + |x|^2)^{-1}$ in $\{n \le |x| \le 2n\}$, with C independent of n, assumption (H1) implies that, for a suitable $\omega(\varepsilon, k, n)$ which goes to 0 as $n \to \infty$ for fixed ε, k, we have

$$\begin{aligned} |I_n| &\le \omega(\varepsilon, k, n) + 2 \int_{\{n \le |x| \le 2n\}} \eta_n \chi_{\{\varepsilon<\rho<k\}} a(D\rho, D\eta_n) \\ &\le \omega(\varepsilon, k, n) + 2 \left(\int_{\mathbf{R}^N} \eta_n^2 \chi_{\{\varepsilon<\rho<k\}} \frac{a(D\rho, D\rho)}{\rho} \right)^{1/2} \left(\int_{\{n \le |x| \le 2n\}} a(D\eta_n, D\eta_n) \rho \right)^{1/2} \\ &\le \omega(\varepsilon, k, n)(1 + \delta^{-1}) + \delta \int_{\mathbf{R}^N} \eta_n^2 \chi_{\{\varepsilon<\rho<k\}} \frac{a(D\rho, D\rho)}{\rho}, \end{aligned}$$

for every $\delta > 0$. From (3.3) we now get, using Young's inequality,

$$
\begin{aligned}
(1-\delta)\int_{\mathbf{R}^N}\eta_n^2\chi_{\{\varepsilon<\rho<k\}}\frac{a(D\rho,D\rho)}{\rho} &\le \omega(\varepsilon,k,n)+C\delta^{-1}\int_{\mathbf{R}^N}|F|^2\rho \\
&\quad +\frac{\delta}{\lambda}\int_{\mathbf{R}^N}\eta_n^2\chi_{\{\varepsilon<\rho<k\}}\frac{a(D\rho,D\rho)}{\rho}
\end{aligned}
$$

and, fixing a sufficiently small δ,

$$
\int_{\mathbf{R}^N}\eta_n^2\chi_{\{\varepsilon<\rho<k\}}\frac{a(D\rho,D\rho)}{\rho}\le\omega(\varepsilon,k,n)+C\int_{\mathbf{R}^N}|F|^2\rho.
$$

Letting $n\to\infty$ and then $\varepsilon\to 0$, $k\to\infty$ we obtain

$$
\int_{\mathbf{R}^N}\frac{a(D\rho,D\rho)}{\rho}\le C\int_{\mathbf{R}^N}|F|^2\rho<\infty.
$$

At this point the previous estimates show that $I_n\to 0$ as $n\to\infty$. Therefore, letting $n\to\infty$ and then $\varepsilon\to 0$, $k\to\infty$ in (3.3) we obtain

$$
\begin{aligned}
\int_{\mathbf{R}^N}\frac{a(D\rho,D\rho)}{\rho} &\le \left(\int_{\mathbf{R}^N}\frac{|D\rho|^2}{\rho}\right)^{1/2}\left(\int_{\mathbf{R}^N}|F|^2\rho\right)^{1/2} \\
&\le \frac{1}{\sqrt{\lambda}}\left(\int_{\mathbf{R}^N}\frac{a(D\rho,D\rho)}{\rho}\right)^{1/2}\left(\int_{\mathbf{R}^N}|F|^2\rho\right)^{1/2}
\end{aligned}
$$

and the statement follows. □

Corollary 3.2 *If $F\in L^2(\mu)$ then $\rho\in W^{1,1}(\mathbf{R}^N)$. Moreover $\rho\in L^{N/(N-2)}(\mathbf{R}^N)$ if $N>2$ and $\rho\in L^p(\mathbf{R}^N)$ for every $p<\infty$ if $N=2$.*

Proof. Since $\sqrt{\rho}\in W^{1,2}(\mathbf{R}^N)$, the Sobolev embedding theorem gives $\rho\in L^{N/(N-2)}(\mathbf{R}^N)$ for $N>2$ and $\rho\in L^p(\mathbf{R}^N)$ for every $p<\infty$ if $N=2$. The integrability of $D\rho$ follows from Hölder's inequality and (3.1). □

We now prove that, assuming $F\in L^k(\mu)$ for some $k>N$, the density ρ belongs to $L^\infty(\mathbf{R}^N)$. The proof relies upon Moser's iteration technique whose starting point is inequality (3.4) proved in the following lemma.

Lemma 3.3 *Assume that $F\in L^k(\mu)$ for some $k>2$ and fix $\beta>0$. If $\rho\in L^{\beta\frac{k}{k-2}+1}(\mathbf{R}^N)$, then*

$$
\lambda^2\int_{\mathbf{R}^N}\rho^{\beta-1}|D\rho|^2\le\int_{\mathbf{R}^N}|F|^2\rho^{\beta+1}<\infty. \tag{3.4}
$$

Proof. First we observe that

$$
\begin{aligned}
\int_{\mathbf{R}^N}|F|^2\rho^{\beta+1} &= \int_{\mathbf{R}^N}|F|^2\rho^{2/k}\rho^{\beta+1-2/k} \\
&\le \left(\int_{\mathbf{R}^N}|F|^k\rho\right)^{2/k}\left(\int_{\mathbf{R}^N}\rho^{\beta\frac{k}{k-2}+1}\right)^{1-2/k}<\infty.
\end{aligned}\tag{3.5}
$$

We use the same strategy and the same notation as in the proof of Theorem 3.1. Inserting $\phi = \eta_n^2((\rho \vee \varepsilon) \wedge k)^\beta$ in (3.2) and observing that $D((\rho \vee \varepsilon) \wedge k)^\beta = \beta \rho^{\beta-1} D\rho \chi_{\{\varepsilon<\rho<k\}}$, we obtain

$$\begin{aligned}
\beta \int_{\mathbf{R}^N} \eta_n^2 \rho^{\beta-1} \chi_{\{\varepsilon<\rho<k\}} a(D\rho, D\rho) &= -2 \int_{\mathbf{R}^N} \eta_n ((\rho \vee \varepsilon) \wedge k)^\beta a(D\rho, D\eta_n) \\
&\quad + \beta \int_{\mathbf{R}^N} \eta_n^2 \rho^\beta \chi_{\{\varepsilon<\rho<k\}} F \cdot D\rho \qquad (3.6) \\
&\quad + 2 \int_{\mathbf{R}^N} \eta_n \rho ((\rho \vee \varepsilon) \wedge k)^\beta F \cdot D\eta_n \\
&= I_n + J_n + K_n. \qquad (3.7)
\end{aligned}$$

Let us first estimate J_n, K_n. We have, with $C = \|D\eta\|_\infty$,

$$\begin{aligned}
|J_n| &\le \beta \int_{\mathbf{R}^N} \eta_n^2 \rho^{(\beta-1)/2} \chi_{\{\varepsilon<\rho<k\}} \rho^{(\beta+1)/2} |F||D\rho| \\
&\le \beta \left(\int_{\mathbf{R}^N} \eta_n^2 \rho^{\beta-1} \chi_{\{\varepsilon<\rho<k\}} |D\rho|^2 \right)^{1/2} \left(\int_{\mathbf{R}^N} |F|^2 \rho^{\beta+1} \right)^{1/2}. \\
|K_n| &\le \frac{Ck^\beta}{n} \int_{\mathbf{R}^N} |F| \rho.
\end{aligned}$$

Observe that $K_n \to 0$ as $n \to \infty$, since $F \in L^1(\mu)$. The term I_n is treated as in Theorem 3.1. Integrating by parts we have

$$\begin{aligned}
I_n &= 2 \int_{\{n \le |x| \le 2n\}} \rho \Bigg(((\rho \vee \varepsilon) \wedge k)^\beta a(D\eta_n, D\eta_n) + \eta_n ((\rho \vee \varepsilon) \wedge k)^\beta \sum_{i,j=1}^N a_{ij} D_{ij} \eta_n \\
&\quad + ((\rho \vee \varepsilon) \wedge k)^\beta \eta_n \sum_{i,j=1}^N D_i a_{ij} D_j \eta_n + \beta \eta_n \rho^{\beta-1} \chi_{\{\varepsilon<\rho<k\}} a(D\rho, D\eta_n) \Bigg).
\end{aligned}$$

Since $|D\eta_n| \le C(1+|x|)^{-1}$, $|D^2\eta_n| \le C(1+|x|^2)^{-1}$ in $\{n \le |x| \le 2n\}$, and C is independent of n, for a suitable $\omega(\varepsilon, k, n)$ which goes to 0 as $n \to \infty$ for fixed ε, k, we have

$$\begin{aligned}
|I_n| &\le \omega(\varepsilon, k, n) \\
&+ \beta \left(\int_{\mathbf{R}^N} \eta_n^2 \rho^{\beta-1} \chi_{\{\varepsilon<\rho<k\}} a(D\rho, D\rho) \right)^{1/2} \left(\int_{\{n \le |x| \le 2n\}} \rho^{\beta+1} \chi_{\{\varepsilon<\rho<k\}} a(D\eta_n, D\eta_n) \right)^{1/2} \\
&\le \omega(\varepsilon, k, n)(1+\delta^{-1}) + \delta \int_{\mathbf{R}^N} \eta_n^2 \rho^{\beta-1} \chi_{\{\varepsilon<\rho<k\}} a(D\rho, D\rho)
\end{aligned}$$

for every $\delta > 0$. We have thus obtained

$$\begin{aligned}
\beta \int_{\mathbf{R}^N} \eta_n^2 \rho^{\beta-1} \chi_{\{\varepsilon<\rho<k\}} a(D\rho, D\rho) &\le \beta \left(\int_{\mathbf{R}^N} \eta_n^2 \rho^{\beta-1} \chi_{\{\varepsilon<\rho<k\}} |D\rho|^2 \right)^{1/2} \left(\int_{\mathbf{R}^N} |F|^2 \rho^{\beta+1} \right)^{1/2} + K_n \\
&+ \omega(\varepsilon, k, n)(1+\delta^{-1}) + \delta \int_{\mathbf{R}^N} \eta_n^2 \rho^{\beta-1} \chi_{\{\varepsilon<\rho<k\}} a(D\rho, D\rho),
\end{aligned}$$

for every $\delta > 0$. Using the ellipticity of the matrix (a_{ij}) and arguing as in Theorem 3.1 we obtain

$$\int_{\mathbf{R}^N} \rho^{\beta-1} a(D\rho, D\rho) < \infty.$$

Therefore $I_n \to 0$ as $n \to \infty$ hence, letting $n \to \infty$ in (3.6), we have

$$\int_{\mathbf{R}^N} \rho^{\beta-1}\chi_{\{\varepsilon<\rho<k\}} a(D\rho, D\rho) \le \left(\int_{\mathbf{R}^N} \rho^{\beta-1}\chi_{\{\varepsilon<\rho<k\}}|D\rho|^2\right)^{1/2} \left(\int_{\mathbf{R}^N} |F|^2\rho^{\beta+1}\right)^{1/2}.$$

Letting $\varepsilon \to 0$, $k \to \infty$ one concludes

$$\lambda \int_{\mathbf{R}^N} \rho^{\beta-1}|D\rho|^2 \le \int_{\mathbf{R}^N} \rho^{\beta-1} a(D\rho, D\rho) \le \left(\int_{\mathbf{R}^N} \rho^{\beta-1}|D\rho|^2\right)^{1/2} \left(\int_{\mathbf{R}^N} |F|^2\rho^{\beta+1}\right)^{1/2}.$$

□

Corollary 3.4 *Assume that $F \in L^k(\mu)$ for some $k > 2$ and fix $\beta > 0$. If $\rho \in L^{\beta\frac{k}{k-2}+1}(\mathbf{R}^N)$, then $\rho^{(\beta+1)/2} \in W^{1,2}(\mathbf{R}^N)$ and*

$$\int_{\mathbf{R}^N} |D\rho^{(\beta+1)/2}|^2 \le \left(\frac{\beta+1}{2\lambda}\right)^2 \left(\int_{\mathbf{R}^N} |F|^k\rho\right)^{2/k} \left(\int_{\mathbf{R}^N} \rho^{\beta\frac{k}{k-2}+1}\right)^{1-2/k} \tag{3.8}$$

Proof. Observe that $\rho^{(\beta+1)/2} \in L^2(\mathbf{R}^N)$, since $1 < \beta+1 < (\beta k)/(k-2)+1$. Estimate (3.8) is immediate from (3.4) and (3.5). □

Theorem 3.5 *If $F \in L^k(\mu)$ for some $k > N$, then $\rho \in L^\infty(\mathbf{R}^N)$.*

Proof. Assume that $N \ge 3$; the case $N = 2$ will be treated separately. Let us first show that the above estimates imply an improvement of the integrability of ρ. To this aim, assume that $\rho \in L^{\beta\frac{k}{k-2}+1}$ for some $\beta > 0$. Using Corollary 3.4 and the Sobolev embedding we obtain $\rho^{(\beta+1)/2} \in L^{2N/(N-2)}(\mathbf{R}^N)$ and

$$\begin{aligned}\left(\int_{\mathbf{R}^N} \rho^{(\beta+1)\frac{N}{N-2}}\right)^{1/2-1/N} &\le C\|D\rho^{(\beta+1)/2}\|_2 \\ &\le C\|F\|_{L^k(\mu)}\frac{\beta+1}{2\lambda}\left(\int_{\mathbf{R}^N} \rho^{(\beta k)/(k-2)+1}\right)^{1/2-1/k},\end{aligned} \tag{3.9}$$

where C depends only upon N. Setting

$$\gamma = \beta\frac{k}{k-2}+1 \qquad \text{and} \qquad \theta = \frac{N}{N-2}\frac{k-2}{k},$$

we have $\theta > 1$ since $k > N$, and the improved integrability exponent can be written $\theta\Big(\gamma+\frac{2}{k-2}\Big)$. We now iterate the above estimate in order to show that the norms $\|\rho\|_{L^p}$ are all uniformly bounded. Let us define

$$\begin{cases} \gamma_{n+1} = \theta\left(\gamma_n + \frac{2}{k-2}\right) \\ \gamma_0 = \frac{N}{N-2} \end{cases} \tag{3.10}$$

and observe that $\rho \in L^{\gamma_0}(\mathbf{R}^N)$, by Corollary 3.2. Then $\gamma_n = \beta_n\frac{k}{k-2}+1$ for some $\beta_n > 0$ and

$$\beta_n + 1 = \frac{N-2}{N}\gamma_{n+1} = \frac{k-2}{k}\left(\gamma_n + \frac{2}{k-2}\right)$$

Setting $C_1 = C\|F\|_{L^k(\mu)}/2\lambda$, inequality (3.9) says that

$$\|\rho\|_{n+1} \le \Big(C_1 \frac{N-2}{N}\gamma_{n+1}\Big)^{\frac{2N}{N-2}\frac{1}{\gamma_{n+1}}} \|\rho\|_n^{\frac{\gamma_n}{\gamma_n+\frac{2}{k-2}}}, \tag{3.11}$$

where $\|\rho\|_n$ denotes the norm of ρ in $L^{\gamma_n}(\mathbf{R}^N)$. Observe that $\gamma_{n+1} \ge \theta^{n+1}\gamma_0$ and that for $\alpha_n = \log\|\rho\|_n$ we have

$$\alpha_{n+1} \le \frac{2N}{N-2}\frac{1}{\gamma_{n+1}}\log(C_2\gamma_{n+1}) + \frac{\gamma_n}{\gamma_n + \frac{2}{k-2}}\alpha_n,$$

$C_2 = (1-2/N)C_1$. These inequalities imply that $\log\|\rho\|_\infty = \lim_{n\to\infty}\alpha_n < \infty$. In fact, if $\alpha_n \to \infty$, then $\alpha_n \ge 0$ for large n and

$$\alpha_{n+1} - \alpha_n \le \frac{2N}{N-2}\frac{1}{\gamma_{n+1}}\log(C_2\gamma_{n+1}) \le C_3\frac{1}{\gamma_{n+1}^{1-\varepsilon}}$$

for some $C_3 > 0$ and any $0 < \varepsilon < 1$. Since the series on the right hand side converges, (α_n) cannot be divergent. This concludes the proof for $N \ge 3$.

Consider now the case $N = 2$, with variables (x, y). Introduce the operator B in $\mathbf{R}^3$, with variables (x, y, z)

$$B = A + D_{zz} - zD_z,$$

and notice that $\exp\{-z^2/2\}\,dz$ is (up to a normalisation constant) the invariant measure of the one-dimensional operator $D_{zz} - zD_z$. Let $d\mu = \rho(x,y)\,dx\,dy$ be the invariant measure of A, and check that $d\nu = \rho(x,y)\exp\{-z^2/2\}\,dx\,dy\,dz$ is invariant for B. In fact, for every $\phi \in C_c^\infty(\mathbf{R}^3)$, using the Fubini theorem and differentiating under the integral sign we have

$$\begin{aligned}\int_{\mathbf{R}^3} B\phi\,d\nu &= \int_{\mathbf{R}} \exp\{-z^2/2\}\left(\int_{\mathbf{R}^2}(A\phi + \phi_{zz} - z\phi_z)\rho(x,y)\,dx\,dy\right)dz \\ &= \int_{\mathbf{R}} \exp\{-z^2/2\}(D_{zz} - zD_z)\left(\int_{\mathbf{R}^2}\phi\,\rho(x,y)\,dx\,dy\right)dz = 0\end{aligned}$$

because the function $z \mapsto \int_{\mathbf{R}^2}\phi\,\rho(x,y)\,dx\,dy$ belongs to $C_c^\infty(\mathbf{R})$. As a consequence of the first part of the proof, the density of ν is bounded in $\mathbf{R}^3$, and taking $z = 0$ this implies that ρ is bounded in $\mathbf{R}^2$. □

In the case $k = N$ we obtain that $\rho \in L^p(\mathbf{R}^N)$ for every $p < \infty$.

Proposition 3.6 *If $F \in L^N(\mu)$, then $\rho \in L^p(\mathbf{R}^N)$ for every $p < \infty$.*

Proof. Assume that $N \ge 3$. Proceeding as in the proof of Theorem 3.5 we obtain that $\rho \in L^{\gamma_n}(\mathbf{R}^N)$ for every n, where

$$\begin{cases} \gamma_{n+1} = \Big(\gamma_n + \frac{2}{N-2}\Big) \\ \gamma_0 = \frac{N}{N-2}\end{cases}$$

Since $\gamma_n \to \infty$ we obtain the statement. The case $N = 2$ is already covered by Corollary 3.2. □

Finally, let us examine the case $2 < k < N$. Observe that the case $k = 2$ is covered by Corollary 3.2.

Proposition 3.7 *If $F \in L^k(\mu)$ with $2 < k < N$ then $\rho \in L^p(\mathbf{R}^N)$ for every $p \le N/(N-k)$.*

Proof. We define (γ_n) as in (3.10). It is easily checked that (γ_n) is increasing and convergent to $N/(N-k)$ and we have only to show that the limit of the sequence $(\|\rho\|_n)$ is finite, where, as in the proof of Theorem 3.5, $\|\rho\|_n$ denotes the norm of ρ in $L^{\gamma_n}(\mathbf{R}^N)$. Suppose, by contradiction, that $\|\rho\|_n \to \infty$. Since $N/(N-2) \le \gamma_n \le N/(N-k)$ and $\frac{\gamma_n}{\gamma_n+2/(k-2)} \le \theta < 1$ from equation (3.11) we obtain

$$\|\rho\|_{n+1} \le C\|\rho\|_n^\theta$$

for large n and a suitable C. However, this easily implies that the sequence $(\|\rho\|_n)$ is convergent. □

Corollary 3.8 *If $F \in L^k(\mu)$ for $k \ge (N+2)/2$, $k > 2$, then $\rho \in W^{1,2}(\mathbf{R}^N)$.*

Proof. We may assume that $k = (N+2)/2$. If $N \ge 3$, Proposition 3.7 gives $\rho \in L^p(\mathbf{R}^N)$ with $p = (2N)/(N-2) = (2k-2)/(k-2)$ for $k = (N+2)/2$. The same is true for $N = 2$ since $\rho \in L^p(\mathbf{R}^N)$ for every $p < \infty$, by Corollary 3.2. We may therefore apply Corollary 3.4 with $\beta = 1$ to conclude the proof. □

If we assume further regularity on F, as we shall do in the next section dealing with $W^{2,p}$ regularity, we can prove global boundedness of ρ assuming that F and $\mathrm{div}\, F$ belong to $L^k(\mu)$ for some $k > N/2$. For simplicity we assume that $F \in W^{1,\infty}_{loc}(\mathbf{R}^N)$ even though less local regularity of F suffices to perform the needed integration by parts.

Theorem 3.9 *Assume that $F \in W^{1,\infty}_{loc}(\mathbf{R}^N)$ and that $F, \mathrm{div}\, F \in L^k(\mu)$ for some $k > N/2$, $k \ge 2$. Then $\rho \in L^\infty(\mathbf{R}^N)$.*

Proof. The proof is similar to that of Theorem 3.5, using Lemma 3.10 below instead of Lemma 3.3. □

Lemma 3.10 *Assume that $F \in W^{1,\infty}_{loc}(\mathbf{R}^N)$ and that $F, \mathrm{div}\, F \in L^k(\mu)$ for some $k > 1$. Fix $\beta > 0$ and suppose that $\rho \in L^{\beta\frac{k}{k-1}+1}(\mathbf{R}^N)$. Then*

$$\lambda(\beta+1)\int_{\mathbf{R}^N} \rho^{\beta-1}|D\rho|^2 \le -\int_{\mathbf{R}^N} \rho^{\beta+1}\mathrm{div}\, F < \infty. \tag{3.12}$$

Proof. We keep the notation of the proof of Lemma 3.3. Multiplying the (distributional) identity $A_0\rho = \mathrm{div}\,(\rho F)$ by $\eta_n^2((\rho\vee\varepsilon)\wedge k)^\beta$ we obtain again (3.6). The estimates for I_n and K_n are similar, whereas J_n is treated as follows.

$$\begin{aligned} J_n &= \frac{\beta}{\beta+1}\int_{\mathbf{R}^N} \eta_n^2 F\cdot D((\rho\vee\varepsilon)\wedge k)^{\beta+1} \\ &= -\frac{\beta}{\beta+1}\int_{\mathbf{R}^N} ((\rho\vee\varepsilon)\wedge k)^{\beta+1}\left(F\cdot D(\eta_n^2) + \eta_n^2\mathrm{div}\, F\right). \end{aligned}$$

Using Hölder's inequality, it is easily seen that $\rho^{\beta+1}F, \rho^{\beta+1}\mathrm{div}\, F \in L^1(\mathbf{R}^N)$ and this implies that

$$J_n \to -\frac{\beta}{\beta+1}\int_{\mathbf{R}^N} ((\rho\vee\varepsilon)\wedge k)^{\beta+1}\mathrm{div}\, F \qquad \text{as} \quad n\to+\infty.$$

From this point on, the proof is similar to that of Lemma 3.3. □

Remark 3.11 Assume that $A = \Delta + F\cdot D$ where $F = -D\Phi$ (which is clearly the case, e.g., if F is radial) and $\Phi \in C^1(\mathbf{R}^N)$ satisfies $e^{-\Phi} \in L^1(\mathbf{R}^N)$. Then $\rho = e^{-\Phi}$ and the assumption $F \in L^k(\mu)$ $(F = -D\Phi)$ is equivalent to $e^{-\Phi/k} \in W^{1,k}(\mathbf{R}^N)$. The integrability statements of Theorem 3.5 and Propositions 3.6, 3.7 are exactly those given by the Sobolev embeddings.

4 Sobolev regularity

In this section we obtain Sobolev regularity results for ρ under the additional hypothesis that $a_{ij} \in C_b^1(\mathbf{R}^N)$. Moreover, we can also deal with the case $F \in L^k(\mu)$ for $1 \le k < 2$, excluded in the previous section. For further reference, let us state a classical L^p-regularity result for uniformly elliptic operaors (see [1]).

Theorem 4.1 *Let $1 < p < \infty$, $a_{ij} \in C_b^1(\mathbf{R}^N)$, $F \in W^{1,\infty}_{loc}(\mathbf{R}^N)$ and set*

$$A = \sum_{i,j=1}^{N} D_i(a_{ij} D_j) + \sum_{i=1}^{N} F_i D_i = A_0 + F \cdot D.$$

(i) Let $u \in L^p(\mathbf{R}^N)$ be such that

$$\Big| \int_{\mathbf{R}^N} u A_0 \phi \Big| \le C \|\phi\|_{W^{1,p'}(\mathbf{R}^N)}$$

for every $\phi \in C_c^\infty(\mathbf{R}^N)$. Then $u \in W^{1,p}(\mathbf{R}^N)$.

(ii) Let $f, u \in L^p_{loc}(\mathbf{R}^N)$ be such that

$$\int_{\mathbf{R}^N} u A \phi = \int_{\mathbf{R}^N} f \phi$$

for every $\phi \in C_c^\infty(\mathbf{R}^N)$. Then $u \in W^{2,p}_{loc}(\mathbf{R}^N)$.

Let us improve the conclusions of Theorem 3.5 and Propositions 3.6, 3.7.

Theorem 4.2 *Assume that $a_{ij} \in C_b^1(\mathbf{R}^N)$.*

(i) *If $F \in L^k(\mu)$ for some $k > N$, then $\rho \in W^{1,p}(\mathbf{R}^N)$ for every $1 \le p \le k$.*

(ii) *If $F \in L^N(\mu)$, then $\rho \in W^{1,p}(\mathbf{R}^N)$ for every $1 \le p < N$.*

(iii) *If $F \in L^k(\mu)$, for $2 \le k < N$ then $\rho \in W^{1,p}(\mathbf{R}^N)$ for every $1 \le p \le N/(N-k+1)$.*

Proof. (i) The invariance of μ yields, for $\phi \in C_c^\infty(\mathbf{R}^N)$,

$$\int_{\mathbf{R}^N} (A_0 \phi) \rho = - \int_{\mathbf{R}^N} (F \cdot D\phi) \rho,$$

where A_0 is defined in (1.1). Since $\rho \in L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ by Theorem 3.5 and $F \in L^k(\mu)$ it follows that $F\rho \in L^k(\mathbf{R}^N)$. Then

$$\Big| \int_{\mathbf{R}^N} (A_0 \phi) \rho \Big| \le C \|\phi\|_{W^{1,k'}(\mathbf{R}^N)}$$

for every $\phi \in C_c^\infty(\mathbf{R}^N)$ and $\rho \in W^{1,k}(\mathbf{R}^N)$, from Theorem 4.1(i). Since $\rho \in W^{1,1}(\mathbf{R}^N)$, by Corollary 3.2, the first statement follows.
(ii) The proof proceeds as in (i). In fact, $\rho \in L^q(\mathbf{R}^N)$ for every $q < \infty$, see Proposition 3.6, and therefore $F \in L^N(\mu)$ implies that $\rho F \in L^p(\mathbf{R}^N)$ for every $p < N$.
(iii) By Proposition 3.7 we know that $\rho \in L^{N/(N-k)}(\mathbf{R}^N)$ and then $\rho F \in L^p(\mathbf{R}^N)$ with $p = N/(N-k+1)$. The same argument as in (i) yields $\rho \in W^{1,p}(\mathbf{R}^N)$. □

Observe that we have obtained $\rho \in W^{1,N/(N-1)}$ when $F \in L^2(\mu)$, whereas Theorem 3.1 yields only $\rho \in W^{1,1}(\mathbf{R}^N)$.

We consider now the case $1 \le k < 2$ where we obtain, however, less precise results. We start by showing that under very weak conditions the function ρ belongs to $L^p(\mathbf{R}^N)$ for $p < N/(N-1)$. We refer the reader to [4] for local versions of the following theorem. We point out that the hypothesis $F \in L^p_{loc}(\mu)$ is not needed in Theorem 4.3 and in Proposition 4.4.

Theorem 4.3 *If $a_{ij} \in C^1_b(\mathbf{R}^N)$ and $F \in L^1(\mu)$, then $d\mu = \rho\, dx$ with $\rho \in L^p(\mathbf{R}^N)$ for every $p < N/(N-1)$.*

Proof. The invariance of μ yields for $\phi \in C_c^\infty(\mathbf{R}^N)$

$$\int_{\mathbf{R}^N} (\phi - A_0\phi)\, d\mu = \int_{\mathbf{R}^N} (\phi + F \cdot D\phi)\, d\mu$$

hence, since $F \in L^1(\mu)$,

$$\left| \int_{\mathbf{R}^N} (\phi - A_0\phi)\, d\mu \right| \le C\|\phi\|_{1,\infty}. \tag{4.1}$$

Fix $1 < p < N/(N-1)$ and let $q = p/(p-1)$ be the conjugate exponent to p. Clearly $q > N$. Given $\psi \in C_c^\infty(\mathbf{R}^N)$, let $w \in W^{2,q}(\mathbf{R}^N)$ be such that $w - A_0 w = \psi$. Then $\|w\|_{2,q} \le C_1\|\psi\|_q$ with C_1 independent of ψ. Moreover, by the Sobolev embedding, $w, Dw \in C_0(\mathbf{R}^N)$ and $\|w\|_{1,\infty} \le C_2\|w\|_{2,q}$.

In order to show that we can insert w in (4.1) we use an approximation procedure. Let $\eta_n = \eta(x/n)$ where $\eta \in C_c^\infty(\mathbf{R}^N)$ satisfies $\eta(x) = 1$ for $|x| \le 1$ and $\eta(x) = 0$ for $|x| \ge 2$. Then $\eta_n w \to w$ in $W^{2,q}(\mathbf{R}^N)$ and $A_0(\eta_n w) \to A_0 w$ in $C_0(\mathbf{R}^N)$. Fix now n and consider $v = \eta_n w$. Setting $v_\varepsilon = v * \xi_\varepsilon$, where ξ is a standard mollifier with compact support, $v_\varepsilon \in C_c^\infty(\mathbf{R}^N)$ and $v_\varepsilon \to v$ in $W^{2,q}(\mathbf{R}^N)$. Moreover,

$$\begin{aligned} A_0 v_\varepsilon &= (A_0 v) * \xi_\varepsilon + \sum_{i,j=1}^N \int_{\mathbf{R}^N} (a_{ij}(x) - a_{ij}(x-y)) D_{ij} v(x-y) \xi_\varepsilon(y)\, dy \\ &\quad + \sum_{i=1}^N \int_{\mathbf{R}^N} (b_i(x) - b_i(x-y)) D_i v(x-y) \xi_\varepsilon(y)\, dy \end{aligned}$$

where $b_i = \sum_j D_j a_{ij}$. Clearly $(A_0 v) * \xi_\varepsilon \to A_0 v$ uniformly, since $A_0 v \in C_0(\mathbf{R}^N)$. The term containing the b_i converges to zero uniformly since the b_i are uniformly continuous and $Dv \in C_0(\mathbf{R}^N)$. Since $|a_{ij}(x) - a_{ij}(x-y)| \le L|y|$, a simple computation using Hölder's inequality (since $q > N$) shows that also the remaining term goes to zero uniformly. Then $A_0 v_\varepsilon \to A_0 v$ uniformly. Summing up, there exists a sequence $(w_n) \subset C_c^\infty(\mathbf{R}^N)$ such that $w_n \to w$ in $W^{2,q}(\mathbf{R}^N)$ and $A_0 w_n \to A_0 w$ uniformly.

Then, passing to the limit, we may insert $\phi = w$ in (4.1) to get

$$\left| \int_{\mathbf{R}^N} \psi d\mu \right| = \left| \int_{\mathbf{R}^N} (w - A_0 w)\, d\mu \right| \le C\|w\|_{1,\infty} \le C_3\|\psi\|_q.$$

Therefore $d\mu = \rho\, dx$ with $\rho \in L^p(\mathbf{R}^N)$. □

Local versions of the following proposition are contained in [5], where regularity results in fractional Sobolev spaces are also obtained for $a_{ij} = \delta_{ij}$.

Proposition 4.4 *If $a_{ij} \in C^1_b(\mathbf{R}^N)$ and $F \in L^k(\mu)$ for some $1 < k < 2$, then $\rho \in W^{1,p}(\mathbf{R}^N)$ for every $1 < p < N/(N-k+1)$.*

Proof. The invariance of μ yields, for $\phi \in C_c^\infty(\mathbf{R}^N)$,

$$\int_{\mathbf{R}^N} (A_0\phi)\rho = -\int_{\mathbf{R}^N} (F \cdot D\phi)\rho.$$

Assume that $\rho \in L^{q_n}(\mathbf{R}^N)$ for some $q_n > 1$. Writing $\rho F = \rho^{1/k}\rho^{1-1/k}F$, Hölder's inequality yields $\rho F \in L^{r_n}(\mathbf{R}^N)$ for

$$\frac{1}{r_n} = \frac{1}{k} + \left(1 - \frac{1}{k}\right)\frac{1}{q_n}, \tag{4.2}$$

hence

$$\left|\int_{\mathbf{R}^N} (A_0\phi)\rho\right| \le C\|\phi\|_{1,r_n'}.$$

Since $1 < r_n < q_n$, Theorem 4.1(i) yields $\rho \in W^{1,r_n}(\mathbf{R}^N)$ hence, by the Sobolev embedding, $\rho \in L^{q_{n+1}}(\mathbf{R}^N)$ with

$$\frac{1}{q_{n+1}} = \frac{1}{r_n} - \frac{1}{N} = \frac{1}{k} - \frac{1}{N} + \left(1 - \frac{1}{k}\right)\frac{1}{q_n}$$

(observe that $r_n < 2 \le N$). We may start an iteration by choosing any $1 < q_0 < N/(N-1)$, by Theorem 4.3, and then it is easily checked that (q_n) in increasing and convergent to $N/(N-k)$. This proves that $\rho \in L^q(\mathbf{R}^N)$ for every $1 \le q < N/(N-k)$ and then, by (4.2), $\rho \in W^{1,p}(\mathbf{R}^N)$ for every $1 < p < N/(N-k+1)$. □

Remark 4.5 Consider again, as in Remark 3.11, the operator $A = \Delta - D\Phi \cdot D$, so that $F \in L^k(\mu)$ is equivalent to $\psi = e^{-\Phi/k} \in W^{1,k}(\mathbf{R}^N)$. Since $\rho = \psi^k$ it is easily seen that Theorem 4.2 gives precise results. However, if $1 \le k < 2$, the limiting cases $p = N/(N-1)$ in Theorem 4.3 and $p = 1$, $p = (NK)/(N-k+1)$ in Proposition 4.4 are excluded. We do not know whether for these values the same results hold.

In order to deal with $W^{2,p}$-regularity of ρ, we observe that Theorem 4.1(ii) yields $\rho \in W^{2,p}_{loc}(\mathbf{R}^N)$ for every $p < \infty$ if $F \in W^{1,\infty}_{loc}(\mathbf{R}^N)$.

Lemma 4.6 *Assume that $a_{ij} \in C^1_b(\mathbf{R}^N)$, $F \in W^{1,\infty}_{loc}(\mathbf{R}^N)$ and that $F, \operatorname{div} F \in L^k(\mu)$ for some $k > N/2$, $k \ge 2$. If $D\rho \in L^q(\mathbf{R}^N)$ for some $1 < q < \infty$, then $\rho \in W^{2,r}(\mathbf{R}^N)$ for every $1 < r \le p$, where*

$$\frac{1}{p} = \left(1 - \frac{2}{k}\right)\frac{1}{q} + \frac{2}{k}. \tag{4.3}$$

Proof. Since $\rho \in W^{2,r}_{loc}(\mathbf{R}^N)$ for every $r < \infty$, it satisfies the equation $A_0\rho = F \cdot D\rho + \rho \operatorname{div} F$. Moreover, since $\rho \in L^\infty(\mathbf{R}^N)$ by Theorem 3.9, it follows that $\rho \operatorname{div} F \in L^p(\mathbf{R}^N)$ for every $p \le k$. Let us deal with the term $F \cdot D\rho$. By Hölder's inequality we have

$$\begin{aligned} \int_{\mathbf{R}^N} |F|^p|D\rho|^p &= \int_{\mathbf{R}^N} |F|^p|D\rho|^{p-\frac{2}{r}}|D\rho|^{2/r}\rho^{-1/r}\rho^{1/r} \\ &\le \left(\int_{\mathbf{R}^N} \frac{|D\rho|^2}{\rho}\right)^{1/r}\left(\int_{\mathbf{R}^N} |F|^{pr}\rho\right)^{1/r}\left(\int_{\mathbf{R}^N} |D\rho|^{(p-\frac{2}{r})s}\right)^{1/s} \end{aligned} \tag{4.4}$$

whenever $r, s > 0$ and $2/r + 1/s = 1$. From (3.1) it follows that the right hand side of (4.4) is finite if $pr = k$ and $(p - 2/r)s = q$. These conditions easily yield (4.3). Since $A_0\rho \in L^p(\mathbf{R}^N)$, the Calderón-Zygmund estimates imply that $\rho \in W^{2,p}(\mathbf{R}^N)$. This proves the statement with $r = p$. If $1 < r < p$, then $r^{-1} = (1-2/k)q_1^{-1} + 2/k$ for some $1 < q_1 < q$ and $D\rho$ belongs to $L^1(\mathbf{R}^N) \cap L^q(\mathbf{R}^N)$, hence to $L^{q_1}(\mathbf{R}^N)$. By the first part of the proof, $\rho \in W^{2,r}(\mathbf{R}^N)$. □

Proposition 4.7 *Assume that $a_{ij} \in C_b^1(\mathbf{R}^N)$, $F \in W^{1,\infty}_{loc}(\mathbf{R}^N)$ and $F, \operatorname{div} F \in L^k(\mu)$, $k \geq 2$.*

(i) *If $k > N$, then $\rho \in W^{2,r}(\mathbf{R}^N)$ for every $1 < r \leq \frac{k^2}{3k-2}$.*

(ii) *If $k = N$ then $\rho \in W^{2,r}(\mathbf{R}^N)$ for every $1 < r < \frac{N^2}{3N-2}$.*

(iii) *If $N/2 < k < N$, then $\rho \in W^{2,r}(\mathbf{R}^N)$ for every $1 < r \leq \frac{kN}{kN-k^2+3k-2}$.*

Proof. Theorem 4.2 allows us to put $q = k$, $q < N$ arbitrary, and $q = N/(N-k+1)$, respectively, in Lemma 4.6, and all the statements follow. □

The above proposition yields, roughly speaking, $\rho \in W^{2,k/3}$ whenever F and $\operatorname{div} F$ belong to $L^k(\mu)$ for some $k > N$. If $k \geq 2N$ we can improve $k/3$ to $k/2$ iterating the procedure of Lemma 4.6.

Theorem 4.8 *Assume that $F \in W^{1,\infty}_{loc}(\mathbf{R}^N)$ and that $F, \operatorname{div} F \in L^k(\mu)$ for some $k \geq 2N$. Then $\rho \in W^{2,p}(\mathbf{R}^N)$ for every $1 < p < k/2$. Moreover, if $k > 2N$, then $\rho \in W^{2,\frac{k}{2}}(\mathbf{R}^N)$.*

Proof. First we show that $D\rho \in L^q(\mathbf{R}^N)$ for every $q < \infty$ if $k \geq 2N$ and that $D\rho \in L^\infty(\mathbf{R}^N)$ if $k > 2N$. Using Lemma 4.6 and setting for every $n \in \mathbf{N}$

$$\frac{1}{p_{n+1}} = \Big(1 - \frac{2}{k}\Big)\frac{1}{q_n} + \frac{2}{k} \qquad \text{and} \qquad \frac{1}{q_{n+1}} = \frac{1}{p_{n+1}} - \frac{1}{N},$$

we deduce that if $D\rho \in L^{q_n}(\mathbf{R}^N)$, then $\rho \in W^{2,p_{n+1}}(\mathbf{R}^N)$. We may take $q_0 = k$, by Theorem 4.2. If $p_n \geq N$ for some n, then $D\rho \in L^q(\mathbf{R}^N)$ for every $q < \infty$. Assume now that $p_n < N$ for every $n \in \mathbf{N}$. Then, by the Sobolev embedding, $D\rho \in L^{q_{n+1}}(\mathbf{R}^N)$. Since $k \geq 2N$ it is easily seen that the sequence (q_n) is increasing, hence it is convergent to some $\ell \geq 0$ such that $\ell^{-1} = (1-2/k)\ell^{-1} + 2/k - 1/N$, whence $\ell = \infty$ and thus $D\rho \in L^q(\mathbf{R}^N)$, for every $q < \infty$, again.

In the case $k > 2N$, arguing as above, the assumption $p_n \leq N$ for every n leads to $\ell < 0$, which is impossible. Hence $p_n > N$ for some n and $D\rho \in L^\infty(\mathbf{R}^N)$.

To show that $\rho \in W^{2,p}(\mathbf{R}^N)$ for $1 < p < k/2$ we use the identity $A_0\rho = F \cdot D\rho + \rho \operatorname{div} F$ and observe that $\rho \operatorname{div} F \in L^p(\mathbf{R}^N)$. Moreover

$$\begin{aligned}
\int_{\mathbf{R}^N} |F|^p |D\rho|^p &= \int_{\mathbf{R}^N} |F|^p |D\rho|^{p(1-2/k)} |D\rho|^{2p/k} \rho^{-p/k} \rho^{p/k} \\
&\leq \left(\int_{\mathbf{R}^N} |D\rho|^{tp(1-2/k)}\right)^{1/t} \left(\int_{\mathbf{R}^N} \frac{|D\rho|^2}{\rho}\right)^{p/k} \left(\int_{\mathbf{R}^N} |F|^k \rho\right)^{p/k} < \infty,
\end{aligned}$$

where $2p/k + 1/t = 1$. This shows that $A_0\rho \in L^p(\mathbf{R}^N)$, hence $\rho \in W^{2,p}(\mathbf{R}^N)$.

If $k > 2N$ then $D\rho \in L^\infty(\mathbf{R}^N)$ and thus

$$\begin{aligned}
\int_{\mathbf{R}^N} |F|^{k/2} |D\rho|^{k/2} &= \int_{\mathbf{R}^N} |F|^{k/2} |D\rho|^{k/2-1} |D\rho| \rho^{-1/2} \rho^{1/2} \\
&\leq \|D\rho\|_\infty^{k/2-1} \left(\int_{\mathbf{R}^N} \frac{|D\rho|^2}{\rho}\right)^{1/2} \left(\int_{\mathbf{R}^N} |F|^k \rho\right)^{1/2} < \infty
\end{aligned}$$

so that $A_0\rho \in L^{k/2}(\mathbf{R}^N)$ and $\rho \in W^{2,\frac{k}{2}}(\mathbf{R}^N)$. □

5 A Harnack inequality

In this section we prove pointwise bounds for $\log\rho$ and $D\rho/\rho$ in terms of F and its derivatives up to the second order. In particular we obtain a quantitative Harnack inequality for ρ. We use these bounds to find conditions under which $|D\log\rho|$ belongs to $L^p(\mu)$ for $1\le p<\infty$.

The following lemma is the main step to the results of this section. Its proof is based on the Bernstein method which requires more regularity on the coefficients in order to differentiate the equation solved by ρ. We refer the reader to [10, Section 7.1.4.b] where similar computations are performed in the parabolic case.

Lemma 5.1 *Assume that $v\in C^3(\mathbf{R}^N)$ solves the equation*

$$Bv+a(Dv,Dv)-H\cdot Dv=G \tag{5.1}$$

where $B=\sum_{i,j}a_{ij}D_{ij}$ and $a_{ij}\in C_b^2(\mathbf{R}^N)$, $H_i\in C^2(\mathbf{R}^N)$, $G\in C^1(\mathbf{R}^N)$ and set

$$\begin{aligned}\Phi(x) &= 1+|H(x)|+|DH(x)|+|D^2H(x)|+|G(x)|+|DG(x)|\\ \Psi(x) &= \sup_{|y-x|\le 1}\Phi(y).\end{aligned}$$

Then $|Dv|\le C\Psi$, where C depends only on the ellipticity constant λ and $\|a_{ij}\|_{C_b^2(\mathbf{R}^N)}$.

Proof. Let $w=a(Dv,Dv)-H\cdot Dv=G-Bv$. Then

$$\begin{aligned}D_hw &= 2\sum_{i,j}a_{ij}D_{ih}vD_jv+\sum_{i,j}D_ha_{ij}D_ivD_jv-\sum_j H_jD_{hj}v-\sum_j D_hH_jD_jv\\ &= D_hG-\sum_{i,j}a_{ij}D_{hij}v-\sum_{i,j}D_ha_{ij}D_{ij}v\end{aligned}$$

and

$$\begin{aligned}Bw &= 2\sum_{i,j,h,k}a_{hk}a_{ij}D_{ih}vD_{jk}v+\sum_j\left(2\sum_i a_{ij}D_iv-H_j\right)\sum_{h,k}a_{hk}D_{hkj}v\\ &\quad +4\sum_{i,j,h,k}a_{hk}D_ha_{ij}D_{ik}vD_jv+\sum_{i,j,h,k}a_{hk}D_{hk}a_{ij}D_ivD_jv\\ &\quad -2\sum_{j,h,k}a_{hk}D_kH_jD_{hj}v-\sum_{j,h,k}a_{hk}D_{hk}H_jD_jv.\end{aligned}$$

Using the identity

$$\sum_{h,k}a_{hk}D_{jhk}v=D_jG-D_jw-\sum_{h,k}D_ja_{hk}D_{hk}v,$$

the ellipticity of the matrix (a_{ij}) and setting $b_j=2\sum_i a_{ij}D_iv-H_j$, $b=(b_1,\dots,b_N)$, we obtain

$$\begin{aligned}-w+Bw+b\cdot Dw &\ge -a(Dv,Dv)+H\cdot Dv+2\lambda^2|D^2v|^2+b\cdot DG-\sum_{j,h,k}b_jD_ja_{hk}D_{hk}v\\ &\quad +4\sum_{i,j,h,k}a_{hk}D_ha_{ij}D_{ik}vD_jv+\sum_{i,j,h,k}a_{hk}D_{hk}a_{ij}D_ivD_jv\\ &\quad -2\sum_{j,h,k}a_{hk}D_kH_jD_{hj}v-\sum_{j,h,k}a_{hk}D_{hk}H_jD_jv.\end{aligned}$$

We fix $x_0 \in \mathbf{R}^N$ and $\eta \in C_c^\infty(\mathbf{R}^N)$ such that $\eta = 1$ in $B(x_0, 1/2)$, $\eta = 0$ outside $B(x_0, 1)$, $0 \le \eta \le 1$ and $|D\eta|, |D^2\eta| \le L$, with L independent of x_0. For $z = \eta^4 w$ we obtain

$$-z+Bz+b\cdot Dz = \eta^4(-w+Bw+b\cdot Dw)+8\eta^3 a(D\eta, Dw)+4\eta^3 wB\eta+12\eta^2 wa(D\eta, D\eta)+4\eta^3 wb\cdot D\eta.$$

Next observe that, denoting by M a generic constant which depends only upon $\|a_{ij}\|_{C_b^2(\mathbf{R}^N)}$ but may change from line to line, the following estimates hold:

(i) $|Dv| \le M(|D^2v|^{1/2} + |H| + |G|^{1/2})$

(ii) $|w| \le M(|D^2v| + |G|)$

(iii) $|b| \le M(|Dv| + |H|) \le M(|D^2v|^{1/2} + |H| + |G|^{1/2})$

(iv) $|Dw| \le M(|Dv||D^2v| + |Dv|^2 + |H||D^2v| + |DH||Dv|)$

Using repeatedly these estimates it follows that for every $\varepsilon > 0$

$$\begin{aligned}
-w + Bw + b\cdot Dw \quad &\ge \quad -M|Dv|^2 - \frac{|H|^2}{2} - \frac{|Dv|^2}{2} + 2\lambda^2|D^2v|^2 - \frac{|b|^2}{2} \\
&\quad -\frac{|DG|^2}{2} - \frac{M}{\varepsilon}|b|^2 - M\varepsilon|D^2v|^2 - \frac{M}{\varepsilon}|Dv|^2 - M\varepsilon|D^2v|^2 \\
&\quad -M|Dv|^2 - \frac{M}{\varepsilon}|DH|^2 - M\varepsilon|D^2v|^2 - \frac{M}{2}|D^2H|^2 - M\frac{|Dv|^2}{2} \\
&\ge \quad (2\lambda^2 - 3M\varepsilon)|D^2v|^2 - \frac{M}{\varepsilon}\left(|D^2v| + |D^2v|^{1/2} + \Phi^2\right).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\eta^3|D\eta||Dw| \quad &\le \quad M\eta^3(|Dv||D^2v| + |Dv|^2 + |H||D^2v| + |DH||Dv|) \\
&\le \quad M\eta^3\Big(|D^2v|(|D^2v|^{1/2} + |H| + |G|^{1/2}) + (|D^2v|^{1/2} + |H| + |G|^{1/2})^2 \\
&\quad +|H||D^2v| + |DH|(|D^2v|^{1/2} + |H| + |G|^{1/2})\Big) \\
&\le \quad M\left(\eta^3|D^2v|^{3/2} + \eta^2|D^2v| + \varepsilon\eta^4|D^2v|^2 + \frac{1}{\varepsilon}\Phi^2\right)
\end{aligned}$$

and also

$$\eta^3|w||B\eta| + 4\eta^2|w|a(D\eta, D\eta) \le M(\eta^2|D^2v| + \Phi)$$

and

$$\begin{aligned}
\eta^3|w||b||D\eta| \quad &\le \quad M\eta^3(|D^2v| + |G|)(|D^2v|^{1/2} + |H| + |G|^{1/2}) \\
&\le \quad M\eta^3(|D^2v|^{3/2} + \Phi|D^2v| + \Phi|D^2v|^{1/2} + \Phi^2) \\
&\le \quad M\left(\eta^3|D^2v|^{3/2} + \eta^2|D^2v| + \varepsilon\eta^4|D^2v|^2 + \frac{1}{\varepsilon}\Phi^2\right).
\end{aligned}$$

Fixing a sufficiently small ε we get for $x \in B(x_0, 1)$

$$\begin{aligned}
-z + Bz + b\cdot Dz \quad &\ge \quad \lambda^2\eta^4|D^2v|^2 - c_1(\eta^3|D^2v|^{3/2} + \eta^2|D^2v|) - c_2\Phi^2 \\
&\ge \quad -K - c_2\Psi^2(x_0)
\end{aligned}$$

where c_1, c_2 depend only upon $\|a_{ij}\|_{C_b^2(\mathbf{R}^n)}, \lambda$ and $-K$ is the minimum of the function $\lambda t^2 - c_1 t^{3/2} - c_1 t$ over $[0, \infty[$. Since $z = 0$ at the boundary of $B(x_0, 1)$, the maximum principle yields $w(x_0) = z(x_0) \leq K + c_2 \Psi^2(x_0) \leq c_3 \Psi^2(x_0)$. Then

$$\lambda |Dv(x_0)|^2 \leq a(Dv(x_0), Dv(x_0)) = w(x_0) + H(x_0) \cdot Dv(x_0) \leq c_4 \Psi^2(x_0) + \frac{\lambda}{2} |Dv(x_0)|^2$$

and the proof is complete. □

We can now estimate $D \log \rho$ in terms of F. Observe that we need the assumption $a_{ij} \in C_b^3(\mathbf{R}^N)$ only since the operator A is written in divergence form.

Theorem 5.2 *Assume that $a_{ij} \in C_b^3(\mathbf{R}^N)$ and that $F \in C^2(\mathbf{R}^N)$ and set*

$$\Gamma(x) = \sup_{\{|y-x| \leq 1\}} \left(1 + |F(y)| + |DF(y)| + |D^2 F(y)|\right). \tag{5.2}$$

Then there exists C depending only on λ and $\|a_{ij}\|_{C_b^3(\mathbf{R}^N)}$ such that

$$\left|\frac{D\rho}{\rho}\right| \leq C\Gamma.$$

PROOF. By local elliptic regularity, $\rho \in C^3(\mathbf{R}^N)$. Set $v = \log \rho$. It is immediately checked that $v \in C^3(\mathbf{R}^N)$ satisfies the equation

$$\sum_{i,j} a_{ij} D_{ij} v + a(Dv, Dv) - H \cdot v = \operatorname{div} \mathrm{F}$$

with $H_j = F_j - \sum_i D_i a_{ij}$. The statement then follows from Lemma 5.1. □

The estimate of the logarithmic derivative of ρ in terms of F leads immediately to a quantitative Harnack inequality. We state it in the next proposition in the simple case where F and its derivatives up to the second order have polynomial growth.

Proposition 5.3 *Assume that $a_{ij} \in C_b^3(\mathbf{R}^N)$ and that $F \in C^2(\mathbf{R}^N)$ satisfies $|F(x)| + |DF(x)| + |D^2 F(x)| \leq C_1(1 + |x|^{\beta-1})$ for some $\beta > 1$. Then*

$$\frac{\rho(y)}{\rho(x)} \leq \exp\left\{K|x-y| \left(1 + (|x| + |y|)^{\beta-1}\right)\right\},$$

where K depends only on C_1, λ and $\|a_{ij}\|_{C_b^3(\mathbf{R}^N)}$.

PROOF. Setting $v = \log \rho$, we have from Theorem 5.2

$$|Dv(x)| \leq C\Gamma(x) \leq C_2(1 + |x|^{\beta-1}).$$

This yields $|v(y) - v(x)| \leq C_3 |x-y| \left(1 + (|x| + |y|)^{\beta-1}\right)$ and the proof is complete. □

6 Pointwise bounds and weighted Sobolev regularity of $\log\rho$

In this section we prove (pointwise) upper and lower bounds on the density ρ. As regards the upper bound, we assume that $V(x)=\exp\{\delta|x|^\beta\}$ is integrable with respect to μ for some $\delta,\beta>0$ and we recall that explicit estimates of δ,β follow from Proposition 2.4 or Corollary 2.5 under assumptions (2.1), (2.2), respectively. We keep the condition $a_{ij}\in C^1_b(\mathbf{R}^N)$ but need the extra assumption that F does not grow more than some exponential, at infinity, in order to integrate $|F|^k$ with respect to μ for every k. Under these assumptions we show that ρ decays exponentially. For the lower bound we need more regularity on a_{ij} and F in order to apply the results of Section 5 and we confine ourselves to the case where F and its derivatives up to the second order have a polynomial growth. Finally, we combine the upper bound on ρ with the Harnack inequality to derive sufficient conditions ensuring that $\log\rho\in W^{2,p}(\mu)$.

Theorem 6.1 *Assume that $a_{ij}\in C^1_b(\mathbf{R}^N)$ and that $V(x)=\exp\{\delta|x|^\beta\}$ is integrable with respect to μ for some $\beta,\delta>0$. Assume moreover that $|F(x)|\le C\exp\{|x|^\gamma\}$ for some $C>0$ and $\gamma<\beta$. Then there exist $c_1,c_2>0$ such that $\rho(x)\le c_1\exp\{-c_2|x|^\beta\}$.*

PROOF. Since $|F(x)|\le C\exp\{|x|^\gamma\}$ for some $C>0$ and $\gamma<\beta$, then $F\in L^k(\mu)$ for every $k<\infty$. The invariance of μ yields

$$\int_{\mathbf{R}^N}(A_0\phi)\rho=-\int_{\mathbf{R}^N}(F\cdot D\phi)\rho$$

for every $\phi\in C^\infty_c(\mathbf{R}^N)$. Taking $\phi=w\psi$ with $\psi\in C^\infty_c(\mathbf{R}^N)$, $0<w\in C^\infty(\mathbf{R}^N)$, $w(x)=\exp\{c_2|x|^\beta\}$ for $|x|\ge 1$, we obtain

$$\int_{\mathbf{R}^N}(A_0\psi)\rho w=-\int_{\mathbf{R}^N}\Big(\psi A_0 w+2\sum_{i,j=1}^N a_{ij}D_i\psi D_j w+wF\cdot D\psi+\psi F\cdot Dw\Big)\rho. \tag{6.1}$$

Let us fix $q>p>N$ and choose $c_2<\delta/q$. It is easily seen that w,Dw,A_0w belong to $L^q(\mu)$. Moreover, since $1/p=1/q+1/k$ for some $k>1$ and $F\in L^k(\mu)$, it follows that $wF,|Dw||F|\in L^p(\mu)$. Since $\rho\in L^\infty$, by Theorem 3.5, we deduce that all the functions $\rho Dw,\rho A_0w,\rho wF$ belong to $L^p(\mathbf{R}^N)$. Then (6.1) yields

$$\Big|\int_{\mathbf{R}^N}(A_0\psi)\rho w\Big|\le L\|\psi\|_{W^{1,p'}(\mathbf{R}^N)}$$

for a suitable L independent of ψ. Since also $\rho w\in L^p(\mathbf{R}^N)$ from Theorem 4.1(i) we infer that ρw belongs to $W^{1,p}(\mathbf{R}^N)$, hence to $L^\infty(\mathbf{R}^N)$, since $p>N$, and the proof is concluded. □

The following result is analogous, but relies upon Theorem 4.8 rather than Theorem 3.5.

Theorem 6.2 *Assume that $a_{ij}\in C^1_b(\mathbf{R}^N)$ and that $V(x)=\exp\{\delta|x|^\beta\}$ is integrable with respect to μ for some $\beta,\delta>0$. Assume moreover that $F\in W^{1,\infty}_{loc}(\mathbf{R}^N)$ and that $|F(x)|\le C\exp\{|x|^\gamma\}$, $|\operatorname{div}F(x)|\le C\exp\{|x|^\gamma\}$ for some $C>0$ and $\gamma<\beta$. Then there exist $c_1,c_2>0$ such that $|D\rho(x)|\le c_1\exp\{-c_2|x|^\beta\}$.*

PROOF. We modify the proof of Theorem 6.1, keeping the notation introduced there. From Theorem 4.8 we obtain that $\rho\in W^{2,p}(\mathbf{R}^N)$ for every $p<\infty$. Since $A_0\rho=F\cdot D\rho+\rho\operatorname{div}F$ we have

$$A_0(\rho w)=w\rho\operatorname{div}F+wF\cdot D\rho+\rho(A_0w)+2\sum_{i,j=1}^N a_{ij}D_iwD_j\rho.$$

As in the proof of Theorem 6.1 one sees that $w\rho\,(\operatorname{div} F), \rho(A_0 w) \in L^p(\mathbf{R}^N)$, where $p > N$ is fixed. To treat the terms containing $D\rho$ we proceed as in Theorem 4.8

$$\begin{aligned}\int_{\mathbf{R}^N} w^p |F|^p |D\rho|^p &= \int_{\mathbf{R}^N} w^p |F|^p |D\rho|^{p-1} |D\rho| \rho^{-1/2} \rho^{1/2} \\ &\leq \|D\rho\|_\infty^{p-1} \left(\int_{\mathbf{R}^N} \frac{|D\rho|^2}{\rho} \right)^{1/2} \left(\int_{\mathbf{R}^N} w^{2p} |F|^{2p} \rho \right)^{1/2}.\end{aligned}$$

If c_2 is small enough, this last integral is finite. Similarly, one estimates the term $|Dw||D\rho|$. Then $A_0(\rho w) \in L^p(\mathbf{R}^N)$, hence $w\rho \in W^{2,p}(\mathbf{R}^N)$ and then $D(w\rho) \in L^\infty(\mathbf{R}^N)$. Since we know that ρDw is bounded, by Theorem 6.1, perhaps taking a smaller c_2, the proof is complete. □

We obtain lower bounds on ρ using the Harnack inequality from Section 5.

Theorem 6.3 *Assume that $a_{ij} \in C_b^3(\mathbf{R}^N)$ and that $F \in C^2(\mathbf{R}^N)$ satisfies $|F(x)| + |DF(x)| + |D^2F(x)| \leq C_1(1 + |x|^{\beta-1})$ for some $\beta > 1$. Then*

$$\rho(x) \geq \exp\{-c_3(1 + |x|^\beta)\},$$

where c_3 depends only on C_1, λ and $\|a_{ij}\|_{C_b^3(\mathbf{R}^N)}$.

Proof. Let $v = \log \rho$. As in the proof of Corollary 5.3 we obtain

$$|Dv(x)| \leq C\Gamma(x) \leq C_2(1 + |x|^{\beta-1})$$

for $v = \log \rho$. Therefore $|v(x)| \leq c_3(1 + |x|^\beta)$ and the statement follows. □

Let us combine the upper and the lower bound to select a class of operators for which the exact decay of ρ can be established.

Corollary 6.4 *Assume that $a_{ij} \in C_b^3(\mathbf{R}^N)$ and that $F \in C^2(\mathbf{R}^N)$ satisfies $|F(x)| + |DF(x)| + |D^2F(x)| \leq C_1(1 + |x|^{\beta-1})$ for some $\beta > 1$. Assume moreover that (2.2) holds, i.e.,*

$$\limsup_{|x|\to\infty} |x|^{1-\beta} F(x) \cdot \frac{x}{|x|} = -c,$$

$0 < c < \infty$. Then

$$\exp\{-c_3(1 + |x|^\beta)\} \leq \rho(x) \leq c_1 \exp\{-c_2(1 + |x|^\beta)\}$$

for suitable $c_1, c_2, c_3 > 0$.

Proof. It is sufficient to use Corollary 2.5 and Theorems 6.1, 6.3. □

The above corollary e.g. applies to $A = \Delta + F \cdot D$ where $F(x) = -|x|^{\beta-2}x + G(x)$ for $\beta > 1$ and $|x| \geq 1$ and $|G| + |DG| + |D^2G| \leq c(1 + |x|^{\beta-1})$. Observe that, if $G = 0$, then ρ is given by $\rho(x) = C \exp\{-|x|^\beta/\beta\}$.

We end this section proving weighted Sobolev regularity results for $\log \rho$. We set

$$W^{k,p}(\mu) = \{u \in W^{k,p}_{loc}(\mathbf{R}^N) : D^\alpha u \in L^p(\mu) \text{ for } |\alpha| \leq k\}$$

and note that, under the hypotheses below, ρ decays exponentially and hence $\log \rho$ belongs to $L^p(\mu)$.

In the next proposition we show a sufficient condition under which $\log \rho$ belongs to $W^{1,p}(\mu)$.

Proposition 6.5 *Assume that $a_{ij} \in C_b^3(\mathbf{R}^N)$ and that $V(x) = \exp\{\delta|x|^\beta\}$ is integrable with respect to μ for some $\beta, \delta > 0$. Assume moreover that $|F(x)| + |DF(x)| + |D^2F(x)| \le C\exp\{|x|^\gamma\}$ for some $C > 0$ and $\gamma < \beta$. Then $D\rho/\rho \in L^p(\mu)$ for every $1 \le p < \infty$.*

Proof. We keep the notation of Section 5 and recall that Γ is defined in (5.2). Since $\Gamma(x) \le c_1\exp\{|x|^{\gamma+\varepsilon}\} \le c_2V(x)$ for $\gamma + \varepsilon < \beta$ the assertion follows from Theorem 5.2. □

Under polynomial growth conditions on F we can prove that $\log\rho \in W^{2,p}(\mu)$.

Theorem 6.6 *Assume that $a_{ij} \in C_b^3(\mathbf{R}^N)$ and that $V(x) = \exp\{\delta|x|^{\beta_1}\}$ is integrable with respect to μ for some $\beta_1, \delta > 0$. Assume moreover that $|F(x)| + |DF(x)| + |D^2F(x)| \le C_1(1 + |x|^{\beta-1})$ for some $C > 0$ and $\beta > 1$ satisfying $\beta - 1 < \beta_1$. Then $\log\rho \in W^{2,p}(\mu)$ for every $1 \le p < \infty$.*

Proof. Using Proposition 6.5 we infer that $\log\rho \in W^{1,p}(\mu)$ for every $1 \le p < \infty$. Setting $v = \log\rho$, then $D_{ij}v = D_{ij}\rho/\rho - (D_i\rho D_j\rho)/\rho^2$ and the last term belongs to $L^p(\mu)$ since $D\rho/\rho$ is in $L^{2p}(\mu)$. Thus, we have to show that $D_{ij}\rho/\rho \in L^p(\mu)$ and, since μ is a finite measure, we may assume that $p > 1$. Using the identity $A_0\rho = F \cdot D\rho + \rho\,\mathrm{div}\,F$ we deduce from Theorem 5.2 the pointwise estimate $|A_0\rho| \le C(1 + \Gamma^2)\rho$ for a suitable $C > 0$.

Let $Q(x,r)$ be a cube of side r centred at x. By the interior estimates for uniformly elliptic operators, see e.g. [11, Theorem 9.11], we obtain

$$\int_{Q(x,1)} |D_{ij}\rho(y)|^p\,dy \le C_1\int_{Q(x,2)} (|A_0\rho(y)|^p + |\rho(y)|^p)\,dy \le C_2\int_{Q(x,2)} (1 + \Gamma^2(y))^p\rho^p(y)\,dy$$

with C_2 independent of x. We use Proposition 5.3 twice and Theorem 6.1 to get

$$\begin{aligned}\int_{Q(x,1)} \frac{|D_{ij}\rho(y)|^p}{\rho(y)^{p-1}}\,dy &\le C_3\frac{\exp\{K_1|x|^{\beta-1}\}}{\rho(x)^{p-1}}\int_{Q(x,2)} (1 + \Gamma^2(y))^p\rho^p(y)\,dy \\ &\le C_4\exp\{K_2|x|^{\beta-1}\}(1 + |x|^{2p(\beta-1)})\rho(x) \\ &\le C_5(1 + |x|^{2p(\beta-1)})\exp\{K_2|x|^{\beta-1} - K_3|x|^{\beta_1}\}\end{aligned}$$

where all the constants are independent of x. At this point we cover $\mathbf{R}^N$ with a sequence of unit cubes $Q(x_n,1)$ whose interiors do not overlap, write the above estimates for each cube $Q(x_n,1)$ and sum over n to conclude the proof. □

Remark 6.7 It is easily seen that Theorem 6.6 holds under the hypotheses of Corollary 6.4. In this case one can take $\beta_1 = \beta$.

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