Spectrum of Ornstein-Uhlenbeck operators in L^p spaces with respect to invariant measures

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Abstract

Let $A = \sum_{i,j=1}^N q_{ij} D_{ij} + \sum_{i,j=1}^N b_{ij} x_j D_i$ be a possibly degenerate Ornstein-Uhlenbeck operator in $\mathbf{R}^N$ and assume that the associated Markov semigroup has an invariant measure μ. We compute the spectrum of A in L^p_μ for $1 \le p < \infty$.

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1 Introduction

In this paper we study the spectrum of the Ornstein-Uhlenbeck operator

$$A = \sum_{i,j=1}^N q_{ij} D_{ij} + \sum_{i,j=1}^N b_{ij} x_j D_i = \mathrm{Tr}(QD^2) + \langle Bx, D\rangle, \qquad x \in \mathbf{R}^N, \tag{1.1}$$

where $Q = (q_{ij})$ is a real, symmetric and nonnegative matrix and $B = (b_{ij})$ is a non-zero real matrix. The associated Markov semigroup $(T(t))_{t\ge 0}$ has the following explicit representation, due to Kolmogorov

$$(T(t)f)(x) = \frac{1}{(4\pi)^{N/2}(\det Q_t)^{1/2}} \int_{\mathbf{R}^N} e^{-<Q_t^{-1}y,y>/4} f(e^{tB}x - y)\, dy, \tag{1.2}$$

where

$$Q_t = \int_0^t e^{sB} Q e^{sB^*}\, ds$$

and B^* denotes the adjoint matrix of B, see for instance [8]. We assume that the spectrum of B is contained in $\mathbf{C}^- = \{\lambda \in \mathbf{C} : \ \mathrm{Re}\,\lambda < 0\}$ and that $\det Q_t > 0$ for any $t > 0$ (that is, Q_t is positive definite). This is clearly true, in particular, if Q is invertible. We point out that the condition $\det Q_t > 0$, $t > 0$, is equivalent to the hypoellipticity of the operator

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$\frac{\partial}{\partial t} - A$ in $(N+1)$ variables $(t, x_1, \ldots, x_N)$, see [16], and it can be also expressed by saying that the kernel of Q does not contain any invariant subspace of B^* (see [16], [17], [19], [23]).

Assuming that $\det Q_t > 0$, in [9, Section 11.2.3] it is proved that $\sigma(B) \subset \mathbf{C}^-$ is equivalent to the existence of an invariant measure μ for $(T(t))_{t\geq 0}$, i.e., a probability measure on $\mathbf{R}^N$ such that

$$\int_{\mathbf{R}^N} \big(T(t)f\big)(x)\, d\mu(x) = \int_{\mathbf{R}^N} f(x)\, d\mu(x)$$

for every $t \geq 0$ and $f \in C_b(\mathbf{R}^N)$, the space of all continuous and bounded functions on $\mathbf{R}^N$. Moreover, the invariant measure μ is unique and it is given by $d\mu(x) = b(x)\, dx$, where

$$b(x) = \frac{1}{(4\pi)^{N/2}(\det Q_\infty)^{1/2}} e^{-<Q_\infty^{-1}x,x>/4} \tag{1.3}$$

and

$$Q_\infty = \int_0^\infty e^{sB} Q e^{sB^*}\, ds.$$

For more information on invariant measures we refer to [10] and [24]. It is well known that $(T(t))_{t\geq 0}$ extends to a strongly continuous semigroup of positive contractions in $L^p_\mu = L^p(\mathbf{R}^N, d\mu)$ for every $1 \leq p < \infty$. Such a semigroup is symmetric in L^2_μ if and only if $QA^* = AQ$, see [6], but we do not assume this condition. Remark that, since $Q_t < Q_\infty$ in the sense of quadratic forms, the integral in (1.2) converges for every $f \in L^p_\mu$ and $x \in \mathbf{R}^N$, so that the extension of $(T(t))_{t\geq 0}$ to L^p_μ is still given by (1.2).

Let us denote by (A_p, D_p) the generator of $(T(t))_{t\geq 0}$ in L^p_μ. The main aim of this paper is the computation of the spectrum of (A_p, D_p) for $1 \leq p < \infty$. If $1 < p < \infty$, it is known that the spectrum is discrete and consists of eigenvalues of finite multiplicities, since the resolvent is compact, see [4]. Let $\lambda_1, \ldots, \lambda_r$ be the eigenvalues of B. For $1 < p < \infty$, we show that $\sigma(A_p) = \{\gamma = \sum_{i=1}^r n_i \lambda_i : \ n_i \in \mathbf{N}\}$ and that all the generalised eigenfunctions are polynomials and form a complete system in L^p_μ. We show that it is possible to reduce the computation of the spectrum of A to that of its drift term $L = \langle Bx, D\rangle$, no matter what the diffusion term $\mathrm{Tr}(QD^2)$ is, see in particular Lemma 3.3. Our method also allows us to compute the algebraic multiplicities of the eigenvalues and to estimate their indices in terms of the spectral properties of the matrix B. In particular, we find that A_p is diagonalisable, i.e., all its eigenvalues have index 1, if and only if the matrix B is diagonalisable. As a by-product of our proof, we also show that the spectrum is independent of $p \in]1, \infty[$ (the p-independence of the spectrum is however a consequence of the compactness of the resolvent, see e.g. [1]).

For $p = 1$ the spectrum is completely different. In fact, the spectrum in L^1_μ is the closed left half-plane and every complex number with negative real part is an eigenvalue. The drastic difference of the spectrum between the cases $p = 1$ and $p > 1$ is the same as for the harmonic oscillator, as shown in [12], see also [11, Section 4.3], and in fact the operator $H = -D^2 + x^2$ on $L^2(\mathbf{R}, dx)$ is unitarily equivalent to a one-dimensional Ornstein-Uhlenbeck operator in L^2_μ, with $d\mu = \pi^{-1/2} e^{-x^2} dx$.

Let us stress that we allow Q to have rank strictly less than N. However our main result seems to be new even in the nondegenerate case, that is when Q is positive definite.

Let us mention another result of the paper. Assuming that Q is nondegenerate, in [14] it is shown that $(T(t))_{t\geq 0}$ is analytic in L^p_μ, $1 < p < \infty$, even in the infinite dimensional setting (see also [3], [8], [18] and [15]). Under our assumptions, in Section 2 we show that the semigroup $(T(t))_{t\geq 0}$ is differentiable in L^p_μ, for $1<p<\infty$. Obviously, it is not so in L^1_μ (see also Corollary 5.2).

We remark that in the particular case $Q=I$, $B=-I$, it is well known that the spectrum in L^2_μ consists of the negative integers and that the Hermite polynomials form a complete system of eigenfunctions, see [2], [22]. More generally, when T_t is symmetric the characterisation of the spectrum of A in L^2_μ also follows from [5, Section 2]. Finally, we refer to [20] for the spectrum of A in $L^p(\mathbf{R}^N, dx)$ and in spaces of continuous functions.

Notation. If C is a linear operator, we denote by $\sigma(C)$, $P\sigma(C)$ and $\rho(C)$, the spectrum, the point-spectrum and the resolvent set of C, respectively. Given $\lambda\in P\sigma(C)$, a vector u is called a generalised eigenvector if $(\lambda - A)^k u = 0$ for some positive integer k. If C has compact resolvent and $\lambda\in\sigma(C)$, we denote by $\nu(\lambda)$ the index of λ, that is the smallest integer ν such that $\mathrm{Ker}(\lambda - C)^\nu = \mathrm{Ker}(\lambda - C)^{\nu+1}$. The subspace $\mathrm{Ker}(\lambda - C)^{\nu(\lambda)}$ of all generalised eigenvectors relative to λ is the spectral subspace associated to λ and its dimension $k(\lambda)$ is the algebraic multiplicity of λ. The spectral bound $s(C)$ is defined by $s(C) = \sup\{\mathrm{Re}\,\lambda : \lambda\in\sigma(C)\}$. $C_b(\mathbf{R}^N)$ stands for the Banach space of all complex continuous and bounded functions on $\mathbf{R}^N$. $C_0(\mathbf{R}^N)$ is the closed subspace of $C_b(\mathbf{R}^N)$ of functions vanishing at infinity, $C_0^\infty(\mathbf{R}^N)$ is the space of C^∞-functions with compact support and $\mathcal{S}(\mathbf{R}^N)$ is the Schwartz class. $\mathcal{P}_n$ is the space of all polynomials of degree less than or equal to n. For $1\leq p<\infty$ and $k\in\mathbf{N}$, $W^{k,p}(\mathbf{R}^N)$ are the usual Sobolev spaces, and we define

$$W^{k,p}_\mu = \{u\in W^{k,p}_{\mathrm{loc}}(\mathbf{R}^N) : D^\alpha u\in L^p_\mu \text{ for } |\alpha|\leq k\}. \tag{1.4}$$

The norm in L^p_μ will be denoted by $\|\cdot\|_p$. Sometimes we write A_p for (A_p, D_p). Throughout this paper $\mathbf{N}$ indicates the set of nonnegative integers and $\mathbf{C}^-$, $\mathbf{C}^+$ the open left and right half-planes, respectively.

2 Properties of $(T(t))_{t\geq 0}$

In this section we collect some properties of $(T(t))_{t\geq 0}$ and of its generator (A_p, D_p) needed in the sequel.

We observe that $C_0^\infty(\mathbf{R}^N)$ is dense in $W^{k,p}_\mu$, $1\leq p<\infty$. Indeed, a simple truncation argument shows that the set of $W^{k,p}_\mu$-functions with compact support is dense and, given $u\in W^{k,p}_\mu$ with compact support, the usual approximating functions $\phi_\varepsilon * u$ converge to u, as $\varepsilon\to 0$, in $W^{k,p}(\mathbf{R}^N)$ and hence in $W^{k,p}_\mu$.

As regards the domains D_p, we remark that $D_p\subset D_q$ if $p\geq q$ and $A_p u = A_q u$ for $u\in D_p$. If Q is non-degenerate and $1<p<\infty$, the domain D_p is nothing but the weighted Sobolev space $W^{2,p}_\mu$ and $A_p u = Au$ for $u\in D_p$ (see [21] and also [18] for $p=2$, [6] and [7] for any p when (A_2, D_2) is self-adjoint).

For our purposes, we only need the following simple lemma.

Lemma 2.1 *Let $1\leq p<\infty$. If $u\in C^\infty(\mathbf{R}^N)$ is such that $D_{ij}u\in L^p_\mu$ for $i,j=1,\ldots,N$ and $|x||Du|\in L^p_\mu$, then $u\in D_p$ and $A_p u = Au$. Moreover, the Schwartz class $\mathcal{S}(\mathbf{R}^N)$ is a core for (A_p, D_p).*

Proof. Observe that $Au\in L^p_\mu$. Let $0\leq\phi\in C_0^\infty(\mathbf{R}^N)$ be such that $\phi(x)=1$ if $|x|\leq 1$ and define $u_n(x)=\phi(x/n)u(x)$. It is easily seen, using dominated convergence, that $u_n\to u$ and $Au_n\to Au$ in L^p_μ. Since $u_n\in C_0^\infty(\mathbf{R}^N)$, it is elementary to check that $(T(t)u_n - u_n)/t\to$

Au_n uniformly (hence in L^p_μ) as $t \to 0$. Therefore, $u_n \in D_p$ and the equality $Au_n = A_p u_n$ holds. Letting $n \to \infty$ we obtain that $u \in D_p$ and that $A_p u = Au$, since (A_p, D_p) is closed. Finally, since $\mathcal{S}(\mathbf{R}^N)$ is contained in D_p and is $T(t)$-invariant, it is a core for (A_p, D_p). $\square$

We discuss now some smoothing properties of $(T(t))_{t\geq 0}$, depending upon the hypoellipticity condition $\det Q_t > 0$. To this purpose, it is useful to recall that the above condition is also equivalent to the well-known Kalman rank condition

$$\text{rank}\left[Q^{1/2}, BQ^{1/2}, \ldots, B^{N-1}Q^{1/2}\right] = N,$$

arising in control theory (see e.g. [26]). In the above formula, the $N \times N^2$ matrix in the left-hand-side is obtained by writing consecutively the columns of the matrices $B^i Q^{1/2}$. Moreover, if $0 \leq m \leq N-1$ is the smallest integer such that $\text{rank}\left[Q^{1/2}, BQ^{1/2}, \ldots, B^m Q^{1/2}\right] = N$, then

$$\|Q_t^{-1/2} e^{tB}\| \leq \frac{C}{t^{1/2+m}}, \qquad t \in (0,1] \tag{2.1}$$

(see [25]). Of course, $m = 0$ if and only if Q is invertible.

The following lemma is a slight modification of a result proved, in the infinite-dimensional setting, in [4, Lemma 3]. We give the proof for completeness. The number m which appears in the statement is that defined above and appearing in (2.1).

Lemma 2.2 *Let $1 < p < \infty$. For every $t > 0$, $T(t)$ maps L^p_μ into $C^\infty(\mathbf{R}^N) \cap W^{k,p}_\mu$ for every $k \in \mathbf{N}$. Moreover, there exists $C = C(k,p) > 0$ such that for every $f \in L^p_\mu$ the inequality*

$$\|D^\alpha T(t) f\|_p \leq \frac{C}{t^{|\alpha|(1/2+m)}} \|f\|_p, \qquad t \in (0,1)$$

holds for every multiindex α with $|\alpha| = k$.

Proof. Let us fix $t > 0$ and set

$$b_t(x) = \frac{1}{(4\pi)^{N/2} (\det Q_t)^{1/2}} e^{-<Q_t^{-1}x, x>/4}.$$

Since $Q_t < Q_\infty$, in the sense of quadratic forms, it is easily seen that there exist $K, \varepsilon > 0$ (depending upon t) such that $b_t(x) \leq K e^{-\varepsilon |x|^2} b(x)$, where b (defined in (1.3)) is the density of μ. It follows that one can differentiate under the integral sign in (1.2) for every $f \in L^p_\mu$ thus obtaining

$$\big(DT(t)f\big)(x) = -\frac{1}{2} \int_{\mathbf{R}^N} e^{tB^*} Q_t^{-1} y f\big(e^{tB}x - y\big) b_t(y)\, dy$$

for every $x \in \mathbf{R}^N$ and hence $T(t)f \in C^1(\mathbf{R}^N)$. By Hölder inequality and (2.1)

$$\begin{aligned}
|(D_i T(t)) f(x)| &\leq \frac{1}{2} \Big(\int_{\mathbf{R}^N} |\langle Q_t^{-1/2} e^{tB} e_i, Q_t^{-1/2} y\rangle|^{p'} b_t(y)\, dy \Big)^{1/p'} \Big((T(t)|f|^p)(x) \Big)^{1/p} \\
&\leq \frac{1}{2} |Q_t^{-1/2} e^{tB} e_i| \Big(\int_{\mathbf{R}^N} |Q_t^{-1/2} y|^{p'} b_t(y)\, dy \Big)^{1/p'} \Big((T(t)|f|^p)(x) \Big)^{1/p} \\
&\leq C_p t^{-1/2-m} \Big((T(t)|f|^p)(x) \Big)^{1/p}
\end{aligned}$$

and the thesis follows for $k = 1$ raising to the power p and integrating the above inequality with respect to μ. The proof for $k \geq 1$ proceeds as in [18, Lemma 3.2] using the equality $DT(t)u = e^{tB^*} T(t) Du$, which holds for every $u \in W^{1,p}_\mu$. This identity is easily verified in $C_0^\infty(\mathbf{R}^N)$ and extends to $W^{1,p}_\mu$ by density. $\square$

The compactness of $(T(t))_{t\geq 0}$ for $p=2$ easily follows from the above lemma and the compactness of the embedding of $W^{1,2}_\mu$ into L^2_μ, see [10]. If $1<p<\infty$, the same holds by interpolation (see [4, Lemma 2]).

If Q is nondegenerate, the analyticity of $(T(t))_{t\geq 0}$ in L^2_μ was proved in [14] (see also [8], [18]). From the Stein interpolation theorem it follows that $(T(t))_{t\geq 0}$ is analytic in L^p_μ for $1<p<\infty$. On the other hand, $(T(t))_{t\geq 0}$ is not analytic in L^2_μ (hence in L^p_μ) if Q is degenerate, see [15]. We show that in any case $(T(t))_{t\geq 0}$ is differentiable in L^p_μ, if $1<p<\infty$. To prove this, we need the following lemma which generalises [18, Lemma 2.1].

Lemma 2.3 *If $1<p<\infty$, for every $h=1,\dots,N$ the map $u\mapsto x_h u$ is bounded from $W^{1,p}_\mu$ to L^p_μ.*

PROOF. It suffices to show that there is a constant K_p such that for every $u\in C_0^\infty(\mathbf{R}^N)$

$$\int_{\mathbf{R}^N} |x_h u(x)|^p\, d\mu(x) \leq K_p \int_{\mathbf{R}^N} \big(|u(x)|^p + |Du(x)|^p\big)\, d\mu(x). \tag{2.2}$$

By a linear change of variables we may assume that Q_∞ is diagonal with eigenvalues $\mu_1,\dots,\mu_N$ and hence that

$$b(x) = \frac{1}{(4\pi)^{N/2}(\mu_1\cdots\mu_N)^{1/2}} \exp\Big\{-\sum_{i=1}^N x_i^2/(4\mu_i)\Big\}.$$

As a first case, assume $p\geq 2$. If $u\in C_0^\infty(\mathbf{R}^N)$, then one has, for $C=2\max\{\mu_1,\dots,\mu_N\}$:

$$\begin{aligned}
&\int_{\mathbf{R}^N} |x_h u(x)|^p\, d\mu(x) \leq -C\int_{\mathbf{R}^N} |u(x)|^p |x_h|^{p-2} x_h\cdot D_h b(x)\, dx\\
=\;& C\int_{\mathbf{R}^N} \big(p x_h u(x)|x_h u(x)|^{p-2} D_h u(x) + (p-1)|x_h|^{p-2}|u(x)|^p\big)\, d\mu(x)\\
\leq\;& C_1\int_{\mathbf{R}^N} |x_h|^{p-2}|u(x)|^p\, d\mu(x) + C_2\Big(\int_{\mathbf{R}^N} |x_h u(x)|^p\, d\mu(x)\Big)^{\frac{p-1}{p}}\Big(\int_{\mathbf{R}^N} |D_h u(x)|^p\, d\mu(x)\Big)^{\frac{1}{p}}\\
\leq\;& \varepsilon\int_{\mathbf{R}^N} |x_h u(x)|^p\, d\mu(x) + C_\varepsilon\int_{\mathbf{R}^N} (|u(x)|^p + |D_h u(x)|^p)\, d\mu(x),
\end{aligned}$$

for every $\varepsilon>0$, with a suitable C_ε (in the last line we have used Young's inequality and the estimate $|x_h|^{p-2}\leq \varepsilon|x_h|^p + C_\varepsilon$). Choosing $\varepsilon<1$ we deduce (2.2).

Let us deal with the case $1<p<2$. We proceed as before but we have to estimate in a different way the term

$$\int_{\mathbf{R}^N} |x_h|^{p-2}|u(x)|^p\, d\mu(x).$$

To simplify the notation, take $h=N$ and write $x'=(x_1,\dots,x_{N-1})$, $b(x)=b'(x')\frac{e^{-x_N^2/4\mu_N}}{(4\pi\mu_N)^{1/2}}$, and $d\mu'=b'(x')dx'$, $d\mu''=(4\pi\mu_N)^{-1/2}\exp\{-x_N^2/\mu_N\}dx_N$, so that

$$\begin{aligned}
\int_{\mathbf{R}^N} |x_N|^{p-2}|u(x)|^p\, d\mu(x) &= \int_{\mathbf{R}^{N-1}} d\mu'(x')\int_{\mathbf{R}} |x_N|^{p-2}|u(x',x_N)|^p\, d\mu''(x_N)\\
&= \int_{\mathbf{R}^{N-1}} d\mu'(x')\int_{|x_N|\geq 1} |x_N|^{p-2}|u(x',x_N)|^p\, d\mu''(x_N)\\
&\quad + \int_{\mathbf{R}^{N-1}} d\mu'(x')\int_{-1}^{1} |x_N|^{p-2}|u(x',x_N)|^p\, d\mu''(x_N)\\
&= J_1 + J_2
\end{aligned}$$

Clearly, $J_1 \leq \int_{\mathbf{R}^N} |u(x)|^p \, d\mu(x)$. Let us estimate J_2. For every $x' \in \mathbf{R}^{N-1}$ we have, by the Sobolev embedding $W^{1,p}(-1,1) \hookrightarrow L^\infty(-1,1)$,

$$\begin{aligned}\int_{-1}^{1} |x_N|^{p-2} |u(x', x_N)|^p \, d\mu''(x_N) &\leq C\Big(\sup_{|x_N|\leq 1} |u(x', x_N)|\Big)^p \int_{-1}^{1} |x_N|^{p-2} \, dx_N \\ &\leq C_1 \int_{-1}^{1} \big(|u(x', x_N)|^p + |D_N u(x', x_N)|^p\big) \, dx_N \\ &\leq C_2 \int_{\mathbf{R}} \big(|u(x', x_N)|^p + |D_N u(x', x_N)|^p\big) \, d\mu''(x_N)\end{aligned}$$

whence, integrating on $\mathbf{R}^{N-1}$,

$$J_2 \leq C_2 \int_{\mathbf{R}^N} (|u(x)|^p + |Du(x)|^p) \, d\mu(x),$$

and this completes the proof. □

It follows, in particular, that the map $Lu = \langle Bx, Du \rangle$ is bounded from $W^{2,p}_\mu$ into L^p_μ for $1 < p < \infty$.

Proposition 2.4 *For $1 < p < \infty$ the semigroup $(T(t))_{t \geq 0}$ is differentiable in L^p_μ.*

Proof. If $f \in \mathcal{S}(\mathbf{R}^N)$ then $T(t)f \in \mathcal{S}(\mathbf{R}^N) \subset D_p$. From Lemmas 2.3, 2.2 it follows as in [18, Proposition 3.3] that

$$\|A_p T(t) f\|_p = \|AT(t)f\|_p \leq \frac{C}{t^{2m+1}} \|f\|_p, \quad 0 < t \leq 1,$$

hence $A_p T(t)$ extends to a bounded operator in L^p_μ and the thesis follows. □

We shall see in Section 4 that the above result is false for $p = 1$.

3 Spectrum in L^p_μ for $1 < p < \infty$

In this section we assume that $1 < p < \infty$. The following estimate is the main step to show that the generalised eigenfunctions of A_p are polynomials.

Lemma 3.1 *Let $k \in \mathbf{N}$ and $\varepsilon > 0$ be given, with $s(B)+\varepsilon < 0$. Then there exists $C = C(k, \varepsilon)$ such that for every $u \in W^{k,p}_\mu$*

$$\sum_{|\alpha|=k} \|D^\alpha T(t) u\|_p \leq C e^{tk(s(B)+\varepsilon)} \sum_{|\alpha|=k} \|D^\alpha u\|_p, \quad t \geq 0. \tag{3.1}$$

Proof. Let $C_1 = C_1(\varepsilon)$ be such that $\|e^{tB^*}\| \leq C_1 e^{t(s(B)+\varepsilon)}$ and recall that $DT(t)u = e^{tB^*} T(t) Du$ for every $u \in W^{1,p}_\mu$. Since $(T(t))_{t\geq 0}$ is contractive in L^p_μ the statement is proved for $k = 1$ with $C = C_1$. Suppose that the statement is true for k with a suitable constant C_k and consider $u \in W^{k+1,p}_\mu$. Then, if $|\alpha| = k$,

$$\begin{aligned}\|DD^\alpha T(t) u\|_p &= \|D^\alpha DT(t)u\|_p = \|D^\alpha e^{tB^*} T(t) Du\|_p \leq C_1 e^{t(s(B)+\varepsilon)} \|D^\alpha T(t) Du\|_p \\ &\leq C_1 C_k e^{t(k+1)(s(B)+\varepsilon)} \|DD^\alpha u\|_p.\end{aligned}$$

□

Observe that $\sigma(A_p) \subset \{\lambda \in \mathbf{C} : \operatorname{Re}\lambda \le 0\}$, since $(T(t))_{t\ge 0}$ is a semigroup of contractions in L^p_μ and that 0 is a simple eigenvalue of A_p. Moreover, every eigenfunction corresponding to the eigenvalue 0 is constant (this holds also for $p = 1$). In fact, if $u \in D_p$ and $A_p u = 0$, then $T(t)u = u$. On the other hand (see [10, Theorem 4.2.1])

$$T(t)u \to \int_{\mathbf{R}^N} u\, d\mu$$

as $t \to \infty$ and therefore u is constant. We now show that all the generalised eigenfunctions are polynomials.

If $r \ge 1$, we denote by $D(A_p^r)$ the domain of A_p^r in L^p_μ.

Proposition 3.2 *Suppose that $u \in D(A_p^r)$ satisfies $(\gamma - A_p)^r u = 0$ for some positive integer r and $\gamma \in \mathbf{C}$. Then u is a polynomial of degree less than or equal to $|\operatorname{Re}\gamma/s(B)|$.*

Proof. Suppose first that $r = 1$, so that $u \in D_p$ is an eigenfunction. Since $T(t)u = e^{\gamma t}u$, from Lemma 2.2 we deduce that $u \in W^{k,p}_\mu \cap \mathcal{C}^\infty(\mathbf{R}^N)$, for every k. Clearly $D^\alpha T(t)u = e^{\gamma t}D^\alpha u$ for every multiindex α. Given $\varepsilon \in (0, |s(B)|)$, from Lemma 3.1 it follows that

$$e^{t\operatorname{Re}\gamma} \sum_{|\alpha|=k} \|D^\alpha u\|_p \le C(k,\varepsilon) e^{tk(s(B)+\varepsilon)} \sum_{|\alpha|=k} \|D^\alpha u\|_p$$

and hence $D^\alpha u = 0$ if $|\alpha||s(B)| \ge |\operatorname{Re}\gamma|$. It follows that u is a polynomial of degree less than or equal to $|\frac{\operatorname{Re}\gamma}{s(B)}|$.

Suppose now that the statement holds for r, and let $u \in D(A_p^{r+1})$ be such that $(\gamma - A_p)^{r+1}u = 0$. Then $v = (A-\gamma)u$ is a polynomial of degree less than or equal to $|\frac{\operatorname{Re}\gamma}{s(B)}|$ and

$$T(t)u = e^{\gamma t}\sum_{j=0}^{r} \frac{t^j}{j!}(A-\gamma)^j u = e^{\gamma t}u + e^{\gamma t}\sum_{j=1}^{r}\frac{t^j}{j!}(A-\gamma)^{j-1}v.$$

If $|\alpha| > \operatorname{Re}\gamma/s(B)$, then $D^\alpha(A-\gamma)^j v = 0$ and hence $D^\alpha T(t)u = e^{\gamma t}D^\alpha u$. At this point one concludes the proof as in the case $r = 1$. □

Let us denote by

$$Lu = \langle Bx, Du\rangle$$

the drift term in (1.1). We reduce the computation of the spectrum of A_p to that of L.

Lemma 3.3 *The following statements are equivalent.*

(i) $\gamma \in \sigma(A_p)$.

(ii) *There exists a homogeneous polynomial $u \ne 0$ such that $Lu = \gamma u$.*

Proof. First we observe that $A_p u = Au$ if u is a polynomial (see Lemma 2.1) and that both A and L map $\mathcal{P}_n$ into itself. Moreover $A = L$ on $\mathcal{P}_1$ and hence we may consider only polynomials of degree greater than or equal to 2.

Suppose that (i) holds and let u be a polynomial of degree $n \ge 2$ such that $A_p u = \gamma u$, that is $\gamma u - \sum_{i,j} q_{ij}D_{ij}u - Lu = 0$. If $\gamma - L$ is bijective on $\mathcal{P}_{n-2}$ we can find $v \in \mathcal{P}_{n-2}$ such that $\gamma v - Lv = \sum_{i,j} q_{ij}D_{ij}u$ and hence $z = u - v \in \mathcal{P}_n$, satisfies $\gamma z - Lz = 0$ and $z \ne 0$. If $\gamma - L$ is not bijective on $\mathcal{P}_{n-2}$ we consider a function z in its kernel. In any case we find $0 \neq z \in \mathcal{P}_n$ such that $\gamma z - Lz = 0$. To find a (nonzero) homogeneous polynomial u such that $\gamma u - Lu = 0$ it is sufficient to observe that L maps homogeneous polynomials into homogeneous polynomials so that all homogeneous addends u of z satisfy $\gamma u - Lu = 0$.

Assume now that (ii) holds with u homogeneous polynomial of degree $n \geq 2$. If $\gamma - A_p$ is not injective on $\mathcal{P}_{n-2}$ clearly (i) is true. Otherwise we find $v \in \mathcal{P}_{n-2}$ such that $\gamma v - Av = \sum_{i,j} q_{ij} D_{ij} u$ and then $0 \neq w = u + v \in \mathcal{P}_n$ satisfies $\gamma w - A_p w = 0$. $\square$

We study now the equation $\gamma u - Lu = 0$ with u polynomial, $\gamma \in \mathbf{C}$. If $B = -I$ this is the well-known Euler equation satisfied by all regular functions homogeneous of degree $(-\gamma)$. If we require that u is a polynomial, we obtain $(-\gamma) \in \mathbf{N}$, hence all negative integers are eigenvalues of L and, for every $n \in \mathbf{N}$, all homogeneous polynomials of degree n are eigenfunctions.

The equation with a general B is much more complicated and we shall not characterise all polynomial solutions but only the values of γ for which such a solution exists. Observe that a differentiable function u satisfies $\gamma u - Lu = 0$ if and only if

$$u(e^{tB}x) = e^{t\gamma}u(x) \qquad t \geq 0, \; x \in \mathbf{R}^N. \tag{3.2}$$

Let u be a (nonzero) homogeneous polynomial of degree n satisfying (3.2): in this case the same equality holds for every *complex* point $x \in \mathbf{C}^N$. Let now M be a non-singular complex $N \times N$ matrix, such that $MBM^{-1} = C$, where C is the canonical Jordan form of B. Introduce a new homogeneous polynomial $v(z) = u(M^{-1}z)$, $z \in \mathbf{C}^N$, so that $u(x) = v(Mx)$. Since $v(Me^{tB}M^{-1}z) = e^{t\gamma}\, v(z)$, we obtain that

$$v(e^{tC}z) = e^{t\gamma}\, v(z), \qquad z \in \mathbf{C}^N,$$

and we find the values of γ for which a solution exists working with the Jordan matrix C. Before proving the main result of this section, we present in a particular case the argument we use in the proof. Let us suppose that C consists of a unique Jordan block of size N relative to an eigenvalue λ, that is

$$C = \begin{pmatrix} \lambda & 1 & \cdots & 0 \\ 0 & \lambda & \cdots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda \end{pmatrix}$$

and write $C = \lambda I + R$ with R nilpotent. Hence e^{tR} has polynomial entries and we obtain

$$e^{t\gamma}v(z) = v(e^{tB}z) = v(e^{t\lambda}e^{tR}z) = e^{n\lambda t}v(e^{tR}z) = e^{n\lambda t}q(t,z) \tag{3.3}$$

where $q(t,z) = \sum_{|\alpha|=n} c_\alpha(t) z^\alpha$ and the $c_\alpha(t)$ are polynomials. Now fix $\hat{z} \neq 0$ in (3.3) such that $v(\hat{z}) \neq 0$ and look at the variable t. It follows that $\gamma = n\lambda$, i.e., the eigenvalues of L are multiples of the (unique) eigenvalue of B. In the general case, we have the following result.

Theorem 3.4 *Let $\lambda_1, \ldots, \lambda_r$ be the (distinct) eigenvalues of B. Then*

$$\sigma(A_p) = \Big\{ \gamma = \sum_{j=1}^{r} n_j \lambda_j : n_j \in \mathbf{N} \Big\}.$$

Moreover, the linear span of the generalised eigenfunctions of A_p is dense in L^p_μ.

Proof. We keep the above notation (recall that M is a non-singular complex $N \times N$ matrix, such that $MBM^{-1} = C$ and C is the canonical Jordan form of B). Let C_j, for $j = 1, \dots r$, be the block of C corresponding to the invariant subspace associated with λ_j, and denote by k_j ($1 \le k_j \le N$, $\sum_{j=1}^r k_j = N$) the size of C_j, i.e., the algebraic multiplicity of λ_j. We may write $C_j = \lambda_j I + R_j$ where R_j is a nilpotent matrix. Let us decompose $\mathbf{C}^N$ into the direct sum of these invariant subspaces and write $z \in \mathbf{C}^N$ in the form $z = (z_1, \dots, z_r)$, with $z_j \in \mathbf{C}^{k_j}$.

Assume that $\gamma \in \sigma(A_p)$. Then, according to Lemma 3.3, there exists a nonzero homogeneous polynomial u such that $Lu = \gamma u$ or, in an equivalent way, $u(e^{tB}x) = e^{\gamma t}u(x)$. Introducing the homogeneous polynomial $v(z) = u(M^{-1}z)$, we know that $v(e^{tC}z) = e^{t\gamma}v(z)$ for every $z \in \mathbf{C}^N$. Let us write v in the following way:

$$v(z) = \sum_{|\alpha_1|+\ldots+|\alpha_r|=n} c_{\alpha_1\ldots\alpha_r} \prod_{j=1}^r z_j^{\alpha_j},$$

and prove that $\gamma = \sum_j \lambda_j|\alpha_j|$, for suitable (α_j). We have

$$\begin{aligned}
e^{t\gamma}v(z) = v(e^{tC}z) &= v(e^{tC_1}z_1, \dots, e^{tC_r}z_r) \\
&= \sum_{|\alpha_1|+\ldots+|\alpha_r|=n} c_{\alpha_1\ldots\alpha_r} \prod_{j=1}^r (e^{tC_j}z_j)^{\alpha_j} \\
&= \sum_{|\alpha_1|+\ldots+|\alpha_r|=n} c_{\alpha_1\ldots\alpha_r}\ e^{t(\lambda_1|\alpha_1|+\ldots+\lambda_r|\alpha_r|)} \prod_{j=1}^r (e^{tR_j}z_j)^{\alpha_j}.
\end{aligned}$$

Now fix $\hat{z} \neq 0$ such that $v(\hat{z}) \neq 0$ and look at the variable t. Since $\prod_{j=1}^r (e^{tR_j}\hat{z}_j)^{\alpha_j}$ is a polynomial in t for any $(\alpha_1, \dots, \alpha_r)$, it follows that there exists some $(\alpha_1, \dots, \alpha_r)$ such that $\gamma = \lambda_1|\alpha_1| + \ldots + \lambda_r|\alpha_r|$. This means that

$$\gamma = \sum_{j=1}^r n_j\lambda_j, \quad n_j \in \mathbf{N}. \tag{3.4}$$

Conversely, let $\gamma = \sum_{j=1}^r n_j\lambda_j$, with arbitrary $n_j \in \mathbf{N}$. Let us write $z \in \mathbf{C}^N$ in the form

$$z = (z_1, \dots, z_r) = (z_1, \dots, z_{k_1}, z_{k_1+1}, \dots, z_{k_1+k_2}, \dots, z_{k_1+\ldots+k_r}).$$

Consider the polynomial

$$v(z) = z_{k_1}^{n_1} \cdot z_{k_1+k_2}^{n_2} \cdots z_{k_1+\ldots k_r}^{n_r},$$

depending only upon the r complex variables $z_{k_1}, z_{k_1+k_2}, \dots, z_{k_1+\ldots k_r}$ (the last variable in each block). It is easy to verify that $v(e^{tC}z) = e^{t\gamma}v(e^{tR_1}z_1, \dots, e^{tR_r}z_r) = e^{t\gamma}v(z)$, $z \in \mathbf{C}^N$. The polynomial $u(z) = v(Mz)$, $z \in \mathbf{C}^N$, satisfies $u(e^{tB}x) = e^{t\gamma}u(x)$, $x \in \mathbf{R}^N$. It follows that $Lu = \gamma u$ and hence $\gamma \in \sigma(A_p)$, by Lemma 3.3.

Finally we show the completeness of the system of the generalised eigenfunctions. Since A_p maps $\mathcal{P}_n$ into itself for every n and $\mathcal{P}_n$ is finite dimensional, the linear span of the generalised eigenfunctions of the restriction of A_p to $\mathcal{P}_n$ is $\mathcal{P}_n$. It follows that the linear span of the generalised eigenfunctions of A_p is the set of all polynomials, hence is dense in L^p_μ. □

4 Eigenfunctions and multiplicities

In this section we still assume that $1 < p < \infty$ and compute the algebraic multiplicity of the eigenvalues of A_p and estimate their index. In particular, we obtain that A_p has semi-simple eigenvalues, that is the index of each eigenvalue is 1, if and only if the matrix B is diagonalisable. We denote by $\nu_{A_p}(\gamma), k_{A_p}(\gamma)$ the index and the algebraic multiplicity of an eigenvalue $\gamma \in \sigma(A_p)$. Similarly, we write $\nu_L(\gamma)$, $k_L(\gamma)$ for the index and the algebraic multiplicity of an eigenvalue γ of the drift operator, regarding this latter as an operator from the space of all polynomials $\mathcal{P} = \cup_n \mathcal{P}_n$ into itself.

If $\lambda_1, \ldots, \lambda_r$ are the distinct eigenvalues of B, we denote by ν_j and k_j the index and the algebraic multiplicity of λ_j, respectively. Finally, we denote by $\mathcal{H}_n$ the space of homogeneous polynomials of degree n (so that $\mathcal{P}_n = \bigoplus_{k=0}^n \mathcal{H}_k$) and by Q_k the canonical projection of $\mathcal{P}$ onto $\mathcal{H}_k$.

We need the following lemma.

Lemma 4.1 *Let $\lambda \notin \sigma(A_p) = \sigma(L)$. Then the following identity holds for $u \in \mathcal{P}$*

$$(\lambda - L)^{-1} u = \sum_j Q_j (\lambda - A_p)^{-1} Q_j u. \tag{4.1}$$

Proof. By linearity, it suffices to prove (4.1) when u is a homogeneous polynomial. Let $u \in \mathcal{H}_k$ and $v = (\lambda - A_p)^{-1} u \in \mathcal{P}_k$. Then

$$u = Q_k u = Q_k(\lambda - A_p)v = Q_k(\lambda - L)v = Q_k(\lambda - L)Q_k v = (\lambda - L)Q_k v.$$

This shows that $(\lambda - L)^{-1} u = Q_k(\lambda - A_p)^{-1} Q_k u$. □

We can compare indices and multiplicities of an eigenvalue γ with respect to A_p and L.

Proposition 4.2 *Let $\gamma \in \sigma(A_p) = \sigma(L)$. Then $k_L(\gamma) = k_{A_p}(\gamma)$ and $\nu_L(\gamma) \le \nu_{A_p}(\gamma)$.*

Proof. Let n be such that the spectral subspaces of L, A_p with respect to γ are contained in $\mathcal{P}_n$ and let Γ be a small circle around γ not containing other eigenvalues. Integrating (4.1) on Γ we obtain in $\mathcal{P}_n$

$$P_L(\gamma) = \sum_{j=1}^n Q_j P_{A_p}(\gamma) Q_j,$$

where $P_L(\gamma)$ and $P_{A_p}(\gamma)$ are the spectral projections of L and A_p associated with the eigenvalue γ. It follows that

$$\begin{aligned} k_L(\gamma) &= \dim P_L(\gamma)(L^p_\mu) = \dim P_L(\gamma)(\mathcal{P}_n) = \text{trace } P_L(\gamma) = \sum_{j=1}^n \text{trace } Q_j P_{A_p}(\gamma) Q_j \\ &= \sum_{j=1}^n \text{trace } Q_j P_{A_p}(\gamma) = \text{trace } P_{A_p}(\gamma) = k_{A_p}(\gamma). \end{aligned}$$

To show that $\nu_L(\gamma) \le \nu_{A_p}(\gamma)$, let us recall that $\nu_L(\gamma)$ and $\nu_{A_p}(\gamma)$ coincide with the orders of the pole $\lambda = \gamma$ for $(\lambda - L)^{-1}$ and $(\lambda - A_p)^{-1}$, respectively. But (4.1) implies that the orders of the poles of $(\lambda - L)^{-1}$ do not exceed the corresponding orders of $(\lambda - A_p)^{-1}$, hence the claim follows. □

We describe now the spectral subspaces of L. To this aim, we employ the same method as in the proof of Theorem 3.4 and assume that B is in the canonical Jordan form. Let C_j, for $j = 1, \dots r$, be the block of B corresponding to the spectral subspace associated with λ_j and observe that k_j (the algebraic multiplicity of λ_j) is the size of C_j. We may write $C_j = \lambda_j I + R_j$ where R_j is a nilpotent matrix. Let us decompose $\mathbf{C}^N$ into the direct sum of these invariant subspaces and write $z \in \mathbf{C}^N$ in the form $z = (z_1, \dots, z_r)$, with $z_j \in \mathbf{C}^{k_j}$. We denote by $\mathcal{H}_{n_1,\dots,n_r}$ the space of all the polynomials depending only on the variables $z_1, \dots, z_r$, which are homogeneous of degree n_j in each group of variables z_j.

Proposition 4.3 *Let $\gamma \in \sigma(L)$. Then the spectral subspace of L associated with γ coincides with*

$$\bigoplus_{n_1\lambda_1+\cdots+n_r\lambda_r=\gamma} \mathcal{H}_{n_1,\dots,n_r}.$$

Proof. Let v be a polynomial

$$v(z) = \sum_{|\alpha_1|+\ldots+|\alpha_r|\le n} c_{\alpha_1\ldots\alpha_r} \prod_{j=1}^{r} z_j^{\alpha_j}.$$

Then

$$v(e^{tB}z) = \sum_{|\alpha_1|+\cdots+|\alpha_r|\le n} c_{\alpha_1\ldots\alpha_r}\ e^{t(\lambda_1|\alpha_1|+\cdots+\lambda_r|\alpha_r|)} \prod_{j=1}^{r}(e^{tR_j}z_j)^{\alpha_j}. \tag{4.2}$$

Assume now that $(\gamma - L)^k v = 0$, $(\gamma - L)^{k-1} v \neq 0$. Then

$$v(e^{tB}z) = e^{\gamma t}\sum_{j=0}^{k-1}\frac{t^j}{j!}\big((L-\gamma)^j v\big)(z). \tag{4.3}$$

Comparing equations (4.2), (4.3) and recalling that the matrices e^{tR_j} have polynomial entries in t, we deduce that $c_{\alpha_1,\dots,\alpha_r} = 0$ if $|\alpha_1|\lambda_1 + \cdots + |\alpha_r|\lambda_r \neq \gamma$ and therefore

$$v(z) = \sum_{|\alpha_1|\lambda_1+\cdots+|\alpha_r|\lambda_r=\gamma} c_{\alpha_1\ldots\alpha_r} \prod_{j=1}^{r} z_j^{\alpha_j} \tag{4.4}$$

and v belongs to $\bigoplus_{n_1\lambda_1+\cdots+n_r\lambda_r=\gamma}\mathcal{H}_{n_1,\dots,n_r}$.

Conversely, fix $n_1, \dots, n_r$ such that $n_1\lambda_1 + \cdots + n_r\lambda_r = \gamma$ and consider $v \in \mathcal{H}_{n_1,\dots,n_r}$. Then

$$v(z) = \sum_{|\alpha_1|=n_1,\dots,|\alpha_r|=n_r} c_{\alpha_1\ldots\alpha_r} \prod_{j=1}^{r} z_j^{\alpha_j}$$

and, from (4.2),

$$v(e^{tB}z) = e^{t\gamma} \sum_{|\alpha_1|=n_1,\cdots,|\alpha_r|=n_r} c_{\alpha_1,\dots,\alpha_r} \prod_{j=1}^{r}(e^{tR_j}z_j)^{\alpha_j}.$$

It follows that the spectrum of the restriction of L to the invariant subspace $\mathcal{H}_{n_1,\dots,n_r}$ consists of the unique point $\{\gamma\}$, hence $\mathcal{H}_{n_1,\dots,n_r}$ is contained in the spectral subspace associated with γ and the proof is complete. □

Theorem 4.4 *Let $\gamma \in \sigma(A_p)$. Then*

$$k_{A_p}(\gamma) = \sum_{n_1\lambda_1+\cdots+n_r\lambda_r=\gamma} \prod_{j=1}^{r} \frac{(k_j+n_j-1)!}{n_j!(k_j-1)!}.$$

PROOF. In fact $k_{A_p}(\gamma) = k_L(\gamma) = \dim \bigoplus_{n_1\lambda_1+\cdots+n_r\lambda_r=\gamma} \mathcal{H}_{n_1,\ldots,n_r}$. Since the sum is direct, the result follows from the equality

$$\dim \mathcal{H}_{n_1,\ldots,n_r} = \sum_{j=1}^{r} \frac{(k_j+n_j-1)!}{n_j!(k_j-1)!}.$$

□

We now compute $\nu_L(\gamma)$ for $\gamma \in \sigma(L)$.

Proposition 4.5 *Let $\gamma \in \sigma(L)$. Then*

$$\nu_L(\gamma) = 1 + \max\,\Big\{\sum_{j=1}^{r} n_j(\nu_j-1) : \sum_{j=1}^{r} n_j\lambda_j = \gamma\Big\}.$$

In particular, $\nu_L(\gamma) = 1$ for every $\gamma \in \sigma(L)$ if and only if $\nu_j = 1$ for every $j = 1, \ldots, r$, that is if and only if B is diagonalisable.

PROOF. Let us define $\eta(\gamma) = 1 + \max\,\{\sum_{j=1}^{r} n_j(\nu_j-1) : \sum_{j=1}^{r} n_j\lambda_j = \gamma\}$. Let v be a generalised eigenfunction relative to γ and assume that $(\gamma - L)^k v = 0$, $(\gamma - L)^{k-1} v \neq 0$. From (4.4) we obtain that

$$v(e^{tB}z) = e^{t\gamma} \sum_{|\alpha_1|\lambda_1+\cdots+|\alpha_r|\lambda_r=\gamma} c_{\alpha_1\ldots\alpha_r} \prod_{j=1}^{r} (e^{tR_j}z_j)^{\alpha_j}.$$

Observing that $\prod_{j=1}^{r}(e^{tR_j}z_j)^{\alpha_j}$ is a polynomial in t of degree less than or equal to

$$\sum_{j=1}^{r} |\alpha_j|(\nu_j-1),$$

we deduce from (4.3) that $k \leq \eta(\gamma)$ and therefore $\nu(\gamma) \leq \eta(\gamma)$.

Conversely, let $n_1, \ldots, n_r$ be such that $\sum_{j=1}^{r} n_j\lambda_j = \gamma$ and $\eta(\gamma) = 1 + \sum_{j=1}^{r} n_j(\nu_j - 1)$. Consider the polynomial

$$v(z) = z_1^{n_1} \cdot z_{k_1+1}^{n_2} \cdots z_{k_1+\ldots+k_{r-1}+1}^{n_r},$$

depending only upon the r complex variables $z_1, z_{k_1+1}, \ldots, z_{k_1+\cdots k_{r-1}+1}$ (the first variable in each block). It is not difficult to check that $e^{-t\gamma}v(e^{tB}z)$ is a polynomial in t of degree exactly $\eta(\gamma) - 1$. Since $v \in \mathcal{H}_{n_1,\ldots,n_r}$ and the spectrum of the restriction of L to $\mathcal{H}_{n_1,\ldots,n_r}$ is $\{\gamma\}$, v is a generalised eigenfunction relative to γ of order $\eta(\gamma)$. This concludes the proof.

□

In order to estimate $\nu_{A_p}(\gamma)$ from above, we deduce an explicit formula for $T(t)u$ when u is a polynomial. To simplify the notation we set

$$d\mu_t(y) = \frac{1}{(4\pi)^{N/2}(\det\, Q_t)^{1/2}} e^{-<Q_t^{-1}y,y>/4}\, dy.$$

It is not difficult to verify that, for every multiindex α with $|\alpha| = 2k$, the following identity holds:

$$\int_{\mathbf{R}^N} y^\alpha d\mu_t(y) = \frac{1}{k!} D_v^\alpha \Big((Q_t v, v)^k \Big)_{|v=0}. \tag{4.5}$$

The proof of the next lemma is straightforward. If α, h are multiindices, we write $h \le \alpha$ if $h_i \le \alpha_i$ for every i.

Lemma 4.6 *Let $u = \sum_{|\alpha| \le n} c_\alpha x^\alpha$. Then*

$$T(t)u(x) = \sum_{|\alpha| \le n} c_\alpha \sum_{h \le \alpha} \binom{\alpha}{h} \left(e^{tB}x\right)^h \int_{\mathbf{R}^N} y^{\alpha - h}\, d\mu_t(y). \tag{4.6}$$

Theorem 4.7 *Let $\gamma \in \sigma(A_p)$ and let $\nu = \max\, \{\nu_1, \ldots, \nu_r\}$. Then*

$$\nu_L(\gamma) \le \nu_{A_p}(\gamma) \le 1 + \frac{\mathrm{Re}\ \gamma}{s(B)}(\nu - 1).$$

PROOF. We have only to prove the second inequality. Let $u \in \mathcal{P}_n$ be such that $(\gamma - A_p)^{\nu_{A_p}(\gamma)} = 0$, $(\gamma - A_p)^{\nu_{A_p}(\gamma)-1} \ne 0$. From Proposition 3.2 we deduce that $n \le |\mathrm{Re}\ \gamma|/|s(B)|$. Next, observe that the entries in the variable t of $e^{tB}Qe^{tB^*}$ are of the form $\sum_k e^{\mu_k t} p_k(t)$ with $\mathrm{Re}\ \mu_k < 0$ and p_k polynomials of degree less than or equal to $2(\nu - 1)$. Therefore the entries of Q_t are of the form $\sum_k (e^{\mu_k t} q_k(t) + c_k)$ with q_k polynomials of degree less than or equal to $2(\nu - 1)$ and $c_k \in \mathbf{C}$. Using (4.5) one sees easily that the integrals

$$\int_{\mathbf{R}^N} y^{\alpha - h}\, d\mu_t(y)$$

(which can be nonzero only if $|\alpha - h|$ is even) are again of the form $\sum_k e^{\tau_k t} r_k(t)$ with $\mathrm{Re}\ \tau_k < 0$ and r_k polynomials of degree less than or equal to $(\nu - 1)(|\alpha| - |h|)$. Now (4.6) shows that $T(t)u(x)$ is of the form $\sum_k e^{v_k t} p_k(t, x)$ with $\mathrm{Re}\ v_k < 0$ and $p_k(t, x)$ polynomials in t of degree less that or equal to $n(\nu - 1)$. Since u satisfies (4.3) with $k = \nu_{A_p}(\gamma)$, it follows that $\nu_{A_p}(\gamma) \le 1 + n(\nu - 1)$, as asserted. □

Corollary 4.8 *The following equivalence holds: $\nu_{A_p}(\gamma) = 1$ for every $\gamma \in \sigma(A_p) = \sigma(L)$ if and only if $\nu_L(\gamma) = 1$ for every $\gamma \in \sigma(A_p) = \sigma(L)$. Moreover, this happens if and only if B is diagonalisable.*

PROOF. We already know that $\nu_L(\gamma) = 1$ for every $\gamma \in \sigma(A_p) = \sigma(L)$ if and only if B is diagonalisable, see Proposition 4.5. Moreover, from Proposition 4.2, we have that $\nu_L(\gamma) \le \nu_{A_p}(\gamma)$. To conclude the proof it suffices to show that if B is diagonalisable, then $\nu_{A_p}(\gamma) = 1$. This is however immediate from the above theorem, since $\nu = 1$. □

5 Spectrum in L^1_μ

We show that the spectrum of A_1 is the left half-plane. To do that, we follow the same method as [12], see p.128, and transform the operator A_1 on L^1_μ into an operator G on $L^1(\mathbf{R}^N, dx)$ via an isometry V between these spaces. Notice that the one-dimensional case of Theorem 5.1 below is in [12, Theorem 3], and that it implies the result for $Au = \Delta + \langle Bx, \nabla u\rangle$, with B symmetric, by separation of variables.

In particular, from our results it follows that $(T(t))_{t\geq 0}$ is not norm-continuous in L^1_μ, hence not analytic, nor differentiable, nor compact (see [13, Ch. II, Sec. 4]). The norm-discontinuity of $(T(t))_{t\geq 0}$ in L^1_μ can also be proved using the methods in [12], where more general situations are discussed for self-adjoint operators.

Theorem 5.1 *The spectrum of* (A_1, D_1) *is the left half-plane* $\{\gamma \in \mathbf{C} : \operatorname{Re}\gamma \leq 0\}$. *Each complex number* γ *with* $\operatorname{Re}\gamma < 0$ *is an eigenvalue.*

Proof. Let b be the density of μ with respect to the Lebesgue measure, given by (1.3), and set $h = 1/b$. Let $V : L^1 = L^1(\mathbf{R}^N, dx) \to L^1_\mu$ be the isometry defined by

$$(Vu)(x) = u(x)h(x), \qquad u \in L^1, \quad x \in \mathbf{R}^N.$$

We define an operator (G, D_G) on L^1 by $D_G = V^{-1}(D_1)$ and $G = V^{-1}A_1V$. If $u \in C_0^\infty(\mathbf{R}^N)$, then $u \in D_G$ and

$$Gu(x) = b(x)\big(A(uh)\big)(x) = Au(x) + 2b(x)\sum_{i,j=1}^N q_{ij}D_ih(x)D_ju(x) + b(x)u(x)Ah(x).$$

A direct computation shows that

$$2b(x)\sum_{i,j=1}^N q_{ij}D_ih(x)D_ju(x) = \langle QQ_\infty^{-1}x, Du(x)\rangle$$

and

$$\begin{aligned} b(x)Ah(x) &= \Big[\frac{1}{2}\operatorname{Tr}(QQ_\infty^{-1}) + \frac{1}{4}\langle QQ_\infty^{-1}x, Q_\infty^{-1}x\rangle + \frac{1}{2}\langle B^*Q_\infty^{-1}x, x\rangle\Big] \\ &= \Big[\frac{1}{2}\operatorname{Tr}(QQ_\infty^{-1}) + \frac{1}{4}\langle QQ_\infty^{-1}x, Q_\infty^{-1}x\rangle + \frac{1}{2}\langle BQ_\infty Q_\infty^{-1}x, Q_\infty^{-1}x\rangle\Big]. \end{aligned}$$

Using the identity $BQ_\infty + Q_\infty B^* = -Q$, which implies $2\langle BQ_\infty x, x\rangle = -\langle Qx, x\rangle$, it follows that $\frac{1}{4}\langle QQ_\infty^{-1}x, Q_\infty^{-1}x\rangle + \frac{1}{2}\langle BQ_\infty Q_\infty^{-1}x, Q_\infty^{-1}x\rangle = 0$ and hence, setting $k = \frac{1}{2}\operatorname{Tr}(QQ_\infty^{-1})$,

$$\begin{aligned} Gu(x) &= Au(x) + \langle QQ_\infty^{-1}x, Du(x)\rangle + ku(x) \\ &= \operatorname{Tr}(QD^2u(x)) + \langle (B + QQ_\infty^{-1})x, Du(x)\rangle + ku(x) \\ &= \operatorname{Tr}(QD^2u(x)) - \langle (Q_\infty B^*Q_\infty^{-1})x, Du(x)\rangle + ku(x). \end{aligned}$$

The operator $G_0 = \operatorname{Tr}(QD^2) - \langle (Q_\infty B^*Q_\infty^{-1})x, D\rangle$, with a suitable domain D_{G_0}, is the generator of an Ornstein-Uhlenbeck semigroup in L^1. Even though an explicit description of D_{G_0} is not known, we point out that $C_0^\infty(\mathbf{R}^N)$ is a core of (G_0, D_{G_0}) (see [20, Proposition 3.2]). The above computation shows that $G = G_0 + kI$ on $C_0^\infty(\mathbf{R}^N)$ and therefore $D_{G_0} \subset D_G$

and $G = G_0 + kI$ on D_{G_0}, since (G, D_G) is closed. On the other hand, if γ is sufficiently large, $\gamma - G$ is invertible on D_G and also on D_{G_0}, because it coincides therein with $G_0 + kI$. Therefore $D_G = D_{G_0}$.

Observe now that the identity $B + Q_\infty B^* Q_\infty^{-1} = -QQ_\infty^{-1}$ yields $\mathrm{Tr}(B) + \mathrm{Tr}(Q_\infty B^* Q_\infty^{-1}) = -\mathrm{Tr}(QQ_\infty^{-1})$ and hence $\mathrm{Tr}(Q_\infty B^* Q_\infty^{-1}) = \mathrm{Tr}(B) = -k$. Moreover G_0 satisfies the hypoellipticity condition. Indeed, if E is an invariant subspace of $Q_\infty^{-1} B Q_\infty$, contained in $\mathrm{Ker}(Q)$, the equation $BQ_\infty + Q_\infty B^* = -Q$ easily implies that $B^*(E) \subset E$. It follows that $E = \{0\}$, since A is hypoelliptic.

Since $\sigma(-Q_\infty B^* Q_\infty^{-1}) = -\sigma(B) \subset \mathbf{C}^+$, from [20, Theorem 4.7] it follows that the spectrum of (G_0, D_{G_0}) is the half-plane

$$\left\{\gamma \in \mathbf{C} : \mathrm{Re}\,\gamma \le \mathrm{Tr}(Q_\infty B^* Q_\infty^{-1}) = -k\right\}$$

and that every complex number γ with $\mathrm{Re}\,\gamma < -k$ is an eigenvalue. Since $G = G_0 + kI$ and the spectra of (A_1, D_1) and (G, D_G) coincide, the proof is complete. $\square$

Observe that the eigenvalues associated to polynomial eigenfunctions are the same for all $p \ge 1$. In fact, assuming that the eigenfunctions are polynomials, the arguments in Section 3 can be used also for $p = 1$ in order to determine the eigenvalues. However in L^1_μ there are nonpolynomial eigenfunctions and the spectrum is much larger. Moreover we have

Corollary 5.2 *The semigroup $(T(t))_{t\ge 0}$ does not map L^1_μ into $W^{1,1}_\mu$, for any $t > 0$.*

Proof. Assume by contradiction that $T(t_0)(L^1_\mu)$ is contained in $W^{1,1}_\mu$ for some $t_0 > 0$. This implies that $T(t)(L^1_\mu) \subset W^{1,1}_\mu$ for every $t \ge t_0$. Proceeding as in Lemma 2.2, we find that $T(t)(L^1_\mu) \subset C^k(\mathbf{R}^N) \cap W^{k,1}_\mu$ for every $k \in \mathbf{N}$, $t \ge kt_0$. Remark that Lemma 3.1 holds also if $p = 1$. Arguing as in Proposition 3.2, we infer that all the eigenfunctions of A_1 are polynomials. Thus, by Lemma 3.3, we deduce that the point spectrum of A_1 is discrete. This is the desired contradiction. $\square$

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References

[1] W. Arendt: Gaussian estimates and interpolation of the spectrum in L^p, *Diff. Int. Eqs.* **7** (1994), 1153-1168.

[2] D. Bakry: L'hypercontractivité et son utilisation en théorie des semi-groupes, in: D. Bakry, R. D. Gill, S. A. Molchanov, *Lectures on probability theory,* Springer LNM 1581, (1992), 1-114.

[3] V. Bogachev, M. Röckner, B. Schmuland: Generalized Mehler semigroups and applications, *Probab. Theory Related Fields* **105** (1996), 193-225.

[4] A. Chojnowska-Michalik, B. Goldys: Existence, uniqueness and invariant measures for stochastic semilinear equations on Hilbert spaces, *Probab. Theory Related Fields* **102** (1995), 331-356.

[5] A. Chojnowska-Michalik, B. Goldys: Nonsymmetric Ornstein-Uhlenbeck semigroup as second quantized operator, *J. Math. Kyoto Univ.* **36** (1996), no. 3, 481-498.

[6] A. Chojnowska-Michalik, B. Goldys: *Symmetric Ornstein-Uhlenbeck generators: characterizations and identification of domains,* preprint (2001).

[7] A. Chojnowska-Michalik, B. Goldys: Generalized Ornstein-Uhlenbeck semigroups: Littlewood-Paley-Stein inequalities and the P. A. Meyer equivalence of norms, *J. Funct. Anal.* **182** (2001), 243-279.

[8] G. Da Prato, A. Lunardi: On the Ornstein-Uhlenbeck operator in spaces of continuous functions, *J. Funct. Anal.* **131** (1995), 94-114.

[9] G. Da Prato, J. Zabczyk: *Stochastic equations in infinite dimensions,* Encyclopedia of Mathematics and its Applications 44, Cambridge U.P., 1992.

[10] G. Da Prato, J. Zabczyk: *Ergodicity for Infinite Dimensional Systems,* Cambridge U.P., 1996.

[11] E. B. Davies: *Heat kernels and spectral theory,* Cambridge U.P., 1989.

[12] E. B. Davies, B. Simon: L^1 properties of intrinsic Schrödinger semigroups, *J. Funct. Anal.* **65** (1986), 126-146.

[13] K. Engel, R. Nagel: *One-parameters Semigroup for Linear Evolution Equations,* Springer Graduate Texts in Mathematics 194, 2000.

[14] M. Fuhrman: Analyticity of transition semigroups and closability of bilinear forms in Hilbert spaces, *Studia Math.,* **115** (1995), 53–71.

[15] B. Goldys: On Analyticity of Ornstein-Uhlenbeck semigroups, *Rend. Mat. Acc. Lincei, s. 9* **10** (1999), 131-140.

[16] L. Hörmander: Hypoelliptic second order differential operators, *Acta Math.,* **119** (1967), 147-171.

[17] E. Lanconelli, S. Polidoro: On a class of hypoelliptic evolution operators, *Rend. Sem. Mat. Univ. Pol. Torino, PDE,s* **52** (1994), 26-63.

[18] A. Lunardi: On the Ornstein-Uhlenbeck operator in L^2 spaces with respect to invariant measures, *Trans. Amer. Math. Soc.* **349** (1997), 155-169.

[19] A. Lunardi: Schauder estimates for a class of degenerate elliptic and parabolic operators with unbounded coefficients, *Ann. Sc. Norm. Sup. Pisa* **24** (1997), 133-164.

[20] G. Metafune: L^p-spectrum of Ornstein-Uhlenbeck operators, *Ann. Sc. Norm. Sup. Pisa,* **30** (2001), 97-124.

[21] G. Metafune, J. Prüss, A. Rhandi, R. Schnaubelt: The domain of the Ornstein-Uhlenbeck operator on an L^p space with an invariant measure, *Ann. Sc. Norm. Sup. Pisa* to appear.

[22] P. A. Meyer: Note sur le processus d'Ornstein-Uhlenbeck, *Séminaire de probabilités XVI,* Springer LNM 920 (1982), 95-133.

[23] E. Priola: The fundamental solution for a degenerate parabolic Dirichlet problem, *Partial differential equations (Torino, 2000), Univ. Torino, Turin (2000),* 137-148.

[24] M. Röckner: L^p analysis of finite and infinite dimensional diffusion operators, in: *Stochastic PDE's and Kolmogorov's equations in infinite dimensions,* (G. Da Prato ed.), Springer LNM 1715 (1999), 65-116.

[25] T. Seidman: How violent are fast controls?, *Control Signals Systems,* **1** (1988), 89-95.

[26] J. Zabczyk: *Mathematical control theory: an introduction,* Birkhäuser, 1992.