GAUSSIAN ESTIMATES FOR ELLIPTIC OPERATORS WITH UNBOUNDED DRIFT

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ABSTRACT. We consider a strictly elliptic operator

$$\mathcal{A}u = \sum_{ij} D_i(a_{ij}D_j u) - b \cdot \nabla u + \mathrm{div}(c \cdot u) - Vu$$

where $0 \le V \in L^\infty_{\mathrm{loc}}, a_{ij} \in C^1_b(\mathbb{R}^N), b, c \in C^1(\mathbb{R}^N, \mathbb{R}^N)$. If $\mathrm{div} b \le \beta V, \mathrm{div} c \le \beta V, 0 < \beta < 1$, then a natural realization of $\mathcal{A}$ generates a positive C_0-semigroup T in $L^2(\mathbb{R}^N)$. The semigroup satisfies pseudo-Gaussian estimates if

$$|b| \le k_1 V^\alpha + k_2, \quad |c| \le k_1 V^\alpha + k_2$$

where $\frac{1}{2} \le \alpha < 1$. If $\alpha = \frac{1}{2}$, then Gaussian estimates are valid. The constant $\alpha = \frac{1}{2}$ is optimal with respect to this property.

INTRODUCTION

We consider a strictly elliptic operator of the form

$$\mathcal{A}u = \sum_{i,j=1}^{N} D_i(a_{ij}D_j u) - b \cdot \nabla u + \mathrm{div}(cu) - Vu$$

on $L^2(\mathbb{R}^N)$ where $a_{ij} \in C^1_b(\mathbb{R}^N)$, $b, c \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ and $V \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ are real coefficients. If b, c, V are bounded, then this is a classical elliptic operator and semigroup properties have been studied extensively. In particular, it is known that the canonical realization of $\mathcal{A}$ in $L^2(\mathbb{R}^N)$ generates a positive C_0-semigroup satisfying Gaussian estimates (see e.g. [AtE97], [Dan00], [Ouh05] and the survey [Are04]). Here we are interested in the case where the drift terms b and c are unbounded. Then one still obtains a semigroup satisfying various regularity properties if the potential V compensates the unbounded drift. We consider the assumption

$$(H_1) \qquad \mathrm{div} b \le \beta V, \quad \mathrm{div} c \le \beta V$$

where $0 < \beta < 1$. Then we show that there is a natural unique realization A of the differential operator $\mathcal{A}$ which generates a minimal

Date: March 19, 2007.

positive semigroup T on $L^2(\mathbb{R}^N)$. This semigroup as well as its adjoint are submarkovian. We say that T satisfies **pseudo-Gaussian** estimates of order $m \geq 2$ if $T(t)$ has a kernel k_t satisfying

$$0 \leq k_t(x,y) \leq c_1 e^{\omega t} t^{-N/2} \exp\{-c_2(|x-y|^m/t)^{1/m-1}\}$$

for all $x, y \in \mathbb{R}^N, t > 0$ and some constants $c_1, c_2 > 0, \omega \in \mathbb{R}$. In the case where $m = 2$ we say that T satisfies **Gaussian estimates**. In order to obtain such pseudo-Gaussian estimates we impose an additional growth condition on the drift terms b and c, namely,

$$(H_2) \qquad |b| \leq k_1 V^\alpha + k_2, \quad |c| \leq k_1 V^\alpha + k_2$$

where $\frac{1}{2} \leq \alpha < 1, k_1, k_2 \geq 0$. If $\alpha = \frac{1}{2}$, then it was proved in [AMP06] that T has Gaussian estimates. The purpose of this paper is to show on one hand that $\alpha = \frac{1}{2}$ is optimal for this property (Section 3). On the other hand, if $\frac{1}{2} < \alpha < 1$, then we show that T still satisfies pseudo-Gaussian estimates even though T need not be holomorphic in that case. Pseudo-Gaussian estimates of order $m > 2$ are still of interest. For instance, they imply that the realizations A_p of A in $L^p(\mathbb{R}^N)$ have all the same spectrum, $1 \leq p \leq \infty$, at least if $m < \frac{2N}{N-2}$. For elliptic operators with moderately growing drift terms but no compensating V such pseudo-Gaussian estimates had been obtained before by Karrmann [Kar01]. Here we do not study regularity properties of the operator A. For this we refer to [AMP06], [MPRS05]. We also mention the works by Liskevich, Sobol and Vogt [LSV02], [LS03], [SV02] where a different approximation is used and spectral properties are studied.

1. Elliptic operators with unbounded drift

In this section we define the realization of an elliptic operator with unbounded drift in $L^2(\mathbb{R}^N)$. The construction is similar to the one in [AMP06] but we ask for less regularity. Moreover, we establish an additional coerciveness property which is used later to prove quasi Gaussian estimates. We assume throughout this section that $a_{ij} \in L^\infty(\mathbb{R})$ and

$$(1.1) \qquad \sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \geq \nu|\xi|^2$$

for all $x \in \mathbb{R}^N, \xi \in \mathbb{R}^N$, where $\nu > 0$ is a fixed constant. Let $b = (b_1, \ldots, b_N), c = (c_1, \ldots, c_N) \in C^1(\mathbb{R}^N, \mathbb{R}^N)$, and let $V \in L^\infty_{\text{loc}}(\mathbb{R}^N)$. We assume in this section that

$$(H_0) \qquad \text{div} b \leq V, \quad \text{div} c \leq V\ .$$

Later in Section 2 we will replace (H_0) by a stronger assumption (H_1) and require more regularity on the diffusion coefficients a_{ij} and positivity of the potential. Define the elliptic operator

$$\begin{aligned}\mathcal{A} &: H^1_{\mathrm{loc}}(\mathbb{R}^N) \to \mathcal{D}(\mathbb{R}^N)' \\ \mathcal{A}u &= \sum_{i,j=1}^{N} D_i(a_{ij}D_ju) - b\cdot\nabla u + \operatorname{div}(cu) - Vu\ ,\end{aligned}$$

i.e., for $u \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$ and $v \in \mathcal{D}(\mathbb{R}^N)$ we have

$$\begin{aligned}-\langle \mathcal{A}u, v\rangle &= \int_{\mathbb{R}^N} \sum_{i,j=1}^{N} a_{ij} D_j u D_i v\, dx \\ &\quad + \int_{\mathbb{R}^N} \{\sum_{j=1}^{N} (b_j D_j uv + c_j u D_j v) + Vuv\}\, dx\ .\end{aligned}$$

We define the maximal operator $A_{\max}$ in $L^2(\mathbb{R}^N)$ by

$$\begin{aligned}D(A_{\max}) &:= \{u \in L^2(\mathbb{R}^N) \cap H^1_{\mathrm{loc}}(\mathbb{R}^N), \mathcal{A}u \in L^2(\mathbb{R}^N)\} \\ A_{\max}u &= \mathcal{A}u\ .\end{aligned}$$

Now we describe the minimal realization of $\mathcal{A}$ in $L^2(\mathbb{R}^N)$ as follows.

Theorem 1.1. *There exists a unique operator A on $L^2(\mathbb{R}^N)$ such that*

(a) *$A \subset A_{\max}$,*
(b) *A generates a positive C_0-semigroup T on $L^2(\mathbb{R}^N)$;*
(c) *if $B \subset A_{\max}$ generates a positive C_0-semigroup S, then $T(t) \le S(t)$ for all $t \ge 0$.*

We call A the **minimal realization** *of $\mathcal{A}$ in $L^2(\mathbb{R}^N)$.*

When giving the proof we also establish important properties of A and of T.

Proposition 1.2 (coerciveness)**.** *One has $D(A) \subset H^1(\mathbb{R}^N)$ and*

$$-(Au|u) \ge \nu\|u\|^2_{H^1} \tag{1.2}$$

for all $u \in D(A)$.

Proposition 1.3 (ultracontractivity)**.** *The semigroup T and its adjoint are submarkovian. Moreover T is* **ultracontractive***, namely*

$$\|T(t)\|_{\mathcal{L}(L^1,L^\infty)} \le c_\nu t^{-N/2} \quad (t > 0) \tag{1.3}$$

where $c_\nu > 0$ depends only on the space dimension and the ellipticity constant ν.

Recall that a C_0-semigroup S on $L^2(\mathbb{R}^N)$ is called **submarkovian** if S is positive and

$$\|S(t)f\|_\infty \leq \|f\|_\infty \quad (t > 0)$$

for all $f \in L^\infty \cap L^2$. If B is an operator on $L^2(\mathbb{R}^N)$ we let

$$\|B\|_{\mathcal{L}(L^p,L^q)} := \sup_{\substack{\|f\|_p \leq 1 \\ f \in L^2}} \|Bf\|_q \ .$$

Since T and T^* are submarkovian, it follows from the Riesz-Thorin Theorem that

$$\|T(t)\|_{\mathcal{L}(L^p)} \leq 1 \quad (t \geq 0)$$

for all $1 \leq p \leq \infty$.

The remainder of this section is devoted to the proofs of Theorem 1.1 and Propositions 1.2, 1.3. As in [AMP06] we approximate the operator A by realizations of $\mathcal{A}$ on balls whose radii go to ∞. However, here we do not study regularity properties of A and we restrict ourselves to the Hilbert space case $L^2(\mathbb{R}^N)$ (whereas $L^p(\mathbb{R}^N)$ was considered in [AMP06]). Our assumptions on V and a_{ij} are more general than in [AMP06]. Denote by $B_r = \{x \in \mathbb{R}^N : |x| < r\}$ the ball of radius $r > 0$. The bilinear form

$$\begin{aligned} a_r(u,v) \ &:= \int_{B_r} \sum_{i,j=1}^N a_{ij} D_j u D_i v \, dx \\ &\quad + \int_{B_r} \{\sum_{j=1}^N (b_j D_j u v + c_j u D_j v) + V uv\} \, dx \end{aligned}$$

is continuous on $H_0^1(B_r)$. We show that

$$a_r(u,u) \geq \nu \int_{B_r} |\nabla u|^2 \, dx \tag{1.4}$$

for all $u \in H_0^1(B_r)$. In fact, let $u \in H_0^1(B_r)$. Then

$$\begin{aligned} a_r(u,u) \ &\geq \ \nu \int_{B_r} |\nabla u|^2 \, dx + \int_{B_r} \{\sum_{j=1}^N (b_j + c_j) \frac{1}{2} D_j u^2 + V u^2\} \, dx \\ &= \ \nu \int_{B_r} |\nabla u|^2 \, dx + \int_{B_r} (-\mathrm{div} \frac{b+c}{2} + V) u^2 \, dx \geq \nu \int_{B_r} |\nabla u|^2 \, dx \ . \end{aligned}$$

□

In view of Poincaré's inequality, (1.4) implies that a_r is coercive. Denote by $-A_r$ the associated operator on $L^2(B_r)$. Then A_r generates a C_0-semigroup T_r on $L^2(B_r)$. Since $u \in H_0^1(B_r)$ implies that $u^+, u^- \in$

$H_0^1(B_r)$ and $a(u^+, u^-) = 0$ the semigroup T_r is positive by the first Beurling-Deny criterion on forms [Ouh05, Theorem 2.6]. Since a_r is coercive, T_r is contractive [Ouh05, Chapter 1]. Next we show that for $0 < r_1 < r_2$

$$T_{r_1}(t) \le T_{r_2}(t) \ , \tag{1.5}$$

or, equivalently,

$$R(\lambda, A_{r_1}) \le R(\lambda, A_{r_2}) \quad (\lambda > 0) \ . \tag{1.6}$$

Here we identify $L^2(B_r)$ with a subspace of $L^2(\mathbb{R}^N)$ and extend an operator B on $L^2(B_r)$ to $L^2(\mathbb{R}^N)$ by defining it as 0 on $L^2(B_r)^\perp = \{u \in L^2(\mathbb{R}^N) : u_{|B_r} = 0\}$. Similarly, we may identify $H_0^1(B_{r_1})$ with a subspace of $H_0^1(B_{r_2})$, see [Bre83, Proposition IX.18].

Proof of (1.6). Let $0 \le f \in L^2(\mathbb{R}^N), \lambda > 0, u_1 = R(\lambda, A_{r_1})f, u_2 = R(\lambda, A_{r_2})f$. We want to show that $u_1 \le u_2$. One has by definition of A_{r_1}, A_{r_2},

$$\begin{aligned}\lambda \int_{B_{r_1}} u_k v \ &+ \int_{B_{r_1}} \sum_{i,j=1}^N a_{ij} D_i u_k D_j v + \int_{B_{r_1}} \sum_{i=1}^N b_i D_i u_k v \\ &+ \int_{B_{r_1}} \sum_{i=1}^N c_i D_i v u_k + \int_{B_{r_1}} V u_k v = \int_{B_{r_1}} f v\end{aligned}$$

for all $v \in H_0^1(B_{r_1}), k = 1, 2$. Since $u_2 \ge 0$ one has $(u_1 - u_2)^+ \le u_1$, hence $(u_1 - u_2)^+ \in H_0^1(B_{r_1})$. Taking $v = (u_1 - u_2)^+$ and subtracting the two identities we obtain

$$\begin{aligned}\lambda \int_{B_{r_1}} (u_1 - u_2)(u_1 - u_2)^+ \ &+ \int_{B_{r_1}} \sum_{i,j=1}^N a_{ij} D_i (u_1 - u_2) \cdot D_j (u_1 - u_2)^+ \\ &+ \int_{B_{r_1}} \sum_{i=1}^N b_i D_i (u_1 - u_2)(u_1 - u_2)^+ \\ &+ \int_{B_{r_1}} \sum_{i=1}^N c_i D_i (u_1 - u_2)^+ (u_1 - u_2) \\ &+ \int_{B_{r_1}} V (u_1 - u_2)(u_1 - u_2)^+ = 0 \ .\end{aligned}$$

Since $D_i(u_1-u_2)(u_1-u_2)^+ = D_i(u_1-u_2)^+(u_1-u_2)^+$ this gives

$$\lambda \int\limits_{B_{r_1}} (u_1-u_2)^{+2} \quad + \quad \int\limits_{B_{r_1}} \nu |\nabla (u_1-u_2)^+|^2 \, dx$$

$$+ \int\limits_{B_{r_1}} \{ \sum_{j=1}^{N} \frac{(b_i+c_i)}{2} D_i (u_1-u_2)^{+2} \quad + \quad V(u_1-u_2)^{+2} \} \leq 0 \, .$$

The third term equals

$$\int\limits_{B_{r_1}} (-\mathrm{div}\frac{b+c}{2} + V)(u_1-u_2)^{+2} \, dx$$

which is ≥ 0 by the hypothesis (H_0). Thus $(u_1-u_2)^+ \leq 0$, hence $u_1 \leq u_2$ on B_{r_1}. $\square$

Next we show that

$$\lim_{r\uparrow\infty} T_r(t)f =: T(t)f \tag{1.7}$$

exists in $L^2(\mathbb{R}^N)$ for all $f \in L^2(\mathbb{R}^N)$ and defines a positive contraction C_0-semigroup whose generator we denote by A.

Proof of (1.7). a) Let $0 \leq f \in L^2(\mathbb{R}^N)$. Since $T_{r_1}(t)f \leq T_{r_2}(t)f$ for $0 < r_1 \leq r_2$ and $\|T_r(t)f\|_2 \leq \|f\|_2$, the limit in (1.7) exists in $L^2(\mathbb{R}^N)$. It follows that $T(t)$ is a positive contraction and $T(t+s) = T(t)T(s)$ for $s,t \geq 0$. In order to show that T is strongly continuous, let $0 \leq f \in \mathcal{D}(\mathbb{R}^N)$. Let $t_n \downarrow 0, f_n = T(t_n)f$. We have to show that $f_n \to f$ in $L^2(\mathbb{R}^N)$ as $n \to \infty$. Let $r > 0$ such that supp $f \subset B_r$. Observe that $0 \leq g_n := T_r(t_n)f \leq f_n$. Since T_r is strongly continuous, $\lim\limits_{n\to\infty} g_n = f$. Moreover, $\|f_n\|_2 \leq \|f\|_2$. Hence $\limsup\limits_{n\to\infty} \|g_n - f_n\|_2^2 = \limsup\limits_{n\to\infty} \{\|g_n\|_2^2 + \|f_n\|_2^2 - 2(g_n|f_n)_2\} \leq \limsup\limits_{n\to\infty}\{2\|f\|_2^2 - 2(g_n|g_n)_2\} = 0$. $\square$

We mention that, by dominated convergence as in [ABHN01, Section 3.6], property (1.7) implies that

$$R(\lambda, A)f = \lim_{r\uparrow\infty} R(\lambda, A_r)f \tag{1.8}$$

for all $\lambda > 0, f \in L^2(\mathbb{R}^N)$. Next we show that

$$D(A) \subset H^1(\mathbb{R}^N) \text{ and } \nu \int\limits_{\mathbb{R}^N} |\nabla u|^2 \, dx \leq (-Au|u) \tag{1.9}$$

for all $u \in D(A)$. Moreover,

$$A \subset A_{\max} \ . \tag{1.10}$$

Proof. a) We prove (1.9). Let $f \in L^2(\mathbb{R}^N), u_n = R(1, A_{r_n})f, u = R(1, A)f$ where $r_n \uparrow \infty$. Then $u_n \to u$ in $L^2(\mathbb{R}^N)$ by (1.8). Since $u_n - A_{r_n}u_n = f$ and $u - Au = f$ in $L^2(B_{r_n})$, it follows that

$$A_{r_n}u_n \to Au \quad \text{in} \quad L^2(\mathbb{R}^N) \ .$$

By (1.4) we have

$$\nu \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx \leq -(A_{r_n}u_n|u_n) \ .$$

Since $-(A_{r_n}u_n|u_n) \to (-Au|u)$ as $n \to \infty$, it follows that

$$\nu \limsup_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx \leq (-Au|u) \ . \tag{1.11}$$

Thus $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^N)$. Considering a subsequence, we may assume that $u_n \to u$ weakly in $H^1(\mathbb{R}^N)$. Let $h = (h_1, \ldots, h_N) \in L^2(\mathbb{R}^N)^N$ such that $\|h\|_2 \leq 1$. Then by (1.11),

$$\begin{aligned} \int_{\mathbb{R}^N} \nabla u \cdot h \, dx &= \lim_{n\to\infty} \int_{\mathbb{R}^N} \nabla u_n \cdot h \, dx \\ &\leq \overline{\lim_{n\to\infty}} \Big(\int_{\mathbb{R}^N} |\nabla u_n|^2 \Big)^{1/2} \\ &\leq [-(Au|u)/\nu]^{1/2} \ . \end{aligned}$$

Hence

$$\begin{aligned} \Big(\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \Big)^{1/2} &= \sup_{\substack{h\in L^2(\mathbb{R}^N)^N \\ \|h\|_2\leq 1}} \int_{\mathbb{R}^N} \nabla u \cdot h \, dx \\ &\leq [-(Au|u)/\nu]^{1/2} \ . \end{aligned}$$

Thus (1.9) is proved.

b) In order to prove (1.10) we keep the notations of a) and have to show that $u \in D(A_{\max})$ and $Au = A_{\max}u$. Let $v \in \mathcal{D}(\mathbb{R}^N)$. Then

$$\begin{aligned}(-A_{r_n}u_n|v) &= \int_{\mathbb{R}^N} \sum_{i,j=1}^N a_{ij} D_j u_n D_i v \, dx \\ &\quad + \int_{\mathbb{R}^N} \{\sum_{j=1}^N (b_j D_j u_n v + c_j u_n D_j v) + V u_n v\} \, dx.\end{aligned}$$

Since $u_n \to u$ weakly in $H^1(\mathbb{R}^N)$ and $A_{r_n}u_n \to Au$ in $L^2(\mathbb{R}^N)$, it follows that $(-Au|v) = (\mathcal{A}u|v)$. □

Next we show the minimality property in Theorem 1.1. Assume that S is a positive semigroup whose generator B satisfies $B \subset A_{\max}$. Then

$$0 \le T(t) \le S(t), \quad (t \ge 0) \ . \tag{1.12}$$

Proof of (1.12). We have to show that

$$R(\lambda, A) \le R(\lambda, B) \tag{1.13}$$

for $\lambda > 0$ sufficiently large. Let $r > 0$; because of (1.8) it suffices to show that

$$R(\lambda, A_r) \le R(\lambda, B) \ . \tag{1.14}$$

Let $f \in L^2(\mathbb{R}^N), f \ge 0, u_1 = R(\lambda, A_r)f, u_2 = R(\lambda, B)f$. Then $0 \le u_1 \in H^1_0(B_r), 0 \le u_2 \in H^1_{\text{loc}}(\mathbb{R}^N)$. We have to show that $u_1 \le u_2$. Since $B \subset A_{\max}$ we have $\lambda u_2 - \mathcal{A}u_2 = f$ in $\mathcal{D}(B_r)'$, and also $\lambda u_1 - \mathcal{A}u_1 = f$ in $\mathcal{D}(B_r)'$ by the definition of A_r. Hence

$$\begin{aligned}\lambda \int_{B_r} (u_1 - u_2) v \, dx \quad &+ \quad \int_{B_r} \sum_{i,j=1}^N a_{ij} D_j (u_1 - u_2) D_i v \, dx \\ + \int_{B_r} \sum_{j=1}^N (b_j D_j (u_1 - u_2) v \quad &+ \quad c_j (u_1 - u_2) D_j v) \, dx \\ + \int_{B_r} V(u_1 - u_2) v dx \quad &= \quad 0\end{aligned}$$

for all $v \in \mathcal{D}(B_r)$. This identity remains true for $v \in H^1_0(B_r)$ by passing to the limit. Since $u_2 \ge 0$ one has $(u_1 - u_2)^+ \le u_1$, hence $(u_1 - u_2)^+ \in H^1_0(B_r)$. Choosing $v = (u_1 - u_2)^+$ in the identity above we obtain

$$
\begin{aligned}
&\lambda \int_{B_r} (u_1 - u_2)^{+2} + \int_{B_r} \sum_{i,j=1}^{N} a_{ij} D_j (u_1 - u_2)^+ D_j (u_1 - u_2)^+ \, dx \\
&+ \int_{B_r} \sum_{j=1}^{N} (b_j D_j (u_1 - u_2)^+ (u_1 - u_2)^+ + c_j D_j (u_1 - u_2)^+ (u_1 - u_2)^+) \, dx \\
&+ \int_{B_r} V (u_1 - u_2)^{+2} \, dx = 0 \ .
\end{aligned}
$$

Consequently

$$
\begin{aligned}
\lambda \int_{B_r} (u_1 - u_2)^{+2} \, dx \quad &+ \quad \nu \int_{B_r} |\nabla (u_1 - u_2)^+|^2 \, dx \\
+ \int_{B_r} (-\mathrm{div}(\frac{b+c}{2}) \quad &+ \quad V)(u_1 - u_1)^{+2} \, dx \leq 0 \ .
\end{aligned}
$$

Since $-\mathrm{div}(\frac{b+c}{2}) + V \geq 0$ this implies that $(u_1 - u_2)^+ = 0$; i.e., $u_1 \leq u_2$. □

The proofs of Theorem 1.1 and Proposition 1.2 are complete.

We now show that T is submarkovian. Because of (1.7), it suffices to show that T_r is submarkovian. By the second criterion of Beurling-Deny-Ouhabaz on forms (see [Ouh05]) this is equivalent to

$$
a_r(u \wedge 1, (u-1)^+) \geq 0 \tag{1.15}
$$

for all $u \in H_0^1(B_r)$.

Proof of (1.15). Since $D_j(u\wedge 1) = D_j u 1_{\{u<1\}}$, $D_j((u-1)^+) = D_j u 1_{\{u>1\}}$ and $D_j u = 0$ a.e. on $\{u = 1\}$, one has

$$\begin{aligned} a_r(u\wedge 1,(u-1)^+) &= \\ \int_{\mathbb{R}^N}\{\sum_{j=1}^N c_j(u\wedge 1)D_j(u-1)^+ &+ V(u\wedge 1)(u-1)^+\}\,dx \\ = \int_{\mathbb{R}^N}\{\sum_{j=1}^N c_j D_j(u-1)^+ &+ V(u-1)^+\}\,dx \\ = \int_{\mathbb{R}^N}(-\mathrm{div}c &+ V)(u-1)^+\,dx \geq 0 \end{aligned}$$

in view of the hypothesis (H_1). □

Next we show that the adjoint semigroup $T^* = (T(t)^*)_{t\geq 0}$ is generated by the minimal realization of the adjoint differential operator $\mathcal{A}^*$ which is defined by replacing a_{ij} by a_{ji} and by interchanging b and c, i.e.

$$\mathcal{A}^*u = \sum_{i,j=1}^N D_i(a_{ji}D_j u) + c\nabla u - \mathrm{div}(bu) - Vu \quad (u \in H^1_{\mathrm{loc}}) \ . \tag{1.16}$$

Lemma 1.4. *The minimal realization in $L^2(\mathbb{R}^N)$ of $\mathcal{A}^*$ is the adjoint A^* of A.*

Proof. The adjoint $-A_r^*$ of $-A_r$ is associated with the form a_r^* defined on $H^1_0(B_r)\times H^1_0(B_r)$ by

$$a_r^*(u,v) = a_r(v,u) \ .$$

The semigroup generated by A_r^* is the adjoint T_r^* of T_r. Let B be the minimal realization of $\mathcal{A}^*$ in $L^2(\mathbb{R}^N)$ and S the semigroup generated by B. Then

$$S(t)f = \lim_{r\uparrow\infty} T_r(t)^*f = T(t)^*f$$

for all $f \in L^2(\mathbb{R}^N)$. □

As a consequence, we deduce that also T^* is submarkovian. Finally, we have to show ultracontractivity. We use the following criterion (cf. [Cou90], [VSC90], [Are04, Section 7], [Rob91]). □

Proposition 1.5. *For each $\delta > 0$ there exists a constant $c_\delta > 0$ such that the following holds. Let S be a C_0-semigroup on $L^2(\mathbb{R}^N)$ such that S and S^* are submarkovian. Assume that the generator B of S satisfies*

(a) $D(B) \subset H^1(\mathbb{R}^N)$;
(b) $(-Bu|u) \geq \delta \|u\|_{H^1}^2 \quad (u \in D(B))$;
(c) $(-B^*u|u) \geq \delta \|u\|_{H^1}^2 \quad (u \in D(B^*))$.

Then

$$\|S(t)\|_{\mathcal{L}(L^1,L^\infty)} \leq c_\delta t^{-N/2} \quad (t > 0)\ . \tag{1.17}$$

The proof of Proposition 1.5 is based on Nash's inequality

$$\|u\|_2^{2+4/N} \leq c_N \|u\|_{H^1}^2 \|u\|_1^{4/N} \tag{1.18}$$

for all $u \in H^1(\mathbb{R}^N)$ and some constant $c_N > 0$, and one may choose $c_\delta = (\frac{c_N \cdot N}{\delta})^{N/2}$.

Proof of Proposition 1.5. i) $D(B) \cap L^1$ is dense in $L^1 \cap L^2$. In fact, the semigroup S extrapolates to a C_0-semigroup on L^1 (see [Dav89], [Are04, Section 7.2]). Hence for $f \in L^1 \cap L^2, \lambda R(\lambda, B) f \to f$ in L^1 and in L^2 as $\lambda \to \infty$. But $\lambda R(\lambda, B) f \in D(B)$.

ii) Now we modify the proof of [AtE97, Proposition 3.8] to show that

$$\|S(t)f\|_2 \leq \left(\frac{Nc_N}{4\delta}\right)^{N/4} t^{-N/4} \|f\|_1 \tag{1.19}$$

for all $f \in D(B) \cap L^1$. Let $f \in D(B) \cap L^1$. Then, by (1.18)

$$\begin{aligned}
\frac{d}{dt}\|S(t)f\|_2^2 &= \big(BS(t)f|S(t)f\big) + \big(S(t)f|B^*S(t)f\big) \\
&\leq -2\delta\|S(t)f\|_{H^1}^2 \leq -\frac{2\delta}{c_N}\frac{\|S(t)f\|_2^{2+4/N}}{\|S(t)f\|_1^{4/N}}.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{d}{dt}\big(\|S(t)f\|_2^2\big)^{-2/N} &= -\frac{2}{N}\|S(t)f\|_2^{2(-2/N-1)}\frac{d}{dt}\|S(t)f\|_2^2 \\
&\geq \frac{4\delta}{Nc_N}\frac{1}{\|S(t)f\|_1^{4/N}} \geq \frac{4\delta}{Nc_N}\frac{1}{\|f\|_1^{4/N}}.
\end{aligned}$$

Integrating, we obtain

$$\big(\|S(t)f\|_2^2\big)^{-2/N} \geq t\frac{4\delta}{Nc_N}\frac{1}{\|f\|_1^{4/N}}$$

which implies(1.19).

It follows from i) that (1.19) remains true for $f \in L^1 \cap L^2$.

iii) Applying b) to S^* instead of S shows that

$$\|S^*(t)f\|_2 \le (\frac{Nc_N}{4\delta})^{N/4} t^{-N/4} \|f\|_1 \tag{1.20}$$

($f \in L^1 \cap L^2$). Hence

$$\|S(t)f\|_\infty \le (\frac{Nc_N}{4\delta})^{N/4} t^{-N/4} \|f\|_2 \tag{1.21}$$

($f \in L^2 \cap L^\infty$). Concluding, for $f \in L^1 \cap L^2$,

$$\begin{aligned}\|S(t)f\|_\infty = \|S(t/2)S(t/2)f\|_\infty &\le (\frac{Nc_N}{4\delta})^{N/4}(t/2)^{-N/4}\|S(t/2)f\|_2 \\ &\le [(\frac{Nc_N}{4\delta})^{N/4}(t/2)^{-N/4}]^2\|f\|_1 = c_\delta t^{-N/2}\|f\|_1 \ .\end{aligned}$$

□

Proposition 1.5 implies the ultracontractivity property (1.3) with $c_\nu = (\frac{c_N \cdot N}{\nu})^{N/2}$ since by (1.9) and Lemma 1.4 the hypotheses (a), (b), (c) in Proposition 1.5 are satisfied for the operator $B = A$. Thus the proofs of Theorem 1.1 and Propositions 1.2, 1.3 are complete.

2. Pseudo-Gaussian Estimates

Let T be a positive C_0-semigroup on $L^2(\mathbb{R}^N)$. We say that T satisfies **pseudo-Gaussian estimates** of type $m \ge 2$ if there exist real constants $c_1 > 0, c_2 > 0, \omega \in \mathbb{R}$ and a measurable kernel $k_t \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ satisfying

$$0 \le k_t(x,y) \le c_1 e^{\omega t} t^{-N/2} \exp(-c_2|x-y|^m/t)^{1/m-1} \tag{2.1}$$

$x, y-$a.e. for all $t > 0$ such that

$$(T(t)f)(x) = \int_{\mathbb{R}^N} k_t(x,y) f(y)\, dy \tag{2.2}$$

$x-$a.e. for all $t > 0, f \in L^2(\mathbb{R}^N)$. If $m = 2$, then we say that T satisfies **Gaussian estimates**.

In fact, the Gaussian semigroup satisfies such an estimate for $m = 2$. It is the best case as the following monotonicity property shows.

Proposition 2.1. *Let $b_1, b_2 > 0$ and let $m_2 > m_1 \ge 2$ be real constant. Then there exists $\omega \ge 0$ such that*

$$\exp(-b_1(|z|^{m_1}/t)^{1/(m_1-1)}) \le \exp(-b_2(|z|^{m_2}/t)^{1/(m_2-1)} e^{\omega t} \tag{2.3}$$

for all $z \in \mathbb{R}^N, t > 0$.

Proof. We have to find a constant ω such that

$$-b_1(|z|^{m_1}/t)^{1/(m_1-1)} \leq -b_2(|z|^{m_2})/t^{-1/(m_2-1)} + \omega t \ .$$

Let $f_t(x) = b_2 x^{m_2/(m_2-1)} t^{-1/(m_2-1)} - b_1 x^{m_1/(m_1-1)} t^{-1/(m_1-1)}, \quad (x \geq 0)$ where $t > 0$. Since $\frac{m_2}{m_2-1} < \frac{m_1}{m_1-1}$, $f_t(\infty) = -\infty$. Moreover, $f_t(0) \leq 0$. Let $x \geq 0$ such that $f_t'(x) = 0$. Then $b_2 \frac{m_2}{m_2-1} x^{\frac{1}{m_2-1}} t^{-\frac{1}{m_2-1}} = b_1 \frac{m_1}{m_1-1} x^{\frac{1}{m_1-1}} t^{-\frac{1}{m_1-1}}$. Hence $\alpha_2 (x/t)^{\frac{1}{m_2-1}} = \alpha_1 (x/t)^{\frac{1}{m_1-1}}$. Thus $\frac{\alpha_2}{\alpha_1} = (\frac{x}{t})^{\frac{1}{m_2-1} - \frac{1}{m_1-1}}$. This implies that $x = \beta t$ for some $\beta > 0$ independent of $t > 0$. Thus $\max\limits_{y>0} f_t(y) = f_t(\beta t) = \widetilde{b_2} t - \widetilde{b_1} t$ where $\widetilde{b_2}, \widetilde{b_1} \in \mathbb{R}$ are constants. Choose $\omega \geq \widetilde{b_2} - \widetilde{b_1}$. $\square$

Pseudo-Gaussian Estimates can be established with the help of a version of Davies' trick which goes as follows. Let

$$\begin{aligned} \mathcal{W} \ := \ & \{\psi \in C^\infty(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) : \\ & \|D_j \psi\|_\infty \leq 1, \|D_i D_j \psi\|_\infty \leq 1, i, j = 1, \dots N\} \ . \end{aligned}$$

Let S be a positive C_0-semigroup on $L^2(\mathbb{R}^N)$. For $\varrho \in \mathbb{R}, \psi \in \mathcal{W}$ we denote by S^ϱ the C_0-semigroup given by

$$S^\varrho(t) f = e^{-\varrho\psi} S(t)(e^{\varrho\psi} f) \ . \tag{2.4}$$

We keep in mind that $S^\varrho(t)$ also depends on ψ, but the estimates should not. In fact, we have the following.

Proposition 2.2. *Let $m \geq 2$ be a real constant. Assume that there exist $c > 0, \omega \in \mathbb{R}$ such that*

$$\|S^\varrho(t)\|_{\mathcal{L}(L^1, L^\infty)} \leq c t^{-N/2} e^{\omega(1+\varrho^m)t} \tag{2.5}$$

for all $\varrho \in \mathbb{R}, \psi \in \mathcal{W}, t > 0$. Then S satisfies pseudo-Gaussian estimates of order m.

We recall the Dunford-Pettis criterion which says that an operator B on $L^2(\mathbb{R}^N)$ is given by a measurable kernel $k \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$ if and only if $\|B\|_{\mathcal{L}(L^1, L^\infty)} < \infty$. In that case,

$$\|k\|_{L^\infty(\mathbb{R}^N \times \mathbb{R}^N)} = \|B\|_{\mathcal{L}(L^1, L^\infty)}.$$

Proof of Proposition 2.2. This is a modification of [AtE97, Proposition 3.3]. It follows from the Dunford-Pettis criterion applied to the operator $S(t)$ that $S(t)$ is given by a measurable kernel k. Consequently, $S^\varrho(t)$ is given by the kernel

$$k^\varrho(t, x, y) = k(t, x, y) e^{\varrho(\psi(y) - \psi(x))}.$$

Since by the Dunford-Pettis criterion again one has

$$k^{\varrho}(t,x,y) \le ct^{-N/2}e^{\omega(1+\varrho^m)t},$$

it follows that

$$k(t,x,y) \le ct^{-N/2}e^{\omega t}e^{\omega\varrho^m t \pm \varrho(\psi(y)-\psi(x))}$$

for all $\varrho \in \mathbb{R}$. Now, $d(x,y) = \sup\{\psi(x)-\psi(y) : \ \psi \in W\}$ defines a metric on $\mathbb{R}^N$ wihic is equivalent to the given metric, see [Rob91, p.200-202]. Hence $d(x,y) \le \beta|x-y|$ for all $x, y \in \mathbb{R}^N$ and some $\beta > 0$. Thus

$$k(t,x,y) \le ct^{-N/2}e^{\omega t}e^{\omega\varrho^m t - \varrho\beta|y-x|}$$

a.e. Choosing

$$\varrho = \left(\frac{\beta|x-y|}{t\omega m}\right)^{\frac{1}{m-1}}$$

we obtain

$$k(t,x,y) \le ct^{-N/2}e^{\omega t}\exp\{-c_2|y-x|^m/t\}^{\frac{1}{m-1}}$$

where $c_2 = \beta^{\frac{m}{m-1}}\left(m^{-\frac{1}{m-1}} - m^{-\frac{m}{m-1}}\right)$. □

Now we have to consider a stronger hypothesis than (H_0), namely

$$(H_1) \qquad \mathrm{div} b \le \beta V, \quad \mathrm{div} c \le \beta V$$

for some constant $0 < \beta < 1$. We also need a condition on the growth of the drift terms b and c with respect to V (assumed nonnegative), namely

$$(H_2) \qquad V \ge 0, \quad |b| \le k_1 V^{\alpha} + k_2, \quad |c| \le k_1 V^{\alpha} + k_2,$$

where $\frac{1}{2} \le \alpha < 1, k_1, k_2 \ge 0$, as well as some more regularity on the diffusion coefficients:

$$(H_3) \qquad a_{ij} \in C_b^1(\mathbb{R}^N).$$

The following result extends [AMP06, Theorem 5.2] from the case $\alpha = \frac{1}{2}$ (i.e., $m = 2$) to $\frac{1}{2} \le \alpha < 1$. Note however, that in contrast to the situation when $\alpha = \frac{1}{2}$, if $\alpha > \frac{1}{2}$ then the operator $-A$ is not associated with a form and the semigroup T may not be holomorphic (see [AMP06, Section 6] and Section 3 below).

Theorem 2.3. *Let A be the minimal realization of the elliptic operator whose coefficients satisfy (1.1), (H_1), (H_2) and (H_3). Let T be the semigroup generated by A. Then T satisfies a pseudo-Gaussian estimate of order $m = \frac{1}{1-\alpha}$.*

Proof. Let $\varrho \in \mathbb{R}, \psi \in \mathcal{W}$. It is obvious that

$$T^{\varrho}(t)f = \lim_{r\uparrow\infty} T^{\varrho}_r(t)f \ .$$

Thus the generator A^{ϱ} of T^{ϱ} is the minimal realization of the elliptic operator $\mathcal{A}^{\varrho}$ with coefficients

$$\begin{aligned} a^{\varrho}_{ij} &= a_{ij} \\ b^{\varrho}_i &= b_i - \varrho \sum_{j=1}^{N} a_{ij}\psi_j \\ c^{\varrho}_i &= c_i + \varrho \sum_{i,j=1}^{N} a_{ki}\psi_k \\ V^{\varrho} &= V - \varrho^2 \sum_{i,j=1}^{N} a_{ij}\psi_i\psi_j + \varrho\sum_{i=1}^{N} b_i\psi_i - \varrho\sum_{i=1}^{N} c_i\psi_i \end{aligned}$$

where $\psi_i = D_i\psi$, cf. [AtE97, Lemma 3.6]. We will find $\omega \in \mathbb{R}$ such that for

$$W^{\varrho} = V^{\varrho} + (1+\varrho^m)\omega$$

one has

$$\text{div} b^{\varrho} \le W^{\varrho}, \quad \text{div} c^{\varrho} \le W^{\varrho} \tag{2.6}$$

where ω is independent of $\varrho \in \mathbb{R}$ and $\psi \in \mathcal{W}$. Then Proposition 1.3 applied to $A^{\varrho} - (1+\varrho^m)\omega$ implies that

$$\|T(t)\|_{\mathcal{L}(L^1,L^\infty)} \le c_\nu t^{-N/2} e^{\omega(1+\varrho^m)t} \quad (t>0) \ . \tag{2.7}$$

Then Proposition 2.2 proves the claim. In order to prove (2.6) we proceed in several steps. We first show that

$$\varrho V^{\alpha} \le \varepsilon^{1/\alpha}\alpha V + (1-\alpha)\varepsilon^{-m}\varrho^m \tag{2.8}$$

for all $\varepsilon > 0$. In fact, let $q = \frac{1}{\alpha}, \frac{1}{p} = 1 - \frac{1}{q}$ and recall that $m = \frac{1}{1-\alpha} = p$. Then by Hölder's inequality

$$\begin{aligned} \varrho V^{\alpha} &= \frac{1}{\varepsilon}\varrho V^{\alpha}\varepsilon \\ &\le \frac{1}{p}\frac{1}{\varepsilon^p}\varrho^p + \frac{1}{q}V^{\alpha q}\varepsilon^q \\ &= (1-\alpha)\varepsilon^{-m}\varrho^m + \alpha V\varepsilon^{1/\alpha} \ . \end{aligned}$$

Next we show that there exists $\omega_1 \in \mathbb{R}$ such that

$$\beta V \le V^{\varrho} + \omega_1(1+\varrho^m) \tag{2.9}$$

for all $\varrho \in \mathbb{R}, \psi \in \mathcal{W}$, where $\beta \in (0,1)$ is the constant in (H_1). In fact, by (H_2) and (2.8),

$$\begin{aligned} V^\varrho &\geq V - k_3\varrho^2 - k_3\varrho V^\alpha - k_4\varrho \\ &\geq V - k_3\varrho^2 - k_3\varepsilon^{1/\alpha}\alpha V - k_3(1-\alpha)\varepsilon^{-m}\varrho^m - k_4\varrho \\ &\geq \beta V - \omega_1(1+\varrho^m) \end{aligned}$$

for suitable constants $k_3, k_4\omega_1$ where $\varepsilon > 0$ is chosen such that $\beta = 1 - k_3\varepsilon^{1/\alpha}\alpha$. Now we show (2.6). One has by (2.9),

$$\begin{aligned} \mathrm{div} b^\varrho &= \mathrm{div} b - \varrho \sum_{i,j=1}^N D_i(a_{ij}\psi_j) \\ &\leq \beta V + k_4\varrho \\ &\leq V^\varrho + \omega_1(1+\varrho^m) + k_5\varrho \\ &\leq V^\varrho + \omega(1+\varrho^m) \end{aligned}$$

for all $\varrho \in \mathbb{R}, \psi \in \mathcal{W}$ where k_5, ω are suitable constants. The estimate for $\mathrm{div} c^\varrho$ is the same. □

Remark 2.4. It is obvious from the definition that a semigroup S satisfies (pseudo-) Gaussian estimates if and only if $(e^{\omega t}S(t))_{t\geq 0}$ does so for some $\omega \in \mathbb{R}$. Thus in Theorem 2.3 we may replace condition (H_1) by the weaker condition

$$(H_1') \qquad \mathrm{div} b \leq \beta V + \beta', \quad \mathrm{div} c \leq \beta V + \beta'$$

where $0 < \beta < 1, \beta' \in \mathbb{R}$ and the result remains valid.

As application we obtain a result on p-independence of the spectrum. Assume that assumptions (1.1) and (H_1) are satisfied. Let A be the minimal realization of the elliptic operator $\mathcal{A}$. Then A generates a C_0-semigroup T on $L^2(\mathbb{R}^N)$ and T as well as T^* are submarkovian. As a consequence there exists a consistent family $T_p = (T_p(t))_{t\geq 0}$ of semigroups on $L^p(\mathbb{R}^N)$ such that $T_2 = T$. Here T_p is a C_0-semigroup if $1 \leq p < \infty$ and T_∞ is a dual C_0-semigroup. We denote by A_p the generator of $T_p, 1 \leq p \leq \infty$.

Corollary 2.5. *Assume that (1.1), (H_1), (H_2) and (H_3) are satisfied. Assume that $\alpha < \frac{N+2}{2N}$. Then $\sigma(A_p) = \sigma(A)$ for all $p \in [1,\infty]$. Here $\frac{1}{2} \leq \alpha < 1$ is the constant occurring in hypothesis (H_2).*

Proof. This follows from a result of Karrmann [Kar01, Corollary 6.2] which in turn is a consequence of a result of Kunstmann [Kun00, Theorem 1.1]. □

The restriction

$$\alpha < \frac{N+2}{2N}$$

is due to the fact that Karrmann proves spectral p-independence in the case of quasi-Gaussian estimates of order m if $m < \frac{2N}{N-2}$. We do not know whether these conditions are optimal.

3. An Example

In order to show that Theorem 2.3 is optimal we consider the 1-dimensional example

$$\mathcal{A}u = u'' - x^3u' + |x|^\gamma u$$

where $\gamma > 2$. Then condition (H_1') is satisfied (see Remark 2.4). Let A be the minimal realization of $\mathcal{A}$ in $L^2(\mathbb{R})$ and let T be the semigroup generated by A. If $\gamma \geq 6$, then it follows from Theorem 2.3 that T satisfies Gaussian estimates. If $6 > \gamma > 3$, then Theorem 2.3 says that T satisfies pseudo-Gaussian estimates of order $m = \frac{\gamma}{\gamma-3}$. We show that T does not satisfy Gaussian estimates in that case.

Proposition 3.1. *Let $3 < \gamma < 6$. Then T does not satisfy Gaussian estimates.*

Proof. Assume that $T(t)$ is given by a kernel k_t satisfying

$$0 \leq k_t(x,y) \leq c_1 e^{\omega t}\frac{1}{\sqrt{t}}e^{-c_2|x-y|^2/t}\ . \tag{3.1}$$

Consider the operator $I_n \in \mathcal{L}(L^2)$ given by

$$(I_n u)(x) = u(\frac{x-n}{\lambda_n})$$

where $\lambda_n = n^{3-\beta}, \gamma < \beta < 6$. Then

$$\|I_n u\|_2 = \sqrt{\lambda_n}\|u\|_2 \quad (u \in L^2(\mathbb{R}))$$

and $(I_n^{-1}u)(x) = u(\lambda_n x + n)$. Define the semigroup T_n on $L^2(\mathbb{R})$ by

$$T_n(t) = I_n^{-1}T(r_n t)I_n$$

where $r_n = n^{-\beta}$. It follows from the Trotter-Kato Theorem that

$$\lim_{n\to\infty} T_n(t)f = S(t)f \tag{3.2}$$

for all $f \in L^2(\mathbb{R})$ where S is the shift semigroup given by $(S(t)u)(x) = u(x-t)$ (see [AMP06, Proposition 6.4]). One has for $f \in L^2(\mathbb{R})$

$$\begin{aligned} T_n(t)f(x) &= (T(r_nt)(I_nf)(n+\lambda_n x) \\ &= \int_{\mathbb{R}} k_{r_nt}(n+\lambda_n x, y) f(\frac{y-n}{\lambda_n})\, dy \\ &= \int_{\mathbb{R}} \lambda_n k_{r_nt}(n+\lambda_n x, n+\lambda_n y) f(y)\, dy \\ &= \int_{\mathbb{R}} k_t^n(x,y) f(y)\ dy \end{aligned}$$

where $k_t^n(x,y) = \lambda_n k_{r_nt}(n+\lambda_n x, n+\lambda_n y)$. By (3.1) we obtain

$$\begin{aligned} k_t^n(x,y) &\le n^{3-\beta} c_1 e^{\omega t r_n} \frac{1}{\sqrt{r_n t}} e^{-c_2\lambda_n^2|x-y|^2/n^{-\beta}t} \\ &= n^{3-\beta/2} c_1 e^{\omega t r_n} \frac{1}{\sqrt{t}} e^{-c_2 n^{6-\beta}|x-y|^2/t}\ . \end{aligned}$$

Denoting by $G = (G(t))_{t\ge 0}$ the Gaussian semigroup, this implies that for $0 \le f \in L^2(\mathbb{R}^N)$,

$$(T_n(t)f)(x) \le ce^{\omega t r_n}(G(t/4c_2 n^{6-\beta})f)(x)\ .$$

Thus

$$\begin{aligned} S(t)f &= \lim_{n\to\infty} T_n(t)f \\ &\le \lim_{n\to\infty} ce^{\omega t r_n} G(t/4c_2 n^{6-\beta})f \\ &= c_1 f\ . \end{aligned}$$

This is a contradiction. □

Remark 3.2. It was shown in [AMP06, Proposition 6.4] that for $2 \le \gamma < 6$, the semigroup T is not holomorphic. It seems not to be known whether Gaussian estimates for positive semigroups imply holomorphy. They do not without positivity assumption as Voigt's example

$$Au = u'' + ix$$

on $L^2(\mathbb{R})$ shows (see Liskevich-Manavi [LM97] for more details).

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