QUASI-STATIC CRACK GROWTH IN HYDRAULIC FRACTURE

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Abstract. We present a variational model for the quasi-static crack growth in hydraulic fracture in the framework of the energy formulation of rate-independent processes. The cracks are assumed to lie on a prescribed plane and to satisfy a very weak regularity assumption.

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1. Introduction

Hydraulic fracture studies the process of crack growth in rocks driven by the injection of high pressure fluids. The main use of hydraulic fracturing is the extraction of natural gas or oil. In these cases, a fluid at high pressure is pumped into a pre-existing fracture through a wellbore, causing the enlargement of the crack.

In the study of hydraulic fracture, all thermal and chemical effects are usually neglected and the fracturing stimulation is performed only by hydraulic forces, not by explosives, thus the inertial effects are negligible. This justifies the use of quasi-static models.

Numerical simulations for this kind of problems have been presented in various papers, coupling the fluid equation, typically Reynolds' equation, and the elasticity system for the rock, see for instance [8, 10, 11]. Particular attention has been given to the tip behaviour of a fluid driven crack, see [5, 7]. Some models, see, e.g., [3, 14, 15], are based on a variational approach introduced by Francfort and Marigo [6] for the *quasi-static* growth of brittle fractures.

While the results of [3, 14, 15] are based on a phase field approximation of the crack introduced by Ambrosio and Tortorelli [1], the model presented in this paper is instead built on the sharp-interface version originally developed in [6].

We assume that the rock fills the whole space $\mathbb{R}^3$ and has an initial crack, lying on a plane Σ passing through the origin. The rock is modelled as a linearly elastic, impermeable material and we allow the crack to grow only within Σ. The fluid is pumped through the origin into the region between the crack lips. It is assumed to be an incompressible fluid.

Since our model is quasi-static, at each time t the fluid and the rock are in equilibrium. This implies that the pressure is uniform in the region occupied by the fluid and exerts a force on the rock through the crack lips. We prove also that the fluid occupies the whole region between the crack lips (see Remarks 3.1 and 4.2). In particular, there is no dry region near the crack edge.

We assume that at every time we know the total volume $V(t)$ of the fluid that has been pumped into the crack up to time t. The mathematical problem is to show that given the function $t \mapsto V(t)$, we can determine at each time the shape and size of the crack, as well as the fluid pressure $p(t)$.

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In Section 3 we discuss a simplified version of our model, where we suppose that the rock is homogeneous and isotropic. This justifies the assumption that the time dependent cracks are circular (*penny-shaped* cracks, see, e.g., [2, 18, 19]). The main result of this section is the existence of a unique *irreversible quasi-static evolution* (see Theorem 3.9) satisfying a global stability condition at each time as well as an energy-dissipation balance, which involves the stored elastic energy, the energy dissipated by the crack, and the power of the pressure forces exerted by the fluid. Moreover, in this simplified setting the solution can be explicitly given as a function of the volume $V(\cdot)$. The uniqueness follows from a careful analysis of the regularity properties satisfied by the solution.

Finally, in Section 4 we discuss a more general model. In this case, the rock is not necessarily homogeneous or isotropic, so we allow the elasticity tensor $\mathbb{C}$ to be a function of the space variable $x \in \mathbb{R}^3$. Because of the lack of homogeneity and isotropy, we do not expect any symmetry for the crack, so we need to define a new class of admissible cracks, which extends the previous one (see Definition 4.1), keeping some regularity properties of the boundary. Also in this case we prove the existence of an *irreversible quasi-static evolution* (see Theorem 4.4) based on a global stability condition and an energy-dissipation balance. The proof relies on a time discretization procedure introduced in [6] and frequently used in the study of rate-independent processes, see [13].

2. Notation and preliminaries

Let us first give some notation and recall some well known results.

Throughout the paper $\mathcal{H}^2$ denotes the 2-dimensional Hausdorff measure in $\mathbb{R}^3$ and $\mathcal{K}$ denotes the set of all compact sets of $\mathbb{R}^3$.

Given $K_1, K_2 \in \mathcal{K}$, the Hausdorff distance $d_H(K_1, K_2)$ between K_1 and K_2 is defined by

$$d_H(K_1, K_2) := \max\left\{ \max_{x \in K_1} d(x, K_2),\ \max_{x \in K_2} d(x, K_1) \right\}.$$

We say that $K_h \to K$ in the Hausdorff metric if $d_H(K_h, K) \to 0$. The following compactness theorem is well known, see, e.g., [17, Blaschke's Selection Theorem].

Theorem 2.1. *Let K_h be a sequence in $\mathcal{K}$. Assume that there exists $H \in \mathcal{K}$ such that $K_h \subseteq H$ for every $h \in \mathbb{N}$. Then there exist a subsequence K_{h_j} and $K \in \mathcal{K}$ such that $K_{h_j} \to K$ in the Hausdorff metric.*

We say that a set function $K : [0,T] \to \mathcal{K}$ is increasing if $K(s) \subseteq K(t)$ for every $0 \le s \le t \le T$. The following two results about increasing set functions can be found for instance in [4].

Theorem 2.2. *Let $H \in \mathcal{K}$ and let $K : [0,T] \to \mathcal{K}$ be an increasing set function such that $K(t) \subseteq H$ for every $t \in [0,T]$. Let $K^- : (0,T] \to \mathcal{K}$ and $K^+ : [0,T) \to \mathcal{K}$ be the functions defined by*

$$K^-(t) := \overline{\textstyle\bigcup_{s<t} K(s)} \qquad \text{for } 0 < t \le T,$$

$$K^+(t) := \textstyle\bigcap_{s>t} K(s) \qquad \text{for } 0 \le t < T.$$

Then

$$K^-(t) \subseteq K(t) \subseteq K^+(t) \qquad \text{for } 0 < t < T.$$

Let Θ be the set of points $t \in (0,T)$ such that $K^+(t) = K^-(t)$. Then $[0,T] \setminus \Theta$ is at most countable and $K(t_h) \to K(t)$ in the Hausdorff metric for every $t \in \Theta$ and every sequence t_h in $[0,T]$ converging to t.

Theorem 2.3. *Let K_h be a sequence of increasing set functions from $[0,T]$ in $\mathcal{K}$. Assume that there exists $H \in \mathcal{K}$ such that $K_h(t) \subseteq H$ for every $t \in [0,T]$ and every $h \in \mathbb{N}$. Then there exist a subsequence, still denoted by K_h, and an increasing set function $K : [0,T] \to \mathcal{K}$ such that $K_h(t) \to K(t)$ in the Hausdorff metric for every $t \in [0,T]$.*

For every open set $\Omega \subseteq \mathbb{R}^3$ we define, as in [12], the space

$$\mathrm{W}^1_{2,6}(\Omega;\mathbb{R}^3) := \{u \in L^6(\Omega;\mathbb{R}^3) : \nabla u \in L^2(\Omega;\mathbb{M}_3)\}$$

equipped with the norm

$$\|u\|_{\mathrm{W}^1_{2,6}(\Omega;\mathbb{R}^3)} := \|u\|_{L^6(\Omega;\mathbb{R}^3)} + \|\nabla u\|_{L^2(\Omega;\mathbb{M}_3)}, \tag{2.1}$$

where $\mathbb{M}_3$ is the space of square matrices of order three with real coefficients. The choice of the exponent 6 is due to the fact that in dimension 3 the exponent 2^* in the Sobolev embedding theorem is equal to 6.

We denote by $\mathrm{E}u$ the symmetric part of the gradient of u, i.e., $\mathrm{E}u = \frac{1}{2}(\nabla u + \nabla u^T)$.

Proposition 2.4. *Let Σ be a plane in $\mathbb{R}^3$ and let $\Omega = \mathbb{R}^3$ or $\Omega = \mathbb{R}^3 \setminus \Sigma$. Then $\mathrm{W}^1_{2,6}(\Omega;\mathbb{R}^3)$ is a Banach space and the norms $\|\nabla u\|_{L^2(\Omega;\mathbb{M}_3)}$ and $\|\mathrm{E}u\|_{L^2(\Omega;\mathbb{M}_3)}$ are equivalent to the norm (2.1), thus $\mathrm{W}^1_{2,6}(\Omega;\mathbb{R}^3)$ is a Hilbert space.*

Proof. When $\Omega = \mathbb{R}^3$ these results are proved in [12, Chapter 1.4], except for the equivalence of the norm (2.1) with $\|\mathrm{E}u\|_{L^2(\mathbb{R}^3;\mathbb{M}_3)}$, which is a consequence of Korn's inequality.

To prove the results for $\Omega = \mathbb{R}^3 \setminus \Sigma$, assume for simplicity that Σ is the plane $x_3 = 0$. Fix $u \in \mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Sigma;\mathbb{R}^3)$. We have $u|_{\mathbb{R}^3_+} \in \mathrm{W}^1_{2,6}(\mathbb{R}^3_+;\mathbb{R}^3)$, where $\mathbb{R}^3_+ := \{x \in \mathbb{R}^3 : x_3 > 0\}$. Extending u by reflection with respect to Σ we obtain a function $\widetilde{u} \in \mathrm{W}^1_{2,6}(\mathbb{R}^3;\mathbb{R}^3)$. Hence, by the previous step,

$$\|u\|_{\mathrm{W}^1_{2,6}(\mathbb{R}^3_+;\mathbb{R}^3)} \leq \|\widetilde{u}\|_{\mathrm{W}^1_{2,6}(\mathbb{R}^3;\mathbb{R}^3)} \leq C\|\nabla\widetilde{u}\|_{L^2(\mathbb{R}^3;\mathbb{M}_3)} = 2C\|\nabla u\|_{L^2(\mathbb{R}^3_+;\mathbb{M}_3)}$$

By the same argument we obtain this estimate also for $u|_{\mathbb{R}^3_-}$.

The statement on $\|\mathrm{E}u\|_{L^2(\mathbb{R}^3\setminus\Sigma;\mathbb{M}_3)}$ can be obtained by Korn's inequality in a half-space. □

From now on we denote by $\mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Sigma)$ the space $\mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Sigma;\mathbb{R}^3)$.

3. Penny-shaped crack

Let us start with the simpler model, the case of *penny-shaped* cracks. The body is supposed to be unbounded, filling $\mathbb{R}^3$ for simplicity. We prescribe a priori the crack path: the admissible cracks lie on the horizontal plane Σ passing through the origin.

In this first model we assume that the initial crack is a circle centered at the origin and contained in Σ, and that the body outside the crack is isotropic, homogeneous, and impermeable. Due to the symmetry conditions, we also assume that the crack is circular at every time.

Moreover, the evolution is governed by linearized elasticity: the material is characterized by its Lamé coefficients $\lambda, \mu \in \mathbb{R}$, so that the stored elastic energy is

$$\int_{\mathbb{R}^3\setminus\Sigma} \Big(\frac{\lambda}{2}\,(\mathrm{div}(u))^2 + \mu|\mathrm{E}u|^2\Big)\,dx,$$

where $u \in \mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Sigma)$ is the displacement and the Lamé coefficients satisfy the usual hypotheses

$$\mu > 0\,, \quad 2\mu + 3\lambda > 0\,. \tag{3.1}$$

To present the results in a way that can be used also for our second model, where the material will be inhomogeneous and anisotropic, it is convenient to introduce the space $\mathbb{M}^s_3$ of symmetric matrices and the linear map $\mathbb{C} : \mathbb{M}^s_3 \to \mathbb{M}^s_3$ defined by

$$\mathbb{C}\mathrm{F} := \lambda\, tr(\mathrm{F})\mathrm{I} + 2\mu\mathrm{F},$$

so that the stored elastic energy becomes

$$\frac{1}{2}\int_{\mathbb{R}^3\setminus\Sigma} \mathbb{C}\mathrm{E}u \cdot \mathrm{E}u\,dx\,,$$

where the dot denotes the scalar product between matrices:

$$\mathrm{F} \cdot \mathrm{G} := tr(\mathrm{F}\mathrm{G}^T) \qquad \forall\, \mathrm{F},\, \mathrm{G} \in \mathbb{M}_3.$$

Conditions (3.1) on the Lamé coefficients make $\mathbb{C}$ positive definite in the sense that there exist two constants $0 < \alpha < \beta < +\infty$ such that

$$\alpha|\mathrm{F}|^2 \leq \mathbb{C}\mathrm{F} \cdot \mathrm{F} \leq \beta|\mathrm{F}|^2 \qquad \text{for every } \mathrm{F} \in \mathbb{M}^s_3. \tag{3.2}$$

We now describe the equilibrium condition for the elastic body when the crack is a circle of radius R in Σ centered at the origin, assuming that the region between the crack lips in the deformed configuration is partially filled by a prescribed volume V of an incompressible fluid. In the spirit of linearized elasticity, in order to simplify the mathematical formulation of the problem, for the volume of the cavity determined by the crack we use the approximate formula

$$\int_\Sigma [u] \cdot \nu_\Sigma \, d\mathcal{H}^2$$

where ν_Σ is the unit normal vector to Σ and $[u]$ denotes the jump of u through Σ, i.e., $[u] := u^+ - u^-$, with u^+ and u^- indicating the traces of u on the two faces Σ^+ and Σ^- of Σ, according to the orientation of ν_Σ. Here we assume that $[u] \cdot \nu_\Sigma \geq 0$ so that the non-interpenetration condition is satisfied.

We define the total energy of the body:

$$\mathcal{E}(u, R) = \frac{1}{2} \int_{\mathbb{R}^3 \setminus \Sigma} \mathbb{C}\mathrm{E}u \cdot \mathrm{E}u \, dx + \kappa\pi R^2, \tag{3.3}$$

where κ is a positive constant related to the fracture toughness. The energy $\mathcal{E}(u, R)$ is the sum of the *stored elastic energy* and of a *surface energy* proportional to the area of the crack. In the framework of Griffith's theory [9], the latter is interpreted as the energy dissipated in the process of crack production.

According to the variational principles of linear elasticity, the body is in equilibrium with a prescribed crack of radius R if the displacement u is the solution of the minimum problem

$$\min_{u \in \mathcal{A}^*(R,V)} \mathcal{E}(u, R), \tag{3.4}$$

where

$$\mathcal{A}^*(R, V) := \Big\{ u \in \mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Sigma) : \{[u] \neq 0\} \subseteq \overline{\mathrm{B}}_R,\, [u] \cdot \nu_\Sigma \geq 0,\, \int_{\mathrm{B}_R} [u] \cdot \nu_\Sigma \, d\mathcal{H}^2 \geq V \Big\}.$$

The choice of the function space $\mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Sigma)$ implies, in a suitable weak sense, that the displacement is zero at infinity. The inclusion in the previous formula reflects the fact that the crack is contained in $\overline{\mathrm{B}}_R$. The former inequality, which is assumed to be satisfied $\mathcal{H}^2$-a.e. on Σ, takes into account the non-interpenetration condition. The latter inequality means that we allow for the presence of a gap between the fluid front and the crack tip, the so called *fluid lag*.

The existence of a solution of (3.4) can be obtained by the direct method of the calculus of variations, taking into account Proposition 2.4. The uniqueness follows from the strict convexity of the functional and the convexity of the constraints.

Remark 3.1. The minimum point of (3.4) satisfies the volume constraint with an equality. Indeed, if $V = 0$, then $u = 0$ is the minimizer. If $V > 0$, assume by contradiction that the minimizer satisfies $\int_{\mathrm{B}_R} [u] \cdot \nu_\Sigma \, d\mathcal{H}^2 > V$. Then there exists $\lambda \in (0, 1)$ such that

$$\int_{\mathrm{B}_R} [\lambda u] \cdot \nu_\Sigma \, d\mathcal{H}^2 \geq V.$$

Since $u \neq 0$ because of the volume constraint, we have $\mathcal{E}(\lambda u, R) < \mathcal{E}(u, R)$, which contradicts the minimality of u. This implies that (3.4) is equivalent to the minimum problem

$$\min_{u \in \mathcal{A}(R,V)} \mathcal{E}(u, R), \tag{3.5}$$

where

$$\mathcal{A}(R, V) := \Big\{ u \in \mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Sigma) : \{[u] \neq 0\} \subseteq \overline{\mathrm{B}}_R,\ [u] \cdot \nu_\Sigma \geq 0,\ \int_{\mathrm{B}_R} [u] \cdot \nu_\Sigma \, d\mathcal{H}^2 = V \Big\}.$$

Proposition 3.2. *Let u be the solution of (3.5) with $V \geq 0$ and $R > 0$. Then for every $v \in \mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Sigma)$ such that $\{[v] \neq 0\} \subseteq \overline{\mathrm{B}}_R$ and $[v] \cdot \nu_\Sigma = 0$ on Σ it holds*

$$\int_{\mathbb{R}^3 \setminus \Sigma} \mathbb{C}\mathrm{E}u \cdot \mathrm{E}v \, dx = 0. \tag{3.6}$$

Moreover, there exists $p \geq 0$ such that for every $\varphi \in C^1(\mathbb{R}^3)$ with $\operatorname{supp}(\nabla\varphi) \subset\subset \mathbb{R}^3$

$$\int_{\mathbb{R}^3 \setminus \Sigma} \mathbb{C}\mathrm{E}u \cdot \mathrm{E}(\varphi u) \, dx = p \int_{\mathrm{B}_R} \varphi [u] \cdot \nu_\Sigma \, d\mathcal{H}^2. \tag{3.7}$$

Proof. When $V = 0$, we have $u = 0$ and we can take $p = 0$.

Assume now $V > 0$. Let $v \in \mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Sigma)$ be such that $\{[v] \neq 0\} \subseteq \overline{\mathrm{B}}_R$ and $[v] \cdot \nu_\Sigma = 0$ on Σ. For every $\varepsilon \in \mathbb{R}$ it holds $u + \varepsilon v \in \mathcal{A}(R, V)$, hence

$$\mathcal{E}(u, R) \leq \mathcal{E}(u + \varepsilon v, R),$$

which implies

$$\int_{\mathbb{R}^3 \setminus \Sigma} \mathbb{C}\mathrm{E}u \cdot \mathrm{E}u \, dx \leq \int_{\mathbb{R}^3 \setminus \Sigma} \mathbb{C}\mathrm{E}(u + \varepsilon v) \cdot \mathrm{E}(u + \varepsilon v) \, dx = \int_{\mathbb{R}^3 \setminus \Sigma} \mathbb{C}\mathrm{E}u \cdot \mathrm{E}u \, dx +$$
$$+ 2\varepsilon \int_{\mathbb{R}^3 \setminus \Sigma} \mathbb{C}\mathrm{E}u \cdot \mathrm{E}v \, dx + \varepsilon^2 \int_{\mathbb{R}^3 \setminus \Sigma} \mathbb{C}\mathrm{E}v \cdot \mathrm{E}v \, dx.$$

By the arbitrariness of ε we get (3.6).

Let us now prove (3.7). We define two linear functionals L, M on the space of functions $\varphi \in C^1(\mathbb{R}^3)$ with $\operatorname{supp}(\nabla\varphi) \subset\subset \mathbb{R}^3$:

$$L(\varphi) := \int_{\mathbb{R}^3 \setminus \Sigma} \mathbb{C}\mathrm{E}u \cdot \mathrm{E}(\varphi u) \, dx,$$
$$M(\varphi) := \int_{\mathrm{B}_R} \varphi [u] \cdot \nu_\Sigma \, d\mathcal{H}^2.$$

For every $\varphi \in C^1(\mathbb{R}^3)$ with $\operatorname{supp}(\nabla\varphi) \subset\subset \mathbb{R}^3$ such that $M(\varphi) = 0$, we consider $(1 + \varepsilon\varphi)u$ with $\varepsilon \in \mathbb{R}$. For $|\varepsilon|$ small enough we have $(1 + \varepsilon\varphi)u \in \mathcal{A}(R, V)$, hence, arguing as before, we get

$$\int_{\mathbb{R}^3 \setminus \Sigma} \mathbb{C}\mathrm{E}u \cdot \mathrm{E}(\varphi u) \, dx = 0.$$

Therefore, $M(\varphi) = 0$ implies $L(\varphi) = 0$, which, by linearity, implies that there exists a constant $p \in \mathbb{R}$ such that $L = pM$, i.e., (3.7) holds. Taking $\varphi = 1$ and recalling that $V > 0$, we get $p > 0$. $\square$

Remark 3.3. Equation (3.6) has two important consequences. Indeed, it means that the function u solution of (3.5) is a weak solution of $\operatorname{div}(\mathbb{C}\mathrm{E}u) = 0$ in $\mathbb{R}^3 \setminus \Sigma$. Moreover, given $v, w \in \mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Sigma)$ such that $[v] = [w]$ on $\Sigma \setminus \overline{\mathrm{B}}_R$ and $[v] \cdot \nu_\Sigma = [w] \cdot \nu_\Sigma$ on Σ, we have that $v - w$ is an admissible test-function for problem (3.6), hence

$$\int_{\mathbb{R}^3 \setminus \Sigma} \mathbb{C}\mathrm{E}u \cdot \mathrm{E}v \, dx = \int_{\mathbb{R}^3 \setminus \Sigma} \mathbb{C}\mathrm{E}u \cdot \mathrm{E}w \, dx. \tag{3.8}$$

Remark 3.4. When $V > 0$, equality (3.7) implies, taking $\varphi = 1$,

$$p = \frac{1}{V} \int_{\mathbb{R}^3 \setminus \Sigma} \mathbb{C}\mathrm{E}u \cdot \mathrm{E}u \, dx.$$

Remark 3.5. Let us now explain why the constant p defined in Proposition 3.2 can be interpreted as the fluid pressure. It is clear that, if u is the solution of (3.5) without the non-interpenetration condition, then p is simply a Lagrange multiplier, hence we have

$$\int_{\mathbb{R}^3 \setminus \Sigma} \mathbb{C}\mathrm{E}u \cdot \mathrm{E}v \, dx = p \int_{\mathrm{B}_R} [v] \cdot \nu_\Sigma \, d\mathcal{H}^2 \tag{3.9}$$

for every $v \in \mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Sigma)$ such that $\{[v] \neq 0\} \subseteq \overline{B}_R$, thus u satisfies the boundary condition $(\mathbb{C}\mathrm{E}u)\nu_\Sigma = -p\nu_\Sigma$ on Σ^+. This means that the force exerted by the fluid on the upper part of the cavity has intensity p and is directed along the normal ν_Σ, so we are allowed to consider p as the fluid pressure. Strictly speaking, according to the formula of the boundary condition, the pressure p is acting along the normal to the crack in the reference configuration rather than in the deformed configuration. This does not affect our interpretation, since we are dealing with a linearized model.

To justify the same interpretation of p when the non-interpenetration constraint is considered, we have to prove that (3.9) holds for a sufficiently large class of functions in $\mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Sigma)$.

Proposition 3.6. *Let u be the solution of* (3.5) *with $V \geq 0$ and $R > 0$. Then* (3.9) *holds for every $v \in \mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Sigma)$ such that $\{[v] \neq 0\} \subseteq \overline{\mathrm{B}}_R$ and $|[v] \cdot \nu_\Sigma| \leq C[u] \cdot \nu_\Sigma$ for some constant $C \geq 0$.*

Proof. When $V = 0$ we have $u = 0$ and the statement is trivial with $p = 0$.

Assume now $V > 0$. Let v be as in the statement of the Proposition. In view of (3.8), we can modify the functions u and v, provided we keep the same values for $[u] \cdot \nu_\Sigma$ and $[v] \cdot \nu_\Sigma$. We fix a function $\hat{u} \in \mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Sigma)$ with compact support, of the form $(0, 0, \hat{u}_3)$, such that $\hat{u} = 0$ for $x_3 < 0$ and $(\hat{u}_3)^+ = [\hat{u}] \cdot \nu_\Sigma = [u] \cdot \nu_\Sigma$ on Σ. Similarly, we fix $\hat{v} \in \mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Sigma)$ with compact support of the form $(0, 0, \hat{v}_3)$ such that $\hat{v} = 0$ for $x_3 < 0$, $(\hat{v}_3)^+ = [\hat{v}] \cdot \nu_\Sigma = [v] \cdot \nu_\Sigma$ on Σ, and

$$|\hat{v}_3| \leq C\hat{u}_3 \quad \text{a.e. on } \mathbb{R}^3 \setminus \Sigma. \tag{3.10}$$

We now need to approximate $\hat{u}$ and $\hat{v}$ by truncations. Let $T_k : \mathbb{R} \to \mathbb{R}$ be the truncation function defined by $T_k(s) = -k$ if $s \leq -k$, $T_k(s) = s$ if $-k \leq s \leq k$, and $T_k(s) = k$ if $s \geq k$. We shall also use the function $S_k : \mathbb{R} \to \mathbb{R}$ defined by $S_k(s) := s - T_k(s)$. We use the same symbols for the maps $T_k, S_k : \mathbb{R}^3 \to \mathbb{R}^3$ defined componentwise by $T_k(s) := (T_k(s_1), T_k(s_2), T_k(s_3))$ and $S_k(s) := (S_k(s_1), S_k(s_2), S_k(s_3))$.

It follows from (3.10) that for every $k \in \mathbb{N}$

$$|S_{1/k}(T_k(\hat{v}_3))| \leq CT_k(\hat{u}_3) \quad \text{a.e. on } \mathbb{R}^3 \setminus \Sigma.$$

In particular $S_{1/k}(T_k(\hat{v}_3)) = 0$ where $\hat{u}_3 < 1/(kC)$.

By the properties of $\hat{u}$, for every $k \in \mathbb{N}$ we have

$$0 \leq [T_k(\hat{u})] \cdot \nu_\Sigma = T_k([\hat{u}]) \cdot \nu_\Sigma \leq [u] \cdot \nu_\Sigma \quad \text{on } \Sigma.$$

Repeating the argument used to prove Proposition 3.2, we deduce that there exists $p_k \in \mathbb{R}$ such that, for every $\varphi \in C^1(\mathbb{R}^3)$ with $\operatorname{supp}(\nabla\varphi) \subset\subset \mathbb{R}^3$,

$$\int_{\mathbb{R}^3 \setminus \Sigma} \mathbb{C}\mathrm{E}u \cdot \mathrm{E}(\varphi T_k(\hat{u})) \, dx = p_k \int_{\mathrm{B}_R} \varphi [T_k(\hat{u})] \cdot \nu_\Sigma \, d\mathcal{H}^2. \tag{3.11}$$

Since $T_k(\hat{u}) \to \hat{u}$ in $\mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Sigma)$, passing to the limit as $k \to +\infty$ in (3.11), and using (3.7) and (3.8), we get $p_k \to p$.

We define the functions

$$w_3^k(x) := \begin{cases} \dfrac{S_{1/k}(T_k(\hat{v}_3(x)))}{T_k(\hat{u}_3(x))} & \text{if } \hat{u}_3(x) \neq 0, \\ 0 & \text{if } \hat{u}_3(x) = 0, \end{cases} \qquad \text{and} \qquad w^k(x) := (0,0,w_3^k(x)).$$

Then $w^k \in \mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Sigma) \cap L^\infty(\mathbb{R}^3 \setminus \Sigma)$ and $\mathrm{supp}(w^k) \subseteq \mathrm{supp}(\hat{v})$. In particular, $w^k = 0$ for $x_3 < 0$ and $w_3^k \in \mathrm{W}^1_{2,6}(\mathbb{R}^3_+) \cap L^\infty(\mathbb{R}^3_+)$. Hence, for every k there exists a sequence $(\varphi_j^k)_j$ in $C_c^\infty(\mathbb{R}^3)$ such that $\|\varphi_j^k\|_{L^\infty(\mathbb{R}^3)} \leq \|w_3^k\|_{L^\infty(\mathbb{R}^3)}$ and $\varphi_j^k \to w_3^k$ strongly in $\mathrm{W}^1_{2,6}(\mathbb{R}^3_+)$ as $j \to +\infty$.

We consider now the sequence $\varphi_j^k T_k(\hat{u})$ in $\mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Sigma)$. By the dominated convergence theorem, we get $\varphi_j^k T_k(\hat{u}) \to S_{1/k}(T_k(\hat{v}))$ strongly in $\mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Sigma)$ as $j \to +\infty$. Since $S_{1/k}(T_k(\hat{v})) \to \hat{v}$ strongly in $\mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Sigma)$ as $k \to +\infty$, by a diagonal argument we find a sequence φ_k in $C_c^\infty(\mathbb{R}^3)$ such that $\varphi_k T_k(\hat{u}) \to \hat{v}$ in $\mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Sigma)$. Since (3.8) and (3.11) hold, using the equalities $[\hat{u}] \cdot \nu_\Sigma = [u] \cdot \nu_\Sigma$ and $[\hat{v}] \cdot \nu_\Sigma = [v] \cdot \nu_\Sigma$ on Σ, we get

$$\begin{aligned}\int_{\mathbb{R}^3\setminus\Sigma} \mathbb{C}\mathrm{E}u \cdot \mathrm{E}v \, dx &= \int_{\mathbb{R}^3\setminus\Sigma} \mathbb{C}\mathrm{E}u \cdot \mathrm{E}\hat{v} \, dx = \lim_k \int_{\mathbb{R}^3\setminus\Sigma} \mathbb{C}\mathrm{E}u \cdot \mathrm{E}(\varphi_k T_k(\hat{u})) \, dx = \\ &= \lim_k p_k \int_{\mathrm{B}_R} \varphi_k [T_k(\hat{u})] \cdot \nu_\Sigma \, \mathcal{H}^2 = p \int_{\mathrm{B}_R} [v] \cdot \nu_\Sigma \, d\mathcal{H}^2,\end{aligned}$$

and this concludes the proof. □

Remark 3.7. Integrating by parts, thanks to Proposition 3.6 we get that the solution u of (3.5) satisfies the boundary condition $(\mathbb{C}\mathrm{E}u)\nu_\Sigma = -p\nu_\Sigma$ on $\{[u] \cdot \nu_\Sigma \neq 0\}$, which is the part of the crack occupied by the fluid. Therefore we can repeat the argument of Remark 3.5 on the set $\{[u] \cdot \nu_\Sigma \neq 0\}$ and we conclude that p can be interpreted as the fluid pressure.

Let us now consider the quasi-static evolution problem. Let $T > 0$ be fixed and for every $t \in [0,T]$ let $V(t)$ be the volume of the fluid present in the crack at time t. It is natural to assume that this quantity can be controlled during the process, so we consider it as a datum of the problem. For technical reasons, we assume that $V \in AC([0,T];[0,+\infty))$, the space of absolutely continuous functions on $[0,T]$ with values in $[0,+\infty)$.

To describe the quasi-static evolution it is convenient to introduce the reduced energy $\mathcal{E}_{min}(R,V)$ defined for every $R \geq 0$ and every $V \geq 0$ by

$$\mathcal{E}_{min}(R,V) := \min_{u \in \mathcal{A}(R,V)} \mathcal{E}(u,R) = \min_{u \in \mathcal{A}(R,V)} \frac{1}{2} \int_{\mathbb{R}^3\setminus\Sigma} \mathbb{C}\mathrm{E}u \cdot \mathrm{E}u \, dx + \kappa\pi R^2. \tag{3.12}$$

In order to make explicit the dependence of $\mathcal{E}_{min}(R,V)$ on R and V, let us denote by u_R the solution to the minimum problem defining $\mathcal{E}_{min}(R,1)$. It is then easy to see that Vu_R is the solution of (3.12) and

$$\mathcal{E}_{min}(R,V) = \frac{V^2}{2} \int_{\mathbb{R}^3\setminus\Sigma} \mathbb{C}\mathrm{E}u_R \cdot \mathrm{E}u_R \, dx + \kappa\pi R^2 \,.$$

Moreover, by the uniqueness of the solution to (3.12) it follows that

$$u_R(x) = \frac{1}{R^2} u_1\Big(\frac{x}{R}\Big) \quad \text{and} \quad \int_{\mathbb{R}^3\setminus\Sigma} \mathbb{C}\mathrm{E}u_R \cdot \mathrm{E}u_R \, dx = \frac{1}{R^3} \int_{\mathbb{R}^3\setminus\Sigma} \mathbb{C}\mathrm{E}u_1 \cdot \mathrm{E}u_1 \, dx \,. \tag{3.13}$$

Therefore

$$\mathcal{E}_{min}(R,V) = K\frac{V^2}{R^3} + \kappa\pi R^2 \,, \tag{3.14}$$

where $K := \dfrac{1}{2} \displaystyle\int_{\mathbb{R}^3\setminus\Sigma} \mathbb{C}\mathrm{E}u_1 \cdot \mathrm{E}u_1 \, dx$. Since

$$\frac{d}{dR}\mathcal{E}_{min}(R,V) = -3K\frac{V^2}{R^4} + 2\kappa\pi R \,, \tag{3.15}$$

we note that the unique minimum point of $R \mapsto \mathcal{E}_{min}(R,V)$ is $R = \left(\frac{3KV^2}{2\kappa\pi}\right)^{1/5}$.

Hence, if we fix $\hat{R} > 0$, the unique solution to the minimum problem

$$\min_{R \geq \hat{R}} \mathcal{E}_{min}(R,V)$$

is given by

$$R_* = \max\left\{\hat{R}, \left(\tfrac{3KV^2}{2\kappa\pi}\right)^{1/5}\right\}.$$

In this simplified setting, since the function $R \mapsto \mathcal{E}_{min}(R,V)$ is convex, Griffith's stability condition expressed by the inequality

$$\frac{d}{dR}\mathcal{E}_{min}(R(t),V(t)) \geq 0 \qquad \text{for every } t \in [0,T] \tag{3.16}$$

is equivalent to the global minimality condition: for every $t \in [0,T]$

$$\mathcal{E}_{min}(R(t),V(t)) \leq \mathcal{E}_{min}(R,V(t)) \qquad \text{for every } R \geq R(t)\,, \tag{3.17}$$

which in this case reduces to

$$R(t) \geq \left(\frac{3KV^2(t)}{2\kappa\pi}\right)^{1/5} \qquad \text{for every } t \in [0,T]\,. \tag{3.18}$$

The use of (3.17) instead of (3.16) allows us to state the problem in a "derivative-free" setting, in the framework of rate-independent evolution processes considered in [13]. This will be useful in the next section where we deal with more general crack shapes.

Since the fracture process is irreversible, we require that $R(\cdot)$ is increasing. Finally, we impose an energy-dissipation balance: the rate of change of the total energy (stored elastic energy plus energy dissipated by the crack) along a solution equals the power of the pressure forces exerted by the fluid.

This leads to the following definition.

Definition 3.8. Let $T > 0$ and $V \in AC([0,T];[0,+\infty))$. We say that a function $R : [0,T] \to (0,+\infty)$ is an *irreversible quasi-static evolution* of the *penny-shaped hydraulic crack problem* if it satisfies the following conditions:

(a) *irreversibility*: R is increasing, i.e., $R(s) \leq R(t)$ for every $0 \leq s \leq t \leq T$;

(b) *global stability*: for every $t \in [0,T]$,

$$\mathcal{E}_{min}(R(t),V(t)) \leq \mathcal{E}_{min}(R,V(t)) \qquad \text{for every } R \geq R(t);$$

(c) *energy-dissipation balance*: the function $t \mapsto \mathcal{E}_{min}(R(t),V(t))$ is absolutely continuous on the interval $[0,T]$ and

$$\frac{d}{dt}\mathcal{E}_{min}(R(t),V(t)) = p(t)\dot{V}(t)$$

for almost every $t \in [0,T]$, where $p(t)$ is the pressure introduced in Proposition 3.2.

While in the tecnological applications to hydraulic fracture it is natural to suppose that V is increasing, the problem makes sense even without this assumption. For instance, if in a time interval V is decreasing, which means that some liquid is removed from the cavity, by the irreversibility assumption we expect that R remains constant in that interval and that the crack opening decreases to accommodate to the volume constraint. This is a direct consequence of the formula (3.20) proved in the next theorem.

We are now ready to state the main result of this section.

Theorem 3.9. *Let $V \in AC([0,T];[0,+\infty))$ and $R_0 > 0$. Assume that (stability at time $t = 0$)*

$$\mathcal{E}_{min}(R_0,V(0)) \leq \mathcal{E}_{min}(R,V(0)) \tag{3.19}$$

for every $R \geq R_0$. Then the unique irreversible quasi-static evolution $R_* : [0,T] \to (0,+\infty)$ of the penny-shaped hydraulic crack problem, with $R(0) = R_0$, is given by

$$R_*(t) = \max\Big\{R_0, \big(\tfrac{3K}{2\kappa\pi}\big)^{1/5} V_*^{2/5}(t)\Big\}, \tag{3.20}$$

where $V_*(t)$ is the smallest monotone increasing function which is greater than or equal to $V(t)$, i.e., $V_*(t) = \max_{0\leq s\leq t} V(s)$.

When V is increasing we recover the explicit solution considered, e.g., in [3], see also [**?**].

Remark 3.10. In view of (3.18) condition (3.19) amounts to

$$R_0 \geq \left(\frac{3KV^2(0)}{2\kappa\pi}\right)^{1/5}.$$

To prove Theorem 3.9 we need the following lemmas. In the first one we prove the absolute continuity of the function V_*.

Lemma 3.11. *Let $V \in AC([0,T];[0,+\infty))$ and for every $t \in [0,T]$ set $V_*(t) = \max_{0\leq s\leq t} V(s)$. Then $V_* \in AC([0,T];[0,+\infty))$ and*

$$\dot V_*(t) = \dot V(t) 1_{\{V=V_*\}}(t) \qquad \textit{for a.e. } t \in [0,T]. \tag{3.21}$$

Proof. As $V \in AC([0,T];[0,+\infty))$, there exist two increasing absolutely continuous functions $V_1, V_2 : [0,T] \to [0,+\infty)$ such that $V = V_1 - V_2$. Note that

$$V_*(t_2) - V_*(t_1) \leq V_1(t_2) - V_1(t_1) \qquad \text{for every } 0 \leq t_1 \leq t_2 \leq T. \tag{3.22}$$

Indeed, for every $t_1 \leq s \leq t_2$

$$\begin{aligned} V(s) - V_*(t_1) &\leq V(s) - V(t_1) = V_1(s) - V_2(s) - V_1(t_1) + V_2(t_1) \\ &\leq V_1(t_2) - V_1(t_1) - (V_2(s) - V_2(t_1)) \leq V_1(t_2) - V_1(t_1), \end{aligned}$$

and by the definition of V_* this implies (3.22). As V_1 is absolutely continuous, from (3.22) we deduce the absolute continuity of V_*.

Since the function V_* is locally constant on the open set $\{t \in [0,T] : V_*(t) > V(t)\}$, we have $\dot V_* = 0$ on this set, while $\dot V_*(\bar t) = \dot V(\bar t)$ for a.e. $\bar t \in \{t \in [0,T] : V_*(t) = V(t)\}$. Therefore (3.21) holds. □

Lemma 3.12. *Let $V \in AC([0,T];[0,+\infty))$ and $R_0 > 0$. Assume that R_0 satisfies* (3.19)*. Then $R_* : [0,T] \to (0,+\infty)$ given by* (3.20) *is the smallest increasing function which satisfies the global stability condition (b), with $R(0) = R_0$.*

Proof. Let $R(t)$ be an increasing function with $R(0) = R_0$ that satisfies the global stability condition (b). In view of (3.18) we have $R(t) \geq \left(\frac{3KV^2(t)}{2\kappa\pi}\right)^{1/5}$ for every $t \in [0,T]$. Since $R(t) \geq R(s)$ for every $s,t \in [0,T]$ with $s \leq t$, we get

$$R(t) \geq \max_{0\leq s\leq t} \left(\frac{3KV^2(s)}{2\kappa\pi}\right)^{1/5} = \left(\frac{3KV_*^2(t)}{2\kappa\pi}\right)^{1/5},$$

which implies $R(t) \geq R_*(t)$.

As $R_*(t)$ satisfies (3.18) for every $t \in [0,T]$, the function $t \mapsto R_*(t)$ satisfies the global stability condition (b). □

We now prove that $R_* : [0,T] \to (0,+\infty)$ defined by (3.20) is an irreversible quasi-static evolution of the penny-shaped hydraulic crack problem.

Proof of Theorem 3.9 (existence). It remains to prove the energy-dissipation balance (c). Let us set

$$\alpha_0 = \left(\frac{2\kappa\pi}{3K}\right)^{1/2} R_0^{5/2} .$$

By (3.20), if $V_*(t) \le \alpha_0$ then $R_*(t) = R_0$. Assume there exists $t \in [0,T]$ such that $V(t) \ge \alpha_0$ and let $\bar{t} := \inf\{t \in [0,T] : V(t) \ge \alpha_0\}$. Then $R_*(t) = R_0$ for $t \in [0,\bar{t}\,]$, while $R_*(t) = \left(\frac{3KV_*^2(t)}{2\kappa\pi}\right)^{1/5}$ and $V_*(t) \in [\alpha_0, +\infty)$ for $t \in [\bar{t}, T]$.

By Lemma 3.11 and the Lipschitz continuity of the function $a \mapsto a^{2/5}$ on $[\alpha_0, +\infty)$ we deduce that $R_*(\cdot)$ is absolutely continuous on $[0,T]$. Then, as

$$\mathcal{E}_{min}(R_*(t), V(t)) = K\frac{V^2(t)}{R_*^3(t)} + \kappa\pi R_*^2(t) \qquad \text{for every } t \in [0,T]\,,$$

it follows that $\mathcal{E}_{min}(R_*(\cdot), V(\cdot))$ is absolutely continuous on $[0,T]$ and

$$\frac{d}{dt}\mathcal{E}_{min}(R_*(t), V(t)) = 2K\frac{V(t)\dot{V}(t)}{R_*^3(t)} + \dot{R}_*(t)\Big(2\kappa\pi R_*(t) - 3K\frac{V^2(t)}{R_*^4(t)}\Big) \qquad \text{for a.e. } t \in [0,T]\,.$$

Since, by Remark 3.4 and (3.14), $p(t) = 2K\frac{V(t)}{R_*^3(t)}$ for every $t \in [0,T]$, while by (3.20) and Lemma 3.11 the product $\dot{R}_*(t)\Big(2\kappa\pi R_*(t) - 3K\frac{V^2(t)}{R_*^4(t)}\Big)$ is equal to 0, we get

$$\frac{d}{dt}\mathcal{E}_{min}(R_*(t), V(t)) = p(t)\dot{V}(t)\,,$$

and this concludes the proof of the existence of an irreversible quasi-static evolution for the penny-shaped hydraulic crack problem. □

The next result establishes some regularity properties of a solution that will be used to prove the uniqueness.

Lemma 3.13. *Let $V \in AC([0,T];[0,+\infty))$ and let $R : [0,T] \to (0,+\infty)$ be an irreversible quasi-static evolution of the penny-shaped hydraulic crack problem with $R(0) = R_0$. Then $R(\cdot)$ is continuous on $[0,T]$ and is absolutely continuous on every compact set contained in*

$$I := \{t \in [0,T] :\ R(t) > R_*(t) \text{ and } V(t) > 0\}.$$

Proof. Let $R : [0,T] \to (0,+\infty)$ be an irreversible quasi-static evolution of the penny-shaped hydraulic crack problem with $R(0) = R_0$. By condition (c) of Definition 3.8 the function $t \mapsto \mathcal{E}_{min}(R(t), V(t))$ is absolutely continuous and by Lemma 3.12 $R(t) \ge R_*(t)$. Let us show by contradiction that R is continuous. Assume $\tilde{t} \in [0,T]$ is a discontinuity point. Since $t \mapsto R(t)$ is increasing and, by (3.14), the function $R \mapsto \mathcal{E}_{min}(R, V(t))$ is strictly increasing for $R \ge R_*(t)$, we have

$$\lim_{s \nearrow \tilde{t}} \mathcal{E}_{min}(R(s), V(s)) = \mathcal{E}_{min}(R(\tilde{t}^-), V(\tilde{t})) < \mathcal{E}_{min}(R(\tilde{t}^+), V(\tilde{t})) = \lim_{s \searrow \tilde{t}} \mathcal{E}_{min}(R(s), V(s))$$

which contradicts the continuity of $t \mapsto \mathcal{E}_{min}(R(t), V(t))$.

Let us define the function $e_R : [0,T] \to \mathbb{R}$ as

$$e_R(t) := K\frac{V^2(0)}{R_0^3} + \kappa\pi R_0^2 + \int_0^t p_R(s)\dot{V}(s)\,ds, \tag{3.23}$$

where $p_R(t)$ is the pressure function introduced in Proposition 3.2 in the case $R = R(t)$ and $V = V(t)$. By Remark 3.4 and (3.13) we get

$$e_R(t) = K\frac{V^2(0)}{R_0^3} + \kappa\pi R_0^2 + 2K\int_0^t \frac{V(s)\dot{V}(s)}{R^3(s)}\,ds.$$

Since $R(t) \geq R_0$ on $[0,T]$ and $V \in AC([0,T];[0,+\infty))$, it follows that $e_R \in AC([0,T];\mathbb{R})$. By the energy-dissipation balance condition (c) of Definition 3.8 and by (3.14)

$$\mathcal{E}_{min}(R(t),V(t)) = K\frac{V^2(t)}{R^3(t)} + \kappa\pi R^2(t) = e_R(t) \qquad \text{for every } t \in [0,T]\,. \tag{3.24}$$

Let $F(B) := \frac{K}{B^3} + \kappa\pi B^2$ for every $B \in (0,+\infty)$. It is easy to see that F belongs to $C^\infty((0,+\infty))$, it is strictly increasing and strictly convex on $J := \Big(\big(\frac{3K}{2\kappa\pi}\big)^{1/5}, +\infty\Big)$. Therefore, $F|_J$ is invertible and F^{-1}, the inverse of $F|_J$, is C^1.

For every $t \in I$ let $B(t) := \frac{R(t)}{V^{2/5}(t)}$. Thus, by (3.24)

$$F(B(t)) = \frac{K}{B^3(t)} + \kappa\pi B^2(t) = \frac{e_R(t)}{V^{4/5}(t)}.$$

Since $t \in I$ we have $B(t) > \frac{R_*(t)}{V^{2/5}(t)} \geq \big(\frac{3K}{2\kappa\pi}\big)^{1/5}$, hence

$$B(t) = F^{-1}\Big(\frac{e_R(t)}{V^{4/5}(t)}\Big) \qquad \text{for every } t \in I. \tag{3.25}$$

Since $\frac{e_R(\cdot)}{V^{4/5}(\cdot)}$ is bounded and absolutely continuous on every compact set contained in I, we deduce that $B(\cdot)$ is absolutely continuous on the same sets and so is $R(\cdot)$. $\square$

To prove the uniqueness of the quasi-static evolution, we need the following lemma on absolutely continuous functions.

Lemma 3.14. *Let $f, g : [a,b] \to \mathbb{R}$ be two functions satisfying the following properties: f is absolutely continuous on $[a,b]$, g is continuous on $[a,b]$, and there exists an open set $A \subset (a,b)$ such that $f = g$ on $(a,b) \setminus A$, and g is constant on each connected component of A. Then g is absolutely continuous on $[a,b]$.*

Proof. Let us fix $\varepsilon > 0$ and choose $\delta > 0$ such that for every finite family of pairwise disjoint intervals $\{(s_i,t_i)\}_{i\in I}$, with $s_i, t_i \in (a,b)$ and $\Sigma_{i\in I}(t_i - s_i) < \delta$, we have $\Sigma_{i\in I}|f(t_i) - f(s_i)| < \epsilon$.

If the interval (s_i,t_i) is contained in a connected component of A then, by our hypotheses on g, we have $g(t_i) = g(s_i)$. Let $I' := \{i \in I : (s_i,t_i) \not\subset A\}$ and let $i \in I'$. Then there exist $s_i', t_i' \in (a,b) \setminus A$ such that $s_i \leq s_i' \leq t_i' \leq t_i$ and $(s_i, s_i'), (t_i', t_i) \subset A$. Indeed, if $s_i \notin A$ we take $s_i' = s_i$. In the opposite case s_i belongs to a connected component (α_i, β_i) of A and we take $s_i' = \beta_i$.

It follows that g is constant on (s_i, s_i') and, by continuity, $g(s_i) = g(s_i') = f(s_i')$, where the last equality holds since $s_i' \in (a,b) \setminus A$. Analogously we get $g(t_i) = g(t_i') = f(t_i')$. Hence $\Sigma_{i\in I}|g(t_i) - g(s_i)| = \Sigma_{i\in I'}|g(t_i) - g(s_i)| = \Sigma_{i\in I'}|f(t_i') - f(s_i')| < \epsilon$, which shows the absolute continuity of the function g on (a,b), and, by continuity, on $[a,b]$. $\square$

We are now ready to prove that R_* defined by (3.20) is the unique irreversible quasi-static evolution of the penny-shaped hydraulic crack problem with $R_*(0) = R_0$.

Proof of Theorem 3.9 (uniqueness). Let $R : [0,T] \to (0,+\infty)$ be an irreversible quasi-static evolution of the penny-shaped hydraulic crack problem with $R(0) = R_0$. By Lemma 3.12 $R(t) \geq R_*(t)$ for every $t \in [0,T]$ and by Lemma 3.13, R is continuous on $[0,T]$.

Let us assume by contradiction that $R \neq R_*$. Then there exists an interval $(a,b) \subset [0,T]$ such that $R(a) = R_*(a)$ and $R(t) > R_*(t)$ for every $t \in (a,b)$. Let

$$A := \{t \in (a,b) : V(t) > 0\}\,. \tag{3.26}$$

By Lemma 3.13, the function R is absolutely continuous on every compact set contained in A, hence it is almost everywhere differentiable on A. Recalling (3.14) and the energy-dissipation balance condition (c) of Definition 3.8, we get

$$\frac{d}{dt}\mathcal{E}_{min}(R(t),V(t)) = p_R(t)\dot V(t) + \dot R(t)\Big(2\kappa\pi R(t) - 3K\frac{V^2(t)}{R^4(t)}\Big) = p_R(t)\dot V(t)$$

for a.e. $t\in A$, hence

$$\dot R(t)\Big(2\kappa\pi R(t) - 3K\frac{V^2(t)}{R^4(t)}\Big) = 0 \qquad \text{for a.e. } t\in A. \tag{3.27}$$

Since $R(t) > R_*(t)$ for every $t\in(a,b)$, by the definition of R_* we have

$$2\kappa\pi R(t) - 3K\frac{V^2(t)}{R^4(t)} = \frac{d}{dR}\mathcal{E}_{min}(R(t),V(t)) > 0 \quad \text{on } A\,.$$

The previous inequality and (3.27) imply that $\dot R(t) = 0$ for a.e. $t\in A$, thus R is constant on each connected component of A.

Moreover, by (3.24), we have $\kappa\pi R^2(t) = e_R(t)$ for every $t\in(a,b)\setminus A$. Hence applying Lemma 3.14 with

$$f = \frac{e_R}{\kappa\pi}, \quad g = R^2, \text{ and the set } A \text{ defined in (3.26)}\,,$$

we obtain that R^2 is absolutely continuous on $[a,b]$.

By (3.23), for every $t\in(a,b)\setminus A$ we have

$$R^2(t) = \frac{e_R(t)}{\kappa\pi} = \frac{1}{\kappa\pi}\Big(K\frac{V^2(0)}{R_0^3} + \kappa\pi R_0^2 + \int_0^t p_R(s)\dot V(s)\,ds\Big).$$

Since $V\in AC([0,T],[0,+\infty))$ and $V(t)=0$ for every $t\in(a,b)\setminus A$, we obtain

$$\frac{d}{dt}R^2(t) = \frac{1}{\kappa\pi}\,p_R(t)\dot V(t) = 0 \qquad \text{for a.e. } t\in(a,b)\setminus A. \tag{3.28}$$

As $\dot R(t) = 0$ for a.e. $t\in A$, we deduce that $\dot R(t) = 0$ for a.e. $t\in(a,b)$, and therefore, being continuous, the function $R(\cdot)$ has to be constant on $[a,b]$. As a consequence, for every $t\in(a,b)$ we have

$$R(a) = R(t) > R_*(t) \geq R_*(a) = R(a),$$

which is a contradiction. Therefore, $R = R_*$ and the proof of uniqueness is concluded. □

4. A more general model

In this Section we present a more general model, where the admissible cracks are not supposed to be circular. As in the previous case, the body is linearly elastic, impermeable, unbounded, for simplicity filling all of $\mathbb{R}^3$, and the crack path is prescribed a priori: the admissible cracks lie on the horizontal plane Σ passing through the origin. We now drop the assumption that the body is isotropic and homogeneous outside the crack, thus we do not expect any symmetry of the fracture. This requires a new class of admissible cracks. For technical reasons, we still need some regularity of the relative boundary of the crack in Σ. This is provided by the *interior ball property* (see condition (c) below).

Definition 4.1. Fix $r>0$. We say that $\Gamma\in Adm_r(\Sigma)$ if it satisfies:

(a) Γ is a compact and connected subset of Σ;
(b) $0\in\Gamma$;
(c) for every $x\in\partial\Gamma$ there exists $y\in\mathring{\Gamma}$ such that $x\in\partial\mathrm{B}_r(y)$ and $\mathrm{B}_r(y)\subseteq\Gamma$.

Here and henceforth, all topological notions (boundary, interior part, balls, etc.) are considered with respect to the relative topology of Σ.

In [16] it is shown that condition (c) implies the existence of a radius $0 < r' < r$ such that every $\Gamma \in Adm_r(\Sigma)$ can be written as the closure of a union of balls of radius r'. In particular, r' can be taken equal to $r/2$. By the Lindelöff's theorem, this union can be assumed to be countable. This fact will be useful in the proof of the continuity of the Hausdorff measure $\mathcal{H}^2$ with respect to the Hausdorff metric in $Adm_r(\Sigma)$ (see Proposition 4.6).

The body outside the crack is supposed to be linearly elastic, with elasticity tensor $\mathbb{C}$. Because of the lack of homogeneity, the elasticity tensor is a function of the space variable, which will be assumed to be measurable. As usual, for almost every $x \in \mathbb{R}^3$ the function $\mathbb{C}(x) : \mathbb{M}^s_3 \to \mathbb{M}^s_3$ is linear, symmetric, and positive definite. We assume that there exist two constants $0 < \alpha < \beta < +\infty$ such that for almost every $x \in \mathbb{R}^3$

$$\alpha|\mathrm{F}|^2 \leq \mathbb{C}(x)\mathrm{F}\cdot\mathrm{F} \leq \beta|\mathrm{F}|^2 \qquad \text{for every } \mathrm{F} \in \mathbb{M}^s_3. \tag{4.1}$$

In this new setting the total energy of the body is given by

$$\mathcal{E}(u,\Gamma) := \frac{1}{2}\int_{\mathbb{R}^3\setminus\Sigma} \mathbb{C}\mathrm{E}u\cdot\mathrm{E}u\,dx + \kappa\mathcal{H}^2(\Gamma),$$

where κ is a positive constant (see the comments following (3.3) for the physical meaning of these terms).

As in Section 3, we assume that the region between the crack lips in the deformed configuration is partially filled by a prescribed volume V of an incompressible fluid. The equilibrium of the elastic body with a crack $\Gamma \in Adm_r(\Sigma)$ is achieved if the displacement u is the solution of the minimum problem

$$\min_{u\in\mathcal{A}^*(\Gamma,V)} \mathcal{E}(u,\Gamma), \tag{4.2}$$

where

$$\mathcal{A}^*(\Gamma,V) := \Big\{u \in \mathrm{W}^1_{2,6}(\mathbb{R}^3\setminus\Sigma) :\ \{[u]\neq 0\}\subseteq\Gamma,\ [u]\cdot\nu_\Sigma \geq 0,\ \int_\Gamma [u]\cdot\nu_\Sigma\,d\mathcal{H}^2 \geq V\Big\}.$$

The existence of a solution of (4.2) can be obtained by the direct method of the calculus of variations, while the uniqueness follows from the strict convexity of the functional and the convexity of the constraints.

Remark 4.2. As in Remark 3.1, the minimum problem (4.2) is equivalent to

$$\min_{u\in\mathcal{A}(\Gamma,V)} \mathcal{E}(u,\Gamma), \tag{4.3}$$

where

$$\mathcal{A}(\Gamma,V) := \Big\{u \in \mathrm{W}^1_{2,6}(\mathbb{R}^3\setminus\Sigma) :\ \{[u]\neq 0\}\subseteq\Gamma,\ [u]\cdot\nu_\Sigma \geq 0,\ \int_\Gamma [u]\cdot\nu_\Sigma\,d\mathcal{H}^2 = V\Big\}.$$

As in Proposition 3.2, if u is the solution of (4.3) with $\Gamma \in Adm_r(\Sigma)$ and $V \geq 0$, then, for every $v \in \mathrm{W}^1_{2,6}(\mathbb{R}^3\setminus\Sigma)$ such that $\{[v]\neq 0\}\subseteq\Gamma$ and $[v]\cdot\nu_\Sigma = 0$ on Σ, we have

$$\int_{\mathbb{R}^3\setminus\Sigma} \mathbb{C}\mathrm{E}u\cdot\mathrm{E}v\,dx = 0.$$

Moreover, there exists a constant $p \geq 0$ such that

$$\int_{\mathbb{R}^3\setminus\Sigma} \mathbb{C}\mathrm{E}u\cdot\mathrm{E}(\varphi u)\,dx = p\int_\Gamma \varphi[u]\cdot\nu_\Sigma\,d\mathcal{H}^2$$

for every $\varphi \in C^1(\mathbb{R}^3)$ with $\mathrm{supp}(\nabla\varphi) \subset\subset \mathbb{R}^3$. In particular, for $\varphi = 1$ we get

$$p = \frac{1}{V}\int_{\mathbb{R}^3\setminus\Sigma} \mathbb{C}\mathrm{E}u\cdot\mathrm{E}u\,dx\,. \tag{4.4}$$

We also have

$$\int_{\mathbb{R}^3\setminus\Sigma} \mathbb{C}\mathrm{E}u\cdot\mathrm{E}v\,dx = p\int_{\Gamma}[v]\cdot\nu_\Sigma\,d\mathcal{H}^2$$

for every $v\in \mathrm{W}^1_{2,6}(\mathbb{R}^3\setminus\Sigma)$ such that $\{[v]\neq 0\}\subseteq\Gamma$ and $|[v]\cdot\nu_\Sigma|\leq C[u]\cdot\nu_\Sigma$ for some $C>0$. We set $p:=0$ when $V=0$. As in Remark 3.7, this constant $p\geq 0$ is interpreted as the fluid pressure.

Let us now describe the quasi-static evolution of hydraulic cracks in this setting. Fixed $T>0$, for every $t\in[0,T]$ we denote by $V(t)$ the volume of the fluid present in the crack at time t. We assume $V\in AC([0,T];[0,+\infty))$.

It is convenient to introduce the reduced energy $\mathcal{E}_{min}(\Gamma,V)$ which is defined for every $\Gamma\in Adm_r(\Sigma)$ and $V\geq 0$ by

$$\mathcal{E}_{min}(\Gamma,V):=\min_{u\in\mathcal{A}(\Gamma,V)}\mathcal{E}(u,\Gamma).$$

The global stability condition is now expressed by

$$\mathcal{E}_{min}(\Gamma(t),V(t))\leq\mathcal{E}_{min}(\Gamma,V(t))\qquad\text{for every }\Gamma\in Adm_r(\Sigma),\ \Gamma\supseteq\Gamma(t).$$

Since the process is irreversible, we require that $\Gamma(\cdot)$ is increasing. Finally, we impose an energy-dissipation balance: the rate of change of the total energy along a solution equals the power of the pressure forces exerted by the fluid.

This leads to the following definition.

Definition 4.3. Let $T>0$ and $V\in AC([0,T],[0,+\infty))$. We say that a set function $\Gamma:[0,T]\to Adm_r(\Sigma)$ is an *irreversible quasi-static evolution* of the *hydraulic crack problem* if it satisfies the following conditions:

(a) *irreversibility*: Γ is increasing, i.e., $\Gamma(s)\subseteq\Gamma(t)$ for every $0\leq s\leq t\leq T$;

(b) *global stability*: for every $t\in[0,T]$,

$$\mathcal{E}_{min}(\Gamma(t),V(t))\leq\mathcal{E}_{min}(\Gamma,V(t))\qquad\text{for every }\Gamma\in Adm_r(\Sigma)\text{ with }\Gamma\supseteq\Gamma(t);$$

(c) *energy-dissipation balance*: the function $t\mapsto\mathcal{E}_{min}(\Gamma(t),V(t))$ is absolutely continuous on the interval $[0,T]$ and

$$\frac{d}{dt}\mathcal{E}_{min}(\Gamma(t),V(t))=p(t)\dot V(t)$$

for almost every $t\in[0,T]$, where $p(t)$ is the pressure introduced in Remark 4.2.

We are now in a position to state the main theorem of this paper.

Theorem 4.4. *Let $V\in AC([0,T],[0,+\infty))$ and $\Gamma_0\in Adm_r(\Sigma)$. Assume that (stability at time $t=0$)*

$$\mathcal{E}_{min}(\Gamma_0,V(0))\leq\mathcal{E}_{min}(\Gamma,V(0))\tag{4.5}$$

for every $\Gamma\in Adm_r(\Sigma)$ such that $\Gamma\supseteq\Gamma_0$. Then there exists an irreversible quasi-static evolution Γ of the hydraulic crack problem, with $\Gamma(0)=\Gamma_0$.

Let us first establish some properties of the admissible cracks.

Proposition 4.5. *The following facts hold:*

(a) $\Gamma=\overline{\mathring{\Gamma}}$ *for every* $\Gamma\in Adm_r(\Sigma)$;

(b) $\Gamma_1,\Gamma_2\in Adm_r(\Sigma)\Longrightarrow\Gamma_1\cup\Gamma_2\in Adm_r(\Sigma)$.

Proof. Property (a) follows immediately from the definition.

Let us prove property (b). Given $\Gamma_1,\Gamma_2\in Adm_r(\Sigma)$, the set $\Gamma_1\cup\Gamma_2$ contains 0 and is closed and connected. Since for every $x\in\partial(\Gamma_1\cup\Gamma_2)$, there exists $i=1,2$ such that $x\in\partial\Gamma_i$, by Definition 4.1, there exists $y_x\in\mathring{\Gamma}_i$ such that $\mathrm{B}_r(y_x)\subseteq\Gamma_i\subseteq\Gamma_1\cup\Gamma_2$ and $x\in\partial\mathrm{B}_r(y_x)$. Hence $\Gamma_1\cup\Gamma_2\in Adm_r(\Sigma)$. □

Proposition 4.6. *Let Γ_n be a sequence in $Adm_r(\Sigma)$ and let K, Γ be compact subsets of Σ such that $\Gamma, \Gamma_n \subseteq K$ for every $n \in \mathbb{N}$ and $\Gamma_n \to \Gamma$ in the Hausdorff metric. Then $\Gamma \in Adm_r(\Sigma)$ and $\mathcal{H}^2(\Gamma_n) \to \mathcal{H}^2(\Gamma)$.*

Proof. Let us first prove that, if $\Gamma_n \to \Gamma$ in the Hausdorff metric, then

$$\lim_n \sup_{y \in \partial\Gamma} d(y, \partial\Gamma_n) = 0. \tag{4.6}$$

By contradiction, suppose that (4.6) is false, then there exist $\varepsilon > 0$ and a subsequence, still denoted by Γ_n, such that $\sup_{y\in\partial\Gamma} d(y, \partial\Gamma_n) > 2\varepsilon$ for every $n \in \mathbb{N}$. We can choose $y_n \in \partial\Gamma$ such that $d(y_n, \partial\Gamma_n) > 2\varepsilon$. Up to another subsequence, we can suppose $y_n \to \overline{y} \in \partial\Gamma$. By the triangle inequality, we can easily prove that $d(\overline{y}, \partial\Gamma_n) > \varepsilon$ for n large enough, hence

$$\mathrm{B}_\varepsilon(\overline{y}) \cap \partial\Gamma_n = \emptyset. \tag{4.7}$$

To show that this is a contradiction, let us fix $z \in \mathrm{B}_\varepsilon(\overline{y}) \setminus \Gamma$. Since $\Gamma_n \to \Gamma$ in the Hausdorff metric, we have $z \notin \Gamma_n$ for n large enough. On the other hand, since $\overline{y} \in \Gamma$, there exists a sequence $\overline{y}_n \to \overline{y}$ with $\overline{y}_n \in \Gamma_n$. For n large enough, $\overline{y}_n \in \mathrm{B}_\varepsilon(\overline{y})$. Since $z \notin \Gamma_n$, in the segment between $\overline{y}_n$ and z there exists a point of $\partial\Gamma_n$ for n large enough. This contradicts (4.7) and proves (4.6).

It is easy to see that Γ contains 0 and is closed and connected. By (4.6), for every $y \in \partial\Gamma$, there exists a sequence $y_n \in \partial\Gamma_n$ such that $y_n \to y$. For every $n \in \mathbb{N}$, by Definition 4.1 we can find $x_n \in \mathring{\Gamma}_n$ such that $\mathrm{B}_r(x_n) \subseteq \Gamma_n$ and $y_n \in \partial\mathrm{B}_r(x_n)$. Up to a subsequence, $x_n \to x \in \Gamma$ and $\overline{\mathrm{B}}_r(x_n) \to \overline{\mathrm{B}}_r(x)$ in the Hausdorff metric. Hence $y \in \partial\mathrm{B}_r(x)$ and $\mathrm{B}_r(x) \subseteq \Gamma$, which gives $\Gamma \in Adm_r(\Sigma)$.

It remains to prove that $\mathcal{H}^2(\Gamma_n) \to \mathcal{H}^2(\Gamma)$. The measure is upper semicontinuous with respect to the Hausdorff metric, so we have only to prove

$$\mathcal{H}^2(\Gamma) \leq \liminf_n \mathcal{H}^2(\Gamma_n).$$

Thanks to [16], we have

$$\Gamma_n = \overline{\bigcup_{k\in\mathbb{N}} \overline{\mathrm{B}}_{\frac{r}{2}}(z_k^n)}, \tag{4.8}$$

for some $z_k^n \in \Gamma_n$.

Consider $\{x_k\} \subseteq \mathring{\Gamma}$ a countable dense set in Γ. By the Hausdorff convergence, for every $k \in \mathbb{N}$ there exists a sequence $x_k^n \in \Gamma_n$ such that $x_k^n \to x_k$. Using (4.8) we deduce that there exists a sequence y_k^n such that $y_k^n \in \Gamma_n$, $\mathrm{B}_{\frac{r}{2}}(y_k^n) \subseteq \Gamma_n$, and $x_k^n \in \overline{\mathrm{B}}_{\frac{r}{2}}(y_k^n)$. Up to a subsequence, we can assume that $y_k^n \to y_k \in \Gamma$ for every $k \in \mathbb{N}$, so that $\overline{\mathrm{B}}_{\frac{r}{2}}(y_k^n) \to \overline{\mathrm{B}}_{\frac{r}{2}}(y_k)$ in the Hausdorff metric and $x_k \in \overline{\mathrm{B}}_{\frac{r}{2}}(y_k) \subseteq \Gamma$. Therefore

$$\Gamma = \overline{\bigcup_{k\in\mathbb{N}} \overline{\mathrm{B}}_{\frac{r}{2}}(y_k)} = \overline{\bigcup_{k\in\mathbb{N}} \mathrm{B}_{\frac{r}{2}}(y_k)}.$$

Let us consider the sets $\Gamma^N := \bigcup_{k=0}^N \overline{\mathrm{B}}_{\frac{r}{2}}(y_k)$, $\Gamma_n^N := \bigcup_{k=0}^N \overline{\mathrm{B}}_{\frac{r}{2}}(y_k^n)$ and the functions

$$\varphi^N := \sum_{k=0}^N \mathbf{1}_{\overline{\mathrm{B}}_{\frac{r}{2}}(y_k)}, \qquad \varphi_n^N := \sum_{k=0}^N \mathbf{1}_{\overline{\mathrm{B}}_{\frac{r}{2}}(y_k^n)}.$$

By the dominated convergence theorem

$$\mathcal{H}^2(\Gamma^N) = \sum_{k=0}^N \int_{\overline{\mathrm{B}}_{\frac{r}{2}}(y_k)} \frac{1}{\varphi^N(x)}\, d\mathcal{H}^2(x) = \lim_n \sum_{k=0}^N \int_{\overline{\mathrm{B}}_{\frac{r}{2}}(y_k^n)} \frac{1}{\varphi_n^N(x)}\, d\mathcal{H}^2(x) =$$
$$= \lim_n \mathcal{H}^2(\Gamma_n^N) \leq \liminf_n \mathcal{H}^2(\Gamma_n).$$

If we pass to the limit as $N \to +\infty$, we get $\mathcal{H}^2(\Gamma^N) \to \mathcal{H}^2\big(\bigcup_{k\in\mathbb{N}} \overline{\mathrm{B}}_{\frac{r}{2}}(y_k)\big)$, so we are led to prove that

$$\mathcal{H}^2\Big(\Gamma \setminus \bigcup_{k\in\mathbb{N}} \overline{\mathrm{B}}_{\frac{r}{2}}(y_k)\Big) = 0. \tag{4.9}$$

Assume, by contradiction, that (4.9) is false. Then there exists $x \in \Gamma \setminus \bigcup_{k\in\mathbb{N}} \overline{\mathrm{B}}_{\frac{r}{2}}(y_k)$ such that

$$\lim_{\rho\to 0^+} \frac{\mathcal{H}^2\big(\mathrm{B}_\rho(x) \cap \Gamma \setminus \bigcup_{k\in\mathbb{N}} \overline{\mathrm{B}}_{\frac{r}{2}}(y_k)\big)}{\mathcal{H}^2(\mathrm{B}_\rho(x))} = 1. \tag{4.10}$$

We can find a ball $\mathrm{B}_{\frac{r}{2}}(y) \subseteq \bigcup_{k\in\mathbb{N}} \overline{\mathrm{B}}_{\frac{r}{2}}(y_k)$ such that $x \in \overline{\mathrm{B}}_{\frac{r}{2}}(y)$, hence

$$\mathrm{B}_\rho(x) \cap \Gamma \setminus \bigcup_{k\in\mathbb{N}} \overline{\mathrm{B}}_{\frac{r}{2}}(y_k) \subseteq \mathrm{B}_\rho(x) \setminus \mathrm{B}_{\frac{r}{2}}(y),$$

so we get

$$\lim_{\rho\to 0^+} \frac{\mathcal{H}^2\big(\mathrm{B}_\rho(x) \cap \Gamma \setminus \bigcup_{k\in\mathbb{N}} \overline{\mathrm{B}}_{\frac{r}{2}}(y_k)\big)}{\mathcal{H}^2(\mathrm{B}_\rho(x))} \le \lim_{\rho\to 0^+} \frac{\mathcal{H}^2(\mathrm{B}_\rho(x) \setminus \mathrm{B}_{\frac{r}{2}}(y))}{\mathcal{H}^2(\mathrm{B}_\rho(x))} = \frac{1}{2},$$

which contradicts (4.10). ☐

Remark 4.7. In this way we get also $\mathbf{1}_{\Gamma_n} \to \mathbf{1}_\Gamma$ in $L^1(\Sigma)$. Indeed, since

$$\mathcal{H}^2\Big(\Gamma \setminus \bigcup_{k\in\mathbb{N}} \mathrm{B}_{\frac{r}{2}}(y_k)\Big) = 0,$$

we have that $\mathbf{1}_{\Gamma_n}(x) \to \mathbf{1}_\Gamma(x)$ for a.e. $x \in \Sigma$ and, by the dominated convergence theorem, we obtain the convergence in $L^1(\Sigma)$.

Proposition 4.8. *Let* $\Gamma \in Adm_r(\Sigma)$. *Then* $diam(\Gamma) \le \frac{8}{\pi r}\mathcal{H}^2(\Gamma) + r$.

Proof. First we prove that $\Gamma \in Adm_r(\Sigma)$ is path-connected. Indeed we can follow the standard proof for open sets and show by contradiction that for every two points $x, y \in \Gamma$, there exists a chain of balls joining them, i.e., there exist $\mathrm{B}_{\frac{r}{2}}(\xi_1), \dots, \mathrm{B}_{\frac{r}{2}}(\xi_k) \subseteq \Gamma$ such that $x \in \overline{\mathrm{B}}_{\frac{r}{2}}(\xi_0)$, $y \in \overline{\mathrm{B}}_{\frac{r}{2}}(\xi_k)$ and $\overline{\mathrm{B}}_{\frac{r}{2}}(\xi_i) \cap \overline{\mathrm{B}}_{\frac{r}{2}}(\xi_{i+1}) \neq \emptyset$ for every $i = 0, \dots, k-1$. Assume that this is not true, then there are two points $x, y \in \Gamma$ for which there is no chain. We define

$$\begin{aligned}\Gamma_1 &:= \{z \in \Gamma : \text{ there exists a chain joining } z, y\},\\ \Gamma_2 &:= \{z \in \Gamma : \text{ there is no chain joining } z, y\}.\end{aligned}$$

Of course $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$. The set Γ_1 is nonempty, since $y \in \Gamma_1$, and closed. Indeed, given z_h in Γ_1 such that $z_h \to z$, then $z \in \Gamma$ and there exists a sequence ξ_h in Γ such that $z_h \in \overline{\mathrm{B}}_{\frac{r}{2}}(\xi_h) \subseteq \Gamma$. We can assume $\xi_h \to \xi$, $\xi \in \Gamma$, hence $\overline{\mathrm{B}}_{\frac{r}{2}}(\xi_h) \to \overline{\mathrm{B}}_{\frac{r}{2}}(\xi)$ in the Hausdorff metric. This implies $z \in \overline{\mathrm{B}}_{\frac{r}{2}}(\xi)$, so $z \in \Gamma_1$. Also the set Γ_2 is nonempty, since $x \in \Gamma_2$, and closed. Let z_h be a sequence in Γ_2 such that $z_h \to z$. We have $z \in \Gamma$. By contradiction, assume that $z \notin \Gamma_2$, then $z \in \Gamma_1$, which implies the existence of a chain joining z and y. For every $h \in \mathbb{N}$ we can find $\xi_h \in \Gamma$ such that $z_h \in \overline{\mathrm{B}}_{\frac{r}{2}}(\xi_h) \subseteq \Gamma$ and $\xi_h \to \xi$. Then $\overline{\mathrm{B}}_{\frac{r}{2}}(\xi_h) \to \overline{\mathrm{B}}_{\frac{r}{2}}(\xi)$ in the Hausdorff metric and $z \in \overline{\mathrm{B}}_{\frac{r}{2}}(\xi) \subseteq \Gamma$. We deduce that $z_h \in \Gamma_1 \cap \Gamma_2$ for h large enough. Hence Γ_2 is closed and Γ is the union of two closed, disjoint and nonempty subset of Γ, which is in contradiction with the fact that Γ is connected. Therefore Γ is path-connected.

Given $x, y \in \Gamma$, we have to estimate the distance $l := d(x, y)$ in term of $\mathcal{H}^2(\Gamma)$. Let $\gamma : [0,1] \to \Gamma$ be a continuous curve such that $\gamma(0) = x$ and $\gamma(1) = y$. We take the lines perpendicular to the segment $[x, y]$ at distance from x a multiple of r and intersecting $[x, y]$. They intersect the segment $[x, y]$ in $x = x_0, x_1, \dots, x_n$. Let us define the segments $I_k := [x_{k-1}, x_k]$ for $k = 1, \dots, n$. For $h \in [0, (n+1)/2] \cap \mathbb{N}$, let ξ_{2h+1} be the middle point of the segment I_{2h+1} and let s_{2h+1} be the line perpendicular to $[x, y]$ passing through ξ_{2h+1}.

These lines intersect the curve γ in ζ_{2h+1}. For every h, there exists a ball $\mathrm{B}_{\frac{r}{2}}(y_{2h+1}) \subseteq \Gamma$ such that $\zeta_{2h+1} \in \overline{\mathrm{B}}_{\frac{r}{2}}(y_{2h+1})$. These balls are mutually disjoint, hence we have

$$\frac{l}{8}\pi r - \frac{\pi}{8}r^2 \leq \left[\frac{l}{2r} + \frac{1}{2}\right]\frac{\pi}{4}r^2 \leq \mathcal{H}^2(\Gamma),$$

which implies

$$l \leq \frac{8}{\pi r}\mathcal{H}^2(\Gamma) + r,$$

and the proof is thus concluded. □

Let us now comment on the initial condition of Theorem 4.4.

Remark 4.9. If the set $\Gamma_0 \in Adm_r(\Sigma)$ does not satisfy the stability condition (4.5), we define Γ_0^* to be a solution of (4.5). In particular, Γ_0^* minimizes $\mathcal{E}_{min}(\Gamma, V(0))$ among all $\Gamma \in Adm_r(\Sigma)$ with $\Gamma \supseteq \Gamma_0^*$.

Therefore we can solve the problem considered in Theorem 4.4 with initial condition $\Gamma(0) = \Gamma_0^*$. A solution of (4.5) can be obtained by the direct method of the calculus of variations. Indeed, a minimizing sequence Γ_k is bounded by Proposition 4.6, so, by Theorem 2.1, we can assume $\Gamma_k \to \Gamma$ in the Hausdorff metric. For every $k \in \mathbb{N}$ there exists a unique $u_k \in \mathcal{A}(\Gamma_k, V(0))$ solution of (4.3). Since u_k is bounded in $\mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Sigma)$ by Proposition 2.4, we have $u_k \rightharpoonup v$ weakly in $\mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Sigma)$, hence $v \in \mathcal{A}(\Gamma, V(0))$ and

$$\mathcal{E}_{min}(\Gamma, V(0)) \leq \mathcal{E}(v, \Gamma) \leq \liminf_k \mathcal{E}_{min}(\Gamma_k, V(0)),$$

which shows that Γ is a minimizer.

To prove Theorem 4.4 we need the following lemma.

Lemma 4.10. *Let $\Gamma_0, \Gamma_k, \Gamma_\infty \in Adm_r(\Sigma)$ be such that $\Gamma_0 \subseteq \Gamma_k$ and $\Gamma_k \to \Gamma_\infty$ in the Hausdorff metric. Let $V_k, V_\infty \geq 0$ with $V_k \to V_\infty$. Assume that*

$$\mathcal{E}_{min}(\Gamma_k, V_k) \leq \mathcal{E}_{min}(\Gamma, V_k) \qquad \text{for every } \Gamma \in Adm_r(\Sigma) \text{ with } \Gamma \supseteq \Gamma_k.$$

Then

$$\mathcal{E}_{min}(\Gamma_\infty, V_\infty) \leq \mathcal{E}_{min}(\Gamma, V_\infty) \qquad \text{for every } \Gamma \in Adm_r(\Sigma) \text{ with } \Gamma \supseteq \Gamma_\infty. \tag{4.11}$$

Let u_k, u_∞ be the solutions of (4.3) corresponding to Γ_k, V_k and Γ_∞, V_∞ and let p_k, p_∞ be the corresponding pressures according to Remark 4.2. Then $u_k \to u_\infty$ in $\mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Sigma)$, $p_k \to p_\infty$, and $\mathcal{E}_{min}(\Gamma_k, V_k) \to \mathcal{E}_{min}(\Gamma_\infty, V_\infty)$.

Proof. Let us fix $w_0 \in \mathcal{A}(\Gamma_0, 1)$.

For every $\Gamma \in Adm_r(\Sigma)$ such that $\Gamma \supseteq \Gamma_\infty$, let $v_\Gamma \in \mathcal{A}(\Gamma, V_\infty)$ be the solution of (4.3). For every k we define $\widehat{\Gamma}_k := \Gamma \cup \Gamma_k$ and

$$v_k := \begin{cases} v_\Gamma + (V_k - V_\infty)\dfrac{v_\Gamma}{V_\infty} & \text{if } V_\infty > 0, \\ V_k w_0 & \text{if } V_\infty = 0. \end{cases}$$

Then $\widehat{\Gamma}_k \in Adm_r(\Sigma)$ by Proposition 4.5, $\Gamma_k \subseteq \widehat{\Gamma}_k$, $\widehat{\Gamma}_k \to \Gamma$ in the Hausdorff metric, and, by Proposition 4.6, $\mathcal{H}^2(\widehat{\Gamma}_k) \to \mathcal{H}^2(\Gamma)$, $v_k \in \mathcal{A}(\widehat{\Gamma}_k, V_k)$ and $v_k \to v_\Gamma$ in $\mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Sigma)$. By hypothesis we have

$$\mathcal{E}_{min}(\Gamma_k, V_k) \leq \mathcal{E}_{min}(\widehat{\Gamma}_k, V_k) \leq \mathcal{E}(v_k, \widehat{\Gamma}_k). \tag{4.12}$$

By Proposition 2.4, this implies that u_k is bounded in $\mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Sigma)$, hence, up to a subsequence, $u_k \rightharpoonup u$ weakly in $\mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Sigma)$ and $u \in \mathcal{A}(\Gamma_\infty, V_\infty)$. By lower semicontinuity, taking also into account (4.12), we have

$$\mathcal{E}_{min}(\Gamma_\infty, V_\infty) \le \mathcal{E}(u, \Gamma_\infty) \le \liminf_k \mathcal{E}_{min}(\Gamma_k, V_k) \le \tag{4.13}$$

$$\le \limsup_k \mathcal{E}_{min}(\Gamma_k, V_k) \le \limsup_k \mathcal{E}(v_k, \widehat{\Gamma}_k) = \mathcal{E}(v_\Gamma, \Gamma) = \mathcal{E}_{min}(\Gamma, V_\infty),$$

which proves (4.11). In particular, taking $\Gamma = \Gamma_\infty$, (4.13) shows that u satisfies

$$\mathcal{E}(u, \Gamma_\infty) = \mathcal{E}_{min}(\Gamma_\infty, V_\infty) = \lim_k \mathcal{E}_{min}(\Gamma_k, V_k) = \lim_k \mathcal{E}(u_k, \Gamma_k).$$

By the uniqueness of the solution of (4.3), the whole sequence u_k converges to u_∞ strongly in $\mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Sigma)$. From this convergence and Remark 4.2, it follows that $p_k \to p_\infty$, when $V_\infty > 0$.

It remains to prove that $p_k \to 0$ if $V_\infty = 0$. It is not restrictive to assume that $V_k > 0$. Since $\Gamma_0 \subseteq \Gamma_k$, we have $V_k w_0 \in \mathcal{A}(\Gamma_k, V_k)$. By the definition of u_k we have

$$\int_{\mathbb{R}^3\setminus\Sigma} \mathbb{C}\mathrm{E}u_k \cdot \mathrm{E}u_k \, dx \le V_k^2 \int_{\mathbb{R}^3\setminus\Sigma} \mathbb{C}\mathrm{E}w_0 \cdot \mathrm{E}w_0 \, dx,$$

hence (4.4) gives

$$p_k \le V_k \int_{\mathbb{R}^3\setminus\Sigma} \mathbb{C}\mathrm{E}w_0 \cdot \mathrm{E}w_0 \, dx. \tag{4.14}$$

Since $V_k \to V_\infty = 0$, this implies that $p_k \to 0$. □

Remark 4.11. By the same argument we can show that the function $V \mapsto \mathcal{E}_{min}(\Gamma, V)$ is continuous for every $\Gamma \in Adm_r(\Sigma)$.

Proof of Theorem 4.4. The proof is based on a time discretization process, see [6]. We choose a subdivision of the interval $[0, T]$ of the form $t_i^k := \frac{iT}{k}$ for $i = 0, \dots, k$. For every k we define Γ_i^k recursively with respect to i: we set $\Gamma_0^k := \Gamma_0$ and, for $i > 0$, Γ_i^k to be a solution of

$$\min\{\mathcal{E}_{min}(\Gamma, V(t_i^k)) : \Gamma \in Adm_r(\Sigma), \text{ with } \Gamma \supseteq \Gamma_{i-1}^k\}, \tag{4.15}$$

whose existence can be proved as in Remark 4.9. We denote by u_i^k the solution of (4.3) for $\Gamma = \Gamma_i^k$ and $V = V(t_i^k)$.

As in the proof of Lemma 4.10, we get that u_i^k are uniformly bounded in $\mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Sigma)$. Moreover, the pressure p_i^k associated to u_i^k according to Remark 4.2 is bounded.

We define the step functions

$$u_k(t) := u_i^k, \quad \Gamma_k(t) := \Gamma_i^k, \quad p_k(t) := p_i^k, \quad \text{for } t_i^k \le t < t_{i+1}^k.$$

We now prove a discrete energy inequality. By (4.15) we have

$$\mathcal{E}_{min}(\Gamma_i^k, V(t_i^k)) \le \mathcal{E}_{min}(\Gamma_{i-1}^k, V(t_i^k)). \tag{4.16}$$

To estimate the right-hand side of the previous inequality, we fix $w_0 \in \mathcal{A}(\Gamma_0, 1)$ and introduce the functions

$$w_i^k := \begin{cases} \dfrac{u_i^k}{V(t_i^k)} & \text{if } V(t_i^k) > 0, \\ w_0 & \text{if } V(t_i^k) = 0. \end{cases}$$

Notice that $w_i^k \in \mathcal{A}(\Gamma_i^k, 1)$, and, by (4.14),

$$\|\mathrm{E}w_i^k\|_{L^2(\mathbb{R}^3\setminus\Sigma;\mathbb{M}_3)} \le M,$$

where $M \ge \|\mathrm{E}w_0\|_{L^2(\mathbb{R}^3\setminus\Sigma;\mathbb{M}_3)}$.

Since $u^k_{i-1} + (V(t^k_i) - V(t^k_{i-1}))w^k_{i-1} \in \mathcal{A}(\Gamma^k_{i-1}, V(t^k_i))$, by (4.16) we get

$$\begin{aligned}\mathcal{E}_{min}\big(\Gamma^k_i, V(t^k_i)\big) &\le \mathcal{E}\big(u^k_{i-1} + (V(t^k_i) - V(t^k_{i-1}))w^k_{i-1}, \Gamma^k_{i-1}\big) = \\ &= \mathcal{E}_{min}\big(\Gamma^k_{i-1}, V(t^k_{i-1})\big) + \frac{(V(t^k_i) - V(t^k_{i-1}))^2}{2}\int_{R^3\setminus\Sigma} \mathbb{C}\mathrm{E}w^k_{i-1}\cdot \mathrm{E}w^k_{i-1}\,dx + \\ &\quad + (V(t^k_i) - V(t^k_{i-1}))\int_{\mathbb{R}^3\setminus\Sigma} \mathbb{C}\mathrm{E}u^k_{i-1}\cdot \mathrm{E}w^k_{i-1}\,dx \le \\ &\le \mathcal{E}_{min}\big(\Gamma^k_{i-1}, V(t^k_{i-1})\big) + \beta M^2 V_k \int_{t^k_{i-1}}^{t^k_i} |\dot V(s)|\,ds + p^k_{i-1}\int_{t^k_{i-1}}^{t^k_i} \dot V(s)\,ds,\end{aligned}$$

where $\beta > 0$ is the constant defined in (4.1) and

$$V_k := \frac{1}{2}\max_{j=1,\dots,k} |V(t^k_j) - V(t^k_{j-1})|. \tag{4.17}$$

Iterating the previous inequality we get

$$\mathcal{E}_{min}\big(\Gamma_k(t), V(t^k_i)\big) \le \mathcal{E}_{min}\big(\Gamma_0, V(0)\big) + \beta M^2 V_k \int_0^T |\dot V(s)|\,ds + \int_0^{t^k_i} p_k(s)\dot V(s)\,ds. \tag{4.18}$$

In particular, (4.18) implies that $\mathcal{H}^2(\Gamma^k(t))$ is uniformly bounded in time, hence, by Proposition 4.8, $\Gamma_k(t)$ is uniformly bounded.

By Theorem 2.3 and Proposition 4.6, up to a subsequence we have $\Gamma_k(t) \to \Gamma(t)$ in the Hausdorff metric for every $t \in [0,T]$ and the set function $\Gamma : [0,T] \to Adm_r(\Sigma)$ is bounded and increasing. Let $u(t)$ be the solution of (4.3) and $p(t)$ be the corresponding pressure. By Lemma 4.10, Γ satisfies the global stability condition (b) and, in addition, $u_k(t) \to u(t)$ strongly in $\mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Sigma)$ and $p_k(t) \to p(t)$ for every $t \in [0,T]$.

To prove the energy-dissipation balance, we first pass to the limit in (4.18) as $k \to +\infty$. The second term in the right-hand side of (4.18) tends to zero as $k \to +\infty$ because V is absolutely continuous. Since p_k is bounded in $L^\infty([0,T])$ and converges pointwise to p, we have $p_k\dot V \to p\dot V$ in $L^1([0,T])$ and we obtain

$$\mathcal{E}_{min}(\Gamma(t), V(t)) \le \mathcal{E}_{min}(\Gamma_0, V(0)) + \int_0^t p(s)\dot V(s)\,ds.$$

For the opposite inequality, for every $t \in [0,T]$ we consider a subdivision of the interval $[0,t]$ of the form $\tau^k_h := \frac{ht}{k}$ defined for every $k, h \in \mathbb{N}$, $k \ne 0$, such that $h \le k$. For every $h = 0, \dots, k$ we set

$$v^k_h := \begin{cases} \dfrac{u(\tau^k_h)}{V(\tau^k_h)} & \text{if } V(\tau^k_h) > 0, \\ w_0 & \text{if } V(\tau^k_h) = 0. \end{cases}$$

Therefore $\|\mathrm{E}v^k_h\|_{L^2(\mathbb{R}^3\setminus\Sigma;\mathbb{M}_3)} \le M$ and $u(\tau^k_{h+1}) + (V(\tau^k_h) - V(\tau^k_{h+1}))v^k_{h+1} \in \mathcal{A}(\Gamma(\tau^k_{h+1}), V(\tau^k_h))$. Since

$$\mathcal{E}_{min}(\Gamma(\tau^k_h), V(\tau^k_h)) \le \mathcal{E}_{min}(\Gamma(\tau^k_{h+1}), V(\tau^k_h)),$$

we have

$$\begin{aligned}\mathcal{E}_{min}\big(\Gamma(\tau^k_h), V(\tau^k_h)\big) &\le \mathcal{E}\big(u(\tau^k_{h+1}) + (V(\tau^k_h) - V(\tau^k_{h+1}))v^k_{h+1}, \Gamma(\tau^k_{h+1})\big) = \\ &= \mathcal{E}_{min}\big(\Gamma(\tau^k_{h+1}), V(\tau^k_{h+1})\big) + \frac{(V(\tau^k_h) - V(\tau^k_{h+1}))^2}{2}\int_{R^3\setminus\Sigma} \mathbb{C}\mathrm{E}v^k_{h+1}\cdot \mathrm{E}v^k_{h+1}\,dx + \\ &\quad + (V(\tau^k_h) - V(\tau^k_{h+1}))\int_{\mathbb{R}^3\setminus\Sigma} \mathbb{C}\mathrm{E}u(\tau^k_{h+1})\cdot \mathrm{E}v^k_{h+1}\,dx \le \\ &\le \mathcal{E}_{min}\big(\Gamma(\tau^k_{h+1}), V(\tau^k_{h+1})\big) + \beta M^2 V_k \int_{\tau^k_h}^{\tau^k_{h+1}} |\dot V(s)|\,ds - \int_{\tau^k_h}^{\tau^k_{h+1}} p(\tau^k_{h+1})\dot V(s)\,ds,\end{aligned}$$

where V_k has been defined in (4.17). Iterating the previous inequality and defining $p^k(s) := p(\tau^k_{h+1})$ for $\tau^k_h < s \le \tau^k_{h+1}$, we get

$$\mathcal{E}_{min}(\Gamma_0, V(0)) \le \mathcal{E}_{min}(\Gamma(t), V(t)) + \beta M^2 V_k \int_0^T |\dot V(s)|\, ds - \int_0^t p^k(s)\dot V(s)\, ds. \tag{4.19}$$

Since $\Gamma(\cdot)$ is an increasing function, by Theorem 2.2 there exists $\Theta \subseteq [0,T]$ such that $[0,T] \setminus \Theta$ is at most countable and $\Gamma(\cdot)$ is continuous at every point of Θ. By Lemma 4.10 we have that $s \mapsto u(s)$ is strongly continuous on $\mathrm{W}^1_{2,6}(\mathbb{R}^3 \setminus \Sigma)$ at every point of Θ and $s \mapsto p(s)$ is continuous at the same points. This implies that $p^k(s) \to p(s)$ for every $s \in \Theta$. By the dominated convergence theorem we get $p^k \dot V \to p\dot V$ in $L^1([0,t])$ and, passing to the limit in (4.19), we obtain

$$\mathcal{E}_{min}(\Gamma_0, V(0)) \le \mathcal{E}_{min}(\Gamma(t), V(t)) - \int_0^t p(s)\dot V(s)\, ds.$$

This concludes the proof of the energy-dissipation balance (c). □

Let $\Gamma : [0,T] \to Adm_r(\Sigma)$ satisfy Theorem 4.4. For every $t \in (0,T]$ we consider $\Gamma^-(t)$ defined, as in Theorem 2.2, by

$$\Gamma^-(t) := \overline{\bigcup_{s<t} \Gamma(s)}\,. \tag{4.20}$$

We have $\Gamma(t) = \Gamma^-(t)$ and $\mathcal{E}_{min}(\Gamma^-(t), V(t)) = \mathcal{E}_{min}(\Gamma(t), V(t))$ for every $t \in (0,T]$ out of a countable set.

Proposition 4.12. *Let $\Gamma : [0,T] \to Adm_r(\Sigma)$ satisfy Theorem 4.4 and let $\Gamma^-(t)$ be given by* (4.20) *for every $t \in (0,T]$. Then*

$$\mathcal{E}_{min}(\Gamma^-(t), V(t)) = \mathcal{E}_{min}(\Gamma(t), V(t)) \qquad \textit{for every } t \in (0,T]\,. \tag{4.21}$$

Moreover,

$$\mathcal{E}_{min}(\Gamma(t), V(t)) \le \mathcal{E}_{min}(\Gamma, V(t)) \qquad \textit{for every } \Gamma \in Adm_r(\Sigma) \textit{ with } \Gamma \supseteq \Gamma^-(t)\,.$$

Proof. Since $\Gamma(s) \to \Gamma^-(t)$ in the Hausdorff metric as $s \nearrow t$, by Lemma 4.10 we get

$$\mathcal{E}_{min}(\Gamma^-(t), V(t)) = \lim_{s \nearrow t} \mathcal{E}_{min}(\Gamma(s), V(s)).$$

By the continuity of $s \mapsto \mathcal{E}_{min}(\Gamma(s), V(s))$ we obtain (4.21).

Fixed $\Gamma \in Adm_r(\Sigma)$ with $\Gamma^-(t) \subseteq \Gamma$, we have $\Gamma(s) \subseteq \Gamma$ for every $0 \le s < t$, hence

$$\mathcal{E}_{min}(\Gamma(s), V(s)) \le \mathcal{E}_{min}(\Gamma, V(s)). \tag{4.22}$$

and passing to the limit as $s \nearrow t$ we get the thesis. □

Remark 4.13. Thanks to Proposition 4.12 we have that if $\Gamma : [0,T] \to Adm_r(\Sigma)$ satisfies Theorem 4.4, the same is true for the function

$$t \mapsto \begin{cases} \Gamma(0) & \text{for } t = 0, \\ \Gamma^-(t) & \text{for } 0 < t \le T, \end{cases}$$

where $\Gamma^-(t)$ is defined in (4.20). We notice that in the energy-dissipation balance we have to replace $p(t)$ with $p^-(t)$ which satisfies Proposition 3.2, extending it at $t = 0$ by $p^-(0) := p(0)$. Then the quasi-static hydraulic crack problem has a left-continuous solution.

Remark 4.14. Repeating the same steps, for every $t \in [0,T)$ we define $\Gamma^+(t)$ as in Theorem 2.2. As in Proposition 4.12 we obtain $\mathcal{E}_{min}(\Gamma(t), V(t)) = \mathcal{E}_{min}(\Gamma^+(t), V(t))$ for every $t \in [0,T)$ and finally, as in Remark 4.13, we define the function

$$t \mapsto \begin{cases} \Gamma^+(t) & \text{for } 0 \le t < T, \\ \Gamma(T) & \text{for } t = T, \end{cases}$$

which satisfies properties (a), (b), and (c) of Definition 4.3. Therefore, we get a right-continuous solution of the problem. Note however that the right-continuous solution does not necessarily satisfy the initial condition.

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