ON A VARIATIONAL APPROACH FOR WATER WAVES

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ABSTRACT. Using a variational method due to Alt and Caffarelli [AC], we study regularity and qualitative properties of local and global minimizers of a functional with a variable domain of integration related to water waves.

1. Introduction

In a classical paper [AC] Alt and Caffarelli studied existence and regularity of minimizers of the functional

$$J(u) = \int_\Omega \left(|\nabla u(\mathbf{x})|^2 + \chi_{\{u>0\}}(\mathbf{x}) Q^2(\mathbf{x}) \right) d\mathbf{x} \tag{1.1}$$

in the class

$$\mathcal{K}_0 := \left\{ u \in L^1_{\text{loc}}(\Omega) : \nabla u \in \left(L^2_{\text{loc}}(\Omega)\right)^2, \, u = u_0 \text{ on } S \right\}.$$

Here Ω is an open connected set of $\mathbb{R}^n$ with locally Lipschitz boundary, $S \subseteq \partial\Omega$ is a measurable set with $\mathcal{H}^{n-1}(S) > 0$ and the Dirichlet datum u_0 on S is a nonnegative function $u_0 \in L^1_{\text{loc}}(\Omega)$ with $\nabla u_0 \in \left(L^2_{\text{loc}}(\Omega)\right)^2$. The identity $u = u_0$ on S is to be understood in the sense of traces.

Under the assumptions that Q is a Hölder continuous function satisfying

$$0 < Q_{\min} \le Q(\mathbf{x}) \le Q_{\max} < \infty, \tag{1.2}$$

Alt and Caffarelli proved, in particular, full regularity of the free boundary $\Omega \cap \partial\{u > 0\}$ of local minimizers for $n = 2$ and partial regularity for $n \ge 3$ (see also the recent paper [CJK] for the case $n = 3$).

Note that when the free boundary $\Omega \cap \partial\{u > 0\}$ is smooth, the Euler-Lagrange equations of (1.1) are given by

$$\begin{aligned} &\Delta u = 0 \quad \text{in } \Omega \cap \{u > 0\}, \\ &u = 0, \quad |\nabla u| = Q \quad \text{on } \Omega \cap \partial\{u > 0\}, \\ &u = u_0 \quad \text{on } S. \end{aligned} \tag{1.3}$$

One of the main purposes of this paper is to study the loss of regularity of the free boundary in the case $Q_{\min} = 0$. Note that in view of Hopf 's boundary point lemma at any regular free boundary point one has $Q = |\nabla u| = -\frac{\partial u}{\partial \nu} > 0$, where ν is the outward normal. Thus, if the free boundary touches a point $\mathbf{x}_0$ at which Q vanishes, then $\mathbf{x}_0$ cannot be a regular point. More precisely, if Q decays like r^α (in spherical coordinates), then by $(1.3)_2$, the function u decays like $r^{\alpha+1}$.

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In dimension $n = 2$ this leads to the formation of angles. Indeed (see e.g., Chapter 4 of [Gris]), the solution of the Dirichlet boundary problem

$$\begin{aligned} \Delta v &= 0 \quad \text{in } A, \\ v &= 0 \quad \text{on } \partial A, \end{aligned}$$

where A is the infinite sector given in polar coordinates by

$$A := \{(r, \theta) : r > 0,\, 0 < \theta < \omega\},$$

is the function

$$v = r^{\frac{\pi}{\omega}} \sin\left(\frac{\pi\theta}{\omega}\right). \tag{1.4}$$

Hence, if u decays like $r^{\alpha+1}$, then a blow-up argument gives $\omega = \frac{\pi}{\alpha+1}$. Note that in [AC] the condition $Q_{\min} > 0$ implies linear decay of minimizers, which corresponds to $\alpha = 0$, that is, $\omega = \pi$, thus one expects a smooth free boundary.

For simplicity in this paper we consider $n = 2$ and the function

$$Q(x, y) = \sqrt{(\lambda - y)_+}, \tag{1.5}$$

where $\mathbf{x} = (x, y) \in \mathbb{R} \times \mathbb{R}$, although all the results can be adapted to cover more general functions of the type

$$Q(x, y) = \left((\lambda - y)_+\right)^b, \quad b > 0.$$

The function (1.5) leads to the formation of angles of $\frac{2}{3}\pi$ at points at which the free boundary touches the line $y = \lambda$.

The motivation for this choice comes from water waves. Indeed, consider periodic steady surface waves in a horizontal channel of uniform rectangular cross-section. We neglect surface tension and we assume that the channel has a flat, rigid bottom and that the water motion is two-dimensional, irrotational and in a vertical plane. We choose the frame of reference so that the velocity field and the free surface profile are time-independent and we assume that the free surface is the graph of a function f. Thus

$$D = \left\{(x, y) \in \mathbb{R}^2 :\, x \in \mathbb{R},\ 0 < y < f(x)\right\},$$

represents the longitudinal section of the water domain, $\Gamma = \{(x, f(x)) :\, x \in \mathbb{R}\}$ is the free surface, and $y = 0$ represents the bottom of the channel. For simplicity, assume that f has period ℓ, has a single crest per wavelength, and is symmetrical about that crest. If $\mathbf{v}$ is the velocity of the fluid and ρ its density, the conservation of mass and the fact that the fluid is irrotational, which are usually given in the form

$$\begin{aligned} \frac{d\rho}{dt} + \operatorname{div}(\rho \mathbf{v}) &= 0 \quad \text{in } D, \\ \operatorname{curl} \mathbf{v} &= 0 \quad \text{in } D, \end{aligned}$$

under the present assumptions simplify to $\operatorname{div} \mathbf{v} = 0$ and $\operatorname{curl} \mathbf{v} = 0$ in D. Hence, we may write $\mathbf{v} = (u_y, -u_x)$, where the potential u satisfies the following problem

$$\Delta u = 0 \quad \text{in } D.$$

Moreover, from Bernoulli's theorem, in the steady motion of an inviscid fluid at every point of the same streamline one has $\frac{p}{\rho} + K =$const, where p is the pressure and K is the energy per unit mass of the fluid. In the present setting this becomes

$$u=0, \quad \frac{1}{2}\left|\nabla u\right|^2+gy=R \quad \text{on } \Gamma,$$

where g is the gravitational acceleration, R is the Bernoulli's constant and K has been taken as the sum of the kinetic and potential energy. Thus we are led to the following free boundary problem:

$$\begin{aligned} &\Delta u=0 \quad \text{in } D,\\ &u\left(x+\ell,y\right)=u\left(x,y\right) \quad \text{in } D,\\ &u=0, \quad \frac{1}{2}\left|\nabla u\right|^2+gy=R \quad \text{on } \Gamma,\\ &u=q \quad \text{on } y=0, \end{aligned} \tag{1.6}$$

where q is the volume rate of flow per unit span (see [AT] and [KK] for more details).

In 1847 Stokes (see [Sto], [Sto2]) assumed that there exists a family of solutions for this problem, which are parametrized by the height H of the wave, $H:=\max f-\min f$, and he conjectured that there exists a wave of greatest height, which is characterized by the fact that its shape is not regular but has sharp crests of included angle $\frac{2}{3}\pi$. He also conjectured that for this wave $f''>0$ in $\left(0,\frac{\ell}{2}\right)$.

Stokes conjectures have been proved in a series of papers. The starting point was a paper of Nekrasov in 1922 (see [MT]), where he used a hodograph transformation to map the region under one period onto the unit circle. More precisely, setting

$$\varphi\left(s\right)=\arctan f'\left(x\right), \tag{1.7}$$

one obtains the nonlinear integral equation

$$\varphi\left(s\right)=\frac{1}{3\pi}\int_0^\pi \frac{\sin\varphi\left(t\right)}{\mu^{-1}+\int_0^t\sin\varphi\left(\tau\right)\,d\tau}\log\left|\frac{\operatorname{sn}\frac{1}{\pi}T\left(s+t\right)}{\operatorname{sn}\frac{1}{\pi}T\left(s-t\right)}\right|dt, \quad 0<s\leq\pi, \tag{1.8}$$

where sn denotes the Jacobian elliptic function whose quarter periods T and iT' satisfy $\frac{T'}{T}=\frac{4h}{\ell}$ and h is the mean depth of the fluid (see Section 1.2 of [AT] for more details). Solutions of (1.8) with $\mu>0$ correspond to *regular waves*, while in the case $\mu=\infty$ one has a *Stokes wave*.

The first existence result for solutions of Nekrasov's integral equation (1.8) is due to Krasovski [K] in 1961, who, using a degree theory argument, proved that for every angle $0<\beta<\frac{\pi}{6}$ there exist $\mu>3\coth\left(2\pi h/\ell\right)$ and a continuous solution φ_μ of (1.8) with

$$\sup_{s\in[0,\pi]}\varphi_\mu\left(s\right)=\beta.$$

In 1978 Keady and Norbury [KN], again using degree theory arguments, proved that for every $\mu>3\coth\left(2\pi h/\ell\right)$ there exists a continuous solution φ_μ of (1.8) with $0\leq\varphi_\mu<\frac{\pi}{2}$, while there are no solutions for $0<\mu\leq3\coth\left(2\pi h/\ell\right)$. These waves have a smooth profile.

Toland [To] in 1978 and McLeod [ML2] in 1979 showed that as $\mu\to\infty$ the regular waves φ_μ converge to a solution φ_0 of the limiting problem $\mu=\infty$ and proved that if the limit $\lim_{s\to0^+}\varphi_0\left(s\right)$ exists, then it must be $\frac{\pi}{6}$. The existence of the limit was proved by Amick, Fraenkel, and Toland [AF2] in 1982, and independently by Plotnikov [Pl] in 1982.

Finally, in 2004 Plotnikov and Toland [PlT] proved that there exist Stokes waves with $f'' > 0$ in $\left(0, \frac{\ell}{2}\right)$.

Although the hodograph transform (1.7) has proved to be quite successful in tackling Stokes conjectures, from an intuitive point of view it is difficult to visualize the qualitative properties of the solutions of the nonlinear integral equation (1.8). In recent years, several papers have addressed water waves using variational approaches. See, for example, the recent work of Burton and Toland [BT] for steady surface waves on flows with vorticity, where existence is proved using a minimax argument, or the paper of Chambolle, Séré, and Zanini [CSZ], where periodic traveling water waves without vorticity, in presence of gravity and surface tension, are found as minimizers of an energy functional.

In this paper we address the free boundary problem (1.6) using the variational approach of Alt and Caffarelli [AC]. Indeed, it follows from (1.1), (1.3), that solutions of (1.6) can be regarded as critical points of the functional

$$J_\lambda(u) := \int_\Omega \left(|\nabla u|^2 + \chi_{\{u>0\}}(\lambda - y)_+ \right) d\mathbf{x} \tag{1.9}$$

defined in the set

$$\mathcal{K}_1 := \Big\{ u \in L^1_{\mathrm{loc}}(\Omega) : \nabla u \in \left(L^2_{\mathrm{loc}}(\Omega)\right)^2,\ u(x,0) = q \text{ for } x \in (-\ell, \ell), \qquad u(x+\ell, y) = u(x,y) \text{ in } \Omega \Big\}, \tag{1.10}$$

where $\Omega := (-\ell, \ell) \times (0, \infty)$ and the parameter $\lambda > 0$ plays the role of the parameter μ in (1.8).

In the first part of the paper, following the work of [AC], we study qualitative properties of local minimizers u of the functional J_λ in the class $\mathcal{K}_1$, including regularity of u and $\partial\{u > 0\}$, decay estimates of u and ∇u near the critical line $y = \lambda$, connectedness properties of the set $\{u > 0\}$. In particular, we show that local minimizers are both "weak solutions" and "variational solutions" of (1.6) in the sense of Varvaruca and Weiss [VW] (see Definition 3.1 and 3.2 in [VW], see also the work of Shargorodsky and Toland [ST]). In particular, the monotonicity formula established in Theorem 3.5 of [VW] applies to local minimizers.

We also prove that if $\mathbf{x}_0 = (x_0, \lambda) \in \Omega \cap \partial\{u > 0\}$, then $|\nabla u(x,y)| \le C\sqrt{(\lambda - y)_+}$ for all $\mathbf{x} = (x,y) \in B_r(\mathbf{x}_0)$, where $r > 0$ is sufficiently small. As in the work of Varvaruca and Weiss (see Proposition 4.7 in [VW]), this decay estimate, together with the monotonicity formula just mentioned, gives a complete characterization of all blow-up solutions at the point $\mathbf{x}_0$ (see Theorem 4.3 below).

In the second part of the paper we focus our attention to absolute minimizers of J_λ in the class $\mathcal{K}_1$. The main drawback here is that absolute minimizers of J_λ in the class $\mathcal{K}_1$ are one-dimensional functions of the type $u = u(y)$ (see Theorem 5.1). In particular, the regular water waves constructed by Keady and Norbury [KN] or the Stokes wave found by Toland [To] and McLeod [ML2] are only critical points of J_λ in the class $\mathcal{K}_1$.

To retain some useful information from absolute minimizers, we restrict our attention near the crest of a water wave. More precisely, we consider a level $y = h_1$, where $0 < h_1$, such that $u(\cdot, h_1) =: w_0(x)$ vanishes at $\pm\ell$ (this is certainly true near the crest if u is a regular water wave or a Stokes wave) and then study the functional J_λ (with Ω replaced by the smaller domain $\Omega_1 := (-\ell, \ell) \times (h_1, \infty)$) in the class of functions (see Figure 1.1)

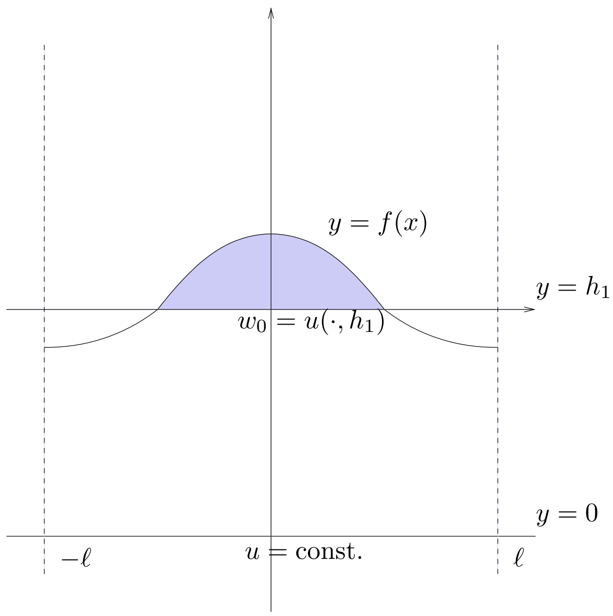

FIGURE 1.1. Wave of Finite Depth

$$\mathcal{K}_3 := \Big\{ u \in L^1_{\text{loc}}(\Omega_1) : \nabla u \in \left(L^2_{\text{loc}}(\Omega_1)\right)^2, \; u(x, h_1) = w_0(x) \text{ for } x \in (-\ell, \ell), \tag{1.11}$$
$$u(-\ell, y) = u(\ell, y) = 0 \text{ for } y \in (h_1, \infty) \Big\}.$$

Inspired by the work of Alt, Caffarelli, and Friedman (see [ACF], [ACF1], [ACF2], [Fr]), we prove the existence of a critical value $\lambda_c \in (h_1, \infty)$ with the property that:

(i) If $u_\lambda \in \mathcal{K}_3$ is an absolute minimizer of the functional J_λ for $\lambda > \lambda_c$, then the support of u_λ remains below the line $y = \lambda$, and thus u_λ and its free-boundary $\partial \{u_\lambda > 0\}$ are smooth by the regularity results of Alt and Caffarelli [AC].

(ii) If $u_\lambda \in \mathcal{K}_3$ is an absolute minimizer of the functional J_λ for $\lambda < \lambda_c$, then the support of u_λ crosses the line $y = \lambda$. We refer to this kind of solutions as *non-physical solutions*.

(iii) We also prove that if $\lambda_n \searrow \lambda_c$ and $\mu_n \nearrow \lambda_c$, then the corresponding sequences of absolute minimizers $\{u_{\lambda_n}\}$ and $\{u_{\mu_n}\}$ converge strongly in $H^1_{\text{loc}}(\Omega_1)$ to two absolute minimizers u^+ and $u^- \in K$ of J_{λ_c}, respectively. Moreover $\operatorname{supp} u^+ \subseteq \{y \le \lambda_c\}$, while $\operatorname{supp} u^-$ intersects the line $y = \lambda_c$. We conjecture that $u^+ = u^-$. Note that if the conjecture were true, then the support of u^+ would touch the line $y = \lambda_c$ and be contained in the set $\{y \le \lambda_c\}$. We have been unable to prove the conjecture.

Note that part (i) shows the existence of a family of local regular solutions, and so, in spirit, could be considered a local version of the result of Keady and Norbury [KN]. However, the main drawback here is that solutions heavily depend on the choice of the initial datum w_0. To recover the result of [KN] one should solve the boundary value problem

$$\Delta v_\lambda = 0 \quad \text{in } (-\ell,\ell)\times(0,h_2)\,,$$
$$v_\lambda(x+\ell,y) = v_\lambda(x,y) \quad \text{in } (-\ell,\ell)\times(0,h_2)\,,$$
$$v_\lambda = u_\lambda \quad \text{on } y=h_2,$$
$$v_\lambda = \text{const} \quad \text{on } y=0$$

for some $h_1 \le h_2$, and then find h_2 and w_0 in such a way that the function

$$w_\lambda := \begin{cases} u_\lambda & \text{in } (-\ell,\ell)\times(h_2,\lambda)\,, \\ v_\lambda & \text{in } (-\ell,\ell)\times(0,h_2] \end{cases}$$

remains harmonic in $(-\ell,\ell)\times(0,\lambda)$. This seems far from trivial.

The same difficulty arises when relating part (iii) to the theorems of Toland [To] and McLeod [ML2] on the existence of a Stokes wave.

We believe, however, that the techniques developed in the proof of (i)-(iii) are of independent interest.

We conclude by mentioning that in ongoing work [FLM] in collaboration with I. Fonseca and M.G. Mora, the second author has shown that, under suitable hypotheses, if u is a smooth critical point of (2.1) with support below $y=\lambda$ and $\partial\{u>0\}$ is given by the graph of a smooth function f, then u is actually a local minimizer of the functional J_λ in the region $(-\ell,\ell)\times(\max f-\varepsilon,\max f)$ provided $\varepsilon>0$ is sufficiently small. The approach consists in deriving new necessary and sufficient minimality conditions by means of second order variations. It was first used to study critical points of the Mumford-Shah functional (see [CMM], [MM]).

Note that this result implies, in particular, that the regular waves constructed by Keady and Norbury [KN] are, near the top of the crest, local minimizers of (2.1).

2. Preliminary Results

In this section we present some preliminary results due to Alt and Caffarelli [AC]. We consider the functional

$$J_\lambda(u) := \int_\Omega \left(|\nabla u|^2 + \chi_{\{u>0\}}(\lambda-y)_+\right)\,d\mathbf{x} \tag{2.1}$$

defined in the set

$$\mathcal{K} := \Big\{u\in L^1_{\text{loc}}(\Omega):\ \nabla u\in\left(L^2_{\text{loc}}(\Omega)\right)^2,\ u=u_0 \text{ on } S,\quad u(x+\ell,y)=u(x,y) \text{ in } \Omega\Big\}, \tag{2.2}$$

where $\lambda>0$, $\Omega:=(-\ell,\ell)\times(0,\infty)$, $S\subseteq\partial\Omega$ is a measurable set with $\mathcal{H}^1(S)>0$, and the Dirichlet datum u_0 on S is a nonnegative function $u_0\in L^\infty(\Omega)$ with $\nabla u_0\in\left(L^2_{\text{loc}}(\Omega)\right)^2$ and $u_0(x+\ell,y)=u_0(x,y)$ in Ω.

Definition 2.1. *Given $u\in\mathcal{K}$ with $J_\lambda(u)<\infty$, we say that*

- *u is an* absolute minimizer *of the functional J_λ if*
$$J_\lambda(u)\le J_\lambda(v)$$
for all $v\in\mathcal{K}$,
- *u is a* local minimizer *of the functional J_λ if there exists $\varepsilon_0>0$ such that*
$$J_\lambda(u)\le J_\lambda(v)$$

for all $v \in \mathcal{K}$ *with*

$$\|\nabla (u - v)\|_{(L^2(\Omega))^2} + \left\|\chi_{\{u>0\}} - \chi_{\{v>0\}}\right\|_{L^1(\Omega)} \leq \varepsilon_0. \tag{2.3}$$

Theorem 2.2 (Theorem 1.3 in [AC])**.** *Assume that* $J_\lambda(u_0) < \infty$*. Then there exists an absolute minimizer* $u \in \mathcal{K}$ *of the functional* J_λ*.*

Proof. Let $\alpha := \inf_{v \in \mathcal{K}} J_\lambda(v)$ and let $\{u_n\} \subset \mathcal{K}$ be a minimizing sequence for J_λ, that is, $J_\lambda(u_n) \to \alpha$ as $n \to \infty$. Since

$$J_\lambda((u_n)_+) \leq J_\lambda(u_n)$$

and $(u_n)_+ \in \mathcal{K}$, without loss of generality we may assume that $u_n \geq 0$ and that

$$J_\lambda(u_n) \leq \alpha + 1 \text{ for all } n \in \mathbb{N}.$$

Then $\{\nabla u_n\}$ is bounded in $\left(L^2(\Omega)\right)^2$.

Let $\Omega_r := (-\ell, \ell) \times (0, r)$, where $r > 0$ is taken so large that $\mathcal{H}^{n-1}(S \cap \partial\Omega_r) > 0$. Since $u_n - u_0 = 0$ on $S \cap \partial\Omega_r$, it follows by Poincaré's inequality that

$$\int_{\Omega_r} |u_n - u_0|^2 \, d\mathbf{x} \leq C(S, \Omega_r) \int_{\Omega_r} |\nabla u_n - \nabla u_0|^2 \, d\mathbf{x}$$

for all $n \in \mathbb{N}$. Therefore $\{u_n\}$ is bounded in $H^1(\Omega_r)$. Since $H^1(\Omega_r)$ is compactly embedded in $L^p(\Omega_r)$, $1 \leq p < \infty$, $\{u_n\}$ admits a subsequence (not relabeled) that converges weakly in $H^1(\Omega_r)$ and strongly in $L^p(\Omega_r)$ to a function $u^r \in H^1(\Omega_r)$.

If we now let $s > r$ and extract a further subsequence we may assume that $u_n \rightharpoonup u^r$ in $H^1(\Omega_r)$ and $u_n \rightharpoonup u^s$ in $H^1(\Omega_s)$. By the uniqueness of the weak limit we have that

$$u^r(\mathbf{x}) = u^s(\mathbf{x}) \text{ for } \mathcal{L}^2 \text{ a.e. } \mathbf{x} \in \Omega_r.$$

Taking a sequence $r_k \,(:= k) \nearrow \infty$ and using a diagonalization argument we may find a subsequence of $\{u_n\}$ (again not relabeled) weakly convergent in $H^1_{\text{loc}}(\Omega)$ to the nonnegative function

$$u(x, y) := u^{r_k}(x, y) \text{ if } r_{k-1} \leq y < r_k,\ k \in \mathbb{N}.$$

Moreover, since $\left\{\chi_{\{u_n>0\}}\right\}$ is bounded in $L^\infty(\Omega)$, we may find a function $\gamma \in L^\infty(\Omega)$, $0 \leq \gamma \leq 1$ and yet another subsequence such that $\left\{\chi_{\{u_n>0\}}\right\}$ converges weakly star to γ in $L^\infty(\Omega)$.

Next we will prove that $\gamma \geq \chi_{\{u>0\}}$. Since $u_n \to u$ in $L^1_{\text{loc}}(\Omega)$ and $\chi_{\{u_n>0\}} \overset{*}{\rightharpoonup} \gamma$ in $L^\infty(\Omega)$, letting $n \to \infty$ in the identity

$$\int_{\Omega_r} u_n(1 - \chi_{\{u_n>0\}}) \, d\mathbf{x} = 0$$

yields

$$\int_{\Omega_r} u(1 - \gamma) \, d\mathbf{x} = 0 \quad \text{for all } r > 0.$$

Since $u \geq 0$ and $\gamma \leq 1$, we conclude that $u(1 - \gamma) = 0$ $\mathcal{L}^2$ a.e. in Ω. Therefore $\gamma = 1$ $\mathcal{L}^2$ a.e. in the set $\{u > 0\}$, and so

$$\gamma \geq \chi_{\{u>0\}}.$$

This is sufficient to show that u is an absolute minimizer. Indeed, for every $r>0$,

$$\begin{aligned}
&\int_{\Omega_r} (|\nabla u|^2 + \gamma\,(\lambda - y)_+)\,d\mathbf{x} \\
&\quad\le \liminf_{n\to\infty} \int_{\Omega_r} |\nabla u_n|^2 \; d\mathbf{x} + \lim_{n\to\infty} \int_{\Omega_r} \chi_{\{u_n>0\}}(\lambda - y)_+ \, d\mathbf{x} \\
&\quad\le \lim_{n\to\infty} J_\lambda(u_n) = \alpha.
\end{aligned}$$

Letting $r\to\infty$ and using the fact that $\gamma \ge \chi_{\{u>0\}}$ yields

$$\begin{aligned}
\alpha \le J_\lambda(u) &= \int_\Omega (|\nabla u|^2 + \chi_{\{u>0\}}(\lambda - y)_+)\,d\mathbf{x} \\
&\le \int_\Omega (|\nabla u|^2 + \gamma(\lambda - y)_+)\,d\mathbf{x} \le \alpha.
\end{aligned}$$

Thus $J_\lambda(u) = \alpha$ and $u \in \mathcal{K}$. □

Lemma 2.3 (Lemma 2.2 in [AC])**.** *Let $u\in\mathcal{K}$ be a local minimizer of J_λ. Then u is subharmonic and for every $\mathbf{x}_0\in\Omega$ and $0<s<r<\operatorname{dist}(\mathbf{x}_0,\partial\Omega)$,*

$$\frac{1}{|B_r(\mathbf{x}_0)|}\int_{B_r(\mathbf{x}_0)} u(\mathbf{y})\,d\mathbf{y} \ge \frac{1}{|B_s(\mathbf{x}_0)|}\int_{B_s(\mathbf{x}_0)} u(\mathbf{y})\,d\mathbf{y}.$$

Remark 2.4. In view of the previous lemma we can work with the precise representative

$$u(\mathbf{x}) := \lim_{r\to 0^+} \frac{1}{|B_r(\mathbf{x})|}\int_{B_r(\mathbf{x})} u(\mathbf{y})\,d\mathbf{y}, \quad \mathbf{x}\in\Omega.$$

Lemma 2.5 (Lemma 2.3 in [AC])**.** *Let $u\in\mathcal{K}$ be a local minimizer of J_λ. Then*

$$0 \le u(\mathbf{x}) \le \sup_\Omega u_0$$

for $\mathcal{L}^2$ a.e. $\mathbf{x}$ in Ω.

3. Regularity of Local Minimizers

In this section we study qualitative properties of local minimizers of J_λ in the class $\mathcal{K}$. To prove regularity of local minimizers, the idea in [AC] is to exploit the competition between the two terms of the functional J_λ. Consider a small ball B_r in Ω and suppose that u is large (in some sense) on the boundary of this ball. Then the minimizer cannot vanish in the ball because in this case $\int_{B_r}|\nabla u|^2\,d\mathbf{x}$ would be too big. Therefore $u>0$ on B_r. Hence, roughly speaking, to minimize the functional in this ball reduces to minimize $\int_{B_r}|\nabla u|^2\,d\mathbf{x}$, therefore u is harmonic. On the other side, if u is small on ∂B_r, then u has to be zero at least on a ball B_{kr}, for some $0<k<1$.

In the following theorems we make this competition precise. We adapt the proofs of Lemmas 3.2 and 3.4 in [AC] to our setting. All the results of this section continue to hold, with some straightforward modifications, when $\mathbb{R}^2$ is replaced by $\mathbb{R}^n$.

The next theorem follows closely Lemma 3.2 of [AC]. In particular, in Steps 1 and 2 there are essentially no changes from the original proof. We present the proof for the convenience of the reader and to follow the precise dependence on the behavior of the function Q.

Theorem 3.1. *There is a constant* $C_{\max} > 0$ *such that for every (small) ball* $B_r(\mathbf{x}_0) \subset \Omega$, $\mathbf{x}_0 = (x_0, y_0)$, *and for every local minimizer* $u \in \mathcal{K}$ *of* J_λ, *if*

$$\frac{1}{r\,|\partial B_r(\mathbf{x}_0)|}\int_{\partial B_r(\mathbf{x}_0)} u\, d\mathcal{H}^1 > C_{\max}\sqrt{(\lambda - y_0 + r)_+},$$

then $u > 0$ *in* $B_r(\mathbf{x}_0)$.

Proof. **Step 1:** For simplicity we denote $B_r(\mathbf{x}_0)$ simply by B_r. Consider the harmonic function v in B_r with boundary values u. Since $u \geq 0$, from the maximum principle we have that $v > 0$ in B_r. Outside B_r define $v := u$. Since $J_\lambda(u) \leq J_\lambda(v)$, we have

$$\int_{B_r}\left(|\nabla u|^2 + \chi_{\{u>0\}}(\lambda - y)_+\right) d\mathbf{x} \leq \int_{B_r}\left(|\nabla v|^2 + (\lambda - y)_+\right) d\mathbf{x}. \tag{3.1}$$

Since $\Delta v = 0$ in B_r and $u - v = 0$ on ∂B_r, by the first Green's formula we have

$$\int_{B_r} \nabla v \cdot (\nabla u - \nabla v)\, d\mathbf{x} = 0,$$

or, equivalently,

$$\int_{B_r} |\nabla v|^2\, d\mathbf{x} = \int_{B_r} \nabla u \cdot \nabla v\, d\mathbf{x}.$$

In turn,

$$\begin{aligned}\int_{B_r}(|\nabla u|^2 - |\nabla v|^2)\, d\mathbf{x} &= \int_{B_r}(|\nabla u|^2 + |\nabla v|^2 - 2\,|\nabla v|^2)\, d\mathbf{x}\\ &= \int_{B_r}(|\nabla u|^2 + |\nabla v|^2 - 2\nabla u\cdot\nabla v)\, d\mathbf{x}\\ &= \int_{B_r}|\nabla u - \nabla v|^2\, d\mathbf{x}.\end{aligned}$$

If we use this in (3.1), we find

$$\int_{B_r(\mathbf{x}_0)} |\nabla u - \nabla v|^2\, d\mathbf{x} \leq \int_{B_r(\mathbf{x}_0)} \chi_{\{u=0\}}(\lambda - y)_+\, d\mathbf{x}. \tag{3.2}$$

We want to control the right-hand side of the previous inequality by the left-hand side.

Step 2: In this step, for simplicity in the notation, we take $\mathbf{x}_0$ to be the origin. For $|\mathbf{z}| \leq \frac{1}{2}r$ consider the transformation $T : B_r \to B_r$ defined by

$$T(\mathbf{x}):\ = \left(1 - \frac{|\mathbf{x}|}{r}\right)\mathbf{z} + \mathbf{x}, \quad \mathbf{x} \in B_r. \tag{3.3}$$

Note that $T(\mathbf{0}) = \mathbf{z}$. For $\mathbf{x} \in B_r$ define

$$u_{\mathbf{z}}(\mathbf{x}) := u(T(\mathbf{x})), \quad v_{\mathbf{z}}(\mathbf{x}) := v(T(\mathbf{x})). \tag{3.4}$$

Consider the set

$$E := \left\{\mathbf{q} \in \partial B_1 :\ u_{\mathbf{z}}(\rho\mathbf{q}) = 0 \text{ for some } \rho \in \left[\frac{1}{8}r, r\right]\right\}$$

and for $\mathbf{q} \in E$ set

$$\rho_{\mathbf{q}} := \inf\left\{\rho \in \left[\frac{1}{8}r, r\right] :\ u_{\mathbf{z}}(\rho\mathbf{q}) = 0\right\}.$$

By a slicing argument for $\mathcal{H}^1$ a.e. $\mathbf{q}\in E$ the function

$$g(\rho):=u_{\mathbf{z}}(\rho\mathbf{q})-v_{\mathbf{z}}(\rho\mathbf{q}),\quad 0<\rho\leq r,$$

is absolutely continuous in $(0,r)$, and so, using also the facts that $g(r)=0$ (since $u=v$ on ∂B_r) and that $u_{\mathbf{z}}(\rho_{\mathbf{q}}\mathbf{q})=0$, we have

$$v_{\mathbf{z}}(\rho_{\mathbf{q}}\mathbf{q})=g(r)-g(\rho_{\mathbf{q}})=\int_{\rho_{\mathbf{q}}}^{r}g'(\rho)\,d\rho=\int_{\rho_{\mathbf{q}}}^{r}\nabla(u_{\mathbf{z}}-v_{\mathbf{z}})(\rho\mathbf{q})\cdot\mathbf{q}\,d\rho.$$

Using Holder's inequality we have

$$v_{\mathbf{z}}(\rho_{\mathbf{q}}\mathbf{q})\leq\sqrt{r-\rho_{\mathbf{q}}}\left(\int_{\rho_{\mathbf{q}}}^{r}\left|\nabla v_{\mathbf{z}}-\nabla u_{\mathbf{z}}\right|^2(\rho\mathbf{q})\,d\rho\right)^{\frac{1}{2}},$$

while, by Poisson's formula for the harmonic function v,

$$\begin{aligned}v_{\mathbf{z}}(\rho_{\mathbf{q}}\mathbf{q})=v(T(\rho_{\mathbf{q}}\mathbf{q}))&=\frac{r^2-\left|T(\rho_{\mathbf{q}}\mathbf{q})\right|^2}{2\pi r}\int_{\partial B_r}\frac{u(\mathbf{y})}{\left|T(\rho_{\mathbf{q}}\mathbf{q})-\mathbf{y}\right|^2}\,d\mathcal{H}^1(\mathbf{y})\\&\geq\frac{r^2-\left|T(\rho_{\mathbf{q}}\mathbf{q})\right|^2}{2\pi r\left(r+\left|T(\rho_{\mathbf{q}}\mathbf{q})\right|\right)^2}\int_{\partial B_r}u(\mathbf{y})\,d\mathcal{H}^1(\mathbf{y})\\&=\frac{r-\left|T(\rho_{\mathbf{q}}\mathbf{q})\right|}{r+\left|T(\rho_{\mathbf{q}}\mathbf{q})\right|}\frac{1}{\left|\partial B_r\right|}\int_{\partial B_r}u(\mathbf{y})\,d\mathcal{H}^1(\mathbf{y})\\&\geq\frac{1}{2}\frac{r-\rho_{\mathbf{q}}}{r}\frac{1}{\left|\partial B_r\right|}\int_{\partial B_r}u(\mathbf{y})\,d\mathcal{H}^1(\mathbf{y}),\end{aligned}$$

where we have used the facts that $\left|T(\rho_{\mathbf{q}}\mathbf{q})\right|=\left|(r-\rho_{\mathbf{q}})\frac{\mathbf{z}}{r}+\rho_{\mathbf{q}}\mathbf{q}\right|\leq\rho_{\mathbf{q}}+\frac{1}{2}(r-\rho_{\mathbf{q}})$, since $|\mathbf{z}|\leq\frac{1}{2}r$ and $|\mathbf{q}|=1$. Squaring and combining the two inequalities we obtain

$$\begin{aligned}(r-\rho_{\mathbf{q}})\left(\frac{1}{r\left|\partial B_r\right|}\int_{\partial B_r}u(\mathbf{y})\,d\mathcal{H}^1(\mathbf{y})\right)^2&\leq C\int_{\frac{r}{8}}^{r}\left|\nabla v_{\mathbf{z}}-\nabla u_{\mathbf{z}}\right|^2(\rho\mathbf{q})\,d\rho\\&\leq\frac{C}{r}\int_{\frac{r}{8}}^{r}\left|\nabla v_{\mathbf{z}}-\nabla u_{\mathbf{z}}\right|^2(\rho\mathbf{q})\rho\,d\rho.\end{aligned}\tag{3.5}$$

Integrating the previous inequality in $\mathbf{q}$ over E yields

$$\begin{aligned}&\int_E(r-\rho_{\mathbf{q}})\left(\frac{1}{r\left|\partial B_r\right|}\int_{\partial B_r}u(\mathbf{y})\,d\mathcal{H}^1(\mathbf{y})\right)^2d\mathcal{H}^1(\mathbf{q})\\&\quad\leq\frac{C}{r}\int_{\partial B_1}\int_{\frac{r}{8}}^{r}\left|\nabla v_{\mathbf{z}}-\nabla u_{\mathbf{z}}\right|^2(\rho\mathbf{q})\rho\,d\rho d\mathcal{H}^1(\mathbf{q})\\&\quad=\frac{C}{r}\int_{B_r\setminus B_{r/8}}\left|\nabla v_{\mathbf{z}}-\nabla u_{\mathbf{z}}\right|(\mathbf{x})^2\,d\mathbf{x},\end{aligned}\tag{3.6}$$

where we have used the fact that $\rho_{\mathbf{q}}$ is bounded from below by $\frac{1}{8}r$. Since the Jacobian of T can be bounded independently of r (see (3.3) and (3.4)), changing variable on the right-hand side gives

$$\begin{aligned}&\int_E(r-\rho_{\mathbf{q}})\,d\mathcal{H}^1(\mathbf{q})\left(\frac{1}{r\left|\partial B_r\right|}\int_{\partial B_r}u(\mathbf{y})\,d\mathcal{H}^1(\mathbf{y})\right)^2\\&\qquad\leq C\frac{1}{r}\int_{B_r}\left|\nabla v-\nabla u\right|^2(\mathbf{x})\,d\mathbf{x}.\end{aligned}$$

Now

$$\begin{aligned}\int_{B_r(\mathbf{0})\setminus B_{\frac{r}{8}}(\mathbf{z})} \chi_{\{u_{\mathbf{z}}=0\}}(\mathbf{x})\, d\mathbf{x} &= \int_{\partial B_1}\int_{r/8}^{r} \chi_{\{u_{\mathbf{z}}=0\}}(\rho\mathbf{q})\rho\, d\rho d\mathcal{H}^1(\mathbf{q}) \\ &= \int_{E}\int_{r/8}^{r} \chi_{\{u_{\mathbf{z}}=0\}}(\rho\mathbf{q})\rho\, d\rho d\mathcal{H}^1(\mathbf{q}) \\ &\leq r\int_E (r-\rho_{\mathbf{q}})\, d\mathcal{H}^1(\mathbf{q}).\end{aligned}$$

Therefore,

$$\begin{aligned}&\left(\int_{B_r(\mathbf{0})\setminus B_{\frac{r}{8}}(\mathbf{z})} \chi_{\{u_{\mathbf{z}}=0\}}\, d\mathbf{x}\right)\left(\frac{1}{r\,|\partial B_r|}\int_{\partial B_r} u\, d\mathcal{H}^1\right)^2 \\ &\leq Cr\int_E (r-\rho_{\mathbf{q}})\, d\mathcal{H}^1\left(\frac{1}{r\,|\partial B_r|}\int_{\partial B_r} u\, d\mathcal{H}^1\right)^2 \\ &\leq C\int_{B_r} |\nabla v-\nabla u|^2\, d\mathbf{x}.\end{aligned}$$

We now need to replace $u_{\mathbf{z}}$ by u on the left-hand side of the previous inequality. We begin by showing that

$$B_r(\mathbf{0})\setminus B_{\frac{3}{16}r}(\mathbf{z}) \subseteq T\left(B_r(\mathbf{0})\setminus B_{\frac{1}{8}r}(\mathbf{z})\right)$$

or, equivalently,

$$T\left(B_{\frac{1}{8}r}(\mathbf{z})\right) \subseteq B_{\frac{3}{16}r}(\mathbf{z}).$$

Indeed,

$$|T(\mathbf{x})-\mathbf{z}| = \left|\left(1-\frac{|\mathbf{x}|}{r}\right)\mathbf{z}+\mathbf{x}-\mathbf{z}\right| \leq |\mathbf{x}|\left(\frac{|\mathbf{z}|}{r}+1\right) \leq \frac{r}{8}\left(\frac{1}{2}+1\right) = \frac{3}{16}r.$$

Hence, by the change of variables $\mathbf{x}=T^{-1}(\mathbf{y})$, we get

$$\begin{aligned}\int_{B_r\setminus B_{\frac{r}{8}}} \chi_{\{u_{\mathbf{z}}=0\}}(\mathbf{x})\, d\mathbf{x} &= \int_{T\left(B_r\setminus B_{\frac{r}{8}}\right)} \chi_{\{u_z=0\}}(T^{-1}(\mathbf{y}))J\left(T^{-1}(\mathbf{y})\right)\, d\mathbf{y} \\ &= \int_{T\left(B_r\setminus B_{\frac{r}{8}}\right)} \chi_{\{u=0\}}(\mathbf{y})J\left(T^{-1}(\mathbf{y})\right)\, d\mathbf{y} \\ &\geq c\int_{T\left(B_r\setminus B_{\frac{r}{8}}\right)} \chi_{\{u=0\}}(\mathbf{y})\, d\mathbf{y},\end{aligned} \tag{3.7}$$

where we used the fact that $J\left(T^{-1}(\mathbf{y})\right)$ is bounded from below by $\frac{1}{2}$, since the Jacobian of T is bounded by 2 (see (3.3)). Therefore,

$$\int_{B_r\setminus B_{\frac{3}{16}r}(\mathbf{z})} \chi_{\{u=0\}}\, d\mathbf{y}\left(\frac{1}{r\,|\partial B_r|}\int_{\partial B_r} u\, d\mathcal{H}^1\right)^2 \leq C\int_{B_r} |\nabla v-\nabla u|^2\, d\mathbf{x}.$$

If we write this inequality for two values of $\mathbf{z}$ such that $B_{\frac{3}{16}r}(\mathbf{z}_1)\cap B_{\frac{3}{16}r}(\mathbf{z}_2)=\emptyset$, say $\mathbf{z}_1=(\frac{r}{2},0)$, $\mathbf{z}_2=(-\frac{r}{2},0)$ and add the two relations, we have

$$\int_{B_r} \chi_{\{u=0\}}\, d\mathbf{y}\left(\frac{1}{r\,|\partial B_r|}\int_{\partial B_r} u\, d\mathcal{H}^1\right)^2 \leq C\int_{B_r} |\nabla v-\nabla u|^2\, d\mathbf{x}.$$

Using this inequality together with (3.2), we get

$$\int_{B_r} \chi_{\{u=0\}}\, d\mathbf{y} \left(\frac{1}{r\,|\partial B_r|} \int_{\partial B_r} u\, d\mathcal{H}^1 \right)^2 \le C \int_{B_r} |\nabla v - \nabla u|^2 \, d\mathbf{x} \le \int_{B_r} \chi_{\{u=0\}} (\lambda - y)_+ \, d\mathbf{x}. \tag{3.8}$$

Step 3: If $B_r(\mathbf{x}_0) \subseteq \{y \ge \lambda\}$, then by (3.2),

$$\int_{B_r(\mathbf{x}_0)} |\nabla v - \nabla u|^2 \, d\mathbf{x} = 0,$$

and so $u = v > 0$ $\mathcal{L}^2$ a.e. in $B_r(\mathbf{x}_0)$. If $B_r(\mathbf{x}_0) \cap \{y < \lambda\} \neq \emptyset$, then by (3.8),

$$\begin{aligned}\int_{B_r(\mathbf{x}_0)} \chi_{\{u=0\}}\, d\mathbf{y} &\left(\frac{1}{r\,|\partial B_r(\mathbf{x}_0)|} \int_{\partial B_r(\mathbf{x}_0)} u\, d\mathcal{H}^1 \right)^2 d\mathbf{y} \\ &\le C \int_{B_r(\mathbf{x}_0)} |\nabla v - \nabla u|^2 \, d\mathbf{x} \\ &\le C(\lambda - y_0 + r)_+ \int_{B_r(\mathbf{x}_0)} \chi_{\{u=0\}} \, d\mathbf{x}.\end{aligned}$$

Hence, if

$$\frac{1}{r\,|\partial B_r(\mathbf{x}_0)|} \int_{\partial B_r(\mathbf{x}_0)} u\, d\mathcal{H}^1 > \sqrt{C(\lambda - y_0 + r)_+},$$

then, necessarily,

$$\int_{B_r(\mathbf{x}_0)} \chi_{\{u=0\}} \, d\mathbf{x} = 0,$$

and so, again

$$\int_{B_r(\mathbf{x}_0)} |\nabla v - \nabla u|^2 \, d\mathbf{x} = 0$$

and we proceed as before to conclude $u = v > 0$ in $B_r(\mathbf{x}_0)$.

We denote $C_{\max} := \sqrt{C}$. □

Remark 3.2. (i) The previous theorem implies that if $B_r(\mathbf{x}_0)$ intersects the free boundary $\partial\{u > 0\}$, then

$$\frac{1}{r\,|\partial B_r(\mathbf{x}_0)|} \int_{\partial B_r(\mathbf{x}_0)} u\, d\mathcal{H}^1 \le C_{\max} \sqrt{(\lambda - y_0 + r)_+}.$$

In particular, if $\lambda - y_0 \le r$, then

$$\frac{1}{|\partial B_r(\mathbf{x}_0)|} \int_{\partial B_r(\mathbf{x}_0)} u\, d\mathcal{H}^1 \le C_{\max} r^{\frac{3}{2}}.$$

(ii) It follows from the previous theorem that if $u(\mathbf{x}_0) > 0$ for some $\mathbf{x}_0 = (x_0, y_0)$ with $y_0 > \lambda$, then u is positive and harmonic in the whole set $(-\ell, \ell) \times (\lambda, \infty)$.

Theorem 3.3. *Let $u \in \mathcal{K}$ be a local (respectively, an absolute) minimizer of J_λ. Then the set $\{u > 0\}$ is open and u is harmonic in $\{u > 0\}$. Moreover, if $u_0 \in C^1(\overline{\Omega})$ and the set $\{\mathbf{x} \in S : u_0(\mathbf{x}) > 0\}$ is connected, then the set $\{u > 0\}$ has finitely many connected components (respectively, is connected).*

Proof. Fix $\mathbf{x}_0 \in \Omega$ such that $u(\mathbf{x}_0) > 0$. If $\mathbf{x}_0 = (x_0, y_0)$, where $y_0 > \lambda$, then we can find a small ball $B_r(\mathbf{x}_0) \subseteq \{y > \lambda\}$, and in view of the last remark, u is positive and harmonic in this ball. Thus in what follows it suffices to assume that $y_0 \leq \lambda$.

By Remark 2.4, for all r sufficiently small

$$\frac{1}{|B_r(\mathbf{x}_0)|}\int_{B_r(\mathbf{x}_0)} u\,d\mathbf{y} > \frac{1}{2}u(\mathbf{x}_0) > 0. \tag{3.9}$$

Fix $r > 0$ so small that (3.9) holds and such that

$$\frac{u(\mathbf{x}_0)}{r} > C_{\max}\sqrt{\lambda - y_0 + r}, \tag{3.10}$$

and define

$$g(\rho) := \int_{\partial B_\rho(\mathbf{x}_0)} u\,d\mathcal{H}^1, \quad 0 < \rho \leq r.$$

Using a slicing argument we have that

$$\frac{1}{r}\int_0^r \frac{1}{\pi r} g(\rho)\,d\rho = \frac{1}{\pi r^2}\int_{B_r(\mathbf{x}_0)} u\,d\mathbf{y} > \frac{1}{2}u(\mathbf{x}_0) > 0,$$

and thus we may find some $0 < \rho < r$ such that $\frac{1}{\pi r}g(\rho) > \frac{1}{2}u(\mathbf{x}_0)$. In turn, since $\frac{1}{\rho} > \frac{1}{r}$,

$$\frac{1}{\pi\rho}\int_{\partial B_\rho(\mathbf{x}_0)} u\,d\mathcal{H}^1 > \frac{1}{2}u(\mathbf{x}_0) > 0.$$

Hence, also by (3.10) and the fact that $\rho < r$, we have

$$\frac{1}{\rho\,|\partial B_\rho(\mathbf{x}_0)|}\int_{\partial B_\rho(\mathbf{x}_0)} u\,d\mathcal{H}^1 \geq \frac{u(\mathbf{x}_0)}{\rho} > C_{\max}\sqrt{\lambda - y_0 + r},$$

It follows by the previous theorem that $u > 0$ in $B_\rho(\mathbf{x}_0)$ and u is harmonic in $B_\rho(\mathbf{x}_0)$.

To prove the last part of the theorem, assume that $u \in \mathcal{K}$ is a local minimizer of J_λ and let $\{A_\alpha\}_\alpha$ be the connected components of the open set $\{u > 0\}$. Assume by contradiction that there exist countably many connected components $\{A_{\alpha_n}\}_{n\in\mathbb{N}}$ whose boundaries do not intersect $\{\mathbf{x} \in S : u_0(\mathbf{x}) > 0\}$. In view of part (ii) of the previous remark, without loss of generality, we may assume that $A_{\alpha_n} \subseteq (-\ell, \ell) \times (0, \lambda)$ for all $n \geq 2$. Let $l < \lambda$ be such that $2\ell(\lambda - l) < \frac{1}{3}\varepsilon_0$, where $\varepsilon_0 > 0$ is the number given in Definition 2.1.

Since

$$\sum_{n=2}^{\infty}\int_{A_{\alpha_n}} \left(|\nabla u|^2 + \chi_{\{u>0\}}(\lambda - y)_+\right) d\mathbf{x} \leq \int_\Omega \left(|\nabla u|^2 + \chi_{\{u>0\}}(\lambda - y)_+\right) d\mathbf{x} < \infty,$$

we may find $m \in \mathbb{N}$ such that

$$\sum_{n=m+1}^{\infty}\int_{A_{\alpha_n}} |\nabla u|^2\,d\mathbf{x} \leq \left(\frac{\varepsilon_0}{3}\right)^2, \quad \sum_{n=m+1}^{\infty}\int_{A_{\alpha_n}} \chi_{\{u>0\}}(\lambda - y)_+\,d\mathbf{x} \leq \frac{\varepsilon_0(\lambda - l)}{3}.$$

Then

$$\begin{aligned}\sum_{n=m+1}^{\infty}\int_{A_{\alpha_n}} \chi_{\{u>0\}}\,d\mathbf{x} &\leq \sum_{n=m+1}^{\infty}\frac{1}{\lambda - l}\int_{A_{\alpha_n}\cap\{y<l\}} \chi_{\{u>0\}}(\lambda - y)_+\,d\mathbf{x} \\ &\quad + 2\ell(\lambda - l) \leq \frac{\varepsilon_0}{3} + \frac{\varepsilon_0}{3}.\end{aligned}$$

Hence, if we define

$$v(\mathbf{x}) := \begin{cases} 0 & \text{if } \mathbf{x} \in A_{\alpha_n},\, n \geq m+1, \\ u(\mathbf{x}) & \text{otherwise in } \Omega, \end{cases}$$

we have that $v \in \mathcal{K}$ satisfies (2.3). Since $J_\lambda(v) < J_\lambda(u)$, we have reached a contradiction.

The proof in the case of absolute minimizers is similar but simpler, so we omit it. □

The next result is based on Corollary 3.3 in [AC].

Theorem 3.4. *Let $u \in \mathcal{K}$ be a local minimizer of J_λ. Then u is locally Lipschitz.*

Proof. Fix $\varepsilon > 0$ and let $\Omega_\varepsilon := (-\ell+\varepsilon, \ell-\varepsilon) \times \left(\varepsilon, \frac{1}{\varepsilon}\right)$. We claim that u is Lipschitz in Ω_ε. To see this, fix $\mathbf{x}_0 \in \Omega_\varepsilon$ such that $u(\mathbf{x}_0) > 0$. Since the set $\{u > 0\} \cap \Omega_\varepsilon$ is open, we may find $B_r(\mathbf{x}_0) \subset \{u > 0\} \cap \Omega_\varepsilon$.

Let

$$\rho_\varepsilon(\mathbf{x}_0) := \sup\{r > 0 : \, B_r(\mathbf{x}_0) \subset \{u > 0\} \cap \Omega_\varepsilon\}. \tag{3.11}$$

By the previous theorem for any $r < \rho_\varepsilon$, u is harmonic in $B_r(\mathbf{x}_0)$. In turn $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are also harmonic in $B_r(\mathbf{x}_0)$, and so, by the mean value and divergence theorems,

$$\frac{\partial u}{\partial x}(\mathbf{x}_0) = \frac{1}{\pi r^2} \int_{B_r(\mathbf{x}_0)} \frac{\partial u}{\partial x}(\mathbf{y})\, d\mathbf{y} = \frac{1}{\pi r^2} \int_{\partial B_r(\mathbf{x}_0)} u \upsilon_1 \, d\mathcal{H}^1, \tag{3.12}$$

where

$$\upsilon(\mathbf{x}) = (\upsilon_1, \upsilon_2) = \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|}.$$

Similarly,

$$\frac{\partial u}{\partial y}(\mathbf{x}_0) = \frac{1}{\pi r^2} \int_{\partial B_r(\mathbf{x}_0)} u \upsilon_2 \, d\mathcal{H}^1. \tag{3.13}$$

There are now two cases. If $B_{\rho_\varepsilon}(\mathbf{x}_0)$ touches $\partial\Omega_\varepsilon$, then by Lemma 2.5 we get

$$\left|\frac{\partial u}{\partial x}(\mathbf{x}_0)\right|, \left|\frac{\partial u}{\partial y}(\mathbf{x}_0)\right| \leq \frac{\sup u_0}{\pi d_\varepsilon(\mathbf{x}_0)}, \tag{3.14}$$

where $d_\varepsilon(\mathbf{x}_0) := \operatorname{dist}(\mathbf{x}_0, \partial\Omega_\varepsilon)$.

A similar estimate holds if $\rho_\varepsilon(\mathbf{x}_0) \geq \frac{1}{8} d_\varepsilon(\mathbf{x}_0)$ (replacing $\sup u_0$ with $8 \sup u_0$ in (3.14)). If $\rho_\varepsilon(\mathbf{x}_0) < \frac{1}{8} d_\varepsilon(\mathbf{x}_0)$, then $B_{\rho_\varepsilon}(\mathbf{x}_0)$ touches $\partial\{u > 0\}$ at some point $\mathbf{x}_1$ in which $u(\mathbf{x}_1) = 0$. Let $r := \frac{1}{4}\rho_\varepsilon(\mathbf{x}_0)$. If $\mathbf{y} \in \partial B_r(\mathbf{x}_0)$, then, since u is subharmonic in $B_s(\mathbf{y})$, where $\frac{5}{4}\rho_\varepsilon(\mathbf{x}_0) < s \leq \frac{3}{2}\rho_\varepsilon(\mathbf{x}_0)$, by Poisson's formula we have

$$u(\mathbf{y}) \leq \frac{1}{|\partial B_s(\mathbf{y})|} \int_{\partial B_s(\mathbf{y})} u \, d\mathcal{H}^1.$$

Since $|\mathbf{x}_1 - \mathbf{y}| \leq |\mathbf{x}_1 - \mathbf{x}_0| + |\mathbf{x}_0 - \mathbf{y}|$, then $\mathbf{x}_1 \in B_s(\mathbf{y})$ and so, by Remark 3.2,

$$\frac{1}{s\, |\partial B_s(\mathbf{y})|} \int_{\partial B_s(\mathbf{y})} u \, d\mathcal{H}^1 \leq C_{\max} \sqrt{(\lambda - y + s)_+},$$

where $\mathbf{y} = (x, y)$. Hence,

$$u(\mathbf{y}) \leq C_{\max} s \sqrt{(\lambda - y + s)}.$$

Thus, from formulas (3.12) and (3.13), we get

$$\begin{aligned}\left|\frac{\partial u}{\partial x}(\mathbf{x}_0)\right| &\le \frac{C_{\max} s}{\pi r^2}\int_{\partial B_r(\mathbf{x}_0)} \sqrt{(\lambda-y+s)_+}\,d\mathcal{H}^1(x,y)\\ &\le \frac{C_{\max}\frac{3}{2}\rho_\varepsilon(\mathbf{x}_0)}{\pi\frac{1}{4}\rho_\varepsilon(\mathbf{x}_0)r}\int_{\partial B_r(\mathbf{x}_0)} \sqrt{(\lambda-y+s)_+}\,d\mathcal{H}^1(x,y)\\ &\le 12C_{\max}\sqrt{(\lambda-y_0+r+s)_+}\\ &\le 12C_{\max}\sqrt{(\lambda-y_0+2\rho_\varepsilon(\mathbf{x}_0))_+}.\end{aligned} \tag{3.15}$$

Similarly,

$$\left|\frac{\partial u}{\partial y}(\mathbf{x}_0)\right| \le 12C_{\max}\sqrt{(\lambda-y_0+2\rho_\varepsilon(\mathbf{x}_0))_+}. \tag{3.16}$$

Since $\rho_\varepsilon(\mathbf{x}_0) < \frac{1}{8}d_\varepsilon(\mathbf{x}_0)$, also by (3.14) we obtain

$$|\nabla u(\mathbf{x}_0)| \le \max\left\{\frac{\sup u_0}{\pi d_\varepsilon(\mathbf{x}_0)}, C\sqrt{\lambda+d_\varepsilon(\mathbf{x}_0)}\right\}.$$

Since $\nabla u(\mathbf{x}) = 0$ for $\mathcal{L}^2$ a.e. $\mathbf{x}\in\Omega_\varepsilon$ such that $u(\mathbf{x}) = 0$, we have proved that u is Lipschitz in Ω_ε.

If now $\Omega' \subset\subset \Omega$ we can choose ε so small that $\Omega' \subset \Omega_\varepsilon$. Then $|\nabla u| \le C(\Omega_\varepsilon)$, and so u is locally Lipschitz continuous. □

Remark 3.5. (i) It follows from the previous proof that if $\mathbf{x}_0 = (x_0, y_0) \in \partial\{u>0\}$ and $r>0$ is sufficiently small, then for every $\mathbf{x} = (x,y)\in B_r(\mathbf{x}_0)$,

$$\lambda - y_0 + r \le \lambda - y + 2r \le \lambda - y_0 + 3r,$$

and so from (3.15) and (3.16),

$$|\nabla u(x,y)| \le C\sqrt{(\lambda-y_0+3r)_+}$$

for all $(x,y)\in B_r(\mathbf{x}_0)$.

(ii) Note that if $\{\mathbf{x}\in\Omega : u(\mathbf{x})>0\} \subset (-\ell,\ell)\times(0,\lambda)$, then again by the previous proof, for every $\mathbf{x}_0 = (x_0,y_0)\in\Omega_\varepsilon$ such that $u(\mathbf{x}_0)>0$ and $\rho_\varepsilon(\mathbf{x}_0) < \frac{1}{8}d_\varepsilon(\mathbf{x}_0)$, we have that $\rho_\varepsilon(\mathbf{x}_0)\le\lambda-y_0$, and so by (3.15) and (3.16),

$$|\nabla u(x_0,y_0)| \le C\sqrt{3(\lambda-y_0)}.$$

Next we adapt the proof of Lemma 3.4 of [AC] to our setting.

Theorem 3.6. *For every $k\in(0,1)$ there exists a positive constant $C(k)$ such that for every local minimizer $u\in\mathcal{K}$ of J_λ and for every (small) ball $B_r(\mathbf{x}_0)\subset\Omega$, if*

$$\frac{1}{r\,|\partial B_r(\mathbf{x}_0)|}\int_{\partial B_r(\mathbf{x}_0)} u\,d\mathcal{H}^1 \le C(k)\sqrt{(\lambda-y_0-kr)_+}, \tag{3.17}$$

then $u=0$ in $B_{kr}(\mathbf{x}_0)$.

Proof. The idea of the proof is that if the average of u on $\partial B_r(\mathbf{x}_0)$ is small, then replacing u by a function w vanishing in $B_{kr}(\mathbf{x}_0)$ will decrease J_λ. All the balls used in this proof are centered at $\mathbf{x}_0$, therefore, for simplicity, we write B_r for $B_r(\mathbf{x}_0)$.

Step 1: In this step we find a lower bound for $\int_{\partial B_{kr}} u\,d\mathcal{H}^1$. Define

$$v(\mathbf{x}) := \frac{\ell_u \sqrt{k}}{\log\left(\frac{1}{\sqrt{k}}\right)} \max\left\{\log\frac{|\mathbf{x}-\mathbf{x}_0|}{kr}, 0\right\}, \quad \mathbf{x} \in B_r,$$

where

$$\ell_u := \frac{1}{\sqrt{k}} \sup_{B_{r\sqrt{k}}} u. \tag{3.18}$$

Note that

$$\nabla v(\mathbf{x}) = \begin{cases} \frac{\ell_u\sqrt{k}}{\log\left(\frac{1}{\sqrt{k}}\right)} \frac{\mathbf{x}-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|^2} & \text{if } |\mathbf{x}-\mathbf{x}_0| > kr, \\ 0 & \text{if } |\mathbf{x}-\mathbf{x}_0| \le kr. \end{cases} \tag{3.19}$$

We claim that

$$\ell_u \le C_1(k) \frac{1}{|\partial B_r|} \int_{\partial B_r} u\,d\mathcal{H}^1 \tag{3.20}$$

for some constant $C_1(k) > 0$ depending only on k. To see this, let V be the harmonic function equal to u on $\partial B_r(x_0)$. By Harnack 's inequality,

$$\begin{aligned}\sup_{B_{r\sqrt{k}}} V &\le C_1(k) \inf_{B_{r\sqrt{k}}} V \le C_1(k)\, V(\mathbf{x}_0) = C_1(k) \frac{1}{|\partial B_r|} \int_{\partial B_r} V\,d\mathcal{H}^1 \\ &= C_1(k) \frac{1}{|\partial B_r|} \int_{\partial B_r} u\,d\mathcal{H}^1.\end{aligned}$$

Since u is subharmonic, we have that $u \le V$, and so, possibly changing $C_1(k)$, we obtain

$$\frac{1}{\sqrt{k}} \sup_{B_{r\sqrt{k}}} u \le \frac{1}{\sqrt{k}} \sup_{B_{r\sqrt{k}}} V \le C_1(k) \frac{1}{|\partial B_r(\mathbf{x}_0)|} \int_{\partial B_r(\mathbf{x}_0)} u\,d\mathcal{H}^1,$$

which proves (3.20).

Define now

$$w := \begin{cases} \min\{u, v\} & \text{in } B_{\sqrt{k}r}, \\ u & \text{outside } B_{\sqrt{k}r}. \end{cases}$$

Since $w \in \mathcal{K}$, we have $J_\lambda(u) \le J_\lambda(w)$, which implies that

$$\int_{B_{\sqrt{k}r}} \left(|\nabla u|^2 + \chi_{\{u>0\}}(\lambda - y)_+\right) d\mathbf{x} \le \int_{B_{\sqrt{k}r}} \left(|\nabla w|^2 + \chi_{\{w>0\}}(\lambda - y)_+\right) d\mathbf{x}.$$

Notice that $w = 0$ in B_{kr} and outside this ball $w = 0$ whenever $u = 0$. Hence,

$$\begin{aligned}\int_{B_{kr}} \left(|\nabla u|^2 + \chi_{\{u>0\}}(\lambda - y)_+\right) d\mathbf{x} &\le \int_{B_{\sqrt{k}r}\setminus B_{kr}} \left(|\nabla w|^2 - |\nabla u|^2\right) d\mathbf{x} \\ &\le 2\int_{B_{\sqrt{k}r}\setminus B_{kr}} (\nabla w - \nabla u)\cdot \nabla w\,d\mathbf{x},\end{aligned} \tag{3.21}$$

where the last inequality follows from the fact that if we move all terms to the right-hand side we obtain a perfect square. On the other hand, since v is harmonic,

$$\begin{aligned}0 &= \int_{B_{\sqrt{k}r}\setminus B_{kr}} (w-u)\,\Delta v\,d\mathbf{x} \\ &= -\int_{B_{\sqrt{k}r}\setminus B_{kr}} (\nabla w - \nabla u)\cdot\nabla v\,dx + \int_{\partial\left(B_{\sqrt{k}r}\setminus B_{kr}\right)} (w-u)\,\nabla v\cdot\nu\,d\mathcal{H}^1.\end{aligned} \tag{3.22}$$

Therefore,

$$\begin{aligned}\int_{B_{\sqrt{k}r}\setminus B_{kr}} (\nabla w - \nabla u)\cdot \nabla v \, d\mathbf{x} &= \int_{\partial\left(B_{\sqrt{k}r}\setminus B_{kr}\right)} (w-u)\,\nabla v\cdot \nu \, d\mathcal{H}^1 = \\ &= -\int_{\partial B_{kr}} u(\nabla v\cdot\nu)\, d\mathcal{H}^1 \\ &\leq \frac{C_2(k)\ell_u}{r}\int_{\partial B_{kr}} u\, d\mathcal{H}^1,\end{aligned} \tag{3.23}$$

where we have used the fact that $w = u$ on ∂B_{kr} and (3.19) and where $C_2(k) > 0$ depends only on k. We also have

$$\int_{B_{\sqrt{k}r}\setminus B_{kr}} (\nabla w - \nabla u)\cdot \nabla w \, d\mathbf{x} = \int_{B_{\sqrt{k}r}\setminus B_{kr}} (\nabla w - \nabla u)\cdot \nabla v \, d\mathbf{x}. \tag{3.24}$$

To see this, we split $B_{\sqrt{k}r}\setminus B_{kr}$ in the two sets $\{u \geq v\}$ and $\{u < v\}$. When $u \geq v$, we have that $w = v$, therefore $\nabla w = \nabla v$ $\mathcal{L}^2$ a.e. in $\{u \geq v\}$. When $u < v$, we have that $w = u$, and so both integrals over the set $\{u < v\}$ are 0.

We conclude from (3.21), (3.22), (3.23), and (3.24) that

$$\int_{B_{kr}} \left(|\nabla u|^2 + \chi_{\{u>0\}}(\lambda - y)_+\right) d\mathbf{x} \leq \frac{C_2(k)\ell_u}{r}\int_{\partial B_{kr}} u\, d\mathcal{H}^1. \tag{3.25}$$

Step 2: Define

$$Q_{\min} := \sqrt{(\lambda - y_0 - kr)_+}. \tag{3.26}$$

If $Q_{\min} = 0$, then the result follows immediately from (3.17) and (3.20). Thus, assume that $Q_{\min} > 0$.

By the trace theorem, (3.18), Young's inequality, and the fact that $Q_{\min} \leq \sqrt{(\lambda - y)_+}$ in B_{kr}, we have

$$\begin{aligned}\int_{\partial B_{kr}} u\, d\mathcal{H}^1 &\leq C_3(k)\left(\frac{1}{r}\int_{B_{kr}} u\, d\mathbf{x} + \int_{B_{kr}} |\nabla u|\, d\mathbf{x}\right) \\ &= C_3(k)\left(\frac{1}{r}\int_{B_{kr}} u\chi_{\{u>0\}}\, d\mathbf{x} + \int_{B_{kr}} |\nabla u|\,\chi_{\{u>0\}}\, d\mathbf{x}\right) \\ &\leq C_3(k)\left[\frac{\ell_u}{rQ_{\min}^2}\int_{B_{kr}} \chi_{\{u>0\}}(\lambda - y)_+\, d\mathbf{x}\right. \\ &\quad \left. + \frac{1}{Q_{\min}}\left(\int_{B_{kr}} |\nabla u|^2\, d\mathbf{x} + \int_{B_{kr}} \chi_{\{u>0\}}(\lambda - y)_+\, d\mathbf{x}\right)\right] \\ &\leq \frac{C_3(k)}{Q_{\min}}\left(\frac{\ell_u}{rQ_{\min}} + 1\right)\int_{B_{kr}} \left(|\nabla u|^2 + \chi_{\{u>0\}}(\lambda - y)_+\right) d\mathbf{x}.\end{aligned}$$

Combined with (3.25), the previous inequality yields

$$\begin{aligned}&\int_{B_{kr}} \left(|\nabla u|^2 + \chi_{\{u>0\}}(\lambda - y)_+\right) d\mathbf{x} \\ &\leq \frac{C_2(k)C_3(k)\ell_u}{rQ_{\min}}\left(\frac{\ell_u}{rQ_{\min}} + 1\right)\int_{B_{kr}} \left(|\nabla u|^2 + \chi_{\{u>0\}}(\lambda - y)_+\right) d\mathbf{x}.\end{aligned} \tag{3.27}$$

It follows from (3.20) that if

$$\frac{1}{rQ_{\min}}\frac{1}{|\partial B_r|}\int_{\partial B_r} u\, d\mathcal{H}^1 \leq a,$$

then

$$\frac{\ell_u}{rQ_{\min}} \leq \frac{C_1(k)}{rQ_{\min}\,|\partial B_r|}\int_{\partial B_r} u\, d\mathcal{H}^1 \leq C_1(k)a$$

and so

$$\frac{C_2(k)C_3(k)\ell_u}{rQ_{\min}}\left(\frac{\ell_u}{rQ_{\min}}+1\right) \leq C_2(k)C_3(k)a\left(C_1(k)a+1\right) < 1$$

for all $a > 0$ sufficiently small (depending only on k). Hence, by taking the constant $C(k)$ in (3.17) sufficiently small, we can ensure that

$$\frac{C_2(k)C_3(k)\ell_u}{rQ_{\min}}\left(\frac{\ell_u}{rQ_{\min}}+1\right) < 1. \tag{3.28}$$

This, together with inequality (3.27), implies that $u = 0$ in B_{kr}. □

Theorem 3.7. *Let $u \in \mathcal{K}$ be a local minimizer of the functional J_λ. Then for every $0 < l < \lambda$, $\partial\{u > 0\}$ is a C^∞ curve locally in $(-\ell, \ell) \times (0, l)$.*

Proof. Since in $(-\ell, \ell) \times (0, l)$,

$$Q(x, y) = \sqrt{\lambda - y} \geq \sqrt{\lambda - l} > 0,$$

we have that condition (1.2) is satisfied, and thus we are in a position to apply Theorem 8.4. in [AC]. □

4. Blow-up limits

Let $u \in \mathcal{K}$ be a local minimizer of J_λ and assume that the point $\mathbf{x}_0 = (x_0, \lambda) \in \Omega$ belongs to the free boundary $\partial\{u > 0\}$. By Remark 3.5(i) we have that

$$|\nabla u(\mathbf{x})| \leq C\sqrt{r}$$

for all $\mathbf{x} \in B_r(\mathbf{x}_0)$, where $r > 0$ is sufficiently small. Hence, if we define the rescaled functions

$$u_n(\mathbf{z}) := \frac{1}{\rho_n^{3/2}} u(\mathbf{x}_0 + \rho_n \mathbf{z}), \tag{4.1}$$

where $\rho_n \to 0^+$, for all n sufficiently large we have that for all $\mathbf{z} \in B_R(\mathbf{0})$, $R > 0$,

$$|\nabla u_n(\mathbf{z})| := \frac{1}{\rho_n^{1/2}} |\nabla u(\mathbf{x}_0 + \rho_n \mathbf{z})|.$$

Since $u_n(\mathbf{0}) = 0$, it follows by a diagonal argument that there exist a subsequence (not relabeled) of $\{u_n\}$ and a function $w \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^2)$ such that for all $R > 0$,

$$u_n \to w \quad \text{in } C^\alpha(B_R(\mathbf{0})) \quad \text{for all } 0 < \alpha < 1, \tag{4.2}$$

$$\nabla u_n \overset{*}{\rightharpoonup} \nabla w \quad \text{in } \left(L^\infty(B_R(\mathbf{0}))\right)^2. \tag{4.3}$$

We call w a *blow-up limit.*

Theorem 4.1. *Let $u \in \mathcal{K}$ be a local minimizer of J_λ and assume that the point $\mathbf{x}_0 = (x_0, \lambda) \in \Omega$ belongs to $\partial\{u > 0\}$. Let $\{u_n\}$ be defined as in (4.1). Then*

i) *$\partial\{u_n > 0\} \to \partial\{w > 0\}$ locally in Hausdorff distance in $\mathbb{R} \times (-\infty, 0)$;*
ii) *$\chi_{\{u_n > 0\}} \to \chi_{\{w > 0\}}$ in $L^1_{\mathrm{loc}}(\mathbb{R} \times (-\infty, 0))$.*

Proof. i) Let $B_r(\mathbf{z}_0) \subset \mathbb{R} \times (-\infty, 0)$ be such that $B_r(\mathbf{z}_0) \cap \partial \{w > 0\} = \emptyset$. Then either $w > 0$ in $B_r(\mathbf{z}_0)$ or $w \equiv 0$ in $B_r(\mathbf{z}_0)$. In the first case, since $u_n \to w$ in $C^\alpha\left(\bar{B}_{\frac{r}{2}}(\mathbf{z}_0)\right)$ and $\min_{\bar{B}_{\frac{r}{2}}(\mathbf{z}_0)} w > 0$, we have that $u_n > 0$ in $\bar{B}_{\frac{r}{2}}(\mathbf{z}_0)$ for all n sufficiently large, and so $B_{\frac{1}{2}r}(\mathbf{z}_0) \cap \partial \{u_n > 0\} = \emptyset$ for all n sufficiently large. If instead $w \equiv 0$ in $B_r(\mathbf{z}_0)$, then using the fact that $u_n \to w$ in $C^\alpha\left(\bar{B}_{\frac{3}{4}r}(\mathbf{z}_0)\right)$ we have that for all $\delta > 0$ there exists $N_\delta \in \mathbb{N}$ such that

$$|u_n(\mathbf{z})| \le \delta$$

for all $\mathbf{z} \in \bar{B}_{\frac{3}{4}r}(\mathbf{z}_0)$ and for all $n \ge N_\delta$. Hence, if $\delta = \delta(r)$ is sufficiently small,

$$\frac{1}{\frac{3}{4}r\left|\partial B_{\frac{3}{4}r}(\mathbf{z}_0)\right|}\int_{\partial B_{\frac{3}{4}r}(\mathbf{z}_0)} u_n \, d\mathcal{H}^1 \le \frac{4\delta}{3r} \le C(2/3)\sqrt{\frac{3}{4}r}$$

for all $n \ge N_\delta$. From the hypothesis and the definition of u_n we have

$$\frac{1}{\frac{3}{4}r\rho_n\left|\partial B_{\frac{3}{4}r\rho_n}(\mathbf{z}_0)\right|}\int_{\partial B_{\frac{3}{4}r}(\mathbf{z}_0)} u\left(\mathbf{x}_0 + \rho_n \mathbf{z}\right)\rho_n d\mathcal{H}^1(\mathbf{z}) \le C(2/3)\sqrt{\frac{3}{4}r\rho_n}$$

and by the change of variables $\mathbf{x} = \mathbf{x}_0 + \rho_n \mathbf{z}$, we get

$$\frac{1}{\frac{3}{4}r\rho_n\left|\partial B_{\frac{3}{4}r\rho_n}\left(\mathbf{x}_0 + \rho_n \mathbf{z}_0\right)\right|}\int_{\partial B_{\frac{3}{4}r\rho_n}(\mathbf{x}_0+\rho_n\mathbf{z}_0)} u(\mathbf{x})\, d\mathcal{H}^1(\mathbf{x}) \le C(2/3)\sqrt{\frac{3}{4}r\rho_n}.$$

It follows from Theorem 3.6 that $u \equiv 0$ in $B_{\frac{3}{4}r\rho_n}\left(\mathbf{x}_0 + \rho_n \mathbf{z}_0\right)$. Therefore, $u_n \equiv 0$ in $B_{\frac{3}{4}r}(\mathbf{z}_0)$. Hence, also in this case,

$$B_{\frac{1}{2}r}(\mathbf{z}_0) \cap \partial \{u_n > 0\} = \emptyset$$

for all n sufficiently large.

On the other hand, if $B_r(\mathbf{z}_0) \subset \mathbb{R} \times (-\infty, 0)$ is such that

$$B_r(\mathbf{z}_0) \cap \partial \{u_n > 0\} = \emptyset$$

for a subsequence, then, up to a further subsequence, we may assume that either $u_n > 0$ in $B_r(\mathbf{z}_0)$ for all $n \in \mathbb{N}$ or $u_n \equiv 0$ in $B_r(\mathbf{z}_0)$. In the first case, by Theorem 3.3, we have that u_n is harmonic in $B_r(\mathbf{z}_0)$, and so w is also harmonic and nonnegative. If w is zero in some point $\mathbf{z}_1 \in B_r(\mathbf{z}_0)$, then by the maximum principle, $w \equiv 0$ in $B_r(\mathbf{z}_0)$. Hence, either $w \equiv 0$ or $w > 0$ in $B_r(\mathbf{z}_0)$, which implies that

$$B_r(\mathbf{z}_0) \cap \partial \{w > 0\} = \emptyset. \tag{4.4}$$

Finally, if $u_n \equiv 0$ in $B_r(\mathbf{z}_0)$ for all $n \in \mathbb{N}$, then $w \equiv 0$ in $B_r(\mathbf{z}_0)$ and so (4.4) continues to hold.

A straightforward compactness argument shows that $\partial \{u_n > 0\} \to \partial \{w > 0\}$ locally in Hausdorff distance in $\mathbb{R} \times (-\infty, 0)$.

ii) We begin by proving that

$$\mathcal{L}^2\left((\mathbb{R} \times (-\infty, 0)) \cap \partial \{w > 0\}\right) = 0. \tag{4.5}$$

Fix $\mathbf{z}_0 \in (\mathbb{R} \times (-\infty, 0)) \cap \partial \{w > 0\}$, $\mathbf{z}_0 = (a_0, y_0)$. Since by part (i), $\partial \{u_n > 0\} \to \partial \{w > 0\}$ locally in Hausdorff distance in $\mathbb{R} \times (-\infty, 0)$, there exists $\mathbf{z}_n \in (\mathbb{R} \times (-\infty, 0)) \cap \partial \{u_n > 0\}$, $\mathbf{z}_n = (x_n, y_n)$, such that $\mathbf{z}_n \to \mathbf{z}_0$. By (4.1), $\mathbf{x}_0 + \rho_n \mathbf{z}_n$ belongs to

$\partial\{u>0\}$ and since $y_0<0$, we have that $\lambda+\rho_n y_n<\lambda$ for all n sufficiently large. Hence, by Theorem 3.6,

$$\frac{1}{s\left|\partial B_s(\mathbf{x}_0+\rho_n\mathbf{z}_n)\right|}\int_{\partial B_s(\mathbf{x}_0+\rho_n\mathbf{z}_n)}u\left(\mathbf{x}\right)\,d\mathcal{H}^1\left(\mathbf{x}\right)\geq C(1/2)\sqrt{(-\rho_n y_n-\frac{1}{2}s)_+}$$

for all $s>0$ sufficiently small (depending on y_0 but not on n). Taking $s=\rho_n r$ and changing variables as in part (i), we get

$$\frac{1}{r\left|\partial B_r(\mathbf{z}_n)\right|}\int_{\partial B_r(\mathbf{z}_n)}u_n\,d\mathcal{H}^1\geq C(1/2)\sqrt{(-y_n-\frac{1}{2}r)_+}$$

and since $u_n\to w$ in $C^\alpha\left(B_R\left(\mathbf{0}\right)\right)$ for all $R>0$, letting $n\to\infty$, we obtain

$$\frac{1}{r\left|\partial B_r(\mathbf{z}_0)\right|}\int_{\partial B_r(\mathbf{z}_0)}w\,d\mathcal{H}^1\geq C(1/2)\sqrt{(-y_0-\frac{1}{2}r)_+}$$

for $r>0$. Similarly by Theorem 3.1,

$$\frac{1}{r\left|\partial B_r(\mathbf{z}_0)\right|}\int_{\partial B_r(\mathbf{z}_0)}w\,d\mathcal{H}^1\leq C_{\max}\sqrt{(-y_0+r)_+}.$$

Note that $-y_0>0$. Hence, for all $\varepsilon>0$ we are in position to apply Theorems 4.3 and 4.5 in [AC] to conclude that for every compact set $K\subset\mathbb{R}\times(-\infty,0)$,

$$\mathcal{H}^1\left(K\cap\partial\left\{w>0\right\}\right)<\infty,$$

which implies (4.5).

Fix $\mathbf{z}_0\in(\mathbb{R}\times(-\infty,0))\setminus\partial\{w>0\}$, $\mathbf{z}_0=(a_0,y_0)$. Since $y_0<0$ and $\partial\{w>0\}$ is closed, there exists $B_r(\mathbf{z}_0)\subset(\mathbb{R}\times(-\infty,0))\setminus\partial\{w>0\}$. Hence, either $w>0$ in $B_r(\mathbf{z}_0)$ or $w\equiv 0$ in $B_r(\mathbf{z}_0)$. By the first part of the proof of i), it follows that in the first case $u_n>0$ in $B_r(\mathbf{z}_0)$ for all n large, while in the second case, $u_n=0$ in $B_r(\mathbf{z}_0)$ for n large. Hence,

$$\chi_{\{u_n>0\}}(\mathbf{z}_0)=\chi_{\{w>0\}}(\mathbf{z}_0).$$

This shows that $\chi_{\{u_n>0\}}\to\chi_{\{w>0\}}$ in $L^1_{\text{loc}}\left(\mathbb{R}\times(-\infty,0)\right)$. □

Theorem 4.2. *Let $u\in\mathcal{K}$ be a local minimizer of J_λ and assume that the point $\mathbf{x}_0=(x_0,\lambda)\in\Omega$ belongs to $\partial\{u>0\}$ and that*

$$\{\mathbf{x}\in\Omega:\ u\left(\mathbf{x}\right)>0\}\subset(-\ell,\ell)\times(0,\lambda)\,.$$

Let $\{u_n\}$ be defined as in (4.1). Then the blow-up limit w of $\{u_n\}$ is a local minimizer for the functional

$$J\left(v\right)=\int_{B_1(\mathbf{0})}\left[\left|\nabla v\right|^2+\chi_{\{v>0\}}\left(-t\right)_+\right]\,d\mathbf{z},\quad \mathbf{z}=(s,t)\in\mathbb{R}^2,$$

defined over all $v\in H^1(B_1(\mathbf{0}))$ such that $v=w$ on $\partial B_1(\mathbf{0})$.

Proof. Consider a function $\eta\in C^1_0\left(B_1(\mathbf{0});[0,1]\right)$ and for every $v\in H^1(B_1(\mathbf{0}))$, with $v=w$ on $\partial B_1(\mathbf{0})$, define

$$v_n(\mathbf{z}):=v(\mathbf{z})+(1-\eta(\mathbf{z}))\left(u_n(\mathbf{z})-w(\mathbf{z})\right),\quad \mathbf{z}\in B_1(\mathbf{0}).$$

Since for $\mathbf{z}\in\partial B_1(\mathbf{0})$,

$$v_n(\mathbf{z})=u_n(\mathbf{z})=\frac{1}{\rho_n^{3/2}}u\left(\mathbf{x}_0+\rho_n\mathbf{z}\right),$$

the function

$$w_n(\mathbf{x}) := \begin{cases} \rho_n^{3/2} v_n\left(\frac{\mathbf{x}-\mathbf{x}_0}{\rho_n}\right) & \text{if } \mathbf{x} \in B_{\rho_n}(\mathbf{x}_0), \\ u(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega \setminus B_{\rho_n}(\mathbf{x}_0), \end{cases}$$

belongs to $\mathcal{K}$, and thus

$$J_\lambda(u) \leq J_\lambda(w_n).$$

This implies that

$$\int_{B_{\rho_n}(\mathbf{x}_0)} \left[|\nabla u|^2 + \chi_{\{u>0\}}(\lambda - y)_+\right] d\mathbf{x} \leq \int_{B_{\rho_n}(\mathbf{x}_0)} \left[|\nabla w_n|^2 + \chi_{\{w_n>0\}}(\lambda - y)_+\right] d\mathbf{x}.$$

After the change of variables $\mathbf{x} = \mathbf{x}_0 + \rho_n \mathbf{z}$, we obtain

$$\int_{B_1(\mathbf{0})} \left[|\nabla u_n|^2 + \chi_{\{u_n>0\}}(-t)_+\right] d\mathbf{z} \leq \int_{B_1(\mathbf{0})} \left[|\nabla v_n|^2 + \chi_{\{v_n>0\}}(-t)_+\right] d\mathbf{z}.$$

Using the facts that

$$(2 - \eta(\mathbf{z}))\,|\nabla u_n(\mathbf{z}) - \nabla w(\mathbf{z})|^2 \geq 0$$

and that

$$\chi_{\{v_n>0\}} \leq \chi_{\{v>0\}} + \chi_{\{\eta<1\}},$$

the previous inequality becomes

$$\begin{aligned} &\int_{B_1(\mathbf{0})} \left[2\nabla u_n \cdot \nabla w - |\nabla w|^2 + \chi_{\{u_n>0\}}(-t)_+\right] d\mathbf{z} \\ &\leq \int_{B_1(\mathbf{0})} \left[|\nabla v|^2 + \left(\chi_{\{v>0\}} + \chi_{\{\eta<1\}}\right)(-t)_+\right] d\mathbf{z} \\ &\quad + \int_{B_1(\mathbf{0})} \left[(\nabla u_n - \nabla w) \cdot (2(1-\eta)\nabla v - 2(1-\eta)(u_n - w)\nabla \eta)\right. \\ &\quad \left. + (u_n - u)\nabla \eta \cdot (2\nabla v + (u_n - w)\nabla \eta)\right] d\mathbf{z}. \end{aligned}$$

Since $u_n \rightharpoonup w$ in $L^2(B_1(\mathbf{0}))$, also by the previous theorem, letting $n \to \infty$ we obtain that

$$\int_{B_1(\mathbf{0})} \left[|\nabla w|^2 + \chi_{\{w>0\}}(-t)_+\right] d\mathbf{z} \leq \int_{B_1(\mathbf{0})} \left[|\nabla v|^2 + \left(\chi_{\{v>0\}} + \chi_{\{\eta<1\}}\right)(-t)_+\right] d\mathbf{z}.$$

Choosing a sequence of functions η_k such that $\eta_k \nearrow 1$, by the Lebesgue dominated convergence theorem, we obtain

$$\int_{B_1(\mathbf{0})} \left[|\nabla w|^2 + \chi_{\{w>0\}}(-t)_+\right] d\mathbf{z} \leq \int_{B_1(\mathbf{0})} \left[|\nabla v|^2 + \chi_{\{v>0\}}(-t)_+\right] d\mathbf{z},$$

which proves the theorem. □

Using some recent results of Varvaruca and Weiss [VW], which rely on a monotonicity formula, we are able to show that there are only two types of blow-up limits. More precisely, we have the following result.

Theorem 4.3. *Let $u \in \mathcal{K}$ be a local minimizer of J_λ and assume that the point $\mathbf{x}_0 = (x_0, \lambda) \in \Omega$ belongs to $\partial\{u > 0\}$ and that*

$$\{\mathbf{x} \in \Omega : u(\mathbf{x}) > 0\} \subset (-\ell, \ell) \times (0, \lambda). \tag{4.6}$$

Then either

$$\lim_{r \to 0^+} \frac{1}{r^3} \int_{B_r(\mathbf{x}_0)} \chi_{\{u>0\}}(\lambda - y)_+ \, d\mathbf{x} = 0,$$

in which case the blow-up limit is $w = 0$, or

$$\lim_{r\to 0^+} \frac{1}{r^3} \int_{B_r(\mathbf{x}_0)} \chi_{\{u>0\}} (\lambda - y)_+ \, d\mathbf{x} = \int_{B_1(\mathbf{0})} (\lambda - y)_+ \chi_{\left\{\cos\left(\frac{3}{2}\left(\theta - \frac{\pi}{2}\right)\right)>0\right\}} \, d\mathbf{x},$$

in which case the blow-up limit is

$$w(\rho, \theta) = \frac{\sqrt{2}}{3} \rho^{3/2} \max\left\{\cos\left(\frac{3}{2}\left(\theta - \frac{3\pi}{2}\right)\right), 0\right\}.$$

Here (ρ, θ) are polar coordinates centered at $\mathbf{x}_0$.

Proof. Let $a := \frac{1}{2} \min\{\ell - x_0, x_0 + \ell\} > 0$. In view of (4.6) and Theorems 3.3 and 3.7, the function

$$v(x, \zeta) := u(x, \lambda - \zeta)$$

is a weak solution (see Definition 3.2 in [VW]) of the problem

$$\begin{aligned} \Delta_{(x,\zeta)} v = 0 &\quad \text{in } A \cap \{v > 0\}, \\ |\nabla v|^2 = \zeta &\quad \text{on } A \cap \partial\{v > 0\}, \end{aligned} \tag{4.7}$$

where $A := \{(x, \zeta) : x \in (x_0 - a, x_0 + a), \zeta \in (-1, \lambda)\}$. Moreover, in view of (4.6), Theorem 3.4, and Remark 3.5 (ii), we have that

$$|\nabla v(x, \zeta)|^2 \le C\zeta_+$$

locally in A. Hence, we are in a position to apply Lemma 3.4 in [VW], and, in turn, Proposition 4.7 in [VW] to conclude that the only possible blow-up limits for the problem (4.7) are $v_0 = 0$, with

$$\lim_{r\to 0^+} \frac{1}{r^3} \int_{B_r((x_0,0))} \zeta_+ \chi_{\{v>0\}} \, dx d\zeta$$

either 0 or $\int_{B_1(\mathbf{0})} \zeta^+ \, dx d\zeta$, or

$$v_0(\rho, \theta) = \frac{\sqrt{2}}{3} \rho^{3/2} \max\left\{\cos\left(\frac{3}{2}\left(\theta - \frac{\pi}{2}\right)\right), 0\right\},$$

with density

$$\lim_{r\to 0^+} \frac{1}{r^3} \int_{B_r((x_0,0))} \zeta_+ \chi_{\{v>0\}} \, dx d\zeta = \int_{B_1(\mathbf{0})} \zeta_+ \chi_{\left\{\cos\left(\frac{3}{2}\left(\theta - \frac{\pi}{2}\right)\right)>0\right\}} \, dx d\zeta.$$

However, in view of Theorem 4.1(ii), we have that in the case $v_0 = 0$ the function χ_0 in the proof of Proposition 4.7 in [VW] is identically zero, and so in this case

$$\lim_{r\to 0^+} \frac{1}{r^3} \int_{B_r((x_0,0))} \zeta_+ \chi_{\{v>0\}} \, dx d\zeta = 0.$$

□

Remark 4.4. (i) For critical points that are not local minimizers, one cannot exclude a priori the case in which the blow-up limit is $w = 0$ but with

$$\lim_{r\to 0^+} \frac{1}{r^3} \int_{B_r(\mathbf{x}_0)} \chi_{\{u>0\}} (\lambda - y)_+ \, d\mathbf{x} = \int_{B_1(\mathbf{x}_0)} (\lambda - y)_+ \, d\mathbf{x}.$$

A significant part of the work of Varvaruca and Weiss ([VW], Sections 6-11) is devoted to treat this additional case.

(ii) We believe that the case $w = 0$ should be excluded. Note that if one could prove that in Remark 3.5 (ii), the constant C is one, that is, that the sharper inequality

$$|\nabla u\,(x,y)|^2 \le (\lambda - y)_+ \tag{4.8}$$

holds, then one could prove that in the previous theorem the case $w = 0$ cannot occur (see Lemma 4.4 in [VW], see also Theorem 5.11 below).

Corollary 4.5. *Let $u \in \mathcal{K}$ be a local minimizer of J_λ and assume that the point $\mathbf{x}_0 = (x_0, \lambda) \in \Omega$ belongs to $\partial\{u > 0\}$ and that (4.6) holds. If*

$$\lim_{r\to 0^+} \frac{1}{r^3}\int_{B_r(\mathbf{x}_0)} \chi_{\{u>0\}}(\lambda - y)_+\, d\mathbf{x} = 0, \tag{4.9}$$

then there is $r > 0$ (small) such that $u = 0$ in $B_{r/8}((x_0, \lambda - r))$.

Proof. Using (4.6), and the Hölder and Poincaré inequalities, we obtain

$$\int_{B_r(\mathbf{x}_0)} u\, d\mathbf{x} \le \pi^{1/2} r\left(\int_{B_r(\mathbf{x}_0)} u^2\, d\mathbf{x}\right)^{1/2} \le Cr^2\left(\int_{B_r(\mathbf{x}_0)} |\nabla u|^2\, d\mathbf{x}\right)^{1/2}$$
$$\le Cr^2\left(\int_{B_r(\mathbf{x}_0)} \chi_{\{u>0\}}(\lambda - y)_+\, d\mathbf{x}\right)^{1/2},$$

where the last inequality follows from Remark 3.5. Hence, also by (4.9),

$$\lim_{r\to 0^+} \frac{1}{r^{7/2}}\int_{B_r(\mathbf{x}_0)} u\, d\mathbf{x} \le C \lim_{r\to 0^+}\left(\frac{1}{r^3}\int_{B_r(\mathbf{x}_0)} \chi_{\{u>0\}}(\lambda - y)_+\, d\mathbf{x}\right)^{1/2} = 0.$$

Since $u \ge 0$ and $B_{r/4}((x_0, \lambda - r/2)) \subset B_r((x_0, \lambda)) = B_r(\mathbf{x}_0)$ it follows, in particular, that

$$\lim_{r\to 0^+} \frac{1}{r^{7/2}}\int_{B_{r/4}((x_0,\lambda - r/2))} u\, d\mathbf{x} = 0. \tag{4.10}$$

Using a slicing argument we have that

$$\frac{8}{r}\int_{r/8}^{r/4} \frac{1}{2\pi\rho^{5/2}}\int_{\partial B_\rho((x_0,\lambda - r/2))} u\, d\mathcal{H}^1\, d\rho \le \frac{C}{r}\int_{r/8}^{r/4} \frac{1}{r^{5/2}}\int_{\partial B_\rho((x_0,\lambda - r/2))} u\, d\mathcal{H}^1\, d\rho$$
$$\le \frac{C}{r^{7/2}}\int_{B_{r/2}((0,\lambda - r/2))} u\, d\mathbf{x}.$$

Fix $k := \frac{1}{2}$and let $C(1/2) > 0$ be the constant given in Theorem 3.6. In view of (4.10), we may find $r_0 > 0$ so small that

$$\frac{8}{r}\int_{r/8}^{r/4} \frac{1}{2\pi\rho^{5/2}}\int_{\partial B_\rho((x_0,\lambda - r/2))} u\, d\mathcal{H}^1\, d\rho < \sqrt{\frac{3}{2}}C(1/2)$$

for all $0 < r < r_0$. Fix any such $r > 0$ and find $r/8 < \rho < r/4$ such that

$$\frac{1}{2\pi\rho^{5/2}}\int_{\partial B_\rho((x_0,\lambda - r/2))} u\, d\mathcal{H}^1 < \sqrt{\frac{3}{2}}C(1/2).$$

Using the fact that $\sqrt{\frac{3}{2}\rho} \leq \sqrt{\frac{r}{2} - \frac{\rho}{2}}$, it follows that

$$\frac{1}{2\pi\rho^2}\int_{\partial B_\rho((x_0,\lambda-r/2))} u\,d\mathcal{H}^1 < C(1/2)\sqrt{\frac{3}{2}\rho} \leq C(1/2)\sqrt{\left(\lambda - \left(\lambda - \frac{r}{2}\right) - \frac{\rho}{2}\right)_+},$$

and so we are in a position to apply Theorem 3.6 to conclude that $u = 0$ in $B_{\rho/2}((x_0, \lambda - r/2))$. □

5. Absolute Minimizers

In this section we study properties of absolute minimizers. The following theorem shows that absolute minimizers of J_λ in the class $\mathcal{K}_1$ (see (1.10)) are one-dimensional functions of the type $u = u(y)$.

Theorem 5.1. *Let $u \in \mathcal{K}_1$ be an absolute minimizer of the functional J_λ. Then $u = u(y)$.*

Proof. **Step 1:** Consider the one-dimensional functional

$$I_\lambda(v) := \int_0^\infty \left(|v'(y)|^2 + \chi_{\{v>0\}}(y)(\lambda - y)_+\right) dy$$

defined in the class

$$\mathcal{K}_4 = \left\{v \in H^1_{\text{loc}}((0,\infty)) : v(0) = c\right\}.$$

We claim that the minimization problem

$$\alpha_1 := \inf_{v\in\mathcal{K}_4} I_\lambda(v)$$

has only one solution. To see this, let v be an absolute minimizer of I_λ, which exists by Theorem 2.2. By Theorem 3.3, the set $\{v > 0\}$ is either $(0,\infty)$ or $(0,a)$ for some $a > 0$. Moreover v is harmonic in $\{v > 0\}$, therefore linear. If $\{v > 0\} = (0,\infty)$, then, necessarily, $v \equiv c$, since otherwise $I_\lambda(v)$ would be infinite. Assume next that $\{v > 0\} = (0,a)$ for some for some $a > 0$. Then

$$v(y) = \begin{cases} \frac{c}{a}(-y+a) & \text{if } 0 < y \leq a, \\ 0 & \text{if } y > a. \end{cases} \tag{5.1}$$

Therefore,

$$\begin{aligned} I_\lambda(v) &= \int_0^a \left(\left(\frac{c}{a}\right)^2 + (\lambda - y)_+\right) dy \\ &= \frac{c^2}{a} + \frac{\lambda^2}{2} - \frac{(\lambda - \min\{a,\lambda\})^2}{2}. \end{aligned}$$

To find the value of a, note that a must be a solution of the infimum infimum problem

$$\inf_{t>0} f(t),$$

where

$$f(t) := \frac{c^2}{t} + \frac{\lambda^2}{2} - \frac{(\lambda - \min\{t,\lambda\})^2}{2}.$$

Note that

$$f'(t) = \begin{cases} -\frac{c^2}{t^2} - t + \lambda & \text{if } 0 < t < \lambda, \\ -\frac{c^2}{t^2} & \text{if } t > \lambda. \end{cases}$$

Since $f(t) \to \infty$ as $t \to 0^+$ and $f'(t) < 0$ for $t > \lambda$, it follows that $0 < a < \lambda$, with $f'(a) = 0$. Hence,

$$a^3 - \lambda a^2 + c^2 = 0.$$

Consider the cubic function

$$g(s) := s^3 - \lambda s^2 + c^2, \quad s \in \mathbb{R}.$$

Since $g(s) \to -\infty$ as $s \to -\infty$ and $g(0) = c^2 > 0$, the cubic equation $g(s) = 0$ has one negative zero, but $a > 0$, and so it must also have two positive zeros $0 < s_1 \le s_2$, and $a = s_1$. Note that g has a local minimum at

$$s_0 = \frac{2\lambda}{3}.$$

Necessarily, $g(s_0) \le 0$, which implies that

$$\lambda \ge 3\left(\frac{c}{2}\right)^{\frac{2}{3}}.$$

In conclusion, we have shown that $v \equiv c$ for all $\lambda < 3(c/2)^{2/3}$, while v is given by (5.1), where $a > 0$ is the first positive zero of g when $\lambda \ge 3(c/2)^{2/3}$.

Step 2: Let $u \in \mathcal{K}_1$ be an absolute minimizer of the functional J_λ. By a standard slicing argument we have that $u(x, \cdot) \in \mathcal{K}_4$ for $\mathcal{L}^1$ a.e. $x \in (-\ell, \ell)$ and so, also by Tonelli's theorem,

$$\begin{aligned} J_\lambda(u) &= \int_{-\ell}^{\ell}\int_0^\infty \left(|\nabla u(x,y)|^2 + \chi_{\{u>0\}}(x,y)(\lambda - y)_+\right) dydx \\ &\ge \int_{-\ell}^{\ell} I_\lambda(u(x,\cdot))\, dy \ge 2\ell\alpha_1. \end{aligned}$$

On the other hand, the minimizer v of I_λ given in the previous step belongs to $\mathcal{K}_1$ and so $J_\lambda(u) \le J_\lambda(v) = 2\ell\alpha_1$. Thus, $J_\lambda(u) = 2\ell\alpha_1$. Since $I_\lambda(u(x,\cdot)) \ge \alpha_1$ for $\mathcal{L}^1$ a.e. $x \in (-\ell, \ell)$, it follows that $I_\lambda(u(x,\cdot)) = \alpha_1$ for $\mathcal{L}^1$ a.e. $x \in (-\ell, \ell)$, which, by the previous step, implies that $u(x,\cdot) = v$ for $\mathcal{L}^1$ a.e. $x \in (-\ell, \ell)$. This concludes the proof. □

In view of the previous theorem, in the remaining of this section we assume that the set S is given by the segment $[-\ell, \ell] \times \{0\}$ and by the two half-lines $\{\pm\ell\} \times [0, \infty)$, and that the datum u_0 satisfies the following properties

$$\begin{aligned} &u_0(-\ell, y) = u_0(\ell, y) = 0 \text{ for } y \in [0, \infty), \\ &u_0(\cdot, 0) \in C_0^1([-\ell, \ell]), \\ &\{x \in (-\ell, \ell) : u_0(\cdot, 0) > 0\} \text{ is connected,} \end{aligned}$$

or equivalently, that

$$\begin{aligned} \mathcal{K} := \Big\{u \in L^1_{\text{loc}}(\Omega) : \nabla u \in \left(L^2_{\text{loc}}(\Omega)\right)^2, u(x,0) = v_0(x) \text{ for } x \in (-\ell, \ell), \\ u(-\ell, y) = u(\ell, y) = 0 \text{ for } y \in (0, \infty)\}, \end{aligned} \tag{5.2}$$

where $v_0 \in C_0^1([-\ell, \ell])$ and the set $\{x \in (-\ell, \ell) : v_0(0) > 0\}$ is connected. As explained in the introduction, by replacing h with a smaller height, this corresponds to localizing our attention near the crest. The drawback is that all the results obtained in this section are only local and depend strongly on the particular choice of the initial datum v_0.

In the next theorem we show that if λ is large, then the support of an absolute minimizer stays below the line $y = \lambda$.

Remark 5.2. If $u \in \mathcal{K}$ is a local minimizer, by extending u to zero in $\mathbb{R} \times [0, \infty) \setminus \overline{\Omega}$, it follows that Theorem 3.6 remains valid if $B_r(\mathbf{x}_0)$ intersects the lateral boundary of Ω.

Theorem 5.3 (Existence of regular solutions)**.** *There exists $\lambda_0 >> 1$, depending on the initial datum v_0, such that for all $\lambda \geq \lambda_0$ and for every absolute minimizer $u \in \mathcal{K}$ of J_λ, the support of u is contained in the set $[-\ell, \ell] \times [0, \lambda)$.*

Proof. Fix $y_0 > 0$, and let $k := \frac{1}{2}$ and $r := \frac{y_0}{2}$. Then for $x_0 \in [-\ell, \ell]$,

$$B_r(x_0, y_0) \subset \mathbb{R} \times \left[\frac{y_0}{2}, \infty\right),$$

and thus we are in a position to apply Remark 5.2. By Lemma 2.5 for every absolute minimizer $u \in K$ of J_λ we have (see (3.18)),

$$\ell_u = \sqrt{2} \sup_{B_{\sqrt{2}r}(x_0, y_0)} u \leq \sqrt{2} \max_{[-\ell,\ell]} v_0,$$

while (see (3.26))

$$Q_{\min} = \sqrt{\left(\lambda - \frac{3}{2} y_0\right)_+}.$$

It now follows from (3.28) that

$$\frac{C(\frac{1}{2})\ell_u}{rQ_{\min}} \left(\frac{\ell_u}{rQ_{\min}} + 1\right) \leq \frac{C(\frac{1}{2})\sqrt{2}\,\|v_0\|_\infty}{\frac{y_0}{2}\sqrt{\left(\lambda - \frac{3}{2}y_0\right)_+}} \left(\frac{\sqrt{2}\,\|v_0\|_\infty}{\frac{y_0}{2}\sqrt{\left(\lambda - \frac{3}{2}y_0\right)_+}} + 1\right) < 1$$

for all $\lambda \geq \lambda_0 = \lambda_0\left(\|v_0\|_\infty, y_0\right)$.

Thus by Theorem 3.6 (see (3.27)), we have that $u \equiv 0$ in $[-\ell, \ell] \times \left[\frac{3y_0}{4}, \frac{5y_0}{4}\right]$, which implies that $u \equiv 0$ in $[-\ell, \ell] \times \left[\frac{3y_0}{4}, \infty\right)$, (recall Theorem 3.3). □

Remark 5.4. Since the support of u remains below the line $y = \lambda$, $\lambda \geq \lambda_0$, we are in a position to apply the regularity results of Alt and Caffarelli (see Theorem 3.7) to conclude that $\partial \{u_\lambda > 0\}$ is locally a C^∞ curve. This gives a family of regular solutions. This result can be considered somehow in the same spirit of the theorem of Keady and Norbury [KN] in the sense that it gives local existence of regular waves (depending however on the initial datum v_0).

In the previous theorem, we have shown that for all λ sufficiently large the support of every absolute minimizer $u \in \mathcal{K}$ of J_λ remains well-below the line $y = \lambda$. Next we prove that for λ very small the support of u crosses the line $y = \lambda$. To highlight the dependence on the parameter λ in what follows we denote by $u_\lambda \in \mathcal{K}$ an absolute minimizer of the functional J_λ. Note that minimizers of J_λ are not necessarily unique.

In what follows, we adapt to our setting ideas from [ACF], [ACF1], [ACF2]. Following Theorem 10.2 in [Fr] we have the following result.

Theorem 5.5 (Monotonicity)**.** *Consider $0 < \mu < \lambda$ and let u_λ, $u_\mu \in \mathcal{K}$ be absolute minimizers of J_λ and J_μ, respectively. Then*

$$\{u_\lambda > 0\} \cap \{y < \lambda\} \subseteq \{u_\mu > 0\} \cap \{y < \lambda\} \tag{5.3}$$

and

$$u_\lambda \le u_\mu. \tag{5.4}$$

Moreover, if u_μ is regular, then $u_\lambda < u_\mu$ in $\{u_\mu > 0\}$.

Proof. Define $v_1 := \min\{u_\lambda, u_\mu\}$ and $v_2 := \max\{u_\lambda, u_\mu\}$. Since v_1 and v_2 belong to $\mathcal{K}$,

$$J_\lambda(u_\lambda) + J_\mu(u_\mu) \le J_\lambda(v_1) + J_\mu(v_2).$$

Let $A_1 := \{u_\mu < u_\lambda\}$ and $A_2 := \{u_\mu \ge u_\lambda\}$. Then the previous inequality becomes

$$\begin{aligned}
&\int_{A_1\cup A_2} \left(|\nabla u_\lambda|^2 + \chi_{\{u_\lambda>0\}}(\lambda - y)_+\right) d\mathbf{x} \\
&\qquad + \int_{A_1\cup A_2} \left(|\nabla u_\mu|^2 + \chi_{\{u_\mu>0\}}(\mu - y)_+\right) d\mathbf{x} \\
&\quad \le \int_{A_1\cup A_2} \left(|\nabla v_1|^2 + \chi_{\{v_1>0\}}(\lambda - y)_+\right) d\mathbf{x} \\
&\qquad + \int_{A_1\cup A_2} \left(|\nabla v_2|^2 + \chi_{\{v_2>0\}}(\mu - y)_+\right) d\mathbf{x}.
\end{aligned}$$

Since $v_1 = u_\mu$, $v_2 = u_\lambda$ in A_1 and $v_1 = u_\lambda$, $v_2 = u_\mu$ in A_2, the integrals containing gradients cancel out. Therefore,

$$\begin{aligned}
&\int_{A_1\cup A_2} \chi_{\{u_\lambda>0\}}(\lambda - y)_+ \, d\mathbf{x} + \int_{A_1\cup A_2} \chi_{\{u_\mu>0\}}(\mu - y)_+ \, d\mathbf{x} \\
&\le \int_{A_1} \chi_{\{u_\mu>0\}}(\lambda - y)_+ \, d\mathbf{x} + \int_{A_2} \chi_{\{u_\lambda>0\}}(\lambda - y)_+ \, d\mathbf{x} \\
&\quad + \int_{A_1} \chi_{\{u_\lambda>0\}}(\mu - y)_+ \, d\mathbf{x} + \int_{A_2} \chi_{\{u_\mu>0\}}(\mu - y)_+ \, d\mathbf{x}.
\end{aligned}$$

The integrals over A_2 cancel out, therefore

$$\begin{aligned}
&\int_{A_1} \chi_{\{u_\lambda>0\}}(\lambda - y)_+ \, d\mathbf{x} + \int_{A_1} \chi_{\{u_\mu>0\}}(\mu - y)_+ \, d\mathbf{x} \\
&\le \int_{A_1} \chi_{\{u_\mu>0\}}(\lambda - y)_+ \, d\mathbf{x} + \int_{A_1} \chi_{\{u_\lambda>0\}}(\mu - y)_+ \, d\mathbf{x},
\end{aligned}$$

which implies

$$\int_{A_1} \left(\chi_{\{u_\lambda>0\}} - \chi_{\{u_\mu>0\}}\right)(\lambda - y)_+ \, d\mathbf{x} \le \int_{A_1} \left(\chi_{\{u_\lambda>0\}} - \chi_{\{u_\mu>0\}}\right)(\mu - y)_+ \, d\mathbf{x},$$

or, equivalently,

$$\int_{A_1} \left(\chi_{\{u_\lambda>0\}} - \chi_{\{u_\mu>0\}}\right)\left((\lambda - y)_+ - (\mu - y)_+\right) d\mathbf{x} \le 0. \tag{5.5}$$

Since $A_1 = \{u_\mu < u_\lambda\}$ and $\mu < \lambda$, we have that the integrand is nonnegative, which implies that it is actually zero $\mathcal{L}^2$ a.e. in A_1. By the continuity of u_μ and u_λ we have that

$$\{u_\lambda > 0\} \cap \{y < \lambda\} \cap \{u_\mu < u_\lambda\} \subseteq \{u_\mu > 0\} \cap \{y < \lambda\} \cap \{u_\mu < u_\lambda\}.$$

On the other hand, since $A_2 = \{u_\mu \ge u_\lambda\}$, we have that

$$\{u_\lambda > 0\} \cap \{u_\mu \ge u_\lambda\} \subseteq \{u_\mu > 0\} \cap \{u_\mu \ge u_\lambda\},$$

and so (5.3) holds.

Since equality holds in (5.5), we have actually proved that

$$J_\lambda(u_\lambda) + J_\mu(u_\mu) = J_\lambda(v_1) + J_\mu(v_2),$$

which implies that $J_\lambda(u_\lambda) = J_\lambda(v_1)$ and $J_\mu(u_\mu) = J_\mu(v_2)$.

Hence, v_1 and v_2 are absolute minimizers for J_λ and J_μ, respectively. In particular, by Theorem 3.3, they are harmonic in the set where they are positive.

If $0 < u_\lambda(\mathbf{x}_0) = u_\mu(\mathbf{x}_0)$ for some $\mathbf{x}_0 = (x_0, y_0) \in (-\ell, \ell) \times (0, \infty)$, then in a neighborhood of $\mathbf{x}_0$ the functions $u_\lambda - v_2 \le 0$ and $u_\mu - v_2 \le 0$ are harmonic and attain a maximum in an interior point. It follows by the maximum principle that $u_\lambda - v_2 = u_\mu - v_2 \equiv 0$ in the connected component of $\{v_2 > 0\}$ that contains $\mathbf{x}_0$. By Theorem 3.3 we have that $u_\mu = u_\lambda$ in Ω.

If u_μ is a regular solution, this is a contradiction. Indeed, at a positive distance below the line $y = \mu$, we can apply Theorem 3.7 to obtain that $\partial\{u_\lambda > 0\}$ is locally a C^∞ curve. By classical regularity results we can write the Euler-Lagrange equations of the functional J_λ to deduce, in particular, that $\frac{\partial u_\lambda}{\partial \nu}(x, y) = \lambda - y$ and $\frac{\partial u_\mu}{\partial \nu}(x, y) = \mu - y$ on $\partial\{u_\lambda > 0\}$, which contradicts the fact that $u_\mu = u_\lambda$ in Ω.

If $u_\lambda(\mathbf{x}) \ne u_\mu(\mathbf{x})$ for all $\mathbf{x} \in \{u_\lambda > 0\} \cup \{u_\mu > 0\}$, then, since $\{u_\mu > 0\}$ is open and connected by Theorem 3.3, we must have that either $u_\mu > u_\lambda$ in $\{u_\mu > 0\}$ or $u_\mu < u_\lambda$ in $\{u_\mu > 0\}$. Assume by contradiction that $u_\mu < u_\lambda$ in $\{u_\mu > 0\}$. Since equality holds in (5.5), it follows from (5.3) that

$$\{u_\lambda > 0\} \cap \{y < \lambda\} = \{u_\mu > 0\} \cap \{y < \lambda\}.$$

In turn, since $\mu \le \lambda$,

$$\int_\Omega \chi_{\{u_\lambda > 0\}} (\lambda - y)_+ \, d\mathbf{x} = \int_\Omega \chi_{\{u_\mu > 0\}} (\lambda - y)_+ \, d\mathbf{x},$$
$$\int_\Omega \chi_{\{u_\lambda > 0\}} (\mu - y)_+ \, d\mathbf{x} = \int_\Omega \chi_{\{u_\mu > 0\}} (\mu - y)_+ \, d\mathbf{x}.$$

Using the facts that $J_\lambda(u_\lambda) \le J_\lambda(u_\mu)$ and $J_\mu(u_\mu) \le J_\mu(v_2)$, it follows that

$$\int_\Omega |\nabla u_\lambda|^2 \, d\mathbf{x} = \int_\Omega |\nabla u_\mu|^2 \, d\mathbf{x}.$$

Consider the function $v := \frac{1}{2}u_\lambda + \frac{1}{2}u_\mu \in \mathcal{K}$. By the strict convexity of the Dirichlet energy

$$\int_\Omega |\nabla v|^2 \, d\mathbf{x} < \int_\Omega |\nabla u_\lambda|^2 \, d\mathbf{x} = \int_\Omega |\nabla u_\mu|^2 \, d\mathbf{x},$$

while $\{v > 0\} \cap \{y < \lambda\} = \{u_\lambda > 0\} \cap \{y < \lambda\}$, so that

$$\int_\Omega \chi_{\{v > 0\}} (\lambda - y)_+ \, d\mathbf{x} = \int_\Omega \chi_{\{u_\mu > 0\}} (\lambda - y)_+ \, d\mathbf{x}.$$

It follows that $J_\lambda(v) < J_\lambda(u_\lambda)$, which is a contradiction. Hence, $u_\mu > u_\lambda$ in $\{u_\mu > 0\}$. This concludes the proof. □

We now prove the existence of a critical level λ_c, which should correspond to solutions whose free boundary forms an angle $\frac{2}{3}\pi$ as in Stokes waves.

Theorem 5.6. *Let*

$$\lambda_c := \inf\{\lambda \ge 0 : \textit{there is an absolute minimizer } u_\lambda \in \mathcal{K} \textit{ of } J_\lambda \textit{ with } \operatorname{supp} u_\lambda \subseteq \{y \le \lambda\}\}.$$

Then $0 < \lambda_c < \infty$*, every absolute minimizer of* J_λ *is a regular solution for* $\lambda > \lambda_c$*, while every absolute minimizer of* J_λ *is a* non-physical solution *for* $\lambda < \lambda_c$*, in the sense that its support crosses the line* $y = \lambda$*.*

Proof. By Theorem 5.3 we have that $\lambda_c \le \lambda_0 < \infty$. If $\lambda > \lambda_c$, by the definition of λ_c there exists $\lambda_c < \mu < \lambda$ such that $\operatorname{supp} u_\mu \subseteq \{y \le \mu\}$ for some absolute minimizer $u_\mu \in \mathcal{K}$ of J_μ. Let $u_\lambda \in \mathcal{K}$ be an absolute minimizer of J_λ. By (5.3),

$$\{u_\lambda > 0\} \cap \{y < \lambda\} \subseteq \{u_\mu > 0\} \cap \{y < \lambda\} \subseteq \{y \le \mu\},$$

which implies that $\operatorname{supp} u_\lambda \subseteq \{y \le \mu\}$. Thus u_λ is a regular solution. Hence, *every* absolute minimizer of J_λ is a regular solution.

On the other hand, if $\lambda_c > 0$ and $0 < \mu < \lambda_c$, let $u_\mu \in \mathcal{K}$ be an absolute minimizer of J_μ. If by contradiction $\operatorname{supp} u_\mu \subseteq \{y \le \mu\}$, then reasoning as in the first part of the proof, we would have that every absolute minimizer of J_λ is a regular solution for $\mu < \lambda < \lambda_c$, which would contradict the definition of λ_c. Thus, *every* absolute minimizer of J_μ is a non-physical solution.

To prove that $\lambda_c > 0$, fix $\lambda > \lambda_c$, let $u_\lambda \in \mathcal{K}$ be an absolute minimizer of J_λ and let $0 < \lambda_1 < \lambda$ be such that the line $y = \lambda_1$ intersects the set $\{u_\lambda > 0\}$. By (5.3) once more, for every $0 < \mu \le \lambda_1 < \lambda$ and for every absolute minimizer $u_\mu \in \mathcal{K}$ of J_μ,

$$\{u_\lambda > 0\} \cap \{y < \lambda\} \subseteq \{u_\mu > 0\} \cap \{y < \lambda\},$$

and, since the line $y = \lambda_1$ intersects the set $\{u_\lambda > 0\}$, it also intersects the set $\{u_\mu > 0\}$. Since the set $\{u_\mu > 0\}$ is connected and $\mu \le \lambda_1$, the line $y = \mu$ intersects the set $\{u_\mu > 0\}$. This shows that $\lambda_c \ge \lambda_1 > 0$. □

Next we prove that as $\lambda \searrow \lambda_c$ and $\lambda \nearrow \lambda_c$, corresponding minimizers u_λ approach two minimizers at level $y = \lambda_c$.

Theorem 5.7. *Let* $\{\lambda_n\} \subset (0, \infty)$ *be a sequence such that* $\lambda_n \to \lambda_c$ *and let* $\{u_{\lambda_n}\} \subseteq \mathcal{K}$ *be absolute minimizers of the functionals* J_{λ_n}*. Then (up to a subsequence)* $\{u_{\lambda_n}\}$ *converges strongly in* $H^1_{\mathrm{loc}}(\Omega)$ *to an absolute minimizer* $u \in \mathcal{K}$ *of* J_{λ_c}*.*

Proof. Extend v_0 to a function $v_0 \in C^1(\overline{\Omega})$ such that $\operatorname{supp} v_0$ is contained in $[-\ell, \ell] \times \left[0, \frac{\lambda_c}{2}\right]$. Let $n_1 \in \mathbb{N}$ be so large that $\lambda_n > \frac{\lambda_c}{2}$ for all $n \ge n_1$. As in the proof of Theorem 2.2 we may extract a subsequence (not relabeled) $\{u_{\lambda_n}\}$ such that $\{u_{\lambda_n}\}$ converges weakly to some function $u_{\lambda_c} \in \mathcal{K}$, while $\left\{\chi_{\{u_{\lambda_n} > 0\}}\right\}$ converges weakly star to a function γ in $L^\infty(\Omega)$ with

$$\gamma(\mathbf{x}) \ge \chi_{\{u_{\lambda_n} > 0\}}(\mathbf{x}) \quad \text{for } \mathcal{L}^2 \text{ a.e. } \mathbf{x} \in \Omega.$$

It remains to show that u_{λ_c} is an absolute minimizer for J_{λ_c}. As in the last part of the proof of the Theorem 2.2, for every $r > 0$ and $u \in \mathcal{K}$, we have

$$\begin{aligned}
\int_{\Omega_r} \left(|\nabla u_{\lambda_c}|^2 + \chi_{\{u_{\lambda_n} > 0\}} (\lambda_c - y)_+\right) d\mathbf{x} &\le \int_{\Omega_r} \left(|\nabla u_{\lambda_c}|^2 + \gamma (\lambda_c - y)_+\right) d\mathbf{x} \\
&\le \liminf_{n \to \infty} \int_{\Omega_r} \left(|\nabla u_{\lambda_n}|^2 + \gamma (\lambda_n - y)_+\right) d\mathbf{x} \\
&\le \liminf_{n \to \infty} J_{\lambda_n}(u_{\lambda_n}) \le \limsup_{n \to \infty} J_{\lambda_n}(u_{\lambda_n}) \\
&\le \limsup_{n \to \infty} J_{\lambda_n}(u) = J_{\lambda_c}(u).
\end{aligned} \tag{5.6}$$

Letting $r \nearrow \infty$, we conclude that

$$J_{\lambda_c}(u_c) \le J_{\lambda_c}(u)$$

for all $u \in \mathcal{K}$. Since $u_{\lambda_c} \in \mathcal{K}$, we have that u_{λ_c} is an absolute minimizer for J_{λ_c}.

Note that taking $u = u_{\lambda_c}$ in (5.6) and letting $r \nearrow \infty$, gives

$$J_{\lambda_c}(u) = \lim_{n\to\infty} \int_\Omega \left(|\nabla u_{\lambda_n}|^2 + \chi_{\{u_{\lambda_n}>0\}} (\lambda_n - y)_+ \right) d\mathbf{x}.$$

On the other hand, since

$$\liminf_{n\to\infty} \int_\Omega |\nabla u_{\lambda_n}|^2 \, d\mathbf{x} \ge \int_\Omega |\nabla u_{\lambda_c}|^2 \, d\mathbf{x}$$

and

$$\liminf_{n\to\infty} \int_\Omega \chi_{\{u_{\lambda_n}>0\}} (\lambda_n - y)_+ \, d\mathbf{x} \ge \int_\Omega \chi_{\{u_{\lambda_c}>0\}} (\lambda_c - y)_+ \, d\mathbf{x},$$

it follows that

$$\lim_{n\to\infty} \int_\Omega |\nabla u_{\lambda_n}|^2 \, d\mathbf{x} = \int_\Omega |\nabla u_{\lambda_c}|^2 \, d\mathbf{x}$$

and

$$\lim_{n\to\infty} \int_\Omega \chi_{\{u_{\lambda_n}>0\}} (\lambda_n - y)_+ \, d\mathbf{x} = \int_\Omega \chi_{\{u_{\lambda_c}>0\}} (\lambda_c - y)_+ \, d\mathbf{x}.$$

It follows that $\{\nabla u_{\lambda_n}\}$ converges strongly to ∇u_{λ_c} in $\left(L^2(\Omega)\right)^2$, and hence $\{u_{\lambda_n}\}$ converges strongly to u_{λ_c} in $H^1_{\mathrm{loc}}(\Omega)$. □

Corollary 5.8. *Let* $\{\lambda_n\}$, $\{\mu_n\} \subset (0,\infty)$ *be such that* $\lambda_n \searrow \lambda_c$ *and* $\mu_n \nearrow \lambda_c$. *Then* $\{u_{\lambda_n}\}$ *and* $\{u_{\mu_n}\}$ *converge strongly in* $H^1_{\mathrm{loc}}(\Omega)$ *and uniformly to two absolute minimizers* u^+ *and* $u^- \in \mathcal{K}$ *of* J_{λ_c}, *respectively. Moreover* $\operatorname{supp} u^+ \subseteq \{y \le \lambda_c\}$, *while* $\operatorname{supp} u^-$ *intersects the line* $y = \lambda_c$.

Proof. By Theorem 5.5 the sequence $\{u_{\lambda_n}\}$ is increasing, while the sequence $\{u_{\mu_n}\}$ is decreasing. Thus for all $\mathbf{x} \in \Omega$ there exist

$$\lim_{n\to\infty} u_{\lambda_n}(\mathbf{x}) = u^+(\mathbf{x}), \qquad \lim_{n\to\infty} u_{\mu_n}(\mathbf{x}) = u^-(\mathbf{x}).$$

It follows by the previous theorem, that u^+ and u^- are absolute minimizers of J_{λ_c}. Since u^+ and u^- are continuous (see Theorem 3.4), by Dini's monotone convergence theorem, the convergence is uniform.

To prove the second part of the statement, assume by contradiction that there exists $\mathbf{x}_0 = (x_0, y_0) \in (-\ell,\ell)\times(\lambda_c,\infty)$ such that $u^+(\mathbf{x}_0) > 0$. Since $\{u_{\lambda_n}\}$ converges uniformly to u^+, we have that

$$u_{\lambda_n}(\mathbf{x}_0) > \frac{u^+(\mathbf{x}_0)}{2}$$

for all n sufficiently large. Since $\lambda_n \searrow \lambda_c$, taking so large that $\lambda_n < y_0$, we have contradicted the fact that $\operatorname{supp} u_{\lambda_n} \subseteq \{y < \lambda_n\}$. Thus $\operatorname{supp} u^+ \subseteq \{y \le \lambda_c\}$.

Next, assume by contradiction that

$$\operatorname{supp} u^- \subseteq \{y < \lambda_c\}.$$

Fix $\varepsilon > 0$ such that $\operatorname{supp} u^- \subseteq \{y < \lambda_c - \varepsilon\}$. Let $C(1/2)$ be the constant given in Theorem 3.6 with $k = \frac{1}{2}$. Since $\mu_n \nearrow \lambda_c$ and $\{u_{\mu_n}\}$ converges uniformly to zero in $[-\ell,\ell] \times [\lambda_c - \varepsilon, \lambda_c]$, we may find n_1 so large that

$$\mu_n > \lambda_c - \frac{\varepsilon}{4}$$

and

$$u_{\mu_n} < C\left(\tfrac{1}{2}\right)\frac{\varepsilon}{4}\sqrt{\frac{3}{8}\varepsilon} \text{ in } [-\ell,\ell]\times[\lambda_c-\varepsilon,\lambda_c] \tag{5.7}$$

for all $n \geq n_1$.

We now apply Theorem 3.6 and Remark 5.2 to u_{μ_n} taking $x_0 \in (-\ell,\ell)$, $y_0 = \lambda_c - \frac{3}{4}\varepsilon$, $r = \frac{\varepsilon}{4}$. By (5.7), for all $n \geq n_1$, we have

$$\begin{aligned}
&\frac{1}{r\,|\partial B_r(\mathbf{x}_0)|}\frac{1}{\sqrt{(\lambda - y_0 - \frac{1}{2}r)_+}}\int_{\partial B_r(\mathbf{x}_0)} u_{\mu_n}\, d\mathcal{H}^1 \\
&= \frac{1}{\frac{\varepsilon}{4}\left|\partial B_{\frac{\varepsilon}{4}}(\mathbf{x}_0)\right|}\frac{1}{\sqrt{(\mu_n - \lambda_c + \frac{5}{8}\varepsilon)_+}}\int_{\partial B_{\frac{\varepsilon}{4}}(\mathbf{x}_0)} u_{\mu_n}\, d\mathcal{H}^1 \\
&< C\left(\tfrac{1}{2}\right)\frac{\sqrt{\frac{3}{8}\varepsilon}}{\sqrt{(\mu_n - \lambda_c + \frac{5}{8}\varepsilon)_+}} \leq C\left(\tfrac{1}{2}\right),
\end{aligned}$$

where in the last inequality we have used (5.7) and the fact that $\mu_n > \lambda_c - \frac{\varepsilon}{4}$.

It follows from Theorem 3.6 and Remark 5.2 that $u_{\mu_n} = 0$ in $B_{\frac{\varepsilon}{4}}\left(x_0, \lambda_c - \frac{\varepsilon}{4}\right)$ for all $x_0 \in (-\ell,\ell)$ and for all $n \geq n_1$. Since $\{u_{\mu_n} > 0\}$ is connected by Theorem 3.3, we have contradicted the fact that $\operatorname{supp} u_{\mu_n}$ meets the line $y = \mu_n$.

Hence, $\operatorname{supp} u^-$ is not contained in $\{y < \lambda_c\}$. □

Conjecture 5.9. *We conjecture that J_{λ_c} has a unique absolute minimizer.*

Note that if the conjecture were true, then $u^+ = u^-$, and so the support of u^+ would touch the line $y = \lambda_c$ and be contained in the set $\{y \leq \lambda_c\}$. This would prove the local existence of a solution behaving like a Stokes wave near the crest. We have been unable to prove the conjecture.

Next we show that if the initial datum v_0 is even and decreasing in $(0,\ell)$, then there exists an absolute minimizer whose boundary is given by the graph of a function $x = g(y)$ for $x > 0$.

Theorem 5.10. *Suppose that the function v_0 in (5.2) is even in $(-\ell,\ell)$ and decreasing in $(0,\ell)$. Then there exists an absolute minimizer $u \in \mathcal{K}$ of the functional J_λ such that $u(x,y) = u(-x,y)$ and the function $x \in [0,\ell] \mapsto u(x,y)$ is decreasing for all $y \geq 0$.*

Proof. **Step 1:** In this step we show the existence of two absolute minimizers that are even in the x-variable. Let u be a minimizer for J. Define

$$w_1(x,y) := \begin{cases} u(x,y) & \text{if } x \geq 0, \\ u(-x,y) & \text{if } x < 0, \end{cases}$$

$$w_2(x,y) := \begin{cases} u(-x,y) & \text{if } x \geq 0, \\ u(x,y) & \text{if } x < 0. \end{cases}$$

Since v_0 is even, it follows that w_1, w_2 belong to $\mathcal{K}$. A simple computation yields

$$J_\lambda(w_1) + J_\lambda(w_2) = 2J_\lambda(u).$$

Since both $J_\lambda(w_1)$ and $J_\lambda(w_2)$ are bigger than $J_\lambda(u)$, we must have

$$J_\lambda(w_1) = J_\lambda(w_2) = J_\lambda(u).$$

Therefore w_1 and w_2 are two minimizers of J_λ that are even in the x-variable.

Step 2: In this step we prove that if an absolute minimizer $u \in \mathcal{K}$ of J_λ is symmetric in x, then the symmetric decreasing rearrangement of u in the variable x (see Corollary 2.14 and Remark 2.32 in [Kaw]), denoted by u^*, is also an absolute minimizer of J_λ. Notice that by Step 1, $\Omega \cap \{u > 0\}$ coincides with its Steiner symmetrization. By Corollary 2.14 in [Kaw], we have

$$\int_\Omega |\nabla u^*(x,y)|^2 \, dxdy \le \int_\Omega |\nabla u(x,y)|^2 \, dxdy, \tag{5.8}$$

while by Fubini's theorem

$$\int_\Omega \chi_{\{u>0\}}(\lambda - y)_+ \, d\mathbf{x} = \int_0^\lambda (\lambda - y)_+ \left(\int_{-\ell}^{\ell} \chi_{\{u>0\}} (x,y) \, dx \right) dy,$$

and similarly

$$\int_\Omega \chi_{\{u^*>0\}}(\lambda - y)_+ \, d\mathbf{x} = \int_0^\lambda (\lambda - y)_+ \left(\int_{-\ell}^{\ell} \chi_{\{u^*>0\}} (x,y) \, dx \right) dy.$$

By the definition of u^* we have for any fixed $y \in \mathbb{R}$,

$$\int_{-\ell}^{\ell} \chi_{\{u>0\}} (x,y) \, dx = \int_{-\ell}^{\ell} \chi_{\{u^*>0\}} (x,y) \, dx,$$

therefore

$$\int_\Omega \chi_{\{u>0\}}(\lambda - y)_+ \, d\mathbf{x} = \int_\Omega \chi_{\{u^*>0\}}(\lambda - y)_+ \, d\mathbf{x},$$

which together with (5.8), implies that $J_\lambda(u^*) \le J_\lambda(u)$. Using the fact that $v_0 = v_0^*$, we have that $u^*(x,0) = u(x,0) = v_0(x)$. □

In view of the previous theorem, if v_0 is even in $(-\ell, \ell)$ and decreasing in $(0, \ell)$, there exists an absolute minimizer $u \in \mathcal{K}$ of J_λ whose free boundary can be described by the graph of a function $x = g(y)$ in $(0,\ell) \times \mathbb{R}$. Indeed, it suffices to define

$$g(y) := \sup \{x \in [-\ell, \ell] : \, u(x,y) > 0\}. \tag{5.9}$$

Next we prove that for the absolute minimizer constructed in the previous theorem there is only one blow-up limit (see Theorem 4.3).

Theorem 5.11. *Assume that v_0 is even in $(-\ell,\ell)$, and decreasing in $(0,\ell)$, and let $u \in \mathcal{K}$ be the absolute minimizer of J_λ given by Theorem 5.10. Assume that the point $(0,\lambda) \in \Omega$ belongs to $\partial\{u > 0\}$ and that*

$$\{\mathbf{x} \in \Omega : \, u(\mathbf{x}) > 0\} \subset (-\ell, \ell) \times (0, \lambda). \tag{5.10}$$

Then the only blow-up limit is

$$w(\rho,\theta) = \frac{\sqrt{2}}{3}\rho^{3/2} \max\left\{ \cos\left(\frac{3}{2}\left(\theta - \frac{3\pi}{2}\right)\right), 0 \right\}$$

and

$$\lim_{r\to 0^+} \frac{1}{r^3} \int_{B_r((0,\lambda))} \chi_{\{u>0\}}(\lambda - y)_+ \, d\mathbf{x} = \int_{B_1(\mathbf{0})} (\lambda - y)_+ \chi_{\left\{\cos\left(\frac{3}{2}\left(\theta - \frac{\pi}{2}\right)\right) > 0\right\}} \, d\mathbf{x}.$$

First proof. In view of Theorem 4.3, it remains to exclude the case in which a blow-up limit is $w = 0$ with

$$\lim_{r\to 0^+} \frac{1}{r^3} \int_{B_r((0,\lambda))} \chi_{\{u>0\}}(\lambda - y)_+ \, d\mathbf{x} = 0. \tag{5.11}$$

If (5.11) holds, then by Corollary 4.5, there exists $r > 0$ such that $u = 0$ in $B_{r/8}((0, \lambda - r))$. However, since the function $x \in [0, \ell] \mapsto u(x, y)$ is decreasing for all $y \geq 0$, it follows that $u(x, \lambda - r) = 0$ for all $x \in (-\ell, \ell)$, which contradicts the fact that $\{\mathbf{x} \in \Omega : \, u(\mathbf{x}) > 0\}$ is connected (see Theorem 3.3). □

We present a second proof of Theorem 4.3, which does not make use of Corollary 4.5.

Second proof. **Step 1.** We claim that the function $y \in [0, \lambda] \mapsto u(0, y)$ is decreasing. Indeed, assume by contradiction that there exist $0 \leq y_1 < y_2 < \lambda$ such that $u(0, y_1) < u(0, y_2)$. Since u is continuous (see Theorem 3.4) and $u(0, \lambda) = 0$ by (5.10), there exists $y_2 < y_3 \leq \lambda$ such that $u(0, y_1) = u(0, y_3)$. Hence, the function $y \in [y_1, y_3] \mapsto u(0, y)$ has an absolute maximum at some $y_0 \in (y_1, y_3)$. In turn, since $u(x, y) = u(-x, y)$ and the function $x \in [0, \ell] \mapsto u(x, y)$ is decreasing for all $y \geq 0$, the point $(0, y_0)$ is a point of absolute maximum for the function u in $[-\ell, \ell] \times [y_1, y_3]$. By replacing $[y_1, y_3]$ with a subinterval containing y_0, without loss of generality, we may assume that $u > 0$ in $[y_1, y_3]$ and that

$$u(0, y_1) = u(0, y_3) < u(0, y_0).$$

Again by continuity, we may assume that $u > 0$ in $[-\ell_1, \ell_2] \times [y_1, y_3]$ for some $0 < \ell_1 \leq \ell$. Since u is harmonic in the set $\{u > 0\}$ (see Theorem 3.3), this contradicts the maximum principle.

Step 2. We claim that

$$u(0, y) \geq C(\lambda - y)^{3/2} \tag{5.12}$$

for all $y \in [0, \lambda]$ and for some $C > 0$. To see this, fix $0 < y_0 < \lambda$. Without loss of generality, we may assume that

$$\lambda - y_0 \leq \min\{\lambda + h, \ell\}.$$

Fix $k := \frac{1}{2}$and let $C(1/2) > 0$ be the constant given in Theorem 3.6. Let $r := \frac{1}{8}(\lambda - y_0)$ and $\mathbf{x}_0 := \left(0, y_0 + \frac{1}{8}(\lambda - y_0)\right)$. If

$$\frac{1}{r\,|\partial B_r(\mathbf{x}_0)|} \int_{\partial B_r(\mathbf{x}_0)} u \, d\mathcal{H}^1 < C(1/2) \sqrt{\left(\lambda - \left(y_0 + \frac{1}{8}(\lambda - y_0)\right)\right)_+},$$

then by Theorem 3.6, $u = 0$ in $\partial B_{r/2}(\mathbf{x}_0)$, which contradicts the fact that $u(0, y) > 0$ in $[0, \lambda)$ by Step 1 and the fact that $(0, \lambda) \in \partial\{u > 0\}$. Hence,

$$\frac{1}{r\,|\partial B_r(\mathbf{x}_0)|} \int_{\partial B_r(\mathbf{x}_0)} u \, d\mathcal{H}^1 \geq C(1/2) \sqrt{\frac{7}{8}(\lambda - y_0)_+}.$$

Since $u(x, y) = u(-x, y)$ and the functions $y \in [0, \lambda] \mapsto u(0, y)$ and $x \in [0, \ell] \mapsto u(x, y)$ are decreasing, it follows that

$$\begin{aligned} u(0, y_0) &\geq \frac{1}{|\partial B_r(\mathbf{x}_0)|} \int_{\partial B_r(\mathbf{x}_0)} u \, d\mathcal{H}^1 \\ &\geq \frac{1}{8} C(1/2) \sqrt{\frac{7}{8}(\lambda - y_0)_+}\,(\lambda - y_0), \end{aligned}$$

which proves the claim.

Using (4.2) and (5.12), we obtain that for every blow-up limit w,

$$w(0,t) \geq C(-t)^{3/2}, \tag{5.13}$$

which implies that w cannot be zero. □

In the next theorem we prove, without using the results of [VW], that if the free boundary forms an angle ω, then, necessarily, $\omega = \frac{2\pi}{3}$.

Theorem 5.12. *Assume that v_0 is even in $(-\ell,\ell)$, and decreasing in $(0,\ell)$, and let $u \in \mathcal{K}$ be the absolute minimizer of J_{λ_c} given by Theorem 5.10 and with* $\operatorname{supp} u \subseteq \{y \leq \lambda_c\}$. *Let g be defined as in* (5.9) *and assume that there exist the limits*

$$\lim_{y\to(\lambda_c)_-} g(y) =: g(\lambda_c) \in [0,\ell), \tag{5.14}$$

and

$$\lim_{y\to(\lambda_c)_-} \frac{g(y)-g(\lambda_c)}{\lambda_c - y} =: \xi \in \mathbb{R}. \tag{5.15}$$

Then $g(\lambda_c) = 0$ and $\xi = \tan\frac{\pi}{3}$.

Proof. **Step 1.** In this step we prove that $g(\lambda_c) = 0$. Indeed, assume by contradiction that $g(\lambda_c) > 0$. By (5.14) and (5.15), the free boundary is contained in a thin sector centered at $(g(\lambda_c), \lambda_c)$. Since the function $x \in [0,\ell] \mapsto u(x,y)$ is decreasing for all $y \geq 0$, we deduce that for some $0 < a < g(\lambda_c)$ and for some small $\varepsilon > 0$, $u > 0$ in $(-a,a)\times(\lambda_c-\varepsilon,\lambda_c)$. Using the fact that $u(x,\lambda_c) = 0$ for all $x \in (-\ell,\ell)$, we have that $\frac{\partial u}{\partial x}(x,\lambda_c) = 0$. Writing the Euler-Lagrange equations in $(-a,a)\times(\lambda_c-\varepsilon,\lambda_c)$, it follows by the Bernoulli's boundary condition $|\nabla u|^2 = \lambda - y$ that $\frac{\partial u}{\partial y}(x,\lambda_c) = 0$ for $x \in (-a,a)$, which contradicts Hopf's lemma.

Step 2. Let w be the blow-up limit of the sequence $\{u_n\}$ defined in (4.1), with $\mathbf{x}_0 = (0,\lambda_c)$. By (5.9), after the change of variables $\mathbf{x} = \mathbf{x}_0 + \rho_n \mathbf{z}$, $\mathbf{z} = (s,t)$, we have that the free boundary of u_n can be described by the function

$$h_n(t) := \frac{1}{\rho_n} g(\lambda_c + \rho_n t), \quad t < 0.$$

Note that by (5.15),

$$\lim_{n\to\infty} h_n(t) = \xi t.$$

On the other hand, since $\partial\{u_n > 0\}$ locally converges in Hausdorff distance to $\partial\{w > 0\}$ in $\mathbb{R}\times(-\infty,0)$, it follows that the function $h(t) = \xi t$, $t < 0$ describes the free boundary of w in $[0,\infty)\times(-\infty,0]$. Since by Theorem 4.2, w is a minimizer of the functional and $(0,0) \in \partial\{w > 0\}$, it follows by Theorem 3.1 that

$$\frac{1}{|\partial B_r(\mathbf{0})|}\int_{\partial B_r(\mathbf{0})} w\, d\mathcal{H}^1 \leq C_{\max} r^{\frac{3}{2}} \tag{5.16}$$

for all $r > 0$ sufficiently small, while by (5.13),

$$w(0,t) \geq C(-t)^{3/2} \tag{5.17}$$

for $t \in [-1,0]$.

Consider the triangle

$$T := \{(s,t) \in \mathbb{R} : \xi t < s < -\xi t, \quad t \in [-1,0]\}$$

and let $\varphi \in C_c^\infty\left(\mathbb{R}^2;[0,1]\right)$ be such that $\varphi \equiv 1$ on $B_\varepsilon(\mathbf{0})$ and $\varphi \equiv 0$ outside $B_\varepsilon(\mathbf{0})$. Then $w\varphi = 0$ on ∂T, while

$$\begin{aligned}\Delta(w\varphi) &= (\Delta w)\varphi + 2\nabla w \cdot \nabla\varphi + w\Delta\varphi \\ &= 2\nabla w \cdot \nabla\varphi + w\Delta\varphi \quad \text{in } T.\end{aligned}$$

It follows by Theorem 4.4.3.7 in Chapter 4 of [Gris] that in a neighborhood U of $\mathbf{0}$,

$$w = w\varphi = \sum_{-1<\lambda_m<0} C_m S_m + v$$

for some $v \in H^2(U)$, where for every $m \in \mathbb{Z}$, $C_m \in \mathbb{R}$,

$$\begin{aligned}\lambda_m &:= \frac{m\pi}{\omega}, \\ S_m &:= \frac{r^{-\lambda_m}}{\sqrt{\omega}\lambda_m}\cos\left(\lambda_m\theta + \frac{\pi}{2}\right)\eta\left(re^{i\theta}\right),\end{aligned}$$

where $\omega \in (0,\pi]$ is the angle at $\mathbf{0}$, (r,θ) are polar coordinates centered at $\mathbf{0}$ and η is a smooth function, which is 1 in a small neighborhood of $\mathbf{0}$ and whose support does not intersect the side of T opposite to $\mathbf{0}$. By (5.16) and (5.17), the singularity of w is of the type $r^{3/2}$ in a neighborhood of the origin. Since $\omega \in [0,\pi]$,

$$-\lambda_m = -\frac{m\pi}{\omega} = \frac{3}{2}$$

can hold only for $m = -1$. From this we conclude that

$$\omega = \frac{2}{3}\pi,$$

which is the expected Stokes angle. □

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