One–dimensional Feller semigroups with reflecting barriers

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Abstract

Given the second order ordinary differential operator $Au = mu'' + qu'$ in the real interval I, possibly degenerate at the boundary, we define $W(x) = \exp\big(-\int_{x_0}^{x} \frac{q(s)}{m(s)} ds\big)$ (where $x_0 \in I$ is arbitrarily fixed) and the domain

$$D_N(A) = \Big\{ u \in C(\overline{I}) \cap C^2(I) : \ Au \in C(\overline{I}), \ \lim_{x \to \partial I} \frac{u'(x)}{W(x)} = 0 \Big\}.$$

Given $f \in D_N(A)$, we consider the initial value problem $u_t = Au$, $u(0) = f$ and study under which conditions on the coefficients a classical solution exists, and when $(A, D_N(A))$ is the generator of a strongly continuous semigroup in the space $C(\overline{I})$.

1 Introduction

Let $I = (r_1, r_2)$ be a (possibly unbounded) open interval and m, q continuous functions, with m positive in I; without any extra assumption on the behaviour of the coefficients at the boundary, consider the second order differential operator $Au = mu'' + qu'$ in I and the abstract Cauchy problem in $C(\overline{I})$

$$u_t = Au \qquad (t > 0) \tag{1}$$

$$u(0) = f \tag{2}$$

with $f \in C(\overline{I})$. A general question is: which is the relation between the behaviour of the coefficients m, q near the endpoints and the boundary conditions that can be imposed in order to have a unique solution of (1), (2) satisfying them, for any initial datum? This problem was posed by W. Feller whose main motivation was the probabilistic interest of equation (1) for the transition probabilities u. In fact, (1) is the backward equation coming from a one-dimensional diffusion process (for this reason, $C(\overline{I})$ is the natural space where the problem should be discussed, keeping its original flavour). Accordingly, many boundary conditions that have been joined to problem (1), (2) have a genuine probabilistic meaning and this reflects also in the terminology introduced by Feller. We refer to [5], [6] and [7] for a discussion of the probabilistic background and of the model, and for a detailed study of the evolution problem (1), (2) in many different situations. In terms of semigroups, assigning boundary conditions amounts to endowing the operator A with a domain $D(A)$ in $C(\overline{I})$; then one investigates whether $(A, D(A))$ is a generator. To begin with, let us introduce the maximal domain of A,

$$D_M(A) = \Big\{ u \in C(\overline{I}) \cap C^2(I) : \; Au \in C(\overline{I}) \Big\}.$$

Moreover, fix once and for all a point $x_0 \in I$, define the Wronskian

$$W(x) = \exp\Big\{ -\int_{x_0}^{x} \frac{q(s)}{m(s)} ds \Big\}$$

and consider the "reflecting barrier" boundary conditions

$$u(t) \in D_N(A) \qquad\qquad t > 0, \tag{3}$$

where

$$D_N(A) = \Big\{ u \in D_M(A) : \lim_{x \to r_1, r_2} \frac{u'(x)}{W(x)} = 0 \Big\}.$$

We point out that the above boundary conditions reduce to the homogeneous Neumann conditions if, *e.g.*, I is bounded and m, q are continuous up to the boundary, with $\inf_I m > 0$. In order to provide a general setting to all the boundary value problems associated to (1), (2), Feller proposed a classification of the boundary points r_1, r_2, which comes from the possible behaviour of the solutions of the equation $\lambda u - Au = 0$. In order to recall Feller's classification of the boundaries we introduce the functions

$$Q(x) = \frac{1}{m(x)W(x)} \int_{x_0}^{x} W(s) ds, \quad R(x) = W(x) \int_{x_0}^{x} \frac{1}{m(s)W(s)} ds; \tag{4}$$

the endpoint r_2 is said to be

regular	if	$Q \in L^1(x_0, r_2)$,	$R \in L^1(x_0, r_2)$;
exit	if	$Q \notin L^1(x_0, r_2)$,	$R \in L^1(x_0, r_2)$;
entrance	if	$Q \in L^1(x_0, r_2)$,	$R \notin L^1(x_0, r_2)$;
natural	if	$Q \notin L^1(x_0, r_2)$,	$R \notin L^1(x_0, r_2)$;

of course, analogous definitions are understood for r_1. If r_2 is regular or exit, then it is called *accessible*, because in terms of Markov processes, there is a positive probability that the particle does reach r_2 in a finite time; otherwise, r_2 is *unaccessible*, see [6]. From this point of view, assigning boundary conditions serves to select (if possible) one among the infinite solutions of (1), (2) only in the case of accessible boundaries. Otherwise, (1), (2) has a unique solution because $(A, D_M(A))$ generates a strongly continuous semigroup (see [5] or [2, Theorem 4.14]). If this is the case and boundary conditions are assigned, the question is whether the solution u enjoys the required properties, and reduces to a kind of regularity control.

In this paper we give necessary and sufficient conditions in order that problem (1), (2), (3), with $f \in D_N(A)$, has a unique classical solution, *i.e.* a function $u \in C([0,\infty), C(\bar{I})) \cap C^1((0,\infty), C(\bar{I}))$ such that $u(t) \in D_N(A)$ for every $t > 0$ and u solves (1), (2). Our main result is the following.

Theorem 1.1 *The initial-boundary value problem* (1), (2), (3) *has a unique classical solution for every $f \in D_N(A)$ if and only if neither r_1 nor r_2 are exit boundaries.*

Theorem 1.1 cannot be reformulated by saying that $(A, D_N(A))$ generates a strongly continuous semigroup if and only if the boundaries are not of exit type: in fact, this last statement is not true. In the language of semigroups, the situation is the following: if the boundaries are of regular or entrance type, then $(A, D_N(A))$ is a generator and the solution of problem (1), (2), (3) is given by the generated semigroup. In the exit case, Theorem 1.1 clearly implies that $(A, D_N(A))$ does not generate; however its closure does, but boundary conditions (3) transforms into conditions of a different kind (see Proposition 3.6). In the natural case the problem is the closedness of $(A, D_N(A))$, that is the continuity of the map $u \mapsto u'/W$ with respect to the graph norm. We shall prove the following result, where we indicate by I_i the interval whose endpoints are x_0 and r_i, for $i = 1, 2$.

Theorem 1.2 *The operator $(A, D_N(A))$ generates a strongly continuous semigroup of positive contractions if and only if at each endpoint r_i one of the following conditions is fulfilled:*

(i) $(mW)^{-1} \in L^1(I_i)$

(ii) $(mW)^{-1} \notin L^1(I_i)$, $W \notin L^1(I_i)$ *and*

$$\sup_{x \in I_i} \left| \int_x^{\sigma(x)} \frac{ds}{m(s)W(s)} \right| < +\infty, \tag{5}$$

where, for every $x \in I_i$, $\sigma(x) \in I_i$ is defined by $\left| \int_x^{\sigma(x)} W(s)ds \right| = 1$.

Notice that condition (ii) above is meaningful only for natural boundaries.

In Section 3 we prove Theorem 1.2 and show that the closure of $(A, D_N(A))$ always generates a strongly continuous semigroup; Section 4 is devoted to characterize when $D_N(A)$ is invariant under the semigroup generated by the closure of $(A, D_N(A))$, and this leads to the proof of Theorem 1.1. Finally, in Section 5 we give some applications to the generation and the regularity of semigroups, under more general boundary conditions, in the case of regular boundaries.

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2 Preliminaries

In this section we collect some results on the behaviour of the solutions of the stationary equation

$$\lambda u - Au = 0, \tag{6}$$

which depends on the type of boundary, *i.e.* on the summability properties of the functions Q, R in (4). All the results we need are proved in [5] and collected in [2, Theorem 4.13], except what is stated in the following lemmas, which are however strongly inspired by the techniques in [5]. Notice that the function W allows us to write A in the form

$$Au = mW\Big(\frac{u'}{W}\Big)'$$

and that the summability of the functions R, Q near *e.g.* r_2 implies the summability of W and $(mW)^{-1}$ respectively. We state and prove the lemmas for r_2; analogous statements hold, of course, for r_1.

Lemma 2.1 *Let r_2 be an exit boundary and u_1, u_2 be positive solutions of* (6) *with u_1 increasing and u_2 decreasing. Then*

$$\lim_{x\to r_2}\frac{u_1'(x)}{W(x)} = +\infty, \qquad \lim_{x\to r_2}\frac{u_2(x)u_1'(x)}{W(x)} = 0.$$

There is no non-zero solution u of (6) *satisfying* $\lim_{x\to r_2}\dfrac{u'(x)}{W(x)} = 0$.

Proof. Integrating (6) with $u = u_1$ we obtain

$$\frac{u_1'(x)}{W(x)} = u_1'(x_0) + \int_{x_0}^{x}\frac{\lambda u_1(s)}{m(s)W(s)}ds \tag{7}$$

and the divergence of the last integral implies that $u_1'(x)/W(x) \to +\infty$ as $x \to r_2$. To prove that $(u_2u_1')/W$ is infinitesimal, notice that by (7)

$$\frac{1}{\lambda}\Big[\frac{u_1'(x)}{W(x)} - u_1'(x_0)\Big] = \int_{x_0}^{x}\frac{u_1(s)}{m(s)W(s)}ds$$

and that u_1 is bounded and u_2 is infinitesimal near r_2 (see [2, Theorem 4.13]), so that it suffices to prove that

$$\lim_{x\to r_2} u_2(x)\int_{x_0}^{x}\frac{1}{m(s)W(s)}ds = 0.$$

To this aim, observe that (7), with u_2 in place of u_1, and $u_2' \le 0$ imply

$$\int_{x_0}^{x}\frac{\lambda u_2(s)}{m(s)W(s)}ds \le |u_2'(x_0)|,$$

and then the integral $\int_{x_0}^{r_2} \frac{u_2(s)}{m(s)W(s)} ds$ is convergent. For $x > y > x_0$ we have

$$u_2(x) \int_{x_0}^{x} \frac{1}{m(s)W(s)} ds \le u_2(x) \int_{x_0}^{y} \frac{1}{m(s)W(s)} ds + \int_{y}^{r_2} \frac{u_2(s)}{m(s)W(s)} ds;$$

since the last integral is convergent and u_2 is infinitesimal, fixing $\varepsilon > 0$, it is possible to choose y such that both quantities in the right hand side are less than ε for x close to r_2 and our claim is proved.

Finally, let $w_0 = u_1'(x_0)u_2(x_0) - u_1(x_0)u_2'(x_0) > 0$ and observe that $u_1'u_2 - u_1u_2' = w_0W$; we obtain $\lim_{x\to r_2} \frac{u_2'(x)}{W(x)} = k < 0$ and since $\lim_{x\to r_2} \frac{u_1'(x)}{W(x)} = +\infty$, we deduce that no non-zero solution u of (6) satisfies $\lim_{x\to r_2} \frac{u'(x)}{W(x)} = 0$. □

Let us now look at the behaviour of u'/W when $(mW)^{-1}$ is summable, an hypothesis that covers regular, entrance and some natural boundaries.

Lemma 2.2 *Let $(mW)^{-1} \in L^1(x_0, r_2)$, then for every $u \in D_M(A)$ the limit $\lim_{x\to r_2} \frac{u'(x)}{W(x)} = \ell$ exists finite. If $W \notin L^1(x_0, r_2)$ then $\ell = 0$. Moreover, the map $u \mapsto u'/W$ is continuous from $D_M(A)$, endowed with the graph norm, into $C(\overline{I})$.*

Proof. Let $f = Au \in C(\overline{I})$. Then we have

$$\frac{u'(x)}{W(x)} = u'(x_0) + \int_{x_0}^{x} \frac{f(s)}{m(s)W(s)} ds \tag{8}$$

and hence $\lim_{x\to r_2} \frac{u'(x)}{W(x)} = \ell$ exists finite. If $W \notin L^1(x_0, r_2)$ then $\ell = 0$ otherwise, if *e.g.* $\ell > 0$, the inequality $\frac{u'(x)}{W(x)} \ge \ell/2$ holds in a neighbourhood of r_2 and implies that u is unbounded near r_2. Finally, (8) gives the continuity of the map $u \mapsto u'/W$, as stated. □

The preceding lemma shows, in particular, that $D_N(A) = D_M(A)$ in case of entrance endpoints. In the natural case, the Corollary to Theorem 13.1 in [5] shows that the maximal domain $D_M(A)$ coincides with the the Ventcel domain, corresponding to "adhesive boundary" conditions

$$D_V(A) = \Big\{ u \in C(\overline{I}) \cap C^2(I) : \lim_{x\to r_1, r_2} Au(x) = 0 \Big\}; \tag{9}$$

we recall that $(A, D_V(A))$ is a generator if and only if both r_1 and r_2 are not entrance boundaries (see [3] or [2, Theorem 4.17]).

We suppose now that r_1, r_2 are not exit boundaries and fix two positive monotonic solutions of (6) u_1, u_2 such that

$$\lim_{x\to r_1} \frac{u_1'(x)}{W(x)} = \lim_{x\to r_2} \frac{u_2'(x)}{W(x)} = 0;$$

the existence of these solutions is proved in [5], see also [2, Theorem 4.13]. We construct the Green function

$$G(x,s) = \begin{cases} \dfrac{u_1(x)u_2(s)}{w_0 W(s)m(s)}, & x \le s, \\ \\ \dfrac{u_1(s)u_2(x)}{w_0 W(s)m(s)}, & x \ge s, \end{cases} \qquad x, s \in I, \tag{10}$$

where we recall that $w_0 = u_1'(x_0)u_2(x_0) - u_1(x_0)u_2'(x_0) > 0$ so that $u_1'u_2 - u_1u_2' = w_0 W$. The integral operator

$$T_\lambda f(x) = \int_{r_1}^{r_2} G(x,s)f(s)ds, \qquad f \in C(\overline{I}). \tag{11}$$

is bounded from $C(\overline{I})$ to $C(\overline{I})$ with $\|T_\lambda\| \le 1/\lambda$ and, for every $f \in C(\overline{I})$, $T_\lambda f \in D_M(A)$ is a solution of the equation $\lambda u - Au = f$. Moreover, $T_\lambda \mathbf{1} = 1/\lambda$ by the choice of u_1, u_2 (see [2, Prop. 4.8]).

3 Generation in $C(\overline{I})$

In this section we prove Theorem 1.2. Let us first show that the operator $(A, D_N(A))$ is always dissipative, independently of the type of boundaries.

Lemma 3.1 *The operator $(A, D_N(A))$ is densely defined and dissipative. Moreover we have for every $u \in D_N(A)$ and $\lambda > 0$:*

$$\sup \lambda u \le \sup(\lambda - A)u \qquad \inf \lambda u \ge \inf(\lambda - A)u.$$

Proof. Let $\sup u = u(c)$ with $c \in \overline{I}$. If $c \in I$, then $Au(c) \le 0$ and $\sup \lambda u = \lambda u(c) \le (\lambda - A)u(c) \le \sup(\lambda - A)u$. If $c = r_i$, $i = 1, 2$, then $Au(r_i) \le 0$ and one concludes as before. Suppose, in fact, $Au(r_2) > 0$; then there are $b < r_2$, $\ell > 0$ such that $Au(x) \ge \ell$ in $[b, r_2[$. Let $b < x < y < r_2$, then

$$\frac{u'(y)}{W(y)} - \frac{u'(x)}{W(x)} \ge \int_x^y \frac{\ell}{m(s)W(s)} ds;$$

letting $y \to r_2$ we obtain

$$-\frac{u'(x)}{W(x)} \ge \int_x^{r_2} \frac{\ell}{m(s)W(s)} ds > 0$$

whence $u'(x) < 0$ in $[b, r_2[$ which is absurd.

Replacing u with $-u$ we obtain $\inf \lambda u \ge \inf(\lambda - A)u$ and the dissipativity follows. □

Observe that for $\lambda > 0$, $(\lambda - A)^{-1}$ is a positive operator on $C(\overline{I})$, when it exists.

Unlike the dissipativity, the generation of a strongly continuous semigroup depends on the type of boundaries. We start our investigation pointing out a simple necessary condition.

Proposition 3.2 *Assume that $(A, D_N(A))$ generates a strongly continuous semigroup in $C(\overline{I})$. Then neither r_1 nor r_2 are exit boundaries.*

PROOF. We suppose that $\lambda - A$ is surjective and show that r_2 is not an exit boundary; to this aim, by Lemma 2.1, it suffices to show that a non-zero solution v of $\lambda u - Au = 0$ exists with v'/W vanishing at r_2. Choose $f \geq 0$, $f \not\equiv 0$, vanishing in $[x_1, r_2[$ and let $u \in D_N(A)$ be the solution of $\lambda u - Au = f$; then $u \geq 0$ by Lemma 3.1 and u is a solution of $\lambda u - Au = 0$ in $]x_1, r_2[$. If we show that $u \not\equiv 0$ in $]x_1, r_2[$, we can take as v the continuation of u to the whole of I and we are done. In fact, we prove that if $J = \{x \in I : u(x) = 0\} \neq \emptyset$, then $u \equiv 0$: hence u never vanishes in I. Assume that $u(x_2) = 0$ and note that $u'(x_2) = 0$, since u does not change sign; from the equation we have $Au \leq \lambda u$, that is

$$\left(\frac{u'}{W}\right)' \leq \frac{\lambda u}{mW}$$

and then for $x \in]x_2, r_2[$

$$u'(x) \leq \lambda W(x) \int_{x_2}^{x} \frac{u(s)}{m(s)W(s)} ds.$$

Choose $\delta < 1$ such that $\lambda W(x) \int_{x_2}^{x} \frac{1}{m(s)W(s)} ds < 1$ for $x \in]x_2, x_2 + \delta[$ and let $M = \sup\{u(x) : x \in]x_2, x_2 + \delta[\}$. Then $u'(x) \leq M$, hence $u(x) \leq M\delta$ for $x \in]x_2, x_2 + \delta[$ and $M \leq M\delta$ implies $M = 0$. A similar argument shows that u vanishes in a left neighbourhood of x_2; therefore J is open, and since it is obviously closed, $J = I$. □

We consider now non-exit boundaries.

Lemma 3.3 *If r_1, r_2 are not exit boundaries, the closure (A, D) of $(A, D_N(A))$ generates a strongly continuous semigroup of positive contractions.*

PROOF. By Lemma 3.1 and the Lumer-Phillips theorem, it suffices to show that $\lambda - A$ has dense range. Suppose that f vanishes in a neighbourhood of r_2; then the function $T_\lambda f$ defined in (11) coincides with a multiple of u_2 near r_2 and hence satisfies the boundary condition at r_2. Since $T_\lambda \mathbf{1} = 1/\lambda$ we deduce that $T_\lambda f$ satisfies the boundary condition at r_2 if f is constant in a neighbourhood of r_2. Repeating the argument for r_1 we obtain that $T_\lambda f \in D_N(A)$ if f is constant in neighbourhoods of the endpoints, hence the range contains these functions and is dense in $C(\overline{I})$. □

Observe that we have shown that T_λ coincides with the resolvent operator of (A, D).

The generation problem is now reduced to characterize when $(A, D_N(A))$ is closed. According to Lemma 2.2, this always happens in the regular or entrance case, and, even more generally, if $(mW)^{-1}$ is summable, a condition that holds also in some natural cases; hence, we now study the case when r_2 is a natural boundary and $(mW)^{-1} \notin L^1(x_0, r_2)$. Let us set

$$D_1 = \Big\{ u \in D_M(A) : \lim_{x \to r_1} \frac{u'(x)}{W(x)} = 0 \Big\};$$

arguing as in Lemma 3.1 (and using that $Au(x) \to 0$ as $x \to r_2$ by [5], Corollary to Theorem 13.1) one sees that (A, D_1) is dissipative. Moreover the function u_2 entering into the definition of T_λ vanishes at r_2 (see again [2, Theorem 4.13]) and repeating the proof of Lemma 3.3 we obtain that the closure of (A, D_1) is a generator and hence coincides with (A, D). Of course (A, D_1) is closed if $(mW)^{-1} \in L^1(r_1, x_0)$ (by Lemma 2.2); if this condition is not satisfied we can repeat the above argument and obtain that the closure of (A, D_1) is the maximal operator $(A, D_M(A))$. At this point, it is clear that $(A, D_N(A))$ is closed if and only if $D_N(A)$ coincides with the closure of D_1 in $D_M(A)$. This remark will be used in the next two lemmas.

Lemma 3.4 *Suppose that r_2 is a natural boundary, that $(mW)^{-1} \notin L^1(x_0, r_2)$ and $W \in L^1(x_0, r_2)$; then the operator $(A, D_N(A))$ is not closed.*

Proof. Let u be a C^2-function vanishing in a neighbourhood of r_1 and equal to $\int_{x_0}^x W(s)ds$ in a neighbourhood of r_2. Then $Au = 0$ and $u'/W = 1$ near r_2 and hence $u \in D_1 \setminus D_N(A)$ and the thesis follows form the above discussion. □

It remains to be considered the case when r_2 is a natural boundary and both the functions W, $(mW)^{-1}$ are not summable near r_2.

In the following lemma we restrict to the particular case $Au = mu''$ with $r_2 = +\infty$ (in this case $W \equiv 1$) and characterize when the condition $u'(x) \to 0$ as $x \to r_2$ is automatically satisfied for every $u \in D_M(A)$; the general case will be reduced to this one through a change of variables.

Lemma 3.5 *Let $Au = mu''$ in $(r_1, +\infty)$; the equality $\lim_{x \to +\infty} u'(x) = 0$ holds for every function $u \in D_M(A)$ if and only if the condition*

$$\sup_{x > x_0} \int_x^{x+1} \frac{ds}{m(s)} < +\infty \tag{12}$$

is satisfied.

Proof. Suppose first that (12) is satisfied. By Taylor's formula we have for every $x > x_0$ every $u \in D_M(A)$

$$u(x+1) - u(x) = u'(x) + \int_x^{x+1} \frac{x+1-s}{m(s)} Au(s)ds.$$

The left hand side of the above equality tends to 0 as $x \to +\infty$ as well as $Au(x)$ by [5], Corollary to Theorem 13.1; then the integral converges to 0 as $x \to +\infty$ by (12) and the same holds for u'.

Suppose now that (12) fails and take points $x_n > x_0$ such that

$$x_{n+1} > x_n + 1 \qquad \text{and} \qquad \int_{x_n}^{x_n+1} \frac{ds}{m(s)} > 2^n;$$

for large n, we divide the interval $[x_n, x_n + 1]$ into n^2 equal parts and consider a subinterval K_n where the integral of $1/m$ is bigger than $2n$. Further, we split K_n into two closed intervals I_n, J_n such that $\int_{I_n} 1/m \ge n$, $\int_{J_n} 1/m \ge n$. Finally we consider continuous functions $\phi_n : [r_1, +\infty[\to \mathbf{R}$ such that

(i) $\phi_n(x) = 0$ for $x \notin K_n$, $\quad \phi_n \ge 0$ in I_n, $\quad \phi_n \le 0$ in J_n;

(ii) $|\phi_n(x)| \le \frac{1}{nm(x)}$;

(iii) $\displaystyle\int_{I_n} \phi_n = -\int_{J_n} \phi_n = 1.$

Put

$$\psi(x) = \sum_{n=1}^{\infty} \int_{r_1}^{x} \phi_n(s)ds;$$

notice that $\psi(x) = 0$ if $x \notin \cup_n K_n$ and that in each interval K_n only one term in the sum is different from zero. Moreover, $\psi \in C^1([r_1, +\infty[)$ and $\psi \in L^1(r_1, +\infty)$ because $\left|\int_{r_1}^x \phi_n\right| \le \chi_{K_n}(x)$ for any $x \ge r_1$, and the length of K_n is $1/n^2$. Finally, since by (ii) $m\psi' \to 0$ as $x \to +\infty$, the function $u(x) = \int_{x_0}^x \psi(s)ds$ belongs to $D_M(A)$ but u' does not have a limit as $x \to +\infty$. □

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. If the boundaries are of exit type both (i) and (ii) fail and, by Proposition 3.2, $(A, D_N(A))$ is not a generator, hence we can exclude the case of exit boundaries.

By Lemma 3.3, we have only to characterize when $(A, D_N(A))$ is closed, *i.e.* when for every sequence $(u_n) \subset D_N(A)$, if $u_n \to u$ with respect to the graph norm then $u'(x)/W(x) \to 0$ as $x \to r_i$, $i = 1, 2$. We discuss only the case of r_2.

If $(mW)^{-1} \in L^1(x_0, r_2)$, Lemma 2.2 ensures that $(A, D_N(A))$ is closed; if $(mW)^{-1} \notin L^1(x_0, r_2)$ and $W \in L^1(x_0, r_2)$, Lemma 3.4 implies that $(A, D_N(A))$ is not closed, and only the case $(mW)^{-1} \notin L^1(x_0, r_2)$, $W \notin L^1(x_0, r_2)$ remains to be treated. Assume first $Au = mu''$ in $(r_1, +\infty)$; by Lemma 3.5 (see also the remarks preceding Lemma 3.4), the boundary condition at r_2 is preserved in $D_N(A)$ by passing to the limit with respect to the graph norm if and only if (12), which coincides with (5) in this particular case, holds.

Let now $Au = mu'' + qu'$; the change of variables

$$s(x) = \int_{x_0}^{x} W(\xi)d\xi, \qquad\qquad u(x) = v(s)$$

maps (r_1, r_2) into $(s(r_1), +\infty)$, transforms the operator Au into $mW^2 v_{ss}$ and the condition $u'/W \to 0$ at r_2 into $v_s \to 0$ at $+\infty$. Changing variables in the integral, it is immediately seen that (5) translates into (12). □

Finally, we come back to the case of exit boundaries and describe the closure of $(A, D_N(A))$. It turns out that the closure of $(A, D_N(A))$ still generates a strongly continuous semigroup, but the Neumann boundary conditions are not preserved, as we shall prove in the next section. For simplicity, we suppose that both r_1 and r_2 are exit boundaries.

Proposition 3.6 *Suppose r_1, r_2 be exit boundaries. Then $(A, D_N(A))$ is not closed; its closure is $(A, D_V(A))$ and generates a strongly continuous semigroup.*

PROOF. First, we show that $D_N(A) \subset D_V(A)$. For, let $u \in D_N(A)$ and $f = Au$. Then, (8) gives $f(r_i) = 0$, $i = 1, 2$, otherwise the non summability of $(mW)^{-1}$ would imply $|u'(x)/W(x)| \to +\infty$ as $x \to r_i$; hence $u \in D_V(A)$.

Let now (A, D) be the closure of $(A, D_N(A))$ and note that D is the closure of $D_N(A)$ in $D_M(A)$ with respect to the graph norm; we show that $D \supset D_V(A)$, proving that every $u \in D_V(A)$ can be approximated by functions in $D_N(A)$. To this aim, let $u \in D_V(A)$ and $f = Au$. Integrating (8) we obtain

$$u(x) - u(x_0) = u'(x_0) \int_{x_0}^{x} W(s)ds + \int_{x_0}^{x} W(s)\Big(\int_{x_0}^{s} \frac{f(\xi)}{m(\xi)W(\xi)} d\xi\Big)ds. \tag{13}$$

Since $f(r_1) = f(r_2) = 0$ and the function $(mW)^{-1}$ is not summable near r_1 and r_2, we can find $a > r_1$, $b < r_2$ and $g \in C(\overline{I})$ with the following properties:

(i) $\operatorname{supp} g \subset [a, b]$;

(ii) $\|g - f\|_\infty \le \varepsilon$;

(iii) $\displaystyle\int_{x_0}^{a} \frac{g}{mW} = \int_{x_0}^{b} \frac{g}{mW} = -u'(x_0)$.

Let v such that $v(x_0) = u(x_0)$ and $Av = g$, *i.e.*

$$v'(x) = W(x)\Big[u'(x_0) + \int_{x_0}^{x} \frac{g(s)}{m(s)W(s)}\Big].$$

Since $g \equiv 0$ in $]r_1, a[$ and $]b, r_2[$, v is constant near r_1, r_2 and belongs to $D_N(A)$; moreover, by (ii), $\|Au - Av\|_\infty = \|f - g\|_\infty \leq \varepsilon$. The equality $v'(x_0) = u'(x_0)$ and (13) imply that

$$u(x) - v(x) = \int_{x_0}^{x} W(s)\Big(\int_{x_0}^{s} \frac{f(\xi) - g(\xi)}{m(\xi)W(\xi)} d\xi\Big) ds$$

whence $\|u - v\|_\infty \leq K\varepsilon$ with $K = \int_{r_1}^{r_2} |R(x)| dx < +\infty$ and hence $D \supset D_V(A)$.

Since, by [3], $(A, D_V(A))$ is a generator and, by Proposition 3.2, $(A, D_N(A))$ is not, we deduce that $(A, D_N(A))$ is not closed. □

We remark that Proposition 3.6 is of local character; in fact, if r_2 is an exit boundary and r_1 arbitrary then each function $u \in D_N(A)$ satisfies $Au(x) \to 0$ as $x \to r_2$ and, conversely, each function $u \in D_M(A)$, zero near r_1 and with Au vanishing at r_2 is the limit (in the graph norm) of a sequence of functions in $D_N(A)$. Using the same techniques as above one can see that the closure of $(A, D_N(A))$ coincides with $(A, D_{NV}(A))$ if r_1 is regular and with $(A, D_{MV}(A))$ if r_1 is entrance or natural where

$$D_{NV}(A) = \Big\{u \in D_M(A) : \lim_{x \to r_1} \frac{u'(x)}{W(x)} = \lim_{x \to r_2} Au(x) = 0\Big\},$$

$$D_{MV}(A) = \Big\{u \in D_M(A) : \lim_{x \to r_2} Au(x) = 0\Big\}.$$

Arguing as in Lemma 3.3 and using a Green function constructed with monotonic solutions u_1, u_2 such that u_i satisfies the boundary condition at r_i, we can prove that $(A, D_{NV}(A))$ and $(A, D_{MV}(A))$ are generators in the cases indicated above (see also [2]). From these observations, Lemma 3.3 and Proposition 3.6, we obtain

Proposition 3.7 *The closure of $(A, D_N(A))$ always generates a strongly continuous semigroup of positive contractions in $C(\overline{I})$.*

Remark 3.8 In particular, if the boundaries are of exit type then the operator $(A, D_N(A))$ is not closed and the evolution leads effectively to its closure, *i.e.*, by Proposition 3.6, to "adhesive boundary", or Ventcel conditions, see (9); from the point of view of diffusion processes, reflecting barrier condition (3) are supposed to correspond to a finite number of reflections near the boundary, and adhesive conditions to an *infinite* number of reflections (in finite time), hence our result seems to support this interpretation, as far as it asserts the possibility of approximating the last condition by the preceding one.

4 The initial-boundary value problem

In this section we prove Theorem 1.1, splitting the proof in two parts: in the first one we assume that neither r_1 nor r_2 are of exit type, and prove that, if $f \in D_N(A)$, a classical solution exists; in the second one we assume for simplicity that both r_1 and r_2 are of exit type and show that a classical solution does not exist whenever the initial datum $f \not\equiv 0$ is positive with compact support in I. Since the closure (A, D) of $(A, D_N(A))$ generates a strongly continuous semigroup $(T(t))_{t\geq 0}$, it is clear that problem (1), (2), (3) with $f \in D_N(A)$ has a (unique) classical solution if and only if $T(t)f \in D_N(A)$ for every $t > 0$. However, since $D_N(A)$ is not closed (in general) neither in $C(\overline{I})$ nor in D, this is not equivalent to say that $(\lambda - A)^{-1} f \in D_N(A)$. To overcome this problem, we use the auxiliary Banach space

$$X = \Big\{ u \in C(\overline{I}) \cap C^1(I) : \ \frac{u'}{W} \in C(\overline{I}) \Big\},$$

endowed with the norm $\|u\|_X = \|u\|_\infty + \|u'/W\|_\infty$.

The operator norm of an operator T acting on a space Z will be denoted by $\|T\|_{L(Z)}$.

Since the proofs of the two parts of Theorem 1.1 are very similar, we give complete details only for Part 1 and indicate the changes needed for Part 2.

Proof of Theorem 1.1, Part 1. Assume that r_1, r_2 are not of exit type and $f \in D_N(A)$; then a (unique) classical solution of problem (1), (2), (3) exists. We write $(\lambda - A)^{-1}$ for the resolvent of (A, D) and recall that it coincides with the operator T_λ defined in (11), as observed after Lemma 3.3.

Step 1. We prove the following: let

$$X_0 = \Big\{ u \in X : \ \frac{u'}{W}(r_1) = \frac{u'}{W}(r_2) = 0 \Big\},$$

then $T_\lambda f \in X_0$ for every $f \in X_0$ and $\|T_\lambda\|_{L(X_0)} \leq 1/\lambda$.

Let $f \in X_0$, and set $F = T_\lambda f$; then $\lambda F - AF = f$, and we have to prove that $F \in X_0$ and to estimate the norm $\|T_\lambda\|_{L(X_0)}$. First, we compute

$$w_0 F'(x) = u_2'(x) \int_{r_1}^{x} \frac{u_1(s) f(s)}{m(s) W(s)} ds + u_1'(x) \int_{x}^{r_2} \frac{u_2(s) f(s)}{m(s) W(s)} ds;$$

since $mW(u_i'/W)' = \lambda u_i$ and $\lim_{x \to r_i} u_i'(x)/W(x) = 0$, integrating by parts the boundary terms vanish and we obtain

$$\frac{F'(x)}{W(x)} = -\frac{1}{\lambda w_0} \Big[\frac{u_2'(x)}{W(x)} \int_{r_1}^{x} u_1'(s) \frac{f'(s)}{W(s)} ds + \frac{u_1'(x)}{W(x)} \int_{x}^{r_2} u_2'(s) \frac{f'(s)}{W(s)} ds \Big] \tag{14}$$

whence

$$\begin{aligned} \Big| \frac{F'(x)}{W(x)} \Big| \ &\leq \ \frac{1}{\lambda w_0 W(x)} \Big[u_1'(x) \big(u_2(x) - u_2(r_2) \big) - u_2'(x) \big(u_1(x) - u_1(r_1) \big) \Big] \Big\| \frac{f'}{W} \Big\|_\infty \\ &\leq \ \frac{1}{\lambda w_0 W(x)} \Big[u_1'(x) u_2(x) - u_1(x) u_2'(x) \Big] \Big\| \frac{f'}{W} \Big\|_\infty = \frac{1}{\lambda} \Big\| \frac{f'}{W} \Big\|_\infty . \end{aligned}$$

This estimate, in connection with $\|T_\lambda\|_{L(C(\overline{I}))} \leq 1/\lambda$, gives $\|T_\lambda\|_{L(X_0)} \leq 1/\lambda$, once we prove that $F \in X_0$. Let us show that $F'(x)/W(x) \to 0$ as $x \to r_2$, the computation for $x \to r_1$ being analogous: to do that, we check that both addends in the right hand side of (14) are infinitesimal as $x \to r_2$. For every $y < x$, taking into account that $u_1' > 0$, $u_2' < 0$ and $u_1' u_2/W \leq w_0$, $|u_1 u_2'|/W \leq w_0$, we have

$$\begin{aligned}
\Big|\frac{u_2'(x)}{W(x)}\int_{r_1}^{x} u_1'(s)\frac{f'(s)}{W(s)}ds\Big| &\leq \Big|\frac{u_2'(x)}{W(x)}\int_{r_1}^{y} u_1'(s)\frac{f'(s)}{W(s)}ds\Big| \\
&\qquad + \frac{|u_1(x)u_2'(x)|}{W(x)} \sup_{y\leq s<r_2}\Big|\frac{f'(s)}{W(s)}\Big| \\
&\leq \Big|\frac{u_2'(x)}{W(x)}\int_{r_1}^{y} u_1'(s)\frac{f'(s)}{W(s)}ds\Big| + w_0 \sup_{y\leq s<r_2}\Big|\frac{f'(s)}{W(s)}\Big|;
\end{aligned}$$

the second addend can be made arbitrarily small by choosing y close enough to r_2, while the first, with y fixed, tends to 0 as $x \to r_2$ since $u_2'(x)/W(x) \to 0$ as $x \to r_2$. As regards the second term in (14), we have, arguing as above

$$\begin{aligned}
\Big|\frac{u_1'(x)}{W(x)}\int_{x}^{r_2} u_2'(s)\frac{f'(s)}{W(s)}ds\Big| &\leq \frac{u_1'(x)u_2(x)}{W(x)} \sup_{x\leq s<r_2}\Big|\frac{f'(s)}{W(s)}\Big| \\
&\leq w_0 \sup_{x\leq s<r_2}\Big|\frac{f'(s)}{W(s)}\Big|,
\end{aligned}$$

which goes to 0 as $x \to r_2$.

Step 2. (Conclusion). Let $(T(t))_{t\geq 0}$ be the semigroup generated by (A, D), and introduce the operator (A_0, D_0) by setting

$$D_0 = \{u \in D \cap X_0 : \ Au \in X_0\}, \qquad A_0 u = Au \text{ for all } u \in D_0.$$

Since $(\lambda - A_0)^{-1} = (\lambda - A)^{-1}\big|_{X_0} = T_\lambda\big|_{X_0}$ is a bounded operator and $D_0 = (\lambda - A)^{-1} X_0$, the operator (A_0, D_0) is closed; moreover, as we shall show in Step 3 below, D_0 is dense in X_0 (for the norm of X_0) hence, by Step 1, we can apply the Hille-Yosida theorem and conclude that (A_0, D_0) generates a strongly continuous semigroup $T_0(t)$ in X_0. Moreover, by the coincidence of the resolvent operators, $T_0(t)f = T(t)f$ for every $f \in X_0$. Since $D_N(A) = D \cap X_0$, we deduce that $D_N(A)$ is invariant under $T(t)$, and the first part of Theorem 1.1 will follow after the next step.

Step 3. (Proof of the density of D_0 in X_0). In order to prove that D_0 is dense in X_0 we consider the dense subspace $Y \subset X_0$ whose elements are the C^∞-functions that are constant in some neighbourhoods of r_1, r_2. If the coefficients m, q are $C^1(I)$, then trivially $Y \subset D_0$, hence the reader who has in mind regular coefficients may skip the rest of the proof. If m, q are assumed only to be continuous, we can prove the density of D_0 in X_0 through a rather indirect argument: given $v \in Y$ and $\varepsilon > 0$, we shall find $u \in D_0$ such that $\|u - v\|_{X_0}$ is of order ε. Let v be locally constant out of a compact interval $J = [s_1, s_2] \subset I$, let $g \in C_0^\infty(J)$ be such that $\|Av - g\|_\infty < \varepsilon$ and

$$\int_{s_1}^{s_2} \frac{u_i(s)Av(s)}{m(s)V(s)} ds = \int_{s_1}^{s_2} \frac{u_i(s)g(s)}{m(s)V(s)} ds, \qquad i = 1, 2 \tag{15}$$

where u_1, u_2 are two solutions of $Au = 0$ verifying the conditions $u_i(s_i) = 0$, $u_i'(s_i) = (-1)^{i-1}$, and $V(x) = u_1'(x)u_2(x) - u_1(x)u_2'(x)$ is their Wronskian. Consider the following regular boundary value problem in J:

$$\begin{cases} Au = g & \text{in } [s_1, s_2] \\ u(s_i) = v(s_i), & i = 1, 2 \end{cases} \tag{16}$$

whose solution is given by

$$\begin{aligned} u(x) \quad = \quad & -\Big[u_2(x) \int_{s_1}^{x} \frac{u_1(s)g(s)}{m(s)V(s)} ds + u_1(x) \int_{x}^{s_2} \frac{u_2(s)g(s)}{m(s)V(s)} ds\Big] \\ & + \frac{v(s_2)}{u_1(s_2)} u_1(x) + \frac{v(s_1)}{u_2(s_1)} u_2(x). \end{aligned} \tag{17}$$

Notice now that $v'(s_1) = v'(s_2) = 0$, and a straightforward computation shows that also $u'(s_1) = u'(s_2) = 0$ holds by virtue of (15); using the equation $Au = g$ we deduce that $u''(s_1) = u''(s_2) = 0$ as well, whence u, continued as a constant in (r_1, s_1) and in (s_2, r_2), belongs to $C^2(\overline{I})$. Moreover, $Au \in C_0^\infty(I)$ and $Au = 0$ in $I \setminus J$ imply that $u \in D_0$. Finally, $\|Au - Av\|_\infty < \varepsilon$ and the continuity of $A^{-1} : C([s_1, s_2]) \longrightarrow C^2([s_1, s_2])$ yield $\|u - v\|_{C^2([s_1,s_2])} \leq C\varepsilon$ and hence $\|u - v\|_{X_0} \leq C\varepsilon$, as claimed.

Proof of Theorem 1.1, Part 2. Assume that r_1, r_2 are both of exit type (for simplicity) and $f \in D_N(A), f \geq 0, f \not\equiv 0$ with compact support; then a classical solution of problem (1), (2), (3) does not exist.

Step 1 Let $(T(t))_{t\geq 0}$ be the semigroup generated by $(A, D_V(A))$ (which is the closure of $(A, D_N(A))$ by Proposition 3.6) and let $(\lambda - A)^{-1}$ its resolvent operator. Fixing two positive monotonic solutions u_1, u_2 of (6) such that $u_i(r_i) = 0$, $i = 1, 2$, we define the Green function as in (10) and obtain

$$(\lambda - A)^{-1} f(x) = \int_{r_1}^{r_2} G(x, s) f(s) ds, \qquad f \in C(\overline{I})$$

(see [2, Theorem 4.17] and [5, Lemma 14.1]). We define

$$X_1 = \Big\{u \in X : \ u(r_1) = u(r_2) = 0\Big\}$$

and prove that $(\lambda - A)^{-1} X_1 \subset X_1$ and that $\|(\lambda - A)^{-1}\|_{L(X_1)} \leq 1/\lambda$. In fact, with $f \in X_1$ and $F = (\lambda - A)^{-1} f$, formula (14) still holds since f vanishes at the boundary and gives $\|(\lambda - A)^{-1}\|_{L(X_1)} \le 1/\lambda$ once we prove that $F \in X_1$. This fact can be verified as in the proof of Part 1 (Step 1), using Lemma 2.1.

STEP 2 Introduce the operator (A_1, D_1) by setting

$$D_1 = \{u \in D_V(A) \cap X_1 : \ Au \in X_1\}, \qquad A_1 u = Au \text{ for all } u \in D_1.$$

The domain D_1 is dense in X_1 (see Step 3 below) and hence, by Step 1, arguing as in the proof of Part 1 (Step 2), we conclude that (A_1, D_1) generates a strongly continuous semigroup $(T_1(t))_{t\ge 0}$ in X_1 such that $T_1(t)f = T(t)f$ for every $f \in X_1$.

Let $f \in D_N(A), f \ge 0, f \not\equiv 0$ with compact support and suppose that problem (1), (2), (3) has a classical solution, that is $T(t)f \in D_N(A)$ for $t \ge 0$. Putting

$$X_2 = \Big\{u \in X_1 : \ \frac{u'}{W}(r_i) = 0, \qquad i = 1, 2\Big\},$$

we deduce that $T_1(t)f \in X_2$ for $t \ge 0$ and hence $(\lambda - A)^{-1} f \in X_2$ for $\lambda > 0$ since X_2 is closed in X_1. We have therefore obtained a solution $u = (\lambda - A)^{-1} f$ of the equation $\lambda u - Au = f$ that belongs to $D_M(A) \cap X_2 \subset D_N(A)$. Arguing as in the proof of Proposition 3.2, we can then construct a non-zero solution v of $\lambda v - Av = 0$ such that $v'(x)/W(X) \to 0$ as $x \to r_2$; this is in contrast with Lemma 2.1.

STEP 3 The proof of the density of D_1 in X_1 is similar to that of D_0 in X_0; let $Y_1 \subset X_1$ be the dense subspace consisting of the C^2-functions that are multiples of $\int_{r_i}^x W(\xi)d\xi$ in some neighbourhoods of r_i, $i = 1, 2$. As above, if the coefficients are regular, then $Y_1 \subset D_1$ and the density follows. Otherwise, we can argue as in Part 1, Step 3, choosing $v \in Y_1$, g as above, and solving again problem (16), where the s_i have the obvious meaning. The solution u is given by (17) and conditions (15) ensure that $u'(s_i) = v'(s_i)$, and the same estimates hold; we omit the details. □

Examples. Let us present some concrete examples of application of the results of Sections 3 and 4.

1. Let $I = (0, 1)$ and $Au = \big(mu'\big)'$; then $(mW)^{-1} = 1$, hence $(A, D_N(A))$ generates a semigroup in $C(\bar{I})$ (this case has been treated also in $L^p(I)$ in [1]).

2. Let $I = (0, +\infty)$, $Au = mu''$ and assume that 0 is a regular point. If $1/m \in L^1(I)$, condition (i) of Theorem 1.2 holds and, if $\inf_I m > 0$, condition (ii) of Theorem 1.2 holds, hence in these cases $(A, D_N(A))$ generates a semigroup. If $m(x) \to 0$ as $x \to +\infty$, both conditions (i) and (ii) above are not satisfied and $(A, D_N(A))$ is not a generator; however $+\infty$ is a natural boundary and problem (1), (2), (3) has a classical solution for any $f \in D_N(A)$.

3. Let $I = (0, 1)$, $Au = mu''$ and assume that 1 is a regular point. If $1/m \in L^1(I)$, condition (i) of Theorem 1.2 holds; if $1/m \notin L^1(I)$ and $\int_{1/2}^x 1/m \in L^1(I)$, 0 is an exit boundary and problem (1), (2), (3) has no solution in general; if $\int_{1/2}^x 1/m \notin L^1(I)$, 0 is a natural boundary, $(A, D_N(A))$ is not a generator because $W \in L^1(I)$, but problem (1), (2), (3) has a classical solution. In particular, for $m(x) = x^\alpha$ (with $\alpha \in \mathbf{R}$), the operator $(A, D_N(A))$ generates a semigroup if and only if $\alpha < 1$, and problem (1), (2), (3) has a classical solution if and only if $\alpha < 1$ or $\alpha \geq 2$. Notice that for $\alpha < 1$ the point 0 is regular, whereas for $\alpha \geq 2$ the point 0 is natural, hence unaccessible. For $1 \leq \alpha < 2$ the point 0 is exit, hence it is accessible as in the regular case, but condition $u'(x) \to 0$ as $x \to 0$ is not preserved in general.

4. Let $I = (0,1)$, $Au = x^{k+1}u'' + bx^k u'$, with $k, b \in \mathbf{R}$. Obviously, 1 is a regular point; as regards the point 0, if $b \leq k < 1$ it is an exit boundary; in all the other cases, a classical solution of problem (1), (2), (3) exists for every $f \in D_N(A)$. Moreover, condition (i) of Theorem 1.2 holds if and only if $b > k$ while condition (ii) holds if and only if $1 \leq b \leq k \leq 2b - 1$: in fact, in this last case, 0 is a natural boundary and it is easy to check condition (12) after the change of variable $s(x) = \int_0^x W(\xi)d\xi$, as in the proof of Theorem 1.2.

5. Let $I = (0, +\infty)$ and $Au = u'' + bx^\alpha u'$, with $\alpha > 0$. If $b < 0$, then condition (i) of Theorem 1.2 holds.

If $b > 0$, then $W(x) = \exp\{-b\frac{x^{\alpha+1}}{\alpha+1}\}$ belongs to $L^1(I)$ and the function $R(x) = W(x)\int_0^x 1/W(\xi)d\xi$ belongs to $L^1(I)$ if and only if $\alpha > 1$; in fact,

$$\int_0^x \exp\Big\{b\frac{\xi^{\alpha+1}}{\alpha+1}\Big\}d\xi \approx C\exp\Big\{b\frac{x^{\alpha+1}}{\alpha+1}\Big\}x^{-\alpha}, \qquad \text{as } x \to +\infty$$

for a suitable $C > 0$. Therefore, for $b > 0$, problem (1), (2), (3) has a classical solution if and only if $\alpha \leq 1$, but the operator $(A, D_N(A))$ is never a generator.

5 Regular boundaries

Let r_1, r_2 be regular boundaries; fixing real numbers α_i, β_i such that $\alpha_i^2 + \beta_i^2 > 0$, $i = 1, 2$, we consider the so-called "elastic barrier" conditions, *i.e.* we put

$$D(A) = \Big\{u \in D_M(A) : \ \alpha_i u(r_i) + \beta_i \lim_{x \to r_i} \frac{u'(x)}{W(x)} = 0, \ i = 1, 2\Big\}.$$

If $\beta_1 = \beta_2 = 0$ we are imposing Dirichlet (or "absorbing barrier") conditions, while if $\alpha_1 = \alpha_2 = 0$ then we are imposing the "reflecting barrier" conditions treated in the previous section. We now show how it is possible to deduce from Theorem 1.2 that $(A, D(A))$ generates a strongly continuous semigroup and study its regularity properties. If the sign condition $(-1)^i\alpha_i\beta_i \geq 0$ holds, the operator $(A, D(A))$ is dissipative and generates a strongly continuous semigroup of positive contractions, as it is shown in [5] (see also [2]). If $\beta_1\beta_2 \neq 0$ we have a dense domain; in the other case, one still obtains a semigroup in the closure of $D(A)$, as it can be easily verified. For example, if $\beta_1 = \beta_2 = 0$, then $\overline{D(A)} = C_0(\overline{I})$ and $D(A) = D_M(A) \cap C_0(\overline{I})$.

Proposition 5.1 *The operator $(A, D(A))$ generates a strongly continuous semigroup.*

Proof. Assume $\beta_1\beta_2 \neq 0$. We perform two changes of variables, in the independent and in the dependent variable, in order to apply Theorem 1.2. First, set $s(x) = \int_{r_1}^x W(\xi)d\xi$, $0 \leq s \leq \ell := s(r_2) < +\infty$, so that the operator $(A, D(A))$ transforms into $(B, D(B))$, where $Bv = pv_{ss}$, with $p = mW^2$, $1/p \in L^1(0, \ell)$, and

$$\begin{aligned} D(B) \quad = \quad & \Big\{v \in C^1([0,\ell]) \cap C^2(]0,\ell[) : \; Bv \in C([0,\ell]), \\ & \alpha_1 v(0) + \beta_1 v'(0) = \alpha_2 v(\ell) + \beta_2 v'(\ell) = 0\Big\}. \end{aligned}$$

Then, let $\phi \in C^2([0,\ell])$ be a strictly positive function such that $\phi(0) = \phi(\ell) = 1$, $\phi'(0) = -\alpha_1/\beta_1$, $\phi'(\ell) = -\alpha_2/\beta_2$ with ϕ linear in neighbourhoods of $0, \ell$. The map $T : C([0,\ell]) \to C([0,\ell])$ given by $Tv = v/\phi$ is a linear isomorphism of $C([0,\ell])$ onto itself and gives $Bv = T^{-1}B_1Tv$, for $v \in D(B)$, where

$$B_1 z = p\Big[z_{ss} + 2\frac{\phi_s}{\phi}z_s + \frac{\phi_{ss}}{\phi}z\Big]$$

for $z \in D(B_1) = \{z \in C^1([0,\ell]) \cap C^2(]0,\ell[) : \; B_1 z \in C([0,\ell]), \; z_s(0) = z_s(\ell) = 0\}$. Observe that $0, \ell$ are regular boundaries for B_1, hence, in particular, condition (i) of Theorem 1.2 is satisfied. Moreover, the term $(p\phi_{ss}/\phi)z$ is a bounded perturbation because $\phi_{ss} = 0$ near the endpoints, hence the operator $(B_1, D(B_1))$ generates a strongly continuous semigroup, and also $(A, D(A))$ by similarity. □

We now prove that the semigroup generated by $(A, D(A))$ is differentiable in $C(\overline{I})$. To this aim we introduce the auxiliary Hilbert space

$$\mathcal{H} = \Big\{u : I \to \mathbf{C} \; : \int_{r_1}^{r_2} \frac{|u|^2}{mW} < +\infty\Big\},$$

endowed with the obvious inner product, and the operator $(A_1, D(A_1))$, where $A_1 u = mu'' + qu'$ for u in

$$D(A_1) = \Big\{u \in \mathcal{H} : u \in H^2_{\text{loc}}(I), A_1 u \in \mathcal{H}, \alpha_i u(r_i) + \beta_i \lim_{x \to r_i} \frac{u'(x)}{W(x)} = 0, i = 1, 2\Big\}.$$

By [8, Sections 17.3 and 19.4] the operator $(A_1, D(A_1))$ is self-adjoint and bounded from above in $\mathcal{H}$, hence it generates an analytic semigroup in $\mathcal{H}$.

We are now in a position to prove the announced regularity result.

Theorem 5.2 *The semigroup generated by $(A, D(A))$ is compact and differentiable in $C(\overline{I})$.*

Proof. Let $(T(t))_{t\geq 0}$, $(T_1(t))_{t\geq 0}$ be the semigroups generated by the operators $(A, D(A))$ and $(A_1, D(A_1))$ respectively and observe that $C(\overline{I}) \subset \mathcal{H}$, $D(A) \subset D(A_1)$ and $Af = A_1 f$ if $f \in D(A)$. If $f \in D(A)$, the uniqueness of the solution of the Cauchy problem $u_t = A_1 u,\ u(0) = f$ implies $T(t)f = T_1(t)f$ for $t \geq 0$ and hence, by density, $T(t)f = T_1(t)f$ for every $f \in C(\overline{I}),\ t \geq 0$.

If $u \in D(A_1)$, by (8), we deduce $|u'(x)| \leq CW(x)$ and $u \in C(\overline{I})$. By the closed graph thorem, the inclusion $D(A_1) \hookrightarrow C(\overline{I})$ is continuous. Since $(T_1(t))_{t\geq 0}$ is analytic, for every $t > 0$ the map $AT_1(t)$ is bounded from $\mathcal{H}$ into $D(A_1)$, hence $AT(t)$ is bounded from $C(\overline{I})$ into itself and the differentiability of $(T(t))_{t\geq 0}$ follows.

To show the compactness of $(T(t))_{t\geq 0}$ it is sufficient to prove that the inclusion $D(A) \hookrightarrow C(\overline{I})$ is compact. For, using (8) again and observing that there is $C > 0$ such that both $|u'(x_0)|$, $\|Au\|_\infty$ are majorized by $C\|u\|_{D(A)}$, we obtain

$$|u'(x)| \leq C\|u\|_{D(A)}[W(x) + |R(x)|];$$

since both W, R are summable, by Ascoli's theorem we deduce the compactness of the above embedding. □

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