# EXISTENCE AND REGULARITY FOR ELLIPTIC EQUATIONS UNDER $p, q$-GROWTH

GIOVANNI CUPINI – PAOLO MARCELLINI – ELVIRA MASCOLO

ABSTRACT. Under general $p, q$-growth conditions, we prove that the Dirichlet problem

$$\begin{cases} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} a^i(x, Du) = b(x) & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega \end{cases}$$

has a weak solution $u \in W^{1,q}_{\text{loc}}(\Omega)$ under the assumptions

$$1 < p \leq q \leq p+1 \quad \text{and} \quad q < p\frac{n-1}{n-p}.$$

More regularity applies. Precisely, this solution is also in the class $W^{1,\infty}_{\text{loc}}(\Omega) \cap W^{2,2}_{\text{loc}}(\Omega)$.

## 1. Introduction

Let $\Omega$ be an open bounded set of $\mathbb{R}^n$, $n \geq 2$. We consider a locally Lipschitz continuous vector field $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ satisfying the *ellipticity* and the *growth conditions*

$$m(1+|\xi|^2)^{\frac{p-2}{2}}|\lambda|^2 \leq \sum_{i,j=1}^{n} a^i_{\xi_j}(x,\xi)\lambda_i\lambda_j\,, \quad \forall\, \xi, \lambda \in \mathbb{R}^n, \tag{1.1}$$

$$\left| a^i_{\xi_j}(x,\xi) \right| \leq M\left(1+|\xi|^2\right)^{\frac{q-2}{2}}, \quad \forall\, \xi \in \mathbb{R}^n, \tag{1.2}$$

for some exponents $q \geq p > 1$ and for constants $M \geq m > 0$. Given a right hand side $b$ and a boundary datum $u_0$, we associate to the vector field $a(x, \xi)$ the *Dirichlet problem*

$$\begin{cases} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} a^i(x, Du) = b(x) & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

A function $u \in W^{1,q}_{\text{loc}}(\Omega)$ is a *weak solution* to the differential equation in (1.3) if

$$\int_\Omega \left\{ \sum_{i=1}^{n} a^i(x, Du)\varphi_{x_i}(x) + b(x)\varphi(x) \right\} dx = 0, \quad \forall\, \varphi \in W^{1,q}_0(\Omega),\ \operatorname{supp}\varphi \Subset \Omega. \tag{1.4}$$

We emphasize that, if $q \neq p$, then the definition of weak solution is well posed only in the class $W^{1,q}_{\text{loc}}(\Omega)$ and it is not sufficient to assume only $u \in W^{1,p}(\Omega)$. This is a main difficulty in the existence theory within this $p, q-$growth context; in fact, the classical existence theory does not apply, due to the *ellipticity* in $W^{1,p}$ and the *growth* in $W^{1,q}$.

We prove that the Dirichlet problem (1.3) has a weak solution under the conditions on $p, q$

$$1 < p \leq q \leq p+1 \quad \text{and} \quad q < p\frac{n-1}{n-p}. \tag{1.5}$$

2000 *Mathematics Subject Classification.* Primary: 35J25; Secondary: 35B65.

*Key words and phrases.* Elliptic equation, existence of solutions, Lipschitz regularity, $p, q$-growth conditions.

*Acknowledgement:* The authors have been supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). .

Moreover this solution $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ is also in the class $W^{1,\infty}_{\mathrm{loc}}(\Omega) \cap W^{2,2}_{\mathrm{loc}}(\Omega)$; precisely see Theorem 2.1. Our bounds on the exponents $p, q$ are quite general and in particular we do not require that they are greater than or equal to 2.

Starting from the pioneering work by De Giorgi [7] (see also the book by Ladyshenskaya-Ural'tseva [12]), the study of the regularity of weak solutions to the elliptic equation in (1.3), under the so-called *natural growth conditions* $p = q$, has been the object of so many papers that it is almost impossible to provide an exhaustive bibliography; here we mention only some relatively more recent and relevant contributions by DiBenedetto [8], Evans [9], Manfredi [14], Tolksdorf [19], the books by Giaquinta [10] and Giusti [11] and the review article by Mingione [18].

The study of problems with $p, q$-growth started in [15] and the following papers [16], [17]. In particular, existence of weak solutions to (1.3) and their local Lipschitz continuity is obtained in [16] whenever $2 \le p \le q < p(n+2)/n$. Differently from [16], we obtain the Lipschitz continuity of the weak solutions into two steps: first by proving a priori the local boundedness of the solutions and then, from that $-$ as a second step $-$ their local Lipschitz continuity. A strategy which gives the existence of Lipschitz solutions of (1.3) under assumptions on $p$ and $q$ substantially more general than those actually known; i.e., in some range the bounds on $p, q$ are new, as described with more details later. In addition, when $q \ge 2$, we prove that locally bounded weak solutions to (1.3) are locally Lipschitz under a less strict condition than $q \le p + 1$, see Remark 6.4.

The condition $q \le p + 1$ (with or without equality), independent of the dimension $n$, seems to be relevant also in other similar contexts; for instance, it appears in the papers by Bildhauer and Fuchs [2], Choe [3], Lee Junjie [13], related to the regularity of locally bounded weak solutions. It also appears in the recent approach to regularity for solutions to parabolic equations and systems by Bögelein, Duzaar and Marcellini [1].

The contents of the paper is described next briefly. Section 2 is devoted to the list of the main assumptions and the precise statement of the existence result. Section 3 is devoted to the *a priori* estimate of the $L^\infty$-norm of $Du$ in terms of $L^p$-norm, by assuming that $u$ is a local *bounded* weak solution. In Section 4 we prove that $u$ is locally bounded; a related result for systems can be found in [5]; see also [4] and [6]. Section 5 is devoted to the proof of the existence result. Finally, in Section 6 we give the specific regularity results when $q \ge 2$.

## 2. Assumptions and existence theorem

We study the existence and the regularity of the solutions to the Dirichlet problem

$$\begin{cases} \displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a^i(x, Du) = b(x) & \text{in } \Omega \\ u = u_0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $b \in L^\infty_{\mathrm{loc}}(\Omega)$ and the functions $a^i(x,\xi)$ for $i = 1, 2, ...n$ are locally Lipschitz-continuous functions in $\Omega \times \mathbb{R}^n$, where $\Omega$ is an open subset of $\mathbb{R}^n$.

Let $1 < p \le q$ and assume that there exist two positive constants $m, M$ such that for every $\xi, \lambda \in \mathbb{R}^n$, for a.e. $x \in \Omega$ and for every $i, j$:

$$m(1 + |\xi|^2)^{\frac{p-2}{2}} |\lambda|^2 \le \sum_{i,j=1}^{n} a^i_{\xi_j}(x,\xi)\lambda_i\lambda_j, \tag{2.2}$$

$$|a^i_{\xi_j}(x,\xi)| \le M(1 + |\xi|^2)^{\frac{q-2}{2}}, \tag{2.3}$$

$$\left| a^i_{\xi_j}(x,\xi) - a^j_{\xi_i}(x,\xi) \right| \le M(1 + |\xi|^2)^{\frac{q+p-4}{4}}, \tag{2.4}$$

$$\left|a^i_{x_j}(x,\xi)\right| \le M(1+|\xi|^2)^{\frac{q+p-2}{4}}. \tag{2.5}$$

Moreover, we assume

$$u_0 \in W^{1,r}(\Omega), \quad \text{with } r = \frac{p(q-1)}{p-1}. \tag{2.6}$$

Under the previous assumptions, $u \in W^{1,q}_{\text{loc}}(\Omega)$ is a weak solution to the Dirichlet problem (2.1) if

$$u - u_0 \in W^{1,p}_0(\Omega) \cap W^{1,q}_{\text{loc}}(\Omega) \tag{2.7}$$

and

$$\int_\Omega \left\{ \sum_{i=1}^n a^i(x,Du)\varphi_{x_i}(x) + b(x)\varphi(x) \right\} dx = 0, \quad \forall \varphi \in W^{1,q}_0(\Omega'), \tag{2.8}$$

where $\Omega'$ is a generic open subset whose closure is contained in $\Omega$.

The following existence and regularity result holds for the Dirichlet problem (2.1):

**Theorem 2.1.** *Let us assume (2.2)-(2.6) with $1 < p \le q \le p+1$ and, if $p < n$, with $q < p\frac{n-1}{n-p}$.*

*Assume that $b \in L^{\frac{p}{p-1}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$. Then there exists a weak solution $u \in W^{1,q}_{\text{loc}}(\Omega)$ to the Dirichlet problem (2.1).*

*In particular, the $W^{1,p}(\Omega)$-norm of $u$ is bounded by a constant depending only to $n$,$p$,$q$,$m$, $M$,$\|Du_0\|_{L^r}$,$\|b\|_{L^{\frac{p}{p-1}}}$.*

*Moreover $u \in W^{1,\infty}_{loc}(\Omega) \cap W^{2,2}_{loc}(\Omega)$ and for all $\Omega' \Subset \Omega$ there exist $C > 0$ and $\alpha, \beta, \gamma > 1$ such that*

$$||u||_{L^\infty(\Omega')} \le C||(1+|Du|^2)^{\frac12}||^\alpha_{L^p(\Omega)},$$

$$||Du||_{L^\infty(\Omega')} \le C||(1+|Du|^2)^{\frac12}||^\beta_{L^p(\Omega)}$$

*and*

$$||D^2u||_{L^2(\Omega')} \le C||(1+|Du|^2)^{\frac12}||^\gamma_{L^p(\Omega)}.$$

*Remark* 2.2. In [16, Theorem 4.1] an analogous result has been proved under the assumptions $n \ge 2$ and $2 \le p \le q < p\frac{n+2}{n}$. It easy to verify, that if $n \ge \frac{p}{2}$ and $p \ge 3$ then the assumptions in Theorem 2.1 are weaker than those in [16].

The proof of this theorem is in Section 5 and it follows from a priori estimates for locally bounded weak solutions to the equation (2.1).

## 3. Lipschitz continuity for locally bounded solutions: a priori estimate

Let us consider the equation

$$\sum_{i=1}^n \frac{\partial}{\partial x_i} a^i(x,Du) = b(x) \qquad \text{in } \Omega \tag{3.1}$$

and let us assume the supplementary assumption: there exists $\epsilon > 0$ such that for every $\xi, \lambda \in \mathbb{R}^n$, for a.e. $x \in \Omega$

$$\epsilon(1+|\xi|^2)^{\frac{q-2}{2}}|\lambda|^2 \le \sum_{i,j=1}^n a^i_{\xi_j}(x,\xi)\lambda_i\lambda_j. \tag{3.2}$$

In this section we prove that the bounded weak solutions of (3.1) are Lipschitz continuous uniformly w.r.t. $\epsilon$ in (3.2).

Let us denote by $B_r(x_0)$ , $B_R(x_0)$ balls compactly contained in $\Omega$ of radii respectively $r, R$ and with the same center. Moreover, we write $V(|Du|^2)$ in place of $(1+|Du|^2)$.

**Lemma 3.1.** *Let $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ be a solution to (3.1). Assume (2.2)-(2.5), (3.2) and $1 < p \le q$. Then there exists a constant $c$ depending on $n, p, q, m, M$, but not on $\epsilon$, such that*

$$\int_\Omega V(|Du|^2)^{\frac{p-2}{2}+\alpha}|D^2u|^2\eta^4\,dx \le \left(1+\alpha+\|b\|_{L^\infty(\mathrm{supp}\,\eta)}\right) c\int_\Omega V(|Du|^2)^{\frac{q}{2}+\alpha}(|\eta|^4+\eta^2|D\eta|^2)\,dx \quad (3.3)$$

*for every $\eta \in C^\infty_c(\Omega)$ and every $\alpha \ge 0$ such that the right hand side is finite.*

*Proof.* By classical regularity results (see for example [11]), by taking into account (3.2) the weak solution $u$ belongs to $W^{2,2}_{\mathrm{loc}}(\Omega)$ when $q \ge 2$ and to $W^{2,q}_{\mathrm{loc}}(\Omega)$ when $1 < q < 2$. Moreover $(1+|Du|^2)^{\frac{q}{4}} \in W^{1,2}_{\mathrm{loc}}(\Omega)$.

By considering as test function $\varphi = \psi_{x_k}$, with $\psi \in C^\infty_c(\Omega)$ and integrating by parts we get

$$\int_\Omega \sum_{i,j=1}^n a^i_{\xi_j}(x,Du)u_{x_jx_k}\psi_{x_i}\,dx = \int_\Omega \left\{-\sum_{i=1}^n a^i_{x_k}(x,Du)\psi_{x_i} + b(x)\psi_{x_k}\right\}dx. \quad (3.4)$$

For $T > 0$ and a.e. $x$ set:

$$V_T(x) = 1 + \min\left\{|Du(x)|^2, T\right\}$$

and consider

$$\psi(x) := V_T^\alpha u_{x_k}[\eta(x)]^4 \qquad \text{with } \alpha \ge 0,$$

where $\eta \in C^\infty_c(\Omega)$.

The function $\psi$ can be inserted in (3.4), that becomes

$$\alpha\int_\Omega \sum_{i,j=1}^n a^i_{\xi_j}(x,Du)u_{x_jx_k}(V_T)_{x_i}u_{x_k}V_T^{\alpha-1}\eta^4\,dx + \int_\Omega \sum_{i,j=1}^n a^i_{\xi_j}(x,Du)u_{x_kx_j}u_{x_kx_i}V_T^\alpha\eta^4\,dx$$

$$= -4\int_\Omega \sum_{i,j=1}^n a^i_{\xi_j}(x,Du)u_{x_kx_j}u_{x_k}\eta^3\eta_{x_i}V_T^\alpha\,dx - \int_\Omega \sum_{i=1}^n a^i_{x_k}(x,Du)\psi_{x_i}(x)\,dx + \int_\Omega b(x)\psi_{x_k}\,dx. \quad (3.5)$$

Let us now consider the first integral at the right hand side:

$$-4u_{x_kx_j}u_{x_k}\eta^3\eta_{x_i}V_T^\alpha = \left\{u_{x_kx_j}\eta^2V_T^{\frac{\alpha}{2}}\right\}\left\{-4u_{x_k}\eta\eta_{x_i}V_T^{\frac{\alpha}{2}}\right\} =: \Lambda_j\Sigma_i.$$

Then, by [16, Lemma 2.4] and Young inequality

$$\left|-4\sum_{i,j=1}^n a^i_{\xi_j}(x,Du)u_{x_kx_j}u_{x_k}\eta^3\eta_{x_i}V_T^\alpha\right| \le c\left(\sum_{i,j=1}^n a^i_{\xi_j}(x,Du)\Lambda_i\Lambda_j\right)^{\frac{1}{2}}[V(|Du|^2)]^{\frac{q-2}{4}}|\Sigma|$$

$$\le \frac{1}{2}\sum_{i,j=1}^n a^i_{\xi_j}(x,Du)\Lambda_i\Lambda_j + c[V(|Du|^2)]^{\frac{q-2}{2}}|\Sigma|^2.$$

By (3.5) we get

$$\alpha\int_\Omega \sum_{i,j=1}^n a^i_{\xi_j}(x,Du)u_{x_jx_k}u_{x_k}(V_T)_{x_i}V_T^{\alpha-1}\eta^4\,dx + \frac{1}{2}\int_\Omega \sum_{i,j=1}^n a^i_{\xi_j}(x,Du)u_{x_kx_j}u_{x_kx_i}V_T^\alpha\eta^4\,dx$$

$$\le c\int_\Omega [V(|Du|^2)]^{\frac{q-2}{2}}V_T^\alpha u_{x_k}^2\eta^2|D\eta|^2\,dx - \int_\Omega \sum_{i=1}^n a^i_{x_k}(x,Du)\psi_{x_i}(x)\,dx + \int_\Omega b(x)\psi_{x_k}\,dx.$$

If we sum on $k = 1, ..., n$, by taking into account that

$$2(V_T)_{x_i}\sum_{k=1}^n u_{x_k}u_{x_jx_k} = (V_T)_{x_i}(V_T)_{x_j}$$

we get

$$\frac{\alpha}{2}\int_\Omega \sum_{i,j=1}^n a^i_{\xi_j}(x,Du)(V_T)_{x_i}(V_T)_{x_j}V_T^{\alpha-1}\eta^4\,dx + \frac{1}{2}\int_\Omega \sum_{i,j,k=1}^n a^i_{\xi_j}(x,Du)u_{x_kx_j}u_{x_kx_i}V_T^\alpha\eta^4\,dx$$

$$\leq c\int_\Omega [V(|Du|^2)]^{\frac{q}{2}}V_T^\alpha\eta^2|D\eta|^2\,dx - \alpha\int_\Omega \sum_{i,k=1}^n a^i_{x_k}(x,Du)V_T^{\alpha-1}(V_T)_{x_i}u_{x_k}\eta^4\,dx$$

$$-\int_\Omega \sum_{i,k=1}^n a^i_{x_k}(x,Du)V_T^\alpha u_{x_kx_i}\eta^4\,dx - 4\int_\Omega \sum_{i,k=1}^n a^i_{x_k}(x,Du)u_{x_k}\eta^3\eta_{x_i}V_T^\alpha\,dx + c\|b\|_{L^\infty(\mathrm{supp}\eta)}\int_\Omega |D\psi|\,dx$$

with $c$ independent of $\alpha$.

By (2.2) and (2.5) the above inequality implies

$$\frac{\alpha m}{2}\int_\Omega [V(|Du|^2)]^{\frac{p-2}{2}}V_T^{\alpha-1}|D(V_T)|^2\eta^4\,dx + \frac{m}{2}\int_\Omega [V(|Du|^2)]^{\frac{p-2}{2}}V_T^\alpha|D^2u|^2\eta^4\,dx$$

$$\leq c\int_\Omega [V(|Du|^2)]^{\frac{q}{2}}V_T^\alpha|D\eta|^2\eta^2\,dx + \alpha c\int_\Omega [V(|Du|^2)]^{\frac{q+p}{4}}V_T^{\alpha-1}|D(V_T)|\eta^4\,dx$$

$$+ c\int_\Omega [V(|Du|^2)]^{\frac{q+p-2}{4}}V_T^\alpha|D^2u|\eta^4\,dx$$

$$+ c\int_\Omega [V(|Du|^2)]^{\frac{q+p}{4}}V_T^\alpha\eta^3|D\eta|\,dx + c\|b\|_{L^\infty(\mathrm{supp}\eta)}\int_\Omega |D\psi|\,dx =: I_1+I_2+I_3+I_4+I_5. \quad (3.6)$$

Let us estimate the right hand side in (3.6).

*Estimate of $I_2$.*

Since $[V(|Du|^2)]^{\frac{q+2}{4}}|D(V_T)| = V_T^{\frac{q+2}{4}}|D(V_T)|$ a.e. then

$$I_2 = \alpha c\int_\Omega \left\{[V(|Du|^2)]^{\frac{p-2}{4}}V_T^{\frac{\alpha-1}{2}}|D(V_T)|\eta^2\right\}\left\{[V_T^{\frac{q+2}{4}+\frac{\alpha-1}{2}}\eta^2\right\}dx$$

$$\leq \frac{\alpha m}{4}\int_\Omega [V(|Du|^2)]^{\frac{p-2}{2}}V_T^{\alpha-1}|D(V_T)|^2\eta^4\,dx + c\alpha\int_\Omega V_T^{\frac{q}{2}+\alpha}\eta^4\,dx. \quad (3.7)$$

*Estimate of $I_3$.*

$$I_3 = c\int_\Omega \left\{[V(|Du|^2)]^{\frac{p-2}{4}}V_T^{\frac{\alpha}{2}}|D^2u|\eta^2\right\}\left\{[V(|Du|^2)]^{\frac{q}{4}}V_T^{\frac{\alpha}{2}}\eta^2\right\}dx$$

$$\leq \frac{m}{4}\int_\Omega [V(|Du|^2)]^{\frac{p-2}{2}}V_T^\alpha|D^2u|^2\eta^4\,dx + c\int_\Omega [V(|Du|^2)]^{\frac{q}{2}}V_T^\alpha\eta^4\,dx. \quad (3.8)$$

*Estimate of $I_5$.*

Taking into account that a.e.

$$|D(V_T)|V_T^{\alpha-1}[V(|Du|^2)]^{\frac{1}{2}} = \left\{|D(V_T)|V_T^{\frac{\alpha-1}{2}}[V(|Du|^2)]^{\frac{p-2}{4}}\right\}\left\{V_T^{\frac{\alpha}{2}}[V(|Du|^2)]^{\frac{2-p}{4}}\right\}$$

then by the Young inequality it holds true that

$$|D(V_T)|V_T^{\alpha-1}[V(|Du|^2)]^{\frac{1}{2}} \leq \frac{m}{8}|D(V_T)|^2V_T^{\alpha-1}[V(|Du|^2)]^{\frac{p-2}{2}} + cV_T^\alpha[V(|Du|^2)]^{\frac{2-p}{2}}.$$

Thus,

$$|D\psi| \leq \alpha V_T^{\alpha-1}|D(V_T)|[V(|Du|^2)]^{\frac{1}{2}}\eta^4 + V_T^\alpha|D^2u|\eta^4 + 4\eta^3|D\eta|V_T^\alpha[V(|Du|^2)]^{\frac{1}{2}}$$

$$\leq \frac{\alpha m}{8} V_T^{\alpha-1}[V(|Du|^2)]^{\frac{p-2}{2}}|D(V_T)|^2\eta^4 + \alpha c\, V_T^{\alpha}[V(|Du|^2)]^{\frac{2-p}{2}}\eta^4$$
$$+ \frac{m}{8} V_T^{\alpha}[V(|Du|^2)]^{\frac{p-2}{2}}|D^2u|^2\eta^4 + cV_T^{\alpha}[V(|Du|^2)]^{\frac{2-p}{2}}\eta^4 + 4\eta^3|D\eta|[V(|Du|^2)]^{\alpha+\frac{1}{2}}. \tag{3.9}$$

Collecting (3.6)–(3.9) we get

$$\frac{\alpha m}{8}\int_\Omega [V(|Du|^2)]^{\frac{p-2}{2}} V_T^{\alpha-1}|D(V_T)|^2\eta^4\,dx + \frac{m}{8}\int_\Omega [V(|Du|^2)]^{\frac{p-2}{2}} V_T^{\alpha}|D^2u|^2\eta^4\,dx$$
$$\leq c\int_\Omega [V(|Du|^2)]^{\frac{q}{2}} V_T^{\alpha}\eta^2|D\eta|^2\,dx + c\,(1+\alpha)\int_\Omega [V(|Du|^2)]^{\frac{q}{2}} V_T^{\alpha}\eta^4\,dx$$
$$+ c\int_\Omega [V(|Du|^2)]^{\frac{q+p}{4}} V_T^{\alpha}\eta^3|D\eta|\,dx$$
$$+ c(1+\alpha)\|b\|_{L^\infty(\mathrm{supp}\eta)}\int_\Omega \left\{[V(|Du|^2)]^{\frac{2-p}{2}}[V(|Du|^2)]^{\alpha}\eta^4 + [V(|Du|^2)]^{\alpha+\frac{1}{2}}4\eta^3|D\eta|\right\}dx$$
$$=: J_1 + J_2 + J_3 + J_4.$$

Taking into account that $2-p\leq q$ and $q>1$ then we can majorize the right hand side as follows

$$J_1+J_2+J_3+J_4 \leq \left(1+\alpha+\|b\|_{L^\infty(\mathrm{supp}\eta)}\right)\, c\int_\Omega [V(|Du|^2)]^{\frac{q}{2}+\alpha}(\eta^4+\eta^2|D\eta|^2)\,dx.$$

By passing to the limit, as $T$ goes to infinity we obtain (3.3). □

From now on, we deal with locally bounded solutions $u$.
Moreover we consider a cut-off function $\eta\in C_c^\infty(B_s(x_0))$ such that

$$B_s(x_0)\subseteq\Omega',\quad 0\leq\eta\leq 1,\quad \eta\equiv 1 \text{ in } B_t(x_0) \text{ with } t<s,\quad |D\eta|\leq\frac{2}{s-t}. \tag{3.10}$$

**Lemma 3.2.** *Let $u\in W^{1,q}_{\mathrm{loc}}(\Omega)\cap L^\infty_{\mathrm{loc}}(\Omega)$ be a weak solution to (3.1). Let $\Omega'\Subset\Omega$. Under the assumptions in Lemma 3.1 there exists $c$, independent of $\epsilon$, such that for every cut-off function $\eta\in C_c^\infty(B_s(x_0))$ satisfying (3.10) and every $\alpha\geq 0$ we have*

$$\int_{B_s}[V(|Du|^2)]^{\frac{p-2}{2}+\alpha}|D^2u|^2\eta^4\,dx$$
$$\leq \frac{c\left(1+\alpha+\|b\|_{L^\infty(\Omega')}\right)^4\left(1+\|u\|^2_{L^\infty(\Omega')}\right)}{(s-t)^4}\int_{B_s}\left([V(|Du|^2)]^{q+\alpha-\frac{p}{2}-1}+[V(|Du|^2)]^{\frac{q}{2}+\alpha-\frac{1}{2}}\right)dx \tag{3.11}$$

*whenever the right hand side is finite.*

*Proof.* For the sake of simplicity, we introduce the notation

$$k_{\alpha,b} := \left(1+\alpha+\|b\|_{L^\infty(\Omega')}\right).$$

Moreover, notice that if $\eta$ is as in (3.10), then $\eta^4+\eta^2|D\eta|^2\leq\frac{4+(\mathrm{diam}(\Omega'))^2}{(s-t)^2}\eta^2$. By Lemma 3.1 (3.3) holds.

Let us estimate the right hand side in (3.3). By an integration by parts we get

$$\frac{k_{\alpha,b}\,c}{(s-t)^2}\int_{B_s}[V(|Du|^2)]^{\frac{q}{2}+\alpha}\eta^2\,dx = \frac{k_{\alpha,b}\,c}{(s-t)^2}\int_{B_s}[V(|Du|^2)]^{\frac{q}{2}+\alpha-1}\left(1+\sum_{k=1}^n u_{x_k}u_{x_k}\right)\eta^2\,dx$$
$$= -\frac{k_{\alpha,b}\,c}{(s-t)^2}\sum_{k=1}^n\int_{B_s}\left([V(|Du|^2)]^{\frac{q}{2}+\alpha-1}u_{x_k}\eta^2\right)_{x_k}u\,dx + \frac{k_{\alpha,b}\,c}{(s-t)^2}\int_{B_s}[V(|Du|^2)]^{\frac{q}{2}+\alpha-1}\eta^2\,dx$$

$$\le\|u\|_\infty \left|\frac{q}{2}+\alpha-1\right| \frac{k_{\alpha,b}\ c}{(s-t)^2}\int_{B_s}[V(|Du|^2)]^{\frac{q}{2}+\alpha-\frac{3}{2}}|D(|Du|^2)|\eta^2\,dx$$
$$+\frac{k_{\alpha,b}\ c}{(s-t)^2}\|u\|_\infty\int_{B_s}[V(|Du|^2)]^{\frac{q}{2}+\alpha-1}|D^2u|\eta^2\,dx$$
$$+\frac{4k_{\alpha,b}\ c}{(s-t)^3}\|u\|_\infty\int_{B_s}[V(|Du|^2)]^{\frac{q}{2}+\alpha-\frac{1}{2}}\eta\,dx+\frac{k_{\alpha,b}\ c}{(s-t)^2}\int_{B_s}[V(|Du|^2)]^{\frac{q}{2}+\alpha-1}\eta^2\,dx \tag{3.12}$$

where $\|u\|_\infty=\|u\|_{L^\infty(\Omega')}$.

By the Young inequality we can estimate the first two integrals in the right hand side. The first one gives

$$\frac{c\,k_{\alpha,b}\|u\|_\infty}{(s-t)^2}\left|\frac{q}{2}+\alpha-1\right|\int_{B_s}[V(|Du|^2)]^{\frac{q}{2}+\alpha-\frac{3}{2}}|D(|Du|^2)|\eta^2\,dx$$
$$=\int_{B_s}\left\{[V(|Du|^2)]^{\frac{p-2}{4}+\frac{\alpha-1}{2}}|D(|Du|^2)|\eta^2\right\}\times$$
$$\times\left\{\frac{c\,k_{\alpha,b}\,\|u\|_\infty}{(s-t)^2}\left|\frac{q}{2}+\alpha-1\right|[V(|Du|^2)]^{\frac{q}{2}+\alpha-\frac{3}{2}-\left(\frac{p-2}{4}+\frac{\alpha-1}{2}\right)}\right\}dx$$
$$\le\frac{1}{16}\int_{B_s}[V(|Du|^2)]^{\frac{p}{2}+\alpha-2}|D(|Du|^2)|^2\eta^4\,dx$$
$$+\frac{ck_{\alpha,b}^2\,\|u\|_\infty^2}{(s-t)^4}\left(\frac{q}{2}+\alpha-1\right)^2\int_{B_s}[V(|Du|^2)]^{q+\alpha-\frac{p}{2}-1}\,dx$$

Thus, by the inequality $|D(|Du|^2)|^2\le 4|Du|^2|D^2u|^2\le 4V(|Du|^2)|D^2u|^2$ and $\left(\frac{q}{2}+\alpha-1\right)^2\le ck_{\alpha,b}^2$, with $c>0$ depending on $q$, but not on $\alpha$, we get

$$\frac{c\,k_{\alpha,b}\|u\|_\infty}{(s-t)^2}\left|\frac{q}{2}+\alpha-1\right|\int_{B_s}[V(|Du|^2)]^{\frac{q}{2}+\alpha-\frac{3}{2}}|D(|Du|^2)|\eta^2\,dx$$
$$\le\frac{1}{4}\int_{B_s}[V(|Du|^2)]^{\frac{p}{2}+\alpha-1}|D^2u|^2\eta^4\,dx+\frac{ck_{\alpha,b}^4\,\|u\|_\infty^2}{(s-t)^4}\int_{B_s}[V(|Du|^2)]^{q+\alpha-\frac{p}{2}-1}\,dx. \tag{3.13}$$

Analogously, the second term in the right hand side of (3.12) gives

$$\frac{c\,k_{\alpha,b}}{(s-t)^2}\|u\|_\infty\int_{B_s}[V(|Du|^2)]^{\frac{q}{2}+\alpha-1}|D^2u|\eta^2\,dx$$
$$\le\int_{B_s}\left\{[V(|Du|^2)]^{\frac{p-2}{4}+\frac{\alpha}{2}}|D^2u|\eta^2\right\}\left\{\frac{c\,k_{\alpha,b}}{(s-t)^2}\|u\|_\infty[V(|Du|^2)]^{\frac{q}{2}+\alpha-1-\left(\frac{p-2}{4}+\frac{\alpha}{2}\right)}\right\}dx$$
$$\le\frac{1}{4}\int_{B_s}[V(|Du|^2)]^{\frac{p}{2}+\alpha-1}|D^2u|^2\eta^4\,dx+\frac{c\,k_{\alpha,b}^2\,\|u\|_\infty^2}{(s-t)^4}\int_{B_s}[V(|Du|^2)]^{q+\alpha-\frac{p}{2}-1}\,dx. \tag{3.14}$$

As far as the last term in the right hand side of (3.12) is concerned, we have

$$\frac{c\,k_{\alpha,b}}{(s-t)^2}\int_{B_s}[V(|Du|^2)]^{\frac{q}{2}+\alpha-1}\eta^2\,dx\le\frac{c\,k_{\alpha,b}\ \operatorname{diam}\Omega'}{(s-t)^3}\int_{B_s}[V(|Du|^2)]^{\frac{q}{2}+\alpha-1}\eta^2\,dx.$$

Therefore, by (3.3) and by (3.12)–(3.14) we get

$$\frac{1}{2}\int_{B_s}[V(|Du|^2)]^{\frac{p-2}{2}+\alpha}|D^2u|^2\eta^4\,dx\le\frac{c\,k_{\alpha,b}^4\,\|u\|_\infty^2}{(s-t)^4}\int_{B_s}[V(|Du|^2)]^{q+\alpha-\frac{p}{2}-1}\,dx$$

$$+\frac{c\,k_{\alpha,b}^2\,\|u\|_\infty^2}{(s-t)^4}\int_{B_s}[V(|Du|^2)]^{q+\alpha-\frac{p}{2}-1}\,dx+\frac{c\,k_{\alpha,b}}{(s-t)^3}\int_{B_s}[V(|Du|^2)]^{\frac{q}{2}+\alpha-\frac{1}{2}}\eta\,dx,$$

that implies (3.11). □

*Remark* 3.3. Observe that when $\alpha=0$ and $q\le p+1$, (3.11) becomes:

$$\int_{B_s}[V(|Du|^2)]^{\frac{p-2}{2}}|D^2u|^2\eta^4\,dx\le c\,\frac{\left(1+\alpha+\|b\|_{L^\infty(\Omega')}\right)^4\left(1+\|u\|^2_{L^\infty(\Omega')}\right)}{(s-t)^4}\int_{B_s}[V(|Du|^2)]^{\frac{q-1}{2}}\,dx.$$

In the following two results $2^*$ is the Sobolev exponent, i.e.,

$$2^*=\begin{cases}\frac{2n}{n-2} & \text{if } n\ge 3\\ \text{any } \mu>2 & \text{if } n=2.\end{cases}\tag{3.15}$$

**Lemma 3.4.** *Let $u\in W^{1,q}_{\rm loc}(\Omega)\cap L^\infty_{\rm loc}(\Omega)$ be a weak solution to (3.1). Let $\Omega'\Subset\Omega$. If the assumptions in Lemma 3.1 hold, then there exists a constant $c$ such that for every cut-off function $\eta\in C^\infty_c(B_s(x_0))$ satisfying (3.10) and every $\alpha\ge 0$ we have*

$$\left\{\int_{B_t}[V(|Du|^2)]^{\left(\frac{p}{2}+\alpha\right)\frac{2^*}{2}}\,dx\right\}^{2/2^*}$$
$$\le\frac{c\,\left(1+\alpha+\|b\|_{L^\infty(\Omega')}\right)^6\left(1+\|u\|^2_{L^\infty(\Omega')}\right)}{(s-t)^4}\int_{B_s}[V(|Du|^2)]^{\alpha+\max\{\frac{p}{2},q-\frac{p}{2}-1,\frac{q-1}{2}\}}\,dx,\tag{3.16}$$

*whenever the right hand side is finite.*

*Proof.* Let $\eta$ be a cut-off function as in Lemma 3.2. By the Sobolev imbedding Theorem

$$\left\{\int_{B_t}[V(|Du|^2)]^{\left(\frac{p}{2}+\alpha\right)\frac{2^*}{2}}\,dx\right\}^{2/2^*}\le\left\{\int_{B_s}\left([V(|Du|^2)]^{\frac{p}{4}+\frac{\alpha}{2}}\eta^2\right)^{2^*}\,dx\right\}^{2/2^*}$$
$$\le\int_{B_s}\left|D\left([V(|Du|^2)]^{\frac{p}{4}+\frac{\alpha}{2}}\eta^2\right)\right|^2\,dx$$
$$\le\frac{c}{(s-t)^2}\int_{B_s}[V(|Du|^2)]^{\frac{p}{2}+\alpha}\eta^2\,dx+c(1+\alpha)^2\int_{B_s}[V(|Du|^2)]^{\frac{p}{2}+\alpha-2}|D(|Du|^2)|^2\eta^4\,dx$$
$$\le\frac{c}{(s-t)^2}\int_{B_s}[V(|Du|^2)]^{\frac{p}{2}+\alpha}\eta^2\,dx+c(1+\alpha)^2\int_{B_s}[V(|Du|^2)]^{\frac{p}{2}+\alpha-1}|D^2u|^2\eta^4\,dx.$$

Thus, using (3.11) to estimate the last integral we get

$$\left\{\int_{B_t}[V(|Du|^2)]^{\left(\frac{p}{2}+\alpha\right)\frac{2^*}{2}}\,dx\right\}^{2/2^*}\le\frac{c}{(s-t)^2}\int_{B_s}[V(|Du|^2)]^{\frac{p}{2}+\alpha}\,dx$$
$$+\frac{c\,(1+\alpha+\|b\|_{L^\infty(\Omega')})^6\,(1+\|u\|^2_{L^\infty(\Omega')})}{(s-t)^4}\int_{B_s}\left\{[V(|Du|^2)]^{q+\alpha-\frac{p}{2}-1}+[V(|Du|^2)]^{\frac{q}{2}+\alpha-\frac{1}{2}}\right\}\,dx$$

and the claim follows. □

Consequence of the above lemma is the Lipschitz regularity estimate for weak solutions to (3.1) under the assumptions (3.2) and $q\le p+1$.

**Theorem 3.5.** *Let $u\in W^{1,q}_{\rm loc}(\Omega)\cap L^\infty_{\rm loc}(\Omega)$ be a weak solution to (3.1), with $1<p\le q\le p+1$. Assume also that (2.2)-(2.5) and (3.2). Then $u\in W^{1,\infty}_{\rm loc}(\Omega)$.*

*Precisely, fixed $\Omega' \Subset \Omega$, there exists a constant $c$ depending on $n,p,q,m,M$, but independent of $\epsilon$, such that for every $B_r(x_0) \subset B_R(x_0) \subseteq \Omega'$ the following estimate holds:*

$$\sup_{B_r(x_0)} |Du| \le c \left( \frac{(1+\|b\|_{L^\infty(B_R(x_0))})^3 (1+\|u\|_{L^\infty(B_R(x_0))})}{(R-r)^2} \right)^{\delta} \left( \int_{B_R(x_0)} \left(1+|Du|^2\right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}}. \quad (3.17)$$

*The exponent $\delta$ is equal to $\frac{n}{p}$ if $n \ge 3$ and it is any number greater than $\frac{2}{p}$ if $n=2$.*

*Proof.* We start using Lemma 3.4, with $\Omega' = B_R(x_0)$. Let us write $\|\cdot\|_\infty$ in place of $\|\cdot\|_{L^\infty(B_R(x_0))}$.

If $q \le p+1$, then

$$q - \frac{p}{2} - 1 \le \frac{q-1}{2} \le \frac{p}{2}.$$

By (3.16) it follows that

$$\left\{ \int_{B_t} [V(|Du|^2)]^{\left(\frac{p}{2}+\alpha\right)\frac{2^*}{2}} dx \right\}^{2/2^*} \le \frac{c\,(1+\alpha)^6\,(1+\|b\|_\infty)^6 (1+\|u\|_\infty^2)}{(s-t)^4} \int_{B_s} [V(|Du|^2)]^{\frac{p}{2}+\alpha} dx. \quad (3.18)$$

Let us define two sequences, $(r_k)$ and $(\alpha_k)$, such that

$$r_k = r + \frac{R-r}{2^{k-1}} \qquad \text{and} \qquad \alpha_k = \frac{p}{2}\left(\frac{2^*}{2}\right)^{k-1} - \frac{p}{2}.$$

In particular, $(\alpha_k)$ is a strictly increasing and positive sequence solution to the difference equation

$$\begin{cases} \frac{p}{2} + \alpha_{k+1} = \left(\frac{p}{2}+\alpha_k\right)\frac{2^*}{2}, \\ \alpha_1 = 0. \end{cases}$$

Let us define $X_k = \|V(|Du|^2)\|_{L^{\frac{p}{2}+\alpha_k}(B_{r_k})}$. Then

$$X_{k+1} = \left\{ \int_{B_{r_{k+1}}} [V(|Du|^2)]^{\frac{p}{2}+\alpha_{k+1}} dx \right\}^{\frac{1}{\frac{p}{2}+\alpha_{k+1}}} = \left\{ \int_{B_{r_{k+1}}} [V(|Du|^2)]^{\left(\frac{p}{2}+\alpha_k\right)\frac{2^*}{2}} dx \right\}^{\frac{2}{2^*}\frac{1}{\frac{p}{2}+\alpha_k}}.$$

Thus, (3.18) can be rewritten as:

$$X_{k+1}^{\frac{p}{2}+\alpha_k} \le \frac{c\,(1+\alpha_k)^6 (1+\|b\|_\infty)^6 (1+\|u\|_\infty^2)}{(r_k - r_{k+1})^4} X_k^{\frac{p}{2}+\alpha_k}.$$

Therefore,

$$X_{k+1} \le c_k X_k, \quad (3.19)$$

where

$$c_k = \left\{ \frac{c\,(1+\|b\|_\infty)^6 (1+\|u\|_\infty^2) 2^{4k}}{(R-r)^4} \left(\frac{2^*}{2}\right)^{6k} \right\}^{\frac{1}{\frac{p}{2}\left(\frac{2^*}{2}\right)^{k-1}}}.$$

By iteration,

$$X_{i+1} \le \left(\Pi_{k=1}^{i} c_k\right) X_1.$$

Notice that

$$\log \Pi_{k=1}^{i} c_k = \sum_{k=1}^{i} \frac{1}{\frac{p}{2}\left(\frac{2^*}{2}\right)^{k-1}} \log \left( \frac{c\,(1+\|b\|_\infty)^6 (1+\|u\|_\infty^2) 2^{4k}}{(R-r)^4} \left(\frac{2^*}{2}\right)^{6k} \right)$$

has a finite limit as $i$ goes to $\infty$. Precisely, since

$$\Pi_{k=1}^{\infty} \left\{(1+\|b\|_\infty)^6 (1+\|u\|_\infty^2)\right\}^{\frac{1}{\frac{p}{2}\left(\frac{2^*}{2}\right)^{k-1}}} = \left\{(1+\|b\|_\infty)^6 (1+\|u\|_\infty^2)\right\}^{\delta},$$

with $\delta = \frac{n}{p}$ if $n \geq 3$ and $\delta$ is any number greater than $\frac{2}{p}$ if $n = 2$, then when $i$ goes to $\infty$ we have

$$\sup_{B_r}[V(|Du|^2)] \leq c\left(\frac{(1+\|b\|_\infty)^6(1+\|u\|_\infty^2)}{(R-r)^4}\right)^\delta \left(\int_{B_R}[V(|Du|^2)]^{\frac{p}{2}}\,dx\right)^{\frac{2}{p}}$$

that implies (3.17). □

## 4. Boundedness and Lipschitz continuity for $W^{1,q}$ solutions

In this section we prove the local boundedness of weak solutions.

**Theorem 4.1.** *Assume (2.2), (2.3) and (2.5), with $1 < p \leq q$. Moreover, if $p < n$ assume also $q < p\frac{n-1}{n-p}$.*

*If $u \in W^{1,q}_{loc}(\Omega)$ is a weak solution to (3.1), then $u$ is locally bounded. Moreover, fixed $\Omega' \Subset \Omega$, there exist $C_1 > 0$ such that for every $B_R(x_0) \subseteq \Omega'$ and $0 < \rho < R$,*

$$\sup_{B_\rho(x_0)} |u| \leq C_1\left(\frac{(1+\|b\|_{L^\infty(\Omega')})^{\frac{1}{p}}}{(R-\rho)^{\frac{q-1}{p-1}}}\right)^{\frac{q}{p^*-q}}\left\{\int_{B_R(x_0)}(1+|Du|)^p\,dx\right\}^{\frac{1+\theta}{p}}, \tag{4.1}$$

*with $\theta = \frac{q}{p}\frac{q-p}{p^*-q}$; here $p^* = \frac{np}{n-p}$, if $p < n$, and $p^*$ is any $\nu > \frac{p(q-1)}{p-1}$, else.*

First we recall the following result, see Lemma 1 in [19].

**Lemma 4.2** (Lemma 1 in [19]). *Assume (2.2), (2.3). Then there exists a positive constant $c$ such that*

$$\sum_{i=1}^n \left(a^i(x,\xi) - a^i(x,\zeta)\right)(\xi_i - \zeta_i) \geq c|\xi-\zeta|^p \quad \textit{if} \quad p \geq 2; \tag{4.2}$$

$$\sum_{i=1}^n \left(a^i(x,\xi) - a^i(x,\zeta)\right)(\xi_i - \zeta_i) \geq c\left(1+|\xi|^2+|\zeta|^2\right)^{\frac{p-2}{2}}|\xi-\zeta|^2 \quad \textit{if} \quad p < 2. \tag{4.3}$$

The above lemma implies that we are considering a monotone operator:

$$\sum_{i=1}^n \left(a^i(x,\xi) - a^i(x,\zeta)\right)(\xi_i - \zeta_i) \geq 0 \qquad \text{for every } \xi, \zeta \in \mathbb{R}^n. \tag{4.4}$$

We are ready to provide a proof of Theorem 4.1.

*Proof of Theorem 4.1.* First of all we prove that (2.3) implies that for fixed $x_0 \in \Omega$ and for every $i = 1, 2, ..., n$, $\eta \in \mathbb{R}^n$ and a.e. $x \in \Omega$

$$|a^i(x,\eta)| \leq \bar{C}(1+|\eta|^2)^{\frac{q-1}{2}} \tag{4.5}$$

with $\bar{C}$ depending on $\Omega, n, q, M$ and $x_0$. Precisely,

$$\bar{C} := |a^i(x_0,0)| + M\left[\operatorname{diam}\Omega + n\max\left\{1, \frac{1}{q-1}\right\}\right].$$

Indeed, consider

$$a^i(x,0) = a^i(x_0,0) + \int_0^1 \langle a^i_x(x_0+t(x-x_0),0), x-x_0\rangle\,dt;$$

thus, by (2.5),

$$\sup_{x\in\Omega}|a^i(x,0)| \leq |a^i(x_0,0)| + M\operatorname{diam}\Omega =: \tilde{M}.$$

Then by (2.3)

$$|a^i(x,\eta)| \le \tilde{M} + \sum_{j=1}^{n} |\eta_j| \int_0^1 |a^i_{\xi_j}(x,t\eta)|\,dt \le \tilde{M} + Mn\,|\eta| \int_0^1 (1+|t\eta|^2)^{\frac{q-2}{2}}\,dt.$$

Then (4.5) holds when $q \ge 2$. If $1 < q < 2$ we get

$$|\eta| \int_0^1 (1+|t\eta|^2)^{\frac{q-2}{2}}\,dt \le |\eta|^{q-1} \int_0^1 t^{q-2}\,dt = \frac{1}{q-1}|\eta|^{q-1}$$

and also in this case (4.5) follows.
Moreover, by Lemma 4.2 for $\eta = 0$, for suitable $0 < \epsilon < c$, we get:

$$\sum_{i=1}^{n} a^i(x,\xi)\xi_i \ge c|\xi|^p + \sum_{i=1}^{n} a^i(x,0)\xi_i \ge (c-\epsilon)|\xi|^p - C\tilde{M}^{\frac{p}{p-1}} \tag{4.6}$$

To construct a sequence of test functions we consider an approximation of the identity function $\mathrm{id} : \mathbb{R}_+ \to \mathbb{R}_+$ by an *increasing* sequence of $C^1$ functions $g_k : \mathbb{R}_+ \to \mathbb{R}_+$, such that

$$g_k(t) = \begin{cases} 0 & \text{for all } t \in [0, \frac{1}{k+1}] \\ k & \text{for all } t \ge k, \end{cases} \qquad 0 \le g_k'(t) \le 2 \quad \text{and} \quad g_k'(t)t \le g_k(t) + \frac{2}{k} \quad \text{in } \mathbb{R}_+. \tag{4.7}$$

The last inequality can be assumed since the restriction of $g_k$ to the interval $\left[\frac{1}{k+1}, k\right]$ can be seen as a smooth approximation of the linear function $G_k(t) = \frac{k(k+1)}{k(k+1)-1}\left(t - \frac{1}{k+1}\right)$, whose graph is the line of the plane connecting $(\frac{1}{k+1}, 0)$ and $(k,k)$ and $G_k$ satisfies $G_k'(t)t \le G_k(t) + \frac{1}{k}$.

Fixed $\nu > 0$ let $\Phi_{k,\nu} : \mathbb{R}_+ \to \mathbb{R}_+$ be the increasing function defined as

$$\Phi_{k,\nu}(t) := g_k(t^{p\nu}).$$

Consider $B_{R_0}(x_0) \Subset \Omega$, $0 < \rho < R \le R_0$ and let $\eta \in C_c^\infty(\Omega)$ be a cut-off function, such that

$$0 \le \eta \le 1, \qquad \eta \equiv 1 \text{ in } B_\rho, \qquad \operatorname{supp} \eta \Subset B_R, \qquad |D\eta| \le \frac{2}{R-\rho}.$$

Let $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ be a weak solution and define the following sequence of test functions:

$$\varphi_{k,\nu}(x) := \Phi_{k,\nu}(|u(x)|)u(x)[\eta(x)]^\mu,$$

where $\mu = \frac{p}{p-1}(q-1)$.
Notice that by (4.7) we easily get

$$\Phi_{k,\nu}'(t)t \le p\nu\left\{\Phi_{k,\nu}(t) + \frac{2}{k}\right\} \le q\nu\left\{\Phi_{k,\nu}(t) + \frac{2}{k}\right\}.$$

Moreover, $\Phi_k$ is in $C^1(\mathbb{R}_+)$, bounded and with bounded derivative; thus $\varphi_{k,\nu} \in W^{1,q}$, with $\operatorname{supp} \varphi_{k,\nu} \Subset B_R$.

From now on, we write $\varphi_k$ and $\Phi_k$ instead of $\varphi_{k,\nu}$ and $\Phi_{k,\nu}$

Let us insert $\varphi_k$ in (2.8), we obtain

$$\begin{aligned}
&\sum_{i=1}^{n} \int_{B_R} a^i(x,Du)u_{x_i}\Phi_k(|u|)\eta^\mu\,dx + \sum_{i=1}^{n} \int_{B_R} a^i(x,Du)u\Phi_k'(|u|)\frac{u}{|u|}u_{x_i}\eta^\mu\,dx \\
&= \mu \sum_{i=1}^{n} \int_{B_R} a^i(x,Du)\Phi_k(|u|)(-u)\eta_{x_i}\eta^{\mu-1}\,dx - \int_{B_R} b(x)\Phi_k(|u|)u\,\eta^\mu\,dx.
\end{aligned}$$

We estimate the first term of the right hand, by applying the monotone property (4.4). For a.e. $x \in B_{R_0} \cap \{\eta \neq 0\}$, by (4.4), we have for $\xi = Du(x)$ and $\zeta = -2\mu u(x)\, \frac{D\eta(x)}{\eta(x)}$,

$$\mu \sum_{i=1}^{n} a^i(x, Du)(-u)\, \eta_{x_i} \eta^{\mu-1} = \frac{\eta^\mu}{2} \sum_{i=1}^{n} a^i(x, Du) \left( \frac{-2\mu u\, \eta_{x_i}}{\eta} \right)$$
$$\leq \frac{\eta^\mu}{2} \left\{ \sum_{i=1}^{n} a^i(x, Du) u_{x_i} + \sum_{i=1}^{n} a^i \left( x, \frac{-2u\, D\eta}{\eta} \right) \left( \frac{-2u\, \eta_{x_i}}{\eta} \right) - \sum_{i=1}^{n} a^i \left( x, \frac{-2u\, D\eta}{\eta} \right) u_{x_i} \right\}.$$

So we obtain

$$\frac{1}{2} \sum_{i=1}^{n} \int_{B_R} a^i(x, Du) u_{x_i} \Phi_k(|u|) \eta^\mu \, dx + \sum_{i=1}^{n} \int_{B_R} a^i(x, Du) u \Phi_k'(|u|) \frac{u}{|u|} u_{x_i} \eta^\mu \, dx$$
$$\leq \frac{1}{2} \sum_{i=1}^{n} \int_{B_R} a^i \left( x, \frac{-2u\, D\eta}{\eta} \right) (\frac{-2u\, D\eta}{\eta}) \eta^\mu \Phi_k(|u|)\, dx + \frac{1}{2} \sum_{i=1}^{n} \int_{B_R} a^i \left( x, \frac{-2u\, D\eta}{\eta} \right) u_{x_i} \eta^\mu \Phi_k(|u|)\, dx$$
$$- \int_{B_R} b(x) \Phi_k(|u|) u\, \eta^\mu \, dx. \tag{4.8}$$

By (4.6) there exist positive constants $c, C$ such that

$$\frac{1}{2} \sum_{i=1}^{n} \int_{B_R} a^i(x, Du) u_{x_i} \Phi_k(|u|) \eta^\mu \, dx + \sum_{i=1}^{n} \int_{B_R} a^i(x, Du) u_{x_i} \Phi_k'(|u|) |u| \eta^\mu \, dx$$
$$\geq c \int_{B_R} |Du|^p \Phi_k(|u|) \eta^\mu \, dx - C \int_{B_R} \left( \Phi_k(|u|) + \Phi_k'(|u|)|u| \right) \eta^\mu \, dx.$$

Therefore, using also (4.5), inequality (4.8) implies that there exists $\tilde{C} > 0$ such that

$$\int_{B_R} |Du|^p \Phi_k(|u|) \eta^\mu \, dx \leq \tilde{C} \int_{B_R} \left( \Phi_k(|u|) + \frac{2}{k} \right) \eta^\mu \, dx$$
$$+ \tilde{C} \int_{B_R} \left( 1 + \left| \frac{uD\eta}{\eta} \right|^2 \right)^{\frac{q}{2}} \Phi_k(|u|) \eta^\mu \, dx + \tilde{C} \int_{B_R} \left( 1 + \left| \frac{uD\eta}{\eta} \right|^2 \right)^{\frac{q-1}{2}} |Du| \Phi_k(|u|) \eta^\mu \, dx$$
$$+ \tilde{C} \|b\|_{L^\infty(\Omega')} \int_{B_R} \Phi_k(|u|) |u| \eta^\mu \, dx$$

Since $\frac{q}{q-1} \leq \frac{p}{p-1}$, we have

$$\tilde{C} \left( 1 + \left| \frac{uD\eta}{\eta} \right|^2 \right)^{\frac{q}{2}} \eta^\mu \leq \tilde{C} \left( 1 + |uD\eta|^2 \right)^{\frac{q}{2}} \eta^{\mu-q} \leq \tilde{C} \max\{1, |D\eta|^{\frac{p(q-1)}{p-1}}\} (1 + |u|^2)^{\frac{p(q-1)}{2(p-1)}}$$

and by Young inequality

$$\tilde{C} \left( 1 + \left| \frac{uD\eta}{\eta} \right|^2 \right)^{\frac{q-1}{2}} |Du| \eta^\mu \leq c|Du| \eta^{\frac{\mu}{p}} \eta^{\mu(1-\frac{1}{p})} + c|Du| \eta^{\frac{\mu}{p}} |uD\eta|^{q-1} \eta^{\mu(1-\frac{1}{p})-q+1}$$
$$\leq \frac{1}{2} |Du|^p \eta^\mu + c\eta^\mu + c|uD\eta|^{\frac{p(q-1)}{p-1}} \eta^{\mu - \frac{p(q-1)}{p-1}},$$

then by the properties of $\eta$ and by recalling that $\mu = \frac{p}{p-1}(q-1)$ we get

$$\int_{B_R} |Du|^p \Phi_k(|u|)\eta^\mu \, dx \le c\int_{B_R} \left(\Phi_k(|u|) + \frac{2}{k}\right)\eta^\mu \, dx + \frac{c}{(R-\rho)^{\frac{p(q-1)}{p-1}}}\int_{B_R} (1+|u|^2)^{\frac{p(q-1)}{2(p-1)}}\Phi_k(|u|)\, dx$$

$$+ \frac{c}{\min\{1,(R-\rho)\}^{\frac{p(q-1)}{p-1}}}\int_{B_R} (1+|u|^2)^{\frac{p(q-1)}{2(p-1)}}\Phi_k(|u|)\, dx + c\|b\|_{L^\infty(\Omega')}\int_{B_R} (1+|u|^2)^{\frac{p(q-1)}{2(p-1)}}\Phi_k(|u|)\, dx$$

Since $\Phi_k(u) \to |u|^{p\nu}$ as $k$ goes to $+\infty$ passing to the limit we get

$$\int_{B_R} (1+|Du|^2)^{\frac{p}{2}}|u|^{p\nu}\eta^\mu \, dx \le \frac{c(1+\|b\|_{L^\infty(\Omega')})}{\min\{1,R-\rho\}^{\frac{p(q-1)}{p-1}}}\int_{B_R} (1+|u|^2)^{\frac{p(q-1)}{2(p-1)}}|u|^{p\nu}\eta^\mu \, dx \tag{4.9}$$

Observe that if $p < n$ then the assumption $q < p\frac{n-1}{n-p}$ implies $p\frac{q-1}{p-1} < p^*$.

Inequality (4.9) is analogous to the inequality (4.33) of [5], then by a careful application of the Sobolev embedding theorem and the classical Moser's iteration method we obtain that $u$ is locally bounded with the following estimate:

$$\sup_{B_\rho(x_0)} |u| \le C_1 \left(\frac{(1+\|b\|_{L^\infty(\Omega')})^{\frac{1}{p}}}{(R-\rho)^{\frac{q-1}{p-1}}}\right)^{\frac{q}{p^*-q}} \left\{\int_{B_R(x_0)} (1+|u|)^{p^*}\, dx\right\}^{\frac{1+\theta}{p^*}}$$

with $\theta$ as in the statement. The Sobolev imbedding gives (4.1). □

By collecting Theorem 3.5 and 4.1, we have

**Theorem 4.3.** *Let $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ be a weak solution to (3.1), with $1 < p \le q \le p+1$ and, if $p < n$, assume also $q < p\frac{n-1}{n-p}$. If (2.2)-(2.5) and (3.2) hold, then $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$.*
*Moreover, fixed $\Omega' \Subset \Omega$, there exist $C, \alpha, \delta, \gamma > 0$, independent of $\epsilon$ in (3.2), such that such that for every $B_r(x_0) \subset B_R(x_0) \subseteq \Omega'$ the following estimate holds:*

$$\sup_{B_r(x_0)} |Du| \le C\frac{(1+\|b\|_{L^\infty(\Omega')})^\alpha}{(R-r)^\delta}\left(\int_{B_R(x_0)} \left(1+|Du|^2\right)^{\frac{p}{2}}\, dx\right)^{\frac{1+\gamma}{p}}. \tag{4.10}$$

## 5. Proof of the existence result

First we state a preliminary result, see also Lemma 4.4 of [16].

**Lemma 5.1.** *Under the assumption (2.2), (2.3) and (2.6) there exists a costant $C$ such that for every $\xi, \eta \in \mathbb{R}^n$ and for a.e. $x \in \Omega$,*

$$|\xi|^p \le C\left\{(1+|\eta|^2)^{\frac{p(q-1)}{2(p-1)}} + \sum_{i=1}^n a^i(x,\xi)(\xi_i - \eta_i)\right\}. \tag{5.1}$$

*Proof.* Fixed $x_0 \in \Omega$, we have that for every $i = 1, 2, ..., n$, every $\eta \in \mathbb{R}^n$ and a.e. $x \in \Omega$, inequality (4.5) holds.

Let $p \ge 2$, by (4.2) and the Young inequality, for all $\epsilon > 0$ we obtain

$$|\xi|^p \le c(|\xi-\eta|^p + |\eta|^p) \le \left\{\sum_{i=1}^n (a^i(x,\xi) - a^i(x,\eta))(\xi_i - \eta_i) + |\eta|^p\right\}$$

$$\le c\left\{|\eta|^p + \sum_{i=1}^n a^i(x,\xi)(\xi_i-\eta_i) + c(n,q,M,x_0,\operatorname{diam}\Omega)(1+|\eta|^2)^{\frac{q-1}{2}}(|\xi|+|\eta|)\right\}$$

$$\leq c\left\{(1+|\eta|^2)^{\frac{p}{2}}+\sum_{i=1}^n a^i(x,\xi)(\xi_i-\eta_i)+c_\epsilon(1+|\eta|^2)^{\frac{p(q-1)}{2(p-1)}}+\epsilon(|\xi|+|\eta|)^p\right\};$$

thus if $\epsilon$ is small enough we get (5.1).

Let now consider $1<p<2$. By the Young inequality with complementary exponents $\frac{2}{p}$ and $\frac{2-p}{2}$ for $\epsilon>0$

$$|\xi|^p \leq c\left(|\xi-\eta|^p+|\eta|^p\right) \leq c\left(|\eta|^p+(|\xi-\eta|^2)^{\frac{p}{2}}(1+|\xi|^2+|\eta|^2)^{\frac{p(p-2)}{4}+\frac{p(2-p)}{4}}\right)$$
$$\leq c\left\{(1+|\eta|^2)^{\frac{p}{2}}+c_\epsilon(1+|\xi|^2+|\eta|^2)^{\frac{p-2}{2}}|\xi-\eta|^2+\epsilon(1+|\xi|^2+|\eta|^2)^{\frac{p}{2}}\right\}.$$

Therefore, by (4.3), for small $\epsilon$ we get

$$|\xi|^p \leq c\left\{(1+|\eta|^2)^{\frac{p}{2}}+\sum_{i=1}^n\left(a^i(x,\xi)-a^i(x,\eta)\right)(\xi_i-\eta_i)\right\}$$

and we conclude by proceeding as above. □

We now turn to prove our existence result.

*Proof of Theorem 2.1.* Fixed $0<\epsilon\leq 1$, let us consider the following Dirichlet problem

$$\begin{cases} \sum_{i=1}^n \frac{\partial}{\partial x_i}\left[a^i(x,Du)+\epsilon(1+|Du|^2)^{\frac{q-2}{2}}u_{x_i}\right]=b(x) & \text{in } \Omega \\ u-u_0\in W_0^{1,q}(\Omega). \end{cases} \tag{5.2}$$

By Lemma 4.2 the differential operator associated to $\left\{a^i\right\}$ is monotone. We can apply the theory of monotone operators to prove the existence of a unique solution $u_\epsilon\in W^{1,q}(\Omega)$ to the problem (5.2).

Now we split the proof into steps:

Step 1. By Lemma 5.1, we prove the boundedness of $\{u_\epsilon\}$ in $W^{1,p}(\Omega)$. More precisely, there exists a constant $C_1$ independent of $\epsilon$, such that

$$||u_\epsilon||_{1,p}\leq C_1. \tag{5.3}$$

In fact, set $a^i_\epsilon(x,\xi)=a^i(x,\xi)+\epsilon(1+|\xi|^2)^{\frac{q-2}{2}}\xi_i$. The functions $a^i_\epsilon$ satisfy the assumptions of Lemma 5.1 with constants $m'=m$, $M'=M+1$. Thus, by Lemma 5.1 applied to $a^i_\epsilon$ with $\xi=Du_\epsilon$, $\eta=Du_0$, give the inequality below with constants independent of $\epsilon$:

$$\int_\Omega |Du_\epsilon|^p\,dx\leq c\left\{\int_\Omega(1+|Du_0|^2)^{\frac{p(q-1)}{2(p-1)}}\,dx+\sum_{i=1}^n a^i_\epsilon(x,Du_\epsilon)D_{x_i}(u_\epsilon-u_0)\,dx\right\}.$$

Since $u_\epsilon$ is the weak solution to (5.2), by Young and Sobolev inequalities we obtain

$$\int_\Omega |Du_\epsilon|^p\,dx\leq c\left\{\int_\Omega(1+|Du_0|^2)^{\frac{p(q-1)}{2(p-1)}}\,dx+c_\tau\int_\Omega|b|^{\frac{p}{p-1}}\,dx+\tau\int_\Omega|u_\epsilon-u_0|^p\,dx\right\}$$
$$\leq c\left\{\int_\Omega(1+|Du_0|^2)^{\frac{p(q-1)}{2(p-1)}}\,dx+c_\tau\int_\Omega|b|^{\frac{p}{p-1}}\,dx+\tau\int_\Omega|Du_\epsilon-Du_0|^p\,dx\right\},$$

for any $\tau>0$; if $\tau$ is small enough the inequality above easily implies (5.3).

Step 2. We claim that for every $\Omega' \Subset \Omega$ the sequence $\{u_\epsilon\}$ is bounded in $L^\infty$. Precisely, by Theorem 4.1 there exists constants $C$ and $\theta$ independent of $\epsilon$ such that,

$$\|u_\epsilon\|_{L^\infty(\Omega')} \le C \left( \int_\Omega \left(1 + |Du_\epsilon|^2\right)^{\frac{p}{2}} dx \right)^{\frac{1+\theta}{p}}.$$

As already noticed, the functions $a^i_\epsilon$ satisfy the assumptions of Lemma 5.1 with constants $m' = m$, $M' = M + 1$. Therefore, by Step 1 the right hand side is bounded uniformly w.r.t. $\epsilon$.

Step 3. Here we prove that for every open sets $\Omega'' \Subset \Omega' \Subset \Omega$ the sequence $\{Du_\epsilon\}$ is bounded in $L^\infty(\Omega'')$. Precisely, there exists a costant $C_2$ independent of $\epsilon$ such that

$$\|Du_\epsilon\|_{L^\infty(\Omega'')} \le C_2 \left(1 + \|u_\epsilon\|_{L^\infty(\Omega')}\right)^\gamma \left( \int_\Omega (1 + |Du_\epsilon|^p)\, dx \right)^{\frac{1}{p}}. \tag{5.4}$$

with the right hand side bounded uniformly w.r.t. $\epsilon$ by the previous steps. The exponent $\gamma$ is positive and it is $\gamma = \frac{n}{p}$ if $n \ge 3$, otherwise $\gamma$ is any number greater than $\frac{2}{p}$.

Indeed, since $a^i_\epsilon(x, \xi)$ satisfy the assumptions of Theorem 3.5 and $\{u_\epsilon\}$ are bounded w.r.t. the $W^{1,p}(\Omega)$ and $L^\infty(\Omega')$ norms, we can apply Theorem 3.5 so obtaining the claim by a covering argument.

Step 4. We claim that for every $\Omega'' \Subset \Omega' \Subset \Omega$ there exists a constant $C_3$ independent of $\epsilon$ such that, if $p \ge 2$,

$$\int_{\Omega''} |D^2 u_\epsilon|^2\, dx \le C_3 \left(1 + \|u_\epsilon\|^2_{L^\infty(\Omega')}\right) \int_\Omega (1 + |Du_\epsilon|^p)\, dx \tag{5.5}$$

and, if $p < 2$,

$$\int_{\Omega''} |D^2 u_\epsilon|^2\, dx \le C_3 \left(1 + \|u_\epsilon\|^2_{L^\infty(\Omega')}\right) \left(1 + \|Du_\epsilon\|^2_{L^\infty(\Omega'')}\right)^{\frac{2-p}{2}} \int_\Omega (1 + |Du_\epsilon|^p)\, dx. \tag{5.6}$$

Also in this case, the constant $C_3$ is independent of $\epsilon$.

This claim follows by Lemma 3.2, precisely by Remark 3.3, and by taking into account that $q \le p + 1$, so we get

$$\int_{\Omega''} (1 + |Du_\epsilon|^2)^{\frac{p-2}{2}} |D^2 u_\epsilon|^2\, dx \le c \left(1 + \|u_\epsilon\|^2_{L^\infty(\Omega')}\right) \int_\Omega (1 + |Du_\epsilon|^2)^{\frac{p}{2}}\, dx.$$

If $p \ge 2$ we immediately conclude. Otherwise, since by Step 3 we have that $\{Du_\epsilon\} \in L^\infty(\Omega'')$, estimate (5.4) implies (5.6).

Step 5. Now, we conclude the proof, by studying the limit $\epsilon \to 0$ of $u_\epsilon$.

By the previous steps the sequence $\{u_\epsilon\}$ is bounded in $W^{2,2}_{\text{loc}}(\Omega) \cap W^{1,\infty}_{\text{loc}}(\Omega)$. Therefore there exists a subsequence, that we still denote by $u_\epsilon$, that converges in the strong topology of $W^{1,2}_{\text{loc}}$ to a function $u$ and we have that

$$u \in (u_0 + W^{1,p}_0) \cap W^{1,\infty}_{\text{loc}} \cap W^{2,2}_{\text{loc}}(\Omega)$$

with $Du_\epsilon$ that converges to $Du$ a.e. in $\Omega$.

Let $\Omega' \Subset \Omega$ and let $\varphi \in W^{1,q}_0(\Omega')$. By definition of weak solution we have that

$$\int_{\Omega'} \left\{ \sum_{i=1}^n a^i_\epsilon(x, Du_\epsilon) \varphi_{x_i}(x) + b(x)\varphi(x) \right\} dx = 0,$$

and we can go to the limit as $\epsilon$ goes to 0. We obtain that $u$ is a locally Lipschitz continuous weak solution to the Dirichlet problem (2.1).

Finally the estimates (5.3), (5.4) and (5.5) hold for $u$ by the lower semicontinuity of the norms. $\square$

## 6. Local Lipschitz regularity of locally bounded solutions

In this section we prove the Local Lipschitz regularity for locally bounded weak solutions to (3.1) when $q \geq 2$ with two type of estimates: in Theorem 6.2 we estimate the $L^\infty$-norm of the gradient with its $L^p$-norm, and in Theorem 6.3 we prove an analogous result, using the $L^q$ norm in place of the $L^p$ one.

The starting point is the following lemma analogous to Lemma 2.8 in [16]; the main difference is that now it can be $1 < p < 2$.

**Lemma 6.1.** *If $q \geq 2$, $1 < p \leq q$ and (2.2)-(2.5) hold then a weak solution $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ to (3.1) satisfies*

$$\int_\Omega \eta^4 \sum_{i=1}^n (1+|u_{x_i}|^2)^{\frac{p}{2}+\gamma-1} |Du_{x_i}|^2 \, dx \leq c\ (1+\gamma) \int_\Omega (\eta^4 + \eta^2 |D\eta|^2) \sum_{i=1}^n (1+|u_{x_i}|^2)^{\frac{q}{2}+\gamma} \, dx$$

*for every $\eta \in C^\infty_c(\Omega)$ and every $\gamma \geq 0$ such that the right hand side is finite.*

*Proof.* Fixed $\gamma \geq 0$ define the odd and Lipschitz function $g_{\gamma,k} : \mathbb{R} \to \mathbb{R}$ by

$$g_{\gamma,k}(t) = t(1+t^2)^\gamma \qquad \text{if } \ |t| \leq k$$

and extended to $\mathbb{R}$ linearly as a function in $C^1(\mathbb{R})$. As a test function in

$$\int_\Omega \left\{ \sum_{i=1}^n a^i(x, Du) \varphi_{x_i} + b(x)\varphi \right\} dx = 0$$

consider the function

$$\varphi = \Delta_{-h}(\eta^4 g_{\gamma,k}(\Delta_h u)),$$

where $\Delta_h$ is the difference quotient in the direction $e_s$ defined by $\Delta_h f(x) = \frac{f(x+he_s)-f(x)}{h}$. Then,

$$\begin{aligned}
&\frac{1}{c} \int_\Omega \int_0^1 \eta^4 g'_{\gamma,k}(\Delta_h u)(1+|Du+th\Delta_h Du|^2)^{\frac{p-2}{2}} |\Delta_h Du|^2 \, dxdt \\
&\leq \int_\Omega \int_0^1 \eta^4 g'_{\gamma,k}(\Delta_h u)(1+|Du+th\Delta_h Du|^2)^{\frac{q}{2}} \, dxdt \\
&+ \int_\Omega \int_0^1 4\eta^3 |D\eta| |g_{\gamma,k}(\Delta_h u)| (1+|Du+th\Delta_h Du|^2)^{\frac{q-1}{2}} \, dxdt \\
&+ \int_\Omega \int_0^1 \eta^2 |D\eta|^2 \frac{g^2_{\gamma,k}}{g'_{\gamma,k}}(\Delta_h u)(1+|Du+th\Delta_h Du|^2)^{\frac{q-2}{2}} \, dx
\end{aligned}$$

for every $\eta \in C^\infty_c(\Omega)$ and every $\gamma \geq 0$ such that the right hand side is finite.

Notice that if $p < 2$ the Young inequality implies

$$|\Delta_h Du|^p \leq c(1+|Du+th\Delta_h Du|^2)^{\frac{p}{2}} + c(1+|Du+th\Delta_h Du|^2)^{\frac{p-2}{2}} |\Delta_h Du|^2.$$

Thus, for any $p > 1$, there exist $Du_{x_s} \in L^{\min\{2,p\}}_{\mathrm{loc}}$ and $D\Delta_h u$ converges a.e. to $Du_{x_s}$. From now on, analogous calculations as those in [16] allow to conclude. $\square$

We state now the first regularity result of this section.

**Theorem 6.2.** *Let $u \in L^\infty_{\mathrm{loc}}(\Omega) \cap W^{1,q}_{\mathrm{loc}}(\Omega)$ be a weak solution to (3.1). Assume (2.2)–(2.5).*

*If $q \geq 2$ and $1 < p \leq q \leq p+1$, then $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$.*
*Moreover, fixed $\Omega' \Subset \Omega$, there exists a constant $c$ depending on the $L^\infty$-norm of $b$ in $\Omega'$, such that for every $B_r(x_0) \subset B_R(x_0) \subseteq \Omega'$ the following estimate holds:*

$$\sup_{B_r(x_0)} |Du| \leq c \left( \frac{(1 + \|u\|_{L^\infty(\Omega')})}{(R-r)^2} \right)^\delta \left( \int_{B_R(x_0)} \left(1 + |Du|^2\right)^{\frac{p}{2}} \, dx \right)^{\frac{1}{p}}.$$

*The exponent $\delta$ is equal to $\frac{n}{p}$ if $n \geq 3$ and it is any number greater than $\frac{2}{p}$ if $n = 2$.*

*Proof.* Consider $x_0 \in \Omega$ and $R > 0$, such that $B_R := B_R(x_0) \subseteq \Omega' \Subset \Omega$. Fix also $0 < r \leq R$. Define $V : [0, \infty) \to [1, \infty)$, $V(t) = (1+t)$ and let $\eta \in C^\infty_c(B_s(x_0))$, $r < s < R$, be a cut-off function satisfying the following assumptions

$$0 \leq \eta \leq 1, \qquad \eta \equiv 1 \text{ in } B_t(x_0) \text{ with } r \leq t < s \qquad |D\eta| \leq \frac{2}{s-t}.$$

We split the proof into different steps.

Step 1. By Lemma 6.1 and the assumptions on $\eta$, we get

$$\int_{B_s} \sum_{i=1}^n (1 + |u_{x_i}|^2)^{\frac{p}{2}+\gamma-1} |D^2 u|^2 \eta^4 \, dx \leq \frac{\bar{c}\,(1+\gamma)}{(s-t)^2} \int_{B_s} \sum_{i=1}^n (1 + |u_{x_i}|^2)^{\frac{q}{2}+\gamma} \eta^2 \, dx \tag{6.1}$$

for some constant $\bar{c}$ possibly depending on $\operatorname{diam} \Omega'$.

Step 2. In this step we prove that there exists $c$ independent of $\gamma$, such that

$$\begin{aligned} \int_{B_s} \sum_{i=1}^n (1 + |u_{x_i}|^2)^{\frac{p}{2}+\gamma-1} |Du_{x_i}|^2 \eta^4 \, dx &\leq \frac{c\,(1+\gamma)^4\,(1+\|u\|^2_\infty)}{(s-t)^4} \times \\ \times \int_{B_s} \left\{ \sum_{i=1}^n (1 + |u_{x_i}|^2)^{q+\gamma-\frac{p}{2}-1} + \sum_{i=1}^n (1 + |u_{x_i}|^2)^{\frac{q}{2}+\gamma-\frac{1}{2}} \right\} dx. \end{aligned} \tag{6.2}$$

Let us estimate the right hand side in (6.1). By an integration by parts we get

$$\begin{aligned} &\frac{\bar{c}\,(1+\gamma)}{(s-t)^2} \int_{B_s} \sum_{i=1}^n (1 + |u_{x_i}|^2)^{\frac{q}{2}+\gamma} \eta^2 \, dx \\ =&\frac{\bar{c}\,(1+\gamma)}{(s-t)^2} \int_{B_s} \sum_{i=1}^n (1 + |u_{x_i}|^2)^{\frac{q}{2}+\gamma-1} \eta^2 u_{x_i} u_{x_i} \eta^2 \, dx + \frac{\bar{c}\,(1+\gamma)}{(s-t)^2} \int_{B_s} \sum_{i=1}^n (1 + |u_{x_i}|^2)^{\frac{q}{2}+\gamma-1} \eta^2 \, dx \\ =&-\frac{\bar{c}\,(1+\gamma)}{(s-t)^2} \sum_{i=1}^n \int_{B_s} \left( (1 + |u_{x_i}|^2)^{\frac{q}{2}+\gamma-1} u_{x_i} \eta^2 \right)_{x_i} u \, dx + \frac{\bar{c}\,(1+\gamma)}{(s-t)^2} \int_{B_s} \sum_{i=1}^n (1 + |u_{x_i}|^2)^{\frac{q}{2}+\gamma-1} \eta^2 \, dx \\ \leq& c\|u\|_\infty \left( \frac{q}{2} + \gamma - 1 \right) \frac{\bar{c}\,(1+\gamma)}{(s-t)^2} \int_{B_s} (1 + |u_{x_i}|^2)^{\frac{q}{2}+\gamma-\frac{3}{2}} |(|u_{x_i}|^2)_{x_i}| \eta^2 \, dx \\ &+ \frac{\bar{c}\,(1+\gamma)}{(s-t)^2} \|u\|_\infty \sum_{i=1}^n \int_{B_s} (1 + |u_{x_i}|^2)^{\frac{q}{2}+\gamma-1} |u_{x_i x_i}| \eta^2 \, dx \\ &+ \frac{4\bar{c}\,(1+\gamma)}{(s-t)^3} \|u\|_\infty \int_{B_s} \sum_{i=1}^n (1 + |u_{x_i}|^2)^{\frac{q}{2}+\gamma-\frac{1}{2}} \eta \, dx + \frac{\bar{c}\,(1+\gamma)}{(s-t)^2} \int_{B_s} \sum_{i=1}^n (1 + |u_{x_i}|^2)^{\frac{q}{2}+\gamma-1} \eta^2 \, dx \end{aligned} \tag{6.3}$$

where $\|u\|_\infty = \|u\|_{L^\infty(\Omega')}$.

Let us now write all the constants as $c$, that may vary from line to line. By the Young inequality we can estimate the first two integrals in the right hand side. The first one gives

$$
\begin{aligned}
& c\|u\|_\infty \left(\frac{q}{2}+\gamma-1\right) \frac{\bar{c}\,(1+\gamma)}{(s-t)^2} \int_{B_s} \sum_{i=1}^n (1+|u_{x_i}|^2)^{\frac{q}{2}+\gamma-\frac{3}{2}} |(|u_{x_i}|^2)_{x_i}| \eta^2 \, dx \\
& = \sum_{i=1}^n \int_{B_s} \left\{ (1+|u_{x_i}|^2)^{\frac{p-2}{4}+\frac{\gamma-1}{2}} |(|u_{x_i}|^2)_{x_i}| \eta^2 \right\} \times \\
& \qquad \times \left\{ \frac{c\,(1+\gamma)\,\|u\|_\infty}{(s-t)^2} \left(\frac{q}{2}+\gamma-1\right) (1+|u_{x_i}|^2)^{\frac{q}{2}+\gamma-\frac{3}{2}-\left(\frac{p-2}{4}+\frac{\gamma-1}{2}\right)} \right\} dx \\
& \le \frac{1}{16} \sum_{i=1}^n \int_{B_s} (1+|u_{x_i}|^2)^{\frac{p}{2}+\gamma-2} |(|u_{x_i}|^2)_{x_i}|^2 \eta^4 \, dx \\
& + \frac{c(1+\gamma)^2\,\|u\|_\infty^2}{(s-t)^4} \left(\frac{q}{2}+\gamma-1\right)^2 \int_{B_s} \sum_{i=1}^n (1+|u_{x_i}|^2)^{q+\gamma-\frac{p}{2}-1} \, dx.
\end{aligned}
$$

Thus, by the inequality $|(|u_{x_i}|^2)_{x_i}|^2 \le 4|u_{x_i}|^2 |u_{x_i x_i}|^2 \le 4(1+|u_{x_i}|^2)|Du_{x_i}|^2$ and $\left(\frac{q}{2}+\gamma-1\right)^2 \le (1+\gamma)^2$ with $c>0$ depending on $q$, but not on $\gamma$, we get

$$
\begin{aligned}
& c\|u\|_\infty \left(\frac{q}{2}+\gamma-1\right) \frac{\bar{c}\,(1+\gamma)}{(s-t)^2} \int_{B_s} \sum_{i=1}^n (1+|u_{x_i}|^2)^{\frac{q}{2}+\gamma-\frac{3}{2}} |(|u_{x_i}|^2)_{x_i}| \eta^2 \, dx \\
& \le \frac{1}{4} \int_{B_s} \sum_{i=1}^n (1+|u_{x_i}|^2)^{\frac{p}{2}+\gamma-1} |Du_{x_i}|^2 \eta^4 \, dx + \frac{c(1+\gamma)^4\,\|u\|_\infty^2}{(s-t)^4} \int_{B_s} \sum_{i=1}^n (1+|u_{x_i}|^2)^{q+\gamma-\frac{p}{2}-1} \, dx. \quad (6.4)
\end{aligned}
$$

Analogously, the second term in the right hand side of (6.3) gives

$$
\begin{aligned}
& \frac{c\,(1+\gamma)}{(s-t)^2} \|u\|_\infty \sum_{i=1}^n \int_{B_s} (1+|u_{x_i}|^2)^{\frac{q}{2}+\gamma-1} |D^2 u| \eta^2 \, dx \\
& \le \sum_{i=1}^n \int_{B_s} \left\{ (1+|u_{x_i}|^2)^{\frac{p-2}{4}+\frac{\gamma}{2}} |u_{x_i x_i}| \eta^2 \right\} \left\{ \frac{c\,(1+\gamma)}{(s-t)^2} \|u\|_\infty (1+|u_{x_i}|^2)^{\frac{q}{2}+\gamma-1-\left(\frac{p-2}{4}+\frac{\gamma}{2}\right)} \right\} dx \\
& \le \frac{1}{4} \sum_{i=1}^n \int_{B_s} (1+|u_{x_i}|^2)^{\frac{p}{2}+\gamma-1} |Du_{x_i}|^2 \eta^4 \, dx + \frac{c\,(1+\gamma)^2\,\|u\|_\infty^2}{(s-t)^4} \sum_{i=1}^n \int_{B_s} (1+|u_{x_i}|^2)^{q+\gamma-\frac{p}{2}-1} \, dx. \quad (6.5)
\end{aligned}
$$

By (6.1) and by (6.3)–(6.5) we get

$$
\begin{aligned}
& \frac{1}{2} \int_{B_s} \sum_{i=1}^n (1+|u_{x_i}|^2)^{\frac{p}{2}+\gamma-1} |Du_{x_i}|^2 \eta^4 \, dx \le \frac{c\,(1+\gamma)^4\,\|u\|_\infty^2}{(s-t)^4} \int_{B_s} \sum_{i=1}^n (1+|u_{x_i}|^2)^{q+\gamma-\frac{p}{2}-1} \, dx \\
& + \frac{c\,(1+\gamma)^2\,\|u\|_\infty^2}{(s-t)^4} \int_{B_s} \sum_{i=1}^n (1+|u_{x_i}|^2)^{q+\gamma-\frac{p}{2}-1} \, dx + \frac{c\,(1+\gamma)}{(s-t)^3} \int_{B_s} \sum_{i=1}^n (1+|u_{x_i}|^2)^{\frac{q}{2}+\gamma-\frac{1}{2}} \eta \, dx \\
& + \frac{c\,(1+\gamma)}{(s-t)^2} \int_{B_s} \sum_{i=1}^n (1+|u_{x_i}|^2)^{\frac{q}{2}+\gamma-1} \eta^2 \, dx,
\end{aligned}
$$

that implies (6.2).

Step 3. In this step we prove that there exists $c$, possibly depending on $R$, but not on $\gamma$, such that

$$\left\{\int_{B_t}\sum_{i=1}^n (1+|u_{x_i}|^2)^{\left(\frac{p}{2}+\gamma\right)\frac{2^*}{2}}\,dx\right\}^{2/2^*}$$
$$\leq c\frac{(1+\gamma)^6(1+\|u\|^2_\infty)}{(s-t)^4}\int_{B_s}\sum_{i=1}^n\left\{(1+|u_{x_i}|^2)^{\frac{p}{2}+\gamma}+(1+|u_{x_i}|^2)^{q+\gamma-\frac{p}{2}-1}+(1+|u_{x_i}|^2)^{\frac{q}{2}+\gamma-\frac{1}{2}}\right\}dx. \tag{6.6}$$

By the Sobolev imbedding Theorem

$$\left\{\int_{B_s}\left((1+|u_{x_i}|^2)^{\frac{p}{4}+\frac{\gamma}{2}}\eta^2\right)^{2^*}dx\right\}^{2/2^*}\leq \int_{B_s}\left|D\left((1+|u_{x_i}|^2)^{\frac{p}{4}+\frac{\gamma}{2}}\eta^2\right)\right|^2dx$$
$$\leq \frac{c}{(s-t)^2}\int_{B_s}(1+|u_{x_i}|^2)^{\frac{p}{2}+\gamma}\eta^2\,dx+c(1+\gamma)^2\int_{B_s}(1+|u_{x_i}|^2)^{\frac{p}{2}+\gamma-2}|D(|u_{x_i}|^2)|^2\eta^4\,dx$$
$$\leq \frac{c}{(s-t)^2}\int_{B_s}(1+|u_{x_i}|^2)^{\frac{p}{2}+\gamma}\eta^2\,dx+c(1+\gamma)^2\int_{B_s}(1+|u_{x_i}|^2)^{\frac{p}{2}+\gamma-1}|Du_{x_i}|^2\eta^4\,dx.$$

Therefore, (6.2) implies

$$\left\{\int_{B_s}\left((1+|u_{x_i}|^2)^{\frac{p}{4}+\frac{\gamma}{2}}\eta^2\right)^{2^*}dx\right\}^{2/2^*}\leq\frac{c}{(s-t)^2}\int_{B_s}(1+|u_{x_i}|^2)^{\frac{p}{2}+\gamma}\eta^2\,dx$$
$$+\frac{c\,(1+\gamma)^6\,(1+\|u\|^2_\infty)}{(s-t)^4}\int_{B_s}\left\{\sum_{i=1}^n(1+|u_{x_i}|^2)^{q+\gamma-\frac{p}{2}-1}+\sum_{i=1}^n(1+|u_{x_i}|^2)^{\frac{q}{2}+\gamma-\frac{1}{2}}\right\}dx.$$

By using the inequality $\sum_{i=1}^n y_i^a\leq(\sum_{i=1}^n y_i)^a$ with $a=2^*/2>1$, the Minkowski's inequality with exponent $2^*/2$, and using (6.2) to estimate the last integral in the chain of inequalities above, we get

$$\int_{B_t}\left\{\sum_{i=1}^n(1+|u_{x_i}|^2)^{\left(\frac{p}{2}+\gamma\right)\frac{2^*}{2}}\,dx\right\}^{2/2^*}\leq\left\{\int_{B_s}\left\{\sum_{i=1}^n\eta^4(1+|u_{x_i}|^2)^{\frac{p}{2}+\gamma}\right\}^{\frac{2^*}{2}}dx\right\}^{2/2^*}$$
$$\leq\sum_{i=1}^n\left\{\int_{B_s}\left((1+|u_{x_i}|^2)^{\frac{p}{4}+\frac{\gamma}{2}}\eta^2\right)^{2^*}dx\right\}^{2/2^*}$$

and the claim follows.

Step 4. *Iteration.*
Since $q\leq p+1$ then

$$q-\frac{p}{2}-1\leq\frac{q-1}{2}\leq\frac{p}{2}.$$

By (6.6) it follows that

$$\left\{\int_{B_t}\sum_{i=1}^n(1+|u_{x_i}|^2)^{\left(\frac{p}{2}+\gamma\right)\frac{2^*}{2}}\,dx\right\}^{2/2^*}\leq\frac{c\,(1+\gamma)^6(1+\|u\|^2_\infty)}{(s-t)^4}\int_{B_s}\sum_{i=1}^n(1+|u_{x_i}|^2)^{\frac{p}{2}+\gamma}\,dx.$$

Now, the proof follows the same scheme of the proof of Theorem 3.5. □

In the next result we prove an estimate of the $L^\infty$-norm of the gradient with its $L^q$-norm. This can be obtained also for some $q>p+1$.

**Theorem 6.3.** *Let $u \in L^\infty_{\rm loc}(\Omega) \cap W^{1,q}_{\rm loc}(\Omega)$ be a weak solution to (3.1) and let (2.2)–(2.5) hold. If $n = 2$, assume $q \le p + 2$; if instead $n \ge 3$,*

$$\begin{cases} q \le p+2 & \text{if } p > n-2 \\ q < \dfrac{n}{n-1}(p+1) & \text{if } p \le n-2. \end{cases}$$

*Then $u \in W^{1,\infty}_{\rm loc}(\Omega)$.*
*Moreover, fixed $\Omega' \Subset \Omega$, there exists a constant $c$ depending on the $L^\infty$-norm of $b$ in $\Omega'$, such that for every $B_r(x_0) \subset B_R(x_0) \subseteq \Omega'$ the following estimate holds:*

$$\sup_{B_r(x_0)} |Du| \le c \left( \frac{1 + \|u\|_{L^\infty(\Omega')}}{(R-r)^2} \right)^{\gamma\Theta} \left( \int_{B_R(x_0)} \left(1 + |Du|^2\right)^{\frac{q}{2}} \, dx \right)^{\frac{\Theta}{q}}$$

*for some $\Theta > 1$.*

*Proof.* If $q \le p + 1$ the thesis follows by Theorem 6.2.
Let us assume that $q > p + 1$. The first four steps of the proof are the same of Theorem 6.2; only the last one changes.

Step 4. *Iteration.*
By the assumption $q > p + 1$,

$$\frac{p}{2} < \frac{q-1}{2} < q - \frac{p}{2} - 1.$$

Thus, by (6.6) we get

$$\left\{ \int_{B_t} [V(|Du|^2)]^{\left(\frac{p}{2}+\gamma\right)\frac{2^*}{2}} \, dx \right\}^{2/2^*} \le \frac{c\,(1+\gamma)^6 (1 + \|u\|^2_\infty)}{(s-t)^4} \int_{B_s} [V(|Du|^2)]^{q - \frac{p}{2} - 1 + \gamma} \, dx. \tag{6.7}$$

Let us denote

$$\lambda := \frac{2}{2^* - 2} \left[ q - \frac{p}{2} \left( 1 + \frac{2^*}{2} \right) - 1 \right],$$

where $2^*$ is the Sobolev exponent (3.15), and define two sequences, $(r_k)$ and $(\gamma_k)$, as follows:

$$r_k = r + \frac{R-r}{2^{k-1}} \qquad \text{and} \qquad \gamma_k = \left( \frac{p-q}{2} + 1 - \lambda \right) \left( \frac{2^*}{2} \right)^{k-1} + \lambda. \tag{6.8}$$

An easy computation shows that $\gamma_k$ solves the difference equation

$$\begin{cases} q - \frac{p}{2} - 1 + \gamma_{k+1} = \left( \frac{p}{2} + \gamma_k \right) \frac{2^*}{2}, \\ \gamma_1 = \frac{p-q}{2} + 1. \end{cases} \tag{6.9}$$

Moreover,

$$\lim_{k \to \infty} \gamma_k = +\infty \quad \Leftrightarrow \quad \frac{p-q}{2} + 1 - \lambda > 0 \quad \Leftrightarrow \quad q < \frac{2 \cdot 2^*}{2^* + 2}(p+1). \tag{6.10}$$

If $n = 2$, since $q \le p + 2$ we can choose $2^*$ as any number $\mu > 2$ such that $q < \frac{2 \cdot \mu}{\mu + 2}(p+1)$; this is possible, because

$$\lim_{\mu \to 2^+} \frac{2 \cdot \mu}{\mu + 2}(p+1) = p + 1, \qquad \lim_{\mu \to +\infty} \frac{2 \cdot \mu}{\mu + 2}(p+1) = 2(p+1).$$

If instead $n \ge 3$ the last inequality in (6.10) becomes $q < \frac{n}{n-1}(p+1)$; which is true by the assumptions on $p$ and $q$.
Moreover $\gamma_k \ge 0$ for all $k$ since $q \le p + 2$.

Let us define

$$X_k := \|V(|Du|^2)\|_{L^{q-\frac{p}{2}-1+\gamma_k}(B_{r_k})} = \|V(|Du|^2)\|_{L^{(\frac{p}{2}+\gamma_{k-1})\frac{2^*}{2}}(B_{r_k})}, \qquad k \geq 1.$$

where $\gamma_0$ is defined coherently with (6.8). We remark that at each step of the iteration below, the $\gamma$'s in (6.7) take the non-negative values $\gamma_k$ with $k \geq 1$, but not the negative value $\gamma_0$. Reasoning as in the proof of the previous theorem, inequality (6.7) can be rewritten as

$$X_{k+1}^{\frac{p}{2}+\gamma_k} \leq \frac{c\,(1+\gamma_k)^6(1+\|u\|_\infty^2)}{(r_k-r_{k+1})^4} X_k^{(\frac{p}{2}+\gamma_{k-1})\frac{2^*}{2}}, \qquad k \geq 1.$$

Taking into account that (6.9) implies

$$\frac{\left(\frac{p}{2}+\gamma_{k-1}\right)\frac{2^*}{2}}{\frac{p}{2}+\gamma_k} = 1 + \frac{q-p-1}{\frac{p}{2}+\gamma_k}$$

we get that

$$X_{k+1} \leq c_k X_k^{1+\theta_k}, \qquad k \geq 1, \tag{6.11}$$

where

$$c_k = \left\{ \frac{c\,2^{4k}(1+\|u\|_\infty^2)}{(R-r)^4}\left(\frac{2^*}{2}\right)^{6k} \right\}^{\frac{1}{\frac{p}{2}+\gamma_k}}, \qquad \theta_k = \frac{q-p-1}{\frac{p}{2}+\gamma_k}.$$

By iteration, (6.11) implies

$$X_{i+1} \leq \left( \Pi_{k=1}^i c_k^{\Pi_{j=k+1}^i(1+\theta_j)} \right) X_1^{\Pi_{j=1}^i(1+\theta_j)} \tag{6.12}$$

with the position $\Pi_{j=i+1}^i(1+\theta_j) = 1$. Without loss of generality we can assume $c_k \geq 1$. Then (6.12) implies

$$X_{i+1} \leq \left( \Pi_{k=1}^i c_k^{\Pi_{j=1}^\infty(1+\theta_j)} \right) X_1^{\Pi_{j=1}^\infty(1+\theta_j)}. \tag{6.13}$$

It is easy to see that

$$\Theta := \Pi_{j=1}^\infty(1+\theta_j) < \infty. \tag{6.14}$$

indeed,

$$\log\left(\Pi_{j=1}^i(1+\theta_j)\right) = \sum_{j=1}^i \log\left(1+\frac{q-p-1}{\frac{p}{2}+\gamma_j}\right);$$

since $\gamma_j$ goes to $+\infty$ due to the assumption $q < p+1+\frac{p}{n-2}$, we obtain

$$\log\left(1+\frac{q-p-1}{\frac{p}{2}+\gamma_j}\right) \sim \frac{q-p-1}{\frac{p}{2}+\gamma_j} = \theta_j \sim \frac{q-p-1}{\left(\frac{p-q}{2}+1-\lambda\right)\left(\frac{2^*}{2}\right)^{j-1}},$$

thus (6.14) follows.

This fact, together with (6.13), implies

$$X_{i+1} \leq \left(\Pi_{k=1}^i c_k^\Theta\right) X_1^\Theta. \tag{6.15}$$

Since

$$\sum_{k=1}^i \log(c_k) = \sum_{k=1}^i \frac{1}{\frac{p}{2}+\gamma_k}\log\left\{ \frac{c\,2^{4k}(1+\|u\|_\infty^2)}{(R-r)^4}\left(\frac{2^*}{2}\right)^{6k} \right\}$$

that obviously converges as $i$ goes to $\infty$, because of the definition of $\gamma_k$, see (6.8). If we define $\gamma := \sum_{k=1}^{\infty} \frac{1}{\frac{p}{2}+\gamma_k}$, by letting $i$ go to $\infty$ in (6.15) we get

$$\sup_{B_r}[V(|Du|^2)] \leq c \left( \frac{1+\|u\|_\infty^2}{(R-r)^4} \right)^{\gamma\Theta} \left( \int_{B_R} [V(|Du|^2)]^{\frac{q}{2}} \, dx \right)^{\frac{2\Theta}{q}}$$

that implies

$$\sup_{B_r} |Du| \leq c \left( \frac{1+\|u\|_{L^\infty(\Omega')}}{(R-r)^2} \right)^{\gamma\Theta} \left( \int_{B_R} \left(1+|Du|^2\right)^{\frac{q}{2}} \, dx \right)^{\frac{\Theta}{q}} .$$

□

*Remark* 6.4. In [16, Theorem 2.1] an analogous Lipschitz estimate has been proved *without* assuming the a priori boundedness, under the assumptions: $n = 2$ and $2 \leq p \leq q$, or $n \geq 3$ and $2 \leq p \leq q < p\frac{n}{n-2}$. For instance, if $n \geq p+2$ the assumptions on $q$ in Theorem 6.3 are weaker than in [16].

## References


[1] V. Bögelein. F. Duzaar, P. Marcellini: Parabolic systems with $p, q$-growth: a variational approach, *Arch. Ration. Mech. Anal.* 210 (2013), 219-267.

[2] M. Bildhauer, M. Fuchs: $C^{1,\alpha}$-solutions to non-autonomous anisotropic variational problems, *Calc. Var. Partial Differential Equations* 24 (2005), 309-340.

[3] H.J. Choe: Interior behaviour of minimizers for certain functionals with nonstandard growth, *Nonlinear Anal.* 19 (1992), 933-945.

[4] G. Cupini, P. Marcellini, E. Mascolo: Regularity under sharp anisotropic general growth conditions, *Discrete Contin. Dyn. Syst. Ser. B* 11 (2009), 66-86.

[5] G. Cupini, P. Marcellini, E. Mascolo: Local boundedness of solutions to quasilinear elliptic systems, *Manuscripta Math.* 137 (2012), 287-315.

[6] G. Cupini, P. Marcellini, E. Mascolo: Local boundedness of solutions to some anisotropic elliptic systems, *Contemp. Math.* 595 (2013), 169-186.

[7] E. DeGiorgi: Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari, *Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat.* 3 (1957), 25-43.

[8] E. DiBenedetto: $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations, *Nonlinear Anal.* 7 (1983), 827-850.

[9] L.C. Evans: A new proof of local $C^{1,\alpha}$ regularity for solutions of certain degenerate elliptic p.d.e., *J. Differential Equations* 45 (1982), 356-373.

[10] M. Giaquinta: *Multiple integrals in the calculus of variations and nonlinear elliptic systems,* Annals of Mathematics Studies, 105. Princeton University Press (1983).

[11] E. Giusti: *Direct methods in the calculus of variations,* World Scientific Publishing Co., Inc., River Edge, NJ (2003).

[12] O. Ladyzhenskaya, N. Ural'tseva: *Linear and quasilinear elliptic equations,* Academic Press, New York and London (1968).

[13] Lee Junjie: Existence of $C^1$-solutions to certain non-uniformly degenerate elliptic equations, *J. Part. Diff. Eq.* 11 (1998), 9-24.

[14] J.J. Manfredi: Regularity for minima of functionals with $p$-growth, *J. Differential Equations* 76 (1988), 203-212.

[15] P. Marcellini: Regularity of minimizers of integrals in the calculus of variations with nonstandard growth conditions, *Arch. Rational Mech. Anal.* 105 (1989), 267-284.

[16] P. Marcellini: Regularity and existence of solutions of elliptic equations with $p, q$-growth conditions, *J. Differential Equations* 90 (1991), 1-30.

[17] P. Marcellini: Regularity for elliptic equations with general growth conditions, *J. Differential Equations* 105 (1993), 296-333.

[18] G. Mingione: Regularity of minima: an invitation to the dark side of the calculus of variations, *Appl. Math.* 51 (2006), 355-426.

[19] P. Tolksdorf: Regularity for a more general class of quasilinear elliptic equations, *J. Differential Equations* 51 (1984), 126-150.

Giovanni Cupini: Dipartimento di Matematica, Università di Bologna, Piazza di Porta S.Donato 5, 40126 - Bologna, Italy.
*E-mail address*: giovanni.cupini@unibo.it

Paolo Marcellini and Elvira Mascolo: Dipartimento di Matematica e Informatica "U. Dini", Università di Firenze, Viale Morgagni 67/A, 50134 - Firenze, Italy.
*E-mail address*: marcellini@math.unifi.it
*E-mail address*: mascolo@math.unifi.it