Periodic homogenization of deterministic control problems via limit occupational measures

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Abstract We consider optimal control problems where the dynamical system and the running cost are affected by fast periodic oscillations of the state variables. We show that, under suitable controllability and structure assumptions, it is possible to describe the limiting optimal control problem. The proofs make use of results in the theory of homogenization and singular perturbations of Hamilton-Jacobi equations.

1 Introduction

We consider an optimal control problem in $\mathbf{R}^N$ in which the dynamics

$$\begin{cases} \dot{x}(s) = f\left(x(s), \frac{x(s)}{\varepsilon}, \alpha(s)\right) \\ x(0) = x \end{cases} \tag{1}$$

and the cost functional

$$J^\varepsilon(t,x,\alpha) := \int_0^t l\left(x(s), \frac{x(s)}{\varepsilon}, \alpha(s)\right) \mathrm{d}s + h(x(t)) \tag{2}$$

undergo fast periodic oscillations. The controls α are measurable functions taking values in a compact metric space A. The vector field $f : \mathbf{R}^N \times \mathbf{R}^N \times A \to \mathbf{R}^N$ is

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bounded, uniformly continuous, and Lipschitz–continuous in x uniformly with respect to α. The running cost $l : \mathbf{R}^N \times \mathbf{R}^N \times A \to \mathbf{R}$ and the terminal cost $h : \mathbf{R}^N \to \mathbf{R}$ are given bounded uniformly continuous functions. Both $f(x,\cdot,\alpha)$ and $l(x,\cdot,\alpha)$ are $\mathbf{Z}^N$–periodic.

We are interested in understanding the behaviour of the solutions of this problem for very small $\varepsilon > 0$. In particular we investigate the existence and the nature of a limit problem, independent of ε, which approximates in some sense the $\varepsilon-$problem. Similar issues in Calculus of Variations have a very wide literature and various notions of convergence were developed in that context, see, for instance, [BDF98] and the references therein. Some particular control problems have been formulated as problems in Calculus of Variations with dynamical constraints and have been studied in that context [BDM82, BDFS97, Fr01], but the problem can be still considered largely open.

Here we study a notion of variational convergence based on the value function of the control problem. We recall that the value function is defined by the infimum of the cost functional among all trajectories, that is,

$$v^{\varepsilon}\big(t,x\big) := \inf\left\{J^{\varepsilon}(t,x,\alpha)\;\middle|\; x(\cdot) \text{ solves } (1)\right\}. \tag{3}$$

We say that the control problem with cost functional J^{ε} defined in (2) and dynamics (1) *converges as* $\varepsilon \to 0$ *to the limit control problem* with cost functional

$$\overline{J}(t,x,\gamma) = \int_0^t \bar{l}\,(x(s),\gamma(s))\,\mathrm{d}s + h(x(t)) \tag{4}$$

and dynamics

$$\dot{x}(s) = \overline{f}\,(x(s),\gamma(s))\,, \tag{5}$$

if the value function $v^{\varepsilon}\big(t,x\big)$ converges locally uniformly to the value function of the limit control problem $v(t,x) := \inf\left\{\overline{J}(t,x,\gamma)\;\middle|\; x(\cdot) \text{ solves } (5)\right\}$.

In the sequel of the paper we look for $\overline{f},\bar{l}$ and a constraint on the control functions $\gamma(\cdot)$ so that this kind of convergence occurs. We split the study in two parts. First we use that v^{ε} solves in viscosity sense the Cauchy problem

$$\begin{cases} \partial_t v^{\varepsilon} + H\left(x, \frac{x}{\varepsilon}, Dv^{\varepsilon}\right) = 0 & \text{in } (0,+\infty)\times\mathbf{R}^N \\ v^{\varepsilon}\big(0,x\big) = h(x) & \text{in } \mathbf{R}^N, \end{cases} \tag{6}$$

where the Hamiltonian H is given by

$$H(x,y,p) = \max_{\alpha\in A}\left\{-p\cdot f\,(x,y,\alpha) - l\,(x,y,\alpha)\right\},$$

see, e.g., [BCD97]. This is a periodic homogenization problem for a Hamilton–Jacobi equation, which consists in finding an *effective Hamiltonian* $\bar{H}(x,p)$ and appropriate conditions such that $v^{\varepsilon}(t,x)$ converges locally uniformly to a function $v(t,x)$, viscosity solution of a limiting Cauchy problem

$$\begin{cases} \partial_t v + \bar{H}(x, Dv) = 0 & \text{in } (0, +\infty) \times \mathbf{R}^N \\ v(0,x) = h(x) & \text{in } \mathbf{R}^N. \end{cases} \tag{7}$$

Results of this type go back to the seminal paper [LPV86] and have been extensively studied in the last decades for many different problems within the theory of viscosity solutions; see [E89], [E92] and also [AB02], [AB03], and [AB10] where it was shown the connection with singular perturbation problems. This is a classical subject in ODEs and control, pioneered by Levinson and Tichonov, see [DZ93], [KKO99], and the references therein. In the context of singular perturbations Artstein and Gaitsgory introduced averaging techniques and the use of invariant measures and limit occupational measures; see [AG97], [A98], [A99], [GL99], [AG00], and also [T11] for connections between such method and the viscosity theory for Hamilton-Jacobi equations.

The second part of our strategy is a representation of the effective Hamiltonian $\bar{H}$ as a Bellman Hamiltonian for suitable dynamics and cost $\overline{f}, \bar{l}$ related to the data f, l of the original problem (1), (2). Then the uniqueness of viscosity solutions to (7) implies that the limit v of v^ε is in fact the value function of the problem (4), (5). Limit occupational measures play a crucial role in the construction of such limit system.

The paper is organized as follows. Section 2 discusses the simple case of uncontrolled dynamics to show the role of invariant measures of ergodic dynamical systems. In Section 3 we reformulate the homogenization problem as singular perturbation, introduce the limit occupational measures, and formulate the main representation result for the limit control problem, under suitable controllability conditions on the system (1). Section 4 deals with vector fields f in (1) that depend on $x(s)/\varepsilon$ but not on $x(s)$, and show that the controllabilty assumptions of the preceding section can be weakened and in some cases the limit problem is very simple. Finally in Section 5 we briefly describe some generalizations to appear in [BT14] together with a more complete theory and detailed proofs.

2 Uncontrolled problem and invariant measures

We assume in this Section that the dynamics and running cost are not controlled, that is

$$f = f\left(x, \frac{x}{\varepsilon}\right) \qquad \text{and} \qquad l = l\left(x, \frac{x}{\varepsilon}\right). \tag{8}$$

Then the value function $v^\varepsilon(t,x)$ coincides with the cost functional $J^\varepsilon(t,x)$ and it solves the inhomogeneous transport equation

$$\begin{cases} \partial_t v^\varepsilon - Dv^\varepsilon \cdot f(x, \frac{x}{\varepsilon}) = l(x, \frac{x}{\varepsilon}) & \text{in } (0, +\infty) \times \mathbf{R}^N \\ v^\varepsilon(0,x) = h(x) & \text{in } \mathbf{R}^N. \end{cases} \tag{9}$$

Classical results in ergodic theory (see [CFS82], [Wal82], [AB10, Sect. 3.1]) ensure that the dynamics

$$\dot{y}(t) = f(x, y(t)) \qquad y(0) = y \qquad x \in \mathbf{R}^N \text{ frozen.} \tag{10}$$

has an invariant probability measure μ_x. Here $y \in \mathbf{R}^N$, but since f and l are $\mathbf{Z}^N$ periodic, we define the averaged vector field and running cost by setting

$$\hat{f}(x) := \int_{\mathbf{T}^N} f(x,y)\mathrm{d}\mu_x(y), \qquad \hat{l}(x) := \int_{\mathbf{T}^N} l(x,y)\mathrm{d}\mu_x(y),$$

where $\mathbf{T}^N = \mathbf{R}^N/\mathbf{Z}^N$.

In the following Proposition we recover within the theory of homogenization of PDEs a result of the classical theory of averaging of ODE's.

Proposition 1. *Consider the problem (1)-(2) with f and l as in (8). Assume that for every x the dynamics (10) has a unique invariant measure μ, independent of x. Then, as $\varepsilon \to 0$, the problem converges to the dynamics*

$$\dot{x}(s) = \hat{f}(x(s)) \tag{11}$$

with cost functional

$$\hat{J}(t,x) := \int_0^t \hat{l}(x(s))ds + h(x(t)). \tag{12}$$

Proof. We look for a solution of (9) of the form $v^\varepsilon(t,x) = u^\varepsilon(t,x,\frac{x}{\varepsilon})$. Then $u^\varepsilon(t,x,y)$ solves

$$\begin{cases} \partial_t u^\varepsilon - \left(D_x u^\varepsilon + \frac{1}{\varepsilon} D_y u^\varepsilon\right) \cdot f(x,y) = l(x,y) & \text{in } (0,+\infty) \times \mathbf{R}^{2N} \\ u^\varepsilon(0,x,y) = h(x) & \text{in } \mathbf{R}^{2N}. \end{cases}$$

A direct computation shows that the function

$$w(t,y) = \int_0^t G(\bar{x}, y(s), \bar{p}, 0)\mathrm{d}s \qquad y(\cdot) \text{ solving (10) with } x = \bar{x}$$

is the unique viscosity solution of the evolutive problem

$$\partial_t w - D_y w \cdot f(\bar{x}, y) + G(\bar{x}, y, \bar{p}, 0) = 0 \quad \text{in } (0,+\infty) \times \mathbf{R}^N, \qquad w(0,y) = 0.$$

where $G(x,y,p,q) := -(p+q) \cdot f(x,y) - l(x,y)$. Since by assumption the invariant measure is unique, it is easy to check that the quotient $w(t,y)/t$ converges to a constant uniformly w.r.t. y as $t \to +\infty$. Then such constant is the appropriate value of the effective Hamiltonian at $(\bar{x}, \bar{p})$, see [AB10, Sect. 2.1]. By definition of invariance we also get

$$\bar{H}(\bar{x}, \bar{p}) = \int_{\mathbf{T}^N} [-\bar{p} \cdot f(\bar{x}, y) - l(\bar{x}, y)]\,\mathrm{d}\mu(y) = -\bar{p} \cdot \hat{f}(x) - \hat{l}(x).$$

Moreover, by the theory of [AB03, AB10], the upper and lower semilimits of u^ε are respectively a subsolution and a supersolution of

$$\begin{cases} \partial_t v - Dv \cdot \hat{f}(x) = \hat{l}(x) & \text{in } (0,+\infty) \times \mathbf{R}^N \\ v(0,x) = h(x) & \text{in } \mathbf{R}^N. \end{cases} \tag{13}$$

Observe that $\hat{f}, \hat{l}$ are averages with respect to a measure independent of x. Then (13) satisfies the comparison principle between viscosity sub- and supersolutions, and its unique solution is the value function associated to problem (11)-(12)

$$\hat{v}(t,x) := \int_0^t \hat{l}(x(s))\mathrm{d}s + h(x(t)), \qquad x(\cdot) \text{ solving (11) with } x(0) = x.$$

Then the upper and lower semilimits of u^ε coincide and we conclude that the convergence of $v^\varepsilon(t,x) = u^\varepsilon(t,x,\frac{x}{\varepsilon})$ to $\hat{v}(t,x)$ unique solution of (13), is locally uniform. □

Remark 1. In Proposition 1 we have assumed that the unique invariant measure of (10) is independent of x. This is verified when $f = f(y)$ and $\dot{y} = f(y)$ is a uniquely ergodic dynamical system. Another case in which this hypothesis is satisfied is when $f = f(x)$ and the following non-resonance condition holds:

$$f(x) \cdot k \neq 0 \quad \forall k \in \mathbf{Z}^N \backslash \{0\}, x \in \mathbf{R}^N.$$

In this case, the unique invariant measure of (10) is the Lebesgue measure, $\hat{f} = f$ and

$$\hat{l}(x) = \int_{\mathbf{T}^N} l(x,y)\mathrm{d}y.$$

3 Controllable dynamics and limit occupational measures

By introducing the additional state variables $y = x/\varepsilon$, we rewrite the dynamics (1) as the singularly perturbed control system

$$\begin{cases} \dot{x}(s) = f(x(s), y(s), \alpha(s)) & x(0) = x \\ \dot{y}(s) = \frac{1}{\varepsilon} f(x(s), y(s), \alpha(s)) & y(0) = x/\varepsilon \end{cases} \tag{14}$$

Rescaling time by $t = \varepsilon s$, the dynamics for the fast variables y in (14) can be approximated by

$$\dot{y}(t) = f(x, y(t), \alpha(t)) \qquad y(0) = y, \tag{15}$$

where the slow variable x is frozen in its initial position. For any choice of control α and initial point $y \in \mathbf{R}^N$ there is a unique solution $y(\cdot)$ of the previous dynamics and we use it to define a measure over $\mathbf{R}^N \times A$ as

$$\mu_t := \frac{1}{t} \int_0^t \delta_{(y(s),\alpha(s))}\mathrm{d}s,$$

where δ is the Dirac's delta. These measures are called *occupational measure*, as they are probability measures giving the percentage of time interval $(0,t)$ spent by a trajectory of (15) in Borel subsets of $\mathbf{R}^N \times A$. We further define the set of *limit occupational measures* [GL99] or *limiting relaxed controls* [AB02] as the set of weak-star limits of occupational measures:

$$Z(x) := \left\{ \mu \;\middle|\; \mu = \lim_{n\to\infty} \mu_{t_n} \text{ weak-star, for some } t_n \to \infty,\ \alpha(\cdot),\ y \right\}.$$

If the dynamics (15) is *bounded-time controllable* - that is, any pair of points in $\mathbf{T}^N$ can be joined by a trajectory of (15) corresponding to a suitable choice of the control, in a uniformly bounded time - the set $Z(x)$ is nonempty, convex and compact with respect to the weak-star topology; see [GL99], [T11]. We define the averaged vector field and running cost by integrating with respect to measures in $Z(x)$:

$$\bar{f}(x,\mu) := \int_{\mathbf{R}^N \times A} f(x,y,\alpha)\mathrm{d}\mu(y,\alpha),$$

$$\bar{l}(x,\mu) := \int_{\mathbf{R}^N \times A} l(x,y,\alpha)\mathrm{d}\mu(y,\alpha), \qquad \mu \in Z(x).$$

Proposition 2. *Assume that*

$$\begin{array}{c} \text{for any } x \text{ exists } \nu(x) > 0 \text{ s.t.} \\ B(0,\nu(x)) \subseteq \overline{\mathrm{co}} f(x,y,A) \text{ for any } y. \end{array} \tag{16}$$

Then the optimal control problem (1)-(2) converges as $\varepsilon \to 0$ *to the problem with cost functional*

$$\bar{J}(t,x,\mu) := \int_0^t \bar{l}(x(s),\mu(s))ds + h(x(t)). \tag{17}$$

and dynamics given by the differential inclusion

$$\dot{x}(s) = \bar{f}(x(s),\mu(s)) \qquad \mu(s) \in Z(x(s)), \qquad x(0) = x. \tag{18}$$

Proof. We look for a solution of (6) of the form $v^\varepsilon(t,x) = u^\varepsilon(t,x,\frac{x}{\varepsilon})$. Then $u^\varepsilon(t,x,y)$ solves in viscosity sense

$$\begin{cases} \partial_t u^\varepsilon + G\left(x,y,D_x u^\varepsilon, \frac{D_y u^\varepsilon}{\varepsilon}\right) = 0 & \text{in } (0,+\infty) \times \mathbf{R}^{2N} \\ u^\varepsilon(0,x,y) = h(x) & \text{in } \mathbf{R}^{2N}, \end{cases} \tag{19}$$

where $G(x,y,p,q) := H(x,y,p+q)$. This PDE corresponds to a singularly perturbed control problem that was studied under the current assumptions in [T11]. We recall here the main steps of the proof. The controllability condition (16) is equivalent to the following coercivity property with respect to q: for every $x, p \in \mathbf{R}^N$,

$$G(x,y,p,q) \geq \nu(x)|q| - C(1+|p|) \quad \text{for any } y,q \tag{20}$$

for some $C > 0$ (see [AB10, Section 6.1]). This entails that G is ergodic, namely, $\bar{H}$ exists and the upper and lower semi-limits of u^ε are respectively a viscosity subsolution and supersolution of (7). Arguing as in [AB02, Theorem 7] it is possible to prove the following representation formula:

$$\bar{H}(x,p) = \max_{\mu \in Z(x)} \left\{ -p \cdot \bar{f}(x,\mu) - \bar{l}(x,\mu) \right\}. \tag{21}$$

Since (20) also implies that $\bar{H}(x,p)$ is Lipschitz continuous (see [AB10, Proposition 6.4]), the comparison principle holds for problem (7) and u^ε converges locally uniformly as $\varepsilon \to 0$ to $v(t,x)$, unique solution of (7); then v^ε also converges to the same function. To complete the proof it is necessary to check that the value function associated to (18)-(17), that is

$$\bar{v}(t,x) := \inf \left\{ \bar{J}(t,x,\mu) \;\middle|\; x(\cdot) \text{ solves } (18) \right\},$$

is a viscosity solution of (7). To see this, one needs to take into account formula (21), to prove that $\bar{v}$ is continuous, that the multivalued map $f(x,Z(x))$ in (18) is Lipschitz continuous, and that the limiting dynamics admits trajectories defined for any positive time: we refer to [T11] for the details. □

Remark 2. Although the controllability condition (16) does not hold in Section 2, the statement of Proposition 1 is consistent with that of Proposition 2 and provides an example in which the set of limiting occupational measures is explicit and it is a singleton.

4 Purely oscillating dynamics

In this section we study simpler expressions for the limiting dynamics when the vector field depends on $x(s)$ only via the oscillating terms $\frac{x(s)}{\varepsilon}$, that is,

$$\begin{cases} \dot{x}(s) = f\left(\frac{x(s)}{\varepsilon}, \alpha(s)\right) \\ x(0) = x. \end{cases} \tag{22}$$

4.1 Weakening the controllability conditions

For systems of the form (22) the dynamics of the oscillating variables in rescaled time is independent of x

$$\dot{y}(t) = f(y(t), \alpha(t)). \tag{23}$$

Consequently, the set of limiting relaxed controls is $Z(x) = Z$ independent of x. This permits to prove the convergence without the additional controllability assumption (16). As before, we set

$$\bar{f}(\mu) := \int_{\mathbf{R}^N \times A} f(y,\alpha) \mathrm{d}\mu(y,\alpha), \qquad \mu \in Z.$$

Proposition 3. *Consider the optimal control problem (22)-(2) and assume that the system (23) is bounded-time controllable. Then, the problem converges as* $\varepsilon \to 0$ *to the one with dynamics*

$$\dot{x}(s) = \bar{f}(\mu(s)), \qquad x(0) = x, \qquad \mu(s) \in Z \tag{24}$$

and cost functional $\bar{J}(t,x,\mu)$ *as in (17).*

Proof. We proceed along the same lines and keep the same notations as in the proof of Proposition 2. The assumed bounded-time controllability of (23) implies that Z is a convex and compact subset of probability measures. Moreover, the upper and lower semi-limits of u^ε are, respectively, a subsolution and a supersolution of (7), with

$$\bar{H}(x,p) = \max_{\mu \in Z} \left\{ -p \cdot \bar{f}(\mu) - \bar{l}(x,\mu) \right\}.$$

Now, since f does not depend on x, $\bar{H}(x,p)$ satisfies regularity properties that guarantee the comparison principle for the effective Cauchy problem (7). Thus, the locally uniform convergence of u^ε - and then that of v^ε - to the value function

$$\bar{v}(t,x) := \inf \left\{ \bar{J}(t,x,\mu) \;\middle|\; x(\cdot) \text{ solves } (24) \right\},$$

unique solution of (7) can be proved without requiring any extra controllability assumption. □

4.2 A simple degenerate limit for special costs.

Here we consider the special case that

$$l = l\left(\frac{x}{\varepsilon}\right), \qquad h(x) = 0,$$

in (2), so that the cost functional is

$$J^\varepsilon(t,x,\alpha) = \int_0^t l\left(\frac{x(s)}{\varepsilon}\right) \mathrm{d}s. \tag{25}$$

We also assume the following controllability condition, much weaker than condition (16),

$$\max_{\alpha \in A} \{-q \cdot f(y,\alpha)\} \geq 0 \quad \text{for any } y, q \in \mathbf{R}^N. \tag{26}$$

The next result says that in this case the limit control problem reduces to the static optimization problem of the running cost with respect to the state variables.

Proposition 4. *Consider the optimal control problem (22)-(25) and assume that the system (23) is bounded-time controllable and satisfies (26). Then, the value function* $v^\varepsilon(t,x) := \inf J^\varepsilon(t,x,\alpha)$ *converges locally uniformly as* $\varepsilon \to 0$ *to*

$$\bar{v}(t) = t \min_{y \in \mathbf{T}^N} l(y).$$

Proof. The value function v^ε is the unique solution of

$$\begin{cases} \partial_t v^\varepsilon + \max_\alpha \left\{ -D_x v^\varepsilon \cdot f\left(\dfrac{x}{\varepsilon}, \alpha\right)\right\} = l\left(\dfrac{x}{\varepsilon}\right) & \text{in } (0,+\infty)\times \mathbf{R}^N \\ v^\varepsilon\big(0,x\big) = 0 & \text{in } \mathbf{R}^N. \end{cases} \tag{27}$$

Set $H(y,q) := \max_\alpha \{-q\cdot f(y,\alpha)\} - l(y)$. Condition (26) implies that $H(y,q) \geq H(y,0)$ for any y,q. Then, arguing by contradiction as in [AB10, Proposition 6.6], we get

$$\bar{H} = \max_{y\in\mathbf{R}^N} H(y,0) = \max_{y\in\mathbf{T}^N}\{-l(y)\} = -l(y_0),$$

for some $y_0 \in [0,1)^N$, since l is continuous and periodic. Then $v^\varepsilon(t,x)$ converges locally uniformly to the unique solution of

$$\partial_t v + \bar{H} = 0 \qquad v\big(0\big) = 0,$$

which is $\bar{v}(t) = t\,l(y_0)$. $\square$

5 Generalizations of the results

We will show in the forthcoming paper [BT14] how the results described in this note can be generalized to prove variational convergence of optimal control problems with dynamics

$$\begin{cases} \dot{x}_1(s) = f_1\left(x(s), \frac{x_2(s)}{\varepsilon}, \alpha_1(s), \alpha_2(s)\right) \\ \dot{x}_2(s) = f_2\left(x(s), \frac{x_2(s)}{\varepsilon}, \alpha_2(s)\right) \\ x(0) = x, \end{cases} \tag{28}$$

and cost functional

$$\int_0^t l\left(x(s), \frac{x_2(s)}{\varepsilon}, \alpha_1(s), \alpha_2(s)\right) \mathrm{d}s + h(x(t)). \tag{29}$$

The state variable x is divided here in two groups, x_1 and x_2. The dynamics for the oscillating variables, x_2, is controlled only by the α_2 component of the control variable (α_1, α_2). Problem (1) considered here corresponds to the particular choice $f_1 \equiv 0$ (then the dynamics for x_1 can be ignored) and $f_2 = f$.

The arguments sketched in the proof of Proposition 1 can be adapted to show convergence of (28)-(29) when the dynamics for x_2 is uncontrolled, i.e. $f_2 = f_2(x, x_2/\varepsilon)$. A representation of the limiting optimal control problem can be provided in terms of invariant measures of the flow associated to the dynamics of fast oscillations.

The strategy described in Section 3 and the averaging result of Proposition 2 can be adapted to show convergence of optimal control problems like (28)-(29); assumption (16) must be satisfied with $f = f_2$ and $A = A_2$, the compact set of values for the α_2 components of the control. An analog of Proposition 3 holds, whenever f_2 depends on x_2/ε but not on x and the dynamics

$$\dot{y}(t) = f_2\left(y(t), \alpha_2(t)\right), \qquad y(0) = y$$

is bounded-time controllable.

The case treated in Section 4.2 can be generalized to (28)-(29) provided $f_i = f_i(x_1, \frac{x_2}{\varepsilon}, \alpha_i)$ $(i = 1,2)$, $l = l(x_1, \frac{x_2}{\varepsilon}, \alpha_1)$, $h = h(x_1)$. We can show that an optimal control problem satisfying this partially decoupled structure admits a variational limit. As $\varepsilon \to 0$, the oscillations x_2/ε play the role of a new control variable valued in the torus. More precisely, the limiting dynamics is governed by the drift $f_1 = f_1(x_1, y, \alpha_1)$ controlled by

$$(y, \alpha_1) : [0, \infty) \to [0,1)^{N_2} \times A_1 \text{ measurable}$$

and the associated cost functional is

$$\int_0^t l(x_1(s), y(s), \alpha_1(s))\mathrm{d}s + h(x_1(t)).$$

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