A GENERAL FORMULA FOR THE ANISOTROPIC OUTER MINKOWSKI CONTENT OF A SET

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Abstract. We generalize to the anisotropic case some classical and recent results on the $(n-1)$-Minkowski content of rectifiable sets in $\mathbb{R}^n$, and on the outer Minkowski content of subsets of $\mathbb{R}^n$. In particular, a general formula for the anisotropic outer Minkowski content is provided and applies to a wide class of sets, stable under finite unions.

Keywords: Minkowski content, outer Minkowski content, anisotropy, rectifiability.

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1. Introduction

The notion of Minkowski content has been introduced by H. Minkowski in order to study an intrinsic definition of $k$-dimensional area of a compact set. Precisely, if $S \subset \mathbb{R}^n$ is a closed set, the $(n-1)$-dimensional Minkowski content of $S$ is defined by

$$\mathcal{M}^{n-1}(S) := \lim_{\varepsilon \to 0} \frac{|\{x \in \mathbb{R}^n : d(x, S) \leq \varepsilon\}|}{2\varepsilon} \quad (1.1)$$

whenever the limit on the right-hand side exists. A natural question arises: is it true that the $(n-1)$-dimensional Minkowski content coincides with the $(n-1)$-dimensional Hausdorff measure? A first result in this direction can be found in [6]: if the set $S$ is compact and $(n-1)$-rectifiable then $\mathcal{M}^{n-1}(S) = \mathcal{H}^{n-1}(S)$. A weaker version of the previous result can be found in [2]; it is sufficient to have compactness, countable $\mathcal{H}^{n-1}$-rectifiability, and a further condition: $\eta(B_r(x)) \geq \gamma r^{n-1}$ holds true for all $x \in S$ and for all $r \in (0, 1)$, for some $\gamma > 0$ and some Radon measure $\eta$ on $\mathbb{R}^n$ absolutely continuous with respect to $\mathcal{H}^{n-1}$. More recently, in [1], motivated by problems arising from stochastic geometry, the notion of outer Minkowski content of a set has been introduced:

$$\mathcal{S}M(E) := \lim_{\varepsilon \to 0} \frac{|\{x \in \mathbb{R}^n : d(x, E) \leq \varepsilon\} \setminus E|}{\varepsilon}.$$ 

In [1] the authors investigated general conditions ensuring the existence of $\mathcal{S}M$: in particular they prove that $\mathcal{S}M(E)$ coincides with $\mathcal{P}(E)$, the perimeter of $E$, whenever $E$ has finite perimeter and $\mathcal{M}^{n-1}(\partial E) = \mathcal{P}(E)$, being $\partial E$ the topological boundary of $E$. A more general formula for the outer Minkowski content of a set has been investigated in [11]: more precisely,
if \( E^0 \) denotes the set of points of density 0 for \( E \),
\[
SM(E) = P(E) + 2H^{n-1}(E^0 \cap \partial E),
\]
holds true for a suitable class of sets which is stable under finite unions, and such a stability is a feature particularly relevant in connection with applications to the study of some stochastic processes (see again [11] for further details). Very recently, an anisotropic variant of the outer Minkowski content of a set has been considered in [4], motivated also by the study of anisotropic perimeters arising from discrete ones (see [3]). The idea introduced in [4] is to look at the limit, whenever it exists, given by
\[
\lim_{\varepsilon \to 0} \frac{|(E + \varepsilon C)| - |E|}{\varepsilon} =: SM_C(E),
\]
where \( C \subset \mathbb{R}^n \) is a convex body. Contrarily with respect to the isotropic case, much less is known about the anisotropic situation. In [4] the authors prove that for sets with finite perimeter and such that \( SM(E) = P(E) \), the anisotropic outer Minkowski content of \( E \) takes the form
\[
SM_C(E) = \int_{F \cap \partial E} h_C(\nu_E) dH^{n-1}
\]
being \( h_C \) is the support function of \( C \) and \( F \cap \partial E \) the reduced boundary of \( E \).

In this paper, first of all we want to find an integral formula also for the anisotropic Minkowski content of a set, and we are able to prove that under suitable conditions on \( S \) similar to what one has to assume for the isotropic case (see Theorems 3.4 and 3.7) the anisotropic Minkowski content of \( S \) takes the form
\[
\frac{1}{2} \int_S (h_C(\nu_S) + h_C(-\nu_S)) dH^{n-1}.
\]
Moreover, we will generalize also formula (1.2) to the anisotropic case: precisely (see Theorem 4.3), we will prove that
\[
SM_C(E) = \int_{F \cap \partial E} h_C(\nu_E) dH^{n-1} + \int_{E^0 \cap \partial E} (h_C(\nu_E) + h_C(-\nu_E)) dH^{n-1}
\]
holds true for a suitable class of sets stable under finite unions.

2. Notation and preliminaries

2.1. Notation. Let \( n \geq 1 \) be integer. Given a measurable set \( A \subset \mathbb{R}^n \), we will denote by \( |A| \) its Lebesgue measure. If \( k \in \{0, \ldots, n\} \), the \( k \)-dimensional Hausdorff measure of \( S \subset \mathbb{R}^n \) will be denoted by \( H^k(S) \). We will use the notation \( x \cdot y \) for the standard scalar product in \( \mathbb{R}^n \) between \( x \) and \( y \), \( B_r(x) \) for the closed ball of radius \( r \) centered in \( x \). For each \( k \in \mathbb{N} \) with \( k \leq n \) we denote by \( G_k \) the set of unoriented \( k \)-planes on \( \mathbb{R}^n \); for any \( \pi \in G_k \) we denote by \( \pi^\perp \in G_{n-k} \) the \( (n-k) \)-plane orthogonal to \( \pi \). Finally, if \( \mu \) is a positive, real or vector
measure on some space $X$ and $f: X \to Y$ is measurable, we define the measure $f_\ast \mu$ on $Y$ as $f_\ast \mu(F) := \mu(f^{-1}(F))$ for any $F$ measurable in $Y$.

2.2. Geometric Measure Theory. In this paragraph we recall some basic notions of geometric measure theory we will need; for all details we refer the reader to [2], [6] and [10]. Let $n \geq 1$ be integer and let $k \in \mathbb{N}$ with $k \leq n$. The following general property of Radon measures holds true; here $\omega_k$ is the volume of the $k$-dimensional unit ball.

**Theorem 2.1.** Let $\Omega \subset \mathbb{R}^n$ be an open set and let $\mu$ be a positive Radon measure on $\Omega$. Then, for any $t > 0$ and for any $B$ Borel set in $\Omega$ the following implications hold:

\[
\limsup_{\rho \to 0} \frac{\mu(B_\rho(x))}{\omega_k \rho^k} \geq t \quad \forall x \in B \Rightarrow \mu \geq t\mathcal{H}^k \blacktriangleleft B,
\]

\[
\limsup_{\rho \to 0} \frac{\mu(B_\rho(x))}{\omega_k \rho^k} \leq t \quad \forall x \in B \Rightarrow \mu \leq 2^k t\mathcal{H}^k \blacktriangleleft B.
\]

A very useful consequence of Theorem 2.1 turns out to be the following fact:

\[B \text{ Borel in } \Omega \text{ with } \mu(B) = 0 \Rightarrow \lim_{\rho \to 0} \frac{\mu(B_\rho(x))}{\omega_k \rho^k} = 0, \text{ for } \mathcal{H}^k\text{-a.e. } x \in B. \quad (2.1)\]

Let now $S \subset \mathbb{R}^n$. We say that $S$ is $k$-rectifiable if there exist a bounded set $B \subset \mathbb{R}^k$ and a Lipschitz function $f: B \to \mathbb{R}^n$ such that $S = f(B)$. We say that $S \subset \mathbb{R}^n$ is countably $\mathcal{H}^k$-rectifiable if there exist countably many Lipschitz functions $f_h: \mathbb{R}^k \to \mathbb{R}^n$ such that

\[\mathcal{H}^k \left(S \setminus \bigcup_{h=0}^{+\infty} f_h(\mathbb{R}^k)\right) = 0.\]

If in adjoint $\mathcal{H}^k(S) < +\infty$ then $S$ is said to be $\mathcal{H}^k$-rectifiable. A classical rectifiability criterium says that a Borel set $S \subset \mathbb{R}^n$ with $\mathcal{H}^k(S) < +\infty$ is $\mathcal{H}^k$-rectifiable if and only

\[\lim_{\rho \to 0} \frac{\mathcal{H}^k(S \cap B_\rho(x))}{\omega_k \rho^k} = 1, \quad \text{for } \mathcal{H}^k\text{-a.e. } x \in S. \quad (2.2)\]

It turns out that if $S$ is countably $\mathcal{H}^k$-rectifiable then for $\mathcal{H}^k$-almost any point $x_0 \in S$ it is well defined the approximate tangent space $\text{Tan}^k(S, x_0) \in \mathbb{G}_k$, that is

\[
\lim_{\rho \to 0} \frac{1}{\rho^k} \int_S \phi \left( \frac{x - x_0}{\rho} \right) d\mathcal{H}^k(x) = \int_{\text{Tan}^k(S, x_0)} \phi(y) d\mathcal{H}^k(y), \quad \forall \phi \in C^\infty_c(\mathbb{R}^n).
\]

In particular, if $k = n - 1$ then $\text{Tan}^{n-1}(S, x_0)^\perp$ is generated by some unit vector denoted by $\nu_S$. We recall that by Lipschitz $k$-graph we mean the graph of a Lipschitz function $\phi: \pi \to \pi^\perp$, where $\pi \in \mathbb{G}_k$. Given a countably $\mathcal{H}^k$-rectifiable set $S$, it is well known that $S$ can be covered, up to a $\mathcal{H}^k$-negligible set, by a countable family of pairwise disjoint compact subsets of $S$ which are contained in some Lipschitz $k$-graph and with finite $k$-dimensional Hausdorff measure. We now recall the notion of $k$-dimensional Jacobian and the area formula. Let $L: \mathbb{R}^k \to \mathbb{R}^n$ be a linear map. The $k$-dimensional Jacobian of $L$ is defined by $J_k L := \sqrt{\det(L^* \circ L)}$ where
$L^* : \mathbb{R}^n \rightarrow \mathbb{R}^k$ denotes the transpose of $L$. The Jacobian is related, as it is well known in the smooth case, to the change of variable formula for multiple integrals: more precisely, if $f : \mathbb{R}^k \rightarrow \mathbb{R}^n$ is Lipschitz then for any measurable set $E \subset \mathbb{R}^k$ the multiplicity function $y \mapsto \mathcal{H}^0(E \cap f^{-1}([y]))$ is measurable and the area formula

$$
\int_{\mathbb{R}^n} \sum_{x \in E \cap f^{-1}([y])} g(x) \, d\mathcal{H}^k(y) = \int_E g(x) \mathbf{J}_k df_x \, dx \tag{2.3}
$$

holds for each $g : E \rightarrow \mathbb{R}$ Borel. There is another useful formula, which is known as coarea formula: if $\Omega$ is open in $\mathbb{R}^n$, $f : \Omega \rightarrow \mathbb{R}$ is Lipschitz continuous and $g : \Omega \rightarrow [0, +\infty]$ is Borel then

$$
\int_{\Omega} g(x) |\nabla f(x)| \, dx = \int_{-\infty}^{+\infty} \int_{\Omega \cap \{f = t\}} g(y) \, d\mathcal{H}^{n-1}(y) \, dt. \tag{2.4}
$$

Let now $E \subset \mathbb{R}^n$ be a measurable set and $\Omega \subset \mathbb{R}^n$ be an open domain; we denote by $\chi_E$ the characteristic function of $E$. We say that $E$ has finite perimeter in $\Omega$ if the distributional derivative of $\chi_E$, denoted by $D\chi_E$, is a $\mathbb{R}^n$-valued Radon measure on $\Omega$ with finite total variation; the perimeter of $E$ in $\Omega$ is defined by $\mathcal{P}(E; \Omega) := |D\chi_E|(\Omega)$, where $|D\chi_E|$ denotes the total variation of $D\chi_E$; we also let $\mathcal{P}(E) := \mathcal{P}(E; \mathbb{R}^n)$. For sufficiently smooth boundaries the perimeter coincides with the $(n - 1)$-dimensional Hausdorff measure of the topological boundary. An interesting situation is the following one: given $f : A \rightarrow \mathbb{R}$ be a Lipschitz map, with $A$ open and bounded in $\mathbb{R}^{n-1}$, the subgraph of $f$ turns out to be a set with finite perimeter in $A \times \mathbb{R}$ and its perimeter coincides with the $(n - 1)$-dimensional Hausdorff measure of the graph of $f$. The upper and lower $n$-dimensional densities of $E$ at $x$ are respectively defined by

$$
\Theta^*(E, x) := \limsup_{\rho \rightarrow 0} \frac{|E \cap B_\rho(x)|}{\omega_n \rho^n}, \quad \Theta^*_n(E, x) := \liminf_{\rho \rightarrow 0} \frac{|E \cap B_\rho(x)|}{\omega_n \rho^n}.
$$

If $\Theta^*(E, x) = \Theta^*_n(E, x)$ their common value is denoted by $\Theta(E, x)$. For every $t \in [0, 1]$ we define $E^t := \{x \in \mathbb{R}^n : \Theta(E, x) = t\}$. The essential boundary of $E$ is defined as $\partial^* E := \mathbb{R}^n \setminus (E^0 \cup E^1)$. It turns out that if $E$ has finite perimeter in $\Omega$, then $\mathcal{H}^{n-1}(\partial^* E \setminus E^{1/2}) = 0$, and $\mathcal{P}(E; \Omega) = \mathcal{H}^{n-1}(\partial^* E \cap \Omega)$. Moreover, one can define a subset of $E^{1/2}$ as the set of points $x$ where there exists a unit vector $\nu_E(x)$ such that

$$
\frac{E - x}{\rho} \rightarrow \{y \in \mathbb{R}^n : y \cdot \nu_E(x) \leq 0\}, \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^n) \text{ as } \rho \rightarrow 0,
$$

and which is referred to as the outer normal to $E$ at $x$. The set where $\nu_E(x)$ exists is called the reduced boundary and is denoted by $\mathcal{F} E$. One can show that $\mathcal{H}^{n-1}(\partial^* E \setminus \mathcal{F} E) = 0$, moreover, one has the decomposition $D\chi_E = (\nu_E)\mathcal{H}^{n-1} \perp \mathcal{F} E$. We also introduce the set $\partial^2 E := \{x \in \partial E \cap \mathbb{R}^n : \exists \text{Tan}^{n-1}(\partial E, x)\}$. Let us collect some elementary properties of sets with countably $\mathcal{H}^{n-1}$-rectifiable boundary and with finite perimeter in $\Omega$; for any $E \subset \mathbb{R}^n$ we let $E^c := \mathbb{R}^n \setminus E$. 


Lemma 2.2. Assume that $E$ has finite perimeter in $\Omega$ and $\partial E$ is countably $\mathcal{H}^{n-1}$-rectifiable. Then the following holds:

$$
\mathcal{H}^{n-1}(FE) = \mathcal{H}^{n-1}(FE^c); \tag{2.5}
$$

$$
\mathcal{H}^{n-1}(\partial E \setminus FE) = \mathcal{H}^{n-1}(\partial^2 E) + \mathcal{H}^{n-1}(\partial^2 E^c); \tag{2.6}
$$

$$
\nu_{E^c}(x) = -\nu_E(x), \quad \text{for any } x \in FE; \tag{2.7}
$$

$$
\mathcal{H}^{n-1}(\partial E) = \mathcal{H}^{n-1}(\partial^2 (\partial E)). \tag{2.8}
$$

Proof. Properties (2.5), (2.7) and (2.8) are trivial. In order to prove (2.6) we notice that

$$
\mathcal{H}^{n-1}(\partial E) = \mathcal{H}^{n-1}(\partial E^c) = \mathcal{H}^{n-1}(FE^c) + \mathcal{H}^{n-1}(\partial E^c \cap (E^c)^1) + \mathcal{H}^{n-1}(\partial E^c \cap (E^c)^0)
$$

$$
= \mathcal{H}^{n-1}(FE) + \mathcal{H}^{n-1}(\partial E \cap (E^c)^0) + \mathcal{H}^{n-1}(\partial^2 E^c)
$$

$$
= \mathcal{H}^{n-1}(FE) + \mathcal{H}^{n-1}(\partial^2 E) + \mathcal{H}^{n-1}(\partial^2 E^c)
$$

Therefore, $\mathcal{H}^{n-1}(\partial E \setminus FE) = \mathcal{H}^{n-1}(\partial^2 E) + \mathcal{H}^{n-1}(\partial^2 E^c)$ which is (2.6). \qed

Remark 2.3. Using Lemma 2.2, we may observe that if $E$ is such that its topological boundary $\partial E$ is a set countably $\mathcal{H}^{n-1}$-rectifiable and bounded, then one of the followings holds for $\mathcal{H}^{n-1}$-a.e. $x \in \partial E$:

(a) $x \in FE$, and the outer normal $\nu_E(x)$ to $E$ at $x$ exists;

(b) $x \in E^1 \cap \partial E$ (in such a case no outer normal exists);

(c) $x \in \partial^2 E$, and two outer normals to $E$ at $x$ exist, say $\nu_E(x)$ and $-\nu_E(x)$.

This is in accordance with already known results in literature for sets with positive reach (for instance, see [5]); namely, it can be shown that the topological boundary of a compact subset $E$ of $\mathbb{R}^n$ with positive reach is $(n-1)$-rectifiable, and that, for $\mathcal{H}^{n-1}$-a.e. $x \in \partial E$, either $x \in FE$ or $x \in \partial^2 E$ (see also [1]).

We recall now the well known Besicovitch’s Theorem.

Theorem 2.4. Let $A \subset \mathbb{R}^n$ be a bounded set and $\rho: A \to (0, +\infty)$ be a function. There exists a set $S \subset A$ at most countable such that

$$
A \subset \bigcup_{x \in S} B_{\rho(x)}(x).
$$

Moreover, every point of $\mathbb{R}^n$ belongs at most to $\xi$ balls $B_{\rho(x)}(x)$ centred in a point of $S$, where $\xi$ is a constant depending only on $n$.

We will need also the following variant of the Vitali covering Theorem concerning Lebesgue measure; if $A \subset \mathbb{R}^n$ we say that $\mathcal{F}$ is a fine cover of $A$ if for each $x \in A$ there exist balls in $\mathcal{F}$ centred in $x$ and with arbitrarily small radii.
Theorem 2.5. Let $A \subset \mathbb{R}^n$ be a bounded and Borel set and let $\mathcal{F}$ be a fine cover of $A$. Then for any positive Radon measure $\mu$ in $\mathbb{R}^n$ there exists a disjoint family $\mathcal{F}' \subset \mathcal{F}$ such that

$$\mu \left( A \setminus \bigcup_{F \in \mathcal{F}'} F \right) = 0.$$ 

3. The anisotropic Minkowski content

Let $C \subset \mathbb{R}^n$ be a convex body, that is a compact and convex subset of $\mathbb{R}^n$ with 0 in its interior; here and in what follows $a, b \in \mathbb{R}$ are such that $0 < a < b$ and $B_a(0) \subset C \subset B_b(0)$. We denote by $h_C$ its support function, i.e.

$$h_C(v) := \sup_{x \in C} x \cdot v, \quad v \in \mathbb{R}^n.$$ 

Define, for any $S \subset \mathbb{R}^n$ closed,

$$M^*_c(S) := \limsup_{\varepsilon \to 0} \frac{|S + \varepsilon C|}{2\varepsilon}, \quad M_*^c(S) := \liminf_{\varepsilon \to 0} \frac{|S + \varepsilon C|}{2\varepsilon}.$$ 

If $M^*_c(S) = M_*^c(S)$ their common value is denoted by $M^c_c(S)$. As we have seen in the introduction, there exists a relation relation between $M^c_{B_1(0)}(\partial E)$ and $SM(E)$. To be more precise, let

$$SM(E; \Omega) := \lim_{\varepsilon \to 0} \frac{|\{ x \in \Omega : d(x, E) \leq \varepsilon \} \setminus E|}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{|(E + \varepsilon B_1(0)) \setminus E|}{\varepsilon}$$

whenever such a limit exists; of course we have $SM(E) = SM(E; \mathbb{R}^n)$. Hence the following Theorem holds (see [1]).

Theorem 3.1. If $E$ has finite perimeter in $\Omega$ and $M^c_{B_1(0)}(\partial E) = \mathcal{P}(E; \Omega)$ then $SM(E; \Omega) = \mathcal{P}(E; \Omega)$.

Moreover, we need to give the precise statement of the main result in [4]; for, let

$$SM^c_c(E; \Omega) := \lim_{\varepsilon \to 0} \frac{|(E + \varepsilon C) \cap \Omega| - |E|}{\varepsilon}.$$ 

Notice that we have also in this case $SM^c_c(E) = SM^c_c(E; \mathbb{R}^n)$.

Theorem 3.2. If $E$ has finite perimeter in $\Omega$ and $SM^c_c(E; \Omega) = \mathcal{P}(E; \Omega)$ then

$$SM^c_c(E; \Omega) = \int_{FE} h_C(\nu_E) d\mathcal{H}^{n-1}.$$ 

Lemma 3.3. Let $A \subset \mathbb{R}^{n-1}$ be open and bounded and let $f : A \to \mathbb{R}$ be Lipschitz continuous. Let $G$ be the graph of $f$, that is $G := \{ (x, y) : x \in A \times \mathbb{R} : y = f(x) \}$. Then $M^c_c(G)$ exists and

$$M^c_c(G) = \int_G \phi_C(\nu_G) d\mathcal{H}^{n-1}$$

being

$$\phi_C(v) := \frac{h_C(v) + h_C(-v)}{2}, \quad \forall v \in \mathbb{R}^n.$$
Proof. Let \( E := \{(x, y) \in A \times \mathbb{R} : y < f(x)\} \) be the subgraph of \( f \). Since \( f \) is Lipschitz, \( E \) has finite perimeter in \( A \times \mathbb{R} \) and \( M^{n-1}(\partial E) = M_{B_1(0)}(\partial E) = \mathcal{P}(E; A \times \mathbb{R}) \). Applying Theorem 3.1 we deduce that also \( SM(E; \Omega) = \mathcal{P}(E; \Omega) \). Thus, using (2.7) and Theorem 3.2 we get
\[
M_C(G) = \frac{SM_C(E; \Omega) + SM_C(\Omega \setminus E; \Omega)}{2} = \int_{\partial E} \frac{h_C(\nu_E) + h_C(-\nu_E)}{2} \, d\mathcal{H}^{n-1}
\]
which yields the conclusion. \( \square \)

We are ready to prove the first main Theorem of this section, which may be seen as the generalization to the anisotropic case of Theorem 2.104 in [2]. Of course, the next proofs owe a bit to ones contained in [2] for the classical Minkowski content, which are anyway not elementary, but for the convenience of the reader we will explain all details.

**Theorem 3.4.** Let \( S \subset \mathbb{R}^n \) be a compact and countably \( \mathcal{H}^{n-1} \)-rectifiable set such that
\[
\eta(B_r(x)) \geq \gamma r^{n-1}
\]
holds for all \( x \in S \) and for all \( r \in (0, 1) \) for some \( \gamma > 0 \) and some Radon measure \( \eta \) on \( \mathbb{R}^n \) absolutely continuous with respect to \( \mathcal{H}^{n-1} \). Then \( M_C(S) \) exists and
\[
M_C(S) = \int_S \phi_C(\nu_S) \, d\mathcal{H}^{n-1}. \tag{3.1}
\]

**Proof.** Let \( \{S_h\}_{h \in \mathbb{N}} \) be a countable family of pairwise disjoint compact subsets of \( S \) which covers \( S \), up to a \( \mathcal{H}^{n-1} \)-negligible set, which are contained in some Lipschitz \( (n-1) \)-graph and with finite \( \mathcal{H}^{n-1} \) measure. Applying Lemma 3.3 and using the subadditivity of the lim inf operator we get, for any \( N \in \mathbb{N} \),
\[
M_{\ast C}(S) \geq \liminf_{\varepsilon \to 0} \left( \frac{\bigcup_{h=1}^N (S_h + \varepsilon C)}{2\varepsilon} \right) \geq \sum_{h=0}^N \liminf_{\varepsilon \to 0} \frac{|S_h + \varepsilon C|}{2\varepsilon} = \sum_{h=0}^N M_{\ast C}(S_h) = \sum_{h=0}^N \int_{S_h} \phi_C(\nu_S) \, d\mathcal{H}^{n-1}.
\]
Passing to the limit as \( N \to +\infty \) we find the estimate from below
\[
M_{\ast C}(S) \geq \int_S \phi_C(\nu_S) \, d\mathcal{H}^{n-1}. \tag{3.2}
\]
The main point of the proof concerns the proof of the estimate from above, that is
\[
M_C(S) \leq \int_S \phi_C(\nu_S) \, d\mathcal{H}^{n-1}. \tag{3.3}
\]
The idea is to use, in a suitable way, a covering argument to control what remains outside a region covered by a finite number of subsets of \( S \) where we can say that \( M_C \) exists by Lemma
3.3. Fix \( \sigma \in (0, 1) \). We can find a finite number \( N \) of pairwise disjoint compact subsets \( S_h \) of \( S \) which are contained in some Lipschitz \((n-1)\)-graph, with finite \( \mathcal{H}^{n-1} \)-measure and such that

\[
\eta(S) \leq \sigma + \sum_{h=1}^{N} \eta(S_h).
\]

Consider the set

\[
E := S \setminus \bigcup_{h=1}^{N} S_h
\]

and, for any \( \varepsilon \in (0, 1) \),

\[
S_{\sigma, \varepsilon} := \{ x \in S : d\left( x, \bigcup_{h=1}^{N} S_h \right) \geq \sigma^{1/n} \varepsilon \}.
\]

Using Besicovitch’s Theorem 2.4 we are able to cover \( S_{\sigma, \varepsilon} \) by many balls \( \{ B_{a\sigma^{1/n} \varepsilon}(x_j) \}_{j \in J} \) (recall that \( a \) has been chosen in such a way that \( B_a(0) \subset C \)) with \( x_j \in S_{\sigma, \varepsilon} \) for each \( j \in J \), and such that, for \( \varepsilon \) small enough, using the assumption on \( \eta \),

\[
\sum_{j \in J} \gamma(a\sigma^{1/n} \varepsilon)_{n-1} \leq \sum_{j \in J} \eta(B_{a\sigma^{1/n} \varepsilon}(x_j)) \leq \xi \eta \left( (S + \sigma^{1/n} C) \setminus \bigcup_{h=1}^{N} S_h \right) \leq \xi \sigma
\]

where \( \xi \) is as in Theorem 2.4. As a consequence we get the estimate

\[
\# J \leq \frac{\xi \sigma^{1/n}}{\gamma a^{n-1} \varepsilon^{n-1}}.
\]

Therefore, we obtain, recalling that \( C \subset B_b(0) \),

\[
|S_{\sigma, \varepsilon} + (1 + \sigma^{1/n}) \varepsilon C| \leq \sum_{j \in J} |B_{b[(1 + 2\sigma^{1/n}) \varepsilon]}(x_j)| \leq \frac{\omega_n b^n (1 + 2\sigma^{1/n} \varepsilon) \sigma^{1/n} \xi \varepsilon}{\gamma a^{n-1}} \leq \frac{\omega_n b^n 3^n \sigma^{1/n} \xi \varepsilon}{\gamma a^{n-1}}.
\]

Now, since it holds

\[
S + \varepsilon C \subset (E + \varepsilon C) \cup \bigcup_{h=1}^{N} (S_h + \varepsilon C) \subset (S_{\sigma, \varepsilon} + (1 + \sigma^{1/n}) \varepsilon C) \cup \bigcup_{h=1}^{N} (S_h + \varepsilon C)
\]

we deduce that, by Lemma 3.3,

\[
\mathcal{M}_C^\varepsilon(S) = \limsup_{\varepsilon \to 0} \frac{|S + \varepsilon C|}{2 \varepsilon} \leq \limsup_{\varepsilon \to 0} \frac{|S_{\sigma, \varepsilon} + (1 + \sigma^{1/n}) \varepsilon C|}{2 \varepsilon} + \sum_{h=1}^{N} \limsup_{\varepsilon \to 0} \frac{|S_h + \varepsilon C|}{2 \varepsilon} \leq \frac{\omega_n b^n 3^n \sigma^{1/n} \xi \varepsilon}{2 \gamma a^{n-1}} + \int_S \phi_C(\nu_S) \, d\mathcal{H}^{n-1}
\]

and (3.3) follows sending \( \sigma \to 0 \). \( \square \)

We now move toward the case of the \((n-1)\)-rectifiability, as the first existence result for the Minkowski content, which can be found in [6], pg. 275:
Theorem 3.5. If $S$ is compact and $(n-1)$-rectifiable then $\mathcal{M}_{B_1(0)}(S) = \mathcal{H}^{n-1}(S)$.

The next Lemma can be found in [2], see Lemma 2.105.

Lemma 3.6. Let $K \subset \mathbb{R}^{n-1}$ be a compact set and $f : K \to \mathbb{R}^n$ be Lipschitz. Assume that $J_{n-1}df_x = 0$ a.e. $x \in K$. Then $\mathcal{M}_{B_1(0)}(f(K)) = 0$.

Theorem 3.7. Let $S \subset \mathbb{R}^n$ be compact and $(n-1)$-rectifiable. Then $\mathcal{M}_C(S)$ exists and

$$\mathcal{M}_C(S) = \int_S \phi_C(\nu_S) \, d\mathcal{H}^{n-1}.$$

Proof. The estimate from below

$$\mathcal{M}_+(S) \geq \int_S \phi_C(\nu_S) \, d\mathcal{H}^{n-1}$$

can be proved as in Theorem 3.4. Let $f : K \to \mathbb{R}^n$ be Lipschitz with $K \subset \mathbb{R}^{n-1}$ compact such that $S = f(K)$ and fix $\sigma \in (0,1)$. Consider the subset of $K$ given by

$$F := \{ x \in K : df_x \text{ exists and } J_{n-1}df_x > 0 \}.$$

Let $K' \subset K \setminus F$ be compact and such that $\mathcal{L}^{n-1}(K \setminus (F \cup K')) < \sigma$, being $\mathcal{L}^{n-1}$ the Lebesgue measure of dimension $n-1$. Moreover, let $S_0 := f(K')$. Combining Theorem 3.5 with Lemma 3.6 we get $\mathcal{H}^{n-1}(S_0) = 0$ and thus we obtain

$$\mathcal{M}_C(S_0) \leq \limsup_{\epsilon \to 0} \frac{|S_0 + \epsilon B_b(0)|}{2\epsilon} = b\mathcal{M}^{n-1}(S_0) = bH^{n-1}(S_0) = 0$$

which means that $\mathcal{M}_C(S_0) = 0$. Now, consider the measure $\eta := f_\sharp(\mathcal{L}^{n-1} \mathcal{L} F)$. By definition, $\eta$ is concentrated on $f(K) = S$; moreover, if $S' \subset S$ is $\mathcal{H}^{n-1}$-negligible then, by area formula we deduce that

$$\int_{F \cap f^{-1}(S')} J_{n-1}df_x \, dx = \int_{S'} \mathcal{H}^0(F \cap f^{-1}(\{y\})) \, d\mathcal{H}^{n-1}(y) = 0.$$

Since, by the very definition of $F$, it holds $J_{n-1}df_x > 0$ on $F$, we get $\mathcal{L}^{n-1}(F \cap f^{-1}(S')) = 0$, which proves that $\eta$ is absolutely continuous with respect to $\mathcal{H}^{n-1}$. Now we are ready to use the same covering argument as in the proof of Theorem 3.4 in order to control the “bad” part of $S$ using the properties of the measure $\eta$. More precisely, we can find a finite number $N$ of pairwise disjoint compact subsets $S_h$ of $S$ which are contained in some Lipschitz $(n-1)$-graph, with finite $\mathcal{H}^{n-1}$-measure and such that

$$\eta(S) \leq \sigma + \sum_{h=1}^N \eta(S_h).$$

Consider the set

$$E := S \setminus \bigcup_{h=0}^N S_h.$$
Notice now that $\eta(E) < \sigma$ and $f^{-1}(E) \setminus F \subset K \setminus (F \cup K')$ since $E \cap S_0 = \emptyset$; we deduce that $\mathcal{L}^{n-1}(f^{-1}(E)) < 2\sigma$. If now $L$ denotes the Lipschitz constant of $f$ and we choose $\bar{\varepsilon} > 0$ such that $\mathcal{L}^{n-1}((K + \bar{\varepsilon}L^{-1}C) \setminus K) < \sigma$, we can consider, for any $\varepsilon \in (0, \bar{\varepsilon})$, the set

$$S_{\sigma, \varepsilon} := \left\{ x \in S : d\left(x, \bigcup_{h=0}^{N} S_h\right) \geq \sigma^{1/n} \varepsilon \right\}.$$ 

Applying Besicovitch’s Theorem we are able to cover $S_{\sigma, \varepsilon}$ by many balls $\{B_{a(1+/\varepsilon)}(x_j)\}_{j \in J}$ centred at points of $S_{\sigma, \varepsilon}$ and such that, for $\varepsilon$ small enough,

$$\sum_{j \in J} \omega_{n-1}(aL^{-1}\sigma^{1/n})^{n-1} \leq \sum_{j \in J} \mathcal{L}^{n-1}((K + \varepsilon L^{-1}C) \cap f^{-1}(B_{a(1+/\varepsilon)}(x_j)))$$

$$\leq \xi \mathcal{L}^{n-1}((K + \varepsilon L^{-1}C) \cap f^{-1}\left(\bigcup_{j \in J} B_{a(1+/\varepsilon)}(x_j)\right))$$

$$\leq \xi (\mathcal{L}^{n-1}(f^{-1}(E)) + \mathcal{L}^{n-1}((K + \varepsilon L^{-1}C) \setminus K))) \leq 3\xi \sigma$$

being $\xi$ is as in the Besicovitch’s Theorem. Therefore

$$\#J \leq \frac{3\xi\sigma^{1/n}}{\omega_{n-1}a^{n-1}L^{1-n}\varepsilon^{n-1}}$$

hence

$$|S_{\sigma, \varepsilon} + (1 + \sigma^{1/n})\varepsilon C| \leq \sum_{j \in J} |B_{b(1+2\sigma^{1/n})}(x_j)| \leq \frac{\omega_{n} b^{n}(1 + 2\sigma^{1/n})^{n}\sigma^{1/n}\xi\varepsilon}{\omega_{n-1}a^{n-1}L^{1-n}} \leq \frac{\omega_{n} b^{n+3n+1+\sigma^{1/n}\xi\varepsilon}}{\omega_{n-1}a^{n-1}L^{1-n}}.$$ 

Using again

$$S + \varepsilon C \subset (E + \varepsilon C) \cup \bigcup_{h=0}^{N} (S_h + \varepsilon C) \subset (S_{\sigma, \varepsilon} + (1 + \sigma^{1/n})\varepsilon C) \cup \bigcup_{h=0}^{N} (S_h + \varepsilon C)$$

we deduce that, by Lemma 3.3,

$$\mathcal{M}^*_C(S) = \limsup_{\varepsilon \to 0} \frac{|S + \varepsilon C|}{2\varepsilon}$$

$$\leq \limsup_{\varepsilon \to 0} \frac{|S_{\sigma, \varepsilon} + (1 + \sigma^{1/n})\varepsilon C|}{2\varepsilon} + \sum_{h=0}^{N} \limsup_{\varepsilon \to 0} \frac{|S_h + \varepsilon C|}{2\varepsilon}$$

$$\leq \frac{\omega_{n} b^{n+3n+1+\sigma^{1/n}\xi\varepsilon}}{2\omega_{n-1}a^{n-1}L^{1-n}} + \int_{S} \phi_{C}(\nu_{S}) \, d\mathcal{H}^{n-1}$$

and the conclusion follows. 

\[\square\]

4. A MORE GENERAL FORMULA FOR $S\mathcal{M}_C$

In this section we prove the generalization of formula (1.2). First of all, let us introduce the class $\mathcal{O}$ and $\mathcal{O}'$.

**Definition 4.1.** Let $\mathcal{O}$ be the class of Borel sets $E$ of $\mathbb{R}^n$ such that:

(a) $\partial E$ is a countably $\mathcal{H}^{n-1}$-rectifiable bounded set;
(b) there exist $\gamma > 0$ and a probability measure $\eta$ in $\mathbb{R}^n$ absolutely continuous with respect to $\mathcal{H}^{n-1}$ such that $\eta(B_r(x)) \geq \gamma r^{n-1}$ for all $x \in \partial E$ and for all $r \in (0, 1)$.

Moreover, let $\mathcal{O}_C'$ be the class of Borel sets $E$ of $\mathbb{R}^n$ such that:

(a') $\partial E$ is a countably $\mathcal{H}^{n-1}$-rectifiable bounded set and

$$\mathcal{M}_C(\partial E) = \int_{\partial E} \phi_C(\nu_{\partial E}) \, d\mathcal{H}^{n-1};$$

(b') there exist $\gamma > 0$ and a probability measure $\eta$ in $\mathbb{R}^n$ such that $\eta(B_r(x)) \geq \gamma r^{n-1}$, for all $x \in \partial E$ and for all $r \in (0, 1)$.

Remark 4.2. We point out that condition (b'), and therefore also condition (b), implies, since Theorem 2.1, that $\mathcal{H}^{n-1}(\partial E)$ is finite; in particular, any set in $\mathcal{O}$ or $\mathcal{O}_C'$ has finite perimeter.

Now we are ready to state the main result of this section.

Theorem 4.3. The class $\mathcal{O}$ and the class $\mathcal{O}_C'$ are stable under finite unions, and for any $E \in \mathcal{O}$ (or $\mathcal{O}_C'$) it holds

$$\mathcal{S}\mathcal{M}_C(E) = \int_{\mathcal{F}_E} h_C(\nu_E) \, d\mathcal{H}^{n-1} + 2 \int_{\partial^2 E} \phi_C(\nu_E) \, d\mathcal{H}^{n-1}. \quad (4.1)$$

Remark 4.4. We notice that If $E \in \mathcal{O}$ (or $E \in \mathcal{O}_C'$) is such that $\mathcal{S}\mathcal{M}(E) = \mathcal{P}(E)$, then by (1.2) it follows $\mathcal{H}^{n-1}(E^0 \cap \partial E) = 0$, and so $\mathcal{H}^{n-1}(\partial^2 E) = 0$. As a consequence, from (4.1) we get

$$\mathcal{S}\mathcal{M}_C(E) = \int_{\mathcal{F}_E} h_C(\nu_E) \, d\mathcal{H}^{n-1},$$

in accordance with Theorem 3.2. Moreover, if

$$\frac{\mathcal{S}\mathcal{M}(E) + \mathcal{S}\mathcal{M}(E^c)}{2} = \mathcal{P}(E),$$

then by (1.2) it follows $\mathcal{H}^{n-1}(E^0 \cap \partial E) + \mathcal{H}^{n-1}(E^1 \cap \partial E) = 0$, and so $\mathcal{H}^{n-1}(\partial^2 E) = \mathcal{H}^{n-1}(\partial^2 E^c) = 0$. In this case, again as a consequence of (4.1), we get

$$\frac{\mathcal{S}\mathcal{M}_C(E) + \mathcal{S}\mathcal{M}_C(E^c)}{2} = \frac{1}{2} \left( \int_{\mathcal{F}_E} h_C(\nu_E) \, d\mathcal{H}^{n-1} + \int_{\mathcal{F}_E^c} h_C(\nu_{E^c}) \, d\mathcal{H}^{n-1} \right)$$

$$= \int_{\mathcal{F}_E} h_C(\nu_E) + h_C(-\nu_E) \, d\mathcal{H}^{n-1},$$

in accordance with Theorem 3.7 in [4].

We are now going to prove Theorem 4.3 by proceeding along the same arguments of the proof of Theorem 3.1 in [11]: for the convenience of the reader we will give complete proofs also for the series of auxiliary lemmas that we needed. First of all, it is easy to observe that $E \cap W \in \mathcal{O}$ for any $E \in \mathcal{O}$ and any closed $W \subset \mathbb{R}^n$; the following lemma implies that the same holds for the class $\mathcal{O}_C'$ as well.
Lemma 4.5. If \( S \subset \mathbb{R}^n \) is a countably \( \mathcal{H}^{n-1} \)-rectifiable compact set such that
\[
\mathcal{M}_C(S) = \int_S \phi_C(\nu_S) \, d\mathcal{H}^{n-1},
\]
then
\[
\mathcal{M}_C(S \cap W) = \int_{S \cap W} \phi_C(\nu_S) \, d\mathcal{H}^{n-1},
\]
for all \( W \subset \mathbb{R}^n \) closed.

Proof. Being \( S \cap W \) countably \( \mathcal{H}^{n-1} \)-rectifiable and compact, by (3.2) we know that
\[
\mathcal{M}^\ast_C(S \cap W) \geq \int_{S \cap W} \phi_C(\nu_S) \, d\mathcal{H}^{n-1}.
\]
Let us show that the opposite inequality holds for \( \mathcal{M}^\ast_C(S \cap W) \). Consider the sequence \( \{W_h\}_{h \in \mathbb{N}} \) of closed sets \( W_h := \{x \in W^c : d(x, W) \geq b/h\} \), which implies that
\[
\left(x + \frac{1}{h} C\right) \cap W = \emptyset
\]
for any \( x \in W^c \), since
\[
x + \frac{1}{h} C \subseteq B_{b/h}(x).
\]
Note that \( W_h \not\supset W^c \) as \( h \) goes to infinity. Let us observe that
\[
S + \varepsilon C \supseteq ((S \cap W) + \varepsilon C) \cup ((S \cap W_h) + \varepsilon C),
\]
and
\[
((S \cap W) + \varepsilon C) \cap ((S \cap W_h) + \varepsilon C) = \emptyset
\]
for all \( \varepsilon \) sufficiently small. Hence, for all \( h \in \mathbb{N} \)
\[
\mathcal{M}^\ast_C(S \cap W) \leq \mathcal{M}^\ast_C(S) - \mathcal{M}^\ast_C(S \cap W_h) \leq \int_S \phi_C(\nu_S) \, d\mathcal{H}^{n-1} - \int_{S \cap W_h} \phi_C(\nu_S) \, d\mathcal{H}^{n-1}
\]
By taking now the limit for \( h \) which goes to infinity, we get
\[
\mathcal{M}^\ast_C(S \cap W) \leq \int_{S \cap W} \phi_C(\nu_S) \, d\mathcal{H}^{n-1}
\]
and so the assertion. \( \square \)

By observing that for each \( G \subset \mathbb{R}^n \) Borel set and for each \( r, \rho > 0 \) it holds
\[
|(\partial G + rC) \cap G \cap B_\rho(x)| \subset |(\partial G + rbB_1(0)) \cap G \cap B_\rho(x)|,
\]
the following assertion is a direct application of Lemma 2 in [1].

Lemma 4.6. Let \( G \subset \mathbb{R}^n \) be a Borel set and assume that there exist \( \gamma > 0 \) and a probability measure \( \eta \) on \( \mathbb{R}^n \) such that \( \eta(B_r(x)) \geq \gamma r^{n-1} \) for all \( x \in \partial G \) and for all \( r \in (0, 1) \). Then
\[
\limsup_{\varepsilon \to 0} \frac{|(\partial G + \varepsilon C) \cap G \cap B_\rho(x)|}{\varepsilon} = o(\rho^{n-1})
\]
for \( \mathcal{H}^{n-1} \)-a.e. \( x \in G^0 \cap \partial G \).
For each $A$ Borel subset of $\mathbb{R}^n$ let $\mathcal{S}M_{aC}(E; A)$ and $\mathcal{S}M_{aC}^* (E; A)$ be given by

$$
\mathcal{S}M_{aC}(E; A) := \liminf_{\varepsilon \to 0} \frac{|((E + \varepsilon C) \setminus E) \cap A|}{\varepsilon}, \quad \mathcal{S}M_{aC}^* (E; A) := \limsup_{\varepsilon \to 0} \frac{|((E + \varepsilon C) \setminus E) \cap A|}{\varepsilon}.
$$

We also let $\mathcal{S}M_{aC}(E) := \mathcal{S}M_{aC}(E; \mathbb{R}^n)$ and $\mathcal{S}M_{aC}^* (E) := \mathcal{S}M_{aC}^* (E; \mathbb{R}^n)$.

**Lemma 4.7.** For any $E \in \mathcal{O}$ (or $\mathcal{O}'$) it holds

$$
\mathcal{S}M_{aC}(E; B_\rho(x)) = o(\rho^{-1}), \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in E^1 \cap \partial E,
$$

$$
\mathcal{S}M_{aC}(E; B_\rho(x)) \geq \int_{E \cap \text{int}B_\rho(x)} h_C(\nu_E) \, d\mathcal{H}^{n-1}, \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in E^{1/2},
$$

$$
\mathcal{S}M_{aC}(E; B_\rho(x)) \geq 2 \int_{\partial E \cap \text{int}B_\rho(x)} \phi_C(\nu_E) \, d\mathcal{H}^{n-1} + o(\rho^{-1}), \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial^2 E.
$$

**Proof.** Equality (4.2) follows directly from Lemma 4.6 with the choice $G := E^c$, and taking into account that

$$
|\partial E + \varepsilon C| = |(E + \varepsilon C) \setminus E| + |(E^c + \varepsilon C) \setminus E^c|.
$$

Equality (4.3) can be found in [4]. It remains to prove (4.4). Since $0 \in \text{int} C$, for any closed set $W \subset\subset B_\rho(x)$ there exists $\tilde{\varepsilon} > 0$ such that $W + \varepsilon C \subset B_\rho(x)$, $\forall \varepsilon < \tilde{\varepsilon}$. So, noticing that $\partial E \cap W$ satisfies the assumption of Theorem 3.4 if $E \in \mathcal{O}$ (or (a') in the definition of the class $\mathcal{O}'$ if $E \in \mathcal{O}'$), we get that

$$
2 \int_{\partial E \cap W} \phi_C(\nu_E) \, d\mathcal{H}^{n-1} = \liminf_{\varepsilon \to 0} \frac{|(\partial E \cap W) + \varepsilon C|}{\varepsilon}
\leq \liminf_{\varepsilon \to 0} \frac{|(\partial E + \varepsilon C) \cap (W + \varepsilon C)|}{\varepsilon}
\leq \liminf_{\varepsilon \to 0} \frac{|(\partial E + \varepsilon C) \cap B_\rho(x)|}{\varepsilon}.
$$

Let $\{W_k\}_{k \in \mathbb{N}}$ be an increasing sequence of closed sets with $W_k \subset\subset B_\rho(x)$ and such that $W_k \not\supset \text{int}B_\rho(x)$. By taking the limit as $k$ tends to $\infty$, we obtain that

$$
\liminf_{\varepsilon \to 0} \frac{|(\partial E + \varepsilon C) \cap B_\rho(x)|}{\varepsilon} \geq 2 \lim_{k \to \infty} \int_{\partial E \cap W_k} \phi_C(\nu_E) \, d\mathcal{H}^{n-1} = 2 \int_{\partial E \cap \text{int}B_\rho(x)} \phi_C(\nu_E) \, d\mathcal{H}^{n-1}.
$$

Finally, we have that

$$
\mathcal{S}M_{aC}(E; B_\rho(x)) = \liminf_{\varepsilon \to 0} \frac{|(\partial E + \varepsilon C) \cap B_\rho(x)| - |(\partial E + \varepsilon C) \cap E \cap B_\rho(x)|}{\varepsilon}
\geq \liminf_{\varepsilon \to 0} \frac{|(\partial E + \varepsilon C) \cap B_\rho(x)|}{\varepsilon} - \limsup_{\varepsilon \to 0} \frac{|(\partial E + \varepsilon C) \cap E \cap B_\rho(x)|}{\varepsilon}.
$$

Thus (4.4) follows from (4.6), Lemma 4.6, and by taking into account that $\mathcal{H}^{n-1}(\partial E \cap E^0) = \mathcal{H}^{n-1}(\partial^2 E)$. \qed
Now we are ready to prove the main result of this section.

Proof of Theorem 4.3. Let \( E \in \mathcal{O} \) (or \( E \in \mathcal{O}_C' \)). Let us show that the following lower bound for \( SM_*(E) \) holds:

\[
SM_*(E) \geq \int_{E} h_C(\nu_E) d\mathcal{H}^{n-1} + 2 \int_{\partial E} \phi_C(\nu_E) d\mathcal{H}^{n-1}. \tag{4.7}
\]

Let \( \mu \) be the measure in \( \mathbb{R}^n \) so defined

\[
\mu(A) := \int_{E \cap A} h_C(\nu_E) d\mathcal{H}^{n-1} + 2 \int_{\partial E \cap A} \phi_C(\nu_E) d\mathcal{H}^{n-1}, \quad A \subset \mathbb{R}^n \text{ Borel.}
\]

By rectifiability we can say that

\[
\lim_{\rho \to 0} \frac{\mathcal{H}^{n-1}(\partial E \cap E^0 \cap B_\rho(x))}{\rho^{n-1}} = \omega_{n-1}, \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial E \cap E^0,
\]

and 0 for \( \mathcal{H}^{n-1}\text{-a.e. } x \in (\partial E \cap E^0)^c \); the same conclusions hold for the quantities

\[
\frac{\mathcal{H}^{n-1}(\partial E \cap E^1 \cap B_\rho(x))}{\rho^{n-1}}, \quad \frac{\mathcal{H}^{n-1}(\partial E \cap B_\rho(x))}{\rho^{n-1}}.
\]

Taking into account all those informations and using Lemma 4.7 we get that, for any \( \varepsilon > 0 \) and for \( \mathcal{H}^{n-1}\text{-a.e. } x \in \partial E \),

\[
\liminf_{\rho \to 0} \frac{SM_*(E; B_\rho(x)) + \varepsilon \mathcal{H}^{n-1}(E \cap \partial E \cap B_\rho(x)) - \mu(\text{int} B_\rho(x))}{\mu(\text{int} B_\rho(x))} \geq 1. \tag{4.8}
\]

(Note that the term \( \varepsilon \mathcal{H}^{n-1}(E^1 \cap \partial E \cap B_\rho(x)) \) in the above fraction is to avoid an indetermination of type 0/0 at points \( x \in E^1 \cap \partial E \).) Since the family of closed balls \( B_\rho(x) \) with \( \mu(\partial B_\rho(x)) = 0 \) is a fine cover of \( \partial E \), by Vitali-Besicovitch covering Theorem 2.5 for any \( \delta > 0 \) there exist finitely many disjoint closed balls \( W_1, \ldots, W_N \) with \( \mu(\partial W_i) = 0 \) such that

\[
\mu\left(\partial E \setminus \bigcup_{i=1}^{N} W_i\right) < \delta.
\]

The balls \( W_i \) can be chosen with centers in \( \partial E \) and such that

\[
SM_*(E; W_i) + \varepsilon \mathcal{H}^{n-1}(E \cap \partial E \cap W_i) \geq (1 - \delta)\mu(W_i), \quad i = 1, \ldots, N.
\]
Then, the following chain of inequalities holds:

\[ \mathcal{S} \mathcal{M}_c(E) + \varepsilon \mathcal{H}^{n-1}(E^1 \cap \partial E) \]

\[ \geq \mathcal{S} \mathcal{M}_c \left( E; \bigcup_{i=1}^N W_i \right) + \varepsilon \mathcal{H}^{n-1} \left( E^1 \cap \partial E \cap \bigcup_{i=1}^N W_i \right) \]

\[ \geq \sum_{i=1}^N \left( \mathcal{S} \mathcal{M}_c(E; W_i) + \varepsilon \mathcal{H}^{n-1}(E^1 \cap \partial E \cap W_i) \right) \]

\[ \geq (1 - \delta) \sum_{i=1}^N \mu(W_i) \]

\[ = (1 - \delta) \left( \mu(\mathbb{R}^n) - \mu \left( \mathbb{R}^n \setminus \bigcup_{i=1}^N W_i \right) \right) \]

\[ \geq (1 - \delta) \left( \int_{\partial E} h_C(\nu_E) \, d\mathcal{H}^{n-1} + 2 \int_{\partial^2 \mathcal{E}} \phi_C(\nu_E) \, d\mathcal{H}^{n-1} - \delta \right). \]

By taking now the limit as \( \delta \to 0 \), and then as \( \varepsilon \to 0 \), we obtain the inequality (4.7).

Observing now that \( E^c \) belongs to \( \mathcal{O} \) (resp. \( \mathcal{O}^c \)), too, we can also claim that

\[ \mathcal{S} \mathcal{M}_c(E^c) \geq \int_{\partial E^c} h_c(\nu_{E^c}) \, d\mathcal{H}^{n-1} + \int_{\partial^2 \mathcal{E}^c} \phi_C(\nu_{E^c}) \, d\mathcal{H}^{n-1}. \quad (4.9) \]

Let us now define, for any \( \varepsilon > 0 \),

\[ a_\varepsilon := \frac{|(E + \varepsilon C) \setminus E|}{\varepsilon}, \quad b_\varepsilon := \frac{|(E^c + \varepsilon C) \setminus E^c|}{\varepsilon}. \]

Observe that by taking into account (2.6) for \( \mathcal{H}^{n-1} \)-a.e. \( x \in \partial E \setminus \mathcal{F}E \)

\[ \int_{\partial E \setminus \mathcal{F}E} \phi_C(\nu_E) \, d\mathcal{H}^{n-1} = \int_{\partial E} \phi_C(\nu_E) \, d\mathcal{H}^{n-1} + \int_{\partial^2 \mathcal{E}^c} \phi_C(\nu_{E^c}) \, d\mathcal{H}^{n-1}. \quad (4.10) \]

Moreover, it holds

\[ \liminf_{\varepsilon \to 0} a_\varepsilon = \mathcal{S} \mathcal{M}_c(E) \quad (4.7) \geq \int_{\mathcal{F}E} h_C(\nu_E) \, d\mathcal{H}^{n-1} + 2 \int_{\partial^2 \mathcal{E}} \phi_C(\nu_E) \, d\mathcal{H}^{n-1} =: a, \]

and, using also (2.5) and (2.7),

\[ \liminf_{\varepsilon \to 0} b_\varepsilon = \mathcal{S} \mathcal{M}_c(E^c) \quad (4.9) \geq \int_{\mathcal{F}E} h_C(-\nu_{E^c}) \, d\mathcal{H}^{n-1} + 2 \int_{\partial^2 \mathcal{E}^c} \phi_C(\nu_{E^c}) \, d\mathcal{H}^{n-1} =: b. \]
By (4.5) and by (3.1) if $E \in \mathcal{O}$ (resp. by (a') in the definition of the class $\mathcal{O}_C'$ if $E \in \mathcal{O}_C'$) it follows that
\[
\limsup_{\varepsilon \to 0} (a_\varepsilon + b_\varepsilon) = \limsup_{\varepsilon \to 0} \frac{|\partial E + \varepsilon C|}{\varepsilon} = 2\mathcal{M}_C(\partial E).
\]
\[
= 2 \int_{\partial E} \phi_C(\nu_{\partial E}) \, d\mathcal{H}^{n-1}
\]
\[
= 2 \int_{\partial E \cap F_E} \phi_C(\nu_E) \, d\mathcal{H}^{n-1} + 2 \int_{\partial E \setminus F_E} \phi_C(\nu_E) \, d\mathcal{H}^{n-1}
\]
\[
= 2 \int_{F_E} (h_C(\nu_E) + h_C(-\nu_E)) \, d\mathcal{H}^{n-1} + 2 \int_{\partial E \setminus F_E} \phi_C(\nu_E) \, d\mathcal{H}^{n-1}
\]
\[
= a + b.
\]
Since
\[
\limsup_{\varepsilon \to 0} (a_\varepsilon + b_\varepsilon) \leq a + b,
\]
\[
\liminf_{\varepsilon \to 0} a_\varepsilon \geq a \in \mathbb{R},
\]
\[
\liminf_{\varepsilon \to 0} b_\varepsilon \geq b \in \mathbb{R}
\]
implies
\[
\lim_{\varepsilon \to 0} a_\varepsilon = a, \quad \lim_{\varepsilon \to 0} b_\varepsilon = b,
\]
equation (4.1) follows.

In order to conclude the proof it remains to show that the class $\mathcal{O}_C'$ is stable under finite unions, since the stability of the class $\mathcal{O}$ under finite unions has been already proved in [11]. Let $E_1, E_2 \in \mathcal{O}_C'$ and let $E := E_1 \cup E_2$. Being $\partial E \subseteq \partial E_1 \cup \partial E_2$, it is clear that $\partial E$ is a countably $\mathcal{H}^{n-1}$-rectifiable bounded set, and that (b') in the definition of the class $\mathcal{O}_C'$ is fulfilled. We know that, from (3.2),
\[
\mathcal{M}_* C(\partial E) \geq \int_{\partial E} \phi_C(\nu_E) \, d\mathcal{H}^{n-1}.
\]
Next, we have to prove that
\[
\mathcal{M}_C^* (\partial E) \leq \int_{\partial E} \phi_C(\nu_E) \, d\mathcal{H}^{n-1}.
\]
We first localize the lower and upper anisotropic Minkowski content: if $S$ is compact and $\mathcal{H}^{n-1}$-rectifiable, $A \subset \mathbb{R}^n$ is closed and $B \subset \mathbb{R}^n$ is open, let
\[
\mathcal{M}_C^* (S; A) := \limsup_{\varepsilon \to 0} \frac{|(S + \varepsilon C) \cap A|}{2\varepsilon}, \quad \mathcal{M}_* C(S; B) := \liminf_{\varepsilon \to 0} \frac{|(S + \varepsilon C) \cap B|}{2\varepsilon}.
\]
Of course, we have $\mathcal{M}_C^* (S; \mathbb{R}^n) = \mathcal{M}_C^* (S)$ and $\mathcal{M}_* C(S; \mathbb{R}^n) = \mathcal{M}_* C(S)$; furthermore, following the same proof of (3.2), we can realize that for any $B \subset \mathbb{R}^n$ open it holds
\[
\mathcal{M}_* C(S; B) \geq \int_{S \cap B} \phi_C(\nu_S) \, d\mathcal{H}^{n-1}.
\]
By observing that
\[ \chi_{\partial E + \varepsilon C} + \chi_{(\partial E_1 \cap \partial E_2) + \varepsilon C} \leq \chi_{\partial E_1 + \varepsilon C} + \chi_{\partial E_2 + \varepsilon C}; \]
we get, for any \( x \in \mathbb{R}^n \) and for \( \mathcal{H}^1 \)-a.e. \( \rho > 0 \), using also (4.13),
\[ \mathcal{M}_C^\varepsilon(\partial E; B_\rho(x)) \]
\[ \leq \int_{\partial E_1 \cap B_\rho(x)} \phi_C(\nu_{E_1}) \, d\mathcal{H}^{n-1} + \int_{\partial E_2 \cap B_\rho(x)} \phi_C(\nu_{E_2}) \, d\mathcal{H}^{n-1} \]
\[ - \int_{\partial E_1 \cap \partial E_2 \cap \text{int} B_\rho(x)} \phi_C(\nu_{E_1 \cap E_2}) \, d\mathcal{H}^{n-1} \]
\[ = \int_{(\partial E_1 \cup \partial E_2) \cap B_\rho(x)} \phi_C(\nu_{E_1 \cup E_2}) \, d\mathcal{H}^{n-1} + \int_{\partial E_1 \cap \partial E_2 \cap B_\rho(x)} \phi_C(\nu_{E_1 \cap E_2}) \, d\mathcal{H}^{n-1} \]
and thus
\[ \mathcal{M}_C^\varepsilon(\partial E; B_\rho(x)) \leq \int_{\partial E \cap B_\rho(x)} \phi_C(\nu_E) \, d\mathcal{H}^{n-1} \]
\[ + \int_{(\partial E_1 \cup \partial E_2) \cap (\partial E) \cap B_\rho(x)} \phi_C(\nu_E) \, d\mathcal{H}^{n-1} \]
\[ + \int_{\partial E_1 \cap \partial E_2 \cap \partial B_\rho(x)} \phi_C(\nu_{E_1 \cap E_2}) \, d\mathcal{H}^{n-1}. \tag{4.14} \]

We notice now that, for \( \mathcal{H}^{n-1} \)-a.e. \( x \in \partial E \),
\[ \int_{(\partial E_1 \cup \partial E_2) \cap (\partial E) \cap B_\rho(x)} \phi_C(\nu_{E_1 \cup E_2}) \, d\mathcal{H}^{n-1} = o(\rho^{n-1}) \]
since we can apply (2.1) to the Radon measure \( \eta \) given by
\[ \eta(D) := \int_{(\partial E_1 \cup \partial E_2) \cap (\partial E) \cap D} \phi_C(\nu_{E_1 \cup E_2}) \, d\mathcal{H}^{n-1}, \quad D \text{ Borel in } \mathbb{R}^n. \]
Moreover, observe that \( \mathcal{H}^{n-1}(\partial E_1 \cap \partial E_2 \cap \partial B_\rho(x)) = 0 \) for \( \mathcal{H}^1 \)-a.e. \( \rho > 0 \). Indeed, by contradiction if for any \( \rho \in A \) with \( \mathcal{L}^1(A) > 0 \) we had \( \mathcal{H}^{n-1}(\partial E_1 \cap \partial E_2 \cap \partial B_\rho(x)) > 0 \), by coarea formula
\[ \mathcal{L}^n(\partial E_1 \cap \partial E_2) = \int_0^{+\infty} \mathcal{H}^{n-1}(\partial E_1 \cap \partial E_2 \cap \partial B_\rho(x)) \, d\rho \geq \int_A \mathcal{H}^{n-1}(\partial E_1 \cap \partial E_2 \cap \partial B_\rho(x)) \, d\rho \]
and therefore \( \mathcal{L}^n(\partial E_1 \cap \partial E_2) > 0 \) which implies that \( \mathcal{H}^{n-1}(\partial E_1 \cap \partial E_2) = +\infty \), which is a contradiction (see Remark (4.2)). Thus,
\[ \int_{\partial E_1 \cap \partial E_2 \cap \partial B_\rho(x)} \phi_C(\nu_{E_1 \cap E_2}) \, d\mathcal{H}^{n-1} = 0, \quad \mathcal{H}^1 \text{-a.e. } \rho > 0. \]
from which we obtain, by (4.14), the key estimate
\[ \mathcal{M}_C^\varepsilon(\partial E; B_\rho(x)) \leq \int_{\partial E \cap B_\rho(x)} \phi_C(\nu_E) \, d\mathcal{H}^{n-1} + o(\rho^{n-1}), \quad \text{for } \mathcal{H}^{n-1} \text{-a.e. } x \in \partial E. \tag{4.15} \]
Now the assertion follows applying Theorem 2.5. For any $D \subset \mathbb{R}^n$ Borel, let
\[ \sigma(D) := \int_{\partial E \cap D} \phi_C(\nu_E) \, d\mathcal{H}^{n-1}. \]
As a consequence of (4.15) and (2.2) we may claim that
\[ \limsup_{\rho \to 0} \frac{M^*_C(\partial E; B_\rho(x))}{\sigma(B_\rho(x))} \leq 1, \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial E. \]
Since $\partial E$ is bounded, for any $\delta > 0$ there exists a finite covering $B_1, \ldots, B_N$ of $\partial E$, being $B_i$ disjoint closed balls in $\mathbb{R}^n$ with
\[ \sigma\left(\partial E \setminus \bigcup_{i=1}^N B_i\right) < \delta; \]
notice that the balls $B_i$ can be assumed to have centers in $\partial E$, and such that $\sigma(\partial B_i) = 0$, and
\[ \frac{M^*_C(\partial E; B_i)}{\sigma(B_i)} \leq 1 + \delta. \]
Finally, let $B := \mathbb{R}^n \setminus \bigcup_{i=1}^N \text{int} B_i$. Notice that $M^*_C(\partial E; B) = 0$ and thus
\[ M^*_C(\partial E) \leq M^*_C(\partial E, B) + M^*_C\left(\partial E, \bigcup_{i=1}^N B_i\right) \]
\[ \leq \sum_{i=1}^N M^*_C(\partial E, B_i) \]
\[ \leq (1 + \delta) \sum_{i=1}^N \sigma(B_i) \]
\[ = (1 + \delta) \sigma\left(\bigcup_{i=1}^N B_i\right) \]
\[ \leq (1 + \delta) \int_{\partial E} \phi_C(\nu_E) \, d\mathcal{H}^{n-1}. \]
Inequality (4.12) follows sending $\delta \to 0$, and this completes the proof.

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