Continuity and differentiability properties of the isoperimetric profile in complete noncompact Riemannian manifolds with bounded geometry

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ABSTRACT. For a complete noncompact connected Riemannian manifold with bounded geometry M^n, we prove that the isoperimetric profile function I_{M^n} is continuous. Here for bounded geometry we mean that M have *Ricci* curvature bounded below and volume of balls of radius 1, uniformly bounded below with respect to its centers. Then under an extra hypothesis on the geometry of M, we apply this result to prove some differentiability property of I_M and a differential inequality satisfied by I_M, extending in this way well known results for compact manifolds, to this class of noncompact complete Riemannian manifolds.

Key Words: Continuity of isoperimetric profile, bounded geometry.

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Contents

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1 Introduction

1.1 Isoperimetric profile

In the remaining part of this paper we always assume that all the Riemannian manifolds (M, g) considered are smooth with smooth Riemannian metric g. We denote by V the canonical Riemannian measure induced on M by g, and by A the $(n-1)$-Hausdorff measure associated to the canonical Riemannian length space metric d of M. When it is already clear from the context, explicit mention of the metric g will be suppressed in what follows. At this point we give the definition of the isoperimetric profile function which is the main object of study in this paper.

Definition 1.1. *The isoperimetric profile function (or briefly, the isoperimetric profile) $I_M : [0, V(M)[\to [0, +\infty[$, is defined by*

$$I_M(v) := \inf\{A(\partial\Omega) : \Omega \in \tau_M, V(\Omega) = v\}, v \neq 0,$$

and $I_M(0) = 0$, where τ_M denotes the set of relatively compact open subsets of M with smooth boundary.

If M is compact, classical compactness arguments of geometric measure theory combined with the direct method of the calculus of variations provide a short proof of continuity of I_M in any dimension n, [AMN13] Proposition 1. Finally, if M is complete, non-compact, and $V(M) < +\infty$, an easy consequence of Theorem 2.1 in [RR04] yields the possibility of extending the same argument and to prove the continuity of the isoperimetric profile. A careful analysis of the Theorem 1 of [Nar14] about the existence of generalized isoperimetric regions, leads to the continuity of the isoperimetric profile I_M in manifolds with bounded geometry satisfying some other assumptions on the geometry of the manifold at infinity, of the kind considered by the second author and A. Mondino in [MN12], i.e., for every sequence of points diverging to infinity, there exists a pointed smooth manifold $(M_\infty, g_\infty, p_\infty)$ such that $(M, g, p_j) \to (M_\infty, g_\infty, p_\infty)$ in C^0-topology. This is not the case for general complete infinite-volume manifolds M. In case of manifolds with density in Proposition 2 of [AMN13] is exhibited an example of a manifold with density having discontinuous isoperimetric profile. The aim of this paper is to prove Theorem 1 in which we give a very short and quite elementary proof of the continuity of I_M when M is a complete noncompact Riemannian manifold of bounded geometry. The reason is that in bounded geometry it is always possible to add or subtract to an isoperimetric region a small ball centered at points of density 0 and 1 respectively. Following this philosophy it is quite easy to show that to have an isoperimetric region in volume v ensures the upper semicontinuity of I_M at v, this is the content of Theorem 2.1. The problems appears when try to prove lower semicontinuity. To prove lower semicontinuity we need some kind of compactness that is expressed here by a bounded geometry condition. Geometrically speaking our assumptions of bounded geometry ensures that the manifold at infinity is not too thin and enough thick to permit to place a small geodesic ball B inside an arbitrary domain D in such a way $V(B)$ is a controlled fraction of $V(D)$ and this fraction depends only on $V(D)$ and the bounds on the geometry n, v_0, k, see Definition 1.2 below for the exact meaning of n, v_0, k. The proof that we present here uses only metric properties of the manifolds with bounded geometry and for this reason it is still valid when suitably reformulated in the context of metric measured spaces. For the full generality of the results we need that the spaces have to be doubling, satisfying a 1-Poincaré inequality and a curvature dimension condition. This class of metric spaces includes for example manifolds with density as well as Subriemannian manifolds.

1.2 Plan of the article

1. Section 1 constitutes the introduction of the paper. We state the main results of the paper.

2. In section 2 we prove the continuity of isoperimetric profile in bounded geometry, i.e., Theorem 1, without assuming existence of isoperimetric regions.

3. In the third and final section 3, we prove Corollary 1 and 2.

1.3 Acknowledgements

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1.4 Main Results

Definition 1.2. *A complete Riemannian manifold (M, g), is said to have* ***bounded geometry*** *if there exists a constant $k \in \mathbb{R}$, such that $Ric_M \geq k(n-1)$ (i.e., $Ric_M \geq (n-1)kg$ in the sense of quadratic forms) and $V(B_{(M,g)}(p,1)) \geq v_0$ for some positive constant v_0, where $B_{(M,g)}(p,r)$ is the geodesic ball (or equivalently the metric ball) of M centered at p and of radius $r > 0$.*

Theorem 1 (Continuity of the isoperimetric profile)**.** *Let M^n be a complete smooth Riemannian manifold with $Ric_M \geq (n-1)k$, $k \in \mathbb{R}$ and $V(B(p,1)) \geq v_0 > 0$. Then I_M is continuous on $[0, V(M)[$.*

Definition 1.3. *For any $m \in \mathbb{N}$, $\alpha \in [0,1]$, a sequence of pointed smooth complete Riemannian manifolds is said to* ***converge in the pointed $C^{m,\alpha}$, respectively C^m topology to a smooth manifold*** *M (denoted $(M_i, p_i, g_i) \to (M, p, g)$), if for every $R > 0$ we can find a domain Ω_R with $B(p,R) \subseteq \Omega_R \subseteq M$, a natural number $\nu_R \in \mathbb{N}$, and C^{m+1} embeddings $F_{i,R} : \Omega_R \to M_i$, for large $i \geq \nu_R$ such that $B(p_i, R) \subseteq F_{i,R}(\Omega_R)$ and $F^*_{i,R}(g_i) \to g$ on Ω_R in the $C^{m,\alpha}$, respectively C^m topology.*

Definition 1.4. *We say that a smooth Riemannian manifold (M^n, g) has* ***$C^{m,\alpha}$-locally asymptotic bounded geometry*** *if it is of bounded geometry and if for every diverging sequence of points (p_j), there exists a subsequence (p_{j_l}) and a pointed smooth manifold $(M_\infty, g_\infty, p_\infty)$ with g_∞ of class $C^{m,\alpha}$ such that the sequence of pointed manifolds $(M, p_{j_l}, g) \to (M_\infty, g_\infty, p_\infty)$, in $C^{m,\alpha}$-topology.*

Corollary 1 (Bavard-Pansu-Morgan-Johnson in bounded geometry)**.** *Let M have C^0-locally asymptotic bounded geometry in the sense of Definition 1.4. Suppose that all the limit manifolds have a metric at least of class C^2. Then I_M is absolutely continuous and twice differentiable almost everywhere. The left and right derivatives $I_M^- \geq I_M^+$ exist everywhere and their singular parts are non-increasing. If $k > 0$ then I_M is strictly concave on $]0, V(M)[$. If $k = 0$, then I_M is just concave* *on* $]0, V(M)[$. *If* $k < 0$, *then* $I_M(v)+C(a,b)v^2$ *is concave, (*I_M *could not be concave). Moreover, we have for every* $k \in \mathbb{R}$ *and almost everywhere*

$$I_M I''_M \leq -\frac{{I'}^2_M}{n-1} - (n-1)k, \tag{1}$$

with equality in the case of the simply connected space form of constant sectional curvature k. *In this case, a generalized isoperimetric region is totally umbilic.*

Corollary 2 (Morgan-Johnson isoperimetric inequality in bounded geometry)**.** *Let* M *have* $C^{2,\alpha}$*-bounded geometry, sectional curvature* K *and Gauss-Bonnet-Chern integrand* G. *Suppose that*

- $K < K_0$, *or*
- $K \leq K_0$, *and* $G \leq G_0$,

where G_0 *is the Gauss-Bonnet-Chern integrand of the model space form of constant curvature* K_0. *Then for small prescribed volume, the area of a region* R *of volume* v *is at least as great as* $A(\partial B_v)$, *where* B_v *is a geodesic ball of volume* v *in the model space, with equality only if* R *is isometric to* B_v.

The proofs of Corollaries 1 and 2 run along the same lines as the corresponding proofs of theorems 3.3 and 4.4 of [MJ00].

2 Continuity of I_M

2.1 Continuity in bounded geometry

To illustrate the proof of theorem 1 we start this section with the easy part of the proof resumed in the next lemma that is straightforward compare [AMN13] Proposition 1.

Theorem 2.1. *Let* M *be a Riemannian manifold (possibly incomplete, or possibly complete not necessarily with bounded geometry). If there exists an isoperimetric region in volume* $v \in]0, V(M)[$ *then* I_M *is upper semicontinuous in* v.

Proof: To prove the theorem it is enough to prove the next two inequalities.

$$\overrightarrow{\lim}_{v' \to v^-} I_M(v') \leq I_M(v). \tag{2}$$

$$\overrightarrow{\lim}_{v' \to v^+} I_M(v') \leq I_M(v). \tag{3}$$

In first we prove (2). If $v_j \nearrow v$, consider an isoperimetric region D in volume $V(D) = v$,

$$I_M(v) = A(\partial D).$$

Then for j sufficiently large one can subtract a small geodesic ball (i.e. of small radius) $B_j = B(p, r'_j)$ of volume $v_j - v$ from D, centered to a point of density 1, to obtain $D'_j := D \setminus B(p, r'_j)$ of volume $V(D'_j) = v_j$ and $A(\partial D'_j) \leq A(\partial D) + A(\partial B_j)$. Observe here that the center p of B_j is fixed with respect to j. Moreover $r'_j \to 0$, and this is always possible to obtain in any Riemannian manifold. So by definition of $I_M(v_j)$, holds

$$I_M(v_j) \leq A(\partial D'_j) \leq A(\partial D) + A(\partial B_j) = I_M(v) + A(\partial B_j),$$

which implies that

$$\overrightarrow{\lim} I_M(v_j) \leq \overrightarrow{\lim} A(\partial D) + A(\partial B_j) \leq I_M(v),$$

since the sequence v_j is arbitrary we get (2). In second, we prove (3). If $v_j \searrow v$, then take an isoperimetric region in volume v, i.e., $V(D) = v$, $A(\partial D) = I_M(v)$ and then add a small ball $B_j := B(p, r_j)$ of volume $v_j - v$ to D outside D to obtain $D'_j := D \mathring{\cup} B_j$ of volume $V(D'_j) = v_j$ and $A(\partial D'_j) = A(\partial D) + A(\partial B_j)$. Observe again that the center p of B_j here is fixed with respect to j and $r_j \to 0$, this is always possible in any Riemannian manifold. By definition of $I_M(v_j)$ we get

$$I_M(v_j) \leq A(\partial D'_j) = A(\partial D) + A(\partial B_j) = I_M(v) + A(\partial B_j),$$

now taking the $\overrightarrow{\lim}$ it follows

$$\overrightarrow{\lim} I_M(v_j) \leq \overrightarrow{\lim}[A(\partial D) + A(\partial B_j)] = I_M(v) + \overrightarrow{\lim} A(\partial B_j) = I_M(v),$$

since the sequence v_j is arbitrary we get (3), which completes the proof. q.e.d.

At this point, we may finish the proof of the main Theorem 1.
Proof: We will prove separately the following four inequalities that together will give the proof of our theorem 1.

$$I_M(v) \leq \underrightarrow{\lim}_{v' \to v^-} I_M(v'). \tag{4}$$

$$I_M(v) \leq \underrightarrow{\lim}_{v' \to v^+} I_M(v'). \tag{5}$$

$$\overrightarrow{\lim}_{v'\to v^-} I_M(v') \leq I_M(v). \tag{6}$$

$$\overrightarrow{\lim}_{v'\to v^+} I_M(v') \leq I_M(v). \tag{7}$$

To prove (4) we want to add a small ball. Let $v_j \nearrow v$, take a domain D_j in volume v_j such that $V(D_j) = v_j$ and $I_M(v_j) \leq A(\partial D_j) + \frac{1}{j}$ then add a small ball $B_j := B(p_j, r_j)$ to D_j outside D_j to obtain D'_j of volume v and $A(\partial D'_j) = A(\partial D_j) + A(\partial B_j)$. This is possible because D_j by the very definition (see Definition 1.1) may be chosen bounded. It is worth to observe here that the centers p_j are variable and not fixed as in the proof of Theorem 2.1. So we need to use Bishop-Gromov's Theorem to bound the area of B_j uniformly w.r.t. the centers. Having in mind the definition of $I_M(v)$ it is easy to see that

$$I_M(v) = I_M(V(D'_j)) \leq A(\partial D_j) = A(\partial D_j) + A(\partial B_j). \tag{8}$$

Now observe that by Lemma 3.2 of [MN12] or Lemma 3.5 of [MJ00] that $A(\partial B_j) \leq A(\partial B_{\mathbb{M}^n_k}(v - v_j))$ where $B_{\mathbb{M}^n_k}(w)$ is a geodesic ball enclosing volume w in $\mathbb{M}^n_k$. As it is easy to check $A(\partial B_{\mathbb{M}^n_k}(w)) \to 0$ when $w \to 0$ because the centers could be chosen fixed in the comparison manifold. Which implies that $A(\partial B_{\mathbb{M}^n_k}(v - v_j)) \to 0$, when $j \to +\infty$ and a fortiori that $\underrightarrow{\lim}_{j\to+\infty} A(\partial B_j) = 0$. Then

$$I_M(v) \leq A(\partial D'_j) \leq I_M(v_j) + \frac{1}{j} + A(\partial B_{\mathbb{M}^n_k}(v - v_j)) \leq \underrightarrow{\lim}\, I_M(v_j). \tag{9}$$

By the arbitrariness of the initial sequence of volumes (v_j), (4) follows readily.

To show (5) the strategy is now to subtract a small ball to an eventually diverging (to infinity) sequence of domains that could become thinner and thinner without leaving the opportunity of placing a small ball of the right value of the volume inside them. To rule out this possibility Lemma 2.5 of [Nar14] is needed. This is a more delicate task with respect to the preceding construction in which we add a small ball to a relatively compact domain.

Remark 2.1. *From the proof of Lemma 2.5 of [Nar14] we argue that when* $|v - v'| \sim r^n << v$, $m'_0 = \frac{1}{2}c_1(n, k, r) = \frac{r^n}{2e^{(n-1)\sqrt{|k|}}}$.

Let D such that $V(D) = v' > v$ and then take r satisfying $\frac{r^n v_0}{2e^{(n-1)\sqrt{k}}} = v' - v$, by Lemma 2.5 of [Nar14] we may take a point $p \in M$ such that for small $v' - v$ one have

$$V(B(p, r) \cap D) > \frac{r^n v_0}{2e^{(n-1)\sqrt{k}}} = v' - v. \tag{10}$$

This is possible because for small $|v - v'|$ we can take r small enough to obtain that the constant m_0' produced by Lemma 2.5 of [Nar14] coincides with the right hand side of the preceding inequality. An easy consequence of (10) is that

$$V(D \setminus B(p,r)) = V(D) - V(B(p,r) \cap D) < v,$$

it follows that we may choose $0 < r' < r$ satisfying $V(D \setminus B(p, r')) = v$. Fix $\eta > 0$ and consider an almost isoperimetric region D in volume v', i.e., such that $V(D) = v'$ and

$$I_M(v') \leq A(\partial D) \leq I_M(v') + \eta, \tag{11}$$

by Bishop-Gromov's theorem it is true that $A(\partial B_M(p, r')) \leq A(\partial B_{\mathbb{M}_k^n}(r'))$, then we have the following

$$\begin{aligned} I_M(v) &\leq A(\partial(D \setminus B_M(p,r'))) \leq A(\partial D) + A(\partial B_M(p,r')) &(12)\\ &\leq I_M(v') + \eta + A(\partial B_{\mathbb{M}_k^n}(r')), &(13) \end{aligned}$$

with $r' < r = \left(2\frac{v'-v}{v_0}e^{(n-1)\sqrt{k}}\right)^{\frac{1}{n}}$. By the arbitrariness of $\eta > 0$ we get

$$I_M(v) \leq I_M(v') + A(\partial B_{\mathbb{M}_k^n}(r')). \tag{14}$$

Taking limits in the last inequality yields

$$I_M(v) \leq \underrightarrow{\lim}_{v' \to v^+} I_M(v'). \tag{15}$$

The last two inequalities are relative to the $\overrightarrow{\lim}$ property and are analogous to the case in which there is existence of an isoperimetric region in volume v, but with the additional difficulty that isoperimetric regions in volume v does not necessarily exists. So we apply the same ideas of the proof of Theorem 2.1 to a minimizing sequence in volume v instead of a genuine isoperimetric region.

Now, we prove (6). If $v_j \nearrow v$, consider an almost minimizer D_j in volume $V(D_j) = v$, i.e.,

$$I_M(v) \leq A(\partial D_j) \leq I_M(v) + \frac{1}{j}.$$

Then subtract a small ball B_j of volume $v - v_j$ to D_j as in the proof of (5), to obtain $D_j' := D_j \setminus B(p_j, r_j')$ of volume $V(D_j') = v_j < v$ and $A(\partial D_j') \leq A(\partial D_j) + A(\partial B_j)$, so by definition it holds

$$I_M(v_j) \leq A(\partial D_j') \leq A(\partial D_j) + A(\partial B_j),$$

which implies (as in the proof of (5)) that

$$\overrightarrow{\lim} I_M(v_j) \leq \overrightarrow{\lim}[A(\partial D_j) + A(\partial B_j)] = I_M(v).$$

Since the sequence (v_j) is arbitrary we get (6).

Finally we prove (7). This last part of the proof is analogous in some respects to the proof of (4), because we add a small ball. If $v_j \searrow v$, then take a minimizing sequence D_j in volume v, i.e., $V(D_j) = v$, $A(\partial D_j) \searrow I_M(v)$ and then add a small ball B_j to D_j outside D_j to obtain D'_j of volume $V(D'_j) = v_j$ and $A(\partial D'_j) = A(\partial D_j) + A(\partial B_j)$,

$$I_M(v_j) \leq A(\partial D'_j) = A(\partial D_j) + A(\partial B_j),$$

now taking the $\overrightarrow{\lim}$ it follows as before

$$\overrightarrow{\lim} I_M(v_j) \leq \overrightarrow{\lim} A(\partial D_j) + A(\partial B_j) = I_M(v) + \overrightarrow{\lim} A(\partial B_j) = I_M(v),$$

since the sequence v_j is arbitrary we get (7), which completes the proof. q.e.d.

3 Differentiability of I_M

Lemma 3.1 (Lemma 3.2 of [MJ00] improved)**.** *Let $f :]a, b[\to \mathbb{R}$ be an upper semicontinuous (resp. lower semicontinuous) function. Then f is concave (resp. convex) if and only if for every $x_0 \in]a, b[$ there exists an open interval $I_{x_0} \subseteq]a, b[$ of x_0 and a concave (resp. convex) C^2 function $g_{x_0} : I_{x_0} \to \mathbb{R}$ such that $g_{x_0} = f(x_0)$ and $f(x) \leq g_{x_0}(x)$ (resp. $f(x) \geq g_{x_0}(x)$) for every $x \in I_{x_0}$.*

We recall here the generalized existence theorem 1 of [Nar14] stated under more general assumptions to check why this is legitimate one can see Remark 2.9 of [MN12], or Remarks 3.1, 3.2.

Theorem 3.1 (Generalized existence)**.** *Let M have C^0-locally asymptotically bounded geometry in the sense of Definition 1.4. Given a positive volume $0 < v < V(M)$, there are a finite number of limit manifolds at infinity such that their disjoint union with M contains an isoperimetric region of volume v and perimeter $I_M(v)$. Moreover, the number of limit manifolds is at worst linear in v.*

Remark 3.1. *The regularity discussion made there in Remark* 2.2 *of [MN12], is necessary in the proof of Corollary 1, where we need to do analysis on the limit manifolds, applying a (by now classical) formula for the second variation of the area functional on those isoperimetric regions which eventually lie in a limit manifold of possibly non-smooth boundary. The assumption of C^0 convergence of the metric tensor in the preceding lemma is due to the necessity of transporting volumes and perimeters in the limit manifold.*

Remark 3.2. *We observe that if $(M_i, g_i, p_i) \to (M, g, p)$ in the pointed Gromov-Hausdorff topology and M_i satisfy $Ric_{g_i} \geq (n-1)k_0 g_i$, it is not true, in general, that $Ric_g \geq (n-1)k_0 g$. Instead, if $(M_i, g_i, p_i) \to (M, g, p)$ in the pointed C^0-topology then $(M_i, g_i, V_i, p_i) \to (M, g, V, p)$ converge in the measured pointed Gromov-Hausdorff topology. Therefore, if all the Riemannian n-manifolds (M_i, g_i) satisfy $Ric_{g_i} \geq (n-1)k_0 g_i$ then also the limit Riemannian manifold (M, g) satisfies $Ric_g \geq (n-1)k_0 g$ (see Section 7 in [AG09]). Notice that for the convergence of the Ricci curvature one should need a stronger convergence of the (M_i, g_i, p_i) to (M, g, p), say in C^2-topology; here we just need the convergence of a lower bound.*

Remark 3.3. *One possible application is to simplify part of the proof of different papers about existence and caracterisation of isoperimetric regions in non compact Riemannian manifolds and prove new theorems of the same kind.*

We can finish now the proof of Corollary 1.

Proof: Using the generalized existence theorem of [Nar14] and evaluating the second variation formula for the area functional on a generalized isoperimetric region $\Omega_{\bar{v}}$ in volume $V(\Omega_{\bar{v}}) = \bar{v}$ we can construct a smooth function $f_{\bar{v}}$ defined in a small neighborhood of $\bar{v}$, that we can compare locally with I_M. Consider the equidistant domains $\Omega_t := \{x \in M : d(x, \Omega_{\bar{v}}) \leq t\}$, if $r_{\bar{v}} \geq t \geq 0$, and $\Omega_t := M \setminus \{x \in M : d(x, M \setminus \Omega_{\bar{v}}) \leq t\}$, if $-r_{\bar{v}} \leq t < 0$, where $r_{\bar{v}} > 0$ is the normal injectivity radius of $\partial\Omega_{\bar{v}}$. Consider the inverse function of $t \mapsto V(\Omega_t)$ as a function of the volume, $v \mapsto t(v)$, and finally set $f_{\bar{v}}(v) := A(\partial\Omega_{t(v)})$ for v belonging to a small neighbourhood $I_{\bar{v}} = [\bar{v} - \varepsilon_{\bar{v}}, \bar{v} + \varepsilon_{\bar{v}}]$. To be rigorous in this construction we have to take care of the singular part of domains Ω_t. This is done, carefully, in Proposition 2.1 and 2.3 of [Bay04]. Here we just ignore this technical complication, to make the exposition simpler to read. We just observe that the proof that we give here works mutatis mutandis also if we consider the case in which Ω is allowed to have a nonvoid singular part. Hence, for every $\bar{v} \in]0, V(M)[$, $f_{\bar{v}}$ gives smooth function $f_{\bar{v}} : [\bar{v}-\varepsilon_{\bar{v}}, \bar{v}+\varepsilon_{\bar{v}}] \to [0, +\infty[$, such that $f_{\bar{v}}(\bar{v}) = I_M(\bar{v})$ and $f_{\bar{v}} \geq I_M$. A standard application of the second variation formula see (V.4.3) [Cha06], or [BP86], shows that

$$f''_{\bar{v}}(v) = -\frac{1}{f^2_{\bar{v}}(v)}\left\{\int_{\partial\Omega_{t(v)}} (|II|^2 + Ricci(\nu))d\mathcal{H}^{n-1}\right\}. \tag{16}$$

$$f''_{\bar{v}}(v) \leq -\frac{(n-1)k}{f_{\bar{v}}(v)}. \tag{17}$$

If $k \geq 0$, then $f_{\bar{v}}$ is concave and a straightforward application of Lemma 3.1 implies that I_M is concave in all $]0, V(M)[$. If $k < 0$ then

$$f''_{\bar{v}}(v) \leq -\frac{(n-1)k}{I_M(v)}, \tag{18}$$

$$C = C(n, k, a, b) := \frac{(n-1)k}{2\delta_{M,a,b}}, \tag{19}$$

where $\delta_{M,a,b} := \inf\{I_M(v) : v \in [a, b]\}$ is strictly positive because by Theorem 1, I_M is continuous. For every $\bar{v} \in]a, b[$ it is easily seen that $I_M(v) + C(a, b)v^2 \leq f_{\bar{v}}(v) + C(a, b)v^2$ and $(f_{\bar{v}}(v) + C(a, b)v^2)'' \leq 0$ for every $v \in]a, b[\cap I_{\bar{v}}$. By Lemma 3.1, for $a, b \in]0, V(M)[$, $I_M(v)+C(a, b)v^2$ is concave in $[a, b]$. Hence, $I_M(v)+C(a, b)v^2$ is locally Lipschitz and it is straightforward to see that I_M is locally Lipschitz too, with $I'^{+} \leq f'_{\bar{v}} \leq I'^{-}$, with equality holding at all but a countable set of points, which are the only points of discontinuity of I'^{+} and I'^{-}. Moreover I'^{+} and I'^{-} are nonincreasing so the set of points at which I_M is nonderivable is at most countable, moreover I'_M or $I'_M + 2Cv$ are respectively monotone nonincresing see for this standard convexity arguments Corollary 2, page 29 of [Bou04] this implies that they are special cases of absolutely continuous functions and for this reason differentiable almost everywhere. So exists $I''_M(v)$ almost everywhere. Now, following [Bay04], for an arbitrary function f, set

$$\overline{D^2 f}(x_0) := \overrightarrow{\lim}_{\delta\to 0}\frac{f(x_0+\delta) + f(x_0-\delta) - 2f(x_0)}{\delta^2}. \tag{20}$$

When f is differentiable two times at x_0 it is straightforward to see that $f''(x_0) = \overline{D^2 f}(x_0)$. From (20) certainly follows

$$I''_M(v) = \overline{D^2 I_M}(v) \leq \overline{D^2 f_{\bar{v}}}(v) = f''_{\bar{v}}(v),$$

for every $v \in I_{\bar{v}}$.

In a point $\bar{v}$ at which I_M is twice differentiable we observe that

$$I''_M(\bar{v}) = \overline{D^2 I_M}(\bar{v}) \le f''_{\bar{v}}(\bar{v}).$$

Hence, (16) yields

$$I_M(\bar{v}) I''_M(\bar{v}) \le I_M(\bar{v}) f''_{\bar{v}}(\bar{v}) \le -I_M(\bar{v}) \left(\frac{I'^2_M(\bar{v})}{n-1} - (n-1)k \right),$$

which is exactly (1), because $|II|^2 \ge \frac{h^2}{n-1}$, where $h = f'_{\bar{v}}(\bar{v})$ by the first variation formula, if equality holds in (1), then $|II|^2 = \frac{h^2}{n-1}$, which is equivalent to say that the regular part of $\partial\Omega_{\bar{v}}$ is totally umbilic. q.e.d.

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