A NOTE ON PETTY'S THEOREM

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Abstract. In this short note we show how, by exploiting the regularity theory for solutions to the Monge-Ampère equation, Petty's equation characterizes ellipsoids without assuming any a priori regularity assumption.

1. Introduction

Affine inequalities play a very important role in the study of the geometry of convex bodies and they also find applications in several different fields (*e.g.* ordinary and partial differential equation, functional analysis). In [13] Petty treated three closely related affine problems, namely the Blaschke-Santalò inequality, the affine isoperimetric inequality and the geominimal surface area inequality [1], and he characterized ellipsoids as the only extremal bodies for these inequalities. In order to establish this characterization he proved that if $K \subset \mathbb{R}^N$ is an extremal convex body for these inequalities, then necessarily there must exist a positive constant c_K such that

$$f_K(\omega) = c_K\, h_K^{-N-1}(\omega), \tag{1.1}$$

for every $\omega \in \mathbb{S}^{N-1}$. Here

$$h_K(\omega) = \max\{\omega \cdot x : x \in K\}$$

denotes the support function of the convex body K and f_K is the *curvature function* of K, see Section 2.1. Petty was then able to show that (1.1) implies that K is an ellipsoid if $N = 2$. If $N \geq 3$ he obtained the same result only under the assumption that K is a C^2-regular convex body or that K is a body of revolution. In any case this was sufficient to prove that extremal sets for the above mentioned problems are ellipsoids since symmetrization techniques allow to reduce to the case of axially symmetric sets.

It remains however an interesting question to understand to which extent (1.1) characterizes ellipsoids without assuming any a priori regularity assumption on K, see for instance [6]. In this short note we prove that every convex body satisfying (1.1) is actually an ellipsoid. More precisely we prove

Theorem 1.1. *Let K be a convex body which possesses a curvature function f_K; if equation* (1.1) *is satisfied for some positive constant c_K, then K is an ellipsoid.*

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[1] For the position of these problems and a comprehensive overview on the subject we refer to [6] and [13] and the references therein.

Besides its own interest Theorem 1.1 will have applications in several problems in which (1.1) characterizes extremal bodies, but for which it is a priori unknown their regularity as well as it can provide new short proofs of some results in which (1.1) appears. As examples let us quote [9, 21, 7, 8] concerning, respectively, *convolution bodies*, *floating bodies* and *K-dense sets*.

In order to prove Theorem 1.1 we closely follow Petty's strategy. Petty's argument was based on the observation that if K is a smooth convex body satisfying (1.1) and we define $\overline{h}_K(x) = |x|h_K(x/|x|)$, then $\overline{h}_K^2$ is a solution of the *Monge-Ampère* equation

$$\det \frac{1}{2} D^2 \overline{h}_K^2(x) = c_K, \qquad x \in \mathbb{R}^N. \tag{1.2}$$

Combining this remark with classical results due to Pogorelov, [14, 15, 16], Petty proved that $\overline{h}_K^2$ is a quadratic polynomial and hence that K is an ellipsoid. By an approximation procedure we show that if a convex set K satisfies (1.1), then its support function still satisfies (1.2) in the Aleksandrov sense, see Section 2.2 for the definition. By relying on standard techniques one can then show that any Aleksandrov solution of (1.2) is smooth and hence, by Pogorelov's Theorem, a quadratic polynomial.

The paper is organized as follows: in Section 2 we recall some preliminaries concerning Convex Geometry and weak solution of the Monge-Ampère equation, in Section 3 we provide a proof of Theorem 1.1.

After we finished writing this note, Prof. Schneider informed us that in the new edition of his book, there is a sketch of the proof of Theorem 1.1 based on Caffarelli's regularity results for the solutions of the Minkowski problem, see the Remark after Theorem 10.5.1 in [20].

2. Prelimanaries

In this section we recall some basic notions concerning convex bodies and solutions of the Monge-Ampère equation.

2.1. **Convex Geometry.** We denote by $\mathcal{K}^N$ the set of convex bodies (a convex body is a compact convex set with nonempty interior) of $\mathbb{R}^N$. We can associate a convex body K with a measure μ_K supported on the unit sphere, called the *surface area measure*, with the property that, for every Borel set $A \subset \mathbb{S}^{N-1}$, $\mu_K(A)$ is the $(N-1)$-dimensional Hausdorff measure of the set of the points in the boundary of K whose normal cone has nonempty intersection with A. More precisely, if for $x \in \partial K$, we define the possibly multivalued map

$$N_K(x) = \big\{\omega \in \mathbb{S}^{N-1} : \omega \cdot (y - x) \le 0 \qquad \text{for all } y \in K\big\},$$

then

$$\mu_K(A) = \mathcal{H}^{N-1}(N_K^{-1}(A)).$$

It is possible to show (see [19, Proposition 4.10]) that such measures are continuous in the K-variable with respect to the Hausdorff convergence.

Namely

$$\lim_i \int_{\mathbb{S}^{N-1}} \varphi \, d\mu_{K_i} = \int_{\mathbb{S}^{N-1}} \varphi \, d\mu_K, \tag{2.3}$$

for every $\varphi \in C(\mathbb{S}^{N-1})$, whenever $K_i \to K$ in the Hausdorff distance.

When K is C^2_+, i.e. if it is C^2-regular body with strictly positive Gauss-Kronecker curvature κ, the surface area measure is absolutely continuous with respect to the Hausdorff measure $\mathcal{H}^{N-1} \llcorner \mathbb{S}^{N-1}$ and its density is given by $\kappa \circ N_k^{-1}$ (note that N_K is single valued and injective since K has differentiable boundary and it is strictly convex).

A convex body K is said to possess a curvature function provided there exists a positive and continuous function $f_K : \mathbb{S}^{N-1} \to \mathbb{R}$ such that[2]

$$\mu_K = f_K \mathcal{H}^{N-1} \llcorner \mathbb{S}^{N-1}.$$

Conversely given a positive and continuous function $f : \mathbb{S}^{N-1} \to \mathbb{R}$, Minkowski existence and uniqueness Theorem, [2, 10, 11, 12, 15], asserts that, provided f fulfills the following (necessary) condition

$$\int_{\mathbb{S}^{N-1}} \omega f(\omega) \, d\mathcal{H}^{N-1}(\omega) = 0\,, \tag{2.4}$$

there exists a unique (up to translation) convex body K whose curvature function equals f.

The above condition leads us to the following observation: while the left-hand side of (1.1) is invariant under translations of K the right-hand is affected by translations. However we shall note that, for every convex body K, there exists a point, say p_K, such that $h_{K-p_K}^{-N-1}$ is a curvature function. To see this let us recall that the *polar reciprocal* of a convex body K with respect to the point p is definend as

$$K_p^* = \big\{x \in \mathbb{R}^N \,:\, x \cdot (y-p) \le 1 \qquad \text{for all } y \in K\big\}.$$

One can show that there exists a unique point, p_K, such that

$$V(K_{p_K}^*) = \min\{V(K_p^*) \,:\, p \in \mathbb{R}^N\},$$

this point is the *Santalò point* of K. We denote by K^* the set $K_{p_K}^*$. It is well-known that the polar reciprocal of a convex body with respect to its Santalò point has its barycenter at the origin (see [17]), this implies that

$$\int_{\mathbb{S}^{N-1}} \omega \rho_{K^*}^{N+1}(\omega) \, d\mathcal{H}^{N-1}(\omega) = 0,$$

where ρ_{K^*} denotes the radial function of the convex set K^*, i.e. $\rho_{K^*}(\omega) = \sup\{\lambda \,:\, \lambda\omega \in K^*\}$. Since one can easily show that $h_K = 1/\rho_{K^*}$, then h_K^{-N-1} satisfies condition (2.4) and hence, by Minkowski Theorem, for every $K \in \mathcal{K}^N$, there exists a body K' such that $f_{K'} = h_{K-p_K}^{-N-1}$.

From these considerations we note that, if we define a map Λ, from the set of convex bodies whose Santalò point is the origin in itself, associating each convex body K the solution of the Minkoski problem with data h_K^{-N-1},

[2]In [13] mixed volumes are used to define curvature functions, however the definition given by Petty coincide with the one above by virtue of [20, Theorem 4.2.3].

then K is a solution of (1.1) if and only if its image $\Lambda(K)$ is a dilation of K. We refer the reader to [6] for more details.

2.2. **Aleksandrov solutions of the Monge-Ampère equation.** In this section we recall the notion of *Aleksandrov solutions* of the Monge-Ampère equation and we summarize the properties of these solutions which we will need in the sequel, see [3, 5] for a more detailed exposition.

Let u be a convex function defined on a convex open domain $\Omega \subset \mathbb{R}^N$, the subdifferential of u, ∂u, is the multi-valued map given by

$$\partial u(x) = \left\{p \in \mathbb{R}^N \,:\, u(y) \geq u(x) + py \cdot (y-x),\, \forall y \in \Omega\right\}.$$

We define a measure ν_u, and we call it *Monge-Ampère measure* of u, as follows: for a Borel set $A \subset \Omega$

$$\nu_u(A) = V(\partial u(A)) := V\big(\bigcup_{x\in A} \partial u(x)\big). \tag{2.5}$$

Note that if $u \in C^2$, the change of variable formula gives that $d\nu_u = \det D^2 u\, dx$. We then call u an *Alexandrov solution* of the equation

$$\det D^2 u = f \tag{2.6}$$

provided $\nu_u = f\, dx$. Among several properties of Aleksandrov solutions we are going to use the following concerning their stability under uniform limit, see [5, Lemma 1.2.3] for a proof.

Lemma 2.1. *If u_k are convex functions defined on an open set Ω and $u_k \to u$ uniformly, then*

$$\nu_{u_k} \overset{*}{\rightharpoonup} \nu_u$$

as Radon measures in Ω, that is

$$\int \varphi\, d\nu_{u_k} \to \int \varphi\, d\nu_u \qquad \forall \varphi \in C^0_c(\Omega).$$

By relying on the uniqueness of the Aleksandrov solution to the Dirichlet problem for the Monge-Ampère equation, [5, Corollary 1.4.7] and on their stability under uniform limits, one can prove the following classical theorem. For the sake of completeness we sketch the main steps of its proof, see also [3, Section 2] for a more detailed account.

Theorem 2.2. *Let u be a* strictly *convex function defined on a convex set Ω satisfying*

$$\nu_u = f\, dx \qquad \text{in } \Omega.$$

If $f \in C^\infty(\Omega)$ and $\lambda \leq f \leq \Lambda$ for some $\lambda, \Lambda > 0$, then for every $\Omega' \Subset \Omega$, $u \in C^\infty(\Omega')$.

Proof. Fix $x_0 \in \Omega'$, $p \in \partial u(x_0)$, and consider the *section* of u at height t defined as

$$S(x,p,t) := \left\{y \in \Omega \,:\, u(y) \leq u(x) + p\cdot(y-x) + t\right\}.$$

Since u is strictly convex we can choose $t > 0$ small enough so that $S(x_0,p,t) \Subset \Omega'$. Then we consider a sequence of smooth uniformly convex sets S_i, converging to $S(x_0,p,t)$ and we apply classical continuity methods in order find a function $v_i \in C^\infty(S_\varepsilon)$ solving

$$\begin{cases} \det D^2 v_i = f * \varrho_{\varepsilon_i} & \text{in } S_i \\ v_i = 0 & \text{on } \partial S_i, \end{cases}$$

where ϱ_{ϵ_i} is a sequence of mollifying kernels, see [3, Theorem 2.11] and [4, Chapter 17]. We apply Pogorelov estimates, see for instance [3, Theorem 2.12], to v_i to infer that

$$|D^2 v_i| \le C \qquad \text{in } S(x_0, p, t/2) \Subset S(x_0, p, t)$$

for a constant C independent on $i \in \mathbb{N}$. Since $S_i \to S(x_0, p, t)$ and $u(x) = u(x_0) + p \cdot x + t$ on $\partial S(x_0, p, t)$, by stability and uniqueness of weak solutions we deduce that $v_i + u(x_0) + p \cdot x + t \to u$ uniformly as $i \to \infty$, hence $|D^2 u| \le C$ in $S(x_0, p, t/2)$. This makes the Monge-Ampère equation uniformly elliptic, hence Evans-Krylov Theorem and Schauder theory imply that $u \in C^\infty(S(x_0, p, t/4))$, see [4, Chapter 17]. By the arbitrariness of x_0 we obtain that $u \in C^\infty(\Omega')$, as desired. $\square$

By a well-known example, strict convexity of u is necessary in order to prove the above Theorem. The following result, due to Caffarelli, implies that the obstruction to strict convexity can only arise from the boundary behavior. In particular every entire solution has to be strictly convex. We recall that x is an extremal point of a convex set K if $x \in \overline{K}$ and $K \setminus \{x\}$ is convex.

Theorem 2.3 ([1]). *Let Ω be an open convex and let u be a convex function such that*

$$\lambda\, dx \le \nu_u \le \Lambda\, dx$$

for some $\lambda, \Lambda > 0$. For every $x \in \Omega$ and $p \in \partial u(x)$, if the set

$$\Gamma_{x,p} := \{y \in \Omega \,:\, u(y) = u(x) + p \cdot (y - x)\}$$

contains more than one point, then it has no extremal points in Ω.

An easy corollary of the above theorem is the following:

Corollary 2.4. *Let $u : \mathbb{R}^N \to \mathbb{R}$ be a convex function such that*

$$\lambda\, dx \le \nu_u \le \Lambda\, dx \tag{2.7}$$

for some $\lambda, \Lambda > 0$, then u is strictly convex.

Proof. Let us assume by contradiction that for some $x_0 \in \mathbb{R}^N$ and $p_0 \in \partial u(x_0)$ the set Γ_{x_0,p_0} contains more than one point, then according to Caffarelli's Theorem it must contain a line. Up to subtracting a linear function and to change the coordinates we can then assume that $u \ge 0$ and $u = 0$ on

$$\ell := \{x \in \mathbb{R}^N : x = (x_1, 0, \dots, 0)\}.$$

This easily implies that $\partial u(\mathbb{R}^N) \subset e_1^\perp$ and hence that $\nu_u = 0$, contradicting (2.7). $\square$

3. Proof of the main Theorem

In this section we prove Theorem 1.1, the argument is based on an approximation procedure in order to show that, for a convex body satisfying (1.1), $\overline{h}_K^2/2$ is an Aleksandrov solution of (1.2). At this point we can apply Corollary 2.4 and Theorem 2.2 to show that $\overline{h}_k^2$ is smooth and hence the classical Pogorelov argument can be applied. More in general we prove the following:

Theorem 3.1. *Let K be a convex body which possesses a curvature function f_K and let $\overline{h}_K$ be the one-homogeneous extension of its support function, $\overline{h}_K = |x|h_K(x/|x|)$, then*

$$\det \frac{1}{2}D^2\overline{h}_K^2 = f_K\Big(\frac{x}{|x|}\Big)h_K^{N+1}\Big(\frac{x}{|x|}\Big)\,dx \qquad \text{in } \mathbb{R}^N \tag{3.8}$$

in the Aleksandrov sense.

In order to prove the above Theorem we need to approximate, in the Hausdorff topology, a convex body with C^2_+ bodies, for which we know that (3.8) holds true at least in $\mathbb{R}^N \setminus \{0\}$. We know from an old theorem by Minkowski that convex sets with analytic boundary are dense in $\mathcal{K}^N$, several years later Schmuckenschläger (see [18]) gave a simple proof of the theorem and showed that one can explicitly write down an approximating sequence with further additional properties; more precisely we have

Theorem 3.2 ([18])**.** *Let K a convex body, there exist a sequence $\{K_i\}_{i\in\mathbb{N}}$, $K_{i+1} \subseteq K_i$, such that*

- *K_i and K_i^* have real analytic boundaries,*
- *The Gaussian curvature of both K and K^* is strictly positive,*
- *$K_i \to K$ in the Hausdorff distance.*

We now discuss the proof of Theorem 3.1.

Proof of Theorem 3.1. We divide the proof in three steps:

• *Step 1: Equation* (3.8) *holds true if $K \in C^2_+$.* Let $K \in C^2_+$, then $h_K \in C^2$ and $\overline{h}_K \in C^2(\mathbb{R}^N \setminus \{0\})$. Then a classical computation, see [13, Lemma 8.4], implies that

$$\det \frac{1}{2}D^2\overline{h}_K^2(x) = f_K\Big(\frac{x}{|x|}\Big)h_K^{N+1}\Big(\frac{x}{|x|}\Big) \qquad \forall x \in \mathbb{R}^N \setminus \{0\}.$$

In particular, by the change of variable formula, if we denote by ν_K the Monge-Ampére measure of $\overline{h}_K^2$

$$\nu_K = f_K\Big(\frac{x}{|x|}\Big)h_K^{N+1}\Big(\frac{x}{|x|}\Big)\,dx$$

as Radon measures on $\mathbb{R}^N\setminus\{0\}$. Moreover since $\overline{h}_K^2$ is homogeneous of degree two, it is differentiable in 0 and $\partial\overline{h}_K^2(0) = \{0\}$. Recalling the definition of Monge-Ampère measure (2.5), we then see that for every Borel set $A \subset \mathbb{R}^n$

$$\begin{aligned}\nu_K(A) &= \nu_K(A \setminus \{0\}) + \nu_K(\{0\})\\ &= \nu_K(A \setminus \{0\}) + V(\{0\}) = \nu_K(A \setminus \{0\}).\end{aligned}$$

Hence (3.2) is valid (as equality between measures) also in $\mathbb{R}^N$.

• *Step 2: Let K_i be a sequence of convex bodies for which Theorem 3.1 is valid and let K be a convex body admitting a curvature function f_K. If $K_i \to K$ in the Hausdroff distance, then the conclusion of the Theorem 3.1 holds true for K.*

Since $K_i \to K$ in the Hausdorff distance, $h_{K_i} \to h_K$ uniformly on $\mathbb{S}^{N-1}$ and $\overline{h}^2_{K_i} \to \overline{h}^2_K$ locally uniformly in $\mathbb{R}^N$. According to Lemma 2.1 it is enough to show that

$$\nu_{K_i} = f_{K_i}\Big(\frac{x}{|x|}\Big) h^{N+1}_{K_i}\Big(\frac{x}{|x|}\Big)\,dx \overset{*}{\rightharpoonup} f_K\Big(\frac{x}{|x|}\Big) h^{N+1}_K\Big(\frac{x}{|x|}\Big)\,dx, \tag{3.9}$$

as Radon measures in $\mathbb{R}^N$. To this end let $\varphi \in C^0_c(\mathbb{R}^N)$ and note that for every $\varrho \in [0, +\infty)$, $\mathbb{S}^{N-1} \ni \omega \mapsto \varphi(\varrho\,\omega)$ is continuous. Since $h_{K_i} \to h_K$ uniformly on $\mathbb{S}^{n-1}$ and

$$\int \varphi\, d\nu_{K_i} = \int_0^\infty \varrho^{N-1} \int_{\mathbb{S}^{N-1}} \varphi(\varrho\,\omega) f_{K_i}(\omega) h_{K_i}(\omega) d\mathcal{H}^{N-1}(\omega)\,,$$

an application of Lebesgue Dominated Convergence Theorem (recall that φ is compactly supported) shows that in order to prove (3.9) it is enough to show that

$$f_{K_i}(\omega)\, d\mathcal{H}^{N-1} \overset{*}{\rightharpoonup} f_K(\omega)\, d\mathcal{H}^{N-1}$$

as Radon measures on $\mathbb{S}^{N-1}$. This however follows by the continuity of curvature measures under the Hausdorff convergence, (2.3).

• *Step 3: Conclusion.* If K is a convex body admitting a curvature function we can apply Theorem 3.2 to approximate it with a sequence of convex bodies $K_i \in C^2_+$, by Step 1 the conclusion of the Theorem holds true for K_i and hence by Step 2 also for K. □

Proof of Theorem 1.1. According to Theorem 3.1, if K is a convex body satisfying (1.1), then

$$\det \frac{1}{2} D^2 \overline{h}^2_K = c_K\, dx \qquad \text{on } \mathbb{R}^N$$

in the Aleksandrov sense. By Corollary 2.4, $\overline{h}^2_K$ is strictly convex and by Theorem 2.2, $\overline{h}^2_K \in C^\infty(\mathbb{R}^N)$. By applying the classical Pogorelov argument (see [5, Theorem 4.3.1] for a proof) $\overline{h}^2_K(x) = Ax \cdot x$ for some positive symmetric matrix A, which immediately implies that K is an ellipsoid. □

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