Trace formulas for some singular differential operators and applications

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Abstract. We characterize the convergence of the series $\sum \lambda_n^{-1}$, where λ_n are the non-zero eigenvalues of some boundary value problems for degenerate second order ordinary differential operators and we prove a formula for the above sum when the coefficient of the zero-order term vanishes. We study these operators both in weighted Hilbert spaces and in spaces of continuous functions. After investigating the boundary behaviour of the eigenfunctions, we give applications to the regularity of the generated semigroups.

1. Introduction

In this paper we study the summability of the series $\sum_n \lambda_n^{-1}$, where (λ_n) is the sequence of the non-zero eigenvalues of some boundary-value problems associated with the general second-order differential operator

$$\mathcal{A}u = \alpha u'' + \beta u' + ru$$

in the space $C([a,b])$, with $-\infty \le a < b \le +\infty$. We assume that the coefficients are real-valued and continuous, with $\alpha > 0$ in the interior and $r \in C([a,b])$, but we allow α and β to vanish or to have no finite limits at a, b.

The asymptotic behaviour of the eigenvalues λ_n of Sturm-Liouville problems has been extensively studied both in the non-degenerate and in the degenerate case. In particular it is well-known that $|\lambda_n| \approx n^2$ in the regular case or even in the singular one under the non-oscillatory limit-circle condition, *i.e.* when all the solutions of the homogeneous equation $\lambda u - \mathcal{A}u = 0$ are square summable and keep a definite sign near the endpoints (see [26]). After assigning the boundary conditions specified below, we give here necessary and sufficient conditions for the convergence of the above sum and we compute explicitly the sum in the case $r \equiv 0$. As regards trace formulas, we can only quote [20] and [21]; this papers contain some results comparable with ours, even though both the motivations and the functional setting seem to be different.

Degenerate operators have been extensively studied since Feller's investigations (see [18], [19]), whose main motivation was the probabilistic interest of the evolution equation $u_t = \mathcal{A}u$ for the transition probabilities u: in fact, the above equation is the

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backward equation coming from a one-dimensional diffusion process, and the coefficients α, β, r have the meaning of *diffusion, drift* and *absorption*, respectively. For this reason, $C([a,b])$ is the space where the problem should be studied in order to have information on the underlying diffusion process. The evolution equation has been studied under many boundary conditions that have a genuine probabilistic meaning and this reflects also in the terminology introduced by Feller. In particular, it is interesting to know whether the operator $\mathcal{A}$, endowed with the domain corresponding to the given boundary conditions, is the generator of a positive semigroup of contractions, possibly even more regular. We refer to the quoted papers of Feller's and to [22] for a discussion of all this matter, and to [7] for a presentation of some aspects of the theory, in a more analytical vein.

In order to discuss the relation between the boundary conditions that can be assigned and the boundary behaviour of the coefficients, after the Wronskian

$$W(x) = \exp\Big\{-\int_c^x \frac{\beta(s)}{\alpha(s)}ds\Big\}, \tag{1.1}$$

Feller introduced the functions

$$Q(x) = \frac{1}{\alpha(x)W(x)}\int_c^x W(s)ds, \qquad R(x) = W(x)\int_c^x \frac{1}{\alpha(s)W(s)}ds,$$

where $c \in (a,b)$ is fixed, and classified the endpoints according to their behaviour as follows. The endpoint b is said to be

$$\begin{array}{lll} \textit{regular} & \text{if} & Q \in L^1(c,b), \quad R \in L^1(c,b); \\ \textit{exit} & \text{if} & Q \notin L^1(c,b), \quad R \in L^1(c,b); \\ \textit{entrance} & \text{if} & Q \in L^1(c,b), \quad R \notin L^1(c,b); \\ \textit{natural} & \text{if} & Q \notin L^1(c,b), \quad R \notin L^1(c,b); \end{array}$$

of course, analogous definitions are understood for a. If b is regular or exit, then it is called *accessible*, because, in terms of Markov processes, the probability that the particle does reach b in a finite time is positive; otherwise, b is *unaccessible*. From this point of view, assigning boundary conditions serves to select (if possible) one among the infinite solutions of $u_t = \mathcal{A}u$, $u(0) = u_0$ (with $u_0 \in C([a,b])$ fixed) only in the case of accessible boundaries. Otherwise, the above Cauchy problem has a unique solution because $\mathcal{A}$ generates a strongly continuous semigroup under its maximal domain (see [18] or [7, Theorem 4.14]).

From [18] and [9] it follows that the evolution equation $u_t = \mathcal{A}u$ gives rise to a C_0-semigroup with Neumann boundary conditions if $(\alpha W)^{-1} \in L^1(a,b)$, and with Ventcel boundary conditions if and only if neither endpoint is an entrance boundary. More precisely, let us define the domains

$$\begin{aligned} D_N &= \Big\{u \in C([a,b]) \cap C^2(a,b):\ \mathcal{A}u \in C([a,b]),\ \lim_{x\to a,b}\frac{u'(x)}{W(x)} = 0\Big\} \\ D_V &= \Big\{u \in C([a,b]) \cap C^2(a,b):\ \mathcal{A}u \in C_0([a,b])\Big\}; \end{aligned}$$

then the operator $(\mathcal{A}, D_N)$ generates C_0-semigroups on $C([a,b])$ when $(\alpha W)^{-1} \in L^1(a,b)$, and $(\mathcal{A}, D_V)$ is a generator when a, b are not entrance boundaries. Notice that, in connection to Markov processes, Neumann conditions correspond to the *reflecting barrier* conditions, and Ventcel's to the *adhesive boundary* condition.

We focus our attention on the distribution of the eigenvalues of $(\mathcal{A}, D_N)$ and $(\mathcal{A}, D_V)$; we point out that if $\lambda \neq r(a)$, $\lambda \neq r(b)$ then for the eigenvalue problem $\mathcal{A}u - \lambda u = 0$ Ventcel boundary conditions and Dirichlet boundary conditions $\lim\limits_{x \to a,b} u(x) = 0$ are equivalent.

The function W allows to write $\mathcal{A}$ in the form

$$\mathcal{A}u = \alpha W \frac{d}{dx}\Big(\frac{u'}{W}\Big) + ru, \tag{1.2}$$

which is formally self-adjoint in the weighted Hilbert space $L^2_{(\alpha W)^{-1}}(a,b)$. For this reason, we are led to study the spectral properties of the general (formally) self-adjoint operators

$$Au = \frac{1}{\varrho}\frac{d}{dx}\big(pu'\big) + ru \tag{1.3}$$

in the weighted Hilbert space $L^2_\varrho(a,b)$. We endow A with Neumann boundary condition if $\varrho \in L^1(a,b)$ and with Dirichlet boundary conditions if $1/p \in L^1(a,b)$. In both cases the operators turn out to be self-adjoint (see Section 2). We assume $r \in L^\infty(a,b)$, ϱ and p strictly positive in the interior of the interval, ϱ continuous and p continuously differentiable. The result for homogeneous Neumann conditions extends a previous result, with $\varrho \equiv 1$, $(a,b) = (0,1)$, see [5].

Coming back to the operators $(\mathcal{A}, D_N)$, $(\mathcal{A}, D_V)$, we prove that their resolvents are compact if and only if $Q \in L^1(a,b)$, $R \in L^1(a,b)$ respectively. Under these conditions, their spectral properties in $C([a,b])$ and in $L^2_{(\alpha W)^{-1}}(a,b)$ are the same and we use the hilbertian results to show that their spectra consist of a sequence of real eigenvalues λ_n tending to $-\infty$. We prove also that both series $\sum \lambda_n^{-1}$ converge and obtain corresponding formulas for their sums in the case $r \equiv 0$. In the last section we study the behaviour of the eigenfunctions of $(\mathcal{A}, D_V)$ near the boundary and deduce the differentiability of the generated semigroup.

Notation. By $u \in AC(I)$ (resp. $u \in AC_{loc}(I)$) we mean that u is absolutely continuous in I (resp. in any compact interval contained in the interior of I). $C([a,b])$ is the Banach space of the continuous functions on the two point compactification of (a,b) for $-\infty \leq a < b \leq +\infty$; $C_0([a,b]) = \{u \in C([a,b]) :\ u(a) = u(b) = 0\}$. $\mathcal{B}(X)$ denotes the space of all bounded linear operators from a Banach space X into itself.

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2. Singular Sturm-Liouville operators

We consider the differential operator A given by (1.3) in (a,b) and

$$A_0 = \frac{1}{\varrho}\frac{d}{dx}\Big(p\frac{d}{dx}\Big).$$

The functions ϱ, p are supposed to be strictly positive in the interior of $[a,b]$, with $\varrho \in C(a,b)$ and $p \in C^1(a,b)$. Both ϱ and $1/p$ will be not always assumed to be summable, hence we define the following auxiliary functions:

$$\mu(x) = \int_a^x \varrho(s)ds, \qquad z(x) = \int_a^x \frac{1}{p(s)}ds \tag{2.1}$$

if $\varrho, 1/p$ are summable, and

$$\nu(x) = \int_c^x \varrho(s)ds, \qquad w(x) = \int_c^x \frac{1}{p(s)}ds \tag{2.2}$$

(where c ia a point in (a,b) fixed once and for all) if $\varrho, 1/p$ are not assumed to be summable. We also introduce the Hilbert space

$$L^2_\varrho = \Big\{u : (a,b) \to \mathbf{C} : \int_a^b |u|^2\varrho < +\infty\Big\} \tag{2.3}$$

endowed with the natural inner product $\langle\ ,\ \rangle_\varrho$ and norm $\|\ \|_\varrho$. Notice that by setting $L^1_\varrho = \big\{u : \int_a^b |u|\varrho < +\infty\big\}$, if $\varrho \in L^1(a,b)$ then $L^2_\varrho \subset L^1_\varrho$. In analogy with the introduction, we define the functions

$$Q = w\varrho, \qquad R = \frac{1}{p}\nu. \tag{2.4}$$

Let us introduce the maximal domain of A:

$$D_M(A) = \Big\{u \in L^2_\varrho :\ u,\ u' \in AC_{loc},\ Au \in L^2_\varrho\Big\}.$$

We present here the main general properties of A: under suitable hypotheses on ϱ and p we study the boundary behaviour of $u \in D_M(A)$ in connection with the assignement of boundary conditions, and prove the self-adjointness of A with homogeneous Neumann or Dirichlet conditions. We shall concentrate on the point a; of course, for $x \to b$ analogous statements hold.

Lemma 2.1. *If $\varrho \in L^1(a,c)$ and $u \in D_M(A)$ then there exists $\ell \in \mathbf{C}$ such that $p(x)u'(x) \to \ell$ as $x \to a$; moreover, if $\ell = 0$ then the integral $\int_a^c p|u'|^2$ is convergent.*

Proof. Let be $f = Au - ru$, and let us integrate the equation between x and c; we have

$$p(x)u'(x) = p(c)u'(c) - \int_x^c f(s)\varrho(s)ds. \tag{2.5}$$

Since $f \in L^2_\varrho(a,c)$ implies $f \in L^1_\varrho(a,c)$, we obtain

$$\lim_{x\to a} p(x)u'(x) = p(c)u'(c) - \int_a^c f(s)\varrho(s)ds = \ell \in \mathbf{C}.$$

and the first assertion is proved. To show the second one, let $x \in (a,c)$; we have

$$\int_x^c (pu')'\bar u = \Big[pu'\bar u\Big]_x^c - \int_x^c p|u'|^2; \tag{2.6}$$

the equality $f\varrho\bar u = (pu')'\bar u$, since f and u belong to L^2_ϱ, implies that the integral in the left hand side of (2.6) converges as $x \to a$. Moreover, the last integral in (2.6) has a non-negative limit (possibly $+\infty$) for $x \to a$, and the existence of the limit of $p(x)u'(x)\bar u(x)$ as $x \to a$ follows, let it be k. Let us prove that if $\ell = 0$ then $k = 0$: so, the thesis follows by letting $x \to a$ in (2.6). If $\ell = 0$ we can write

$$|p(x)u'(x)|^2 = \Big|\int_a^x f\varrho\Big|^2 \le \mu(x)\int_a^x |f|^2\varrho,$$

and if $k \neq 0$ then $|u(x)|^2 \ge C'/|p(x)u'(x)|^2 \ge C/\mu(x)$ as $x \to a$, whence

$$\int_a^c |u|^2\varrho \ge C\int_a^c \frac{\varrho(x)}{\mu(x)}dx;$$

the divergence of the last integral gives a contradiction with $u \in L^2_\varrho$, hence $k = 0$ and we achieve the thesis. □

Remark 2.2. If $\ell \neq 0$ in the above lemma then in general the integral $\int_a^c p|u'|^2$ can be divergent; in fact, the function $u(x) = \log x$ verifies $(xu')' = 0$ in $(0,1)$, $\ell = 1$ and the above integral is divergent.

Lemma 2.3. *If $1/p \in L^1(a,c)$ and $u \in D_M(A)$ then the integral $\int_a^c p|u'|^2$ is convergent; as a consequence, there exists $\ell \in \mathbf{C}$ such that $u(x) \to \ell$ as $x \to a$; moreover, if $\varrho \notin L^1(a,c)$ then $\ell = 0$.*

Proof. By (2.5) we deduce

$$\begin{aligned} p(x)|u'(x)| &\le p(c)|u'(c)| + \Big|\int_c^x |f(s)|\varrho(s)ds\Big| \\ &\le p(c)|u'(c)| + \big|\nu(x)\big|^{1/2}\Big(\int_c^x |f(s)|^2\varrho(s)ds\Big)^{1/2}. \end{aligned}$$

If $\varrho \in L^1(a,c)$ then ν is bounded, $p|u'| \le C$, $p|u'|^2 \le C^2/p \in L^1(a,c)$ and we are done. If conversely $\varrho \notin L^1(a,c)$, then in a fixed neighbourhood of a we have

$$|p(x)u'(x)|^2 \le C|\nu(x)|.$$

By (2.6) the existence of the limit of $p(x)u'(x)\bar u(x)$ as $x \to a$ follows as in Lemma 2.1, let it be k. If $k \neq 0$ we obtain as $x \to a$

$$|u(x)|^2\varrho(x) \ge \frac{C'\varrho(x)}{|p(x)u'(x)|^2} \ge \frac{C''\varrho(x)}{|\nu(x)|},$$

whence

$$\int_a^c |u|^2\varrho \ge C''\int_a^c \frac{\varrho(x)}{|\nu(x)|}dx$$

and again the divergence of the last integral gives a contradiction with $u \in L^2_\varrho$. Then $k=0$ and by (2.6) the integral $\int_a^c p|u'|^2$ converges. Finally notice that by the estimate

$$\int_a^c |u'| \leq \Big(\int_a^c p|u'|^2\Big)^{1/2}\Big(\int_a^c \frac{1}{p}\Big)^{1/2}$$

it follows that $u' \in L^1(a,c)$ and the limit of $u(x)$ as $x \to a$ exists; if $\varrho \notin L^1(a,c)$ then it must obviously vanish because $u \in L^2_\varrho$. □

Let us now define the domain of A according to the boundary conditions to be imposed; Dirichlet conditions lead to the domain

$$D_0 = \Big\{u \in D_M(A) : \lim_{x\to a,b} u(x) = 0\Big\}; \tag{2.7}$$

while Neumann conditions to

$$D_1 = \Big\{u \in D_M(A) : \lim_{x\to a,b} p(x)u'(x) = 0\Big\}. \tag{2.8}$$

Observe that by Lemma 2.3, if $1/p \in L^1(a,b)$ and $\varrho \notin L^1(a,c)$, $\varrho \notin L^1(c,b)$, then $D_0 = D_M(A)$.

Proposition 2.4. *If $\varrho \in L^1(a,b)$ then (A, D_1) is self-adjoint on L^2_ϱ; if $r \leq= 0$, it is non-positive.*

Proof. We assume first that $r \equiv 0$. Let be $u, v \in D_1$; then

$$\int_a^b (Au)\bar{v}\varrho = \lim_{\substack{\alpha\to a\\ \beta\to b}} \int_\alpha^\beta (Au)\bar{v}\varrho = \lim_{\substack{\alpha\to a\\ \beta\to b}} \Big(\Big[pu'\bar{v}\Big]_\alpha^\beta - \int_\alpha^\beta pu'\bar{v}'\Big).$$

By Lemma 2.1 the integrals $\int_a^b p|u'|^2$ and $\int_a^b p|v'|^2$ are finite, so that $\int_a^b pu'\bar{v}'$ is convergent and the above equality shows that $p(x)u'(x)\bar{v}(x)$ has a finite limit as $x \to a$ or $x \to b$. Arguing as in Lemma 2.1 these limits must vanish and we obtain

$$\int_a^b (Au)\bar{v}\varrho = -\int_a^b pu'\bar{v}'. \tag{2.9}$$

Exchanging the role of u and v we obtain

$$\langle Au, v\rangle_\varrho = \int_a^b (Au)\bar{v}\varrho = -\int_a^b pu'\bar{v}' = \int_a^b u(A\bar{v})\varrho = \langle u, Av\rangle_\varrho$$

and (A, D_1) is symmetric. Let now $v \in D(A^*)$ and $f = A^*v \in L^2_\varrho$; then

$$\int_a^b (Au)\bar{v}\varrho = \int_a^b u\bar{f}\varrho$$

for any $u \in D_1$, and in particular for $u \in C_0^\infty(a,b)$. As a consequence, $Av = f$ in the sense of distributions, hence v belongs to the Sobolev space $H^2_{loc}(a,b)$, $Av = f$ a.e. in (a,b) and $v \in D_M$. Therefore we have

$$\int_a^b (Au)\bar{v}\varrho = \int_a^b u(A\bar{v})\varrho$$

for every $u \in D_1$; on the other hand, integrating by parts twice, we obtain

$$\int_a^b (Au)\bar{v}\varrho = \lim_{\substack{\alpha\to a\\ \beta\to b}} \left(\Big[pu'\bar{v} - p\bar{v}'u\Big]_\alpha^\beta + \int_\alpha^\beta u(A\bar{v})\varrho \right) \tag{2.10}$$

and the boundary terms must vanish for every $u \in D_1$. By choosing $u \equiv 1$ near a, $u \equiv 0$ near b, we see that necessarily $p(x)v'(x) \to 0$ as $x \to a$; analogously, $p(x)v'(x) \to 0$ as $x \to b$, and $v \in D_1$; therefore, $D(A^*) = D_1$ and (A, D_1) is self-adjoint. Finally, (2.9) with $v = u$ gives $\langle Au, u\rangle_\varrho \le 0$. The general case with $r \not\equiv 0$ is immediate consequence of the previous one. □

Proposition 2.5. *If $1/p \in L^1(a,b)$ then (A, D_0) is self-adjoint on L^2_ϱ; if $r \le 0$, it is non-positive.*

Proof. We assume again that $r \equiv 0$. Arguing as in the above proposition and using Lemma 2.3 instead of Lemma 2.1, we check that (A, D_0) is symmetric and non-positive, *i.e.*, for $u, v \in D_0$,

$$\int_a^b (Au)\bar{v}\varrho = \int_a^b u(A\bar{v})\varrho \qquad \text{and} \qquad \int_a^b (Au)\bar{u}\varrho \le 0. \tag{2.11}$$

For $v \in D(A^*)$ and $f = A^*v \in L^2_\varrho$ we can argue as in Proposition 2.4 and we infer that $v \in D_M(A)$ and $Av = f$ a.e. If $\varrho \notin L^1(a,c)$ then Lemma 2.3 shows that necessarily $v(x) \to 0$ as $x \to a$; if $\varrho \in L^1(a,c)$ then we can use again (2.10) with $u \in D_0$ such that $p(x)u'(x) \equiv 1$ near a and $u \equiv 0$ near b to obtain $v(x) \to 0$ as $x \to a$; arguing in the same way near b, we obtain that $v \in D_0$in any case. Therefore, $D(A^*) = D_0$ and (A, D_0) is self-adjoint. □

In the singular cases we shall deal with boundedness and compactness of some integral operators on weighted Hilbert spaces. Many general results are known (see *e.g.* [23], [16]). Nevertheless, in our cases it is easier to obtain meaningful conditions on the kernels giving direct proofs rather than deducing them from the general theory. Our approach is based on the following Hardy-type inequality in L^2_ϱ, whose proof is inspired by [13, §5.3].

Lemma 2.6. *Let $-\infty \le \alpha < \beta \le +\infty$ and $\varrho > 0$ in (α, β) with $\varrho \in L^1(\alpha, x)$ for every $x \in (\alpha, \beta)$. Setting*

$$\sigma(x) = \int_\alpha^x \varrho(s)ds,$$

the operator

$$Hf(x) = \frac{1}{\sigma(x)} \int_\alpha^x f(s)\varrho(s)ds$$

is bounded in $L^2_\varrho(\alpha, \beta)$.

Proof. We may suppose $f \ge 0$ and $f(x) = 0$ in a neighbourhood of α and β. Let $g(x) = \int_\alpha^x f(s)\varrho(s)ds$, and $\phi(x) = [\sigma(x)]^{-1/2}$. Notice that $\phi'\phi^{-3} = (-1/2)\varrho$ and

$|\phi'|^2\phi^{-2}\varrho^{-1} = (1/4)\varrho\sigma^{-2}$. Then

$$\begin{aligned}
\|f\|_\varrho^2 &= \int_\alpha^\beta |g'|^2\varrho^{-1} = \int_\alpha^\beta \frac{1}{\varrho\phi^2}\Big[(\phi g)' - \phi' g\Big]^2 \\
&\geq \int_\alpha^\beta \frac{|\phi'|^2 g^2}{\varrho\phi^2} - 2\int_\alpha^\beta \frac{1}{\varrho\phi^2}\Big[(\phi g)'\phi' g\Big] \\
&= \frac{1}{4}\int_\alpha^\beta \frac{g^2\varrho}{\sigma^2} + \int_\alpha^\beta (\phi g)'(\phi g) = \frac{1}{4}\|Hf\|_\varrho^2 + \frac{1}{2}\Big[(\phi g)^2\Big]_\alpha^\beta \geq \frac{1}{4}\|Hf\|_\varrho^2,
\end{aligned}$$

since $(\phi g)^2(\beta) \geq 0$ and $(\phi g)(\alpha) = 0$. □

Remark 2.7. We shall apply the above lemma with $(\alpha, \beta) = (a, b)$, $\sigma = \mu$ in the Neumann case and with $(\alpha, \beta) = (a, c)$ or (c, b), $\sigma = \nu$ in the Dirichlet case.

3. Neumann boundary conditions

In the present section we study the operator A with homogeneous Neumann boundary conditions, *i.e.* in the domain D_1 defined in (2.8). According to the discussion in Section 2, we assume $\varrho \in L^1(a, b)$, so that (A, D_1) is self-adjoint in L^2_ϱ. We denote by A_0 the operator A when $r \equiv 0$. Since A differs from A_0 by a bounded perturbation, the main properties of the resolvent of A actually depend on A_0, hence we concentrate on the invertibility of A_0 and on the properties of its inverse. Let us first state some elementary facts. The functions $\mathbf{1}$ and w defined by (2.2) are two linearly independent solutions of $A_0 u = 0$ and $Ker(A_0, D_1)$ reduces to constant functions. Moreover, if $u \in D_1$ and $f = A_0 u$ then

$$p(x)u'(x) = \int_a^x f(s)\varrho(s)ds, \tag{3.1}$$

hence the boundary conditions imply $\int_a^b f\varrho = 0$. Set

$$E = \Big\{ f \in L^2_\varrho : \int_a^b f\varrho = 0 \Big\}$$

and let P be the orthogonal projection along E. By the above remarks it follows that $A_0 : D_1 \cap E \to E$ is injective. To study the surjectivity, we begin by writing, for $u \in D_1$ and $f = A_0 u$:

$$u'(x) = \frac{1}{p(x)}\int_a^x f(s)\varrho(s)ds.$$

Integrating by parts and taking into account that $\int_a^b f\varrho = 0$, we obtain

$$\begin{aligned}
u(x) - u(c) &= w(x)\int_a^x f(s)\varrho(s)ds + \int_x^c w(s)f(s)\varrho(s)ds \\
&= \int_a^b K(x, s)f(s)\varrho(s)ds = T_K f(x),
\end{aligned}$$

where

$$K(x,s) = \begin{cases} w(x) & a < s \le x < c, \\ w(s) & a < x \le s < c, \\ -w(x) & c < x \le s < b, \\ -w(s) & c < s \le x < b, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

Conversely, if $f \in L^2_\varrho$ then $u(x) = T_K f(x)$ is well defined for $x \in (a,b)$ and satisfies $A_0 u(x) = f(x)$. Moreover (3.1) holds and the boundary condition at a is always satisfied, while that one at b is satisfied if and only if $f \in E$.

At this point, it is clear that $A_0 : D_1 \cap E \to E$ is invertible if and only if $T_K f \in L^2_\varrho$ for every $f \in E$, that is (by the closed graph theorem) if and only if the operator T_K is bounded from E to L^2_ϱ; in this case, we call T_G the inverse of A_0 on E, and we have $T_G f = (I-P)T_K f$ for every $f \in E$.

We extend the operator T_G as a self-adjoint operator on the whole of L^2_ϱ setting $T_G = (I-P)T_K(I-P)$. Furthermore, it is non-positive, since for every $f \in L^2_\varrho$ we have

$$\langle (I-P)T_K(I-P)f, f\rangle_\varrho = \langle T_K g, g\rangle_\varrho = \langle u, A_0 u\rangle_\varrho \le 0,$$

where $g = (I-P)f$ and $u = T_K g$.

Since $T_K f|_{(a,c)}$ and $T_K f|_{(c,b)}$ depend only on $f|_{(a,c)}$ and $f|_{(c,b)}$ respectively, it is easily seen that T_K is bounded on E if and only if T_K is bounded on L^2_ϱ, which in turn is equivalent to the boundedness in $L^2_\varrho(a,c)$ and $L^2_\varrho(c,b)$.

The following lemma shows that in order to study the compactness of the resolvent it suffices to look at the behaviour of A_0 on E.

Lemma 3.1. *The resolvent of A_0 is compact if and only if $A_0 : D_1 \cap E \to E$ is invertible and its inverse T_G is compact.*

Proof. Assume that A_0 has compact resolvent; then, the same holds for $A_0|_E$ and the thesis follows, because 0 does not belong to the point spectrum of $A_0|_E$.

Conversely, let T_G be compact. Note that $D_1 = (D_1 \cap E) \oplus E^\perp$, where $E^\perp$ consists of the constant functions and A_0 vanishes on $E^\perp$. It follows that, for $\lambda \ne 0$, $\lambda - A_0$ is always invertible on $E^\perp$ and its inverse is compact. Therefore for $\lambda \ne 0$ the operator $\lambda - A_0$ is invertible on the whole of D_1 if and only if it is so on $D_1 \cap E$; in this case the inverse is compact since the resolvent of $A_0|_E$ is compact. □

Let us now characterize the boundedness and the compactness of the operators T_K and T_G (see also [20] in connection with (ii)); recall that the function μ is defined in (2.1).

Proposition 3.2. *The following statements hold:*

(i) *T_K and T_G are bounded if and only if the function $\mu(x)[\mu(b) - \mu(x)]w(x)$ is bounded.*

(ii) *T_K and T_G are compact if and only if the following conditions hold:*

$$\lim_{x\to a} \mu(x)w(x) = \lim_{x\to b} [\mu(b) - \mu(x)]w(x) = 0. \tag{3.3}$$

Proof. Let us first prove the sufficiency of the conditions in (i), (ii); to this aim, let $B : L^2_\varrho \to L^2_\varrho$ be the operator $f \mapsto Bf(x) = w(x) \int_a^x f(s)\varrho(s)ds$; if the function $\mu(x)[\mu(b) - \mu(x)]w(x)$ is bounded, the operator B is bounded on $L^2_\varrho(a, c)$ by Lemma 2.6, and so is T_K, because $T_K = B + B^*$; by a similar argument T_K is bounded on $L^2_\varrho(c, b)$ and the sufficient part of (i) follows. Moreover, if $\mu(x)w(x) \to 0$ as $x \to a$, we write

$$Bf(x) = \frac{m(x)}{\mu(x)} \int_a^x f(s)\varrho(s)ds,$$

with $m(x) \to 0$ as $x \to a$; taking a sequence of functions m_ε which vanish in a neighbourhood of a and converge uniformly to m, we construct operators B_ε with m_ε in place of m. Let $\mathrm{supp} m_\varepsilon \subset [a_\varepsilon, c]$, with $a_\varepsilon > a$. The estimate

$$\Big|\int_x^y f(s)\varrho(s)ds\Big| \le |\mu(y) - \mu(x)|^{1/2} \|f\|_\varrho,$$

together with Ascoli's Theorem, shows that for any $\varepsilon > 0$ the operator $B_\varepsilon : L^2_\varrho(a, c) \to C([a_\varepsilon, c])$ is compact, whence $B_\varepsilon : L^2_\varrho(a, c) \to L^2_\varrho(a, c)$ is so. Since $\|B_\varepsilon - B\| \to 0$ as $\varepsilon \to 0$, we obtain the compactness of B in $L^2_\varrho(a, c)$ and by the same argument in $L^2_\varrho(c, b)$; therefore, T_K is compact in $L^2_\varrho(a, b)$.

As regards the necessity, assume that T_K is bounded on L^2_ϱ, fix $\delta > a$ sufficiently close to a, and consider the function $f_\delta(x) = \mu(\delta)^{-1/2}\chi_{(a,\delta)}(x)$; we have $\|f_\delta\|_\varrho = 1$ and, for $x \in (\delta, c]$, also $|T_K f_\delta(x)| \ge \mu(\delta)^{1/2}|w(x)|$. Fixing δ' in such a way that $\mu(\delta') = 2\mu(\delta)$, and using the fact that $|w|$ is decreasing in (a, c), we obtain

$$\begin{aligned} \|T_K\|^2 &\ge \int_a^c |T_K f_\delta(x)|^2 \varrho(x)dx \ge \mu(\delta) \int_a^{\delta'} |w(x)|^2 \varrho(x)dx \\ &\ge \mu(\delta)|w(\delta')|^2 \mu(\delta') = \frac{1}{4}\Big[\mu(\delta')w(\delta')\Big]^2 \end{aligned}$$

which implies that $\mu(x)|w(x)|$ is bounded in (a, c).

Finally, assuming that T_K is compact, since $f_\delta \to 0$ weakly, we have that $\|T_K f_\delta\|_\varrho \to 0$ as $\delta \to a$, we deduce as above that $\mu(x)w(x) \to 0$ as $x \to a$. The same arguments apply in (c, b). □

Now we come back to the general operator A and characterize when its resolvent is a Hilbert-Schmidt operator on L^2_ϱ.

Theorem 3.3. *The resolvent of A is Hilbert-Schmidt if and only if*

$$\mu(x)[\mu(b) - \mu(x)]|w(x)|^2 \in L^1_\varrho. \tag{3.4}$$

Proof. It is sufficient to prove the statement for A_0 because A and A_0 differ only by a bounded perturbation. By the above discussion, this reduces to prove that T_G is Hilbert-Schmidt if and only if condition (3.4) holds.

Observe that if (3.4) holds, the function $|\mu(x)[\mu(b) - \mu(x)]w(x)|$ is bounded: in fact, since $|w|$ is decreasing in (a, c),

$$|w(x)\mu(x)|^2 = 2|w(x)|^2 \int_a^x \mu(s)\varrho(s)ds \le 2 \int_a^x |w(s)|^2 \mu(s)\varrho(s)ds$$

and the right hand side is bounded by (3.4); hence $\mu|w|$ is bounded in (a,c) and the same argument in (c,b) allows to conclude that $|\mu(x)[\mu(b)-\mu(x)]w(x)|$ is bounded in (a,b). On the other hand, this last condition follows from the compactness of the resolvent of A by Lemma 3.1 and Proposition 3.2 (ii), hence we may assume that $|\mu(x)[\mu(b)-\mu(x)]w(x)| \le C$.

It is clear that T_G is Hilbert-Schmidt on E if and only if it is so on L^2_ϱ and that if T_K is Hilbert-Schmidt then T_G is so. Let us show that the converse holds, and the thesis will follow at once. A direct computation yields (formally)

$$T_G f(x) = \int_a^b G(x,s) f(s)\varrho(s)ds,$$

where

$$\begin{aligned} (3.5)\quad G(x,s) \;&=\; K(x,s) - \frac{1}{\mu(b)}\int_a^b K(x,\eta)\varrho(\eta)d\eta \\ &-\frac{1}{\mu(b)}\int_a^b K(\xi,s)\varrho(\xi)d\xi + \frac{1}{\mu(b)^2}\int_a^b\int_a^b K(\xi,\eta)\varrho(\xi)\varrho(\eta)d\xi d\eta \end{aligned}$$

and by assumption $G \in L^2_\varrho((a,b)^2)$, *i.e.*

$$\int_a^b\int_a^b |G(x,s)|^2\varrho(x)\varrho(s)dxds < +\infty.$$

We compute the intermediate integrals in (3.5) by parts and we obtain for the second one

$$\int_a^b K(\xi,s)\varrho(\xi)d\xi = \begin{cases} \displaystyle\int_c^x \frac{\mu(\xi)}{p(\xi)}d\xi & s \le c, \\ \displaystyle -\int_c^x \frac{\mu(b)-\mu(\xi)}{p(\xi)}d\xi & s \ge c. \end{cases} \qquad (3.6)$$

A similar formula holds for the third integral. By the preceding discussion we have $\mu(x) \le C/|w(x)|$, hence the integrals above can be estimated by

$$|\log|w(x)|| = O(|\log[\mu(x)(\mu(b)-\mu(x))]|)$$

and $K \in L^2_\varrho((a,b)^2)$ (the last term is a real number). This justifies (3.5) and proves that T_K is Hilbert-Schmidt. Finally, compute:

$$\begin{aligned} (3.7)\qquad &\int_a^b\int_a^b |K(x,s)|^2\varrho(x)\varrho(s)dxds = \\ &2\int_a^c |w(x)|^2\varrho(x)\mu(x)dx + 2\int_c^b |w(x)|^2\varrho(x)[\mu(b)-\mu(x)]dx; \end{aligned}$$

by (3.7), $K \in L^2_\varrho((a,b)^2)$ is equivalent to (3.4) and the proof is complete. □

Remark 3.4. Notice that in the proof of the preceding theorem we have shown in particular that if one among T_G, T_K and the resolvent of A is Hilbert-Schmidt the same holds for the other operators.

We prove the main theorem of this section; recall (see (2.4)) that $Q = w\varrho$, hence the condition $Q \in L^1(a,b)$ reads $w \in L^1_\varrho$. According to Feller's terminology, this means that a, b are regular or entrance boundary points.

Theorem 3.5. *Suppose that condition* (3.3) *holds, and let* (λ_n) *be the sequence of the non-zero eigenvalues of* A*, counted with their multiplicities; then the sum* $\sum_n \frac{1}{\lambda_n}$ *is finite if and only if* $Q \in L^1$*. In the case* $r \equiv 0$ *we have the equality*

$$-\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \frac{1}{\mu(b)} \int_a^b \frac{\mu(x)[\mu(b) - \mu(x)]}{p(x)} dx. \tag{3.8}$$

Proof. Condition (3.3) ensures that the resolvent of A is compact; moreover, both the convergence of the series $\sum \lambda_n^{-1}$ and the condition $w \in L^1_\varrho$ (*i.e.* $Q \in L^1(a,b)$) imply that T_G is Hilbert-Schmidt. In fact, if the above series converges then necessarily the resolvent of A is Hilbert-Schmidt, and so is T_G by Remark 3.4. Conversely, if $w \in L^1_\varrho$ then by (3.3) condition (3.4) holds. Therefore we can assume that T_G is Hilbert-Schmidt.

Suppose now that $r \equiv 0$ and prove that (3.8) holds as an equality in $\mathbf{R} \cup \{+\infty\}$. This will follow from the equality

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \int_a^b G(x,x)\varrho(x) dx, \tag{3.9}$$

and the computation of the last integral.

As in [5], equality (3.9) can be proved by adapting the proof of the classical Mercer's Theorem. The Green's function G can be expanded as follows:

$$G(x,s) = \sum_{n=1}^{\infty} \frac{v_n(x)v_n(s)}{\lambda_n}$$

where (v_n) is a complete orthonormal system of eigenfunctions in E and the series converges in $L^2_\varrho\big((a,b)^2\big)$. By (3.2) and the proof of Theorem 3.3 we deduce that $G(x,s)$ is continuous on $(a,b)^2$ and for every compact subset $Q \subset (a,b)$ there are constants α, β such that

$$|G(x,s)| \le \alpha |\log \mu(x)[\mu(b) - \mu(x)]| + \beta \tag{3.10}$$

for every $s \in Q$, $x \in (a,b)$.

For every $N \in \mathbf{N}$ consider the integral operator T_{G_N} defined by the kernel

$$G_N(x,s) = G(x,s) - \sum_{n=1}^{N} \frac{v_n(x)v_n(s)}{\lambda_n}.$$

Since both G and all the eigenfuntions v_n are continuous, G_N is also continuous on $(a,b)^2$. The operators T_G and T_{G_N} are self-adjoint and non-positive. Then, taking into account that the weight ϱ is continuous and strictly positive in (a,b), as in the proof of Mercer's Theorem (see *e.g.* [28]) we deduce that $G(x,x) \le 0$ and $G_N(x,x) \le 0$ for $a < x < b$. Letting $N \to +\infty$ we obtain for $a < x < b$:

$$G(x,x) \le \sum_{n=1}^{\infty} \frac{v_n^2(x)}{\lambda_n} \le 0.$$

The inequality

$$\sum_{n=1}^{\infty} \Big| \frac{v_n(x)v_n(s)}{\lambda_n} \Big| \le \Big| \sum_{n=1}^{\infty} \frac{v_n^2(x)}{\lambda_n} \Big|^{1/2} \Big| \sum_{n=1}^{\infty} \frac{v_n^2(s)}{\lambda_n} \Big|^{1/2} \le \big| G(x,x) \big|^{1/2} \Big| \sum_{n=1}^{\infty} \frac{v_n^2(s)}{\lambda_n} \Big|^{1/2}$$

implies that for any fixed $a < \bar{s} < b$ the series $\sum_n \lambda_n^{-1} v_n(x) v_n(\bar{s})$ is uniformly convergent on any compact subset of (a,b). By (3.10) and the continuity of v_n on (a,b) we deduce that the equality

$$\int_a^b G(x,s) v_n(x) \varrho(x) dx = \frac{v_n(s)}{\lambda_n} \tag{3.11}$$

holds pointwise for $a < s < b$. Hence the expansion

$$G(x,\bar{s}) = \sum_{n=1}^{\infty} \frac{v_n(x)v_n(\bar{s})}{\lambda_n}$$

is valid in L^2_ϱ for every $a < \bar{s} < b$. Therefore for every $a < \bar{s} < b$ the functions $G(x,\bar{s})$ and $\sum_n \lambda_n^{-1} v_n(x) v_n(\bar{s})$ are continuous on (a,b) and coincide a.e. It follows that

$$G(x,s) = \sum_{n=1}^{\infty} \frac{v_n(x)v_n(s)}{\lambda_n} \tag{3.12}$$

for *every* $(x,s) \in (a,b)^2$; in particular

$$G(x,x) = \sum_{n=1}^{\infty} \frac{v_n^2(x)}{\lambda_n}. \tag{3.13}$$

By Dini's Theorem in (3.13) the convergence is locally uniform, hence (3.12) holds uniformly on compact subsets of $(a,b)^2$ and this implies (3.9) by integration on the diagonal.

Let us compute the integral; by (3.5) we have

$$\int_a^b G(x,x)\varrho(x)dx = \int_a^b K(x,x)\varrho(x)dx - \frac{1}{\mu(b)} \int_a^b \int_a^b K(x,s)\varrho(x)\varrho(s)dxds, \tag{3.14}$$

and the convergence of the integral on the left hand side depends only on the convergence of the first one on the right, because the last one is finite according to the discussion in the proof of Theorem 3.3. Integrating by parts and using (3.2):

$$\begin{aligned} \int_a^b K(x,x)\varrho(x)dx &= \int_a^c w(x)\varrho(x)dx - \int_c^b w(x)\varrho(x)dx \\ &= -\int_a^c \frac{\mu(x)}{p(x)}dx - \int_c^b \frac{\mu(b)-\mu(x)}{p(x)}dx \end{aligned}$$

since the boundary terms vanish by (3.3). Arguing similarly, we now compute the last integral in (3.14), using (3.6):

$$\int_a^b \int_a^b K(x,s)\varrho(x)\varrho(s)dxds = -\int_a^c \frac{\mu(x)^2}{p(x)}dx - \int_c^b \frac{(\mu(b)-\mu(x))^2}{p(x)}dx$$

and (3.8) is proved.

The convergence of the integral in (3.8) is equivalent to the convergence of the integrals

$$\int_a^c \frac{\mu(x)}{p(x)}dx, \qquad \int_c^b \frac{\mu(b)-\mu(x)}{p(x)}dx$$

and, by integrating by parts, this is in turn equivalent to $w \in L^1_\varrho$, *i.e.* to $Q \in L^1(a,b)$. This concludes the proof in the case $r \equiv 0$; the general case follows by the compactness of the resolvent of A and a minimax argument on the eigenvalues. Let (λ_n) be the eigenvalues of A and (λ'_n) be the eigenvalues of A_0; from [13, Th. 4.5.1] the inequalities

$$\lambda'_n + \inf_{(a,b)} r \le \lambda_n \le \lambda'_n + \sup_{(a,b)} r \qquad \forall\, n \in \mathbf{N}$$

follow and the proof is complete. □

Remark 3.6. If $Q \in L^1(a,b)$ then condition (3.3) is automatically satisfied since w is monotone in (a,c) and in (c,b); hence, as a consequence of the above theorem, the resolvent of A is a trace-class operator on L^2_ϱ if and only if $Q \in L^1(a,b)$, that is $w \in L^1_\varrho$. Observe that this condition is more general than the limit circle condition (see [26]), which in our case reads $w \in L^2_\varrho$.

4. Dirichlet Boundary Conditions

This section is devoted to the study of the differential operator A with homogeneous Dirichlet conditions. According to the discussion in Section 2, we assume $1/p \in L^1(a,b)$, so that the operator (A, D_0) is self-adjoint and non-positive on L^2_ϱ. We use the notation introduced therein, recalling that ν is defined in (2.2), and z in (2.1). As in Section 3 we focus on the operator A_0. Two linearly independent solutions of $A_0u = 0$ are

$$u_1(x) = z(x), \qquad u_2(x) = z(x) - z(b). \tag{4.1}$$

Since neither u_1 nor u_2 belong to D_0, the operator A_0 is injective and this gives some simplifications with respect to Section 3. In particular, it is no longer necessary to introduce the subspace E and the projection P.

Lemma 4.1. *If $A_0 : D_0 \to L^2_\varrho$ is invertible, then $u_1 \in L^2_\varrho(a,\gamma)$ and $u_2 \in L^2_\varrho(\gamma,b)$ for every $\gamma \in (a,b)$.*

Proof. Let $a < \alpha < \beta < b$ and fix a continuous function f strictly positive in (α,β) and vanishing in $(a,\alpha) \cup (\beta,b)$; let further $u \in D_0$ be the solution of $A_0u = f$.

Then $u \in C^2(a,b)$ and by the boundary conditions $u(x) = c_1 u_1(x)$ in (a,α) and $u(x) = c_2 u_2(x)$ in (β, b). Were it $c_1 = 0$, we would obtain $u(\alpha) = u'(\alpha) = 0$ and, by the equation, pu' increasing in (a,b) and strictly increasing in (α,β), so that $u' > 0$ in (α, b): this is in contrast with $u(\alpha) = u(b) = 0$. Arguing in the same way near b, it follows that $c_1 \neq 0$, $c_2 \neq 0$ and the assertion follows. □

Assuming $u_1 \in L^2_\varrho(a,\gamma)$ and $u_2 \in L^2_\varrho(\gamma, b)$ for every $\gamma \in (a,b)$, let us write the Green's function

$$G(x,s) = \frac{1}{z(b)} \begin{cases} u_1(x)u_2(s) & a < x \leq s < b, \\ \\ u_1(s)u_2(x) & a < s \leq x < b. \end{cases} \tag{4.2}$$

Observe that the integral operator

$$T_G f(x) = \int_a^b G(x,s) f(s) \varrho(s) ds$$

is well defined for every $x \in (a,b)$ and for every $f \in L^2_\varrho$. Moreover, setting $u = T_G f$, we have $u, u' \in AC_{loc}(a,b)$ and $A_0 u = f$ a.e. Let us show that A_0 is invertible if and only if T_G is bounded from L^2_ϱ into itself. In fact, if T_G is bounded, then $u \in D_M(A)$ and satisfies the boundary conditions, by a direct computation if $\varrho \in L^1(a,b)$ (see the explicit expression (4.3) below), and using Lemma 2.3 if $\varrho \notin L^1(a,b)$. Conversely, if A_0 is invertible, taking $f \in C_0^\infty(a,b)$ we see that $T_G f$ is equal to $c_1 u_1$ near a and to $c_2 u_2$ near b, hence it belongs to D_0 and coincides with $A_0^{-1} f$. Therefore, $A_0^{-1} = T_G$ on $C_0^\infty(a,b)$ and by approximation the boundedness of T_G easily follows.

We are now in a position to state the main properties of the resolvent of A (see also [20] in connection with (ii)).

Theorem 4.2. *The following statements hold:*

(i) *the operator A_0 is invertible if and only if the function*

$$z(x)[z(x) - z(b)]\nu(x)$$

is bounded;

(ii) *the operator A has compact resolvent if and only if*

$$\lim_{x \to a} z(x)\nu(x) = \lim_{x \to b} [z(x) - z(b)]\nu(x) = 0;$$

(iii) *the resolvent of A is Hilbert-Schmidt if and only if*

$$z^2(x)[z(b) - z(x)]^2 \nu(x) \in L^1_\varrho.$$

Proof. By (4.2) and (4.1) the operator T_G is

$$\begin{aligned} z(b) T_G f(x) &= [z(x) - z(b)] \int_a^x z(s) f(s) \varrho(s) ds \\ &\quad + z(x) \int_x^b [z(s) - z(b)] f(s) \varrho(s) ds = B_1 f(x) + B_2 f(x), \end{aligned} \tag{4.3}$$

hence for positive f

$$|B_1 f(x)| \leq z(b)|T_G f(x)| \leq |B_1 f(x)| + |B_2 f(x)|,$$

since $z(x)$ and $z(x)-z(b)$ have constant sign in (a,b). Observe also that formally $B_2 = B_1^*$, hence T_G is bounded (resp. compact) if and only if B_1 is so.
(i) Assume first that $z(x)[z(x)-z(b)]\nu(x)$ is bounded, and take $f \in L^2_\varrho$; then we have for $x \in (a,c)$

$$|B_1 f(x)| \leq C\int_a^x |f(s)|\frac{\varrho(s)}{|\nu(s)|}ds = CH^*(|f|)(x)$$

and for $x \in (c,b)$

$$\begin{aligned}|B_1 f(x)| \quad &\leq \quad C\Big\{\|f\|_\varrho |z(x)-z(b)| + \frac{1}{|\nu(x)|}\int_c^x |f(s)|\varrho(s)ds\Big\}\\ &\leq \quad C\Big\{\|f\|_\varrho |z(x)-z(b)| + H(|f|)(x)\Big\}\end{aligned}$$

where $C>0$ is a suitable constant. The operator H (defined according to Remark 2.7) is bounded by Lemma 2.6 and H^* is the adjoint of H in $L^2_\varrho(a,c)$. The boundedness of B_1 follows.

Conversely, fix $\delta > a$ and choose $\delta' > \delta$ such that $\nu(\delta) = 2\nu(\delta')$; set further $f_\delta(x) = |\nu(\delta')|^{-1/2}\chi_{(\delta,\delta')}(x)$ (so that $\|f_\delta\|_\varrho = 1$); assuming that B_1 is bounded we have, using the monotonicity of z,

$$\|B_1\|^2 \geq \int_{\delta'}^c |B_1 f_\delta(x)|^2 \varrho(x)dx \geq Cz^2(\delta)\nu^2(\delta), \tag{4.4}$$

hence $z(x)\nu(x)$ is bounded near a; in the same way one can see that $[z(x)-z(b)]\nu(x)$ is bounded near b.
(ii) Let us prove the sufficiency of the condition; assuming $z(x)\nu(x) \to 0$ as $x \to a$, and setting $m = z\nu$, the operator

$$f \mapsto \frac{m}{\nu}\int_x^c f\varrho$$

turns out to be compact in $L^2_\varrho(a,c)$ by the same argument as in Proposition 3.2 (ii). But the above operator is (up to the bounded multiplier $z(b)-z(x)$) the adjoint of B_1 in $L^2_\varrho(a,c)$, which is therefore compact. As usual, the same argument works in $L^2_\varrho(c,b)$.

As regards the necessity, we can use the same functions f_δ as above; since $f_\delta \to 0$ weakly as $\delta \to a$, $\|B_1 f_\delta\| \to 0$ and the last inequality in (4.4) implies that $z(x)\nu(x) \to 0$ as $x \to a$.
(iii) It is sufficient to characterize when T_G is Hilbert-Schmidt. In fact, a straightforward computation gives

$$\int_a^b\int_a^b |G(x,s)|^2\varrho(x)\varrho(s)dxds = \frac{2}{z^2(b)}\int_a^b u_2^2(x)\varrho(x)\Big(\int_a^x u_1^2(s)\varrho(s)ds\Big)dx$$

and the thesis follows splitting the integral over (a,b) in two integrals over (a,c) and (c,b); in fact, the integral in the right hand side converges *e.g.* in (a,c) if and only if the integral

$$\int_a^c \varrho(x)\Big(\int_a^x z^2(s)\varrho(s)ds\Big)dx = \int_a^c z^2(x)\nu(x)\varrho(x)dx$$

is finite. □

We now state the main result of this section; notice that, according to Feller's terminology, the condition $R \in L^1(a,b)$ means that a, b are regular or exit boundary points.

Theorem 4.3. *Suppose that condition* (ii) *in Theorem* 4.2 *holds, and let* (λ_n) *be the sequence of the non-zero eigenvalues of* A*, counted with their multiplicities; then the sum* $\sum_n \frac{1}{\lambda_n}$ *is finite if and only if* $R \in L^1(a,b)$*. In the case* $r \equiv 0$ *we have the equality*

$$-\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \frac{1}{z(b)} \int_a^b z(x)[z(b) - z(x)]\varrho(x)dx. \tag{4.5}$$

Proof. The proof is similar to that one of Theorem 3.5, with some simplifications; we deal only with the case $r \equiv 0$, because the general case can be obtained by a minimax argument as in Theorem 3.5. Expanding the Green's function through a complete orthonormal system (v_n) of continuous eigenfunctions we obtain

$$G(x,s) = \sum_{n=1}^{\infty} \frac{v_n(x)v_n(s)}{\lambda_n} \tag{4.6}$$

with $G(x,s)$ continuous in $(a,b)^2$ by (4.2). Observe that for every $s \in (a,b)$ and $n \in \mathbf{N}$ the function $\int_a^b G(x,s)v_n(x)\varrho(x)dx$ is continuous: in fact for $s \in [\alpha,\beta] \subset (a,b)$ we have

$$|G(x,s)| \le \frac{1}{z(b)} \begin{cases} |u_1(x)| \sup\limits_{\alpha\le t\le\beta} |u_2(t)| & x \in (a,\alpha) \\ \sup\limits_{\alpha\le t,\tau\le\beta} |u_1(t)u_2(\tau)| & x \in [\alpha,\beta] \\ |u_2(x)| \sup\limits_{\alpha\le t\le\beta} |u_1(t)| & x \in (\beta,b) \end{cases}$$

and $G(x,s)v_n(x)\varrho(x)$ is dominated by a L^1-function uniformly with respect to $s \in (a,b)$. As a consequence, equality (3.11) holds again and (4.6) is uniform on compact subsets of $(a,b)^2$. Integrating on the diagonal we obtain the trace formula (4.5) for $r \equiv 0$. The convergence of the integral is readily seen to be equivalent to the condition $R \in L^1(a,b)$. □

Remark 4.4. As in the Neumann case, $R \in L^1(a,b)$ implies that the resolvent of A is compact; hence we obtain again that the resolvent of (A, D_0) is of the trace class on L^2_ϱ if and only if $R \in L^1(a,b)$, *i.e.* $z(x)[z(x) - z(b)]$ is in L^1_ϱ.

Remark 4.5. In the regular case, *i.e.* if both ϱ and $1/p$ are summable in (a,b), then our methods can be applied under more general boundary conditions (see [17]); if one imposes

$$\begin{cases} \cos\alpha u(a) + \sin\alpha p(a) u'(a) = 0 \\ \\ \cos\beta u(b) + \sin\beta p(b) u'(b) = 0 \end{cases}$$

for some $\alpha, \beta \in \mathbf{R}$, and defines accordingly the domain, then the computations that lead to Theorem 4.3 are simplified and give the following result if $r \equiv 0$:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \quad = \quad & \frac{1}{D}\Big[\mu(b)\sin\alpha(\sin\beta + z(b)\cos\beta) - \sin(\alpha+\beta)\int_a^b z(x)\varrho(x)dx \\ & -\cos\alpha\cos\beta\int_a^b z(x)\big[z(b) - z(x)\big]\varrho(x)dx\Big], \end{aligned}$$

where $D = \sin(\beta - \alpha) + z(b)\cos\alpha\cos\beta$ is assumed not to vanish. Notice that the case of Dirichlet boundary conditions corresponds to $\alpha = \beta = 0$.

5. Degenerate operators in spaces of continuous functions

We now turn our attention to the operator

$$\mathcal{A}u = \alpha u'' + \beta u' + ru \tag{5.1}$$

(denoted by $\mathcal{A}_0$ if $r \equiv 0$) in (a,b), allowing degeneracy at the endpoints. We assume $\alpha, \beta \in C(a,b)$ with $\alpha(x) > 0$ and $r \in C([a,b])$. The operator $\mathcal{A}$ will be studied in the space $C([a,b])$ with Neumann or Ventcel boundary conditions, depending on the behaviour of the functions W, Q, R defined in the introduction.

The function W allows to write the operator $\mathcal{A}$ in the form (1.2), *i.e.* as a formally self-adjoint operator in the space L^2_ϱ with $\varrho = (\alpha W)^{-1}$. Hence the results proved in the previous sections can be applied assuming $\varrho \in L^1(a,b)$ or $W = 1/p \in L^1(a,b)$, respectively. As a consequence, we have (see (2.1))

$$\mu(x) = \int_a^x \frac{1}{\alpha(s)W(s)}ds, \qquad z(x) = \int_a^x W(s)ds.$$

If the weight $(\alpha W)^{-1}$ belongs to $L^1(a,b)$ as in Section 3, then we define the domain of $\mathcal{A}$ imposing (degenerate) homogeneous Neumann boundary conditions, *i.e.* we consider the domain D_N given by

$$D_N = \Big\{u \in C([a,b]) \cap C^2(a,b) : \ \mathcal{A}u \in C([a,b]), \ \lim_{x\to a,b} \frac{u'(x)}{W(x)} = 0\Big\}. \tag{5.2}$$

It follows from [18] (see also [6]) that $(\mathcal{A}, D_N)$ generates a C_0-semigroup. Notice that in the genuinely singular case $W \notin L^1(a,c)$ we have

$$\mathcal{A}u \in C([a,b]) \quad \Longrightarrow \quad \lim_{x\to a} \frac{u'(x)}{W(x)} = 0.$$

In fact, setting $f = \mathcal{A}u - ru$, we obtain

$$\frac{u'(x)}{W(x)} - u'(c) = \int_c^x \frac{f(s)}{\alpha(s)W(s)} ds. \tag{5.3}$$

The convergence of the integral as $x \to a$ implies the convergence of $\lim_{x\to a} \frac{u'(x)}{W(x)}$, which in turn must be 0 since u is continuous. The same argument holds for $x \to b$ if $W \notin L^1(c,b)$. In particular, if both the endpoints are entrance points, then D_N coincides with the maximal domain of $\mathcal{A}$, *i.e.*

$$D_N = \Big\{u \in C([a,b]) \cap C^2(a,b) : \ \mathcal{A}u \in C([a,b])\Big\}.$$

If $W \in L^1(a,b)$, the results of Section 4 can be applied to the operator $\mathcal{A}$ with Ventcel boundary conditions: we consider the domain D_V given by

$$D_V = \Big\{u \in C_0([a,b]) \cap C^2(a,b) : \ \mathcal{A}u \in C([a,b])\Big\} \tag{5.4}$$

and recall that under the above assumption, by [9], $(\mathcal{A}, D_V)$ generates a C_0-semigroup. Note also that 0 is an eigenvalue of $(\mathcal{A}, D_V)$ and that $\mathbf{1}$ and $\int_c^x W(s)ds$ are two linearly independent eigenfunctions. If $\lambda \neq 0$ and $\mathcal{A}u - \lambda u = 0$, then $u \in D_V$ if and only if $u \in C_0([a,b])$, hence Ventcel and Dirichlet boundary conditions are equivalent for eigenfunctions relative to non-zero eigenvalues. This leads us in the frame of Section 4.

In this section we characterize the compactness of the resolvents of the operators $(\mathcal{A}, D_N)$ and $(\mathcal{A}, D_V)$ and show that whenever they are compact the series $\sum_n \frac{1}{\lambda_n}$ is convergent. In the next section we study the regularity of the eigenfunctions and of the semigroup generated by $(\mathcal{A}, D_V)$.

Lemma 5.1. *Let $\mathcal{A}$, D_N and D_V be as above. Then*

(i) *the operator $(\mathcal{A}, D_N)$ has compact resolvent if and only if $Q \in L^1(a,b)$,* i.e. *if and only if both a and b are either regular or entrance boundary points.*

(ii) *the operator $(\mathcal{A}, D_V)$ has compact resolvent if and only if $R \in L^1(a,b)$,* i.e. *if and only if both a and b are either regular or exit boundary points.*

Proof. (i) Let $Q \in L^1(a,b)$, $u \in D_N$ and $f = \mathcal{A}u - ru$; by (5.3) we obtain

$$u'(c) = \int_a^c \frac{f(s)}{\alpha(s)W(s)} ds \quad \text{and} \quad u'(x) = W(x) \int_a^x \frac{f(s)}{\alpha(s)W(s)} ds,$$

whence for $x \in (a,c)$

$$|u'(x)| \le \|u\|_{D_N} W(x) \int_a^x \frac{1}{\alpha(s)W(s)} ds = \|u\|_{D_N} h(x),$$

with

$$\|h\|_{L^1(a,c)} = \int_a^c W(x) \int_a^x \frac{1}{\alpha(s)W(s)} ds = \int_a^c |Q(x)| dx < +\infty.$$

Arguing in the same way in (c,b), for $u \in D_N$ we obtain that $u'(x) \le \|u\|_{D_N} h(x)$ in (c,b) with $h(x) = W(x)\int_x^b \frac{1}{\alpha(s)W(s)}ds$. Hence, by Ascoli's Theorem, D_N is compactly embedded in $C([a,b])$ and the thesis follows, as $(\mathcal{A}, D_N)$ has non-empty resolvent set.

Conversely, suppose that $(\mathcal{A}_0, D_N)$ has compact resolvent, and note that this implies that $\mathcal{A}_0$ is invertible between $D_N \cap E$ and $E = \{f \in C([a,b]) : \int_a^b f\varrho = 0\}$. Then from the equation $\mathcal{A}_0 u = f$ we obtain

$$u'(x) = W(x)\int_a^x \frac{f(s)}{\alpha(s)W(s)}ds;$$

choosing $f \in E$ with $f \equiv 1$ in (a,c) and using Fubini's Theorem we obtain the summability of Q in (a,c).

Proof of (ii). Let $R \in L^1(a,b)$ and observe that $W \in L^1(a,b)$ as well. For $u \in D_V$ and $f = \mathcal{A}u - ru$ the formula

$$u'(x) = u'(c)W(x) + W(x)\int_c^x \frac{f(s)}{\alpha(s)W(s)}ds$$

implies that

$$|u'(x)| \le C\|u\|_{D_V}\left[W(x) + |R(x)|\right],$$

so D_V is compactly embedded in $C([a,b])$ (by Ascoli's Theorem again) and the thesis follows.

Conversely, suppose that $(\mathcal{A}_0, D_V)$ has compact resolvent and remark that $C_0([a,b])$ is invariant under $\mathcal{A}_0$. Then, since $\mathcal{A}_0$ is injective on $C_0([a,b])$, it in invertible therein. Let us prove first that $W \in L^1(a,b)$. For, if $W \notin L^1(a,c)$, then no non-zero solution $u \in C_0([a,b])$ of $\mathcal{A}_0 u = 0$ satisfies Ventcel condition at a and a contradiction follows by arguing as in Lemma 4.1. If $\mathcal{A}_0 u = f$ then

$$u'(x) = u'(c)W(x) + W(x)\int_c^x \frac{f(s)}{\alpha(s)W(s)}ds$$

and for any $f \in C_0([a,b])$ increasing in (a,c) the inequality

$$W(x)\int_x^c \frac{f(s)}{\alpha(s)W(s)}ds \ge f(x)|R(x)|$$

implies that $f|R| \in L^1(a,c)$ because the left hand side is summable; by the arbitrariness of f it follows that $R \in L^1(a,c)$. □

From now on, in the discussion of $(\mathcal{A}, D_N)$ and $(\mathcal{A}, D_V)$, we assume $Q \in L^1(a,b)$ and $R \in L^1(a,b)$, respectively. In order to apply the results of Sections 3 and 4 we introduce the operators $(\mathcal{A}, D_1)$ and $(\mathcal{A}, D_0)$, where $\mathcal{A}u$ is given by (1.2), D_1 and D_0 are given by (2.8) and (2.7) respectively, with $\varrho = (\alpha W)^{-1}$, $p = 1/W$. To this aim, we need to know that the spectral properties of $\mathcal{A}$ are the same in the domains D_N and D_1 and respectively in D_V and D_0. By a result of W. Arendt's (see [2, Prop. 2.6]) we know that the spectra coincide, but in order to apply the trace formulas (3.8) and (4.5) we have to check also that the multiplicities are the same. This can be obtained by adapting Arendt's argument in a quite general setting as follows.

Let E, F be Banach spaces with $E \cap F$ dense both in E and in F and assume that E and F are continuously embedded in some topological vector space G; let $A_E : D(A_E) \to E$, $A_F : D(A_F) \to F$ be densely defined closed linear operators with non empty resolvent set such that $A_E = A_F$ in $D(A_E) \cap D(A_F)$.

Proposition 5.2. *Let E, F, A_E, A_F be as above. Assume that A_E and A_F have compact resolvents and that for some $\lambda^* \in \rho(A_E) \cap \rho(A_F)$ the equality $(\lambda^* - A_E)^{-1} = (\lambda^* - A_F)^{-1}$ holds in $E \cap F$. Then, $\sigma(A_E) = \sigma(A_F)$ and if λ_0 is in the spectrum, λ_0 is a pole of the same order for $(\lambda - A_E)^{-1}$ and $(\lambda - A_F)^{-1}$; moreover, the spectral subspaces of A_E and A_F relative to λ_0 coincide, hence the multiplicity of λ_0.*

Proof. By [2, Prop. 2.2 and 2.6] and the compactness of the resolvents it follows that $\rho(A_E) = \rho(A_F)$ and that $(\lambda - A_E)^{-1}$ and $(\lambda - A_F)^{-1}$ agree on $E \cap F$ for every λ in the resolvent set. Let us consider the Laurent's expansions of the resolvents of A_E and A_F:

$$(\lambda - A_E)^{-1} = \sum_{k=-\infty}^{+\infty} A_k(\lambda_0 - \lambda)^k, \qquad (\lambda - A_F)^{-1} = \sum_{k=-\infty}^{+\infty} B_k(\lambda_0 - \lambda)^k,$$

where

$$A_k = -\frac{1}{2\pi i}\int_\gamma \frac{(\lambda - A_E)^{-1}}{(\lambda_0 - \lambda)^{k+1}} d\lambda, \qquad B_k = -\frac{1}{2\pi i}\int_\gamma \frac{(\lambda - A_F)^{-1}}{(\lambda_0 - \lambda)^{k+1}} d\lambda$$

and γ is a small circle around λ_0, oriented counterclockwise. Since the resolvents coincide on $E \cap F$, A_k and B_k coincide on $E \cap F$ as well, and by density $A_k = 0$ if and only if $B_k = 0$. Hence, the order of λ_0 as a pole is the same in both cases. Moreover, since $A_{-1}(E)$ and $B_{-1}(F)$ are finite-dimensional and $E \cap F$ is dense both in E and F, we deduce $A_{-1}(E) = A_{-1}(E \cap F) = B_{-1}(E \cap F) = B_{-1}(F)$: therefore, the spectral subspaces are the same and also the multiplicities. □

We are now in a position to obtain the main result of this section.

Theorem 5.3. *Let $\mathcal{A}$, D_N and D_V be given by* (5.1), (5.2) *and* (5.4)*.*

(i) *Assume $Q \in L^1(a, b)$ and let (λ_n) be the sequence of the non-zero eigenvalues of $(\mathcal{A}, D_N)$. Then the sum $\sum_n \frac{1}{\lambda_n}$ is convergent, and in the case $r \equiv 0$ we have the equality*

$$-\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \Big(\int_a^b \frac{1}{\alpha W}\Big)^{-1} \int_a^b W(x)\Big(\int_a^x \frac{1}{\alpha W}\Big)\Big(\int_x^b \frac{1}{\alpha W}\Big) dx$$

(ii) *Assume $R \in L^1(a, b)$, and let (λ_n) be the sequence of the non-zero eigenvalues of $(\mathcal{A}, D_V)$. Then the sum $\sum_n \frac{1}{\lambda_n}$ is convergent, and in the case $r \equiv 0$ we have the equality*

$$-\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \Big(\int_a^b W\Big)^{-1} \int_a^b \frac{1}{\alpha(x)W(x)}\Big(\int_a^x W\Big)\Big(\int_x^b W\Big) dx$$

Proof. Observe that by Lemma 5.1 and Remarks 3.6 and 4.4 all the operators have compact resolvents. We shall deduce (i) and (ii) from Theorems 3.5 and 4.3 through Proposition 5.2.

Proof of (i). Proposition 5.2 can be applied with $F = C([a,b]) \subset E = L^2_\varrho$, and $D(A_F) = D_N \subset D(A_E) = D_1$: we only have to check that the resolvents agree on $C([a,b])$; to this aim, if $\lambda^* \in \rho(A_E) \cap \rho(A_F)$ and $\mathcal{A}u - \lambda^* u = \mathcal{A}v - \lambda^* v = f$ with $u \in D_N$, $v \in D_1$ and $f \in C([a,b])$, then $u - v \in D_1$ and $(\mathcal{A} - \lambda^*)(u - v) = 0$ implies $u = v$.

Proof of (ii). We apply again Proposition 5.2, with $E = L^2_\varrho$, $F = C_0([a,b])$, $D_E = D_0$ and $D_F = D_V \cap C_0([a,b])$. Let us take $\lambda^* \in \rho(A_E) \cap \rho(A_F)$, $f \in L^2_\varrho \cap C_0([a,b])$ and $u \in D_0$ such that $\mathcal{A}u - \lambda^* u = f$. Then $u \in C^2(a,b)$ and, since u and f tend to 0 as $x \to a, b$, it follows that $\mathcal{A}_0 u(x) \to 0$ as $x \to a, b$, *i.e.* $u \in D_V \cap C_0([a,b])$. □

6. Regularity of eigenfunctions and semigroups

In this section we study the boundary behaviour of the eigenfunctions and the regularity of the semigroups generated in $C([a,b])$ by the operators of the previous section. If the endpoints are regular boundaries, the differentiability of the semigroup is proved in [6] under general boundary conditions.

If a, b are entrance boundaries, the domain D_N given by (5.2) coincides with the maximal domain. Hence, the associated evolution problem is free from boundary conditions and regularity results on the generated semigroups can be obtained by using stochastic methods or by transforming the operator into one of Schrödinger type. To be precise, suppose that $\mathcal{A}u = u'' + \beta u'$ on $(-\infty, +\infty)$ and observe that $W^{-1} \in L^1(-\infty, +\infty)$. Stochastic methods lead to the differentiability of the semigroup both in $L^2_{W^{-1}}(-\infty, +\infty)$ and in $C([-\infty, +\infty])$; we remark that, in this framework, the measure $W^{-1}dx$ has an intrinsic probabilstic meaning, being the invariant measure of the related stochastic process (see *e.g.* [8] and [11] for the general framework).

The reduction to a Schrödinger operator is done as follows. Consider again $\mathcal{A}u = u'' + \beta u'$ in $L^2_{W^{-1}}(-\infty, +\infty)$ and observe that the map $u \mapsto Vu = \phi u$, with $\phi = W^{-1/2}$, is a unitary isomorphism between $L^2_{W^{-1}}(-\infty, +\infty)$ and $L^2(-\infty, +\infty)$. Then the operator $\mathcal{A}$ is similar to $\mathcal{B} = V\mathcal{A}V^{-1}$ and a straightforward computation shows that $\mathcal{B}f = f'' - \frac{\phi''}{\phi} f$. Although the similarity above is valid only for the L^2-spaces above, the investigation can be carried on to L^1-spaces and, by duality, to the case of continuous functions. We refer to [14] and [15] for the detailed explanation and applications of this method.

Therefore we confine ourselves to $(\mathcal{A}, D_V)$, with D_V given by (5.4), and exit boundaries: in this case, regularity results cannot be obtained using the above methods, because the boundary conditions are effective. A statement similar to Theorem 6.2 can be proved also in the entrance case and, as far as we know, it is not covered by the above quoted methods. However such a result does not cover the most meaningful cases and so we omit it.

Let us first consider the operator $Bu = \alpha u'' + ru$ in the bounded interval $(0, \ell)$: we shall see later that the general case can be reduced to this one. Assume that α and

r fulfil all the hypotheses stated at the beginning of the preceding section, and that $R \in L^1(0,\ell)$, *i.e.*

$$\int_0^\ell \frac{x(1-x)}{\alpha(x)}dx < +\infty.$$

Hence Theorem 5.3(ii) applies to $(B, D_V(B))$ with

$$D_V(B) = \big\{u \in C([0,\ell]) \cap C^2(0,\ell) : \ Bu \in C_0([0,\ell])\big\}.$$

As in Section 5, the operator B can be regarded in $L^2_{1/\alpha}$, and the corresponding domain is

$$D_0(B) = \big\{u \in L^2_{1/\alpha} : \ u, u' \in AC_{loc}(0,\ell), \ Bu \in L^2_{1/\alpha}, \ \lim_{x\to 0,\ell} u(x) = 0\big\}.$$

We recall that, apart from the eigenvalue $\lambda = 0$, the operators $(B, D_V(B))$ and $(B, D_0(B))$ have the same eigenvalues (λ_n) and the same eigenfunctions (ϕ_n), which we assume to be normalized, *i.e.* $\int_0^\ell |\phi_n|^2/\alpha = 1$. Using the first equality in (2.11) with $p \equiv 1$ and the equation $\alpha\phi_n'' + r\phi_n = \lambda_n\phi_n$ we obtain

$$\int_0^\ell |\phi_n'|^2 dx = -\int_0^\ell \phi_n\phi_n'' dx \le \int_0^\ell (|\lambda_n| + |r|)\frac{\phi_n^2}{\alpha}dx \le (|\lambda_n| + \|r\|_\infty).$$

Taking into account that $\phi_n(0) = \phi_n(\ell) = 0$, Hölder's inequality gives

$$\begin{aligned} |\phi_n(x)| &= \Big|\int_0^x \phi_n'(s)ds\Big| \le \sqrt{x}\Big(\int_0^x |\phi_n'(s)|^2 ds\Big)^{1/2}, \\ |\phi_n(x)| &= \Big|\int_x^\ell \phi_n'(s)ds\Big| \le \sqrt{\ell - x}\Big(\int_x^\ell |\phi_n'(s)|^2 ds\Big)^{1/2}, \end{aligned}$$

and the following pointwise estimate can be deduced:

$$|\phi_n(x)| \le \sqrt{2/\ell}\,(|\lambda_n| + \|r\|_\infty)^{1/2}\,[x(\ell - x)]^{1/2}. \tag{6.1}$$

In the next lemma we show that, under stronger hypotheses on α, the above estimate can be improved. We denote by $[z]$ the integer part of z.

Lemma 6.1. *Assume that for some $\varepsilon > 0$*

$$M = \int_0^\ell \frac{x^{1-\varepsilon}(\ell - x)^{1-\varepsilon}}{\alpha(x)}dx < +\infty; \tag{6.2}$$

then ϕ_n belongs to the Sobolev space $W^{2,1}(0,\ell)$ and there exists a constant C, depending only on ε, M, such that for $N = \big[\frac{1}{2\varepsilon}\big]$ the following inequalities hold:

(i) $|\phi_n(x)| \le C(|\lambda_n| + \|r\|_\infty)^{N+3/2} x(1-x), \quad x \in [0,\ell]$;

(ii) $|\phi_n'(x)| \le C(|\lambda_n| + \|r\|_\infty)^{N+3/2}, \quad x \in [0,\ell]$;

(iii) $\displaystyle\int_0^\ell |\phi_n''(x)| dx \le C(|\lambda_n| + \|r\|_\infty)^{N+3/2}$;

(iv) $\displaystyle\int_0^\ell \frac{|\phi_n(x)|}{\alpha(x)}dx \le C(|\lambda_n| + \|r\|_\infty)^{N+1/2}$.

Proof. For simplicity's sake, we write ϕ and λ instead of ϕ_n and λ_n, respectively. Let us prove that if for some $\gamma \in (0,1)$ we have $|\phi(x)| \le c_\gamma[x(\ell-x)]^\gamma$ for any $x \in [0,\ell]$, then if $\gamma + \varepsilon \le 1$

$$|\phi(x)| \le (c_\gamma/\ell)(|\lambda| + \|r\|_\infty)[x(\ell-x)]^{\gamma+\varepsilon} \qquad \text{for any } x \in [0,\ell]. \tag{6.3}$$

For, write $\phi(x) = \int_0^\ell \Gamma(x,s)\phi''(s)ds$, where

$$\Gamma(x,s) = -\frac{1}{\ell}\begin{cases} s(\ell-x) & 0 \le s \le x \le \ell \\ x(\ell-s) & 0 \le x \le s \le \ell \end{cases}$$

is the Green's function of the operator u'' with Dirichlet boundary conditions. Hence for $0 \le x \le \ell/2$:

$$\begin{aligned} |\phi(x)| &\le \int_0^\ell \left|\Gamma(x,s)\frac{[\lambda - r(s)]\phi(s)}{\alpha(s)}\right|ds \le c_\gamma(|\lambda| + \|r\|_\infty)\int_0^\ell \frac{|\Gamma(x,s)|}{\alpha(s)}[s(\ell-s)]^\gamma ds \\ &\le \frac{c_\gamma}{\ell}(|\lambda| + \|r\|_\infty)\left[\int_0^x \frac{s(\ell-x)}{\alpha(s)}[s(\ell-s)]^\gamma ds + \int_x^\ell \frac{x(\ell-s)}{\alpha(s)}[s(\ell-s)]^\gamma ds\right] \\ &\le \frac{c_\gamma}{\ell}(|\lambda| + \|r\|_\infty)x^{\gamma+\varepsilon}\left[\int_0^x \frac{s^{1-\varepsilon}(\ell-s)^{1+\gamma}}{\alpha(s)}ds + \int_x^\ell \frac{s^{1-\varepsilon}(\ell-s)^{1+\gamma}}{\alpha(s)}ds\right] \\ &\le \frac{c_\gamma}{\ell}(\ell/2)^{\gamma+\varepsilon}(|\lambda| + \|r\|_\infty)Mx^{\gamma+\varepsilon}. \end{aligned}$$

Notice that the above estimate relies on the inequality $xs^\gamma \le x^{\gamma+\varepsilon}s^{1-\varepsilon}$ for $x \le s \le \ell$, which is true if $\gamma + \varepsilon \le 1$.

Arguing in the same way for $\ell/2 \le x \le \ell$, we get $|\phi(x)| \le (c_\gamma/\ell)(\ell/2)^{\gamma+\varepsilon}(|\lambda| + \|r\|_\infty)M(\ell-x)^{\gamma+\varepsilon}$ and finally

$$|\phi(x)| \le \frac{c_\gamma}{\ell}(|\lambda| + \|r\|_\infty)M[x(\ell-x)]^{\gamma+\varepsilon}.$$

We know that, according (6.1), we can start with $\gamma = 1/2$ and $c_{1/2} = \sqrt{2/\ell}(|\lambda| + \|r\|_\infty)^{1/2}$, and we may iterate the above procedure until $N\varepsilon + 1/2 \le 1 < (N+1)\varepsilon + 1/2$, *i.e.* $N = \left[\frac{1}{2\varepsilon}\right]$ times. At this point, we have

$$|\phi(x)| \le C'(|\lambda| + \|r\|_\infty)^{N+1/2}[x(\ell-x)]^{N\varepsilon+1/2}, \qquad \text{with } C' = \frac{M}{\ell^N}.$$

Since $[x(\ell-x)]^{N\varepsilon+1/2} \le C''[x(\ell-x)]^{1-\varepsilon}$, we have

$$\int_0^\ell \frac{|\phi(x)|}{\alpha(x)}dx \le C'C''(|\lambda| + \|r\|_\infty)^{N+1/2}\int_0^\ell \frac{[x(\ell-x)]^{1-\varepsilon}}{\alpha(x)}dx \le C(|\lambda| + \|r\|_\infty)^{N+1/2}$$

and (iv) is proved. Since $\alpha\phi'' = (\lambda - r)\phi$, the above estimate implies (iii) and shows that $\phi \in W^{2,1}(0,\ell)$, whence $\phi \in C^1([0,\ell])$. Since $\phi(0) = \phi(\ell) = 0$, by Rolle's Theorem,

$\phi'(x_0) = 0$ for some $x_0 \in (0, \ell)$, and

$$|\phi'(x)| = \Big|\int_{x_0}^x \phi''(s)ds\Big| \le \int_0^\ell |\phi''(s)|ds \le C(|\lambda| + \|r\|_\infty)^{N+3/2} \qquad \forall\, x \in [0, \ell].$$

Finally, assertion (i) immediately follows from the inequality above, again by integration. □

Let us see how the estimates in the preceding lemma read for the complete operator $\mathcal{A}u = \alpha u'' + \beta u' + ru$ in the interval (a, b).

Theorem 6.2. *Let, as usual, $z(x) = \int_a^x W(s)ds$, and assume that for some $\varepsilon > 0$*

$$L = \int_a^b \frac{z(x)^{1-\varepsilon}[z(b) - z(x)]^{1-\varepsilon}}{\alpha(x)W(x)} dx < +\infty. \tag{6.4}$$

Let (λ_n) and (v_n) be the sequences of the non-zero eigenvalues and the corresponding eigenfunctions of the operator $(\mathcal{A}, D_V)$. Assume that $\displaystyle\int_a^b \frac{|v_n|^2}{\alpha W} = 1$ for every natural n. Then, there exists a constant C, depending only on ε, L, such that for $N = \left[\frac{1}{2\varepsilon}\right]$ the following inequalities hold:

(i) $|v_n(x)| \le C(|\lambda_n| + \|r\|_\infty)^{N+3/2} z(x)[z(b) - z(x)], \quad x \in [a, b]$;

(ii) $\dfrac{|v_n'(x)|}{W(x)} \le C(|\lambda_n| + \|r\|_\infty)^{N+3/2}, \quad x \in [a, b]$;

(iii) $\displaystyle\int_a^b \frac{|\alpha v_n''(x) + \beta v_n'(x)|}{\alpha(x)W(x)} dx \le C(|\lambda_n| + \|r\|_\infty)^{N+3/2}$;

(iv) $\displaystyle\int_a^b \frac{|v_n(x)|}{\alpha(x)W(x)} dx \le C(|\lambda_n| + \|r\|_\infty)^{N+1/2}$.

Proof. The change of variable $z(x) = \int_a^x W(s)ds$ maps the interval (a, b) into $(0, \ell)$, with $\ell = z(b)$. Setting $\phi(z) = u(x)$, an elementary computation shows that $\mathcal{A}u$ translates into the operator $B\phi = \tilde\alpha\phi_{zz} + \tilde r\phi_z$, where $\tilde\alpha = \alpha W^2$, $\tilde r = r$ (as functions of the new independent variable z). Moreover, the boundary conditions $\mathcal{A}u = 0$ translate into $B\phi = 0$, *i.e.* the domain D_V changes into $D_V(B)$, and hypothesis (6.4) translates into (6.2). Hence, Lemma 6.1 applies to the operator B and all the assertions in the present theorem follow from the corresponding ones in Lemma 6.1, coming back to the variable $x \in (a, b)$. □

We apply now the preceding theorem to show that the semigroup $(T(t))_{t\ge 0}$ generated by $(\mathcal{A}, D_V)$ in $C([a, b])$ is differentiable. Notice that the change of variable used in the proof of the above theorem allows to deduce the inequality

$$\|v_n\|_\infty \le C(|\lambda_n| + \|r\|_\infty)^{1/2} \tag{6.5}$$

from (6.1), under the only hypothesis $R \in L^1(a, b)$.

Theorem 6.3. *Under the hypothesis* (6.4)*, the semigroup* $(T(t))_{t\geq 0}$ *generated by* $(\mathcal{A}, D_V)$ *in* $C([a,b])$ *is differentiable; moreover, the map* $t \mapsto T(t)$ *is real analytic from* $]0,+\infty]$ *to* $\mathcal{B}\big(C([a,b])\big)$.

Proof. Since $\mathcal{A}\mathbf{1} = \mathcal{A}z = 0$, it follows $T(t)\mathbf{1} = \mathbf{1}$, $T(t)z = z$ and hence it is sufficient to prove the statement in $C_0([a,b])$. Let $(S(t))_{t\geq 0}$ be the semigroup generated by $(\mathcal{A}, D_0)$ in $L^2_{(\alpha W)^{-1}}(a,b)$, where D_0 is given by (2.7). For $Re\lambda > 0$, the resolvents of $(\mathcal{A}, D_V)$ and $(\mathcal{A}, D_0)$ coincide on $C_0([a,b]) \cap L^2_{(\alpha W)^{-1}}(a,b)$ (see the proof of Theorem 5.3(ii)) and hence $T(t)v = S(t)v$ for every $t \geq 0$, $v \in C_0([a,b]) \cap L^2_{(\alpha W)^{-1}}(a,b)$. Since the operator $(\mathcal{A}, D_0)$ is self-adjoint we have for $v \in L^2_{(\alpha W)^{-1}}(a,b)$

$$S(t)v = \sum_{n=1}^{\infty} e^{\lambda_n t}\langle v, v_n\rangle_{(\alpha W)^{-1}} v_n$$

where (λ_n) are the eigenvalues and (v_n) the (normalized) eigenfunctions of $(\mathcal{A}, D_0)$. If $v \in C_0([a,b])$ and $Re\ t > 0$, from Theorem 6.2(iv) and (6.5) we infer

$$\begin{aligned}\sum_{n=1}^{\infty} |e^{\lambda_n t}|\ |\langle v, v_n\rangle_{(\alpha W)^{-1}}|\ \|v_n\|_\infty &\leq \|v\|_\infty \sum_{n=1}^{\infty} e^{\lambda_n Re\ t}\Big(\int_a^b \frac{|v_n|}{\alpha W}\Big)\|v_n\|_\infty \\ &\leq C^2\|v\|_\infty \sum_{n=1}^{\infty} e^{\lambda_n Re\ t}(|\lambda_n| + \|r\|_\infty)^{N+1}.\end{aligned}$$

The last series is uniformly convergent in $\{Re\ t \geq \delta\}$ for every $\delta > 0$, since $|\lambda_n| \to \infty$ at least linearly (recall that $|\lambda_n| \leq |\lambda_{n+1}|$ and that $\sum_n \frac{1}{|\lambda_n|} < \infty$); hence we deduce that for $t > 0$

$$T(t)v = \sum_{n=1}^{\infty} e^{\lambda_n t}\langle v, v_n\rangle_{(\alpha W)^{-1}} v_n \tag{6.6}$$

for every $v \in C_0([a,b]) \cap L^2_{(\alpha W)^{-1}}(a,b)$, the series being convergent in the norm of $C_0([a,b])$. The density of $C_0([a,b]) \cap L^2_{(\alpha W)^{-1}}(a,b)$ in $C_0([a,b])$ implies that (6.6) holds for every $v \in C_0([a,b])$. Finally, the uniform convergence of (6.6) in $\{Re\ t \geq \delta\}$ for every $\delta > 0$ shows that the map $t \mapsto T(t)$ is real analytic from $]0,+\infty]$ to $\mathcal{B}\big(C([a,b])\big)$ and, in particular, that the semigroup is differentiable. □

Remark 6.4. (i) The statements of Theorem 6.2 are false, in general, if one only assumes that $R \in L^1(a,b)$, *i.e.* (6.4) with $\varepsilon = 0$. For, consider the operator $\mathcal{A}u = u'' + x^3u'$ in $(-\infty,+\infty)$ and observe that $W(x) = e^{-x^4/4}$. Let $u \in D_V$ be such that $\mathcal{A}u = \lambda u$. Writing $u(x) = v(x)e^{-x^4/8}$ we see that v satisfies the equation

$$v'' = (-\frac{x^6}{4} + \frac{x^3}{2} + \frac{3x^2}{2} + \lambda)v.$$

By [27, Theorem 2.2 p.196] we obtain $v(x) \approx x^{-3/2}$ as $x \to +\infty$ and hence $u(x) \approx x^{-3/2}e^{-x^4/8}$ as $x \to +\infty$. Therefore (i), (iii), (iv) are false.

We conjecture that also the conclusion of Theorem 6.3 fails for this operator.

(ii) A particular case of Theorem 6.3 is stated in [3] for the operator $\alpha u'' + \beta u'$ in $[0,1]$, under assumptions equivalent to (6.4) with $\varepsilon = 1/2$. The assertion is deduced from the asymptotics

$$-\frac{\lambda_n}{\pi n^2} \to \int_0^1 \frac{1}{\sqrt{\alpha}}$$

which is quoted from [26]. However, the above asympotics is proved in [26] only in the *limit circle* case that, in this situation, reads $\displaystyle\int_0^1 \frac{1}{\alpha} < \infty$, a condition stronger than the assumptions.

Examples. Let us present some concrete examples of application of the results of Sections 5 and 6.

1. Consider the operator with first-order degeneracy in the unit interval given by

$$\mathcal{A}_1 u = mu'' + \frac{b(x)}{x(1-x)} u' + ru$$

in the space $C([0,1])$; the functions m, b and r are real-valued, continuous on $[0,1]$ and m is strictly positive; moreover we assume that b satisfies a Hölder condition at the endpoints. The function W reads

$$W(x) = \exp\Big\{-\int_{1/2}^{x} \frac{b(s)}{s(1-s)} ds\Big\}$$

hence $Q \in L^1(0,1)$ if and only if $b(0) > -1$ and $b(1) < 1$, while $R \in L^1(0,1)$ if and only if $b(0) < 1$ and $b(1) > -1$.

Assuming $Q \in L^1(0,1)$, Theorem 5.3 holds for $(\mathcal{A}_1, D_N(\mathcal{A}_1))$ with

$$D_N(\mathcal{A}_1) = \Big\{u \in C([0,1]) \cap C^2(0,1) : \ \mathcal{A}_1 u \in C([0,1]), \ \lim_{x\to 0,1} \frac{u'(x)}{W(x)} = 0\Big\}.$$

We point out that the above boundary conditions are equivalent to the classical homogeneuos Neumann boundary conditions $u'(0) = u'(1) = 0$, that is

$$u \in D_N(\mathcal{A}_1) \quad \iff \quad u \in C^2([0,1]) \text{ and } u'(0) = u'(1) = 0.$$

For, assume $u \in D_N(\mathcal{A}_1)$, and let $f = m^{-1}(\mathcal{A}_1 u - ru)$; then from the expression of W and

$$u'(x) = W(x) \int_0^x \frac{f(s)}{W(s)} ds$$

we obtain $u'(0) = 0$ and

$$\lim_{x\to 0} \frac{u'(x)}{x} = \begin{cases} f(0) & \text{if } -1 < b(0) \le 0 \\[2ex] \dfrac{f(0)}{b(0)+1} & \text{if } b(0) \ge 0 \end{cases}$$

hence u is twice differentiable at $x = 0$. Arguing in the same way for $x = 1$ and observing that $\mathcal{A}_1 u$ is continuous, we obtain that $u \in C^2([0,1])$. The converse implication easily follows since $x(1-x)W(x) \to 0$ as $x \to 0, 1$. As regards the regularity of the generated semigroup, we refer to [1], where the analyticity in $C([0,1])$ is shown.

If $R \in L^1(0,1)$ we impose Ventcel conditions, *i.e.* we define

$$D_V(\mathcal{A}_1) = \Big\{ u \in C([0,1]) \cap C^2(0,1) : \ \mathcal{A}_1 u \in C_0([0,1]) \Big\}$$

and Theorem 5.3(ii) holds. In this case Theorem 6.2 applies to the eigenfunctions. The differentiability of the generated semigroup then follows from Theorem 6.3; however, this last result actually follows from [4] and [24], where the analyticity is proved with methods inspired to [1].

2. Let

$$\mathcal{A}_2 u = q\big[y(1-y)u'' + gu'\big] + ru,$$

with the same hypotheses on the coefficients as in Example 1. The change of variable $y = (1-\cos \pi x)/2$ transforms the operator $\mathcal{A}_2$ into $\mathcal{A}_1$. Hence, the above discussion can be repeated and $Q \in L^1(0,1)$ if and only if $g(0) > 0$ and $g(1) < 0$, while $R \in L^1(0,1)$ if and only if $g(0) < 1$ and $g(1) > -1$. Accordingly, we consider $(\mathcal{A}_2, D_N(\mathcal{A}_2))$ or $(\mathcal{A}_2, D_V(\mathcal{A}_2))$ and we can apply Theorem 5.3. We point out also that, using the above change of variable and [24, §3], we can write the domain $D_N(\mathcal{A}_2)$ as follows:

$$u \in D_N(\mathcal{A}_2) \quad \Longleftrightarrow \quad u \in C^1([0,1]) \cap C^2(0,1), \ \lim_{y \to 0,1} y(1-y)u''(y) = 0.$$

Theorems 6.2 and 6.3 apply to $(\mathcal{A}_2, D_V(\mathcal{A}_2))$.

3. Fix p, q, r, s, with $0 \le p, q < 2$ and $r, s \in \mathbf{R}$ arbitrary and define

$$\begin{aligned} B_1 u &= \frac{d}{dx}[m x^p |\log x|^r (1-x)^q |\log(1-x)|^s u'], \\ B_2 u &= m x^p |\log x|^r (1-x)^q |\log(1-x)|^s u''; \end{aligned}$$

with $\inf_{[0,1]} m > 0$. Then, Theorem 5.3 applies to $(B_1, D_N(B_1))$ and $(B_2, D_V(B_2))$, and the trace formulas read

$$-\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \int_0^1 \frac{x^{1-p}|\log x|^r (1-x)^{1-q} |\log(1-x)|^s}{m(x)} dx.$$

In particular, if $r = s = 0$ and $m \equiv 1$, the above integral equals $\mathbf{B}(2-p, 2-q)$, where $\mathbf{B}$ is the Euler Beta function.

As regards the regularity of eigenfunctions and semigroups, Theorems 6.2 and 6.3 apply to $(B_2, D_V(B_2))$.

4. Consider the operator $Lu = u'' + bx|x|^{p-1}u'$ in $\mathbf{R}$, with real $b \neq 0$ and $p > 1$. If $b < 0$ then $Q \in L^1(\mathbf{R})$ and Theorem 5.3(i) applies to $(L, D_N(L))$. To see that Q is summable, observe that $W(x) = \exp\{-b\frac{|x|^{p+1}}{p+1}\}$ and $Q(x) = (1/W(x))\int_0^x W(\xi)d\xi$. Then,

$$\int_0^x e^{-b\frac{|\xi|^{p+1}}{p+1}} d\xi \approx C|x|^{-p} e^{-b\frac{|x|^{p+1}}{p+1}}, \qquad \text{as } x \to \pm\infty \tag{6.7}$$

for a suitable $C > 0$ and the summability of Q follows, as $p > 1$. The semigroup generated by $(L, D_N(L))$ turns out to be differentiable in $C([-\infty, +\infty])$; this result can be deduced from [15], transforming L into a Schrödinger operator, as described at the beginning of this section.

Arguing as in (6.7), it can be checked that $R \in L^1(\mathbf{R})$ in the case $b > 0$. Hence, Theorem 5.3(ii) applies to $(L, D_V(L))$. The regularity of the generated semigroup cannot be proved by our methods (see Remark 6.4(i)).

5. Consider now the case $p = 1$ of example 4, *i.e.* the Ornstein-Uhlenbeck operator $Lu = \frac{1}{2}u'' + bxu'$ in $\mathbf{R}$.

In this case both Q and R are not summable, hence the endpoints are natural boundaries and L, endowed with the maximal domain $D_M(L)$, is the generator of a strongly continuous semigroup in $C([-\infty, +\infty])$. By Lemma 5.1 the resolvent operator is not compact. The semigroup is neither differentiable, nor even norm-continuous (see [10] and [25]).

The operator L can be also regarded in the weighted Hilbert space L^2_ϱ, with $\varrho(x) = \exp(bx^2)$. According to Section 2, we endow L with the Neumann domain D_1 if $b < 0$, and the Dirichlet domain D_0 if $b > 0$. Theorems 3.3 and 4.2(iii) show that in both cases the resolvent operators are Hilbert-Schmidt. The conditions in the above quoted theorems can be checked as in example 4. For instance, let us check (3.4) if $b < 0$, *i.e.* the summability of $\mu|w|^2\varrho$ near $-\infty$. In this situation $\varrho(x) = e^{b|x|^2}$ and we have for $x \to -\infty$

$$w(x) = \int_0^x e^{-b|\xi|^2} d\xi \approx C|x|^{-1}e^{-b|x|^2}, \qquad \mu(x) = \int_{-\infty}^x e^{b|\xi|^2} d\xi \approx C|x|^{-1}e^{b|x|^2},$$

whence $\mu|w|^2\varrho \approx c|x|^{-3}$, as $x \to -\infty$.

The spectral properties of L in $C([-\infty, +\infty])$ and in L^2_ϱ are very different. In fact the spectrum of L in L^2_ϱ is a discrete subset of $]-\infty, 0]$ while it is the half-plane $\{Re\ \lambda \le 0\}$ in $C([-\infty, +\infty])$. If $b = -1$ this fact is consequence of Section 4.3 of [12] since, with the methods outlined at the beginning of this section, L translates into the harmonic oscillator operator $Hf = 1/2[f'' - x^2 f + f]$, but the result is true for every b.

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