LONG TIME BEHAVIOR OF HEAT KERNELS OF OPERATORS WITH UNBOUNDED DRIFT TERMS

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ABSTRACT. Given a second-order elliptic operator on $\mathbb{R}^d$, with bounded diffusion coefficients and unbounded drift, which is the generator of a strongly continuous semigroup on $L^2(\mathbb{R}^d)$ represented by an integral, we study the time behavior of the integral kernel and prove estimates on its decay at infinity. If the diffusion coefficients are symmetric, a local lower estimate is also proved.

1. INTRODUCTION

We consider the differential operator

$$A = -\mathrm{div}(a(x)\nabla) + b\cdot\nabla = -\sum_{j,k=1}^{d} \partial_k(a_{kj}\partial_j) + \sum_{k=1}^{d} b_k\partial_k, \tag{1}$$

on $L^2(\mathbb{R}^d)$, under the following assumptions on the coefficients:

$$a_{kj} \in L^\infty(\mathbb{R}^d), \qquad \exists\, \nu > 0 : \sum_{j,k=1}^{d} a_{kj}(x)\xi_j\xi_k \geq \nu|\xi|^2 \quad \forall\, x,\, \xi \in \mathbb{R}^d \tag{2}$$

$$b = (b_j) \in C^1(\mathbb{R}^d, \mathbb{R}^d) \text{ with } \mathrm{div}\, b \leq w, \tag{3}$$

where w is a constant in $\mathbb{R}$. Notice that the drift coefficients b_j are not assumed to be bounded, the condition on the divergence is only a one-side bound and the diffusion coefficients are not assumed to be symmetric. In [1], [16] (see also [12]), under stronger regularity hypotheses on the coefficients, it is proved that A, endowed with its maximal domain, generates a semigroup in $L^p(\mathbb{R}^d)$ and $C_b(\mathbb{R}^d)$. Here, following basically the same ideas but relying upon the sesquilinear form method rather than Sobolev estimates on the resolvent equation, we obtain a realization of A which is (minus) the generator of a strongly continuous semigroup on $L^2(\mathbb{R}^d)$ (see the beginning of the next section for more details). We also denote by A this realization and by $(e^{-tA})_{t\geq 0}$ the corresponding semigroup on $L^2(\mathbb{R}^d)$. We shall see that e^{-tA} is given by an integral kernel $p_t(x,y)$, usually referred to as the heat kernel or the transition probability, taking into account the stochastic counterpart of the diffusion process described by A. The present paper deals with

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estimates of $p_t(x,y)$. It is a well known fact that if the drift term b is bounded then $p_t(x,y)$ is bounded with respect to (x,y). One even has a Gaussian upper bound

$$p_t(x,y) \le Ct^{-d/2}e^{\delta t}e^{-c\frac{|x-y|^2}{t}}$$

for some positive constants C, c and δ. See for example Chapter 6 in [17] and [1] and the references therein. The latter paper deals with some cases with unbounded drift terms. Note also that Gaussian lower bounds (again when b is bounded) are proved in [7].
Our aim here is to study short and long time upper estimates of $p_t(x,y)$ as well as lower bounds for unbounded b. One of the main motivations is to prove a precise long time behaviour of $p_t(x,y)$. Our main results are the following.

Theorem 1. *Assume that* (2) *and* (3) *are satisfied.*
1) There exists a positive constant M_0 depending only on the dimension d and the ellipticity constant ν such that, if $w \neq 0$,

$$0 \le p_t(x,y) \le M_0 e^{\frac{w}{2}t}\left(\frac{d}{|w|}\right)^{-d/2}\left(\cosh\left(\frac{wt}{d}\right)-1\right)^{-d/4}, \tag{4}$$

and if $w=0$,

$$0 \le p_t(x,y) \le M_0 t^{-d/2}. \tag{5}$$

These estimates hold for all $t>0$ and $x,y \in \mathbb{R}^d$.
2) Suppose that $w \neq 0$. Then there exists a constant M_1 depending only on the dimension d and the ellipticity constant ν such that, for all $t > \frac{2}{|w|}$,

$$0 \le p_t(x,y) \le M_1 e^{t\min(w,0)}. \tag{6}$$

Note that in all cases ($w=0$, $w<0$ or $w>0$) the above estimates imply the bound $p_t(x,y) \le Ct^{-d/2}$ for small $t>0$. If $w<0$, (6) gives an exponential decay for long t of the heat kernel which is of course much better than the usual polynomial decay.
Denote by $\|T\|_{1,\infty}$ the norm of an operator T from L^1 to L^∞. Suppose now that $w>0$ and apply Proposition 1.3 in [1] with $V=w$ and $c=0$. We obtain

$$\|e^{-tA}\|_{1,\infty} = \|e^{-t(A+w)}e^{wt}\|_{1,\infty} \le Ct^{-d/2}e^{wt}.$$

In the case where $w<0$, we apply the same proposition with $V=0$ and obtain

$$\|e^{-tA}\|_{1,\infty} \le Ct^{-d/2}.$$

In both cases, these two estimates are less sharp than those given in our theorem. If one considers for example the Ornstein-Uhlenbeck operator

$A = -\Delta - x\nabla$ on $L^2(\mathbb{R})$, the precise expression of the kernel gives (see (22) below)

$$\sup_{x,y} p_t(x,y) = C\left(\frac{2}{e^{2t}-1}\right)^{1/2}.$$

This means that $p_t(x,y)$ behaves like e^{-t} for large t. This is exactly what we obtain from (6) since div $b = -1$. This shows that the long time behavior given in the previous theorem is sharp.

We also mention that our proof gives explicit constants M_0 and M_1 in terms of the dimension d, the ellipticity constant ν and the constant c_0 in Nash's inequality, see (16) below, (which in turn depends only on d).

Note that short time estimates (say $t \in (0,1]$) are obtained in [15], [8] and related results are proved in [10], [22] and [3]. In these works the hypotheses on the drift term are different from ours. Growth conditions on b and its derivatives as well as smoothness of the coefficients a_{kj} are assumed in [15] and [8]. In [22] and [3] b is assumed to be in a certain Kato class. There are many other interesting works on operators with drift terms dealing with different questions. We mention for example [2], [11], [21] dealing with regularity properties and existence of an invariant measure, [19], [8] and the references therein for a description of the domain on L^p, and [13], [14] for the functional calculus and harmonic analysis of such operators.
For operators without a drift term (i.e., $b = 0$), precise estimates of the heat kernel as well as their consequences to spectral theory and harmonic analysis are known. The literature in this case is vast, we refer to the monographs [5], [17], [20] and the references therein.

Concerning lower estimates of heat kernels of operators with drift terms less is known. As mentioned above, a Gaussian lower bound is proved in [7] provided b is bounded. Here we prove the following result.

Theorem 2. *Assume that* (2) *and* (3) *are satisfied, and that the diffusion matrix is symmetric, i.e.,* $a_{kj} = a_{jk}$ *for all* j, k. *Then there exists a constant* $C > 0$ *such that*

$$\liminf_{t\to 0} t^{d/2} p_t(x,x) \geq C$$

for all $x \in \mathbb{R}^d$.

Notation. In this paper we will use the following notation. For a linear operator T, $\|T\|_{\mathcal{L}(L^p)} := \sup\{\|Tf\|_p, \|f\|_p \leq 1\}$ and $\|T\|_{p,q} := \sup\{\|Tf\|_q, \|f\|_p \leq 1\}$ are the $L^p - L^p$ and $L^p - L^q$ norms, respectively. For any measurable real-valued function u, $u^+ := \max(u,0)$, $u^- := \max(-u,0)$, $1 \wedge u = \min(u,1)$. The spaces $W^{1,p}$ and $W_0^{1,p}$ are the usual Sobolev spaces.

2. Proof of the upper bounds

We first explain what is the realization of the operator A defined in (1) which generates a strongly continuous semigroup $(e^{-tA})_{t\geq 0}$ on $L^2(\mathbb{R}^d)$. The ideas we explain here are taken from [1]. For $n \geq 1$, we denote by B_n the open Euclidean ball of $\mathbb{R}^d$ of centre 0 and radius n and define on $L^2(B_n)$ the sesquilinear form

$$\mathfrak{a}_n(u,v) = \int_{B_n} a(\nabla u, \nabla v)\, dx + \int_{B_n} b\cdot \nabla u v\, dx,\ \ D(\mathfrak{a}_n) = W_0^{1,2}(B_n), \tag{7}$$

where we have set

$$a(\xi,\eta) = \sum_{j,k=1}^{d} a_{kj}\xi_j\eta_k, \qquad \xi,\eta \in \mathbb{R}^d. \tag{8}$$

Each form $\mathfrak{a}_n$ is bounded from below (by $-\frac{w}{2}$). This follows from the obvious equality

$$\int_{B_n} b\cdot \nabla u\, u\, dx = \frac{1}{2}\int_{B_n} b\cdot \nabla |u|^2 = -\frac{1}{2}\int_{B_n} \operatorname{div} b|u|^2 dx, \quad u \in W_0^{1,2}(B_n)$$

and condition (3). In addition, since $\mathfrak{a}_n$ has bounded coefficients, it is clear that by adding a positive constant if necessary, we obtain a positive and closed form. We can then associate with $\mathfrak{a}_n$ an operator A_n such that $-A_n$ generates a strongly continuous semigroup $(e^{-tA_n})_{t\geq 0}$ on $L^2(B_n)$. It follows from Corollary 4.3 in [17] that $(e^{-tA_n})_{t\geq 0}$ is a positive semigroup and from Proposition 4.23 in [17] that $(e^{-tA_n})_{t\geq 0}$ is dominated by $(e^{-tA_{n+1}})_{t\geq 0}$, that is $e^{-tA_n}f \leq e^{-tA_{n+1}}f$ in B_n for every non-negative $f \in L^2(\mathbb{R}^d)$. Since $\mathfrak{a}_n$ is bounded from below by $-\frac{w}{2}$ we have

$$\|e^{-tA_n}\|_{\mathcal{L}(L^2)} \leq e^{\frac{w}{2}t},\ t > 0. \tag{9}$$

Hence for non-negative $f \in L^2(\mathbb{R}^d)$, the sequence (defined for large n) $e^{-tA_n}f$ is non-decreasing and bounded in L^2. From this it follows that $e^{-tA_n}f$ converges as $n \to \infty$. It is not difficult to show that the limit of $e^{-tA_n}f$ allows to define a strongly continuous semigroup on $L^2(\mathbb{R}^d)$. We denote by $(e^{-tA})_{t\geq 0}$ this semigroup and by $-A$ its generator, formally given by (1). For further details, see [1].

Proposition 3. *Assume* (2) *and* (3). *Then*

$$\|e^{-tA}\|_{\mathcal{L}(L^2)} \leq e^{\frac{w}{2}t},\ t > 0.$$

Proof. As explained above, $e^{-tA}f$ is the limit in L^2 of $e^{-tA_n}f$. The proposition is then an immediate consequence of (9). □

Proposition 4. *Assume* (2) *and* (3). *Then the semigroup* $(e^{-tA})_{t\geq 0}$ *is positive and extends from* $L^2 \cap L^p$ *to a semigroup on* $L^p(\mathbb{R}^d)$ *for* $1 \leq p \leq \infty$ *which is strongly continuous for* $1 \leq p < \infty$. *In addition,*

$$\|e^{-tA}\|_{\mathcal{L}(L^p)} \leq e^{\frac{w}{p}t},\ t > 0 \tag{10}$$

for all $p \in [1, \infty]$.

Proof. The positivity of $(e^{-tA_n})_{t\geq 0}$ (which we explained above) and the strong convergence to $(e^{-tA})_{t\geq 0}$ imply the positivity of the latter semigroup.
By well-known properties of Sobolev functions, see e.g. Proposition 4.11 in [17], $1 \wedge u$ and $(u-1)^+$ belong to $W_0^{1,2}(B_n)$ for $u \in W_0^{1,2}(B_n)$ and

$$\nabla(1 \wedge u) = \chi_{\{u<1\}} \nabla u, \qquad \nabla(u-1)^+ = \chi_{\{u>1\}} \nabla u.$$

Therefore, for $0 \leq u \in W_0^{1,2}(B_n)$, $\mathfrak{a}_n(1 \wedge u, (u-1)^+) = 0$. By Beurling-Deny criterion [17], Theorem 2.15 (or Corollary 2.17) we conclude that $(e^{-tA_n})_{t\geq 0}$ is a sub-Markovian semigroup. This means that

$$\|e^{-tA_n}\|_{\mathcal{L}(L^\infty)} \leq 1, \ t > 0.$$

By the Riesz-Thorin interpolation theorem (see e.g. [5], p. 3) it follows from the latter estimate and (9) that for $p \in (2, \infty)$

$$\|e^{-tA_n}\|_{\mathcal{L}(L^p)} \leq e^{\frac{w}{p} t}, \ t > 0. \tag{11}$$

In order to prove the estimate for $p < 2$ we proceed by duality. We have for $0 \leq u \in W_0^{1,2}(B_n)$,

$$\begin{aligned} \mathfrak{a}_n((u-1)^+, 1 \wedge u) &= \int_{B_n} a(\nabla(u-1)^+, \nabla(1 \wedge u)) dx \\ &\quad + \int_{B_n} (1 \wedge u)\, b \cdot \nabla(u-1)^+ dx \\ &= -\int_{B_n} \nabla(1 \wedge u) \cdot b\, (u-1)^+ dx \\ &\quad - \int_{B_n} (\text{div } b)(1 \wedge u)(u-1)^+ dx \\ &\geq -w \int_{B_n} (1 \wedge u)(u-1)^+ dx. \end{aligned}$$

Using again Theorem 2.15 in [17], we obtain

$$\|e^{-wt} e^{-tA_n^*}\|_{\mathcal{L}(L^\infty)} \leq 1 \text{ for } t \geq 0, \tag{12}$$

where A^*, A_n^* are the adjoint operators on $\mathbb{R}^d$ and B_n formally given by

$$-\sum_{j,k=1}^{d} \partial_k(a_{jk}\partial_j) - \sum_{k=1}^{d} b_k \partial_k - \operatorname{div} b.$$

By duality, (12) gives

$$\|e^{-tA_n}\|_{\mathcal{L}(L^1)} \leq e^{wt} \text{ for } t \geq 0. \tag{13}$$

Again, by the Riesz-Thorin interpolation theorem and the L^2-estimate (9) we obtain (11) for $p \in [1, 2]$.

Since $e^{-tA_n}f$ converges to $e^{-tA}f$, (10) easily follows. Finally, the strong continuity of the semigroup $(e^{-tA})_{t\geq 0}$ on $L^p(\mathbb{R}^d)$, for $1 \leq p < \infty$, follows as in [17], pp. 56-57. □

Proof of Theorem 1. We proceed in five steps.
Step 1: $L^1 - L^2$ estimate. We assume that $b \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ is bounded with $\operatorname{div} b \leq w$. Then for all $t > 0$, we are going to prove that e^{-tA} is bounded from L^1 to L^2 with

$$\|e^{-tA}\|_{1,2} \leq Ce^{\frac{w}{2}t}\left(\frac{d}{2w}(1 - e^{\frac{-2wt}{d}})\right)^{-d/4} \tag{14}$$

if $w \neq 0$. If $w = 0$ we have for all $t > 0$

$$\|e^{-tA}\|_{1,2} \leq Ct^{-d/4}. \tag{15}$$

In both estimates C is a positive constant depending only on d, ν and c_0 for which Nash's inequality (16) holds.
We follow the same argument as [5] (p. 79) or [17] (p. 158) using the Nash inequality

$$\|u\|_2^{2+4/d} \leq c_0\|u\|_1^{4/d}\int_{\mathbb{R}^d}|\nabla u|^2dx, \quad u \in L^1(\mathbb{R}^d) \cap W^{1,2}(\mathbb{R}^d). \tag{16}$$

Since for $u \in W^{1,2}(\mathbb{R}^d)$

$$\mathfrak{a}(u,u) \geq \nu\int_{\mathbb{R}^d}|\nabla u|^2dx - \frac{w}{2}\int_{\mathbb{R}^d}|u|^2dx$$

we obtain from (16) that for $u \in L^1(\mathbb{R}^d) \cap W^{1,2}(\mathbb{R}^d)$

$$\|u\|_2^{2+4/d} \leq \frac{c_0}{\nu}\|u\|_1^{4/d}\left[\mathfrak{a}(u,u) + \frac{w}{2}\int_{\mathbb{R}^d}|u|^2dx\right]. \tag{17}$$

Fix $f \in L^1 \cap L^2$ and set $\varphi(t) := \|e^{-t(A+\frac{w}{2})}f\|_2^2$. We have from (17)

$$\begin{aligned}-\frac{d\varphi}{dt}(t) &= 2(\mathfrak{a} + \frac{w}{2})(e^{-t(A+\frac{w}{2})}f, e^{-t(A+\frac{w}{2})}f)\\ &\geq \frac{2\nu}{c_0}\varphi(t)^{1+2/d}\|e^{-t(A+\frac{w}{2})}f\|_1^{-4/d}.\end{aligned}$$

Using Proposition 4 with $p = 1$ yields for all $t > 0$

$$-\frac{d\varphi}{dt}(t) \geq \frac{2\nu}{c_0}\varphi(t)^{1+2/d}\|f\|_1^{-4/d}e^{-\frac{2wt}{d}}.$$

In other words

$$\frac{d}{dt}\left(\varphi(t)^{-2/d}\right) \geq \frac{4\nu}{dc_0}\|f\|_1^{-4/d}e^{-\frac{2wt}{d}}.$$

Hence

$$\varphi(t)^{-2/d} \geq \varphi(t)^{-2/d} - \varphi(0)^{-2/d} \geq \frac{4\nu}{dc_0}\|f\|_1^{-4/d}\int_0^t e^{-\frac{2ws}{d}}ds.$$

This gives

$$\varphi(t)^{-2/d} \geq \frac{4\nu}{dc_0} \|f\|_1^{-4/d} t \qquad \text{if } w = 0$$

and

$$\varphi(t)^{-2/d} \geq \frac{4\nu}{dc_0} \|f\|_1^{-4/d} \frac{d}{2w}(1 - e^{-\frac{2wt}{d}}) \qquad \text{if } w \neq 0.$$

This gives (15) and (14) and with $C = \left(\frac{dc_0}{4\nu}\right)^{d/4}$.

Step 2: $L^2 - L^\infty$ estimate. Assume that $b \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ is bounded with $\operatorname{div} b \leq w$. In order to estimate the $L^2 - L^\infty$ norm of e^{-tA} we argue by duality. For all $t > 0$, the adjoint semigroup e^{-tA^*} is bounded from L^1 to L^2 with

$$\|e^{-tA}\|_{2,\infty} = \|e^{-tA^*}\|_{1,2} \leq C e^{\frac{w}{2}t} \left(\frac{d}{2w} (e^{\frac{2wt}{d}} - 1) \right)^{-d/4} \tag{18}$$

if $w \neq 0$. If $w = 0$ we have for all $t > 0$

$$\|e^{-tA}\|_{2,\infty} = \|e^{-tA^*}\|_{1,2} \leq C t^{-d/4}. \tag{19}$$

Again, in both estimates C is a positive constant depending only on d, c_0, ν.
The proof is similar to the previous one. The adjoint operator A^* is associated with the form

$$\mathfrak{a}^*(u, v) = \mathfrak{a}(v, u), \ \ u, v \in W^{1,2}(\mathbb{R}^d).$$

In particular, $\mathfrak{a}^*(u, u) = \mathfrak{a}(u, u)$ and hence (17) holds with $\mathfrak{a}^*$. We fix $f \in L^1 \cap L^2$ and define $\varphi(t) := \|e^{-t(A^* + \frac{w}{2})} f\|_2^2$. We have as above

$$\begin{aligned} -\frac{d\varphi}{dt}(t) &= 2(\mathfrak{a}^* + \frac{w}{2})(e^{-t(A^*+\frac{w}{2})} f, e^{-t(A^*+\frac{w}{2})} f) \\ &\geq \frac{2\nu}{c_0} \varphi(t)^{1+2/d} \|e^{-t(A^*+\frac{w}{2})} f\|_1^{-4/d}. \end{aligned}$$

From Proposition 4 (with $p = \infty$) we have $\|e^{-tA^*} f\|_1 \leq \|f\|_1$ for all $t > 0$. Hence

$$-\frac{d\varphi}{dt}(t) \geq \frac{2\nu}{c_0} \varphi(t)^{1+2/d} \|f\|_1^{-4/d} e^{\frac{2wt}{d}}.$$

Integrating with respect to t, this gives the desired estimates again with $C = \left(\frac{dc_0}{4\nu}\right)^{d/4}$.

Step 3: $L^1 - L^\infty$ estimate. Assume that $b \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ is bounded with $\operatorname{div} b \leq w$. Then for all $t > 0$, the semigroup e^{-tA} is bounded from L^1 to L^∞ with

(20)

$$\|e^{-tA}\|_{1,\infty} \leq \left(\frac{dc_0}{4\nu}\right)^{d/2} 2^{-d/4} e^{\frac{w}{2}t} \left(\frac{d}{|w|}\right)^{-d/2} \left(\cosh\left(\frac{wt}{d}\right) - 1\right)^{-d/4}$$

if $w \neq 0$. If $w = 0$ we have for all $t > 0$

$$\|e^{-tA}\|_{1,\infty} \leq \left(\frac{dc_0}{4\nu}\right)^{d/2} t^{-d/2}. \tag{21}$$

This follows from the previous steps and the classical estimate

$$\|e^{-tA}\|_{1,\infty} \leq \|e^{-\frac{t}{2}A}\|_{1,2}\|e^{-\frac{t}{2}A}\|_{2,\infty}.$$

Step 4. Assume that $b \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ is bounded with $\operatorname{div} b \leq w$ for some $w \neq 0$.
Suppose that $w > 0$. For $t > \frac{2}{w}$, we have by the semigroup property

$$\|e^{-tA}\|_{1,\infty} \leq \|e^{-\frac{1}{w}A}\|_{1,2}\|e^{-(t-\frac{1}{w})A}\|_{2,\infty}.$$

Now we use the estimates in Steps 1 and 2 to obtain for $t > \frac{2}{w}$

$$\begin{aligned}\|e^{-tA}\|_{1,\infty} &\leq C^2(1-e^{-2/d})^{-d/4} e^{\frac{w}{2}t}\left(e^{2(t-\frac{1}{w})w/d}-1\right)^{-d/4} \\ &\leq C^2(1-e^{-2/d})^{-d/4}(e^{-2/d}-e^{-4/d})^{-d/4} = M_1.\end{aligned}$$

Suppose now that $w < 0$. For $t > \frac{2}{-w}$ we use

$$\|e^{-tA}\|_{1,\infty} \leq \|e^{-(t+\frac{1}{w})A}\|_{1,2}\|e^{\frac{1}{w}A}\|_{2,\infty}.$$

Applying again our previous $L^1 - L^2$ and $L^2 - L^\infty$ estimates, we obtain

$$\|e^{-tA}\|_{1,\infty} \leq C^2(1-e^{-2/d})^{-d/4}(e^{-2/d}-e^{-4/d})^{-d/4}e^{wt}.$$

Step 5. To finish the proof of the theorem it remains to show that the boundedness assumption of b is not needed in the results of the previous steps. In order to see this we can argue as follows. Consider as in the beginning of this section the sesquilinear forms $\mathfrak{a}_n(u, v)$ defined in (7). Propositions 3 and 4 hold for the corresponding semigroups $(e^{-tA_n})_{t\geq 0}$. We can now use the arguments in Steps 1 and 2 for A_n and obtain the same $L^1 - L^2$ estimates for the semigroups e^{-tA_n} and $e^{-tA_n^*}$ as we had for e^{-tA} and e^{-tA^*}. These estimates hold with the constant $C = \left(\frac{dc_0}{4\nu}\right)^{d/4}$. In particular, they are uniform with respect to n. Therefore, by the L^2-convergence $e^{-tA_n}f \to e^{-tA}f$ for every $f \in C_c^\infty(\mathbb{R}^d)$ (for the adjoint semigroup, we have at least the weak convergence $(e^{-tA_n^*}f, g) = (f, e^{-tA_n}g) \to (f, e^{-tA}g) = (e^{-tA^*}f, g)$) we obtain the same L^1-L^2 estimates for e^{-tA} and e^{-tA^*}. This gives L^1-L^∞ estimates as in (20) and (21). By Dunford-Pettis theorem, see e.g. [9, Section XI.1] the boundedness of e^{-tA} from $L^1(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$ implies that an integral kernel $p_t(x, y)$ exists such that the representation

$$e^{-tA}f(x) = \int_{\mathbb{R}^d} p_t(x, y)f(y)dy$$

holds. The equality

$$\|e^{-tA}\|_{1,\infty} = \sup_{x,y\in\mathbb{R}^d} p_t(x, y)$$

concludes the proof. □

In the following example we construct a smooth drift b such that the heat kernel of the operator $A = -D^2 + bD$ is unbounded. The corresponding semigroup acts on $C_b(\mathbb{R})$, the space of continuous and bounded functions on $\mathbb{R}$.

Example 5. We recall that the Ornstein-Uhlenbeck operator is

$$A = -D^2 + \omega x D,$$

where $\omega \in \mathbb{R}$. The operator $-A$ generates a semigroup in $L^p(\mathbb{R})$ and in $C_b(\mathbb{R})$ which is given by the kernel

$$p_t(x,y) = C \left(\frac{2\omega}{1-e^{-2\omega t}} \right)^{1/2} \exp\left\{ -\frac{\omega}{2(1-e^{-2\omega t})} |e^{-\omega t}x - y|^2 \right\}, \tag{22}$$

where $C = 1/2\sqrt{\pi}$. This is the Kolmogorov expression, see e.g. [4]. Fix $t = 1$ and note that

$$\sup_{x,y} p_1(x,y) \geq c\sqrt{\omega} \tag{23}$$

for $\omega \geq 1$, with c independent of ω.

Denote now by A_n the operator A with Dirichlet boundary conditions on $[-n,n]$. For $f \in L^2$ with bounded support, $e^{-tA}f = \lim_{n\to\infty} e^{-tA_n}f$ (in the L^2 sense) and hence it follows that the sequence of corresponding kernels $p_t^n(x,y)$ converges monotonically to $p_t(x,y)$, see [16, Theorem 4.4]. Using now (23) we obtain n, depending on ω, such that

$$\sup_{x,y\in[-n,n]} p_1^n(x,y) \geq (c/2)\sqrt{\omega}.$$

Changing x with $x-a$ one sees that for every $a \in \mathbb{R}$ the heat kernel $p_t^{a,\omega}$ of $-D^2 + b_{a,\omega}D$, where $b_{a,\omega}(x) = \omega(x-a)$, satisfies

$$\sup_{x,y\in[a-n,a+n]} p_1^{a,\omega}(x,y) \geq (c/2)\sqrt{\omega}.$$

Fix now $\omega_k = k$ and construct, using the above facts, a sequence of disjoint intervals I_k, affine drifts in I_k and operators $A_k = -D^2 + b_k D$ whose heat kernels satisfy

$$\sup_{x,y\in I_k} p_1^k(x,y) \geq (c/2)\sqrt{k}.$$

Consider a smooth function b such that $b = b_k$ in I_k. Denote by $p_t(x,y)$ the kernel of the minimal semigroup generated by $-D^2 + bD$, see [16]. The kernel $p_t(x,y)$ dominates p_t^k in I_k. In particular, for $t = 1$

$$\sup_{x,y\in I_k} p_1(x,y) \geq \sup_{x,y\in I_k} p_1^k(x,y) \geq \sup_{x,y\in I_k} (c/2)\sqrt{k}.$$

Since this is true for all k, $p_t(x,y)$ is unbounded for $t = 1$.

3. Proof of the lower bound

Proof of Theorem 2. Fix $x \in \mathbb{R}^d$, and set $\Omega = B(x,1)$.
Consider $A_\Omega := -\sum_{jk} \partial_k(a_{kj}\partial_j) + b\cdot\nabla$ on $L^2(\Omega)$ with Dirichlet boundary conditions. The operator A_Ω is defined via the form

$$\mathfrak{a}_\Omega(u,v) = \int_\Omega a(\nabla u, \nabla v)\,dx + \int_\Omega b\cdot\nabla u v\,dx, \qquad D(\mathfrak{a}_\Omega) = W_0^{1,2}(\Omega).$$

Fix n such that $\Omega \subset B_n$ and consider any non-negative function $f \in L^2(\mathbb{R}^d)$. Arguing as in the beginning of the previous section, we see that

$$0 \le e^{-tA_\Omega} f \le e^{-tA_n} f \le e^{-tA} f$$

in Ω. Therefore, if $p_t^\Omega(x,y)$ denotes the heat kernel of A_Ω, then

$$0 \le p_t^\Omega(x,y) \le p_t(x,y) \tag{24}$$

for all $t > 0$ and a.e. $x, y \in \Omega$.
Denote by H_Ω the operator $-\sum_{jk}\partial_k(a_{kj}\partial_j)$ with Dirichlet boundary conditions on $L^2(\Omega)$ and by $k_t^\Omega(x,y)$ the heat kernel of $-H_\Omega$. Then by Lemma 3.3.3 in [5], we obtain

$$k_t^\Omega(x,x) \ge Ct^{-d/2} \;\; \forall\, t \in (0,1], \tag{25}$$

where the constant C is independent of x.
Our task now is to obtain a lower estimate for the non-symmetric heat kernel p_t^Ω using the previous estimate for the symmetric one k_t^Ω. For this, we look for $L^1 - L^\infty$ estimates of the difference $e^{-tA_\Omega} - e^{-tH_\Omega}$. Since the operators A_Ω and H_Ω have the same domain, we can apply the variation of constants formula, see [18, Corollary 4.2.2], observing that $u(t) = e^{-tH_\Omega} f$ for f in the domain of H_Ω is a classical solution of $u_t = -A_\Omega u + b\cdot\nabla u$, $u(0) = f$, to obtain

$$e^{-tA_\Omega} - e^{-tH_\Omega} = -\int_0^t e^{-(t-s)A_\Omega} b\cdot\nabla e^{-sH_\Omega} ds$$

$$= -\int_0^{t/2} e^{-(t-s)A_\Omega} b\cdot\nabla e^{-sH_\Omega} ds - \int_{t/2}^t e^{-(t-s)A_\Omega} b\cdot\nabla e^{-sH_\Omega} ds. \tag{26}$$

Fix $p > 1$ close to 1. Since b is bounded on Ω the semigroup e^{-tA_Ω} is bounded from L^1 into L^∞ with bound $Ct^{-d/2}$ for $t \in (0,1]$ (see Steps 1 and 2 in the proof of Theorem 1. The constant C can be chosen independent of x). This and the Riesz-Thorin interpolation theorem imply

$$\begin{aligned}\|e^{-(t-s)A_\Omega} b\cdot\nabla e^{-sH_\Omega}\|_{1,\infty} &\le \|e^{-(t-s)A_\Omega}\|_{p,\infty}\|b\|_{L^\infty(\Omega)}\|\nabla e^{-sH_\Omega}\|_{1,p}\\ &\le C(t-s)^{-d/(2p)}\|b\|_{L^\infty(\Omega)}\|\nabla e^{-s/2H_\Omega}\|_{\mathcal{L}(L^p)}\|e^{-s/2H_\Omega}\|_{1,p}\\ &\le C_1\|b\|_{L^\infty(\Omega)}(t-s)^{-d/(2p)}s^{-1/2}s^{-\frac{d}{2}(1-1/p)}.\end{aligned}$$

Here we used the fact that for all $t > 0$

$$\|\nabla e^{-tH_\Omega}\|_{\mathcal{L}(L^p)} \le C_2 t^{-1/2}, \tag{27}$$

for some constant C_2. A reason for it is the fact that the Riesz transforms $\nabla H_\Omega^{-1/2}$ are bounded on $L^p(\Omega)$ for all $p \in (1,2]$ (see Section 7.7 in [17]). Indeed,

$$\begin{aligned}\|\nabla e^{-tH_\Omega}\|_{\mathcal{L}(L^p)} &= \|\nabla H_\Omega^{-1/2} H_\Omega^{1/2} e^{-tH_\Omega}\|_{\mathcal{L}(L^p)}\\ &\le C\|H_\Omega^{1/2} e^{-tH_\Omega}\|_{\mathcal{L}(L^p)}\\ &\le C' t^{-1/2},\end{aligned}$$

which gives (27). Note that the standard inequality $\|H_\Omega^{1/2} e^{-tH_\Omega}\|_{\mathcal{L}(L^p)} \le Ct^{-1/2}$ is a consequence of the analyticity of the semigroup e^{-tH_Ω} on L^p.

We shall also need (27) for A_Ω for all $t \in (0,1)$. This holds for the same reason as above (the Riesz transform $\nabla(\delta I + A_\Omega)^{-1/2}$ is bounded on $L^p(\Omega)$ for $p \in (1,2]$ for some positive constant δ). See Chapter 8 in [17] for Riesz transforms of non-symmetric operators.

Using the above estimate for $\|e^{-(t-s)A_\Omega} b\cdot\nabla e^{-sH_\Omega}\|_{1,\infty}$ we obtain

$$\begin{aligned}&\Big\|\int_0^{t/2} e^{-(t-s)A_\Omega} b\cdot\nabla e^{-sH_\Omega} ds\Big\|_{1,\infty}\\ &\le C_1\|b\|_{L^\infty(\Omega)} t^{-d/(2p)} \int_0^{t/2} s^{-1/2-\frac{d}{2}(1-1/p)} ds\\ &\le C_1'\|b\|_{L^\infty(\Omega)} t^{-d/2p+1/2}.\end{aligned}$$

Now we estimate the $L^1 - L^\infty$ norm of the second term in (26) using similar ideas. Fixing $q \in (2,\infty)$, for p such that $\frac{1}{q} + \frac{1}{p} = 1$ and $t \in (0,1]$ we have

$$\begin{aligned}&\|e^{-(t-s)A_\Omega} b\cdot\nabla e^{-sH_\Omega}\|_{1,\infty}\\ &\le \|e^{-(t-s)/2A_\Omega}\|_{q,\infty}\|e^{-(t-s)/2A_\Omega} b\cdot\nabla\|_{\mathcal{L}(L^q)}\|e^{-sH_\Omega}\|_{1,q}\\ &= \|e^{-(t-s)/2A_\Omega}\|_{q,\infty}\|\mathrm{div}(be^{-(t-s)/2A_\Omega})\|_{\mathcal{L}(L^p)}\|e^{-sH_\Omega}\|_{1,q}\\ &\le C_3(t-s)^{-d/(2q)} s^{-\frac{d}{2}(1-1/q)}\|e^{-sH_\Omega}\|_{1,q}\times\\ &\big[\|b\|_{L^\infty(\Omega)}\|\nabla e^{-(t-s)/2A_\Omega})\|_{\mathcal{L}(L^p)} + \|\mathrm{div}\, b\|_{L^\infty(\Omega)}\|e^{-(t-s)/2A_\Omega})\|_{\mathcal{L}(L^p)}\big]\\ &\le C(\|b\|_{L^\infty(\Omega)} + \|\mathrm{div}\, b\|_{L^\infty(\Omega)})\,(t-s)^{-d/(2q)} s^{-1/2} s^{-\frac{d}{2}(1-1/q)}.\end{aligned}$$

This gives for all $t \in (0,1]$

$$\Big\|\int_{t/2}^{t} e^{-(t-s)A_\Omega} b\cdot\nabla e^{-sH_\Omega} ds\Big\|_{1,\infty} \le C'' t^{-d/2+1/2}. \tag{28}$$

Therefore, there exists a constant C_3 such that for all $t \in (0,1]$

$$\|e^{-tA_\Omega} - e^{-tH_\Omega}\|_{1,\infty} \le C_3 t^{-d/2+1/2}. \tag{29}$$

This implies that for a.e. $x \in \Omega$

$$|p_t^\Omega(x,x) - k_t^\Omega(x,x)| \le C_3 t^{-d/2+1/2}.$$

Using this and (25) and (24) we obtain

(30)
$$p_t(x,x) \geq p_t^\Omega(x,x) \geq k_t^\Omega(x,x) - C_3 t^{-d/2+1/2} \geq Ct^{-d/2} - C_3 t^{-d/2+1/2}$$

and the thesis follows. □

Remark 6. Notice that the only point where we need the hypothesis that the matrix (a_{jk}) is symmetric is in (25), which we do not know if it is true in the nonsymmetric case.
A careful inspection of its proof shows that Theorem 2 can be rephrased in different ways: one is

for every compact $K \subset \mathbb{R}^d$ there is a constant C such that

$$p_t(x,x) \geq Ct^{-d/2}$$

for all $t \in (0,1)$ and $x \in K$.

In fact, one can take $x \in K$ and argue as above with $\Omega_x = B(x,1)$ depending on x and replacing the L^∞-norms of b, $\operatorname{div} b$ in Ω with the respective norms in $K' = \cup_{x\in K}\overline{\Omega_x}$.
Another way of stating Theorem 2 is the following:

for every $x \in \mathbb{R}^d$ there is $t_x > 0$ such that

$$p_t(x,x) \geq \frac{C}{2} t^{-d/2}$$

for all $t \in (0,t_x)$, where C is the constant in theorem 2.

This follows from (30), choosing t_x e.g. such that $C_3 t^{-d/2+1/2} \leq \frac{C}{2} t^{-d/2}$ for $t \in (0,t_x)$ (notice that C_3 depends upon x through L^∞-norms of b, $\operatorname{div} b$ in $\Omega = B(x,1)$, while C does not).
Finally, the following statement folows directly from (30)

$$\exists\, C > 0 \textit{ such that } \sup_{x\in\mathbb{R}^d} p_t(x,x) \geq Ct^{-d/2} \textit{ for } 0 < t \leq 1.$$

Remark 7. If the coefficients a_{kj} are smooth, say $C^{1,\alpha}$, then a simpler proof of Theorem 2 can be given. In fact, in this case the following estimates hold for the operator H_Ω:

$$\|\nabla e^{-tH_\Omega}\|_{\mathcal{L}(L^1)} \leq Ct^{-1/2}, \tag{31}$$

$$\|\nabla e^{-tH_\Omega}\|_{\mathcal{L}(L^\infty)} \leq Ct^{-1/2}, \tag{32}$$

see [6, Theorem 9.3] for (31) and [12], Theorem 6.1.7 for (32). As a consequence, we can estimate the first integral in (26) directly with

$p = 1$, and the second as follows:

$$\begin{aligned}&\Big\| \int_{t/2}^{t} e^{-(t-s)A_\Omega} b \cdot \nabla e^{-sH_\Omega} f ds \Big\|_\infty \\ &\le \|b\|_{L^\infty(\Omega)} \int_{t/2}^{t} \|\nabla e^{-(s/2)H_\Omega} e^{-(s/2)H_\Omega} f\|_\infty ds \\ &\le C\|b\|_{L^\infty(\Omega)} \|f\|_1 \int_{t/2}^{t} s^{-1/2} s^{-d/2} ds \\ &\le C\|b\|_{L^\infty(\Omega)} \|f\|_1 t^{-d/2+1/2},\end{aligned}$$

where we have used estimate (21).

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