

BV FUNCTIONS ON CONVEX DOMAINS IN WIENER SPACES

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ABSTRACT. We study functions of bounded variation defined in an abstract Wiener space X , relating the variation of a function u on a convex open set $\Omega \subset X$ to the behavior near $t = 0$ of $T(t)u$, $T(t)$ being the Ornstein–Uhlenbeck semigroup in Ω .

1. INTRODUCTION

This paper is devoted to bounded variation (BV) functions in open sets of infinite dimensional separable Banach spaces endowed with Gaussian measures. BV functions defined in the whole space X have been introduced in [19] and studied also in [20, 4]. As in the finite dimensional case, they are strongly related to geometric measure theory and in particular to the theory of perimeters, see [5, 22, 13].

We use notation and results from [9], to which we refer for the general theory of Gaussian measures. Our setting is an abstract Wiener space, i.e., a separable Banach space X , with a nondegenerate centred Gaussian measure γ and the induced Cameron–Martin Hilbert space $H \subset X$ (with inner product $[\cdot, \cdot]_H$ and norm $|\cdot|_H$). See Subsection 2.2 for more details.

A basic result of the theory of BV functions in Wiener spaces is the characterisation of the BV functions in X as the elements of the Orlicz space $L \log L^{1/2}(X, \gamma)$ such that

$$\lim_{t \rightarrow 0} \int_X |\nabla_H T_t u(x)|_H d\gamma(x) < +\infty. \quad (1.1)$$

In this case, the gradient $D_\gamma u$ of u is an H -valued measure on X and the above limit is just the total variation of $D_\gamma u$ in X (see for instance in [19, Proposition 4.1], [20, Proposition 3.6] and [4, Thm 4.1]). Here, $(T_t)_{t \geq 0}$ denotes the Ornstein–Uhlenbeck semigroup

$$T_t u(x) = \int_X u(e^{-t}x + \sqrt{1 - e^{-2t}}y) d\gamma(y), \quad t > 0, \quad (1.2)$$

that, as well known, plays the role of the heat semigroup in the context of Wiener spaces. A main feature in the proof is the monotonicity of the function $t \mapsto \int_X |\nabla_H T_t u(x)|_H d\gamma(x)$.

If $X = \mathbb{R}^d$ is endowed with the Lebesgue measure, under some regularity assumptions on Ω there are bounded extension operators from $W^{1,p}(\Omega)$ and $BV(\Omega)$ to $W^{1,p}(\mathbb{R}^d)$ and $BV(\mathbb{R}^d)$, respectively. In infinite dimensions, the lack of analogous results makes the study of Sobolev and BV functions on domains more difficult.

In this paper we deal with BV functions (and sets with finite perimeter) on convex open sets of abstract Wiener spaces. The theory is still at its very beginning, see [22, 23, 24, 14, 10]. However, convex open sets have finite perimeter by [13].

We propose a definition of BV functions on domains in X through an integration by parts formula against a suitable class of test functions, rather than merely as restrictions of BV functions on the whole space. This is not a trivial issue because of the lack of smooth bump

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functions (for general X) on the one hand, and of bounded extension operators on the other hand.

Moreover, we relate the variation of a function and the short time behaviour of the Ornstein–Uhlenbeck semigroup; besides the interest of extending similar results available in different contexts, see [6, 11, 21], such a relation has proved to be useful to describe fine properties of BV functions in Wiener spaces, see [1, 2].

In particular, we get a characterisation of BV functions on a convex open set Ω similar to (1.1). We consider the Ornstein–Uhlenbeck operator L associated with the Dirichlet form

$$\mathcal{E}(u, v) = \int_{\Omega} [\nabla_H u(x), \nabla_H v(x)]_H d\gamma(x), \quad u, v \in W^{1,2}(\Omega, \mu),$$

and the semigroup $(T_t)_{t \geq 0}$ generated by the realisation of L in $L^2(\Omega, \gamma)$. The main result of this paper is the next Theorem.

Theorem 1.1. *Let $\Omega \subset X$ be an open convex set, and let $u_0 \in BV(X, \gamma) \cap L^2(X, \gamma)$ be such that $|D_\gamma u_0|(\partial\Omega) = 0$. Then, for any $t > 0$,*

$$\int_{\Omega} |\nabla_H T_t u_0(x)|_H d\gamma(x) \leq |D_\gamma u_0|(\Omega)$$

and

$$\lim_{t \rightarrow 0} \int_{\Omega} |\nabla_H T_t u_0(x)|_H d\gamma(x) = |D_\gamma u_0|(\Omega). \quad (1.3)$$

The study of the Ornstein–Uhlenbeck semigroup on domains is less straightforward than in the whole X , since no explicit formula like (1.2) is available for $T(t)$; nevertheless, the function $t \mapsto \int_{\Omega} |\nabla_H T_t u_0(x)|_H d\gamma(x)$ is still monotone. In the proof of monotonicity the convexity of Ω plays an essential role.

The paper is organised as follows. In Section 2 we recall basic notations and results, both in the finite and infinite dimensional case. In Section 3 we prove Theorem 1.1. In the Appendix we collect some useful properties of convex domains.

2. NOTATIONS AND PRELIMINARIES

In this section we introduce the basic notation and recall some preliminary results, both in finite and in infinite dimensional spaces. Let us recall that $u : \mathbb{R}^d \rightarrow \mathbb{R}$ is in $BV_{\text{loc}}(\mathbb{R}^d)$ if $u \in L^1_{\text{loc}}(\mathbb{R}^d)$ and

$$V(u, \mathcal{O}) = \sup \left\{ \int_{\mathcal{O}} u(x) \sum_{j=1}^d \frac{\partial \varphi_j}{\partial x_j}(x) dx : \varphi \in C_c^1(\mathcal{O}, \mathbb{R}^d), |\varphi(x)| \leq 1 \ \forall x \in \mathcal{O} \right\} < \infty$$

for every bounded open set $\mathcal{O} \subset \mathbb{R}^d$. In this case, the distributional gradient of u is a locally finite Radon measure with values in \mathbb{R}^d . For more information about BV functions and their fine properties in finite dimensions we refer to [3]. In the next subsection we describe our finite dimensional Gaussian framework and the related class of BV functions. We refer to [9] for a comprehensive presentation of infinite dimensional gaussian analysis, to [25] for a survey on BV functions in infinite dimensions.

2.1. BV functions in finite dimensions. Let \mathcal{O} be an open set in a d -dimensional Euclidean space F endowed with the *standard* Gaussian measure,

$$d\gamma_F(y) = \frac{1}{(2\pi)^{d/2}} \exp(-|y|^2/2) dy = G_d(y) dy.$$

We define the (negative) formal adjoint of the partial derivative ∂_{y_j} by $\partial_{y_j}^* \varphi(y) = \partial_{y_j} \varphi(y) - y_j \varphi(y)$, and the Gaussian divergence by

$$\operatorname{div}_F \varphi = \sum_{j=1}^d \partial_{y_j}^* \varphi_j, \quad \varphi = \{\varphi_1, \dots, \varphi_d\} \in C^1(\mathcal{O}, F). \quad (2.1)$$

Given a function $u \in L^1(\mathcal{O}, \gamma_F)$, we define its Gaussian total variation by

$$|D_{\gamma_F} u|(\mathcal{O}) = \sup \left\{ \int_{\mathcal{O}} u(y) \operatorname{div}_F \varphi(y) d\gamma_F(y) : \varphi \in C_c^1(\mathcal{O}, F), |\varphi(y)| \leq 1 \forall y \in \mathcal{O} \right\}. \quad (2.2)$$

The space $BV(\mathcal{O}, \gamma_F)$ is defined as the set of functions $u \in L^1(\mathcal{O}, \gamma_F)$ with $|D_{\gamma_F} u|(\mathcal{O}) < +\infty$. This space is characterised by the fact that the distributional gradient of any $u \in BV(\mathcal{O}, \gamma_F)$ is a vector valued finite measure $\mu = (\mu_1, \dots, \mu_d)$, namely

$$\int_{\mathcal{O}} u(y) \partial_{y_i}^* g(y) d\gamma_F(y) = \int_{\mathcal{O}} g(y) d\mu_i(y), \quad \forall g \in C_c^1(\mathcal{O}), i = 1, \dots, d.$$

For such a measure we have $|\mu|(\mathcal{O}) = |D_{\gamma_F} u|(\mathcal{O})$; moreover for any open set $A \subset \mathcal{O}$ we have

$$|D_{\gamma_F} u|(A) = \sup \left\{ \int_A u(y) \operatorname{div}_F \varphi(y) d\gamma_F(y) : \varphi \in C_c^1(A, F), |\varphi(y)| \leq 1 \forall y \in A \right\}.$$

The space $BV(\mathcal{O}, \gamma_F)$ is a Banach space with the norm

$$\|u\|_{BV} = \|u\|_{L^1(\mathcal{O}, \gamma_F)} + |D_{\gamma_F} u|(\mathcal{O}). \quad (2.3)$$

We refer to [15] and to [12] for more details on sets with finite perimeter and functions with bounded variation in Gaussian spaces. Let us point out that, due to the regularity and non degeneracy of the standard Gaussian density, local and fine properties of Gaussian BV functions do not differ from those of classical BV_{loc} functions and indeed $BV(\mathcal{O}) \subset BV(\mathcal{O}, \gamma_F) \subset BV_{\text{loc}}(\mathcal{O})$. As a matter of fact, a Gaussian BV function defines a finite measure that we denote by $D_{\gamma_F} u$ for which the following integration by parts formula holds,

$$\int_{\mathcal{O}} u(x) \operatorname{div}_F \varphi(x) d\gamma_F(x) = \int_{\mathcal{O}} \langle \varphi, D_{\gamma_F} u \rangle, \quad \forall \varphi \in C_c^1(\mathcal{O}; \mathbb{R}^d).$$

Such a measure is absolutely continuous with respect to the standard total variation measure of u and $D_{\gamma_F} u = G_d Du$. Of course, global properties are very different, as the Gaussian density vanishes at infinity. Moreover, if $u \in W^{1,1}(\mathcal{O}, \gamma_F)$, then $u \in BV(\mathcal{O}, \gamma_F)$ and

$$|D_{\gamma_F} u|(\mathcal{O}) = \int_{\mathcal{O}} |\nabla u(y)| d\gamma_F(y). \quad (2.4)$$

To some extent the above norm (2.3) is too strong, for instance smooth functions are not dense in $BV(\mathcal{O}, \gamma_F)$. However BV functions may be approximated by smooth functions in the sense of the so called approximation in variation, as the next lemma shows. The proof we present is based on classical ideas that require a minor adaptation to the present situation.

Here and in the following, we denote by $B_R(x)$ the open ball centred at x with radius R .

Lemma 2.1. *Let $v_0 \in BV(\mathcal{O}, \gamma_F) \cap L^2(\mathcal{O}, \gamma_F)$. Then for any $\varepsilon > 0$ there exists $v_\varepsilon \in W^{1,2}(\mathcal{O}, \gamma_F)$ such that*

$$\|v_0 - v_\varepsilon\|_{L^2(\mathcal{O}, \gamma_F)} < \varepsilon \quad \text{and} \quad \left| \int_{\mathcal{O}} |\nabla v_\varepsilon(x)| d\gamma_F(x) - |D_{\gamma_F} v_0|(\mathcal{O}) \right| < \varepsilon.$$

Proof. The proof is a modification of the classical Meyers-Serrin arguments; we refer to [3, Thm 3.9] for more details. First of all we can assume that v_0 has compact support in F . Indeed, for every $R > 1$ let ϑ_R be a cut-off function such that $0 \leq \vartheta_R \leq 1$, $|\nabla \vartheta_R(x)| \leq 2$ for every x and

$$\vartheta_R \equiv 1 \text{ on } B_{R-1}(0), \quad \text{supp}(\vartheta_R) \subset B_R(0).$$

Setting $v_R = v_0 \vartheta_R$, we have

$$\limsup_{R \rightarrow +\infty} \|v_0 - v_R\|_{L^2(\mathcal{O}, \gamma_F)} \leq \lim_{R \rightarrow +\infty} \|v_0\|_{L^2(\mathcal{O} \setminus B_{R-1}(0), \gamma_F)} = 0$$

and, by the obvious equality $\text{div}_F(\vartheta_R \varphi) = \vartheta_R \text{div}_F \varphi + \langle \nabla \vartheta_R, \varphi \rangle$,

$$|D_{\gamma_F} v_R|(\mathcal{O}) \leq 2 \int_{\mathcal{O} \cap (B_R(0) \setminus B_{R-1}(0))} |v_0| d\gamma_F + |D_{\gamma_F} v_0|(\mathcal{O} \cap B_R(0)).$$

This implies that v_R converges to v_0 in variation as $R \rightarrow \infty$. Precisely, for any $\varepsilon > 0$ there is $R > 0$ such that $\|v_R - v_0\|_{L^2(\mathcal{O}, \gamma_F)} < \varepsilon$ and

$$\left| |D_{\gamma_F} v_0|(\mathcal{O}) - |D_{\gamma_F} v_R|(\mathcal{O}) \right| < \varepsilon.$$

From now on, we consider $v = v_R$ with this choice of R in place of v_0 . We can consider a sequence of open sets \mathcal{O}_j with compact closure in \mathcal{O} such that any point of \mathcal{O} belongs to at most four sets \mathcal{O}_j ; a possible choice is

$$\mathcal{O}_1 = \{x \in \mathcal{O} \cap B_2(0) : \text{dist}(x, \partial \mathcal{O}) > 1/2\}$$

and for $j \geq 2$

$$\mathcal{O}_j = \left\{ x \in \mathcal{O} \cap B_{j+1}(0) \setminus \overline{B_{j-1}(0)} : \frac{1}{j+1} < \text{dist}(x, \partial \mathcal{O}) < \frac{1}{j-1} \right\}.$$

Let $\{\varphi_j\}_j$ be a partition of unity associated with such a covering, and let ϱ be a standard mollifier. For every $j \in \mathbb{N}$ fix $\varepsilon_j < \varepsilon$ such that $\text{supp}((v\varphi_j) * \varrho_{\varepsilon_j}) \subset \mathcal{O}_j$ and

$$\|(v\varphi_j) * \varrho_{\varepsilon_j} - v\varphi_j\|_{L^2(\mathcal{O}, \gamma_F)} + \int_{\mathcal{O}} |(v\nabla \varphi_j) * \varrho_{\varepsilon_j} - v\nabla \varphi_j| d\gamma_F < \frac{\varepsilon}{2j}.$$

The approximation of v is then defined by

$$v_\varepsilon = \sum_{j \in \mathbb{N}} (v\varphi_j) * \varrho_{\varepsilon_j}.$$

v_ε is indeed smooth, and

$$\|v_\varepsilon - v\|_{L^2(\mathcal{O}, \gamma_F)} \leq \sum_{j \in \mathbb{N}} \|(v\varphi_j) * \varrho_{\varepsilon_j} - v\varphi_j\|_{L^2(\mathcal{O}, \gamma_F)} < \varepsilon.$$

On the other hand, see [3, Proposition 3.2]

$$\nabla v_\varepsilon = \sum_{j \in \mathbb{N}} (\varphi_j Dv) * \varrho_{\varepsilon_j} + \sum_{j \in \mathbb{N}} ((v\nabla \varphi_j) * \varrho_{\varepsilon_j} - v\nabla \varphi_j).$$

Here we have used the fact that $BV(\mathcal{O}, \gamma_F) \subset BV_{\text{loc}}(\mathcal{O})$, hence Dv is a measure with locally finite variation and the convolution above is well defined. From these considerations, we then obtain

$$\begin{aligned}
|D_{\gamma_F} v_\varepsilon|(\mathcal{O}) &= \int_{\mathcal{O}} |\nabla v_\varepsilon(x)| d\gamma_F(x) \\
&\leq \sum_{j \in \mathbb{N}} \int_{\mathcal{O}} |v \nabla \varphi_j * \varrho_{\varepsilon_j}(x) - v(x) \nabla \varphi_j(x)| G_d(x) dx \\
&\quad + \sum_{j \in \mathbb{N}} \int_{\mathcal{O}} dx \int_{B_\varepsilon(x)} G_d(x) \varphi_j(y) \varrho_{\varepsilon_j}(x-y) d|Dv|(y) \\
&\leq \varepsilon + \sum_{j \in \mathbb{N}} \int_{\mathcal{O}} dx \int_{B_\varepsilon(x)} \frac{G_d(x)}{G_d(y)} \varphi_j(y) \varrho_{\varepsilon_j}(x-y) G_d(y) d|Dv|(y) \\
&\leq \varepsilon + \sum_{j \in \mathbb{N}} e^{\varepsilon_j R + \varepsilon_j^2/2} \int_{\mathcal{O}} \varphi_j(y) d|D_{\gamma_F} v|(y) \leq \varepsilon + e^{\varepsilon R + \varepsilon^2/2} |D_{\gamma_F} v|(\mathcal{O}),
\end{aligned}$$

where we have used the fact that the support of v is contained in $B_R(0)$ and for $y \in B_{\varepsilon_j}(x)$

$$\frac{G_d(x)}{G_d(y)} = e^{\frac{|y|^2 - |x|^2}{2}} \leq e^{\varepsilon_j R + \varepsilon_j^2/2}.$$

□

2.2. Abstract Wiener spaces. We consider an infinite dimensional separable Banach space X (whose topological dual we denote by X^*), endowed with the Borel σ -algebra $\mathcal{B}(X)$ and a centred and non degenerate Gaussian measure γ with nondegenerate covariance operator $Q \in \mathcal{L}(X^*, X)$ uniquely determined by the relation

$$y^*(Qx^*) = \int_X x^*(x) y^*(x) d\gamma(x) \quad \forall x^*, y^* \in X^*.$$

If we consider the operator $R : L^2(X, \gamma) \rightarrow X$ given by the Bochner integral

$$R\varphi = \int_X x\varphi(x) d\gamma(x), \quad \varphi \in L^2(X, \gamma),$$

it is easily seen that its adjoint $R^* : X^* \rightarrow L^2(X, \gamma)$ is just the embedding operator, $(R^*x^*)(x) = x^*(x)$, $x \in X$, and the equality $Q = RR^*$ follows.

The Cameron-Martin space H is given by $R(\mathcal{H})$, where \mathcal{H} is the closure of X^* in $L^2(X, \gamma)$. It coincides with the set of all $h \in X$ such that there exists $\hat{h} \in \mathcal{H}$ for which

$$\int_X x^*(x) \hat{h}(x) d\gamma(x) = x^*(h), \quad x^* \in X^*.$$

In this case, we have $R(\hat{h}) = h$, and $R|_{\mathcal{H}} : \mathcal{H} \rightarrow H$ is an isometry if we endow H with the norm $|\cdot|_H$ associated with the inner product $[h, k]_H = \langle \hat{h}, \hat{k} \rangle_{L^2(X, \gamma)}$. The space $Q(X^*)$ is dense in H , and H is continuously and densely embedded in X .

The symbol $\mathcal{F}C_b^1(X)$ denotes the space of bounded continuously differentiable cylindrical functions with bounded derivatives, that is, $u \in \mathcal{F}C_b^1(X)$ if

$$u(x) = \varphi(x_1^*(x), \dots, x_m^*(x))$$

for some $\varphi \in C_b^1(\mathbb{R}^m)$ and $x_1^*, \dots, x_m^* \in X^*$.

We fix once and for all an orthonormal basis $(h_j)_{j \in \mathbb{N}}$ of H , with $h_j = Qh'_j$, $h_j \in X^*$ (such a basis exists, see [9, Corollary 2.10.10]). We denote by $\pi_m : X \rightarrow \text{span}\{h_1, \dots, h_m\}$ the projection $\pi_m x = \sum_{j=1}^m \hat{h}_j(x) h_j$ and by X_m , X_m^\perp the range and the kernel of π_m , respectively. Note that the restriction of π_m to H is the orthogonal projection on the linear span of h_1, \dots, h_m .

π_m induces the canonical factorisation $\gamma = \gamma_m \otimes \gamma_m^\perp$, where $\gamma_m = \gamma \circ \pi_m^{-1}$ and $\gamma_m^\perp = \gamma \circ (I - \pi_m)^{-1}$ are the pull-back measures on X_m and X_m^\perp .

For every function $u \in L^1(X, \gamma)$ we define its canonical cylindrical approximations $\mathbb{E}_m u$ by

$$\mathbb{E}_m u(x) = \int_X u(\pi_m x + (I - \pi_m)y) d\gamma(y) = \int_{X_m^\perp} u(\pi_m x + y') d\gamma_m^\perp(y'), \quad (2.5)$$

Then, $\lim_{m \rightarrow \infty} \mathbb{E}_m u = u$ in $L^1(X, \gamma)$ and γ -a.e. (see e.g. [9, Corollary 3.5.2]). Moreover $\mathbb{E}_m u$ is invariant under translations along all the vectors in X_m^\perp , hence $\mathbb{E}_m u(x) = v(\pi_m x)$ for some function v .

Let us recall the notation for the partial derivative along $h \in H$ and for its formal adjoint. For $f \in C_b^1(X)$ we set

$$\partial_h f(x) = \lim_{t \rightarrow 0} \frac{f(x + th) - f(x)}{t}, \quad \partial_h^* f(x) = \partial_h f(x) - f(x) \hat{h}(x)$$

where $h = R\hat{h} \in H$ with $\hat{h} \in \mathcal{H}$. The gradient along H , $\nabla_H f : X \rightarrow H$ of f is defined as

$$\nabla_H f(x) = \sum_{j \in \mathbb{N}} \partial_{h_j} f(x) h_j$$

and it is the unique element $y \in H$ such that, for every $h \in H$, $\partial_h f(x) = [y, h]_H$. Notice that if $f(x) = g(\pi_m x)$ with $g \in C^1(\mathbb{R}^m)$, then

$$\partial_h f(x) = \langle \nabla g(\pi_m x), \pi_m h \rangle_{\mathbb{R}^m}.$$

The operator ∂_h^* defined by $\partial_h^* \varphi = -\partial_h \varphi + \hat{h} \varphi$ is (up to a change of sign) the formal adjoint of ∂_h with respect to $L^2(X, \gamma)$, namely

$$\int_X \varphi \partial_h f d\gamma = - \int_X f \partial_h^* \varphi d\gamma \quad \forall \varphi, f \in \mathcal{F}C_b^1(X).$$

We define the space $\mathcal{F}C_b^1(X, H)$ of cylindrical H -valued functions as the vector space spanned by functions φh , where φ runs in $\mathcal{F}C_b^1(X)$ and h in H . With this notation, the divergence operator is defined for $\varphi \in \mathcal{F}C_b^1(X, H)$ as

$$\text{div}_\gamma \varphi = \sum_{j \in \mathbb{N}} \partial_j^* [\varphi, h_j]_H,$$

and we have the integration by parts formula

$$\int_X [\nabla_H f, \varphi]_H d\gamma = - \int_X f \text{div}_\gamma \varphi d\gamma \quad f \in \mathcal{F}C_b^1(X), \varphi \in \mathcal{F}C_b^1(X; H).$$

If we fix a finite dimensional space $F \subset Q(X^*) \subset H$ with $\dim F = d$, we identify F with \mathbb{R}^d and we denote by div_F the divergence on F defined according to (2.1) with respect to any orthonormal basis $\{h_1, \dots, h_d\}$ of F . Moreover, since $F \subset H$, there is an orthogonal projection of H onto F . According to Theorem 2.10.11 in [9] there is a unique (up to equivalence) measurable projection $\pi_F : X \rightarrow F$ which extends it.

We denote by $\mathcal{M}(X, H)$ the space of all H -valued measures μ with finite total variation on $\mathcal{B}(X)$. The total variation measure $|\mu|$ of μ is defined by

$$|\mu|(B) = \sup \left\{ \sum_{j=1}^{\infty} |\mu(B_j)|_H : B = \bigcup_{j \in \mathbb{N}} B_j \right\},$$

where $B_j \in \mathcal{B}(X)$ for every j and $B_j \cap B_i = \emptyset$ for $j \neq i$. Moreover, using the polar decomposition $\mu = \sigma|\mu|$, the total variation of μ can be obtained as

$$|\mu|(\Omega) = \sup \left\{ \int_{\Omega} [\sigma, \varphi]_H d|\mu| : \varphi \in C_b(\Omega, H), \|\varphi\|_{\infty} \leq 1 \right\}. \quad (2.6)$$

Indeed, in the real valued case this is a direct consequence of the isometry between the space of real measures on and open set Ω and the dual space of $C_b(\Omega)$, see [17, Section IV.6]. The finite dimensional case follows immediately because a vector-valued measure is just an n -tuple of real-valued measures. If μ has infinitely many components $\mu_j = [\mu, h_j]_H$, then $\mu \in \mathcal{M}(X, H)$ if and only if

$$\sup_m |(\mu_1, \dots, \mu_m)|(X) < \infty.$$

In fact, setting $\lambda = \sup_m |(\mu_1, \dots, \mu_m)|$, the inequality $\lambda \leq |\mu|$ is obvious. Conversely, since $\mu_j \ll \lambda$ for every $j \in \mathbb{N}$, there is a sequence of λ -measurable functions (f_j) such that $\mu_j = f_j \lambda$ and $\sum_{j=1}^m |f_j|^2 \leq 1$ λ -a.e. for every $m \in \mathbb{N}$, whence $\|(f_j(x))\|_{\ell^2} \leq 1$ λ -a.e., $\sum_{j=1}^{\infty} f_j h_j \in L^1(X, \lambda; H)$ and $\mu = \sum_{j=1}^{\infty} f_j h_j \lambda \in \mathcal{M}(X, H)$.

Definition 2.2. Let $u \in L^2(X, \gamma)$. We say that u has bounded variation in X and we write $u \in BV(X, \gamma)$ if there exists $\mu \in \mathcal{M}(X, H)$ such that for any $\varphi \in \mathcal{F}C_b^1(X)$ we have

$$\int_X u(x) \partial_j^* \varphi(x) d\gamma(x) = - \int_X \varphi(x) d\mu_j(x) \quad \forall j \in \mathbb{N}, \quad (2.7)$$

where $\mu_j = [h_j, \mu]_H$. In this case we set $D_{\gamma}u = \mu$.

Even though in this paper we deal with BV functions defined in the whole space, it is interesting to point out that an intrinsic definition of $BV(\Omega, \gamma)$ is possible, using a suitable class of test functions. By (2.2), we notice that in finite dimension the natural class of test functions is that of boundedly supported smooth functions. In infinite dimensions compactly supported smooth functions are not adequate and for H -valued measures the following result holds.

Lemma 2.3. Let $\Omega \subset X$ be open and let $\mu \in \mathcal{M}(\Omega, H)$ be an H -valued Radon measure. Then, denoting by $|\mu|$ the total variation measure and using the polar decomposition $\mu = \sigma|\mu|$ we have

$$|\mu|(\Omega) = \sup \left\{ \int_{\Omega} [\sigma, \varphi]_H d|\mu| : \varphi \in \text{Lip}_0(\Omega, H), \|\varphi\|_{\infty} \leq 1 \right\},$$

where $\text{Lip}_0(\Omega, H)$ denotes the space of H -valued functions defined on X , Lipschitz continuous with respect to the X -norm and vanishing in $X \setminus \Omega$.

Proof. We recall that in our framework all Borel measures on X are Radon measures. Therefore, for every $\varepsilon > 0$ there are a function $\varphi_{\varepsilon} \in C_b(\Omega, H)$ with $\|\varphi_{\varepsilon}\|_{\infty} \leq 1$ and a compact set $K \subset \Omega$ with $|\mu|(\Omega \setminus K) < \varepsilon$ such that

$$|\mu|(\Omega) \leq \int_{\Omega} [\sigma, \varphi_{\varepsilon}]_H d|\mu| + \varepsilon \leq \int_K [\sigma, \varphi_{\varepsilon}]_H d|\mu| + 2\varepsilon.$$

Let us now approximate σ : there is $\sigma_\varepsilon \in C(K, H)$ with finite dimensional range (just write $\sigma = \sum_j [\sigma, h_j]_H h_j$ and take a suitable finite dimensional projection) such that

$$\|\sigma - \sigma_\varepsilon\|_{L^1(K, |\mu|)} < \varepsilon \quad \text{and} \quad |\mu|(\Omega) \leq \int_K [\sigma_\varepsilon, \varphi_\varepsilon]_H d|\mu| + 3\varepsilon.$$

Notice that, since σ_ε has finite dimensional range, only finitely many components of $\varphi_\varepsilon = \sum_j [\varphi_\varepsilon, h_j]_H h_j$ are involved in the above integral. We may therefore argue component by component to show that φ_ε can be approximated by Lip_0 functions uniformly on K . To this end, let us first remark that by the Stone-Weierstrass theorem the class of the restrictions to K of $\mathcal{F}C_b^1(X)$ functions is dense in $C_b(K)$, hence there is a function $g_\varepsilon \in \mathcal{F}C_b^1(X, H)$ with finite dimensional range such that $\|g_\varepsilon - \varphi_\varepsilon\|_{L^\infty(K)} < \varepsilon$. Moreover, the function

$$f(x) = \left(1 - \frac{2}{\delta} \text{dist}(x, K)\right)^+, \quad x \in H,$$

with $\delta = \text{dist}(K, \partial\Omega)$, belongs to $\text{Lip}_0(\Omega)$, so that, setting

$$G(h) = \begin{cases} h & \text{if } |h|_H \leq 1 \\ \frac{h}{|h|_H} & \text{if } |h|_H > 1 \end{cases}, \quad h \in H,$$

we have that $\psi_\varepsilon(x) = f(x)(G \circ g_\varepsilon)(x) \in \text{Lip}_0(\Omega, H)$, $\|\psi_\varepsilon\|_{L^\infty(K, |\mu|)} \leq 1$ and $\|\varphi_\varepsilon - \psi_\varepsilon\|_{L^\infty(K)} \leq 2\varepsilon$, whence

$$\begin{aligned} |\mu|(\Omega) &\leq \int_K [\sigma_\varepsilon, \psi_\varepsilon]_H d|\mu| + 5\varepsilon \leq \int_\Omega [\sigma, \psi_\varepsilon]_H d|\mu| + 6\varepsilon \\ &\leq \sup \left\{ \int_\Omega [\sigma, \psi]_H d|\mu| : \psi \in \text{Lip}_0(\Omega, H), \|\psi\|_\infty \leq 1 \right\} + 6\varepsilon. \end{aligned}$$

By the arbitrariness of ε , the proof is complete. \square

In the next lemma we extend the integration by parts formula (2.7) with $u \in BV(X, \gamma)$ to $\text{Lip}_0(\Omega, H)$ functions.

Lemma 2.4. *For every $\varphi \in \text{Lip}_0(\Omega, H)$ and $u \in BV(X, \gamma)$ the following equality holds:*

$$\int_\Omega u \text{div}_\gamma \varphi d\gamma = - \int_\Omega [\varphi, D_\gamma u]_H. \quad (2.8)$$

Proof. Let us show (2.8) arguing component by component. Fix h_j , an element of the given orthonormal basis in H , and consider the projection $\pi_j : H \rightarrow \text{span } h_j$, $\pi_j(x) = \hat{h}_j(x)h_j$. Then write $x = y + th_j$, $u_y(t) = u(y + th_j)$. Setting $X_j^\perp = (I - \pi_j)(X)$ and $\varphi_j = [\varphi, h_j]_H$, we have

$$\begin{aligned} \int_X u \partial_j^* \varphi_j d\gamma &= \int_{X_j^\perp} d\gamma_{h_j^\perp}^\perp \int_{\mathbb{R}} u_y(t) \partial_t^* (\varphi_j)_y(t) d\gamma_1(t) \\ &= - \int_{X_j^\perp} d\gamma_{h_j^\perp}^\perp \int_{\mathbb{R}} (\varphi_j)_y(t) dD_{\gamma_1} u_y(t) = - \int_X \varphi_j d[D_\gamma u, h_j], \end{aligned}$$

where we have used the notation $\gamma = \gamma_1 \oplus \gamma_{h_j^\perp}^\perp$ for the factorisation of γ induced by the decomposition of X into $\pi_j(X) \oplus X_j^\perp$ and in the second line for any $y \in X_j^\perp$ the integral on \mathbb{R} is with respect to the measure $D_{\gamma_1} u_y$, the measure derivative of the section u_y of u , see [4]. \square

An easy but useful consequence is the following lower semicontinuity property of the total variation, see also [1, Proposition 2.5] for a different proof.

Corollary 2.5. *Let $u \in BV(X, \gamma)$ and let $\Omega \subset X$ be any open set such that $u|_\Omega \in L^2(\Omega, \gamma)$. If a sequence $(u_n)_n$ converges to u in $L^2(\Omega, \gamma)$, then*

$$|D_\gamma u|(\Omega) \leq \liminf_{n \rightarrow \infty} |D_\gamma u_n|(\Omega).$$

Proof. By Lemmas 2.3 and 2.4,

$$\begin{aligned} |D_\gamma u|(\Omega) &= \sup \left\{ \int_\Omega [\varphi, D_\gamma u]_H : \varphi \in \text{Lip}_0(\Omega, H), \|\varphi\|_\infty \leq 1 \right\} \\ &= \sup \left\{ \int_\Omega u \operatorname{div}_\gamma \varphi \, d\gamma : \varphi \in \text{Lip}_0(\Omega, H), \|\varphi\|_\infty \leq 1 \right\}. \end{aligned}$$

On the other hand, $\varphi \in \text{Lip}_0(\Omega, H)$ the functional $u \mapsto \int_X u \operatorname{div}_\gamma \varphi \, d\gamma$ is continuous in $L^2(X, \gamma)$ and therefore, by Lemma 2.4, the functional $u \mapsto |D_\gamma u|(\Omega)$ is lower semicontinuous in $L^2(X, \gamma)$, as it is the supremum of continuous functionals. \square

It is not hard to see that if $u \in BV(X, \gamma)$ then

$$|D_\gamma u|(X) = \sup \left\{ \int_X [\varphi, D_\gamma u]_H : \varphi \in \mathcal{F}C_b^1(X, H), |\varphi(x)|_H \leq 1 \, \forall x \in X \right\},$$

see [4]. A useful consequence of Lemma 2.3 is that the canonical cylindrical approximations $\mathbb{E}_m u$ defined in (2.5) (which are known to converge to u in variation, i.e.

$$|D_\gamma u|(X) = \lim_{m \rightarrow +\infty} |D_\gamma \mathbb{E}_m u|(X),$$

see equality (34) in [4]), verify the inequality $|D_\gamma \mathbb{E}_m u|(A) \leq |D_\gamma u|(A)$ for all open sets A . In fact,

$$\begin{aligned} |D_\gamma \mathbb{E}_m u|(A) &= \sup \left\{ \int_A \mathbb{E}_m u \operatorname{div}_\gamma \varphi \, d\gamma, \varphi \in \text{Lip}_0(A, H), \|\varphi\|_\infty \leq 1 \right\} \tag{2.9} \\ &= \sup \left\{ \int_A [\varphi, dD\mathbb{E}_m u]_H, \varphi \in \text{Lip}_0(A, H), \|\varphi\|_\infty \leq 1 \right\} \\ &= \sup \left\{ \int_A \langle \pi_m \varphi, dD\mathbb{E}_m u \rangle, \varphi \in \text{Lip}_0(A, H), \|\varphi\|_\infty \leq 1 \right\} \\ &= \sup \left\{ \int_A \langle \pi_m \varphi, d(\pi_m D_\gamma u) \rangle, \varphi \in \text{Lip}_0(A, H), \|\varphi\|_\infty \leq 1 \right\} \\ &\leq \sup \left\{ \int_A [\varphi, dD_\gamma u]_H, \varphi \in \text{Lip}_0(A, H), \|\varphi\|_\infty \leq 1 \right\} \\ &= \sup \left\{ \int_A u \operatorname{div}_\gamma \varphi \, d\gamma, \varphi \in \text{Lip}_0(A, H), \|\varphi\|_\infty \leq 1 \right\} = |D_\gamma u|(A). \end{aligned}$$

2.3. Sobolev spaces and the Ornstein–Uhlenbeck semigroup on convex domains.

There are several equivalent ways of defining Sobolev spaces on Wiener spaces, see [9, Section 5.2]. If X is replaced by a domain $\Omega \subset X$, the equivalence of different definitions is not obvious. Here we adopt the definition of [14], that works for sublevel sets $\Omega = \{G < 0\}$ of Sobolev functions $G \in W^{1,p}(X, \gamma)$ for some $p > 1$. Since we are interested in a convex Ω , we fix any $x_0 \in \Omega$ and we define the Minkowski function

$$\mathfrak{m}(x) := \inf \{ \lambda \geq 0 : x - x_0 \in \lambda(\Omega - x_0) \}$$

which is Lipschitz continuous; then $\Omega = \{G < 0\}$ with $G(x) = \mathfrak{m}(x) - 1 \in W^{1,p}(X, \gamma)$ for every $p > 1$. By [14, Lemma 2.2], the operator $\text{Lip}(\Omega) \rightarrow L^2(\Omega, \gamma; H)$ defined by $u \mapsto \nabla_H \tilde{u}|_\Omega$, where

\tilde{u} is any Lipschitz continuous extension of u to the whole X , is closable. The space $W^{1,2}(\Omega, \gamma)$ is defined as the domain of its closure, still denoted by ∇_H . Therefore, it is a Hilbert space for the inner product

$$\langle u, v \rangle_{W^{1,2}(\Omega, \gamma)} = \int_{\Omega} uv \, d\gamma + \int_{\Omega} [\nabla_H u, \nabla_H v]_H \, d\gamma$$

which induces the graph norm of ∇_H . The associated quadratic form in the gradient,

$$\mathcal{E}(u, v) = \int_{\Omega} [\nabla_H u(x), \nabla_H v(x)]_H \, d\gamma(x), \quad u, v \in W^{1,2}(\Omega, \gamma),$$

is used to define the Ornstein–Uhlenbeck operator $L : D(L) \subset L^2(\Omega, \gamma) \rightarrow L^2(\Omega, \gamma)$ by setting

$$D(L) = \left\{ u \in W^{1,2}(\Omega, \gamma) : \exists f \in L^2(\Omega, \gamma) \text{ s.t.} \right. \\ \left. \mathcal{E}(u, v) = - \int_{\Omega} f v \, d\gamma, \quad \forall v \in W^{1,2}(\Omega, \gamma) \right\},$$

and $Lu = f$. The operator $(L, D(L))$ is self-adjoint in $L^2(\Omega, \gamma)$ and dissipative (namely, $\langle Lu, u \rangle_{L^2(\Omega, \gamma)} \leq 0$ for every $u \in D(L)$), hence it is the infinitesimal generator of an analytic contraction semigroup $(T_t)_{t \geq 0}$ in $L^2(\Omega, \gamma)$.

For the moment we have considered only real valued functions. In the sequel we use also the complexification of L in the space $L^2(\Omega, \gamma; \mathbb{C})$, which is the operator associated with the sesquilinear form $(u, v) \mapsto \int_{\Omega} [\nabla_H u, \overline{\nabla_H v}]_H \, d\gamma$ defined for $u, v \in W^{1,2}(\Omega, \gamma; \mathbb{C})$. The semigroup generated by the complexification agrees with $(T_t)_{t \geq 0}$ on real valued functions, and we use its representation formula as a Dunford integral along a complex path.

3. PROOF OF THEOREM 1.1

The proof of Theorem 1.1 is divided in several steps and each step is discussed in a subsection.

3.1. Monotonicity in finite dimensions. Let \mathcal{O} be a convex open set with smooth boundary in a finite dimensional space F , with inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$. We denote by $\nu^{\mathcal{O}}(x)$ the exterior unit normal vector at $x \in \partial\mathcal{O}$. Let $(T_t^F)_{t \geq 0}$ be the semigroup generated by the Ornstein-Uhlenbeck operator L defined by the Dirichlet form

$$\mathcal{E}_{\mathcal{O}}(u, v) = \int_{\mathcal{O}} \langle \nabla u, \nabla v \rangle \, d\gamma_F, \quad u, v \in W^{1,2}(\mathcal{O}, \gamma_F),$$

as explained in Subsection 2.3. By [16], $D(L) \subset W^{2,2}(\mathcal{O}, \gamma_F)$ and the elements of $D(L)$ satisfy the Neumann boundary condition $\partial u / \partial \nu^{\mathcal{O}} = 0$ at $\partial\mathcal{O}$. Moreover for every $u \in D(L)$ we have $Lu(x) = \Delta u(x) - \langle x, \nabla u(x) \rangle$. Since L is a realisation of an elliptic operator with smooth coefficients, and the boundary of \mathcal{O} is smooth, the function $(t, x) \mapsto T_t^F u_0(x)$ is smooth in $(0, +\infty) \times \mathcal{O}$ for every $u_0 \in L^2(\mathcal{O}, \gamma_F)$.

For any $v_0 \in L^2(\mathcal{O}, \gamma_F)$ let us introduce the function $\mathcal{F}_{v_0} : (0, +\infty) \rightarrow [0, +\infty]$ defined as

$$\mathcal{F}_{v_0}(t) = \int_{\mathcal{O}} |\nabla T_t^F v_0(x)| \, d\gamma_F(x).$$

Then the following result holds.

Proposition 3.1. *For each $v_0 \in BV(\mathcal{O}, \gamma_F) \cap L^2(\mathcal{O}, \gamma_F)$ the function \mathcal{F}_{v_0} is decreasing in $(0, \infty)$. Moreover*

$$\mathcal{F}_{v_0}(t) \leq |D_{\gamma_F} v_0|(\mathcal{O}), \quad \forall t > 0$$

and

$$\lim_{t \rightarrow 0} \mathcal{F}_{v_0}(t) = |D_{\gamma_F} v_0|(\mathcal{O}).$$

Proof. In order to avoid integrability problems, we introduce a family of cut-off functions ϑ_R such that $0 \leq \vartheta_R \leq 1$, $\vartheta_R \equiv 1$ in $B_R(0)$, $\text{supp}(\vartheta_R) \subset B_{2R}(0)$ and $|\nabla \vartheta_R(x)| \leq 2/R$ for every x . Analogously, in order to overcome the lack of regularity of the function $|\nabla T_t^F v_0(x)|$ at its zeroes, we replace it by $\sqrt{|\nabla T_t^F v_0(x)|^2 + 1/R}$. We then define

$$\mathcal{F}_{R,v_0}(t) = \int_{\mathcal{O}} \vartheta_R(x) \sqrt{|\nabla T_t^F v_0(x)|^2 + \frac{1}{R}} d\gamma_F(x).$$

As a first step, we prove that \mathcal{F}_{R,v_0} is differentiable. Since T_t^F is analytic, $t \mapsto T_t^F v_0$ is differentiable with values in $D(L)$, and

$$\partial_t \sqrt{|\nabla T_t^F v_0(x)|^2 + \frac{1}{R}} = \frac{1}{\sqrt{|\nabla T_t^F v_0(x)|^2 + \frac{1}{R}}} \langle \nabla T_t^F v_0(x), \nabla L T_t^F v_0(x) \rangle$$

so that

$$\partial_t \sqrt{|\nabla T_t^F v_0(x)|^2 + \frac{1}{R}} \leq |\nabla L T_t^F v_0(x)|.$$

Then we can differentiate under the integral, and recalling that $\partial_i \partial_j^* \varphi = \partial_j^* \partial_i \varphi - \varphi \delta_{ij}$ we get

$$\begin{aligned} \mathcal{F}'_{R,v_0}(t) &= \int_{\mathcal{O}} \frac{\vartheta_R(x)}{\sqrt{|\nabla T_t^F v_0(x)|^2 + \frac{1}{R}}} \langle \nabla T_t^F v_0(x), \nabla L T_t^F v_0(x) \rangle d\gamma_F(x) \\ &= \sum_{i,j=1}^d \int_{\mathcal{O}} \frac{\vartheta_R(x)}{\sqrt{|\nabla T_t^F v_0(x)|^2 + \frac{1}{R}}} \partial_i T_t^F v_0(x) \partial_i \partial_j^* \partial_j T_t^F v_0(x) d\gamma_F(x) \\ &= \sum_{i,j=1}^d \int_{\mathcal{O}} \frac{\vartheta_R(x)}{\sqrt{|\nabla T_t^F v_0(x)|^2 + \frac{1}{R}}} \partial_i T_t^F v_0(x) \partial_j^* \partial_{ij}^2 T_t^F v_0(x) d\gamma_F(x) + \\ &\quad - \int_{\mathcal{O}} \vartheta_R(x) \frac{|\nabla T_t^F v_0(x)|^2}{\sqrt{|\nabla T_t^F v_0(x)|^2 + \frac{1}{R}}} d\gamma_F(x) \\ &= \sum_{i,j=1}^d \int_{\partial \mathcal{O}} \frac{\vartheta_R(x)}{\sqrt{|\nabla T_t^F v_0(x)|^2 + \frac{1}{R}}} \partial_i T_t^F v_0(x) \partial_{ij}^2 T_t^F v_0(x) \nu_j^\mathcal{O}(x) G_d(x) d\mathcal{H}^{d-1}(x) \\ &\quad - \sum_{i,j=1}^d \int_{\mathcal{O}} \partial_{ij}^2 T_t^F v_0(x) \partial_j \left(\frac{\vartheta_R(x)}{\sqrt{|\nabla T_t^F v_0(x)|^2 + \frac{1}{R}}} \partial_i T_t^F v_0(x) \right) d\gamma_F(x) \\ &\quad - \int_{\mathcal{O}} \vartheta_R(x) \frac{|\nabla T_t^F v_0(x)|^2}{\sqrt{|\nabla T_t^F v_0(x)|^2 + \frac{1}{R}}} d\gamma_F(x) \\ &= \int_{\partial \mathcal{O}} \frac{\vartheta_R(x)}{\sqrt{|\nabla T_t^F v_0(x)|^2 + \frac{1}{R}}} \langle D^2 T_t^F v_0(x) \nu^\mathcal{O}(x), \nabla T_t^F v_0(x) \rangle G_d(x) d\mathcal{H}^{d-1}(x) \end{aligned}$$

$$\begin{aligned}
& - \sum_{i,j=1}^d \int_{\mathcal{O}} \frac{\vartheta_R(x)}{\sqrt{|\nabla T_t^F v_0(x)|^2 + \frac{1}{R}}} \left[(\partial_{ij}^2 T_t^F v_0(x))^2 + \right. \\
& - \left. \frac{1}{|\nabla T_t^F v_0(x)|^2 + \frac{1}{R}} \partial_{kj}^2 T_t^F v_0(x) \partial_k T_t^F v_0(x) \partial_i T_t^F v_0(x) \partial_{ij}^2 T_t^F v_0(x) \right] d\gamma_F(x) + \\
& - \int_{\mathcal{O}} \frac{1}{\sqrt{|\nabla T_t^F v_0(x)|^2 + \frac{1}{R}}} \langle D^2 T_t^F v_0(x) \nabla \vartheta_R(x), \nabla T_t^F v_0(x) \rangle d\gamma_F(x) + \\
& - \int_{\mathcal{O}} \vartheta_R(x) \frac{|\nabla T_t^F v_0(x)|^2}{\sqrt{|\nabla T_t^F v_0(x)|^2 + \frac{1}{R}}} d\gamma_F(x) \\
= & - \int_{\partial \mathcal{O}} \frac{\vartheta_R(x)}{\sqrt{|\nabla T_t^F v_0(x)|^2 + \frac{1}{R}}} \langle J\nu^{\mathcal{O}}(x) \nabla T_t^F v_0(x), \nabla T_t^F v_0(x) \rangle G_d(x) d\mathcal{H}^{d-1}(x) \\
& + \int_{\mathcal{O}} \frac{\vartheta_R(x)}{\sqrt{|\nabla T_t^F v_0(x)|^2 + \frac{1}{R}}} \left(\frac{|\nabla T_t^F v_0(x)|^2}{|\nabla T_t^F v_0(x)|^2 + \frac{1}{R}} \left| D^2 T_t^F v_0(x) \frac{\nabla T_t^F v_0(x)}{|\nabla T_t^F v_0(x)|} \right|^2 + \right. \\
& - \left. \|D^2 T_t^F v_0(x)\|_2^2 \right) d\gamma_F(x) + \\
& - \int_{\mathcal{O}} \frac{1}{\sqrt{|\nabla T_t^F v_0(x)|^2 + \frac{1}{R}}} \langle D^2 T_t^F v_0(x) \nabla \vartheta_R(x), \nabla T_t^F v_0(x) \rangle d\gamma_F(x) + \\
& - \int_{\mathcal{O}} \vartheta_R(x) \frac{|\nabla T_t^F v_0(x)|^2}{\sqrt{|\nabla T_t^F v_0(x)|^2 + \frac{1}{R}}} d\gamma_F(x),
\end{aligned}$$

where we have denoted by $\|D^2 T_t^F v_0(x)\|$ the Euclidean norm of the matrix $D^2 T_t^F v_0(x)$. The second integral in the right hand side is negative because

$$\left| D^2 T_t^F v_0(x) \frac{\nabla T_t^F v_0(x)}{|\nabla T_t^F v_0(x)|} \right|^2$$

is bounded by the square of the largest eigenvalue of $D^2 T_t^F v_0(x)$, while $\|D^2 T_t^F v_0(x)\|_2^2$ is the sum of the square of all the eigenvalues. In the first integral we have denoted by $J\nu^{\mathcal{O}}(x)$ the Jacobian matrix of $\nu^{\mathcal{O}}$ at x , and we have used the fact that $\nabla T_t^F v_0(x)$ is orthogonal to $\partial \mathcal{O}$ and any tangential derivative of $\langle \nabla T_t^F v_0(x), \nu^{\mathcal{O}}(x) \rangle$ is equal to 0, that is

$$\begin{aligned}
0 & = \langle \nabla \left(\langle \nabla T_t^F v_0(x), \nu^{\mathcal{O}}(x) \rangle \right), \nabla T_t^F v_0(x) \rangle \\
& = \langle D^2 T_t^F v_0(x) \nu^{\mathcal{O}}(x), \nabla T_t^F v_0(x) \rangle + \langle J\nu^{\mathcal{O}}(x) \nabla T_t^F v_0(x), \nabla T_t^F v_0(x) \rangle.
\end{aligned}$$

The convexity of $\partial \mathcal{O}$ implies

$$\langle J\nu^{\mathcal{O}} \xi, \xi \rangle \geq 0, \quad \forall \xi \in (\nu^{\mathcal{O}}(x))^{\perp}$$

and so we can conclude that

$$\begin{aligned}
\mathcal{F}_{R,v_0}^t & \leq - \int_{\mathcal{O}} \frac{1}{\sqrt{|\nabla T_t^F v_0(x)|^2 + \frac{1}{R}}} \langle D^2 T_t^F v_0(x) \nabla \vartheta_R(x), \nabla T_t^F v_0(x) \rangle d\gamma_F(x) \\
& \leq \frac{2}{R} \| |D^2 T_t^F v_0| \|_{L^2(\mathcal{O}, \gamma_F)}^2,
\end{aligned}$$

where the last inequality holds with $|D^2T_t^F v_0|$ the operator norm of $D^2T_t^F v_0$. As a consequence, for any $t_1 < t_2$

$$\mathcal{F}_{R,v_0}(t_2) = \mathcal{F}_{R,v_0}(t_1) + \int_{t_1}^{t_2} \mathcal{F}'_{R,v_0}(s) ds. \leq \mathcal{F}_{R,v_0}(t_1) + \frac{2}{R} \int_{t_1}^{t_2} \| |D^2T_s^F v_0| \|_{L^2(\mathcal{O}, \gamma_F)}^2 ds;$$

Letting $R \rightarrow +\infty$ we obtain the monotonicity of \mathcal{F}_{v_0} , since

$$\mathcal{F}_{v_0}(t_2) = \lim_{R \rightarrow +\infty} \mathcal{F}_{R,v_0}(t_2) \leq \lim_{R \rightarrow +\infty} \left(\mathcal{F}_{R,v_0}(t_1) + \frac{2}{R} \int_{t_1}^{t_2} \| |D^2T_s^F v_0| \|_{L^2(\mathcal{O}, \gamma_F)}^2 ds \right) = \mathcal{F}_{v_0}(t_1).$$

To prove the second part of the statement, let us fix $w \in W^{1,2}(\mathcal{O}, \gamma_F)$. Since $W^{1,2}(\mathcal{O}, \gamma_F)$ is the domain of $(I - L)^{1/2}$, then T_t^F is strongly continuous in $W^{1,2}(\mathcal{O}, \gamma_F)$. It follows that $|\nabla T_t^F w|$ converges to $|\nabla w|$ in $L^1(\mathcal{O}, \gamma)$ as $t \rightarrow 0$, and hence by (2.4)

$$\lim_{t \rightarrow 0} \int_{\mathcal{O}} |\nabla T_t^F w(x)| d\gamma_F(x) = \int_{\mathcal{O}} |\nabla w(x)| d\gamma_F(x) = |D_{\gamma_F} w|(\mathcal{O}).$$

Therefore, for any $t > 0$ and for $w \in W^{1,2}(\mathcal{O}, \gamma_F)$

$$\mathcal{F}_w(t) \leq \mathcal{F}_w(0^+) = \lim_{t \rightarrow 0} \int_{\mathcal{O}} |\nabla T_t^F w(x)| d\gamma_F(x) = |D_F w|(\mathcal{O}).$$

Now let $v_0 \in BV(\mathcal{O}, \gamma_F) \cap L^2(\mathcal{O}, \gamma_F)$. Thanks to Lemma 2.1, there exists a sequence of functions $(w_j)_{j \in \mathbb{N}} \subset W^{1,2}(\mathcal{O}, \gamma_F)$ such that

$$\lim_{j \rightarrow \infty} \|w_j - v_0\|_{L^2(\mathcal{O}, \gamma_F)} = 0, \quad \lim_{j \rightarrow \infty} \int_{\mathcal{O}} |\nabla w_j(x)| d\gamma_F(x) = |D_{\gamma_F} v_0|(\mathcal{O}).$$

Then, for every $t > 0$ we have $\lim_{j \rightarrow \infty} |\nabla T_t^F w_j| = |\nabla T_t^F v_0|$ in $L^2(\mathcal{O}, \gamma_F)$, and by the first part of the proof

$$\int_{\mathcal{O}} |\nabla T_t^F w_j(x)| d\gamma_F(x) \leq \int_{\mathcal{O}} |\nabla w_j(x)| d\gamma_F(x), \quad j \in \mathbb{N},$$

so that

$$\begin{aligned} \int_{\mathcal{O}} |\nabla T_t^F v_0(x)| d\gamma_F(x) &= \lim_{j \rightarrow +\infty} \int_{\mathcal{O}} |\nabla T_t^F w_j(x)| d\gamma_F(x) \\ &\leq \liminf_{j \rightarrow +\infty} \int_{\mathcal{O}} |\nabla w_j(x)| d\gamma_F(x) = |D_{\gamma_F} v_0|(\mathcal{O}). \end{aligned}$$

□

Remark 3.2. *Some arguments of this section may be related to the approach of Bakry and Émery [7], that has been widely developed in the last years also in the metric space setting, and in fact results of this type in such framework can be found in [28], to which we refer for the details. Monotonicity of $t \mapsto \|\nabla T_t f\|_{L^1(\mathcal{O})}$ follows also from the pointwise estimate*

$$|\nabla T_t f(x)| \leq e^{-t} T_t |\nabla f|(x) \tag{3.1}$$

and the L^1 contractivity of T_t . Inequality (3.1) is known in several contexts. In particular, it has been proved in [8] and, by probabilistic methods, in [27], [29] for bounded convex domains with smooth boundary and in [30] for convex domains with nonsmooth boundary. However, (3.1) is not needed for our purposes, and therefore here we give a simple purely analytic proof of the monotonicity of $t \mapsto \|\nabla T_t f\|_{L^1(\mathcal{O})}$.

3.2. Convex sets as countable intersection of cylindrical convex sets. In this section we consider a convex open set $\Omega \subset X$ and a sequence $(\Omega_n)_n$ of open convex cylindrical sets of the form $\Omega_n = \pi_{F_n}^{-1}(\mathcal{O}_n)$, where, for every $n \in \mathbb{N}$, π_{F_n} is a finite dimensional projection from X onto F_n , such that $\Omega_n \subset \Omega_{n+1}$, $\partial\mathcal{O}_n$ is regular, $\Omega \subset \Omega_n$ and

$$\bar{\Omega} = \bigcap_{n \in \mathbb{N}} \bar{\Omega}_n. \quad (3.2)$$

We give a construction of the approximating sets Ω_n in the Appendix.

Since Ω and Ω_n are open convex sets, by [13, Prop. 4.2] we know that $\gamma(\partial\Omega) = \gamma(\partial\Omega_n) = 0$. Moreover, $\Omega \subset \Omega_n$ for any $n \in \mathbb{N}$ and

$$\gamma\left(\bigcap_{n \in \mathbb{N}} \Omega_n \setminus \Omega\right) = 0.$$

3.3. Approximations of semigroups. We recall that L is the Ornstein-Uhlenbeck operator in $L^2(\Omega, \gamma)$ defined in Section 2.3. Similarly, L_n are the Ornstein-Uhlenbeck operators in $L^2(\Omega_n, \gamma)$ associated with the Dirichlet forms

$$\mathcal{E}^{(n)}(u, v) = \int_{\Omega_n} [\nabla_H u, \nabla_H v]_H d\gamma, \quad u, v \in W^{1,2}(\Omega_n, \gamma).$$

The semigroups generated by L_n are denoted by $(T_t^{(n)})_{t \geq 0}$. We denote by $R(\lambda, A)$ the resolvent of the operator A . In the next proposition we deal with complex-valued functions.

Proposition 3.3. *Under the above assumptions, for any $f \in L^2(X, \gamma)$ and for any $\lambda \in \mathbb{C} \setminus (-\infty, 0]$,*

$$\lim_{n \rightarrow \infty} \left(R(\lambda, L_n)(f|_{\Omega_n}) \right)|_{\Omega} = R(\lambda, L)(f|_{\Omega}) \quad \text{in } W^{1,2}(\Omega, \gamma).$$

It follows

$$(T_t^{(n)} u_0|_{\Omega_n})|_{\Omega} \rightarrow T_t u_0|_{\Omega} \quad \text{in } W^{1,2}(\Omega, \gamma)$$

for any $u_0 \in L^2(X, \gamma)$ and $t > 0$.

Proof. Fix $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ and set $\vartheta = \arg \lambda$. Since each L_n is self-adjoint and dissipative, $\lambda \in \rho(L_n)$ and

$$\|R(\lambda, L_n)\|_{\mathcal{L}(L^2(\Omega_n, \gamma))} \leq \frac{1}{|\lambda| \cos \frac{\vartheta}{2}},$$

$$\|L_n R(\lambda, L_n)\|_{\mathcal{L}(L^2(\Omega_n, \gamma))} = \|\lambda R(\lambda, L_n) - I\|_{\mathcal{L}(L^2(\Omega_n, \gamma))} \leq \frac{1}{\cos \frac{\vartheta}{2}} + 1.$$

Setting $u = R(\lambda, L)(f|_{\Omega})$ and $u_n = R(\lambda, L_n)(f|_{\Omega_n})$, such estimates imply

$$\|u_n\|_{L^2(\Omega_n, \gamma)} \leq \frac{1}{|\lambda| \cos \frac{\vartheta}{2}} \|f\|_{L^2(X, \gamma)}$$

and

$$\int_{\Omega_n} |\nabla_H u_n|_H^2 d\gamma = - \int_{\Omega_n} u_n L_n u_n d\gamma \leq \frac{1 + \cos \frac{\vartheta}{2}}{|\lambda| \cos^2 \frac{\vartheta}{2}} \|f\|_{L^2(X, \gamma)}^2.$$

Recalling that $\Omega \subset \Omega_n$, we arrive at the estimate

$$\|u_n|_{\Omega}\|_{W^{1,2}(\Omega, \gamma)} \leq \|u_n\|_{W^{1,2}(\Omega_n, \gamma)} \leq \frac{\|f\|_{L^2(X, \gamma)}}{\cos \frac{\vartheta}{2}} \left(\frac{1}{|\lambda|} + \sqrt{\frac{1 + \cos \frac{\vartheta}{2}}{|\lambda|}} \right).$$

Then the sequence $(u_n|_{\Omega})$ is bounded in $W^{1,2}(\Omega, \gamma)$, so that it admits a subsequence weakly convergent to some function u_{∞} . By definition, $u_n = R(\lambda, L_n)(f|_{\Omega_n})$ means

$$\lambda \int_{\Omega_n} u_n \varphi d\gamma + \int_{\Omega_n} [\nabla_H u_n, \nabla_H \varphi]_H d\gamma = \int_{\Omega_n} f \varphi d\gamma, \quad \forall \varphi \in W^{1,2}(\Omega_n, \gamma). \quad (3.3)$$

Fix any $\varphi \in W^{1,2}(X, \gamma)$. Notice that

$$\left| \int_{\Omega_n \setminus \Omega} f \varphi d\gamma \right| \leq \|f\|_{L^2(X, \gamma)} \|\varphi\|_{L^2(\Omega_n \setminus \Omega, \gamma)}$$

where $\lim_{n \rightarrow \infty} \|\varphi\|_{L^2(\Omega_n \setminus \Omega, \gamma)} = 0$ since $\gamma(\Omega_n \setminus \Omega)$ vanishes as $n \rightarrow \infty$. Then,

$$\lim_{n \rightarrow +\infty} \int_{\Omega_n \setminus \Omega} f \varphi d\gamma = 0.$$

Recalling that $\|u_n\|_{W^{1,2}(\Omega_n, \gamma)}$ is bounded by a constant independent of n , the same argument yields

$$\lim_{n \rightarrow +\infty} \int_{\Omega_n \setminus \Omega} u_n \varphi d\gamma = \lim_{n \rightarrow +\infty} \int_{\Omega_n \setminus \Omega} [\nabla_H u_n, \nabla_H \varphi]_H d\gamma = 0.$$

We conclude that

$$\begin{aligned} \lambda \int_{\Omega} u_{\infty} \varphi d\gamma + \int_{\Omega} [\nabla_H u_{\infty}, \nabla_H \varphi]_H d\gamma &= \lim_{n \rightarrow +\infty} \left(\lambda \int_{\Omega_n} u_n \varphi d\gamma + \int_{\Omega_n} [\nabla_H u_n, \nabla_H \varphi]_H d\gamma \right) \\ &= \lim_{n \rightarrow +\infty} \int_{\Omega_n} f \varphi d\gamma = \int_{\Omega} f \varphi d\gamma. \end{aligned}$$

Since the restrictions to Ω of elements of $W^{1,2}(X, \gamma)$ are dense in $W^{1,2}(\Omega, \gamma)$, we obtain

$$\lambda \int_{\Omega} u_{\infty} \varphi d\gamma + \int_{\Omega} [\nabla_H u_{\infty}, \nabla_H \varphi]_H d\gamma = \int_{\Omega} f \varphi d\gamma, \quad \forall \varphi \in W^{1,2}(\Omega, \gamma).$$

Therefore, the limit function u_{∞} coincides with $u = R(\lambda, L)(f|_{\Omega})$ and the whole sequence $u_n|_{\Omega}$ weakly converges in $W^{1,2}(\Omega, \gamma)$ to u , with no need of subsequences.

Let us now show that $u_n|_{\Omega}$ converges strongly to u in $W^{1,2}(\Omega, \gamma)$. To this aim it is enough to show that

$$\limsup_{n \rightarrow +\infty} \|u_n\|_{W^{1,2}(\Omega, \gamma)} \leq \|u\|_{W^{1,2}(\Omega, \gamma)}.$$

To see this we use (3.3) with $\varphi = \bar{u}_n$ and obtain

$$\lambda \int_{\Omega_n} |u_n|^2 d\gamma + \int_{\Omega_n} |\nabla_H u_n|_H^2 d\gamma = \int_{\Omega_n} f \bar{u}_n d\gamma.$$

Letting $n \rightarrow +\infty$, as before we obtain

$$\begin{aligned} \lim_{n \rightarrow +\infty} \left(\lambda \int_{\Omega_n} |u_n|^2 d\gamma + \int_{\Omega_n} |\nabla_H u_n|_H^2 d\gamma \right) &= \int_{\Omega} f \bar{u} d\gamma \\ &= \lambda \int_{\Omega} |u|^2 d\gamma + \int_{\Omega} |\nabla_H u|_H^2 d\gamma \end{aligned} \quad (3.4)$$

since $\lambda u - Lu = f$.

We write $\lambda = \alpha + i\beta$ with $\alpha, \beta \in \mathbb{R}$. If $\beta \neq 0$, taking the imaginary parts in (3.4) we get

$$\lim_{n \rightarrow +\infty} \beta \int_{\Omega_n} |u_n|^2 d\gamma = \beta \int_{\Omega} |u|^2 d\gamma,$$

that is

$$\lim_{n \rightarrow +\infty} \int_{\Omega_n} |u_n|^2 d\gamma = \int_{\Omega} |u|^2 d\gamma,$$

and we deduce

$$\lim_{n \rightarrow +\infty} \int_{\Omega_n} |\nabla_H u_n|_H^2 d\gamma = \int_{\Omega} |\nabla_H u|_H^2 d\gamma.$$

Therefore,

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \left(\int_{\Omega} |u_n|^2 d\gamma + \int_{\Omega} |\nabla_H u_n|_H^2 d\gamma \right) &\leq \limsup_{n \rightarrow +\infty} \left(\int_{\Omega_n} |u_n|^2 d\gamma + \int_{\Omega_n} |\nabla_H u_n|_H^2 d\gamma \right) \\ &= \int_{\Omega} |u|^2 d\gamma + \int_{\Omega} |\nabla_H u|_H^2 d\gamma. \end{aligned}$$

If $\beta = 0$, since $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, we have $\alpha > 0$; (3.4) gives

$$\lim_{n \rightarrow \infty} \alpha \|u_n\|_{L^2(\Omega_n, \gamma)}^2 + \|\nabla_H u_n\|_{L^2(\Omega_n, \gamma)}^2 = \alpha \|u\|_{L^2(\Omega, \gamma)}^2 + \|\nabla_H u\|_{L^2(\Omega, \gamma)}^2,$$

and since the norm $u \mapsto (\alpha \|u\|_{L^2(\Omega, \gamma)}^2 + \|\nabla_H u\|_{L^2(\Omega, \gamma)}^2)^{1/2}$ is equivalent to the norm of $W^{1,2}(\Omega, \gamma)$ we are done.

Convergence of resolvents implies convergence of semigroups. Indeed, it is sufficient to use the Dominated Convergence Theorem in the canonical representation formula,

$$T_t f|_{\Omega} = \frac{1}{2\pi i} \int_{\Gamma} R(\lambda, L) f|_{\Omega} d\lambda = \lim_{n \rightarrow +\infty} \frac{1}{2\pi i} \int_{\Gamma} R(\lambda, L_n) f|_{\Omega_n} d\lambda = \lim_{n \rightarrow +\infty} T_t^{(n)} f|_{\Omega_n},$$

where Γ is any of the usual integration paths for analytic semigroups. \square

Remark 3.4. *One of the referees suggested an alternative proof of the above convergence result along the following lines. We do not enter into all the details.*

Define the forms

$$\begin{aligned} \tilde{\mathcal{E}}^{(n)}(u, v) &= \int_{\Omega_n} [\nabla_H u, \nabla_H v]_H d\gamma, \quad u, v \in D(\tilde{\mathcal{E}}^{(n)}) = \{u \in L^2(X, \gamma) : u|_{\Omega_n} \in W^{1,2}(\Omega_n, \gamma)\} \\ \tilde{\mathcal{E}}(u, v) &= \int_{\Omega} [\nabla_H u, \nabla_H v]_H d\gamma, \quad u, v \in D(\tilde{\mathcal{E}}) = \{u \in L^2(X, \gamma) : u|_{\Omega} \in W^{1,2}(\Omega, \gamma)\}, \end{aligned}$$

the associated operators \tilde{L}_n, \tilde{L} and the semigroups $\tilde{T}_t^{(n)}, \tilde{T}_t$ in $L^2(X, \gamma)$. Notice that the sequence $\{\tilde{\mathcal{E}}^{(n)}, D(\tilde{\mathcal{E}}^{(n)})\}$ is decreasing in the sense of bilinear forms, i.e., $D(\tilde{\mathcal{E}}^{(n)}) \subset D(\tilde{\mathcal{E}}^{(n+1)})$ and $\tilde{\mathcal{E}}^{(n)}(u) \geq \tilde{\mathcal{E}}^{(n+1)}(u)$ for all $n \in \mathbb{N}$ and $u \in D(\tilde{\mathcal{E}}^{(n)})$. Moreover, due to (3.2), we have

$$D(\tilde{\mathcal{E}}) \supset D := \bigcup_{n=1}^{\infty} D(\tilde{\mathcal{E}}^{(n)}) \quad \text{and} \quad \tilde{\mathcal{E}}(u, u) = \lim_{n \rightarrow \infty} \tilde{\mathcal{E}}^{(n)}(u, u) \quad \forall u \in D.$$

To apply [26, Thm S.16] and deduce that $\tilde{L}_n \rightarrow \tilde{L}$ in the strong resolvent sense we need to prove that $D(\tilde{\mathcal{E}})$ is contained in the closure of D with respect to the norm $[\|\cdot\|_{L^2(X, \gamma)}^2 + \mathcal{E}(\cdot)]^{1/2}$. Since Ω is convex, this follows from the important approximation result of [23]. It then follows from [31, Thm 1, §IX.12] that $\tilde{T}_t^{(n)} u \rightarrow \tilde{T}_t u$ in $L^2(X, \gamma)$, uniformly for t in compact intervals, for all $u \in D(\tilde{\mathcal{E}})$. Checking that $(\tilde{T}_t^{(n)} u)|_{\Omega_n} = T_t^{(n)}(u|_{\Omega_n})$ and the same for \tilde{T}_t, T_t allows to conclude that $T_t^{(n)} u \rightarrow T_t u$ in $L^2(\Omega, \gamma)$ for every $u \in W^{1,2}(\Omega, \gamma)$. Moreover, the monotonicity of the forms yields

$$\lim_{n \rightarrow \infty} \tilde{\mathcal{E}}_n(\tilde{T}_t^{(n)} u) = \tilde{\mathcal{E}}(\tilde{T}_t u)$$

for every $t > 0$. From one side, this shows that $(T_t^n u)|_\Omega = (\tilde{T}_t^n u)|_\Omega$ is uniformly bounded in $W^{1,2}(\Omega, \gamma)$ and therefore it converges weakly to $T_t u$ in $W^{1,2}(\Omega, \gamma)$ (since convergence in $L^2(\Omega, \gamma)$ has already been checked). On the other hand, the decreasing property of the forms yields

$$\limsup_{n \rightarrow \infty} \int_{\Omega} |\nabla_H(T_t^n u)|_H^2 d\gamma \leq \limsup_{n \rightarrow \infty} \tilde{\mathcal{E}}_n(\tilde{T}_t^n u) = \tilde{\mathcal{E}}(\tilde{T}_t u) = \int_{\Omega} |\nabla_H(T_t u)|_H^2 d\gamma$$

From the convergence of the norm one gets the strong convergence in $W^{1,2}(\Omega, \gamma)$.

3.4. Conclusion: approximation by finite dimensional estimates. In this Subsection we complete the proof of Theorem 1.1.

Proof. First of all, by lower semicontinuity (Corollary 2.5) we know that

$$|D_\gamma u_0|(\Omega) \leq \liminf_{t \rightarrow 0} |D_\gamma T_t u_0|(\Omega) = \liminf_{t \rightarrow 0} \int_{\Omega} |\nabla_H T_t u_0|_H d\gamma$$

by the strong continuity in $L^2(\Omega, \gamma)$ of the semigroup $(T_t)_{t \geq 0}$.

Next, we prove the estimate

$$\int_{\Omega_n} |\nabla_H T_t^{(n)} u_0|_{\Omega_n}|_H d\gamma \leq |D_\gamma u_0|(\Omega_n)$$

where Ω_n is the approximation of Ω constructed in the Appendix, $(T^{(n)})_{t \geq 0}$ is the semigroup associated with the Dirichlet form $\mathcal{E}^{(n)}$ in $L^2(\Omega_n, \gamma)$.

For any n , we choose a finite dimensional space $F_n \subset Q(X^*)$ such that $\mathcal{O}_n \subset F_n$ and fix an orthonormal basis $(h_j)_{j \in \mathbb{N}}$ of H such that F_n is a subspace of $\text{span}\{h_1, \dots, h_n\}$ (indeed, $\dim F_n \leq n$ by construction). Let also $v_j = \mathbb{E}_j u$ be the sequence of canonical cylindrical approximations of u_0 , converging to u_0 in variation. We shall let $j \rightarrow \infty$ and then $n \rightarrow \infty$. So, we fix n , set $F = F_n$ in order to simplify the notation, and take j big enough so that F is a subspace of $G := \text{span}\{h_1, \dots, h_j\}$. Setting $d = j - \dim F$, we consider the open convex set $\mathcal{O} := \mathcal{O}_n \times \mathbb{R}^d \subset G$, the associated Ornstein–Uhlenbeck semigroup operator $T^G(t)$ and its generator L^G studied in Section 3.1. We remark that $v_j(x) = w_j(\pi_G(x))$ for a function $w_j \in W^{1,2}(G, \gamma_j)$.

We claim that $(T_t^{(n)} v_j|_{\Omega_n})(x) = (T^G(t) w_j)(\pi_G(x))$, for a.e. $x \in \Omega_n$. Indeed, the function $t \mapsto T^G(t) w_j$ belongs to $C([0, \infty); L^2(\mathcal{O}, \gamma_j)) \cap C^1((0, \infty); L^2(\mathcal{O}, \gamma_j)) \cap C((0, \infty); D(L^G))$ and it is the unique classical solution to

$$\begin{cases} u'(t) = L^G u(t), & t > 0, \\ u(0) = w_j|_{\mathcal{O}}, \end{cases}$$

in the space $L^2(\mathcal{O}, \gamma_j)$. It follows immediately that the function $g : [0, +\infty) \rightarrow L^2(\Omega_n, \gamma)$, $g(t)(x) = T^G(t) w_j(\pi_G(x))$, belongs to $C([0, \infty); L^2(\Omega_n, \gamma)) \cap C^1((0, \infty); L^2(\Omega_n, \gamma))$. Moreover, using the definition of L^G and L_n we deduce that for every $w \in D(L^G)$, the function $w \circ \pi_G$ belongs to $D(L_n)$, and $L_n(w \circ \pi_G) = (L^G(w)) \circ \pi_G$. Therefore, $g \in C((0, \infty); D(L_n))$ and it satisfies

$$\begin{cases} g'(t) = L_n g(t), & t > 0, \\ g(0) = v_j|_{\Omega_n}. \end{cases}$$

So, g is a classical solution to the above Cauchy problem in the space $L^2(\Omega_n, \gamma)$. Therefore, $g(t) = T_t^{(n)}v_j|_{\Omega_n}$.

Applying Proposition 3.1 we get

$$\int_{\Omega_n} |\nabla_H T_t^{(n)}v_j|_{\Omega_n}|_H d\gamma = \int_{\mathcal{O}} |\nabla T^G(t)w_j| d\gamma_j \leq |D_{\gamma_j}w_j|(\mathcal{O}) = |D_{\gamma}v_j|(\Omega_n).$$

Let us recall that $v_j = \mathbb{E}_j u \rightarrow u$ in $L^2(X, \gamma)$ and in variation. Therefore, taking into account the above estimate and (2.9) we obtain

$$\begin{aligned} \int_{\Omega_n} |\nabla_H T_t^{(n)}u_0|_{\Omega_n}|_H d\gamma &= \lim_{j \rightarrow +\infty} \int_{\Omega_n} |\nabla_H T_t^{(n)}v_j|_{\Omega_n}|_H d\gamma \\ &\leq \liminf_{j \rightarrow +\infty} |D_{\gamma}v_j|(\Omega_n) = |D_{\gamma}u_0|(\Omega_n) \leq |D_{\gamma}u_0|(\overline{\Omega}_n). \end{aligned}$$

Now, as a consequence of Proposition 3.3 and the hypothesis $|D_{\gamma}u_0|(\partial\Omega) = 0$, we obtain

$$\begin{aligned} \int_{\Omega} |\nabla_H T_t u_0|_H d\gamma &= \lim_{n \rightarrow +\infty} \int_{\Omega} |\nabla_H T_t^{(n)}u_0|_{\Omega_n}|_H d\gamma \leq \limsup_{n \rightarrow +\infty} \int_{\Omega_n} |\nabla_H T_t^{(n)}u_0|_{\Omega_n}|_H d\gamma \\ &\leq \lim_{n \rightarrow +\infty} |D_{\gamma}u_0|(\overline{\Omega}_n) = |D_{\gamma}u_0|(\Omega), \end{aligned} \tag{3.5}$$

which finishes the proof of the Theorem. \square

Remark 3.5. *It is worth noticing that the proof of Theorem 1.1, estimate (3.5), yields that also in the infinite dimensional setting the map*

$$t \mapsto \int_{\Omega} |\nabla_H T_t u_0(x)|_H d\gamma(x)$$

is monotone decreasing, for any $u_0 \in BV(X, \gamma) \cap L^2(X, \gamma)$.

APPENDIX A. FINITE DIMENSIONAL CONVEX ANALYSIS

This section is devoted to recall some properties of convex sets and convex functions in Euclidean spaces. Most of these results can likely be found in the literature, but we recall here some of the proofs for the reader's convenience. Let $C \subset \mathbb{R}^d$ be a closed convex set with interior part $C^\circ \neq \emptyset$. Possibly translating C , without loss of generality we may assume that $0 \in C^\circ$.

If C is unbounded, then there exists $\nu \in \mathbb{S}^{d-1}$ such that $t\nu \in C$ for all $t \geq 0$. Indeed, if $(x_j)_j \subset C$ is a sequence with $\|x_j\| \rightarrow +\infty$, then

$$\nu_j = \frac{x_j}{\|x_j\|} \subset \mathbb{S}^{d-1}$$

admits an accumulation point ν ; convexity and closedness of C imply that

$$t\nu \in C, \quad \forall t \geq 0.$$

We set

$$\mathbb{S}_C^{d-1} = \{\nu \in \mathbb{S}^{d-1} : t\nu \in C, \quad \forall t \geq 0\}$$

and we define the maximal cone with vertex at 0 contained in C ,

$$K_C = \{t\nu : t \geq 0, \nu \in \mathbb{S}_C^{d-1}\},$$

while we set $K_C = \{0\}$ if C is bounded.

We define the map

$$m(x) = \inf\{\lambda \geq 0 : x \in \lambda C\}.$$

If $x \in K_C$ we have $\mathfrak{m}(x) = 0$. If $x \notin K_C$ there exists a unique point $y \in \partial C$ such that

$$x = \mathfrak{m}(x)y.$$

We set

$$y = p_C(x).$$

Proposition A.1. *Let $C \subset \mathbb{R}^d$ be a closed convex set and let $0 \in C^\circ$. Setting*

$$r = \sup\{t > 0 : B_t(0) \subset C\},$$

\mathfrak{m} is convex, $\frac{1}{r}$ -Lipschitz continuous and $C = \{\mathfrak{m} \leq 1\}$. In addition, if ∂C is C^1 , \mathfrak{m} is differentiable at any point $x \notin \partial K_C$; at such points $\langle \nabla \mathfrak{m}(x), x \rangle = \mathfrak{m}(x)$.

Proof. First of all we remark that \mathfrak{m} is positively homogeneous, namely $\mathfrak{m}(tx) = t\mathfrak{m}(x)$ for every $t > 0$ and $x \in \mathbb{R}^d$.

Let us show that \mathfrak{m} is convex. As a first step we show that for any $y_1, y_2 \in \mathbb{R}^d$

$$\mathfrak{m}(y_1 + y_2) \leq \mathfrak{m}(y_1) + \mathfrak{m}(y_2).$$

Indeed, for all $t_i > \mathfrak{m}(y_i)$, $i = 1, 2$, we have $y_i \in t_i C$. Since C is convex, then $y_1 + y_2 \in (t_1 + t_2)C$, i.e., $\mathfrak{m}(y_1 + y_2) \leq t_1 + t_2$.

Let now $x_1, x_2 \in \mathbb{R}^d$ and $\lambda \in (0, 1)$. Using the above inequality and recalling that \mathfrak{m} is homogeneous, we obtain

$$\mathfrak{m}(\lambda x_1 + (1 - \lambda)x_2) \leq \mathfrak{m}(\lambda x_1) + \mathfrak{m}((1 - \lambda)x_2) = \lambda \mathfrak{m}(x_1) + (1 - \lambda)\mathfrak{m}(x_2).$$

Let us show that \mathfrak{m} is Lipschitz continuous. For any $t < r$ and any $x \in \mathbb{R}^d$,

$$t \frac{x}{\|x\|} \in B_t(0) \subset C,$$

that is $x \in \frac{\|x\|}{t}C$, whence in particular $\mathfrak{m}(x) \leq \frac{\|x\|}{t}$, and letting $t \rightarrow r$, we obtain $\mathfrak{m}(x) \leq \frac{1}{r}\|x\|$. As a consequence, for any $x, y \in \mathbb{R}^d$,

$$\mathfrak{m}(x) = \mathfrak{m}(y + x - y) \leq \mathfrak{m}(y) + \mathfrak{m}(x - y) \leq \mathfrak{m}(y) + \frac{1}{r}\|x - y\|,$$

which implies

$$|\mathfrak{m}(x) - \mathfrak{m}(y)| \leq \frac{1}{r}\|x - y\|, \quad \forall x, y \in \mathbb{R}^d.$$

Let us prove the statements about the regularity of \mathfrak{m} . Every $p \in \partial C$ has a neighbourhood U such that $\partial C \cap U$ is the zero level of a C^1 function f whose gradient does not vanish at ∂C . The function of $(d + 1)$ variables

$$g(x, \lambda) = f\left(\frac{1}{\lambda}x\right),$$

is well defined in a neighbourhood of $(p, 1)$. g implicitly defines the Minkowski functional \mathfrak{m} , since for every x outside K_C and $\lambda > 0$, $x/\lambda \in \partial C$ iff $\lambda = \mathfrak{m}(x)$. Moreover, $\partial g(x, \lambda)/\partial \lambda = -\lambda^{-2}\langle x, \nabla f(x/\lambda) \rangle$ does not vanish at any $(x, 1)$ with $x \in \partial C$, otherwise the tangent hyperplane at x would contain the origin, which is impossible since C is convex.

This shows that \mathfrak{m} is C^1 outside \overline{K}_C . Since $\mathfrak{m} \equiv 0$ in K_C , it follows that \mathfrak{m} is C^1 outside ∂K_C . The equality $\langle \nabla \mathfrak{m}(x), x \rangle = \mathfrak{m}(x)$ at such points follows from the Euler Theorem on homogeneous functions. \square

We state the following technical lemma that is used in the proof of Lemma A.3.

Lemma A.2. Let $C_n, C \subset \mathbb{R}^d$ be closed convex sets with C_n converging in $L^1_{\text{loc}}(\mathbb{R}^d)$ to $C \neq \emptyset$, that is

$$\lim_{n \rightarrow +\infty} \mathcal{L}^d((C_n \Delta C) \cap \bar{B}_R) = 0, \quad \forall R > 0;$$

then for every $r > 0$ there exists $\alpha(r) > 0$ such that for any bounded sequence $x_n \in \partial C_n$

$$\mathcal{L}^d(C_n \cap B_r(x_n)) \geq \alpha(r).$$

Proof. Let us assume that there exists a sequence $x_n \in \partial C_n \cap B_{\frac{R}{2}}$ with $R > 2r$ such that

$$\mathcal{L}^d(C_n \cap B_r(x_n)) \leq \frac{1}{n};$$

we define the sets

$$S_n = \left(\frac{C_n - x_n}{r} \right) \cap \mathbb{S}^{d-1}$$

and the cone

$$K_n = \{x_n + t\nu : \nu \in S_n, t \geq 0\}.$$

Since C is convex, then

$$K_n \cap B_r(x_n) \subset C_n \cap B_r(x_n), \quad C_n \setminus B_r(x_n) \subset K_n \setminus B_r(x_n).$$

So in particular we have

$$\frac{1}{n} \geq \mathcal{L}^d(C_n \cap B_r(x_n)) \geq \mathcal{L}^d(K_n \cap B_r(x_n)) = \int_0^r \mathcal{H}^{d-1}(tS_n) dt = \frac{r^d}{d} \mathcal{H}^{d-1}(S_n),$$

and then $\mathcal{H}^{d-1}(S_n) \leq \frac{d}{nr^d}$. On the other hand, if we set

$$K_{n,R} = \{x_n + t\nu : \nu \in S_n, 0 \leq t \leq 2R\} = K_n \cap B_{2R}(x_n),$$

we also have $C_n \cap (\bar{B}_R \setminus B_r(x_n)) \subset K_n \cap (\bar{B}_R \setminus B_r(x_n)) \subset K_{n,R} \setminus B_r(x_n)$ and then, since

$$\mathcal{L}^d(K_{n,R} \setminus B_r(x_n)) = \int_r^{2R} \mathcal{H}^{d-1}(tS_n) dt = \frac{2^d R^d - r^d}{d} \mathcal{H}^{d-1}(S_n),$$

we obtain

$$\begin{aligned} \mathcal{L}^d(C \cap \bar{B}_R) &= \lim_{n \rightarrow +\infty} \left(\mathcal{L}^d(C_n \cap \bar{B}_r(x_n)) + \mathcal{L}^d(C_n \cap B_R \setminus \bar{B}_r(x_n)) \right) \\ &\leq \lim_{n \rightarrow +\infty} \left(\frac{1}{n} + \frac{2^d R^d - r^d}{d} \mathcal{H}^{d-1}(S_n) \right) \\ &\leq \lim_{n \rightarrow +\infty} \frac{1}{n} \left(1 + \frac{2^d R^d - r^d}{r^d} \right) = 0, \end{aligned}$$

and this is a contradiction. \square

In the next lemma we show the connection between the L^1 convergence of characteristic functions of convex sets and the convergence of boundaries. We recall that the Hausdorff distance between two sets $A, B \subset F$ is defined as

$$d_{\mathcal{H}}(A, B) = \inf\{t : A \subset (B)_t \text{ and } B \subset (A)_t\},$$

where $(A)_t = \{x \in F : \text{dist}(x, A) < t\}$. On compact sets this distance induces the Kuratowski convergence. A sequence of compact sets K_j converges to a set K in the sense of Kuratowski if

- (1) for any sequence $(x_j)_j$ of elements $x_j \in K_j$, if $x_j \rightarrow x$, then $x \in K$;

(2) for any $x \in K$, there exists a sequence $(x_j)_j$ of elements $x_j \in K_j$ such that $x_j \rightarrow x$. Indeed, if $x_j \in K_j$ for every j and $x_j \rightarrow x$, then $x \in K$ because for every $\varepsilon > 0$ the points x_j definitively belong to K_ε . Moreover, fixed $x \in K$, for every $j \in \mathbb{N}$ there is ν_j such that $K_i \subset K_{1/j}$ for $i \geq \nu_j$, hence we may select a sequence of points $x_j \in K_j$ converging to x .

Lemma A.3. *Let $C \subset \mathbb{R}^d$ be a convex set and let $(C_n)_n \subset \mathbb{R}^d$ be a sequence of convex sets such that*

$$\lim_{n \rightarrow +\infty} \gamma_F(C \Delta C_n) = 0.$$

Then ∂C_n converges uniformly on compact sets to ∂C , that is for every compact set K the sequence $\partial C_n \cap K$ converges to $\partial C \cap K$ in the Hausdorff distance.

Proof. It suffices to prove the statement for $K = \overline{B_R(0)}$; we have

$$\mathcal{L}^d((C_n \Delta C) \cap B_{R+1}(0)) \leq (2\pi)^{\frac{d}{2}} e^{\frac{(R+1)^2}{2}} \gamma_F(C_n \Delta C).$$

Assume by contradiction that there exists $\varepsilon_0 > 0$ such that for infinitely many $n \in \mathbb{N}$, either $\partial C_n \cap \overline{B_R(0)} \not\subset (\partial C)_{\varepsilon_0} \cap \overline{B_R(0)}$, or $\partial C \cap \overline{B_R(0)} \not\subset (\partial C_n)_{\varepsilon_0} \cap \overline{B_R(0)}$. In the first case there are infinitely many $n \in \mathbb{N}$ for which there exists $x_n \in \partial C_n \cap \overline{B_R(0)}$ but $x_n \notin (\partial C)_{\varepsilon_0} \cap \overline{B_R(0)}$; we have two possibilities, either $B_{\varepsilon_0}(x_n) \subset C^\circ$ or $B_{\varepsilon_0}(x_n) \subset \mathbb{R}^d \setminus C$. If $B_{\varepsilon_0}(x_n) \subset C^\circ$, then

$$\mathcal{L}^d((C_n \Delta C) \cap B_{R+1}(0)) \geq \mathcal{L}^d(B_{\varepsilon_0}(x_n) \setminus C_n) \geq \frac{1}{2} \omega_d \varepsilon_0^d.$$

If $B_{\varepsilon_0}(x_n) \subset \mathbb{R}^d \setminus C$, then by Lemma A.2

$$\mathcal{L}^d((C_n \Delta C) \cap B_{R+1}(0)) \geq \mathcal{L}^d(B_{\varepsilon_0}(x_n) \cap C_n) \geq \alpha(\varepsilon_0);$$

In both cases

$$\limsup_{n \rightarrow +\infty} \mathcal{L}^d((C_n \Delta C) \cap B_{R+1}(0)) > 0.$$

Similarly, if there exists $x \in \partial C \cap \overline{B_R(0)}$ such that for infinitely many $n \in \mathbb{N}$, $B_{\varepsilon_0}(x) \cap \partial C_n = \emptyset$, then either $B_{\varepsilon_0}(x) \subset C_n^\circ$ or $B_{\varepsilon_0}(x) \subset \mathbb{R}^d \setminus C_n$, and then again either

$$\mathcal{L}^d((C_n \Delta C) \cap B_{R+1}(0)) \geq \mathcal{L}^d(B_{\varepsilon_0}(x) \setminus C) \geq \frac{1}{2} \omega_d \varepsilon_0^d,$$

or

$$\mathcal{L}^d((C_n \Delta C) \cap B_{R+1}(0)) \geq \mathcal{L}^d(B_{\varepsilon_0}(x) \cap C),$$

so that, again,

$$\limsup_{n \rightarrow +\infty} \mathcal{L}^d((C_n \Delta C) \cap B_{R+1}(0)) > 0.$$

contradicting the fact that $\mathcal{L}^d((C_n \Delta C) \cap B_{R+1}(0)) \rightarrow 0$ as $n \rightarrow +\infty$. \square

Proposition A.4. *Let $C \subset \mathbb{R}^d$ be a closed convex set. Then for any $\delta > 0$ there exists a closed convex set C_δ such that $C \subset C_\delta^\circ$, ∂C_δ is smooth and*

$$\lim_{\delta \rightarrow 0} \gamma_F(C_\delta \setminus C) = 0.$$

Proof. Fix $\delta > 0$ and set

$$(C)_\delta = \{y : d(y, C) \leq \delta\}.$$

Then $(C)_\delta$ is convex and contains C . A result of Federer [18] implies that the boundary of $(C)_\delta$ is $C^{1,1}$ if δ is sufficiently small; however that is not enough for our aims.

Fix $\delta > 0$, let \mathfrak{m} be the Minkowski function of $(C)_\delta$ and let $\varrho \in C_c^\infty(\mathbb{R}^d)$ be a standard mollifier. For $\eta > 0$ define as usual $\varrho_\eta(x) = \varrho(x/\eta)/\eta^d$,

$$\mathfrak{m}_\eta = \mathfrak{m} * \varrho_\eta,$$

and consider the set

$$C_\delta = \{x \in \mathbb{R}^d : \mathfrak{m}_\delta(x) \leq 1\}.$$

Since \mathfrak{m} is convex, each \mathfrak{m}_η is convex too. Indeed,

$$\begin{aligned} \mathfrak{m}_\eta(\lambda x_1 + (1-\lambda)x_2) &= \int \mathfrak{m}(\lambda x_1 + (1-\lambda)x_2 - y) \varrho_\eta(y) dy \\ &= \int \mathfrak{m}(\lambda(x_1 - y) + (1-\lambda)(x_2 - y)) \varrho_\eta(y) dy \\ &\leq \lambda \int \mathfrak{m}(x_1 - y) \varrho(y) dy + (1-\lambda) \int \mathfrak{m}(x_2 - y) \varrho(y) dy \\ &= \lambda \mathfrak{m}_\eta(x_1) + (1-\lambda) \mathfrak{m}_\eta(x_2). \end{aligned}$$

Therefore, C_δ is a convex set.

Let us prove that $C_\delta^\circ \supset C$. For every $x \in C$, the ball $B_\delta(x)$ is contained in $(C)_\delta$, and then

$$\mathfrak{m}_\delta(x) = \int_{B_\delta(x)} \mathfrak{m}(y) \varrho_\delta(x-y) dy \leq 1$$

since $\mathfrak{m}(y) \leq 1$ for all $y \in B_\delta(x)$. This shows that $C \subset C_\delta$. To prove the inclusion $C \subset C_\delta^\circ$ we remark that $\overline{B_\delta(x)} \cap C$ has positive Lebesgue measure and it is contained in $(C)_\delta^\circ$, therefore the restriction of \mathfrak{m} to $\overline{B_\delta(x)} \cap C$ has maximum strictly less than 1 and the integral above is strictly less than 1. This shows that $C \subset C_\delta^\circ$.

Let us prove that if δ is sufficiently small then the boundary of C_δ is smooth. We have only to show that the gradient of \mathfrak{m}_δ does not vanish at the boundary. To this aim it is sufficient to show that for every x such that $\mathfrak{m}_\delta(x) = 1$ we have $\langle \nabla \mathfrak{m}_\delta(x), x \rangle \neq 0$.

Let $r > 0$ be such that $B_r(0) \subset C$ and let

$$(i) \ \delta < r/4, \quad (ii) \ \delta \int_{\mathbb{R}^d} |u| \varrho(u) du < r/2.$$

For every x such that $\mathfrak{m}_\delta(x) = 1$ there exists $\bar{x} \in B_\delta(x)$ such that $\mathfrak{m}(\bar{x}) \geq 1$ (otherwise we would get $\mathfrak{m}_\delta(x) < 1$). Since $B_r(0) \subset (C)_\delta$, then \mathfrak{m} is $1/r$ -Lipschitz, so that for every $y \in \mathbb{R}^d$ we have $\mathfrak{m}(y) \geq \mathfrak{m}(\bar{x}) - \|y - \bar{x}\|/r$, and hence by (i)

$$\mathfrak{m}(y) \geq \frac{1}{2}, \quad \forall y \in B_\delta(x).$$

Consequently,

$$\begin{aligned} \langle \nabla \mathfrak{m}_\delta(x), x \rangle &= \int_{\mathbb{R}^d} \langle \nabla \mathfrak{m}(y), x - y \rangle \varrho_\delta(x-y) dy + \int_{\mathbb{R}^d} \langle \nabla \mathfrak{m}(y), y \rangle \varrho_\delta(x-y) dy \\ &= \int_{\mathbb{R}^d} \langle \nabla \mathfrak{m}(y), x - y \rangle \varrho_\delta(x-y) dy + \int_{B_\delta(x)} \mathfrak{m}(y) \varrho_\delta(x-y) dy. \end{aligned}$$

The modulus of the first integral does not exceed

$$\frac{1}{r} \int_{\mathbb{R}^d} |x - y| \varrho_\delta(x-y) dy = \frac{\delta}{r} \int_{\mathbb{R}^d} |u| \varrho(u) du < \frac{1}{2}$$

while the second integral is $\geq 1/2$. Therefore, $\langle \nabla \mathfrak{m}_\delta(x), x \rangle > 0$.

To prove the last statement it is sufficient to show that for every $x \notin C$ we have $\mathfrak{m}_\delta(x) > 1$ if δ is small enough. Indeed, in this case $\mathbb{1}_{C_\delta \setminus C}$ goes to 0 pointwise as $\delta \rightarrow 0$, so that $\gamma_F(C_\delta \setminus C)$ vanishes as $\delta \rightarrow 0$. Let $\delta_0 = \text{dist}(x, C) > 0$ and let $\delta < \delta_0/2$. Then $\overline{B_\delta(x)} \cap (C)_\delta = \emptyset$, so that $\mathfrak{m} > 1$ in $\overline{B_\delta(x)} \cap (C)_\delta$ and $\min\{\mathfrak{m}(y) : y \in \overline{B_\delta(x)} \cap (C)_\delta\} > 1$. Consequently,

$$\mathfrak{m}_\delta(x) = \int_{B_x(\delta)} \mathfrak{m}(y) \varrho_\delta(x-y) dy > 1.$$

□

We conclude this section approximating an infinite dimensional open convex set by finite dimensional regular open convex sets.

Proposition A.5. *Let $\Omega \subset X$ be an open convex set. Then there exists a sequence of open convex cylindrical sets $\Omega_n \supset \Omega_{n+1} \supset \Omega$ with smooth boundaries, such that*

$$\lim_{n \rightarrow +\infty} \gamma(\Omega_n \setminus \Omega) = 0.$$

Proof. Since Ω is an open convex set, then $\gamma(\partial\Omega) = 0$. Since $\overline{\Omega}$ is a closed convex set and X is separable, by the Lindelöf theorem, see e.g. [17, Thm I.4.14] we have

$$\overline{\Omega} = \bigcap_{j \in \mathbb{N}} \overline{S}_j,$$

where

$$S_j = S(x_j^*, a_j) = \{x \in X : x_j^*(x) < a_j\}$$

with $x_j^* \in X^* \setminus \{0\}$, $a_j \in \mathbb{R}$, are open half-spaces containing Ω . The set

$$A_n = S_1 \cap \dots \cap S_n$$

is an open convex set containing Ω , and $\overline{A}_n = \overline{S}_1 \cap \dots \cap \overline{S}_n$ contains $\overline{\Omega}$. Then,

$$\gamma(\Omega) = \gamma(\overline{\Omega}) = \lim_{n \rightarrow +\infty} \gamma(\overline{A}_n).$$

We denote by F_n the linear span of the vectors x_1^*, \dots, x_n^* , which is a subspace of H of dimension $d \leq n$. We fix an orthonormal (along H) basis $\{h_1, \dots, h_d\}$ of F_n contained in $Q(X^*)$ and we define the projection $\Pi_n : X \rightarrow F_n$, $\Pi_n(x) = \sum_{j=1}^d \hat{h}_j(x) h_j$. The induced measure $\gamma \circ \Pi_n^{-1}$ in F_n is denoted by γ_n ; if F_n is identified with \mathbb{R}^d through the isomorphism $h \mapsto ([h, h_1]_H, \dots, [h, h_d]_H)$, then γ_n is just the standard Gaussian measure in \mathbb{R}^d .

Then, $\overline{A}_n = \Pi_n^{-1}(C_n)$, C_n is a polyhedral closed convex set in F_n with $\gamma(\overline{A}_n) = \gamma_n(C_n)$. By Proposition A.4, for any n we find a smooth open convex set \mathcal{O}_n with smooth boundary such that $C_n \subset \mathcal{O}_n$ and

$$\gamma_n(\mathcal{O}_n \setminus C_n) \leq \frac{1}{n}.$$

We may then define $\Omega_n = \Pi_n^{-1}(\mathcal{O}_n)$. Such sets are open cylindrical convex sets, and $\Omega \subset \Omega_n$ for any $n \in \mathbb{N}$. If $F_n = F_{n+1}$, i.e. x_{n+1}^* is a linear combination of x_1^*, \dots, x_n^* , then $C_{n+1} \subset C_n$, otherwise $F_{n+1} = F_n \times \mathbb{R}$ and $C_{n+1} \subset C_n \times \mathbb{R}$. To get the inclusion $\Omega_{n+1} \subset \Omega_n$, it suffices to apply Proposition A.4 with decreasing sequences (δ_n) in place of δ .

Moreover,

$$\gamma(\Omega) \leq \liminf_{n \rightarrow +\infty} \gamma(\Omega_n) = \liminf_{n \rightarrow +\infty} \gamma_n(\mathcal{O}_n) \leq \lim_{n \rightarrow +\infty} \left(\gamma_n(C_n) + \frac{1}{n} \right) = \lim_{n \rightarrow +\infty} \gamma(\overline{A}_n) = \gamma(\overline{\Omega}) = \gamma(\Omega),$$

then the conclusion follows. □

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