A compatible-incompatible decomposition of symmetric tensors in L^p with application to elasticity

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Abstract

In this paper, we prove the Saint-Venant compatibility conditions in L^p for $p \in (1, +\infty)$, in a simply-connected domain of any space dimension. As a consequence, alternative, simple and direct proofs of some classical Korn inequalities in L^p are provided. We also use the Helmholtz decomposition in L^p to show that every symmetric tensor in a smooth domain can be decomposed in a compatible part, which is the symmetric part of a displacement gradient, and in an incompatible part, which is the incompatibility of a certain divergence-free tensor. Moreover under suitable Dirichlet boundary condition, this Beltrami-type decomposition is proved to be unique. This decomposition result has several applications, one of which being in dislocation models, where the incompatibility part is related to the dislocation density and where $1 < p < 2$. This justifies the need to generalize and prove these rather classical results in the Hilbertian case ($p = 2$), to the full range $p \in (1, +\infty)$.

Keywords: Elasticity, Korn inequality, compatibility conditions, strain decomposition

1 Introduction

1.1 Intrinsic and displacement-based approaches in elasticity

The classical variational formulation of three-dimensional finite elasticity problems can be formulated as follows. Let $\Omega \subseteq \mathbb{R}^3$ be a domain, i.e. an open, bounded, connected and Lipschitz set which is the reference configuration of a hyperelastic and homogeneous body. We say that Ω is a smooth domain if its boundary is C^∞. Let $\phi : \overline{\Omega} \to \mathbb{R}^3$ be a deformation, i.e. a sufficiently smooth map (for example ϕ is in the Sobolev space $H^1(\Omega, \mathbb{R}^3)$), globally injective on Ω and which preserves the orientation, i.e. $\det\nabla\phi > 0$ almost everywhere in Ω. The set $\phi(\overline{\Omega})$ is the current configuration of the body.

The minimization problem of three dimensional elasticity consists in looking for a solution of

$$\min_{\phi\in\mathcal{A}} I(\phi), \tag{1.1}$$

where $\mathcal{A}$ is a family of deformations and

$$I(\phi) := \int_\Omega W(\nabla\phi)\mathrm{d}x - \int_\Omega f \cdot \phi \mathrm{d}x \tag{1.2}$$

is the potential energy. Here $W : \mathbb{M}^3_+ := \{A \in \mathbb{M}^3 : \det A > 0\} \to \mathbb{R}$ is the density of the elastic energy and $f : \Omega \to \mathbb{R}^3$ is the density of the volume force applied to Ω. If the energy W is polyconvex and satisfies some growth conditions, a classical result due to J. Ball (see [3], see also [2] and [8]) shows the existence of minimizers for the functional I.

An alternative way to study this problem, sometimes referred to in literature as the intrinsic approach (see for example [10]), consists in choosing as problem unknown the Green St-Venant tensor E instead of the deformation ϕ. This physical quantity is the change of metric from the reference to the current configuration. By a constitutive law, it is also related to the strain, which, being measured in the current configuration, turns out to be an intrinsic quantity. We can write E in function of the displacement field $u := \phi - \mathrm{Id}$ as $E = \nabla^S u + \frac{1}{2}\nabla u^T \nabla u$, where $\nabla^S u := \frac{1}{2}(\nabla u + \nabla u^T)$ is the symmetric part of the Jacobian matrix ∇u. The issue of passing from one description to the other, can be formulated as follows: *given two functional spaces $\mathcal{B}$ and $\mathcal{C}$ and given $E \in \mathcal{B}$ a prescribed symmetric tensor, is there any $u \in \mathcal{C}$ such that*

$$E = \nabla^S u + \frac{1}{2}\nabla u^T \nabla u? \tag{1.3}$$

Equivalently, *given two functional spaces $\mathcal{B}$ and $\mathcal{C}$ and given $g \in \mathcal{B}$ a symmetric and positive-definite tensor in Ω (i.e., a Riemannian metric), is there any $\phi \in \mathcal{C}$ such that*

$$g = (\nabla\phi)^T \nabla\phi? \tag{1.4}$$

If $\mathcal{B}$ is the space of smooth functions C^∞, then (1.4) is true if and only if the Riemann curvature tensor R_{ijkl} of the manifold Ω is zero and in this case we say that the metric is compatible. The result is still valid if $\mathcal{B} = C^1$ or $\mathcal{B} = W^{2,\infty}$ (for a proof, we refer to [22] and [23]).The case $\mathcal{B} = C^2$, which states the Strong Saint-Venant compatibility conditions, is recalled in Theorem 3.13.

In linearized elasticity one can also pass from a displacement-based approach to an intrinsic approach. More precisely let $\mathcal{D}$ be a family of displacements and let

$$j(u) := \frac{1}{2}\int_\Omega \mathbb{C}\nabla^S u \cdot \nabla^S u \mathrm{d}x - \int_\Omega f \cdot u \mathrm{d}x, \tag{1.5}$$

be the linearized functional associated to the potential energy I defined in (1.2), where $\mathbb{C} := D^2W(I)$ is the elasticity tensor. Our aim is to minimize j on the set $\mathcal{D}$. For example, if $\mathcal{D} = H^1(\Omega, \mathbb{R}^3)$, $f \in L^2(\Omega, \mathbb{R}^3)$ and $\int_\Omega f \cdot r \mathrm{d}x = 0$ for every rigid displacement r, then there exists a unique minimizer of j. If we want to study this Neumann problem from another point of view, by using an intrinsic approach, the new unknown will be the strain tensor

$$e := \mathbb{C}^{-1}\sigma,$$

with σ the Cauchy stress tensor. In order to pass from one description to the other, the question in this simpler setting is whether e is the linearized part of the Green St-Venant tensor E, that is: *given two functional spaces $\mathcal{B}$ and $\mathcal{C}$ and a symmetric tensor $e \in \mathcal{B}$, is there any displacement u such that*

$$e = \nabla^S u? \tag{1.6}$$

Observe that the problem to establish when e is the symmetric part of the gradient of a displacement u is in some sense similar to determining whenever a vector field $h \in C^1(\Omega, \mathbb{R}^3)$ is conservative. Indeed, Poincaré Lemma tells us that if Ω is simply-connected, there exists a scalar function $p \in C^2(\Omega)$ such that $h = \nabla p$ if and only if h is irrotational, i.e. $\mathrm{Curl}\ h = 0$. Let $\mathbb{S}^n$ be the space of all symmetric matrices of order n. Then Ph.G.Ciarlet and P.Ciarlet in [10] proved that if $\mathcal{B} = L^2(\Omega, \mathbb{S}^3)$, then (1.6) is true if and only if

$$R_{ijkl}(e) := \partial_i\partial_k e_{jl} + \partial_j\partial_l e_{ik} - \partial_j\partial_k e_{il} - \partial_i\partial_l e_{jk} = 0 \text{ in } H^{-2}(\Omega, \mathbb{S}^3). \tag{1.7}$$

These are exactly the Weak Saint-Venant compatibility conditions in linearized elasticity.

1.2 Article outline and main results

This article is organized as follows. In Section 2.2 we recall the classical problem of reconstructing a displacement from a given smooth symmetric tensor. An easy computation shows that the displacement (and the rotation) can be rewritten as recursive line integrals depending on the strain tensor e, and its curl $\mathrm{Curl}\ e$. In Proposition 2.2 we observe that this integral is well defined if and only if the incompatibility tensor inc e of the strain e, viz.,

$$\mathrm{inc}\, e := \mathrm{Curl}\,(\mathrm{Curl}\ e)^T, \tag{1.8}$$

is zero. From this fact, we easily deduce in Corollary 2.4 the Strong Saint-Venant compatibility conditions in linearized elasticity in the well-known smooth case. In Section 3.2 we give a geometrical interpretation of the concept of incompatibility in linearized elasticity. If, for $\eta > 0$, we define the metric

$$g^\eta := I + 2\eta e,$$

then Proposition 3.11 shows that $R_{ijkl}(e)$ is exactly the first order term of the Taylor expansion at $\eta = 0$ of the Riemannian curvature tensors R^{η}_{ijkl} associated to g^{η}. This geometric linearization justifies the definition of $R_{ijkl}(e)$ as the Riemann curvature tensor in linearized elasticity. Moreover it is seen that $R_{ijkl}(e) = \epsilon_{ijs}\epsilon_{klr}(\text{inc } e)_{sr}$, so that it vanishes if and only if inc e vanishes (see Remark 3.12). This observation allows us to rewrite the weak Saint-Venant compatibility tensor of Ciarlet in terms of the incompatibility tensor inc e (see Theorem 3.14).

The new contributions of our work are found in Section 3. The first main result of our work is Theorem 3.17, about Eq. (1.6), which extends the result of Ph.Ciarlet when $e \in L^p$, with $1 < p < +\infty$. This extension may be interesting if the body presents some defects such as dislocations, since the involved energies are not quadratic (see [25]) and one must consider exponents in the range $1 < p < 2$. In Ph.Ciarlet's proof, the most important tool is the weak Poincaré lemma, which basically states that every irrotational field $h \in H^{-1}(\Omega, \mathbb{R})$ is conservative. This theorem was proved in an elegant manner by Kesavan in [19]. Its proof relies substantially on the existence of solution for the Stokes equations if the force f belongs to the space $H^{-1}(\Omega, \mathbb{R}^3)$. In order to prove the same kind of result in the non-Hilbertian case, i.e. with p possibly different from 2, other techniques have to be used. Our main idea is to define a tensor T as a sum of e and of a skewsymmetric tensor ω constructed, in a general form, from the curl of e (see Lemma 3.15). Then we conclude be noticing that T is irrotational and applying the weak Poincaré Lemma in L^p. Let us recall that the pioneer work about this topic is due to E. Cesaro [7].

Our second main result is the structure Theorem 3.19, which contains a decomposition of a symmetric tensor $e \in L^p$, with $p \in (1, +\infty)$ in a sum of the type

$$e = \nabla^S u + \text{inc}F, \tag{1.9}$$

where $u \in W^{1,p}(\Omega, \mathbb{R}^3)$ is a Sobolev vector field and $F \in L^p(\Omega, \mathbb{M}^3)$ is a divergence-free tensor field with appropriate boundary conditions. We call $\nabla^S u$ the compatible part of the decomposition and inc$F$ its incompatible part. Here, a crucial lemma will consist in exhibiting those symmetric gradients which write also as the incompatibility of a tensor (see Lemma 3.17). Such a decomposition has potentially many applications. In particular, it was used in [29] to study the mathematical properties of countable families of dislocation lines.

Such a decomposition is often named after E. Beltrami for his pioneer article [4]. Let us precise that at the best of our knowledge such a decomposition, though regularly mentioned in the physical literature since [27], was not given a mathematical proof. Here the proof holds not only for the Hilbertian case $p = 2$ but for the whole range $p \in (1, +\infty)$.

1.3 Application to Korn inequality in L^p

A related issue is the Korn inequality, whose study is the object of the Section 4. By using the same argument as in [10], we show in an alternative way two classical Korn inequalities in L^p (see Theorem 4.2 and 4.3) in a simply-connected domain. Korn inequalities are of utmost importance in linear and nonlinear theories of elasticity. Let us recall that Korn inequality basically asserts that if Ω is a bounded domain and $p \in (1, +\infty)$ there exists a constant $C > 0$ such that

$$\| u \|_{W^{1,p}} \leq C \| \nabla^S u \|_{L^p}, \tag{1.10}$$

for all $u \in W_0^{1,p}(\Omega, \mathbb{R}^3)$. For $p = 1$ or $p = \infty$, some counterexamples show that this result does not hold. However, Korn inequality is very important in elasticity not only in the case $p = 2$, where it allows one to show that the functional (1.5) is coercive, but also for an exponent p different from 2. For example, it is essential to prove a geometric rigidity estimate (see, e.g., [11]) asserting that there exist a constant $C > 0$ and a rotation $Q \in \mathrm{SO}(n)$ such that

$$\| \nabla\phi - Q \|_{L^p} \leq C \| \operatorname{dist}(\nabla\phi, \mathrm{SO}(n)) \|_{L^p}, \tag{1.11}$$

for every deformation $\phi \in W^{1,p}(\Omega, \mathbb{R}^n)$.

There exists several proofs of Korn inequality in the literature for $p = 2$. The most classical one (see for example [8]), valid in a domain $\Omega \subseteq \mathbb{R}^n$, is based on the J.-L. Lions Lemma, which says that every distribution v whose derivatives are in $L^2(\Omega)$, belongs to the Sobolev space $H^1(\Omega)$. An other proof of (1.10) for $p = 2$ was provided for open sets with cone property by Nitsche in [24]. Here the idea is to construct an extension operator from Ω to the whole space $\mathbb{R}^n$ which preserves the strain. If $p \neq 2$, the proof is more complicated. If $\Omega \subseteq \mathbb{R}^2$, i.e. Ω is a plane domain, a Korn inequality type was proved by Wang (see [32]) in a quite simple way. The proof is based on the existence of solutions $\phi_0 \in W_0^{1,p}(\Omega, \mathbb{R}^3)$ for the equation $\operatorname{div}\phi_0 = f$, where $f \in L^p(\Omega)$ with null average. For a general proof in L^p in arbitrary dimension we refer to [20] (see also [12] in the case of $\mathcal{C}^2$ domains). In the present paper we propose a simple and direct proof of Korn inequality in L^p which is a direct consequence of our results (cf. Theorem 3.19 and in particular, Theorem 3.17).

2 Preliminaries

2.1 Notations and conventions

Assumption 2.1. *Unless otherwise specified, the considered domain Ω is a open connected and bounded subset of $\mathbb{R}^3$ with Lipschitz boundary and outward unit normal N.*

Smoothness of the boundary is not a too strong assumption for this problem, as discussed in Remark 3.5.

The space of the square matrices of order n is denoted by $\mathbb{M}^n$. Let $B \in \mathbb{M}^n$ be a square matrix. Then $B^S := \frac{B+B^T}{2}$ is the symmetric part of B, while $B^A := \frac{B-B^T}{2}$ is its skew-symmetric part of B. Moreover $\mathbb{S}^n$ is the space of all symmetric matrices of order n, while $\mathbb{A}^n$ is the space of all skew-symmetric matrices of order n.

Here δ_{ij} is the Kronecker symbol, while ϵ_{ijk} is the Levi-Civita symbol. We will use the relation

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}.$$

Let $\Omega \subseteq \mathbb{R}^n$ be an open set. We will use the following functional spaces:

- $\mathcal{C}^k(\Omega, \mathbb{R}^n)$ is the space of continuous function whose derivative up to order k are continuous;
- $L^p(\Omega, \mathbb{R}^n)$ with $1 \leq p \leq +\infty$, and the usual norm $\| \,.\, \|_{L^p}$ is the usual Lebesgue space;
- $W^{m,p}(\Omega, \mathbb{R}^n)$ with $1 \leq p \leq +\infty$, $m \in \mathbb{N}$ and the usual norm $\| \,.\, \|_{W^{m,p}}$ is the usual Sobolev space;
- $\mathcal{D}(\Omega, \mathbb{R}^n)$ is the topological vector space of indefinitely differentiable functions with compact support on Ω;
- $\mathcal{D}'(\Omega)$ is the space of distributions on Ω;
- $W_0^{1,p}(\Omega, \mathbb{R}^n)$, where $1 \leq p < +\infty$ is the closure of $\mathcal{D}(\Omega, \mathbb{R}^n)$ with respect to the usual Sobolev norm;
- $W^{-m,q}(\Omega, \mathbb{R}^n) := (W_0^{m,p}(\Omega, \mathbb{R}^n))'$, where $1 \leq p + \infty$ and $q = \frac{p}{p-1}$, is the dual of the usual Sobolev space;
- $W^{\frac{1}{p},p}(\partial\Omega, \mathbb{R}^n)$ denotes the set of all Sobolev functions which are trace of a function $u \in W^{1,p}(\Omega, \mathbb{R}^n)$.

The divergence of a vector v and of a tensor T are defined componentwise as follows:

- $(\,\mathrm{div} v) := \partial_j v_j$; $(\,\mathrm{div} T)_i := \partial_j T_{ij}$,

where sum is intended on the repeated indices. The curl of a vector v and of a tensor T are defined componentwise as follows:

- $(\,\mathrm{Curl}\ v)_i := \epsilon_{ikl}\partial_k v_l$; $(\,\mathrm{Curl}\ T)_{ij} := \epsilon_{jkl}\partial_k T_{il}$.

The incompatibility of a tensor E is defined componentwise as follows:

- $(\,\mathrm{inc} E)_{ij} := (\,\mathrm{Curl}\ (\,\mathrm{Curl}\ E)^{\mathrm{T}})_{ij} = \epsilon_{ikm}\epsilon_{jln}\partial_k\partial_l E_{mn}$.

Moreover, we will use the following spaces:

$$
\begin{aligned}
L^p_{\mathrm{div}}(\Omega, \mathbb{R}^{3\times 3}) &:= \{F \in L^p(\Omega, \mathbb{M}^3) \quad \text{s.t.} \quad \mathrm{div} F = 0\} \\
&= \mathrm{adh}_{L^p}\{F \in \mathcal{C}^\infty(\bar{\Omega}, \mathbb{M}^3) \quad \text{s.t.} \quad \mathrm{div} F = 0\}, \qquad (2.1) \\
\mathcal{X}^p_{\mathrm{div}}(\Omega) &:= \left\{V \in L^p_{\mathrm{div}}(\Omega, \mathbb{M}^3) \text{ s.t. } \mathrm{Curl}\ V \in L^p(\Omega, \mathbb{M}^3)\right\}, \\
\mathcal{V}^p(\Omega) &:= \{V \in \mathcal{X}^p_{\mathrm{div}}(\Omega) \text{ s.t. } V \times N = 0 \text{ on } \partial\Omega\}\,, \qquad (2.2) \\
\tilde{\mathcal{V}}^p(\Omega) &:= \{V \in \mathcal{X}^p_{\mathrm{div}}(\Omega) \text{ s.t. } VN = 0 \text{ on } \partial\Omega\}\,. \qquad (2.3)
\end{aligned}
$$

Let $(M, (g_{ij}))$ be a Riemannian manifold and let ∇ be a Levi-Civita connection on M. We denote by Γ^k_{ij} the Christoffel symbols and with symbol R_{ijkl} the Riemannian curvature tensor, with the convention that $\Gamma_{ijk} := \Gamma^l_{ij} g_{kl}$.

2.2 A classical result: Michell-Cesaro-Volterra decomposition

As a first step, let us recall the problem of reconstructing a displacement from a given symmetric tensor. In linearized elasticity, if all the functions involved are smooth enough, we prove that the displacement field u is completely defined in terms of the linearized strain tensor e by a recursive integral formula (cf. (2.5)), which we compute explicitly.

Let $e \in C^\infty(\Omega, \mathbb{M}^3)$ be a symmetric tensor field such that $\operatorname{inc} e = 0$ on Ω. Let us fix $x_0, x \in \Omega$, and let $\gamma \in C^1([0,1], \Omega)$ be a curve in Ω such that $\gamma(0) = x_0$ and $\gamma(1) = x$. We define the following quantities:

$$w_i(x;\gamma) := w_i(x_0) + \int_\gamma \epsilon_{ipn} \partial_p e_{mn}(y) \mathrm{d}y_m \tag{2.4}$$

$$u_i(x;\gamma) := u_i(x_0) + \int_\gamma (e_{il}(y) - \epsilon_{ilk} w_k(y))\, dy_l. \tag{2.5}$$

Let us now prove that the quantities $w(x)$ and $u(x)$ defined in (2.4) and (2.5) do not depend on the choice of the path from x_0 to x. We will show that this is a consequence of the fact that inc e $= 0$. In such a case the quantities w and u define two C^∞ functions on Ω that will be called the rotation and the displacement vectors associated to the strain e, respectively. In order to prove this fact, we compute the jump of w and u between two arbitrary curves with the same endpoints, and observe that this quantity is zero if and only if the incompatibility tensor vanishes. These are exactly the well known Saint-Venant compatibility relations.

The rotation and displacement jumps are defined as

$$[\![w_i]\!](x;x_0) := w_i(x;\gamma) - w_i(x;\tilde{\gamma}), \tag{2.6}$$

$$[\![u_i]\!](x;x_0) := u_i(x;\gamma) - u_i(x;\tilde{\gamma}), \tag{2.7}$$

respectively.

Proposition 2.2. *Let $\Omega \subseteq \mathbb{R}^3$ be a simply-connected domain, let $x_0 \in \Omega$ be prescribed, and let $w, u \in C^\infty(\Omega, \mathbb{R}^3)$ be the functions defined in* (2.4) *and* (2.5)*, respectively. Then the following formulae hold:*

$$[\![w_i]\!](x;x_0) = \int_{S_{\gamma-\tilde{\gamma}}} (\operatorname{inc} e(y))_{im} \mathrm{d}S_m(y), \tag{2.8}$$

$$[\![u_i]\!](x;x_0) = \int_{S_{\gamma-\tilde{\gamma}}} (y_m - x_m) \epsilon_{imk} (\operatorname{inc} e(y))_{qk} \mathrm{d}S_q(y), \tag{2.9}$$

for all $x \in \Omega$, and where $S_{\gamma-\tilde{\gamma}}$ is a surface enclosed by the the closed path $\gamma - \tilde{\gamma}$. In particular,

$$[\![w_i]\!], [\![u_i]\!] = 0 \text{ for each couple of curves } \gamma,\ \tilde{\gamma} \iff \operatorname{inc} e = 0.$$

Remark 2.3. *As a consequence of* inc e $= 0$, (2.4) *and* (2.5) *do not depend on the choice of the curve $\gamma \in \mathcal{C}^1([0,1], \Omega)$ connecting x_0 to x. In particular, the vector fields $w \in \mathcal{C}^\infty(\Omega, \mathbb{R}^3)$ and $u \in \mathcal{C}^\infty(\Omega, \mathbb{R}^3)$ are univoquely defined. Thus, in* (2.4) *and* (2.5)*, one can use the notation* $\int_\gamma = \int_{x_0}^x$.

Proof. Let us first compute $[\![w_i]\!]$. The domain being simply-connected, there is always a surface $S := S_{\gamma-\tilde{\gamma}}$ in Ω which has as boundary the closed path $\gamma - \tilde{\gamma}$ (here $-\tilde{\gamma} : [0,1] \to \Omega$ is the curve defined by $-\tilde{\gamma}(t) := \tilde{\gamma}(1-t)$ for all $t \in [0,1]$). Then By Stokes formula it results that

$$[\![w_i]\!](x; x_0) = \int_S \epsilon_{mqr}\epsilon_{ipn}\partial_q\partial_p e_{rn} \mathrm{d}S_m(y) = \int_S (\mathrm{inc} e(y))_{im} \mathrm{d}S_m(y).$$

This proves the formula for $[\![w_i]\!]$. Since the closed path $\gamma - \tilde{\gamma}$ is arbitrary and the domain simply-connected, it results that $[\![w_i]\!] = 0$ if and only if $\mathrm{inc}\ e = 0$.

Now, observe first that (2.5) rewrites by part integration as

$$\begin{aligned} u_i(x; \gamma_{x_0,x}) = \int_\gamma &(e_{il}(y) + (y_m - (x_0)_m)\epsilon_{imk}\partial_l w_k(y))\, \mathrm{d}y_l \\ &- \epsilon_{imk} w_k(x)(x_m - (x_0)_m). \end{aligned} \tag{2.10}$$

For $[\![u_i]\!]$, apply again Stokes formula to deduce that

$$\begin{aligned} [\![u_i]\!](x; x_0) = &\int_S \epsilon_{qpl}\partial_p \left[e_{il}(y) + (y_m - (x_0)_m)\epsilon_{imk}\partial_l w_k(y)\right] \mathrm{d}S_q(y) \\ - \epsilon_{imk}&[\![w_k]\!](x)(x_m - (x_0)_m) \\ &= \int_S (\epsilon_{qpl}\partial_p e_{il}(y) + \epsilon_{qml}\epsilon_{imk}\partial_l w_k(y) \\ + (y_m - (x_0)_m)&\epsilon_{qpl}\epsilon_{imk}\partial_p\partial_l w_k(y) \mathrm{d}S_q(y) - \epsilon_{imk}(x_m - (x_0)_m) \int_S (\mathrm{inc} e(y))_{kq} \mathrm{d}S_q. \end{aligned}$$

We have already proved that $w_i(x; \gamma) = w_i(x; x_0) := \int_{x_0}^x \epsilon_{ipn}\partial_p e_{mn}(y) \mathrm{d}y_m$ in the simply-connected domain Ω, and hence the relation $\partial_i w_q(x) = \epsilon_{qpl}\partial_p e_{il}(x)$ holds. Since $\partial_l w_l = 0$,

$$\epsilon_{qml}\epsilon_{imk}\partial_l w_k = (\delta_{qi}\delta_{lk} - \delta_{qk}\delta_{li})\partial_l w_k = \delta_{qi}\partial_l w_l - \partial_i w_q = -\partial_i w_q,$$

and hence, by identity $\partial_l w_k = \epsilon_{krn}\partial_r e_{ln}$ and the fact that $(\mathrm{inc}\ e)_{qk} = \epsilon_{qpl}\epsilon_{krn}\partial_p\partial_r e_{ln}$, it holds

$$\begin{aligned} [\![u_i]\!](x; x_0) = &\int_S (y_m - (x_0)_m)\epsilon_{imk}\epsilon_{qpl}\epsilon_{krn}\partial_p\partial_r e_{ln}(y) \mathrm{d}S_q(y) \\ &- \epsilon_{imk}(x_m - (x_0)_m) \int_S (\mathrm{inc}\ e(y))_{kq} \mathrm{d}S_q(y) \\ &= \int_S (y_m - x_m)\epsilon_{imk}(\mathrm{inc}\ e(y))_{qk} \mathrm{d}S_q(y). \end{aligned}$$

This achieves the proof that $[\![w_i]\!], [\![u_i]\!] = 0$ if and only if $\mathrm{inc}\ e = 0$. ☐

Now it is straightforward to prove the following result:

Corollary 2.4 (Saint-Venant compatibility conditions in C^∞)**.** *Let Ω be a simply-connected and bounded open set in $\mathbb{R}^3$ and let $e \in C^\infty(\Omega, \mathbb{M}^3)$ be a symmetric tensor field. Then there exists $u \in C^\infty(\Omega, \mathbb{R}^3)$ (given by (2.5)) such that $e = \nabla^s u$, if and only if*

$$\mathrm{inc}\ e = 0. \tag{2.11}$$

Proof. If e is the symmetric gradient of a displacement it is straightforward that its incompatibility vanishes. Then let us prove the converse. Therefore assume $\mathrm{inc}\, e = 0$ and define the vector fields w and u as in (2.4) and (2.5). Proposition 2.2 shows that w and u are independent of the path γ, so that we can write

$$\begin{aligned} w_i(x) &= w_i(x_0) + \int_{x_0}^{x} \epsilon_{ipn}\partial_p e_{jn}(\xi)\mathrm{d}\xi_j, \\ u_i(x) &= u_i(x_0) + \int_{x_0}^{x} \left(e_{ij}(\xi) - \epsilon_{ijk}w_k(\xi)\right)\mathrm{d}\xi_j, \end{aligned} \tag{2.12}$$

thus $\partial_j w_i = \epsilon_{ipn}\partial_p e_{jn}$ and $\partial_j u_i = e_{ij} - \epsilon_{ijk}w_k$, from which the thesis follows since e is symmetric, and $\epsilon_{ijk}w_k$ skewsymmetric. □

The L^p counterpart of Corollary 2.4 will be proved in Theorem 3.17 with other techniques. It represents the cornerstone of the proof of Korn inequality in L^p. Now, the following classical quantities can be introduced:

Definition 2.5. *Let $u : \Omega \to \mathbb{R}^3$ be a smooth displacement field. Let us introduce the following quantities:*

(i) $e_{ij} := \frac{1}{2}\left(\partial_j u_i + \partial_i u_j\right)$ *is said strain tensor (it is the linear part of Green St-Venant tensor* $E_{ij} := e_{ij} + \partial_i u_k \partial_k u_i$*) ;*

(ii) $\omega_{ij} := \frac{1}{2}\left(\partial_j u_i - \partial_i u_j\right)$ *is said rotation tensor;*

(iii) $w_i := \frac{1}{2}\epsilon_{ijk}\omega_{kj}$ *is said rotation vector.*

Remark 2.6. *A simple computation allows us to express the rotation tensor w_{ij} in terms of the rotation vector w_i, since*

$$\begin{aligned} \epsilon_{ijk}w_k &= \frac{1}{2}\epsilon_{ijk}\epsilon_{kmn}\omega_{nm} \\ &= \frac{1}{2}\left(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\right)\omega_{nm} = \frac{1}{2}\left(\omega_{ji} - \omega_{ij}\right) = -\omega_{ij}. \end{aligned}$$

3 Decomposition of a symmetric tensor in L^p

3.1 Some preliminary results

Let us recall some results and remarks which will be used in the sequel.

Lemma 3.1 (Helmholz-Weyl-Hodge-Yanagisawa)**.** *Let $1 < p < \infty$ and let Ω be a smooth domain in $\mathbb{R}^3$. For every $F \in L^p(\Omega, \mathbb{M}^3)$, there exists $u_0 \in W_0^{1,p}(\Omega, \mathbb{R}^3)$ and a solenoidal $V \in \tilde{\mathcal{V}}^p(\Omega)$, such that*

$$F = Du_0 + \text{Curl } V, \quad \left(L^p(\Omega, \mathbb{M}^3) = \nabla W_0^{1,p}(\Omega, \mathbb{R}^3) \oplus \text{Curl } \tilde{\mathcal{V}}^p(\Omega)\right), \tag{3.1}$$

Alternatively, there exists $u \in W^{1,p}(\Omega, \mathbb{R}^3)$ and a solenoidal $V_0 \in \mathcal{V}^p(\Omega)$, such that

$$F = Du + \text{Curl } V_0, \quad \left(L^p(\Omega, \mathbb{M}^3) = \nabla W^{1,p}(\Omega, \mathbb{R}^3) \oplus \text{Curl } \mathcal{V}^p(\Omega)\right). \tag{3.2}$$

Moreover the decompositions are unique, in the sense that u_0, V, V_0 are uniquely determined, while u is unique up to a constant, and it holds $\|Du_0\|_p, \|Du\|_p \leq C\|F\|_p$, respectively.

Remark 3.2. *When F is smooth with compact support, decompositions such as* (3.1) *and* (3.2) *are classically given [31, 5] by explicit formulae involving the divergence and the curl of F. Notice that no boundary data for F is here given.*

Remark 3.3. *Let $F \in \mathcal{C}^1$. In the particular case* $\mathrm{Curl}\ F = 0$ *the Helmholtz decomposition is trivial when Ω is a simply-connected domain. Indeed a well-known consequence of the Stokes theorem is that in such a case there exists $u \in \mathcal{C}^2(\Omega, \mathbb{R}^3)$ satisfying $F = Du$. This result extends for $F \in L^p$ with $1 < p < +\infty$ as shown in [15]. See [21] for a complete treatment of Helmholtz decomposition in L^p, relying on the pioneer paper [18].*

Remark 3.4. *Let Ω be a smooth simply-connected domain, and let $F \in L^p$ with $1 < p < +\infty$. If* $\mathrm{div} F = 0$ *then, by Lemma 3.1,* $F = \mathrm{Curl}\ V$ *with $V \in \tilde{\mathcal{V}}^p(\Omega)$. Remark that for smooth functions F, this result holds for any simply-connected domain.*

Remark 3.5. *Smoothness of the boundary is a strong requirement which is needed for the following reason:* (3.1) *and* (3.2) *require to solve a Poisson equation* $\Delta u = \mathrm{div} F$ *with the right-hand side in some distributional (i.e., Sobolev-Besov) space for which smoothness of the boundary is needed. It is known [17] that for a Lipschitz boundary the solution holds for $3/2 - \epsilon \leq p \leq 3 + \epsilon$, where $\epsilon = \epsilon(\Omega) > 0$. Note that for $p = 2$ a Lipschitz boundary would be sufficient.*

Lemma 3.6. *Let Ω be a domain, and H be a function in $L^p(\Omega)$ with $\int_\Omega H dx = 0$. Then there exists a function $h \in W^{1,p}(\Omega, \mathbb{R}^3)$ solution of*

$$\begin{cases} \mathrm{div} h = H & \text{in } \Omega, \\ h = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

satisfying $\|h\|_{W^{1,p}} \leq \|H\|_p$.

A proof of this Lemma can be found in [15, Theorem III.3.3]. Moreover it also holds that if $H \in C_c^\infty(\Omega)$, then

$$\|h\|_{C^k} \leq C_k \|H\|_{C^{k-1}}, \tag{3.4}$$

for every $k \in \mathbb{N}_*$ (see [15, Theorem III.3.5]). The following estimate can be found in [21].

Lemma 3.7 (Kozono-Yanagisawa)**.** *Let $F \in \mathcal{V}^p(\Omega)$ or $F \in \tilde{\mathcal{V}}^p(\Omega)$. Then $F \in W^{1,p}(\Omega, \mathbb{R}^{3\times 3})$ and it holds*

$$\|\nabla F\|_p \leq C \left(\|\, \mathrm{Curl}\ F\|_p + \|F\|_p \right). \tag{3.5}$$

This shows that $\mathcal{V}^p(\Omega)$ and $\tilde{\mathcal{V}}^p(\Omega)$ are closed subspaces in $W^{1,p}(\Omega, \mathbb{R}^{3\times 3})$. By virtue of Lemma 3.7 and for simply-connected and bounded domains, a better estimate, found in [31], reads as follows. Again, this classical result for smooth functions with compact support is less standard in our setting.

Lemma 3.8 (von Wahl)**.** *Let $F \in \mathcal{V}^p(\Omega)$ or $F \in \tilde{\mathcal{V}}^p(\Omega)$. Then it holds*

$$\|\nabla F\|_p \leq C \|\, \mathrm{Curl}\ F\|_p. \tag{3.6}$$

As a direct consequence the following result holds.

Lemma 3.9. *Let $F \in \mathcal{V}^p(\Omega)$ or $F \in \tilde{\mathcal{V}}^p(\Omega)$. Then* Curl $F = 0 \iff F = 0$.

Let us now state the linear elasticity problem in L^p. Let $\Omega \subseteq \mathbb{R}^3$ be a smooth domain, let $1 < p < +\infty$ and let $e := \nabla^S u$ be the linearized strain tensor and $f \in L^p(\Omega, \mathbb{R}^3)$, a volume force. The elasticity system reads

$$\begin{cases} \mathrm{div}\mathbb{C}e + f = 0 & \text{on } \Omega, \\ u = U & \text{on } \partial\Omega. \end{cases}$$

with $U \in W^{1/p,p}(\partial\Omega)$, the prescribed boundary datum. Alternatively the Neumann problem is associated with the boundary condition $\left(\mathbb{C}\nabla^S u\right) N = g$ on $\partial\Omega$, with $g \in W^{-1/p,p}(\partial\Omega)$ the exerted boundary force. If the material is homogeneous and isotropic and its reference configuration is a natural state, it is well known that the constitutive relations depend only on the Lamé constants of the material λ and μ and are given by the formula $\mathbb{C}e = 2\mu e + \lambda(\mathrm{tr}e)I$. These two Lamé constants satisfy the relations $\mu > 0$ and $3\lambda + 2\mu > 0$: in this case $\mathbb{C}$ is a coercive tensor, i.e. there exists $\alpha > 0$ such that $\mathbb{C}A \cdot A \geq \alpha \parallel A \parallel^2$ for all symmetric 3×3 matrix A. Therefore

$$\begin{aligned} \mathrm{div}\mathbb{C}e &= \mathrm{div}(2\mu e + \lambda(\mathrm{tr}e)I) = 2\mu\mathrm{div}e + \lambda\nabla\mathrm{tr}e \\ &= 2\mu\mathrm{div}(\frac{\nabla u + \nabla u^T}{2}) + \lambda\nabla\mathrm{tr}(\frac{\nabla u + \nabla u^T}{2}) = (\lambda + \mu)\nabla\mathrm{div}u + \mu\Delta u, \end{aligned}$$

and the Dirichlet problem in this particular case reads

$$\begin{cases} (\lambda + \mu)\nabla\mathrm{div}u + \mu\Delta u + f = 0 & \text{in } \Omega \\ u = U & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

By the regularity theory for partial differential elliptic equations, this problem admits a strong solution $u \in W^{2,p}(\Omega)$ in Ω. The same fact also holds for the Neumann problem.

3.2 Geometrical view of compatibility in nonlinear and linearized elasticity

Definition 3.10. *Let $\Omega \subseteq \mathbb{R}^3$ be an open subset and let $\eta > 0$ be a real number. Then a family of elastic metrics on Ω is given by*

$$g^\eta := I + 2\eta e, \tag{3.8}$$

where I is the identity matrix and e a smooth symmetric tensor.

Now let us compute the Riemannian curvature tensor of an elastic metric.

Proposition 3.11. *Let $\Omega \subseteq \mathbb{R}^3$ be an open set, $\eta > 0$ be a real number, and g^η a family of elastic metrics in Ω. Then the Riemann curvature tensor R^η_{ijkl} associated to the elastic metrics g^η is given in terms of e by the formula*

$$R^\eta_{ijkl} = \eta\epsilon_{ijs}\epsilon_{klr}(\mathrm{inc}\ e)_{sr} + o(\eta). \tag{3.9}$$

Proof. Let $g^{ij,\eta}$ be the inverse matrix of $g^\eta = (g^\eta_{ij})$. It is given by

$$g^{ij,\eta} = \delta^{ij} - 2\eta e^{ij} + o(\eta). \tag{3.10}$$

The Christoffel symbols of the Riemannian metric g^η_{ij} reads

$$\Gamma^\eta_{ijk} = \eta(\partial_j e_{ik} + \partial_i e_{jk} - \partial_k e_{ij}) + o(\eta). \tag{3.11}$$

Thus $\partial_i \Gamma^\eta_{jkl} = \eta(\partial_i\partial_j e_{kl} + \partial_i\partial_k e_{jl} - \partial_i\partial_l e_{jk}) + o(\eta)$, and $\Gamma^\eta_{isk}\Gamma^{s,\eta}_{jl} = o(\eta)$. Hence

$$R^\eta_{ijkl} = \eta(\partial_i\partial_k e_{jl} + \partial_j\partial_l e_{ik} - \partial_j\partial_k e_{il} - \partial_i\partial_l e_{jk}) + o(\eta). \tag{3.12}$$

Let us finally rewrite R^η_{ijkl} in terms of the incompatibility tensor inc e:

$$\begin{aligned} R^\eta_{ijkl} &= \eta\epsilon_{ijm}\epsilon_{mab}(\partial_a\partial_k e_{bl} - \partial_a\partial_l e_{bk}) + o(\eta) \\ &= \eta\epsilon_{ijm}\epsilon_{mab}\epsilon_{kln}\epsilon_{ncd}\partial_a\partial_c e_{bd} + o(\eta) = \eta\epsilon_{ijm}\epsilon_{kln}(\mathrm{inc}\ e)_{sr} + o(\eta). \end{aligned} \tag{3.13}$$

This concludes the computations and the proof. ☐

The above result suggests the following definition.

Remark 3.12. *Let $\Omega \subseteq \mathbb{R}^3$ be a domain and let $e \in L^p(\Omega, \mathbb{M}^3)$ be a tensor. Then it follows from* (1.7) *that the Riemann curvature tensor is a distribution fourth-order tensor whose components are given by*

$$R_{ijkl}(e) = \epsilon_{ijm}\epsilon_{kln}(\mathrm{inc}\ e)_{mn}. \tag{3.14}$$

Therefore (3.9) *reads*

$$R^\eta = \eta R + o(\eta).$$

It is clear that inc $e = 0$ *if and only if $R_{ijkl}(e) = 0$. In fact it is easy to rewrite the incompatibility tensor in terms of $R_{ijkl}(e)$ as* $(\mathrm{inc}\ e)_{ij} = \epsilon_{ikl}\epsilon_{jmn}R_{klmn}(e)$. *Observe that* inc e *is a second-grade tensor, whereas $R(e)$ is fourth grade.*

We conclude this section recalling two classical results about Saint-Venant compatibility conditions in finite and in linearized elasticity.

Theorem 3.13 (Saint-Venant compatibility conditions in finite elasticity [9].)**.** *Let $\Omega \subseteq \mathbb{R}^3$ be an open and simply-connected domain, let $C = (g_{ij}) \in \mathcal{C}^2(\Omega, \mathbb{S}^3_>)$ be a symmetric and positive definite tensor and let $E := \frac{1}{2}(C - I)$. Then the following conditions are equivalent:*

1. *there exists a map $\Phi \in \mathcal{C}^3(\Omega, \mathbb{R}^3)$ with $\det \nabla\Phi > 0$ on Ω such that*

$$C = \nabla\Phi^T\nabla\Phi.$$

2. *there exists a vector field $u \in \mathcal{C}^3(\Omega, \mathbb{R}^3)$ such that*

$$E = \frac{1}{2}(\nabla u + \nabla u^T) + \nabla u^T\nabla u.$$

3. *The Riemann curvature tensor vanishes, i.e., $R_{ijkl} = 0$.*

Lemma 3.11 has shown that inc e is the counterpart for linearized elasticity of the Riemann curvature tensor. Formula (3.14) allows us to rewrite the weak Saint-Venant compatibility conditions in L^2, proved in [10], in the following way:

Theorem 3.14 (Saint-Venant compatibility conditions in linearized elasticity)**.** *Let $\Omega \subseteq \mathbb{R}^3$ be a simply-connected domain and let $e \in L^2(\Omega, \mathbb{S}^3)$ be a symmetric tensor. Then there exists a displacement field $u \in H^1(\Omega, \mathbb{R}^3)$ such that $e = \nabla^S u$ if and only if* inc $e = 0$ *in $H^{-2}(\Omega, \mathbb{S}^3)$. Moreover u is unique up to rigid displacements.*

3.3 First main result: Saint-Venant compatibility conditions in L^p

We want to extend Theorem 3.14 for $p \neq 2$. As we said, if $p = 2$ the proof is based on the existence of solutions for the Stokes equations when the external force f is in $H^{-1}(\Omega, \mathbb{R}^3)$. If we want to extend its result in L^p, we have to use other techniques. The following lemma is essential to the proof of our first main result. No smoothness of the boundary is assumed.

Lemma 3.15. *Let $\Omega \subset \mathbb{R}^3$ be a simply-connected domain and let $G \in L^p(\Omega, \mathbb{M}^3)$ be such that* inc $G = 0$ *in $\mathcal{D}'(\Omega, \mathbb{M}^3)$. Then, there exists $w \in L^p(\Omega, \mathbb{R}^3)$ such that $\nabla w = (\,\mathrm{Curl}\ G)^{\mathrm{T}}$ in $\mathcal{D}'(\Omega, \mathbb{M}^3)$.*

Proof. Let w be defined by

$$\langle w, \varphi \rangle := -\langle (\,\mathrm{Curl}\ G)^{\mathrm{T}}, \psi \rangle, \tag{3.15}$$

for every test function $\varphi \in \mathcal{D}(\Omega, \mathbb{R}^3)$, where $\psi \in \mathcal{C}_0^\infty(\Omega, \mathbb{M}^3)$ is a solution of (3.3) with $H = \varphi - \hat{\varphi}$, $\hat{\varphi}$ being the mean value of φ on Ω. Let us prove that w is well defined as a distribution. First of all fix φ, and let us check that $\langle w, \varphi \rangle$ does not depend on the choice of ψ. If ψ_1, $\psi_2 \in \mathcal{C}_0^\infty(\Omega, \mathbb{M}^3)$ are such that $\mathrm{div}\psi_1 = \mathrm{div}\psi_2 = \varphi - \hat{\varphi}$, then $\mathrm{div}(\psi_1 - \psi_2) = 0$ and there exists $\zeta \in \mathcal{C}_c^\infty(\Omega, \mathbb{M}^3)$ such that $\mathrm{Curl}\ \zeta = \psi_1 - \psi_2$ (see Remark 3.4). Hence by assumption,

$$\langle (\,\mathrm{Curl}\ G)^{\mathrm{T}}, \psi_1 - \psi_2 \rangle = \langle\, \mathrm{inc}\, G, \zeta \rangle = 0.$$

Moreover w is clearly linear, while if $\varphi_n \to 0$ in $\mathcal{D}(\Omega, \mathbb{R}^3)$, then, denoting by ψ_n a solution of (3.3) with $H = \varphi - \hat{\varphi}$, we have that $\psi_n \to 0$ in $\mathcal{D}(\Omega, \mathbb{M}^3)$ thanks to estimate (3.4). This proves that w is a distribution. Now, for every test function ψ, $-\langle \nabla w, \psi \rangle = \langle w, \mathrm{div}\psi \rangle = -\langle (\,\mathrm{Curl}\ G)^{\mathrm{T}}, \psi \rangle$, by (3.15), so the thesis will follow as soon as we prove that $w \in L^p(\Omega, \mathbb{R}^3)$. Let $\varphi \in L^q(\Omega, \mathbb{R}^3)$ be a function with zero average such that $\|\varphi\|_q \leq 1$, where $\frac{1}{q} = 1 - \frac{1}{p}$. By Lemma 3.6 there exists $\psi \in W^{1,q}(\Omega, \mathbb{M}^3)$ with $\|\psi\|_{W^{1,q}} \leq C\|\varphi\|_q \leq C$ such that $\mathrm{div}\psi = \varphi$. Thus,

$$|\langle w, \varphi \rangle| \leq |\langle G, (\,\mathrm{Curl}\ \psi)^{\mathrm{T}} \rangle| \leq C\|\varphi\|_q \|G\|_p,$$

and the Lemma is proved, observing that the linear functional w vanishes on the finite dimensional subspace of $L^q(\Omega, \mathbb{R}^3)$ of constant functions.

$\square$

Remark 3.16. *The distributional gradient ∇w of Lemma 3.15 generalizes in the L^p-case the gradient found in the path integral of* (2.4) *for smooth fields. Moreover, w is divergence-free.*

With this Lemma we are now ready to state and prove our first main result.

Theorem 3.17 (Saint-Venant compatibility conditions in L^p)**.** *Let $\Omega \subseteq \mathbb{R}^3$ be a simply-connected domain, let $1 < p < +\infty$, and let $e \in L^p(\Omega, \mathbb{S}^3)$ be a symmetric tensor. Then*

$$\mathrm{inc}\ e = 0 \ \textit{in}\ W^{-2,p}(\Omega, \mathbb{S}^3) \iff e = \nabla^S u$$

for some $u \in W^{1,p}(\Omega, \mathbb{R}^3)$. Moreover, u is unique up to rigid displacements.

Proof. Let us assume that inc e = 0. Let w be defined by Lemma 3.15 with $G = e$ and define also $\omega_{ij} := -\epsilon_{ijk} w_k$. Using the relation $\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$ and the fact that e is symmetric, we compute $\mathrm{Curl}\,\omega = -\mathrm{Curl}\, e$, so that $T := e + \omega \in L^p(\Omega, \mathbb{M}^3)$ satisfies $\mathrm{Curl}\, T = 0$. Hence by Lemma 3.1 and Remark 3.3, it results that $T = \nabla u$ for some $u \in W^{1,p}(\Omega, \mathbb{R}^3)$. Observing that ω is skewsymmetric and recalling that $\nabla^S v = 0$ if and only if v is a rigid displacement (see [8]) concludes the proof. □

3.4 Second main result: structure theorem - Beltrami decomposition

The second main result of this work stands on the following lemma:

Lemma 3.18. *Let Ω be a domain and $u_0 \in W^{1/p',p}(\partial\Omega, \mathbb{R}^3)$, and let u be the solution of the system*

$$\begin{cases} \nabla \mathrm{div} u + \Delta u = 0 & \textit{in } \Omega, \\ u = u_0 & \textit{on } \partial\Omega. \end{cases} \tag{3.16}$$

Then there exists $\tilde F \in L^p(\Omega, \mathbb{M}^3)$ with $\mathrm{Curl}\ \tilde F \in L^p(\Omega, \mathbb{M}^3)$, $\mathrm{div}\tilde F = 0$ in Ω and $\tilde F N = 0$ on $\partial\Omega$, and such that

$$\nabla^s u = \mathrm{inc}\tilde F. \tag{3.17}$$

Proof. From (3.16) we see that $\nabla^s u$ is divergence free. By Remark 3.4, there exists a divergence-free G such that

$$\nabla^s u = \mathrm{Curl}\ G \qquad \Big((\nabla^s u)_{ij} = \epsilon_{jlk}\partial_l G_{ik}\Big). \tag{3.18}$$

Let H be the zero-average solution of

$$\begin{cases} \Delta H = -\partial_i\partial_j G_{ij} = 0 & \text{in } \Omega \\ \partial_N H = -\partial_i G_{ij} N_j & \text{on } \partial\Omega \end{cases}, \tag{3.19}$$

and let h be the solution of the problem (3.3). We then define $G' := G + \nabla h$. We obviously have

$$\nabla^s u = \mathrm{Curl}\ G' \qquad \Big((\nabla^s u)_{ji} = \epsilon_{ilk}\partial_l G'_{jk}\Big). \tag{3.20}$$

Since $\nabla^s u$ is symmetric and divergence-free, we have also

$$0 = \partial_j(\nabla^s u)_{ji} = \epsilon_{ilk}\partial_l\partial_j G'_{jk}. \tag{3.21}$$

Moreover by the definition of G', H, and by (3.19), we have

$$\partial_i \partial_j G'_{ji} = \partial_i \partial_j G_{ij} + \partial_i \partial_j \partial_j h_i = \Delta H = 0. \tag{3.22}$$

Equation (3.19) implies also

$$N_i \partial_j G'_{ji} = N_i \partial_j G_{ji} + \partial_N H = 0, \tag{3.23}$$

so that from (3.21), (3.22), and (3.23), we see that $F = F_i := \partial_j G'_{ji}$ is the solution of the system $\text{Curl } F = \text{div} F = 0$ of Lemma 3.9, and then it holds $\partial_j G'_{ji} = 0$ in Ω. From Remark 3.4 again, we entail the existence of $\tilde{F}$ satisfying $\text{div}\tilde{F} = 0$ in Ω and $\tilde{F}N = 0$ on $\partial\Omega$ such that

$$G'_{ij} = \epsilon_{ilk} \partial_l \tilde{F}_{jk} \qquad \Big(G' = (\text{ Curl } \tilde{F})^T\Big),$$

and the thesis follows by (3.20). $\square$

We are now ready to state and prove our second main result.

Theorem 3.19 (Structure theorem-Beltrami decomposition)**.** *Let $\Omega \subseteq \mathbb{R}^3$ be a simply-connected domain with smooth boundary, let $p \in (1, +\infty)$ be a real number and let $e \in L^p(\Omega, \mathbb{S}^3)$ be a symmetric tensor. Then,*

1) There exist a vector field $\bar{u} \in W^{1,p}(\Omega, \mathbb{R}^3)$ and a tensor $F^0 \in L^p(\Omega, \mathbb{S}^3)$ with $\text{Curl } F^0 \in L^p(\Omega, \mathbb{S}^3)$, $\text{inc} F^0 \in L^p(\Omega, \mathbb{S}^3)$, $\text{div} F^0 = 0$ and $F^0 N = 0$ on $\partial\Omega$ such that

$$e = \nabla^S \bar{u} + \text{inc} F^0. \tag{3.24}$$

2) For any $U \in W^{1/p,p}(\partial\Omega)$, there exists $u \in W^{1,p}(\Omega, \mathbb{R}^3)$ with $u = U$ on $\partial\Omega$ and $F \in L^p(\Omega, \mathbb{S}^3)$ with $\text{Curl } F \in L^p(\Omega, \mathbb{R}^{3\times 3})$, $\text{inc} F \in L^p(\Omega, \mathbb{S}^3)$ and $\text{div} F = 0$ and $FN = 0$ on $\partial\Omega$ such that

$$e = \nabla^S u + \text{inc} F. \tag{3.25}$$

This pair $\{u, F\}$ is unique.

We call $\nabla^S u$ the compatible part and $\text{inc} F$ the incompatible part of the decomposition (3.24) and (3.25).

Proof. 1) By Helmholtz decomposition (3.2) there exist two vector fields $\tilde{u} \in W^{1,p}(\Omega, \mathbb{R}^3)$ and $v \in \mathcal{V}^p(\Omega)$ such that

$$e = \nabla \tilde{u} + \text{ Curl } v,$$

or componentwise

$$e_{ij} = \partial_j \tilde{u}_i + \epsilon_{jkl} \partial_k v_{il}.$$

Apply again Helmholtz decomposition to v^T_{li} to deduce the existence of $\tilde{W} \in W^{1,p}(\Omega, \mathbb{R}^3)$ and $W \in \mathcal{V}^p(\Omega)$ such that

$$v^T_{li} = v_{il} = \partial_i \tilde{W}_l + \epsilon_{ipq} \partial_p W_{lq}.$$

Moreover by the decomposition in symmetric and skew-symmetric parts, it holds that:

$$W_{lq} = W^S_{lq} + W^A_{lq} = W^S_{lq} + \frac{1}{2} \epsilon_{lqm} \epsilon_{mij} W_{ij}.$$

Since the rotation vector is given by $\omega_m = \frac{1}{2}\epsilon_{mji}W_{ij}$, then

$$W_{lq} = W^S_{lq} + W^A_{lq} = W^S_{lq} - \epsilon_{lqm}\omega_m.$$

Let us define $F := W^S$. Therefore

$$\begin{aligned}
\epsilon_{ipq}\partial_p W_{lq} &= \epsilon_{ipq}\partial_p F_{lq} - \epsilon_{ipq}\epsilon_{lqm}\partial_p\omega_m \\
&= \epsilon_{ipq}\partial_p F_{lq} + (\delta_{il}\delta_{pm} - \delta_{im}\delta_{pl})\partial_p\omega_m \\
&= \epsilon_{ipq}\partial_p F_{lq} + \delta_{il}\mathrm{div}\omega - \partial_l\omega_i.
\end{aligned}$$

Now we want to show that there exists $\phi \in W^{2,p}(\Omega, \mathbb{R}^3)$ such that the tensor $F^0_{lq} := F_{lq} + \frac{1}{2}(\partial_l\phi_q + \partial_q\phi_l)$, satisfy $\operatorname{Curl} F^0 \in L^p(\Omega, \mathbb{S}^3)$, $F^0 N = 0$ on $\partial\Omega$ and $\mathrm{div}F^0 = 0$ on Ω. The first requirement on the Curl is true by definition of F_0. Since $F \in W^{1,p}(\Omega, \mathbb{S}^3)$ by Remark 3.7, the Neumann problem

$$\begin{cases}
\mathrm{div}\nabla^S\phi = \nabla\mathrm{div}\phi + \Delta\phi = -\mathrm{div}F & \text{in } \Omega, \\
\left(\nabla^S\phi\right) N = 0 & \text{on } \partial\Omega,
\end{cases} \tag{3.26}$$

admits a strong solution $\phi \in W^{2,p}(\Omega, \mathbb{R}^3)$ a.e. in Ω by Remark 3.7 (here $\lambda = 0$ and $\mu = 1$). This solution is unique up to rigid displacements. Hence F^0 satisfy all the conditions required. Therefore it results that

$$\begin{aligned}
e_{ij} &= \partial_j\tilde{u}_i + \epsilon_{jkl}\partial_k v_{il} = \partial_j\tilde{u}_i + \epsilon_{jkl}\partial_k(\partial_i\tilde{W}_l + \epsilon_{ipq}\partial_p W_{lq}) \\
&= \partial_j\tilde{u}_i + \epsilon_{jkl}\partial_k(\partial_i\tilde{W}_l + \epsilon_{ipq}\partial_p F_{lq} + \delta_{il}\mathrm{div}\omega - \partial_l\omega_i) \\
&= \partial_j\tilde{u}_i + \epsilon_{jkl}\partial_k\partial_i\tilde{W}_l + \epsilon_{jkl}\epsilon_{ipq}\partial_k\partial_p F_{lq} + \epsilon_{ijk}\partial_k(\mathrm{div}\omega) \\
&= \partial_j\tilde{u}_i + \epsilon_{jkl}\partial_k\partial_i\tilde{W}_l + \epsilon_{jkl}\epsilon_{ipq}\partial_k\partial_p F^0_{lq} + \epsilon_{ijk}\partial_k(\mathrm{div}\omega) \\
&- \frac{1}{2}\epsilon_{jkl}\epsilon_{ipq}\partial_k\partial_p\partial_q\phi_l - \frac{1}{2}\epsilon_{jkl}\epsilon_{ipq}\partial_k\partial_p\partial_l\phi_q \\
&= \partial_j\tilde{u}_i + \epsilon_{jkl}\partial_k\partial_i\tilde{W}_l + \epsilon_{jkl}\epsilon_{ipq}\partial_k\partial_p F^0_{lq} + \epsilon_{ijk}\partial_k(\mathrm{div}\omega).
\end{aligned}$$

Since e is symmetric, then inc e is also symmetric. Thus

$$\begin{aligned}
e_{ij} &= \frac{e_{ij} + e_{ji}}{2} \\
&= \frac{\partial_j\tilde{u}_i + \partial_i\tilde{u}_j}{2} + (\mathrm{inc}F^0)_{ij} + \frac{1}{2}\epsilon_{jkl}\partial_k\partial_i\tilde{W}_l + \frac{1}{2}\epsilon_{ikl}\partial_k\partial_j\tilde{W}_l.
\end{aligned} \tag{3.27}$$

Let us define the vector field $\bar{u} = (\bar{u}_i)_i \in W^{1,p}(\Omega, \mathbb{R}^3)$ as

$$\bar{u}_i := \tilde{u}_i + \epsilon_{ikl}\partial_k\tilde{W}_l.$$

Therefore

$$e_{ij} = \frac{\partial_j\bar{u}_i + \partial_i\bar{u}_j}{2} + (\mathrm{inc}F^0)_{ij},$$

proving 1).

Let us now prove 2). Let w be the solution of (3.16) with boundary datum $v = \bar{u} - U$. Then Lemma 3.18 provides $\tilde{F}$ such that $\nabla^s w = \mathrm{inc}\tilde{F}$. So that setting $u := \bar{u} - w$, we find $e = \nabla^s u + \mathrm{inc}F$ with $F := \tilde{F} + F^\circ$, and the sought decomposition follows. To see uniqueness of such u, consider another $\hat{u}$ and another $\hat{F}$ satisfying $e = \nabla^S \hat{u} + \mathrm{inc}\hat{F}$ and $\hat{u} = U$ on $\partial\Omega$. Then its difference $v := u - \hat{u}$ satisfies (3.16) with $v = 0$ on $\partial\Omega$ and hence $u = \hat{u}$. Let us now prove that F is unique. Assume that there exist another such $\hat{F}$ and define $G := F - \hat{F}$. It holds $\mathrm{inc}\ G = 0$. By virtue of Theorem 3.17 and taking the divergence of $G = \nabla^S \zeta$, it holds $\Delta\zeta + \nabla\,\mathrm{div}\zeta = 0$ in Ω and $(\nabla^S\zeta)N = 0$ on $\partial\Omega$. Thus ζ is a rigid displacement and $G = 0$.

□

Remark 3.20. *It is easily verified that taking* $\mathrm{inc}\ e = 0$ *in* (3.24) *and since* F *is divergence free yields the PDE* $\Delta\Delta F = 0$. *The issue of finding appropriate, well-posed boundary conditions for this problem can be addressed by recalling the classical theory by Agmon, Douglis and Nirenberg [1] (see also[16]) of complementary boundary conditions. For instance it results from this analysis that the following system is well posed (see [30] for detail):*

$$\begin{cases} \Delta^2 F & = f & \text{in} & \Omega \\ FN & = \varphi^1 & \text{on} & \partial\Omega \\ \mathrm{div}F & = g & \text{on} & \partial\Omega \\ \partial_N\,\mathrm{div}F & = h & \text{on} & \partial\Omega \\ (\partial_N F \times N)^t \times N & = \varphi^2 & \text{on} & \partial\Omega \end{cases} . \tag{3.28}$$

Taking $f = \mathrm{inc}\ e$, $h = g = 0$, *it is immediately seen that such* F *is divergence free, since* $\mathrm{div}\ F$ *is the solution of the homogeneous Dirichlet problem with vanishing RHS .*

4 Application to the Korn inequalities in L^p

Our first main result allows us to deduce an alternative proof of Korn inequalities, which are crucial in elasticity. We follow the same procedure as of P. Ciarlet and Ph.G. Ciarlet in [10]. Let Ω be a domain and define the spaces

$$\begin{aligned} E^p(\Omega) &:= \{e \in L^p(\Omega, \mathbb{S}^3) : \mathrm{inc}\ e = 0\}, \\ R(\Omega) &:= \{u(x) = Ax + b : A \in \mathbb{A}^3, b \in \mathbb{R}^3\}, \\ \dot{W}^{1,p}(\Omega, \mathbb{R}^3) &:= W^{1,p}(\Omega, \mathbb{R}^3) \big/ R(\Omega), \end{aligned}$$

where $R(\Omega)$ is the closed subspace of $W^{1,p}(\Omega, \mathbb{R}^3)$ consisting of rigid displacements. $\dot{W}^{1,p}(\Omega, \mathbb{R}^3)$ turns out to be a Banach space if endowed with the norm

$$\|u\|_{\dot{W}^{1,p}} := \inf_{r \in R(\Omega)} \|u' + r\|_{W^{1,p}},$$

where u' is a representative of the class of u.

Theorem 3.17 has the following consequence:

Corollary 4.1. *Let* $\Omega \subseteq \mathbb{R}^3$ *be a simply-connected domain, let* $p \in (1, +\infty)$. *Let us define the linear application*

$$\mathcal{F}_p : E^p(\Omega) \to \dot{W}^{1,p}(\Omega, \mathbb{R}^3)$$

given by

$$\mathcal{F}_p(e) = \dot{u} \text{ for all } e \in E^p(\Omega),$$

where $\dot{u} \in \dot{W}^{1,p}(\Omega, \mathbb{R}^3)$ *is the unique element such that* $\nabla^S \dot{u} = e$. *Then* $\mathcal{F}_p$ *is bijective, continuous and its inverse is continuous.*

Proof. First of all we recall that $E_p(\Omega)$ is a closed subspace of $L^p(\Omega)$, therefore a Banach space. Moreover $\mathcal{F}$ is a bijection by Theorem 3.17. If we prove that $\mathcal{F}_p^{-1}$ is continuous, thesis follows from open mapping Theorem. But

$$\| \mathcal{F}_p^{-1}(\dot{u}) \|_{L^p} = \| \nabla^S \dot{u} \|_{L^p} \leq \| \dot{u} \|_{\dot{W}^{1,p}},$$

then $\mathcal{F}_p^{-1}$ is continuous. □

Now it is trivial to prove a Korn inequality in the quotient space $\dot{W}^{1,p}(\Omega, \mathbb{R}^3)$, where $p \in (1, +\infty)$. Remark that no smoothness of the boundary is required.

Theorem 4.2 (Korn inequality in $\dot{W}^{1,p}$)**.** *Let $\Omega \subseteq \mathbb{R}^3$ be a simply-connected domain and let $p \in (1, +\infty)$. Then there exists $C > 0$ such that*

$$\| \dot{u} \|_{\dot{W}^{1,p}} \leq C \| \nabla^S \dot{u} \|_{L^p} \quad \textit{for all } \dot{u} \in \dot{W}^{1,p}(\Omega, \mathbb{R}^3). \tag{4.1}$$

Proof. By Corollary 4.1 $\mathcal{F}_p$ is a continuous map. Then there exists $C > 0$ such that

$$\| \mathcal{F}_p(e) \|_{\dot{W}^{1,p}} = \| \dot{u} \|_{\dot{W}^{1,p}} \leq C \| \nabla^S \dot{u} \|_{L^p}$$

□

Now we want to prove another useful Korn inequality. Let $\Gamma_0 \subseteq \mathbb{R}^3$ be a subset of $\partial\Omega$ with $\mathcal{H}^2(\Gamma_0) > 0$, where $\mathcal{H}^2$ denotes the two-dimensional Hausdorff measure. Let us define the spaces

$$\begin{aligned} &W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3) = \{u \in W^{1,p}(\Omega, \mathbb{R}^3), u = 0 \text{ on } \Gamma_0\}, \\ &E^p_{\Gamma_0}(\Omega) := \{e := \nabla^S u : u \in W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3)\}, \end{aligned}$$

the latter being a closed subspace of $E^p(\Omega)$ since the trace operator is continuous on $W^{1,p}(\Omega, \mathbb{R}^3)$. The linear application

$$\hat{\mathcal{F}}_p : E^p_{\Gamma_0}(\Omega) \to W^{1,p}_{\Gamma_0}(\Omega, \mathbb{R}^3),$$

given by

$$\hat{\mathcal{F}}_p(e) = u,$$

for every $e \in E^p_{\Gamma_0}(\Omega)$, is well defined by Theorem 3.17, since there is a unique $u \in W_{\Gamma_0}{}^{1,p}(\Omega, \mathbb{R}^3)$ such that $\nabla^S u = e$. Indeed, suppose $\nabla^S u_1 = \nabla^S u_2 = e$, then $\nabla^S(u_1 - u_2) = 0$, and hence u_1 and u_2 differ by a rigid displacement, that is well known to be zero by the assumption that $u = 0$ on Γ_0 with $\mathcal{H}^2(\Gamma_0) > 0$. Thus $u_1 - u_2 = 0$ on Ω. Moreover it is straightforward that $\hat{\mathcal{F}}_p$ is a bijection. The continuity of $\hat{\mathcal{F}}_p^{-1}$ readily follows from the fact that

$$\| e \|_{L^p} = \| \nabla^s u \|_{L^p} \leq \| u \|_{W^{1,p}} .$$

Then the open mapping Theorem implies the following result:

Theorem 4.3 (Korn inequality in $W^{1,p}_{\Gamma_0}$)**.** *Let $\Omega \subseteq \mathbb{R}^3$ be a simply-connected domain and let $p \in (1, +\infty)$. Then there exists $C > 0$ such that*

$$\| u \|_{W^{1,p}} \leq C \| \nabla^s u \|_{L^p} \quad \textit{for every } u \in W_{\Gamma_0}{}^{1,p}(\Omega, \mathbb{R}^3). \tag{4.2}$$

These proofs of Korn inequalities are valid in simply-connected domains. It is easy to extend them to the case of more general domains. For instance, Theorem 4.3, is valid for domains which are finite union of simply-connected open sets (each one with a nonnegligible part of the boundary intersecting Γ_0). For a path-connected and locally simply-connected domain, we can argue as follows: we first split the domain in a countable union of disjoint simply-connected open sets Ω_i (plus a negligible set), and then obtain the Korn inequality for each one of them by an approximation argument. This can be done adding to Ω_i a small open neighborhood of a path connecting it to Γ_0, and then letting the width of it go to zero. Korn inequalities are classical results already proved for general domains with other methods (see, e.g., [20, Theorem 8]).

5 Concluding remarks

The aim of this paper was to be on the one hand to write a brief survey on the intrinsic approach in elasticity, emphasizing the role of the incompatibility operator in linearized elasticity. On the other hand, our aim was to provide and prove new results on general and incompatibility-free symmetric tensors in L^p for for $p \in (1, +\infty)$. The obvious mechanical interpretation of these tensors are the strain tensors, related to the stress tensor by a constitutive law, a linear law in most cases, and providing the deformation of the elastic body under analysis.

Saint-Venant-type relations are well known in the Mechanical literature, and the Hilbertian case was long established. However, to the knowledge of the authors, it had not been demonstrated in L^p for any space dimension. Therefore, generalizations to $p \in (1, +\infty)$ was the first motivation of the present work. The structure theorem is also a new contribution of this paper, and might be seen as a generalization of Saint-Venant result in L^p. Remark that both results are intimately related, but none follows directly from the other in our setting. Let us observe that the first main result holds for any simply-connected domain, whereas the structure theorem was proved with an additional smoothness assumption of the domain boundary. In a second stage, the first main result has been applied to suggest another proof of certain Korn inequalities in L^p. From an application viewpoint, it should be stressed that the structure theorem is useful in dislocation models since it can be proved (see, e.g., [28, 29]) that in the presence of dislocations $\operatorname{inc} e$ is related to the curl of the dislocation density κ, and hence the field F is a dislocation-induced tensor satisfying $\operatorname{inc}\operatorname{inc} F = f(\operatorname{Curl} \kappa)$, whereas u is related to the mechanical equilibrium equations, and f a certain material-dependent function.

It should in fact be emphasized that in dislocation models, first, it is not true that the strain $e = \mathbb{C}^{-1}\sigma$ equals a symmetric gradient everywhere. Second, the structure theorem would split $e = \mathbb{C}^{-1}\sigma$ in two contributions, each with a clear physical interpretation. Indeed, u is identified with a displacement provided by equilibrium of Newtonian forces, while F is a defect variable obeying specific transport-diffusion-reaction equations for the dislocation density. Moreover, in a complete elasto-plastic model, this incompatible part is undoubtly related to plasticity. This also justifies the interest of our structure theorem with a view to modeling. To conclude let us justify the study of the case $p \neq 2$ by recalling that dislocations are singularities that prohibit for intrinsic physical reasons the use of quadratic energies (see [25], [26]). Indeed, in [30] simple approach based on linear PDEs is suggested, where the classical Lamé system of elasticity is solved, though with a variable u which is not primarily the displacement field, rather originating from a Beltrami-strain decomposition which also provides a dislocation-dependent field F. This latter field solves an incompatibility-based PDE, as given in Remark 3.20. Specifically, being f, g and U the body and surface forces, and the prescribed boundary load, the following result will is proved in [30]: there exists $u \in W^{1,p}(\Omega)$ with $1 \leq p \leq 3/2$ such that

$$\left\{ \begin{array}{ccc} -\operatorname{div}\left(A\nabla^S u\right) & = & f + \mathcal{F} \quad \text{in} \quad \Omega \\ \left(A\nabla^S u\right) N & = & g + \mathcal{G} \quad \text{on} \quad \Gamma_1 \\ u & = & U \quad \text{on} \quad \Gamma_0, \end{array} \right. , \tag{5.1}$$

where $\mathcal{F}$ and $\mathcal{G}$ are dislocation-induced body and surface forces in $W^{-1,p}(\Omega)$ and $W^{-1/p,p}(\Gamma_1)$, respectively, with $1 \leq p \leq 3/2$, depending on F and vanishing as soon as the dislocation density vanishes.

References

[1] S. Agmon, A. Douglis, and L. Nirenberg. *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II.* Commun. Pure Appl. Math., 17:35–92.

[2] S. S. Antman, *Nonlinear problems of elasticity*, Springer-Verlag,1995,Berlin.

[3] J. Ball, *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Rational Mech. Anal.,63:337-403, 1977.

[4] E. Beltrami, *Osservazioni sulla nota precedente*, Atti Accad. Lincei Rend. (5) 1 141–142; Opere 4 (1892) 510–512.

[5] J. Bolik and W. von Wahl. *Estimating ∇u in terms of div u, curl u, either (ν, u) or $\nu \times u$ and the topology.* Math. Methods Appl. Sci., 20(9):737–744, 1997.

[6] H. Brezis *Functional analysis, Sobolev spaces and partial differential equations*, Springer, 2010.

[7] E. Cesàro, *Sulle formule di Volterra, fondamentali nella teoria della distorsioni elastiche*, Rend. Napoli (3a) 12 (1906) 311–321.

[8] Ph. G. Ciarlet, *Three dimensional elasticity*, first ed., vol. 1, North-Holland, 1988.

[9] Ph. G. Ciarlet *An introduction to differential geometry with applications to elasticity*, Springer-Verlag, 2005.

[10] Ph. G. Ciarlet; P. Ciarlet, *Another approach to linearized elasticity and a new proof of Korn inequality*, Math. Models Methods Appl. Sci. (2004), no.15, 259-271.

[11] S. Conti, G. Dolzmann, S. Müller, *Korn's second inequality and geometric rigidity with mixed growth conditions* Calc.Var.Partial Differential Equations 50 (2014), 437-454.

[12] F. Demengel; G. Demengel, *Functional spaces for the theory of elliptic partial differential equations*, Springer, 2012.

[13] B. Dubrovin, A. Fomenko, S. Novikov, *Modern geometry: methods and applications*, part 1, second. ed., Springer, 1992.

[14] L. Evans, *Partial Differential Equations*, second. ed., AMS, 2010.

[15] G. P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations. Steady-state problems. 2nd ed.* New York, NY: Springer., 2011.

[16] I. M. Gelfand. On elliptic equations. *Uspehi Mat. Nauk.*, 15(3 (93)):121–132, 1960.

[17] E. Fabes, O. Mendez, and M. Mitrea. *Boundary layers on Sobolev-Besov spaces and Poisson's equation for the Laplacian in Lipschitz domains. J. Funct. Anal.*, 159(2):323–368, 1998.

[18] D. Fujiwara and H. Morimoto. *An L_r-theorem of the Helmholtz decomposition of vector fields. J. Fac. Sci., Univ. Tokyo, Sect. I A*, 24:685–700, 1977.

[19] S. Kesavan, *On Poincaré's and J.-L. Lions Lemmas*, C.R. Math. Acad. Sci. Paris 340 (2005), no.1, 27-30.

[20] V. A. Kondrat'ev and O. A. Oleinik. *Boundary value problems for a system of the theory of elasticity in unbounded domains. Korn inequalities.* 1988.

[21] H. Kozono, T. Yanagisawa, *L^r variational inequality for vector fields and the Helmholtz-Weyl decomposition in bounded domains*, Indiana Univ. Math. no. 58(4) (2009), 1853-1920.

[22] S. Mardare, *Inequality of Korn's type on compact surfaces without boundary conditions*, Chinese Annals Math. Vol. 24B (2003), 191-204.

[23] S. Mardare, *On isometric immersions of a Riemannian space with little regularity*, Analysis and applications Vol.2 (2004), 193-226.

[24] J. A. Nitsche, *On Korn's second inequality*, RAIRO An. Num. Vol.15, No.3 (1981), 237-248.

[25] R. Scala, N. Van Goethem, *Currents and dislocations at the continuum scale*, (submitted), 2014.

[26] R. Scala, N. Van Goethem, *Graphs of torus-valued harmonic maps, with application to a variational model for dislocations*, (submitted), 2015.

[27] H. Schaefer, *Die Spannungsfunktionen des dreidimensionalen Kontinuums und des elastischen Körpers*, Z. Angew. Math. Mech. 33 (1953) 356–362.

[28] N. Van Goethem. *Fields of bounded deformation for mesoscopic dislocations. Math. Mech. Solids*, 19 (5), 579-600, 2014.

[29] N. Van Goethem and F. Dupret. *A distributional approach to the geometry of $2D$ dislocations at the continuum scale.* Ann. Univ. Ferrara, 58(2):407–434, 2012.

[30] N. Van Goethem. *Cauchy elasticity with dislocations in the small strain assumption.* Appl. Math. Letters, (to appear) (2015).

[31] W. von Wahl. *Estimating ∇u by div u and curl u. Math. Methods Appl. Sci.*, 15(2):123–143, 1992.

[32] L. H. Wang, *On Korn inequality*, Journal of Comp. Math. Vol.21, No.3 (2003), 321-324.