A RIGIDITY RESULT FOR GLOBAL MUMFORD-SHAH MINIMIZERS IN DIMENSION THREE

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ABSTRACT. We study global Mumford-Shah minimizers in $\mathbb{R}^N$, introduced by Bonnet as blow-up limits of Mumford-Shah minimizers. We prove a new monotonicity formula for the energy of u when the singular set K is contained in a smooth enough cone. We then use this monotonicity to prove that for any reduced global minimizer (u, K) in $\mathbb{R}^3$, if K is contained in a half-plane and touching its edge, then it is the half-plane itself. This partially answers to a question of Guy David.

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1. INTRODUCTION

Global Mumford-Shah minimizers was introduced first by Bonnet [4] in dimension 2 as blow-up limit of minimizers of the Mumford-Shah functional, and are crucial in the regularity result of [4]. They are couples (u, K), with $K \subset \mathbb{R}^N$ a closed set and $u \in \bigcap_{R>0} W^{1,2}(B(0, R) \setminus K)$, that locally minimizes in $\mathbb{R}^N$ the Mumford-Shah energy. More precisely, for any ball $B \subset \mathbb{R}^N$, (u, K) minimizes the sum

$$\int_{B \setminus K} |\nabla u|^2 dx + \mathcal{H}^{N-1}(K \cap B),$$

among all compettiting couples (u', K') that are equal to (u, K) outside B, and that keep the topology of K (see Definition 2.4 below for more details). The k-dimensional Hausdorff measure $\mathcal{H}^k$ generalizes the natural notion of k-area and will be precisely defined in Section 2.

The list of global minimizers is nearly complete in dimension 2: if K is connected then it is either a line, a half-line, or three half-lines meeting at the origin by 120°. Knowing this without assuming "connected" would solve the famous Mumford-Shah conjecture. Some improvements in this direction can be found in [14] and [10].

But on the other hand, almost nothing is known about global minimizers in dimension $N > 2$. A first discussion is given in Chapter H.76. of Guy David's book [8], and a few more results can be found in the paper [15].

For instance it is known in dimension 3 that, if u is locally constant, then K must be a minimal cone (i.e. a Plane, three half planes meeting by 120°, or a cone over the regular tetrahedron). This fact looks very natural but is surprisingly not obvious at all, and a rigorous proof is contained in [9]. It is also known that a half-plane is a minimizer with the corresponding function $u = cracktip \times \mathbb{R}$, where the cracktip function is $\sqrt{2r/\pi} \sin(\theta/2)$ in polar coordinates. These are the only known examples of global minimizers in $\mathbb{R}^3$, and we refer to [15] and [8] for more details.

There is also counter-examples. For instance in [15] it is proved that an angular sector cannot be a global minimizer, unless it is a half-plane or a plane. In [18] a tentative construction for an extra global minimizer of particular type is proved to fail.

Notice that adding to K a piece of set negligible for $\mathcal{H}^{N-1}$ does not change the fact of being a Mumford-Shah minimizer. Therefore, it is convenient to work with *reduced* minimizers only, for which one cannot remove from K more pieces of set of zero measure without being enable to extend the associated function u as a $W^{1,2}$ function. For instance, reduced minimizers enjoy nice properties as Ahlfors-regularity, or local C^1 regularity at points where the energy of u is small enough, that non-reduced minimizers can obviously not expect to have in general. As proved in [8], it is always possible to replace a minimizer by a reduced one (there is even a debate between 3 different possibilities, the one used in this paper is called "topological reduction" in [8]), but anyway in the case when the the global minimizer is coming

from a blow-up procedure, then it is automatically reduced [8, Proposition 40.9]. We refer to [8, Section 8] for more details about reduced minimizers.

In this paper we focus our attention on the following conjecture which is stated in [8, Page 571].

Conjecture 1.1 (Guy David). *Let (u, K) be a reduced global minimizer in $\mathbb{R}^3$ such that K is contained in a plane and not empty. Then K must be either a plane or a half-plane.*

In this paper we give a partial answer to this conjecture. We prove the following.

Theorem 1.2. *Let (u, K) be a reduced global minimizer in $\mathbb{R}^3$ such that K is contained in a half-plane, which contains a point of K on its edge. Then K must be the whole half-plane.*

The major obstacle which prevents us to prove the full conjecture will be explain a the end of the introduction. It comes from geometrical requirements to control the first eigenvalue of the Neumann-Laplace-Beltrami operator on spherical domains in order to get a monotonicity formula for the normalized energy, with the right power. We are able to control the first eigenvalue of $\mathbb{S}^{N-1} \setminus K$ when K is contained in a half-plane, but not larger.

We stress that in dimension $N = 2$, similar results have been obtained by J-C Léger [14]. For instance Proposition 25 in [14] says that when $N = 2$, if K is contained in a line then K is either the whole line, a half-line, or the empty set. The proof of Léger relies strongly on complex analysis. In particular the beautifull (and planar) formula showing the link between the set K and the complex derivative of u is one of the key ingredient in [14]. The second main ingredient is the monotonicity formula of Bonnet [4].

Therefore, the challenge in Conjecture 1.1 is to find a new strategy in higher dimension, without complex analysis, and prioritary try to find a monotonicity formula like the one of Bonnet valid in higher dimensions. The latter being also a question in page 572 of [8].

In the present paper we indeed propose a monotonicity formula to replace the one of Bonnet. It relies on the famous argument from Alt-Caffarelli-Friedman, and is obtained here for solutions of the Neumann problem in the complement of a set $K \subseteq \mathbb{R}^N$ contained in a cone. According to us, this monotonicity result is probably as interesting as our main result itself since, as far as we know, it is the first time that some monotonicity is shown for the Neumann problem in dimension $N > 2$. On the other hand to be valid, the singular set K must be contained in a cone, which could be slightly restrictive but turns out to be exactly the case in Conjecture 1.1.

Let us recall the monotonicity formula of Bonnet [4] which says that if $K \subset \mathbb{R}^2$ is a closed connected set and u is a local energy minimizer in $\mathbb{R}^2 \setminus K$ (i.e. u is harmonic with Neumann boundary conditions on K), then

$$r \mapsto \frac{1}{r} \int_{B_r \setminus K} |\nabla u|^2 dx$$

is nondecreasing (see also [8, Section E.47] or [6]). The proof of Bonnet seems not to be directly extendable in higher dimensions.

On the other hand, using a classical argument from Alt-Caffarelli-Friedman [2], it is easy to show that, for the Dirichlet problem, if the first eigenvalue of the Dirichlet Laplacian is uniformly bounded from below by $(2N-3)/4$ on $\partial B_r \setminus K$ for a.e. $r > 0$, then

$$r \mapsto \frac{1}{r} \int_{B_r \setminus K} \frac{|\nabla u|^2}{|x|^{N-2}} dx$$

is nondecreasing. See for instance [17, Lemma 16] for a proof. Notice that for $N = 2$ we recover exactly Bonnet's energy.

Both in Bonnet [4] and Alt-Caffarelli-Friedman [2], the monotonicity formula is established through an integration by parts which transforms the energy $f(r) = \int_{B_r} |\nabla u|^2 / |x|^{N-2} dx$ as an integral over ∂B_r, and then estimate the later by Poincaré-Wirtinger inequality to get $f(r) \leq Cf'(r)$, where the constant C is related to eigenvalue on the sphere.

Now if one tries to use the proof of Alt-Caffarelli-Friedman [2] for the Neumann problem in $\mathbb{R}^N \setminus K$, while performing the integration by parts, a boundary term of type

$$(1.1) \qquad \int_{K \cap B_r} u^2 \frac{\partial}{\partial \nu} \left(\frac{1}{|x|^{N-2}} \right) d\mathcal{H}^{N-1}$$

remains, which a priori prevents us to prove any monotonicity (here the integration over K has to be understood as an integration on both sides of K and u may take a different value on each side). Of course this term does not occur when $u = 0$ on K, which happens for the Dirichlet problem.

Now the key point in our proof relies on the following simple observation: if for instance K is contained in a vectorial $N - 1$-plane, then the normal vector on K at point x must be orthogonal to the vector x itself. By consequence we have $\frac{\partial}{\partial \nu}(|x|^{2-N}) = (2 - N)|x|^{-N} x \cdot \nu = 0$ on K, and the above extra term disappears. So follows the monotonicity, and it works the same way if K is contained in a more general $N - 1$ dimensional cone.

Moreover, as for Bonnet's monotonicity formula, the argument here is sharp too, in the sense that constant normalized energy happens if and only if u is positively homogeneous of right degree.

Even if the proof of Theorem 1.2 needs the monotonicity result for half-planes only (and $N = 3$), we will prove it in any dimension for more general cones that we will call Neumann cones, and that we define as follows.

Definition 1.3. *If $N \geq 3$, a closed set $K \subset \mathbb{R}^N$ will be called a Neumann cone if the three following properties hold:*
  (1) *K is a cone.*
  (2) *$K \cap \mathbb{S}^{N-1}$ is $(N-2)$-rectifiable.*
  (3) *The embedding $W^{1,2}(\mathbb{S}^{N-1} \setminus K) \to L^2(\mathbb{S}^{N-1})$ is compact.*

If $N = 2$ we keep the same definition but replacing (2) by

(2)' $K \cap \mathbb{S}^1$ *is a finite number of singletons.*

It is easy to check that a vectorial hyperplane in $\mathbb{R}^N$ is a Neumann cone, or that a half-hyperplane is a Neumann cone. We recall that for instance, the so-called cone property guarantees the compactness of $W^{1,2}(\Omega) \to L^2(\Omega)$ (see [1, Theorem 6.2. p.144]). Moreover, when K is a Neumann cone then the first positive eigenvalue of the Neumann Laplacian, denoted by $\lambda_1(\mathbb{S}^2 \setminus K)$, is well defined. This is also the case for any set K contained in a Neumann cone. We refer to Section 4 for the exact definition of rectifiability and basic properties of Neumann cones.

Now here is an example of statement for the monotonicity result (see Lemma 7.5 for the general case where assumption (2) of Lemma 1.4 is replaced by a more general topological assumption). We call energy minimizer in $\mathbb{R}^N \setminus K$, a function u that locally minimizes $\int |\nabla u|^2 dx$ in $\mathbb{R}^N \setminus K$ (see Definition 2.1). This function is harmonic in $\mathbb{R}^N \setminus K$ with a Neumann boundary condition on K, i.e. zero normal derivative on K, in a weak sense.

Lemma 1.4. *Let $K \subset \mathbb{R}^N$ be a closed set satisfying the following assumptions.*

(1) *K is contained in a Neumann cone.*
(2) *$\partial B_r \setminus K$ is connected for a.e. $r > 0$.*
(3) *$\exists \gamma > 0$ s.t. for a.e. $r > 0$, the first positive eigenvalue of the Neumann-Laplace-Beltrami operator on $\partial B_r \setminus K$ satisfies $\lambda_1(\partial B_r \setminus K) \geq \gamma/r^2$.*

Then for every local energy minimizer u in $\mathbb{R}^N \setminus K$ we have that

$$(1.2) \qquad \varphi : r \mapsto \frac{1}{r^\alpha} \int_{B_r} \frac{|\nabla u|^2}{|x|^{N-2}} \, dx$$

is nondecreasing, where α is defined through

$$\alpha = \alpha(N, \gamma) = \sqrt{(N-2)^2 + 4\gamma} - (N-2).$$

Moreover, if $\varphi(r)$ is constant on an interval $[a, b]$ then for a.e. $r \in [a, b]$ the value γ/r^2 is the first positive eigenvalue for the Neumann-Laplace-Beltrami operator on $\partial B_r \setminus K$ and the restritction of u on $\partial B_r \setminus K$ is an associated eigenfunction.

For instance when $N = 3$, the eigenvalue $\gamma = 3/4$ provides a monotonicity with power $\alpha = 1$. The monotonicity is then employed to characterize blow-up and blow-in limits of a global minimizer when K is contained in a half-plane, and this is the key ingredient to prove Theorem 1.2.

The assumption of existing a point of K on the edge of the half-plane is here the guarantee the existence of a point at which the normalized energy (i.e. the quantity denoted by $\varphi(r)$ in Lemma 1.4 with $\alpha = 1$) will not converge to 0 when r goes to zero. At this step of the proof we shall need the regularity result of [16]. For an arbitrary closed K contained in a half-plane, it is tempting to try to find a point on the edge of another suitable half-plane by moving the first one. But notice that for instance the closed set $K := \{(x, y, 0) \in \mathbb{R}^3; y \geq e^{-|x|}\}$ is contained in the half-plane

$\{z = 0$ and $y \geq -1\}$ but does not touch the edge of any half-plane containing it, thus we cannot avoid our assumption using this kind of strategy.

It is well known that the natural power in the normalized energy of a Mumford-Shah minimizer is indeed $\alpha = 1$, because of the standard energy estimate $\int_{B_r} |\nabla u|^2 dx \leq Cr^{N-1}$. To this respect, let us emphasis that regarding to the exponent α in the monotonicity formula, the situation in dimension 3 is pretty much different than the situation in dimension 2. This is due to the fact that $\lambda_1(\mathbb{S}^2 \setminus K)$ is decreasing with respect to increasing cracks on the sphere, at the contrary of what happens in dimension 2, where $\lambda_1(\mathbb{S}^1 \setminus K)$ increases while removing increasing arc of circles K (see Remark 6.1 and 6.2 for more details).

By consequence, in dimension 2, the more $K \cap \partial B(0,r)$ is large, better is the decay of energy, and you always have $\alpha \geq 1$ as soon K meets $\partial B(0,r)$ at least at one point (for instance when K is connected and r is small enough). In dimension 3, it is not so easy to obtain $\alpha \geq 1$, because to get this you need $\lambda_1 \geq 3/4$ on the sphere, which is not trivially verified in practice. It depends on the geometry of the set $K \cap B(0,r)$, which can now be complicated even with zero $\mathcal{H}^{N-1}$-measure, whereas consisting essentially in a finite number of points in dimension 2.

But the situation is even worst than that: if the value $3/4$ is achieved for some $K \cap \partial B(0,r)$ (for instance when K is a half-plane), then any larger set $K' \supset K$ (with zero $\mathcal{H}^{N-1}$-measure on the sphere) would fail to have $\lambda_1 \geq 3/4$, and consequently the larger set would imply a smaller exponent α. This phenomenon is somehow counter-intuitive regarding to what happens in dimension 2.

All the above explains why we have the restriction of K being contained in a half-plane in the statement of Theorem 1.2. By this way we guarantee that assumption (3) of Lemma 1.4 is satisfied with $\gamma = 3/4$, thus the monotonicity holds with $\alpha = 1$. If K was larger than a half-plane then we would have $\gamma < 3/4$ and the monotonicity with $\alpha < 1$ would be useless regarding to blow-in or blow-down limits of a Mumford-Shah minimizer.

2. Preliminaries

We will denote by $\mathscr{L}^N$ the Lebesgue measure on $\mathbb{R}^N$. Sometimes we use the notation $|A|$ for $\mathscr{L}^N(A)$ and the symbol dx in a integral will mean $d\mathscr{L}^N(x)$, and will be sometimes omitted if no confusion arizes. We denote by $B(x,R)$ the open ball of center x and radius R. The closed ball will be denoted by $\overline{B}(x,R)$, and when $x = 0$ we will sometimes simply use the notation B_R and $\overline{B}_R$. The topological boundary of a set $A \subset \mathbb{R}^N$ will be denoted by by ∂A, and the unit sphere of $\mathbb{R}^N$ will be denoted either by $\partial B(0,1)$ or $\mathbb{S}^{N-1}$.

Let E be a subset of $\mathbb{R}^N$. The k-dimensional Hausdorff measure of E is

$$\mathcal{H}^k(E) = \lim_{\tau \to 0^+} \mathcal{H}^k_\tau(A),$$

where

$$\mathcal{H}_\tau^k(A) := \inf \left\{ c_k \sum_{i=1}^{+\infty} \operatorname{diam}(E_i)^k \quad ; \quad E \subset \bigcup_{i=1}^{+\infty} E_i \ \text{ and } \ \operatorname{diam}(E_i) \leq \tau \right\},$$

and $c_k = \mathscr{L}^k(B(0,1))/2^k$. It is well known that $\mathcal{H}^k$ is an outer measure on $\mathbb{R}^N$ for which the Borel sets are measurable sets. Moreover, its restriction to k-dimensional spaces coincides with the Lebesgue measure $\mathscr{L}^k$ for $k \geq 1$, and $\mathcal{H}^0$ is the standard counting measure.

Sometimes we will need to integrate over spheres of $\mathbb{R}^N$, and this will mean endowed with the Hausdorff measure $\mathcal{H}^{N-1}$ but sometimes abbreviated by $d\omega$, or even omitted when there is no possible ambiguity.

Definition 2.1. *Let $K \subset \mathbb{R}^N$ be a closed set and $u \in \bigcap_{R>0} W^{1,2}(B_R \setminus K)$. We say that u is a local energy minimizer in $\mathbb{R}^N \setminus K$ if for every $R > 0$, u is a solution for the problem*

$$\min \left\{ \int_{B_R \setminus K} |\nabla v|^2 dx; v \in W^{1,2}(B_R \setminus K) \text{ and } v = u \text{ on } \partial B_R \setminus K \right\}.$$

Remark 2.2. It is standard to check that u will satisfy $-\Delta u = 0$ in $B_R \setminus K$, and $\frac{\partial u}{\partial \nu} = 0$ on K, in a weak sense. In particular u is a C^∞ function outside K.

We introduce the set of admissible pairs

$$\mathcal{A} := \left\{ (u, K); \ K \subseteq \mathbb{R}^N \text{ is closed and } u \in \bigcap_{R>0} W^{1,2}(B_R \setminus K) \right\}.$$

Following [4] and [8] we introduce the following definitions.

Definition 2.3. *Let $(u, K) \in \mathcal{A}$ and $B \subset \mathbb{R}^N$ be a ball. A competitor for the pair (u, K) in the ball B is a pair $(v, L) \in \mathcal{A}$ such that*

$$\left. \begin{array}{c} u = v \\ K = L \end{array} \right\} \ \text{in } \mathbb{R}^N \setminus B$$

and satisfying furthermore the following topological condition: if x and y are two points in $\mathbb{R}^N \setminus (B \cup K)$ that lie in different connected components of $\mathbb{R}^N \setminus K$, then they also lie in different connected components of $\mathbb{R}^N \setminus L$.

We can now define what is a global minimizer.

Definition 2.4. *A global minimizer in $\mathbb{R}^N$ is a pair $(u, K) \in \mathcal{A}$ such that for every ball B in $\mathbb{R}^N$ and every competitor (v, L) in B we have*

$$\int_{B \setminus K} |\nabla u|^2 dx + \mathcal{H}^{N-1}(K \cap B) \leq \int_{B \setminus L} |\nabla v|^2 dx + \mathcal{H}^{N-1}(L \cap B).$$

The minimizer (u, K) is called reduced if whenever $(\tilde{K}, \tilde{u})$ is a competitor in a ball B with $\tilde{K} \subset K$ and $\tilde{u}$ is an extension of u in $W^{1,2}(B \setminus \tilde{K})$ then $\tilde{K} = K$.

Remark 2.5. In particular, if (u, K) is a global minimizer in $\mathbb{R}^N$ then u is a local energy minimizer in $\mathbb{R}^N \setminus K$.

We end this section with a regularity result that will be needed later. We recall that a minimal cone in $\mathbb{R}^3$ is a set belonging to the following list.

- A plane.
- A union of three half-planes meeting along their edges with 120°.
- A cone over the edges of a regular tetrahedron.

We shall need the use of the following 3-dimensional "ε-regularity" result. There is no comparable theorem in higher dimensions (essentially because the minimal cones for $N \geq 4$ are unknown), and this is the principal reason why our main result will be proved in dimension 3 only.

Theorem 2.6. [16, Theorem 9] *We can find some absolute positive constants $\varepsilon_0 > 0$, $\alpha \in (0, 1)$ and $c < 1$ such that the following holds. Let (u, K) be a reduced global minimizer in $\mathbb{R}^3$. Let $x \in K$ and $r > 0$ be such that*

$$\frac{1}{r^2} \int_{B(x,r) \setminus K} |\nabla u|^2 dx \leq \varepsilon_0.$$

Then there is a diffeomorphism ϕ of class $C^{1,\alpha}$ from $B(0, cr)$ to its image, verifying $\phi(0) = x$, $|\phi(y) - (y + x)| \leq 10^{-3} cr$, and there is a minimal cone Z such that $K \cap B(x, cr) = \phi(Z) \cap B(x, cr)$.

3. BLOW-UP AND BLOW-IN LIMITS OF MINIMIZERS

In this section we recall some known facts about blow-up or blow-in limits of Mumford-Shah minimizers. Extracting sequences which converge to a minimizers in a fairly weak sense is not very difficult. The most difficult part is to show that this convergence holds strongly in $L^2_{loc}(\mathbb{R}^N)$ for the gradients.

If K and K' are two compact subsets of $\mathbb{R}^N$ the Hausdorff distance between K and K' is

$$d_H(K, K') = \max \left(\sup_{x \in K} dist(x, K'), \sup_{x \in K'} dist(x, K) \right).$$

when a sequence of compact sets K_n is said to converge to a set K, it will always refer to the convergence in the sense of the Hausdorff distance.

Subsequently, the convergence of a sequence of minimizers (u_k, K_k) will be understood in the following way.

Definition 3.1. *We say that a sequence of couples $(u_k, K_k) \in \mathcal{A}$ converges to some $(u_0, K_0) \in \mathcal{A}$ if the following holds.*

(1) $K_k \cap \overline{B}_M \to K_0 \cap \overline{B}_M$ *for every $M > 0$.*
(2) *for any connected component $U \subseteq \mathbb{R}^N \setminus K_0$, there exists a sequence of numbers a_k such that $u_k - a_k$ converges to u_0 strongly in $L^1(H)$, for every compact set $H \subset U$.*
(3) $\nabla u_k \to \nabla u_0$ *strongly in $L^2(B_M)$, for every $M > 0$.*

The following statement summarizes the properties that we shall need later. It is mainly coming from gluing together several Propositions contained in the book [8], applied to the special case of global minimizers.

Theorem 3.2. *Let (u, K) be a reduced global minimizer in $\mathbb{R}^N$ and for all $r > 0$ let (u_r, K_r) be the new global minimizer defined by*

$$u_r(x) := \frac{1}{\sqrt{r}}u(rx) \text{ and } K_r := \frac{1}{r}K.$$

Then, there exists two sequences $r_k \to 0$ and $R_k \to +\infty$ such that (u_{r_k}, K_{r_k}) converges to some (u_0, K_0) and (u_{R_k}, K_{R_k}) converges to some (u_∞, K_∞) in the sense of Definition 3.1. Moreover (u_0, K_0) and (u_∞, K_∞) are reduced global minimizers in $\mathbb{R}^N$.

Proof. The convergence of K_r and u_r, together with a weak convergence for ∇u_r in $L^2(B_M)$ are consequences of standard compactness results, as explained for instance in [8, Proposition D.37.8.]. The starting point is the fact that u_r have a uniform Dirichlet energy in B_M, due to the classical energy estimate for Mumford-Shah minimizers $\int_{B_r} |\nabla u|^2 dx \leq Cr^{N-1}$, obtained by taking $(u1_{\mathbb{R}^N \setminus B_r}, (K \setminus B_r) \cup \partial B_r)$ as competitor. This "weak" convergence is enough to obtain that (u_0, K_0) and (u_∞, K_∞) are again reduced global minimizers. This fact is quite not obvious but a proof is, for instance, given in [8, Theorem D.40.9]. In addition, by [8, Proposition D.37.18.] (which is just semicontinuity with respect to the weak convergence), we get

$$\int_{B_M \setminus K_0} |\nabla u_0|^2 dx \leq \liminf_k \int_{B_M \setminus K_{r_k}} |\nabla u_{r_k}|^2 dx, \text{ and}$$

$$\int_{B_M \setminus K_\infty} |\nabla u_0|^2 dx \leq \liminf_k \int_{B_M \setminus K_{R_k}} |\nabla u_{R_k}|^2 dx.$$

Finally, the reverse inequality in the above with a limsup is again not obvious, but follows from [8, Corollary D.38.48.], which together with the weak convergence implies strong convergence in $L^2(B_M)$ for the gradients. $\square$

Remark 3.3. Actually, since u_r is a sequence of harmonic functions, the convergence in compact sets of $\mathbb{R}^N \setminus K_0$ is even better: it is a uniform convergence [8, Proposition D.37.25.], but we will not need it.

4. GEOMETRIC PROPERTIES OF NEUMANN CONES

The purpose of this section is to give a rigorous formulation and justification of the fact that the normal vector to a Neumann cone K at some point $x \in K$, is always orthogonal to the vector x itself, which is one of the key ingredient to prove the monotonicity Lemma of Section 7.

If $E \subset \mathbb{R}^N$ and $1 \leq k \leq N$, we say that E is k-rectifiable if there exists at most countably many Lipschitz mappings $f_i : \mathbb{R}^k \to \mathbb{R}^N$ such that

$$\mathcal{H}^k(E \setminus \cup_i f_i(\mathbb{R}^k)) = 0.$$

We denote the Grassmannian by $G(N, k)$, i.e. the set of all vectorial subspaces of $\mathbb{R}^N$ of dimension k.

Let $1 \leq k \leq N$ and $K \subset \mathbb{R}^N$ be a set with locally finite $\mathcal{H}^k$ measure. We say that a plane $P \in G(N, k)$ is an approximate tangent plane of K at point x if

$$\int_{\frac{1}{r}(K-x)} \varphi(y) d\mathcal{H}^k(y) \xrightarrow[r \to 0]{} \int_P \varphi(y) d\mathcal{H}^k(y) \qquad \forall \varphi \in C_c^0(\mathbb{R}^N).$$

We shall denote this plane by $T_x(K)$. If $K \subset \mathbb{R}^N$ is k-rectifiable then it well known that it admits an approximate tangent plane at $\mathcal{H}^k$ a.e. point [3, Theorem 2.83 (i)].

The following proposition is an easy exercise, but we provide a detailed proof since it appears as a crucial fact regarding to our main result.

Proposition 4.1. *Assume $N \geq 3$. Let $E \subset \mathbb{S}^{N-1}$ be $(N-2)$-rectifiable and $C \subset \mathbb{R}^N$ be the cone generated by E, i.e.*

$$C := \{tw \in \mathbb{R}^N \ ; \ t \geq 0 \ and \ w \in E\}.$$

Then C is $(N-1)$-rectifiable and for $\mathcal{H}^{N-1}$-a.e. $x \in C$, if $T_x(C) \in G(N, N-1)$ is the approximate tangent plane to C at point x and ν_x is a normal vector to $T_x(C)$, then $x \cdot \nu_x = 0$.

Proof. Since the cone $\mathbb{S}^{N-1} \times \mathbb{R}^+$ is, away from the origin, locally bi-Lipschitz equivalent to an orthogonal cartesian product of type $\mathbb{R}^{N-1} \times \mathbb{R}$, it follows from [11, Theorem 3.2.23] that C is $(N-1)$-rectifiable in $\mathbb{R}^N$. By consequence C admits an approximative tangent plane $T_x(C) \in G(N, N-1)$ at $\mathcal{H}^{N-1}$-a.e. point.

Let $E := C \cap \mathbb{S}^{N-1}$. By assumption, E is a $(N-2)$-rectifiable set. Let $S \subset E$ be the $\mathcal{H}^{N-2}$-negligible set such that $E \setminus S$ admits a $(N-2)$-dimensional approximate tangent plane $T_w(E) \in G(N, N-2)$ for every $x \in E \setminus S$.

In virtue of [11, Theorem 2.10.45], the cone generated by S, namely $\bigcap_{t \geq 0} tS$ is $\mathcal{H}^{N-1}$-negligible. We deduce that for $\mathcal{H}^{N-1}$-a.e. $x \in C$, we can assume that both $T_x(C)$ exists and moreover $T_{x/|x|}(E) \in G(N, N-2)$ also does exist. But then it is easy to check that $T_x(C) = T_{x/|x|}(E) \times vect\{x\}$.

Indeed, let $x_0 \in C$ be given and let us seek for the approximate tangent plane at point x_0. Since C is invariant under dilatation, we can assume without loss of generality that $|x_0| = 1$. For $r > 0$ we set $C_r := \frac{1}{r}(C - x_0)$. Then for every $\varphi \in C_c^0(\mathbb{R}^N)$, the coarea formula [3, (2.72)] applied on the rectifiable set C_r with the r-Lipschitz function $f : x \mapsto r|x + x_0|$, for which under the notation of [3] the

coarea factor is equal to $\mathbf{C}_1 d^P f_x = r$ for all $P \in G(N, N-1)$, yields

$$
\begin{aligned}
\int_{C_r} \varphi(y) d\mathcal{H}^{N-1}(y) &= \frac{1}{r} \int_{\mathbb{R}+} \left(\int_{C_r \cap \partial B(-x_0, t/r)} \varphi(w) d\mathcal{H}^{N-2}(w) \right) dt \\
\text{(4.1)} \qquad &= \frac{1}{r} \int_{\mathbb{R}+} \left(\int_{\frac{1}{r}(C \cap \partial B_t - x_0)} \varphi(w) d\mathcal{H}^{N-2}(w) \right) dt
\end{aligned}
$$

But since C is conical, we have that $C \cap \partial B_t = tE$ for all t and from the fact that $\mathcal{H}^{N-2}(tE) = t^{N-2}\mathcal{H}^{N-2}(E)$ we deduce that

$$
\int_{\frac{1}{r}(tE - x_0)} \frac{\varphi(w)}{r} d\mathcal{H}^{N-2}(w) = \int_{\frac{1}{r}(E - x_0)} \frac{1}{r}\varphi\big(tz + x_0(t-1)/r\big) t^{N-2} d\mathcal{H}^{N-2}(z).
$$

Let us denote by μ_r the σ-finite measure on $\mathbb{R}^N$ defined by $\mu_r := \mathcal{H}^{N-2}|_{\frac{1}{r}(E - x_0)}$, that we know to converge weakly to $\mu_0 := \mathcal{H}^{N-2}|_{T_{x_0}(E)}$. Returning to (4.1) and applying Fubini Theorem we arrive to

$$
\int_{C_r} \varphi(y) d\mathcal{H}^{N-1}(y) = \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}+} \frac{1}{r}\varphi\big(tz + x_0(t-1)/r\big) t^{N-1} dt \right) d\mu_r(z)
$$

or, using the change of variable $u = (t-1)/r$ in the integral on the t variable, we can also write it as

$$
\begin{aligned}
\int_{C_r} \varphi(y) d\mathcal{H}^{N-1}(y) &= \int_{\mathbb{R}^N} \left(\int_{-\frac{1}{r}}^{+\infty} \varphi\big((ru+1)z + x_0 u\big)(ru+1)^{N-1} du \right) d\mu_r(z) \\
\text{(4.2)} \qquad &= \int_{\mathbb{R}^N} \left(\int_{-\infty}^{+\infty} \varphi\big((ru+1)z + x_0 u\big)(ru+1)^{N-1} du \right) d\mu_r(z),
\end{aligned}
$$

the latter being true for r small enough, depending on the size of the support of φ, i.e. the integration being in reality computed on a compact domain only. Now we notice that, for $0 \le r \le 1$ the family of continuous functions

$$
g_r : z \mapsto \int_{-\infty}^{+\infty} \varphi\big((ru+1)z + x_0 u\big)(ru+1)^{N-1} du
$$

are all supported on a same compact set, and converges uniformly as $r \to 0$ to the function

$$
g_0(z) := \int_{\mathbb{R}} \varphi\big(z + x_0 u\big) du.
$$

Therefore, passing to the limit in (4.2) we obtain that

$$
\int_{C_r} \varphi(y) d\mathcal{H}^{N-1}(y) \xrightarrow[r \to 0]{} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}} \varphi\big(z + x_0 u\big) du \right) d\mu_0(y) = \int_{T_{x_0}(E) \times \mathbb{R}x_0} \varphi d\mathcal{H}^{N-1}
$$

and this is exactly saying that $T_{x_0}(C) = T_{x_0}(E) \times vect\{x_0\}$, as desired. $\qquad\square$

Remark 4.2. A direct consequence of Proposition 4.1 is that Neumann cones are $(N-1)$-rectifiable.

Corollary 4.3. *Let $N \geq 2$, $C \subset \mathbb{R}^N$ be a Neumann cone and $K \subset C$ a closed set. Then K is $(N-1)$-rectifiable and for $\mathcal{H}^{N-1}$a.e. $x \in K$ we have that $x \cdot \nu_x = 0$ for any vector ν_x orthogonal to $T_x(C)$.*

Proof. The statement is trivial for $N = 2$ thus we assume that $N \geq 3$. From the fact that $K \subset C$ we directly get that K is $(N-1)$-rectifiable (because C is), and moreover $T_x(K)$ must be contained in $T_x(C)$. But since they have same dimension we must have $T_x(K) = T_x(C)$. The conclusion then follows from Proposition 4.1.   □

5. INTEGRATION BY PARTS FOR SBV FIELDS

Let us recall some standard definitions and properties of the functional spaces BV and SBV that will be used later, and that one can find for instance in [3].

5.1. **Short review on SBV theory.** For any open set $\Omega \subset \mathbb{R}^N$, the space $BV(\Omega)$ is the class of all functions $u \in L^1_{loc}(\Omega, \mathbb{R})$ such that Du (the derivative of u in the distributional sense) is a finite measure. If $u \in (BV(\Omega))^N$ is a BV vector field, a point $z \in \mathbb{R}^N$ is an approximate limit for u at point x if

$$\lim_{\rho \to 0} \frac{1}{|B(x, \rho)|} \int_{B(x,\rho)} |u(y) - z| dy = 0.$$

The set S_u of points where this property does not hold is called the *approximate discontinuity set* of u, and the points z for which the limit exists is called an approximative limit of u at point x and is denoted by $\tilde{u}(x)$. A remarkable result of Federer and Vol'pert (see [3, Th. 3.78.]) says that when $u \in (BV(\Omega))^N$, then S_u is $(N-1)$-rectifiable and $D^s u$ (the singular part of Du with respect to $\mathscr{L}^N$ in the Radon-Nikodym decomposition $Du = D^a u + D^s u$) restricted to S_u is absolutely continuous with respect to $\mathcal{H}^{N-1}$. We will say that $u \in (SBV(\Omega))^N$ when $D^s u$ is actually *concentrated* on S_u.

The density of the regular part $D^a u$ of Du with respect to $\mathscr{L}^N$, denoted by ∇u, coincides $\mathscr{L}^N$-a.e. with the *approximate differential* of u (see [3, Th. 3.83.]). A function u is approximately differentiable at x if there exists a matrix $\nabla u(x)$ such that

$$\lim_{\rho \to 0} \frac{1}{|B(x, \rho)|} \int_{B(x,\rho)} \frac{|u(y) - \tilde{u}(x) - \nabla u(x).(y - x)|}{\rho} dy = 0.$$

In the sequel we will also use the notion of *trace* of u on the singular set S_u. Since S_u is rectifiable, one can fix an orientation $\nu_u : S_u \to \mathbb{S}^{N-1}$ in such a way that for $\mathcal{H}^{N-1}$-a.e. $x \in S_u$ the approximate tangent plane to S_u at x is orthogonal to the vector $\nu_u(x)$. Then for any $x \in S_u$ and $\rho > 0$ we define $B(x, \rho)^+ := B(x, \rho) \cap \{y; \langle y, \nu_u(x) \rangle \geq 0\}$ and $B(x, \rho)^- := B(x, \rho) \cap \{y; \langle y, \nu_u(x) \rangle \leq 0\}$. For $\mathcal{H}^{N-1}$-a.e. $x \in S_u$, Theorem 3.77. of [3] provides the existence of traces $u^+(x)$ and $u^-(x)$ satisfying

$$(5.1) \qquad \lim_{\rho \to 0} \frac{1}{|B(x, \rho)^\pm|} \int_{B(x,\rho)^\pm} |u(y) - u^\pm(x)| dy = 0.$$

The set of points $x \in S_u$ where $u^{\pm}(x)$ exist is called the *jump set* and is denoted by J_u. It can be shown that $\mathcal{H}^{N-1}(S_u \setminus J_u) = 0$ and for $x \in J_u$ the quantity $(u^+(x) - u^-(x))$ is called the jump of u at point x, which sign depends on the orientation of S_u. Moreover for any $u \in (SBV(\Omega))^N$ the representation

$$(5.2) \qquad Du = D^a u + D^s u = \nabla u \, \mathcal{L}^N + (u^+ - u^-) \otimes \nu_u \, \mathcal{H}^{N-1}|S_u,$$

holds.

Let us also mention that a simple approximation argument [3, Proposition 4.4.] says that, if $K \subset \Omega$ is closed and $\mathcal{H}^{N-1}(K) < +\infty$, then any $u \in L^\infty(\Omega) \cap W^{1,1}(\Omega \setminus K)$ belongs to $SBV(\Omega)$ and $\mathcal{H}^{N-1}(S_u \setminus K) = 0$.

5.2. **Sets of finite perimeter.**
If $U \subset \mathbb{R}^N$ is open we say that U has finite perimeter if $\mathbf{1}_U \in BV(\mathbb{R}^N)$. The reduced boundary $\partial^* U$ is the set of points $x \in \mathbb{R}^N$ such that the limit $\lim_{r \to 0} D(\mathbf{1}_U)(B_r(x))/|B_r(x)| =: -\nu_U(x)$ exists and satisfies $|\nu_U(x)| = 1$ (here we chosed the normal to be pointing outwise). In particular when $\mathcal{H}^{N-1}(\partial \Omega) < +\infty$, then $\mathbf{1}_U \in SBV(\mathbb{R}^N)$ and $\partial^* U$ is equal to the jump set of $\mathbf{1}_\Omega$. By consequence the following Gauss-Green formula holds

$$\int_U \operatorname{div} \varphi \, dx = \int_{\partial^* U} \nabla \varphi \cdot \nu_U \, d\mathcal{H}^{N-1}(x) \qquad \forall \varphi \in (C_c^1(\mathbb{R}^N))^N.$$

5.3. **A Gauss-Green Formula for SBV fields.**

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^N$ be open and $F \in L^\infty \cap (SBV(\Omega))^N$. Let $\operatorname{div} F^a$ be the trace of the absolutely continuous part of DF with respect to the Lebesgue measure and S_F be the singular set of F. Then for any open set U such that*

    (1) $\overline{U} \subset \Omega$
    (2) $\mathcal{H}^{N-1}(\partial U) < +\infty$
    (3) $\mathcal{H}^{N-1}(S_F \cap \partial^* U) = 0$

we have that

$$(5.3) \quad \int_U \operatorname{div} F^a \, dx = \int_{\partial^* U} \tilde{F} \cdot \nu_U \, d\mathcal{H}^{N-1} + \int_{S_F \cap U} (F^- - F^+) \cdot \nu_F \, d\mathcal{H}^{N-1},$$

where $\tilde{F}$ denotes the approximative limit of F, which exists $\mathcal{H}^{N-1}$-a.e. on the rectifiable set $\partial^* U$.

Proof. When $F \in (SBV(\Omega))^N$ we recall that

$$(5.4) \qquad DF = \nabla F \, \mathcal{L}^N + (F^+ - F^-) \otimes \nu_F \, \mathcal{H}^{N-1}|_{J_F}.$$

By consequence $\operatorname{div} F$ is a measure and taking the trace in the above identity yields

$$(5.5) \qquad \operatorname{div} F = \operatorname{div} F^a \, \mathcal{L}^N + (F^+ - F^-) \cdot \nu_F \, \mathcal{H}^{N-1}|_{J_F}.$$

Now let U be such that $\overline{U} \subset \Omega$. It is well known that $L^\infty \cap (SBV(\Omega))^N$ is an algebra, i.e. stable by products, as a consequence of the Chain-Rule in BV (see [3, Example 3.97.]). By consequence, $G = \mathbf{1}_U F \in (SBV(\Omega))^N$ and $S_G \subseteq S_F \cup \partial U$

(Remember that $\mathbf{1}_U \in SBV$ because $\mathcal{H}^{N-1}(\partial U) < +\infty$). Since by assumption $\mathcal{H}^{N-1}(S_F \cap \partial U) = 0$, we obtain that the decomposition of $\operatorname{div} G$ as a measure is the following

$$(5.6) \quad \operatorname{div} G = \operatorname{div} F^a \mathcal{L}^N|_U + (F^+ - F^-) \cdot \nu_F \, \mathcal{H}^{N-1}|_{J_F \cap U} - \tilde{F} \cdot \nu_U \, \mathcal{H}^{N-1}|_{\partial^* U}$$

(see again [3, Example 3.97.] for the characterization of G^+ and G^- on the jump set in terms of $F^\pm$ and $\mathbf{1}_U^\pm$).

Now let us prove (5.3). For this purpose we take a test function $\psi \in C_c^\infty(\Omega)$ such that $\psi = 1$ on $\overline{U}$. Then in the sense of distributions, $\langle \operatorname{div} G, \psi \rangle = \langle G, \operatorname{div} \psi \rangle = \int_\Omega G \operatorname{div} \psi \, dx = 0$. But now returning to (5.6) we get

$$\begin{aligned}
0 &= \langle \operatorname{div} G, \psi \rangle = \int_\Omega \psi \, d(\operatorname{div} G) \\
&= \int_U \operatorname{div} F^a \, dx - \int_{\partial^* U} \tilde{F} \cdot \nu_U \, d\mathcal{H}^{N-1} + \int_{S_F \cap U} (F^+ - F^-) \cdot \nu_F \, d\mathcal{H}^{N-1}.
\end{aligned}$$

$\square$

6. Spectrum of the Laplacian on a fractured sphere

The purpose of this section is to study a Neumann eigenvalue problem on $\mathbb{S}^{N-1} \setminus K$ when K is a Neumann cone, or more generally contained in a Neumann cone.

We first assume that K is a Neumann cone. We will call Neumann-Laplace-Beltrami operator on $\mathbb{S}^{N-1} \setminus K$, the operator associated to the quadratic form $Q(u) = \int_{\mathbb{S}^{N-1} \setminus K} |\nabla u|^2 \, d\mathcal{H}^{N-1}$ on the Hilbert space $L^2(\mathbb{S}^{N-1})$, with domain $W^{1,2}(\mathbb{S}^{N-1} \setminus K)$. Since by assumption the embedding $W^{1,2}(\mathbb{S}^{N-1} \setminus K) \to L^2(\mathbb{S}^{N-1})$ is compact, the quadratic form $Q(u) + \|u\|_2^2$ has a compact resolvent and it follows that the eigenvalues of the Neumann-Laplace-Beltrami operator on $\mathbb{S}^{N-1} \setminus K$ is a countable and discrete set starting from 0 and going to $+\infty$, denoted by $\lambda_k(\mathbb{S}^{N-1} \setminus K)$, $k \in \mathbb{N}$. Actually the eigenvalue problem can be studied separately in each connected component of $\mathbb{S}^{N-1} \setminus K$, in which the eigenvalue $\lambda_0(\mathbb{S}^{N-1} \setminus K) = 0$ is simple. If U is one connected component and γ denotes the first positive eigenvalue in this domain, then the following Wirtinger inequality holds

$$(6.1) \quad \int_U (u(x) - m)^2 d\mathcal{H}^{N-1}(x) \le \frac{1}{\gamma} \int_U |\nabla_\tau u(x)|^2 d\mathcal{H}^{N-1}(x), \quad \forall u \in W^{1,2}(U),$$

where m denotes the average of u on U and ∇_τ is the tangential gradient on the sphere.

Notice that if C is a Neumann cone and if K is an arbitrary closed set such that $K \subset C$, then the embedding $W^{1,2}(\mathbb{S}^{N-1} \setminus K) \to L^2(\mathbb{S}^{N-1})$ is still compact because $\|u\|_{W^{1,2}(\mathbb{S}^{N-1} \setminus C)} = \|u\|_{W^{1,2}(\mathbb{S}^{N-1} \setminus K)}$. By consequence all the above applies with K.

We will use several times the following simple fact.

Remark 6.1. Let $E = \mathbb{S}^2 \cap \{(x, y, z) \; ; \; z = 0\}$ be the equator, and let F and G be two subsets of E verifying $F \subseteq G \subseteq E$. Then

$$\lambda_1(\mathbb{S}^2 \setminus G) \leq \lambda_1(\mathbb{S}^2 \setminus F).$$

This is a simple consequence of the Rayleigh formula, which says that

$$\lambda_1(\mathbb{S}^2 \setminus G) = \min_{v \in W^{1,2}(\mathbb{S}^2 \setminus G)} \frac{\int_{\mathbb{S}^2 \setminus G} |\nabla v|^2}{\int_{\mathbb{S}^2 \setminus G} v^2}.$$

The claim then follows from the fact that $W^{1,2}(\mathbb{S}^2 \setminus F) \subseteq W^{1,2}(\mathbb{S}^2 \setminus G)$, thus the eigenfunction in $\mathbb{S}^2 \setminus F$ is a competitor in the minimum above, and its Rayleigh quotient in $\mathbb{S}^2 \setminus G$ is still equal to $\lambda_1(\mathbb{S}^2 \setminus F)$ because $G \setminus F$ has zero measure.

Remark 6.2. Notice that the monotonicity described in Remark 6.1 is the exact opposite of what happens in dimension 2 ! Indeed, if $I \subset \mathbb{S}^1$ is an arc of circle, then $\lambda_1(I) = (\pi/|I|)^2$, where $|I|$ is the length of the arc I. Therefore if $F \subset G$ are two arcs in $\mathbb{S}^1$, then $\lambda_1(\mathbb{S}^1 \setminus G) > \lambda_1(\mathbb{S}^1 \setminus F)$. What does not work in the above argument in dimension 2 is that the L^2 norm is not preserved in the respective two domains so that the Rayleigh quotient is not the same in the two domains, even if the inclusion of the spaces $W^{1,2}$ still holds.

The eigenvalue in the special case of a half-plane is explicit. We sumarize some known facts in the following proposition.

Proposition 6.3. *Let*

$$S_\beta := \{(-\cos(\varphi), 0, \sin(\varphi)) \in \mathbb{S}^2; \; \varphi \in [-\beta, \beta]\}$$

be an arc of circle of aperture 2β on the unit sphere (vertical and left sided according to our parametrization). Then $\lambda_1(\mathbb{S}^2 \setminus S_{\frac{\pi}{2}}) = \frac{3}{4}$ is simple and the eigenspace is generated by the restriction on $\mathbb{S}^2 \setminus S_{\frac{\pi}{2}}$ of the following cracktip function written in cylindrical coordinates

(6.2) $$f(r, \theta, z) = \sqrt{r} \sin(\theta/2), \quad r > 0 \text{ and } \theta \in (-\pi, \pi).$$

In addition for β close to $\frac{\pi}{2}$ we have the following expansion :

(6.3) $$\lambda_1(\mathbb{S}^2 \setminus S_\beta) = \frac{3}{4} + \frac{2}{\pi} \cos(\beta) + o(\cos(\beta)).$$

By consequence $\lambda_1(\mathbb{S}^2 \setminus S_\beta) = 3/4$ if and only if $\beta = \frac{\pi}{2}$ (i.e. S_β is a half-equator).

Proof. The fact that $\lambda_1(\mathbb{S}^2 \setminus S_{\frac{\pi}{2}}) = \frac{3}{4}$ is simple and the eigenspace is generated by the function in (6.2) is part of [7, Lemma 4.1.]. Notice that a completely independent and somehow exotic proof can be found in [15, Section 3].

The expansion (6.3) has been proved by Robert Legendre in [13] (see alternatively [12, page 53] or [5]). More precisely, Legendre found the expansion $\alpha = \frac{1}{2} + \frac{1}{\pi} \cos \theta + o(\cos \theta)$ for the first degree of homogeneity of an homogeneous Harmonic function

with Neumann condition on a plane sector with aperture 2θ. Since the degree α is linked to the eigenvalue λ_1 through the relation

$$\alpha = \frac{1}{2}(\sqrt{1 + 4\lambda_1} - 1),$$

we deduce (6.3) from Legendre's expansion.

Finally, the fact that $\lambda_1(\mathbb{S}^2 \setminus S_\beta) = 3/4$ if and only if $\beta = \frac{\pi}{2}$ follows by monotonicity of the eigenvalue with respect to the length of S_β (i.e. Remark 6.1), together with (6.3) which proves a strict monotonicity near the value $3/4$. $\qquad\square$

Our main result is intimely related to the knowledge of eigenvalues on the sphere. For instance the following rigidity result will be pivotal.

Proposition 6.4. *Let $E \subseteq \mathbb{S}^2$ be a half-equator and $K \subseteq E$ a closed set such that $\lambda_1(\mathbb{S}^2 \setminus K) = 3/4$. Then $K = E$.*

Proof. We already know by Proposition 6.3 that $\lambda_1(\mathbb{S}^2 \setminus E) = 3/4$. We also know that $\lambda_1(\mathbb{S}^2 \setminus C) > 3/4$ for any arc of circle $C \subseteq E$ with $C \neq E$.

Now assume that $K \subseteq E$ is closed and $K \neq E$. Even if all cases could be treated the same way, we shall give a separated and simpler argument in the case when, denoting by d_E the geodesic distance in E,

$$\max_{x,y \in K} d_E(x, y) = \alpha < \pi.$$

If this occurs then there exists an arc of circle C such that $\mathcal{H}^1(C) = \alpha < \pi$ and $K \subseteq C$. We deduce that $\lambda_1(\mathbb{S}^2 \setminus K) \geq \lambda_1(\mathbb{S}^2 \setminus C) > 3/4$ (see Proposition 6.3), a contradiction.

Thus we are left with the case when

$$\max_{x,y \in K} d_E(x, y) = \pi.$$

Since K is closed and $K \neq E$, there exists an open interval $I \subset E \setminus K$. Then $\lambda_1(\mathbb{S}^2 \setminus K) \geq \lambda_1(\mathbb{S}^2 \setminus (E \setminus I))$. Now we will prove that $\lambda_1(\mathbb{S}^2 \setminus (E \setminus I)) > 3/4$. Assume by contradiction that $\lambda_1(\mathbb{S}^2 \setminus (E \setminus I)) = 3/4$, and let φ be an eigenfunction. Then $\varphi \in W^{1,2}(\mathbb{S}^2 \setminus E)$, and

$$\frac{3}{4} = \frac{\int_{\mathbb{S}^2 \setminus E} |\nabla \varphi|^2}{\int_{\mathbb{S}^2 \setminus E} \varphi^2} = \min_{v \in W^{1,2}(\mathbb{S}^2 \setminus E)} \frac{\int_{\mathbb{S}^2 \setminus E} |\nabla v|^2}{\int_{\mathbb{S}^2 \setminus E} v^2}.$$

We deduce that φ is an eigenfunction for the Neumann-Laplacian in $\mathbb{S}^2 \setminus E$ as well, but since the corresponding eigenspace have dimension 1, φ must be a multiple of the first eigenfunction in $\mathbb{S}^2 \setminus E$, which is given in (6.2) (up to a suitable choice of cylindrical coordinates). We deduce that φ must have a jump on I, in other words does not belong to $W^{1,2}(\mathbb{S}^2 \setminus (E \setminus I))$. This is a contraction. $\qquad\square$

7. A MONOTONICITY LEMMA

Definition 7.1. *Let $K \subset \mathbb{R}^N$ be a closed set. We say that K satisfies the topological condition (T) if the following holds. For almost every $r > 0$ such that $\partial B_r \setminus K$ is not connected, and for every connected component U of $\partial B_r \setminus K$, there exists an open set $D \subset B_r \setminus K$ such that $\partial D \setminus U \subset K$.*

Remark 7.2. For instance if $\partial B_r \setminus K$ is connected for a.e. $r > 0$, then K satisfies the topological condition (T). In particular this happens when K is contained in a half-plane in $\mathbb{R}^3$.

The monotonicity Lemma relies on the following integration by parts formula.

Lemma 7.3. *Let $K \subset \mathbb{R}^N$ be a closed set satisfying the following assumptions.*
   (1) *K satisfies the topological condition (T).*
   (2) *K is contained in a Neumann cone.*
Then for every local energy minimizer u in $\mathbb{R}^N \setminus K$ (as in Definition 2.1) and for a.e. $r > 0$ we have that

$$(7.1) \quad \int_{B_r \setminus K} \frac{|\nabla u|^2}{|x|^{N-2}} dx \leq \sum_{i \in I} r^{2-N} \int_{U_i} (u - m_i) \frac{\partial u}{\partial \nu} d\omega + \frac{N-2}{2r^{N-1}} \int_{U_i} (u - m_i)^2 d\omega,$$

where $(U_i)_{i \in I}$ denotes the connected components of $\partial B_r \setminus K$ and m_i is the average of u on U_i.

Proof. This is just an integration by parts, but since the domain is not smooth, the full details are a bit technical.

We first regularize $|x|^{2-N}$ in the following way. For $\varepsilon > 0$ we define

$$|x|_\varepsilon := \sqrt{x_1^2 + x_2^2 + \cdots + x_N^2 + \varepsilon}$$

so that $|x|_\varepsilon$ is a C^∞ function on $\mathbb{R}^N$. A simple computation shows that

$$\Delta(|x|_\varepsilon^{2-N}) = (2 - N)N \frac{\varepsilon}{|x|_\varepsilon^{N+2}} \leq 0,$$

in other words $|x|_\varepsilon^{2-N}$ is superharmonic. Let us write, to lighten the notation, $h_\varepsilon := |x|_\varepsilon^{2-N}$.

We will successively integrate two times by parts. The first one can be treated by a variational argument, using the fact that u is an energy minimizer. Indeed, let $\delta > 0$ be a small parameter, and let $f(x) \in \mathrm{Lip}(\mathbb{R}^+, \mathbb{R})$ be the piecewise affine function such that $f(x) = 1$ for all $x \in [0, r - \delta]$, $f(x) = 0$ for all $x \geq r$ and $f'(x) \leq \frac{1}{\delta}$ a.e. on $\mathbb{R}^+$. Then let $\varphi_\delta(x) := f(|x|)$.

Next, using that for any $t \in \mathbb{R}$, the function $u + t(\varphi_\delta u h_\varepsilon)$ is a competitor for u in $W^{1,2}(B_r \setminus K)$, we deduce that

$$\int_{B_r \setminus K} \nabla u \cdot \nabla(\varphi_\delta \, u \, h_\varepsilon) \, dx = 0,$$

which implies

$$\int_{B_r \setminus K} |\nabla u|^2 \, \varphi_\delta \, h_\varepsilon \, dx + \int_{B_r \setminus K} u \, h_\varepsilon \, \nabla u \cdot \nabla \varphi_\delta \, dx + \int_{B_r \setminus K} \varphi_\delta \, u \, \nabla u \cdot \nabla h_\varepsilon \, dx = 0.$$

Letting $\delta \to 0$, we observe that φ_δ converges increasingly to 1 so that the first term in the above converges by the monotone convergence theorem. For the convergence of the second term we notice that $\nabla \varphi_\delta = -\frac{1}{\delta} \left(\frac{x}{|x|} \right) 1_{B_r \setminus B_{r-\delta}}$ thus from Lebesgue's differentiation theorem we deduce that

$$\int_{B_r \setminus K} u \, h_\varepsilon \, \nabla u \cdot \nabla \varphi_\delta \, dx \to_{\delta \to 0} - \int_{\partial B_r \setminus K} u \, h_\varepsilon \, \nabla u \cdot \nu,$$

for almost every $r > 0$. Finally, the last term easily converges by the dominated convergence theorem so in total we arrive to

$$(7.2) \quad \int_{B_r \setminus K} |\nabla u|^2 \, h_\varepsilon \, dx - \int_{\partial B_r \setminus K} u \, h_\varepsilon \, \nabla u \cdot \nu \, dx + \int_{B_r \setminus K} u \, \nabla u \cdot \nabla h_\varepsilon \, dx = 0.$$

Now we want to apply a second integration by parts on the last term. Observe that we cannot argue as before because, even if $|x|^{2-N}$ is harmonic outside the origin, it is not an energy minimizer since it has infinite energy. So we need to integrate by parts for real, and for this we shall use the SBV formula of Section 5.

We first assume $u \in L^\infty$ and we remark that $F := u^2 \nabla h_\varepsilon \in W^{1,1}(B_r \setminus K) \cap L^\infty(B_r)$ and therefore belongs to $SBV(B_r)$ due to [3, Proposition 4.4.]. In addition since F is smooth outside K we have that $S_F \subset K$ and $\mathcal{H}^{N-1}(K \cap \partial B_r) = 0$ because it is contained in a Neumann cone. By consequence, applying Lemma 5.1 to F with $U = B_r$ yields

$$(7.3) \quad \int_{B_r} \mathrm{div} F^a dx = \int_{\partial B_r} F \cdot \nu \, d\mathcal{H}^{N-1} + \int_{K \cap B_r} (F^- - F^+) \cdot \nu_F \, d\mathcal{H}^{N-1}.$$

Now let us identify each term: since F is smooth outside K it directly follows that

$$\mathrm{div} F^a = 2u \nabla u \cdot \nabla h_\varepsilon + u^2 \Delta h_\varepsilon \text{ a.e. in } B_r,$$

and

$$F \cdot \nu = u^2 \frac{\partial h_\varepsilon}{\partial \nu} \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial B_r.$$

Now we claim that

$$(F^- - F^+) \cdot \nu_F = 0 \quad , \quad \mathcal{H}^{N-1}\text{-a.e. on } K.$$

This is because $F^\pm = (u^2)^\pm \nabla h_\varepsilon$, $\mathcal{H}^{N-1}$- a.e. on K and $\nabla h_\varepsilon \cdot \nu_F = 0$. Indeed, by definition, $\nu_F(x)$ must be orthogonal to the approximative tangent plane of K at point x. But since K is contained in a Neumann cone, Corollary 4.3 says that $x \cdot \nu_F(x) = 0$. Therefore, (7.3) simply becomes

$$(7.4) \quad \int_{B_r} 2u \nabla u \cdot \nabla h_\varepsilon + u^2 \Delta h_\varepsilon \, dx = \int_{\partial B_r} u^2 \frac{\partial h_\varepsilon}{\partial \nu} \, d\mathcal{H}^{N-1}.$$

This was assuming $u \in L^\infty$, but if not we could replace u by

$$u^M := \min(\max(u(x), -M), M),$$

establish (7.4) for u^M and then let $M \to +\infty$ to get (7.4) for u (notice that $u \in L^2$, $\nabla u \in L^2$, $\nabla h_\varepsilon \in L^\infty$ and that $|u^M| \le |u|$, $|\nabla u^M| \le |\nabla u|$ so that $u^M \nabla u^M \cdot \nabla h_\varepsilon$ strongly converges in L^1 to $u \nabla u \cdot \nabla h_\varepsilon$ as $M \to +\infty$ by the dominated convergence theorem. The other terms are treated by a similar way).

Gathering (7.4) together with (7.2) we obtain that

$$\int_{B_r \setminus K} |\nabla u|^2 \, h_\varepsilon \, dx = \int_{\partial B_r \setminus K} u \, h_\varepsilon \, \frac{\partial u}{\partial \nu} - \frac{1}{2} \int_{\partial B_r \setminus K} u^2 \frac{\partial h_\varepsilon}{\partial \nu} + \int_{B_r \setminus K} u^2 \Delta h_\varepsilon dx.$$

and since $\Delta(|x|_\varepsilon^{2-N}) \le 0$, we deduce that

$$(7.5) \qquad 0 \le \int_{B_r \setminus K} |\nabla u|^2 \, h_\varepsilon \, dx \le \int_{\partial B_r \setminus K} u \, h_\varepsilon \, \frac{\partial u}{\partial \nu} - \frac{1}{2} \int_{\partial B_r \setminus K} u^2 \frac{\partial h_\varepsilon}{\partial \nu}.$$

Now if we write the explicit expressions of h_ε and $\frac{\partial h_\varepsilon}{\partial \nu}$ on the sphere ∂B_r we get

$$0 \le \int_{B_r \setminus K} |\nabla u|^2 h_\varepsilon dx \le (r^2 + \varepsilon)^{\frac{2-N}{2}} \left(\int_{\partial B_r} u \frac{\partial u}{\partial \nu} \right) + (N-2) \frac{r}{2(r^2 + \varepsilon)^{\frac{N}{2}}} \left(\int_{\partial B_r} u^2 \right).$$

Finally, letting $\varepsilon \to 0$ we obtain

$$0 \le \int_{B_r \setminus K} \frac{|\nabla u|^2}{|x|^{N-2}} \, dx \le r^{2-N} \left(\int_{\partial B_r \setminus K} u \frac{\partial u}{\partial \nu} \right) + \frac{N-2}{2 r^{N-1}} \left(\int_{\partial B_r \setminus K} u^2 \right).$$

Indeed notice that $\int_{B_r \setminus K} |\nabla u|^2 h_\varepsilon \, dx \to \int_{B_r \setminus K} \frac{|\nabla u|^2}{|x|^{N-2}} \, dx$ by the monotone convergence theorem.

We are almost done, excepted the subtraction by the constants m_i. But this follows easily by applying all the above argument to the function $\tilde{u} := u - \sum_{i \in I} \mathbf{1}_{D_i}(x) m_i$, where the D_i are given by the topological condition (T). Notice that the domains D_i are necessarily disjoint and that $\tilde{u}$ remains an energy minimizer in $B_r \setminus K$ because we subtracted a locally constant function. $\qquad \square$

Remark 7.4. Observe that, from the fact that $u \in W^{1,2}(B \setminus K)$ for all B, the right hand side of the inequality is bounded for a.e. $r > 0$, thus a free consequence of Lemma 7.3 is that $\int_{B_1} |\nabla u|^2 |x|^{2-N} dx < +\infty$, which is not obvious since a priori ∇u is only in $L^2(B(0,1))$. This says that ∇u is more integrable at the origin than expected, and could be understood as a regularity result for energy minimizers outside a set K contained in a Neumann cone.

Lemma 7.5. *Let $K \subset \mathbb{R}^N$ be a closed set satisfying the following assumptions.*

(1) *K satisfies the topological condition (T).*
(2) *K is contained in a Neumann cone.*

(3) $\exists \gamma > 0$ *s.t. for a.e. $r > 0$ and for every connected component U of $\partial B_r \setminus K$, the first positive eigenvalue of the Neumann-Laplace-Beltrami operator on U satisfies $\lambda_1(U) \geq \gamma/r^2$.*

Then for every local energy minimizer u in $\mathbb{R}^N \setminus K$ we have that

$$(7.6) \qquad \varphi : r \mapsto \frac{1}{r^\alpha} \int_{B_r} \frac{|\nabla u|^2}{|x|^{N-2}} \, dx$$

is nondecreasing, where α is defined by

$$\alpha = \alpha(N, \gamma) = \sqrt{(N-2)^2 + 4\gamma} - (N-2).$$

Moreover, if $\varphi(r)$ is positively constant on an interval $[a, b]$ then for a.e. $r \in [a, b]$ and for every connected component U of $\partial B_r \setminus K$, the value γ/r^2 is the first positive eigenvalue for the Neumann-Laplace-Beltrami operator on U and the restritction of u on U is an associated eigenfunction.

Proof. We first apply Lemma 7.3 and then use Cauchy-Schwarz and the elementary inequality $ab \leq \frac{1}{2\delta} a^2 + \frac{\delta}{2} b^2$ to write

$$\int_{B_r \setminus K} \frac{|\nabla u|^2}{|x|^{N-2}} \leq \sum_{i \in I} r^{2-N} \int_{U_i} (u - m_i) \frac{\partial u}{\partial \nu} d\mathcal{H}^{N-1} + \frac{N-2}{2r^{N-1}} \int_{U_i} (u - m_i)^2 d\mathcal{H}^{N-1}$$

$$\leq \sum_{i \in I} r^{2-N} \left(\int_{U_i} (u - m_i)^2 \right)^{\frac{1}{2}} \left(\int_{U_i} \left(\frac{\partial u}{\partial \nu} \right)^2 \right)^{\frac{1}{2}} + \frac{N-2}{2r^{N-1}} \int_{U_i} (u - m_i)^2$$

$$\leq \sum_{i \in I} r^{2-N} \frac{r}{\sqrt{\lambda_1(U_i)}} \left(\int_{U_i} |\nabla_\tau u|^2 \right)^{\frac{1}{2}} \left(\int_{U_i} \left(\frac{\partial u}{\partial \nu} \right)^2 \right)^{\frac{1}{2}} + \frac{(N-2)r^2}{2r^{N-1}\lambda_1(U_i)} \int_{U_i} |\nabla_\tau u|^2$$

$$\leq \sum_{i \in I} r^{2-N} \frac{r}{\sqrt{\gamma}} \left(\int_{U_i} |\nabla_\tau u|^2 \right)^{\frac{1}{2}} \left(\int_{U_i} \left(\frac{\partial u}{\partial \nu} \right)^2 \right)^{\frac{1}{2}} + \frac{(N-2)r^2}{2r^{N-1}\gamma} \int_{U_i} |\nabla_\tau u|^2$$

$$\leq \sum_{i \in I} \frac{r^{3-N}}{\sqrt{\gamma}} \left(\frac{\delta}{2} \int_{U_i} |\nabla_\tau u|^2 + \frac{1}{2\delta} \int_{U_i} \left(\frac{\partial u}{\partial \nu} \right)^2 \right) + \frac{(N-2)r^{3-N}}{2\gamma} \int_{U_i} |\nabla_\tau u|^2$$

$$\leq \sum_{i \in I} r^{3-N} \left(\frac{\delta}{2\sqrt{\gamma}} + \frac{(N-2)}{2\gamma} \right) \int_{U_i} |\nabla_\tau u|^2 + \frac{r^{3-N}}{2\delta\sqrt{\gamma}} \int_{U_i} \left(\frac{\partial u}{\partial \nu} \right)^2.$$

Then we choose $\delta > 0$ so that

$$\left(\frac{\delta}{2\sqrt{\gamma}} + \frac{(N-2)}{2\gamma} \right) = \frac{1}{2\delta\sqrt{\gamma}}.$$

This gives the value

$$\delta = \frac{\sqrt{(N-2)^2 + 4\gamma} - (N-2)}{2\sqrt{\gamma}}$$

which implies the inequality,

$$(7.7) \qquad \int_{B_r \setminus K} \frac{|\nabla u|^2}{|x|^{N-2}} \leq \frac{1}{\alpha} r^{3-N} \int_{\partial B_r \setminus K} |\nabla u|^2,$$

with the α defined in the statement of the Lemma. But this exactly says that

$$E(r) \leq \frac{1}{\alpha} r E'(r)$$

with

$$E(r) = \int_{B_r \setminus K} \frac{|\nabla u|^2}{|x|^{N-2}},$$

in other words $\frac{d}{dr}(r^{-\alpha} E(r)) \geq 0$ and $r^{-\alpha} E(r)$ is nondecreasing.

Finally to finish the proof, we observe that when φ is constant, the derivative is zero, in other words we must have $E(r) = \frac{1}{\alpha} r E'(r)$ for a.e. $r > 0$, thus, since all the inequalities that we used to prove the monotonicity (starting from the second line) are nonnegative, or sum of nonnegative terms, they must all be equalities. In particular the equality in the third line says that the restriction of u to almost every sphere must be the optimal function in the Poincaré-Wirtinger inequality associated with the constant r^2/γ, and so follows the Lemma. $\qquad \square$

Remark 7.6. Here for our convenience we considered energy minimizers, but the statement of Lemma 7.5 (or a simple variant) would still be true for stationary solutions, i.e. weak solution of a Neumann problem in all the $B(0, R) \setminus K$. Furthermore, arguing like in [6], it would be possible to prove a similar monotonicity result for solutions of more general elliptic equations of the form $-\text{div} A \nabla u = f$ for some $f \in L^p$ and Hölder regular coefficients $A(x)$.

8. Proof of main result

Proof of Theorem 1.2. Without loss of generality, we can assume that the origin is situated on the edge of the half-plane, and contained in K. Thanks to Remark 6.1 and Proposition 6.3, we are exactly under the hypothesis of Lemma 7.5 with $\gamma = 3/4$ because K is contained in a half plane. Therefore

$$\varphi(r) := \frac{1}{r} \int_{B_r \setminus K} \frac{|\nabla u|^2}{|x|^{N-2}}$$

is nondecreasing, and the limit in 0 and $+\infty$ exists, let us denote them respectively by f_0 and f_∞. We claim that f_0 and f_∞ are finite. To check this we shall use the following elementary inequality valid for any global Mumfors-Shah minimizer (u, K) in $\mathbb{R}^N$,

$$\int_{B_r \setminus K} |\nabla u|^2 dx \leq \omega_N r^{N-1} \quad \forall r > 0,$$

obtained simply by taking $(u \mathbf{1}_{\mathbb{R}^N \setminus B_r}, K \cup \partial B_r)$ as a competitor. Notice that the constant is only dimensional: ω_N is the measure of the $N - 1$-dimensional unit

sphere. Now using Fubini and Chebychev, for any $r > 0$ we can chose $r_0 \in (r, 2r)$ such that

$$\int_{\partial B_{r_0} \setminus K} |\nabla u|^2 d\omega \leq \frac{1}{r} \int_{(B_{2r} \setminus B_r) \setminus K} |\nabla u|^2 dx \leq \omega_N (2r)^{N-2}.$$

From (7.7) we deduce that

$$(8.1) \quad \int_{B_r \setminus K} \frac{|\nabla u|^2}{|x|^{N-2}} \leq \int_{B_{r_0} \setminus K} \frac{|\nabla u|^2}{|x|^{N-2}} \leq \frac{1}{\alpha} r_0^{3-N} \int_{\partial B_{r_0} \setminus K} |\nabla u|^2 \leq C(N, \alpha) r,$$

with $C(N, \alpha) = \alpha^{-1} 2^{N-1} \omega_N$. This implies that $\varphi(r)$ is uniformly bounded for $r > 0$ thus f_0 and f_∞ are finite, and we readily have that $f_0 \leq f_\infty < +\infty$.

Let us check moreover that $f_0 > 0$. Indeed if not, then one would have, for $r > 0$ small enough,

$$\frac{1}{r^{N-1}} \int_{B_r \setminus K} |\nabla u|^2 \leq \frac{1}{r} \int_{B_r \setminus K} \frac{|\nabla u|^2}{|x|^{N-2}} \leq \varepsilon_0,$$

where ε_0 is the same as the statement of Theorem 2.6. But then $K \cap B_{cr}$ must be the image of a minimal cone by a C^1-smooth map, containing 0 in its interior. This is a contradiction with the fact that the origin lies on the edge of the half-plane containing K. We therefore conclude that $f_0 > 0$.

Now we take blow-up and blow-in limits. We begin with the blow-up. Let $u_k := \frac{1}{\sqrt{r_k}} u(r_k x)$ and $K_k := \frac{1}{r_k} K$. Then, by Theorem 3.2, up to a subsequence (not relabeled), as $r_k \to 0$ the sequence (u_k, K_k) converges to some (u_0, K_0) (in the sense of Definition 3.1), and (u_0, K_0) is still a reduced global minimizer, with K_0 still satisfying assumptions $(1), (2)$, and (3) of Theorem 1.2. Moreover for any $R > 0$ and $k \in \mathbb{N}$ we have that

$$(8.2) \quad \frac{1}{R} \int_{B_R} \frac{|\nabla u_k|^2}{|x|^{N-2}} dx = \frac{1}{r_k R} \int_{B_{r_k R}} \frac{|\nabla u|^2}{|x|^{N-2}} dx \xrightarrow[k \to +\infty]{} f_0.$$

On the other hand, we know that ∇u_k converges to ∇u_0 in $L^2(B_M)$ for any $M > 0$. We actually claim that the following stronger convergence holds true

$$(8.3) \quad \int_{B_R \setminus K} \frac{|\nabla u_k|^2}{|x|^{N-2}} dx \xrightarrow[k \to +\infty]{} \int_{B_R \setminus K} \frac{|\nabla u_0|^2}{|x|^{N-2}} dx.$$

To prove the claim, we use the estimate (8.1) and write

$$\left| \int_{B_R} \frac{|\nabla u_k|^2}{|x|^{N-2}} dx - \int_{B_R} \frac{|\nabla u_0|^2}{|x|^{N-2}} dx \right| \leq \left| \int_{B_R \setminus B_\varepsilon} \frac{|\nabla u_k|^2}{|x|^{N-2}} dx - \int_{B_R \setminus B_\varepsilon} \frac{|\nabla u_0|^2}{|x|^{N-2}} dx \right|$$

$$+ \int_{B_\varepsilon} \frac{|\nabla u_k|^2}{|x|^{N-2}} dx + \int_{B_\varepsilon} \frac{|\nabla u_0|^2}{|x|^{N-2}} dx$$

$$\leq \left| \int_{B_R \setminus B_\varepsilon} \frac{|\nabla u_k|^2}{|x|^{N-2}} dx - \int_{B_R \setminus B_\varepsilon} \frac{|\nabla u|^2}{|x|^{N-2}} dx \right|$$

$$+ 2C\varepsilon$$

where $C = C(N, \alpha)$. Passing to the limsup and using the strong convergence of $|\nabla u_k|$ in L^2 we deduce that

$$\limsup_{k \to +\infty} \left| \int_{B_R} \frac{|\nabla u_k|^2}{|x|^{N-2}} dx - \int_{B_R} \frac{|\nabla u_0|^2}{|x|^{N-2}} dx \right| \leq 2C\varepsilon.$$

The claim then follows letting $\varepsilon \to 0$.

Returning now to (8.2), we obtain that

$$\frac{1}{R} \int_{B_R} \frac{|\nabla u_0|^2}{|x|^{N-2}} dx = f_0, \quad \forall R > 0.$$

But, since $f_0 \neq 0$, the last conclusion of Lemma 7.5 then says that $\lambda_1(\partial B_R \setminus K_0) = 3/4$ for a.e $R > 0$, and therefore Lemma 6.4 implies that, up to a $\mathcal{H}^{N-1}$ negligible set G, K_0 is a half-plane. Since K_0 is reduced, then we claim that it must be the half-plane itself. Indeed, applying Proposition 19 of [15] to the couple $(K_0 \cup G, u_0)$ we get that u must be the function $cracktip \times \mathbb{R}$, where cracktip is the function $\sqrt{2r/\pi} \sin(\theta/2) + C$ in suitable polar coordinates. But then the only reduced global minimizer in the equivalent class of couples (K, u) for which K coincides with the half plane a.e. and $u = cracktip \times \mathbb{R}$ a.e., is the whole half plane (because any "hole" in the half plane would create an irreversible jump for the function u which would not be admissible in $W^{1,2}$).

Then we do exactly the same for u_∞ and conclude that, akin to the blow-up limit, the normalized energy associated to the blow-down limit must be constant as well. Thus up to an additive constant, u_∞ is the same function $\sqrt{2r/\pi} \sin(\theta/2)$ as u_0 (with the same constant in front), and therefore $f_0 = f_\infty$. But then returning to the function u and in virtue of the monotonicity of $\varphi(r)$, we deduce that $\varphi(r)$ is constant for all $r > 0$, and by consequence K must be a half-plane itself. $\square$

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