TRANSPORT DENSITY IN MONGE-KANTOROVICH PROBLEMS WITH DIRICHLET CONDITIONS

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Abstract. We study the properties of the transport density measure in the Monge-Kantorovich optimal mass transport problem in the presence of so-called Dirichlet constraint, i.e. when some closed set is given along which the cost of transportation is zero. The Hausdorff dimension estimates, as well as summability and higher regularity properties of the transport density are studied. The uniqueness of the transport density is proven in the case when the masses to be transported are represented by measures absolutely continuous with respect to the Lebesgue measure.

1. Introduction

Let φ^+, φ^- be two given finite nonnegative Borel measures supported in some open bounded set $\Omega \subset \mathbb{R}^n$, and $\Sigma \subset \bar{\Omega}$ be a given closed set. Equip $\bar{\Omega}$ with the geodesic distance d_Ω defined by

$$d_\Omega(x,y) := \inf\left\{ \int_0^1 |\phi'(t)|\,dt \;:\; \phi:[0,1]\to\bar{\Omega} \text{ Lipschitz curve in } \bar{\Omega}, \phi(0)=x, \phi(1)=y \right\},$$

which of course reduces to the usual Euclidean distance when Ω is convex. Introduce now the new semi-distance $d_{\Omega,\Sigma}$ in $\bar{\Omega}$ by the formula

$$d_{\Omega,\Sigma}(x,y) := d_\Omega(x,y) \wedge \delta_{\Omega,\Sigma}(x,y), \text{ where}$$
$$\delta_{\Omega,\Sigma}(x,y) := \inf\left\{ d_\Omega(x,k_x) + d_\Omega(k_y,y) \;:\; (k_x,k_y)\in\Sigma\times\Sigma \right\}.$$

Note that $d_{\Omega,\Sigma}(x,y) = 0$ when $x\in\Sigma$ and $y\in\Sigma$. In this paper we consider the generalized Monge-Kantorovich optimal transport problem, which consists in finding a Borel measure γ_{opt} over $\bar{\Omega}\times\bar{\Omega}$ (called *optimal plan*) minimizing the functional

$$I_\Sigma(\gamma) := \int_{\bar{\Omega}\times\bar{\Omega}} d_{\Omega,\Sigma}(x,y)\,d\gamma(x,y)$$

over all Borel measures γ in $\bar{\Omega}\times\bar{\Omega}$ satisfying

$$\pi^+_\#\gamma - \pi^-_\#\gamma = \varphi^+ - \varphi^- \text{ over } \bar{\Omega}\setminus\Sigma \tag{1}$$

(called *admissible* measures). Such a problem will be referred to as the Monge-Kantorovich problem with Dirichlet condition and denoted $MK(\varphi^+,\varphi^-,\Sigma)$, if $\Sigma\neq\emptyset$, and as the Monge-Kantorovich problem without Dirichlet condition with the notation $MK(\varphi^+,\varphi^-)$, if $\Sigma=\emptyset$. Note that in the latter simpler case (1) implies $\varphi^+(\bar{\Omega}) = \varphi^-(\bar{\Omega})$ for the problem to be well posed, while if $\Sigma\neq\emptyset$, then the problem is well posed for arbitrary φ^+ and φ^-. In the sequel, in all the notation we omit

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the reference to Σ when $\Sigma = \emptyset$. From the physical point of view, Σ is a zone where the cost of transporting the measure φ^+ to φ^- vanishes.

Let $G_\Omega(x,y)$ stand for the set of closed geodesic segments with respect to d_Ω, which join x with y, $g_\Omega(x,y)$ being the elements of $G_\Omega(x,y)$, and let

$$G_{\Omega,\Sigma}(x,y) := \begin{cases} G_\Omega(x,y), & \text{if} \quad \delta_{\Omega,\Sigma}(x,y) > d_\Omega(x,y), \\ G_\Omega(x,y) \cup D_{\Omega,\Sigma}(x,y), & \text{if} \quad \delta_{\Omega,\Sigma}(x,y) = d_\Omega(x,y), \\ D_{\Omega,\Sigma}(x,y), & \text{if} \quad \delta_{\Omega,\Sigma}(x,y) < d_\Omega(x,y), \end{cases}$$

where $D_{\Omega,\Sigma}(x,y) := \{g_\Omega(x,k_x) \cup g_\Omega(y,k_y)\}$ and $k_x, k_y \in \Sigma$ denote such points that $d_{\Omega,\Sigma}(x,y) = d_\Omega(x,k_x) + d_\Omega(y,k_y)$. Let $g_{\Omega,\Sigma}$ be any Borel measurable selector of $G_{\Omega,\Sigma}$ (the existence of such a selector is an easy consequence of proposition 4.2 (iii) from [2]). Then the measure μ over $\bar{\Omega}$ introduced in [2]) and defined by

$$\mu(B) := \int_{\bar{\Omega}\times\bar{\Omega}} \mathcal{H}^1(g_{\Omega,\Sigma}(x,y) \cap B)\, d\gamma(x,y) \tag{2}$$

for every Borel $B \subset \bar{\Omega}$, where γ is an optimal plan and $\mathcal{H}^1$ stands for the one-dimensional Hausdorff measure, is called a *transport density.*

In this paper we study the properties of transport densities such as Hausdorff dimension, summability and higher regularity (differentiability) in relation with the properties of the data, i.e. of $\varphi^\pm$ and Σ. We will show, that although clearly the presence of the Dirichlet constraint $\Sigma \neq \emptyset$ changes significantly the geometry of the problem, still much similarly to the classical case $\Sigma = \emptyset$ the transport density is unique when $\varphi^\pm$ are absolutely continuous with respect to the Lebesgue measure. Moreover, in this case the transport density has local summability properties and is Sobolev differentiable along some directions. The global summability properties also hold when Σ satisfies some regularity assumption. The results we present in this paper are sharp, which we show on a number of examples.

2. Preliminaries: dual problem and Monge-Kantorovich equation

We denote by $\mathrm{Lip}_{1,\Sigma}$ the set of Lipschitz functions in $\mathbb{R}^n$ with Lipschitz constant less than or equal to 1 which are zero over Σ. It has been proven in [2] that if μ is a transport density for the problem $MK(\varphi^+,\varphi^-,\Sigma)$ defined by (2), then there is a $u \in \mathrm{Lip}_{1,\Sigma}$ called *Kantorovich potential,* such that the pair (μ, u) satisfies the system of equations

$$\begin{cases} -\mathrm{div}\,(\mu\nabla_\mu u) &= & \varphi^+ - \varphi^- \text{ in } \bar{\Omega}\setminus\Sigma, \\ |\nabla_\mu u| &= & 1 \qquad \mu-\text{a.e. in } \bar{\Omega}, \\ \mu(\Sigma) &= & 0, \end{cases} \tag{3}$$

where ∇_μ is the tangential gradient operator with respect to μ defined in [2]. Furthermore, the Kantorovich potential u solves the dual problem to $MK(\varphi^+,\varphi^-,\Sigma)$ of finding

$$\sup\left\{\int_{\bar{\Omega}} v\, d\varphi^- - \int_{\bar{\Omega}} v\, d\varphi^+ \;:\; v \in \mathrm{Lip}_{1,\Sigma}\right\}.$$

The system (3) is called the *Monge-Kantorovich equation.* In [1] it has been proven for the case $\Sigma = \emptyset$ and $\Omega = \mathbb{R}^n$ that in fact every solution of the Monge-Kantorovich equation corresponds to a transport density for the respective optimal transport problem.

In [2] it is proven that there is a one-to-one correspondence between the solutions to the Monge-Kantorovich equation and solutions of some shape optimization problems. Namely, the pair (μ, u) solves (3), if and only if μ gives the conductivity distribution of a conducting material in $\bar\Omega$ which minimizes the Joule heating induced by the current flow under the electric charge distributions φ^+ and φ^- given the zone Σ where the electrostatic potential u is set to be equal to zero.

In the sequel we will need the following useful statement.

Lemma 2.1. *Let γ be an optimal transport plan for the Monge-Kantorovich problem $MK(\varphi^+,\varphi^-)$ and let $D \subset \bar\Omega \times \bar\Omega$ be a Borel set. Then $\gamma_D := \gamma \llcorner D$ is an optimal plan for the Monge-Kantorovich problem $MK(\varphi_D^+,\varphi_D^-)$, where $\varphi_D^\pm := \pi_\#^\pm \gamma_D$.*

PROOF: Supposing the contrary we would have the existence of a better plan γ'_D with the same marginals (i.e. $\pi^\pm_\# \gamma'_D = \varphi^\pm_D$), which would then give rise to the better plan

$$\gamma' := \begin{cases} \gamma'_D & \text{over } D, \\ \gamma & \text{over } (\bar\Omega \times \bar\Omega) \setminus D \end{cases}$$

to the original problem, which contradicts the optimality of γ. □

3. Problem reduction

The purpose of this section is to reduce the Monge-Kantorovich problem with the Dirichlet condition (i.e. with $\Sigma \neq \emptyset$) to an "equivalent" one without this condition (i.e. with $\Sigma = \emptyset$).

Let us remark first of all that without loss of generality we may assume that the measures $\varphi^\pm$ are concentrated outside of Σ. Denote

$$\begin{aligned} \mathcal{R}_+ &:= \{(x,y) \in \bar\Omega \times \bar\Omega \,:\, \delta_{\Omega,\Sigma}(x,y) > d_\Omega(x,y)\}, \\ \mathcal{R}_0 &:= \{(x,y) \in \bar\Omega \times \bar\Omega \,:\, \delta_{\Omega,\Sigma}(x,y) = d_\Omega(x,y)\}, \\ \mathcal{R}_- &:= \{(x,y) \in \bar\Omega \times \bar\Omega \,:\, \delta_{\Omega,\Sigma}(x,y) < d_\Omega(x,y)\}, \end{aligned}$$

and consider an arbitrary Borel set $\mathcal{R} \subset \bar\Omega \times \bar\Omega$ satisfying

$$\mathcal{R}_+ \subset \mathcal{R} \subset \mathcal{R}_+ \cup \mathcal{R}_0.$$

Observe that $d_{\Omega,\Sigma}(x,y) = d_\Omega(x,y)$ over $\mathcal{R}$. Set now

$$\varphi_r^\pm := \pi_\#^\pm \gamma \llcorner \mathcal{R}, \qquad \varphi_s^\pm := (\pi_\#^\pm \gamma \llcorner (\bar\Omega \times \bar\Omega) \setminus \mathcal{R}) \llcorner (\bar\Omega \setminus \Sigma),$$

where γ is an optimal plan for the Monge-Kantorovich problem $MK(\varphi^+,\varphi^-,\Sigma)$. It is worth noting that since according to our assumption $\varphi^\pm$ are concentrated outside of Σ, then one has $\varphi^\pm = \varphi_r^\pm + \varphi_s^\pm$.

Define $K^\pm$: $\operatorname{supp}\varphi_s^\pm \multimap \Sigma$ to be the set-valued projections to Σ according to the formulae

$$K^\pm(x) := \{y \in \Sigma \,:\, d_\Omega(x,y) = \operatorname{dist}_\Omega(x,\Sigma)\},$$

where $\operatorname{dist}_\Omega(x,\Sigma)$ stands for the distance between x and Σ, i.e.

$$\operatorname{dist}_\Omega(x,\Sigma) := \inf_{y\in\Sigma} d_\Omega(x,y).$$

The following simple assertion is valid.

Lemma 3.1. *The maps $K^\pm$ admit Borel measurable selections $k^\pm$: $\operatorname{supp}\varphi_s^\pm \to \Sigma$.*

PROOF: According to theorem III.6 and proposition III.11 of [4] it suffices to verify the following two conditions for each of the set-valued maps $K^\pm$:

(i) $K^\pm(x)$ are closed for every $x \in \operatorname{supp}\varphi_s^\pm$;
(ii) $(K^\pm)_+^{-1}(C) := \{x \,:\, K^\pm(x) \cap C \neq \emptyset\}$ are closed for every closed $C \subset \Sigma$.

Then, (i) is verified immediately since for every sequence $\{y_\nu\} \subset K^\pm(x)$ one has $d_\Omega(x, y_\nu) = \mathrm{dist}_\Omega(x, \Sigma)$, and hence $y_\nu \to y$ implies $d_\Omega(x, y) = \mathrm{dist}_\Omega(x, \Sigma)$, that is, $y \in K^\pm(x)$.

To verify (ii), assume that $C \subset \Sigma$ is closed and let a sequence $\{x_\nu\} \subset (K^\pm)^{-1}_+(C)$ be convergent, namely, $x_\nu \to x$. Let $y_\nu \in C$ be such that $y_\nu \in K^\pm(x_\nu)$. In view of compactness of C one has $y_\nu \to y \in C$ up to a subsequence (not relabeled). Since the functions $d_\Omega(x_\nu, \cdot)\colon \Sigma \to \mathbb{R}$ are uniformly convergent to $d_\Omega(x, \cdot)\colon \Sigma \to \mathbb{R}$, and y_ν are absolute minimizers of the former, then y is an absolute minimizer of the latter, or, in other words, $y \in K^\pm(x)$. The latter means $x \in (K^\pm)^{-1}_+(C)$, which concludes the proof. $\square$

Let now $k^\pm\colon \operatorname{supp}\varphi^\pm_s \to \Sigma$ be any Borel measurable selections of $K^\pm$. Setting $\psi^\mp := k^\pm_\# \varphi^\pm_s$, we define a new measure

$$\tilde\gamma := \begin{cases} \gamma & \text{over} \quad \mathcal{R}, \\ \gamma^+ & \text{over} \quad \operatorname{Graph} k^+, \\ (\gamma^-)^T & \text{over} \quad (\operatorname{Graph} k^-)^T, \end{cases}$$

where

$$\gamma^\pm(e) := \varphi^\pm_s(\{x \,:\, (x, k^\pm(x)) \in e\})$$

for all Borel sets $e \subset \bar\Omega \times \bar\Omega$, the notation A^T and α^T standing for

$$A^T := \big\{(y, x) \in \bar\Omega \times \bar\Omega \,:\, (x, y) \in A\big\} \text{ and } \alpha^T(A) := \alpha(A^T)$$

for every Borel set $A \subset \bar\Omega \times \bar\Omega$ and for every Borel measure α over $\bar\Omega \times \bar\Omega$. We can claim now the following result which will be our main tool in the sequel.

Theorem 3.1. *The following assertions are valid.*

(i) *If γ is optimal for the problem $MK(\varphi^+, \varphi^-, \Sigma)$, then $\tilde\gamma$ is optimal for $MK(\varphi^+ + \psi^+, \varphi^- + \psi^-)$.*

(ii) *Suppose μ is a transport density for $MK(\varphi^+, \varphi^-, \Sigma)$ corresponding to the optimal measure γ. There exists such a choice of $\mathcal{R}$, that if $\tilde\mu$ is the transport density for the respective problem $MK(\varphi^+ + \psi^+, \varphi^- + \psi^-)$, which corresponds to the measure $\tilde\gamma$, then*

$$\mu = \tilde\mu = \tilde\mu^0(e) + \tilde\mu^+(e) + \tilde\mu^-(e), \tag{4}$$

where $\tilde\mu^0(e)$ is an optimal measure for the problem $MK(\varphi^+_r, \varphi^-_r)$, $\tilde\mu^+(e)$ is an optimal measure for $MK(\varphi^+_s, \psi^-)$ and $\tilde\mu^-(e)$ is an optimal measure for $MK(\psi^+, \varphi^-_s)$.

PROOF:
Step 1. To prove (i), we first show

$$I(\tilde\gamma) = I_\Sigma(\gamma). \tag{5}$$

In fact,

$$\begin{aligned}
I(\tilde\gamma) &= I^0(\gamma)+I^+(\gamma^+)+I^-(\gamma^-),\\
I_\Sigma(\gamma) &= I^0_\Sigma(\gamma)+I'_\Sigma(\gamma), \text{ where}\\
I^0(\gamma) &:= \int_{\mathcal{R}} d_\Omega(x,y)\,d\gamma(x,y),\\
I^0_\Sigma(\gamma) &:= \int_{\mathcal{R}} d_{\Omega,\Sigma}(x,y)\,d\gamma(x,y),\\
I^\pm(\gamma^\pm) &:= \int_{\mathrm{Graph}\,k^\pm} d_\Omega(x,y)\,d\gamma^\pm(x,y),\\
I'_\Sigma(\gamma) &:= \int_{(\bar\Omega\times\bar\Omega)\setminus\mathcal{R}} d_{\Omega,\Sigma}(x,y)\,d\gamma(x,y).
\end{aligned}$$

Since $d_{\Omega,\Sigma}=d_\Omega$ over $\mathcal{R}$, one has $I^0(\gamma)=I^0_\Sigma(\gamma)$. To show (5), it remains to observe that

$$\begin{aligned}
I'_\Sigma(\gamma) \;=\; & \int_{\bar\Omega\times\bar\Omega} d_\Omega(x,K^+(x))\,d\gamma_{\mathcal{R}^c}(x,y)+\int_{\bar\Omega\times\bar\Omega} d_\Omega(y,K^-(y))\,d\gamma_{\mathcal{R}^c}(x,y)=\\
& \int_{\bar\Omega} d_\Omega(x,K^+(x))\,d(\varphi_s^+ +\tilde\psi^+)(x)+\int_{\bar\Omega} d_\Omega(y,K^-(y))\,d(\varphi_s^- +\tilde\psi^-)(y)=\\
& \int_{\bar\Omega\setminus\Sigma} d_\Omega(x,K^+(x))\,d\varphi_s^+(x)+\int_{\bar\Omega\setminus\Sigma} d_\Omega(y,K^-(y))\,d\varphi_s^-(y)=\\
& I^+(\gamma^+)+I^-(\gamma^-),
\end{aligned}$$

where $\gamma_{\mathcal{R}^c}:=\gamma\,\llcorner(\bar\Omega\times\bar\Omega\setminus\mathcal{R})$ and $\tilde\psi^\pm$ are some measures concentrated on Σ.

Step 2. Assume now that γ is optimal for $MK(\varphi^+,\varphi^-,\Sigma)$. If the statement (i) of the theorem being proven were wrong, then there would exist another measure $\tilde\gamma'$ with the same marginals such that

$$I(\tilde\gamma')<I(\tilde\gamma)=I_\Sigma(\gamma).$$

But then $\tilde\gamma'$ is also admissible for $MK(\varphi^+,\varphi^-,\Sigma)$, while $I(\tilde\gamma')\geq I_\Sigma(\tilde\gamma')$, which together with the above inequality implies

$$I_\Sigma(\tilde\gamma')<I_\Sigma(\gamma),$$

contradicting the optimality of γ for $MK(\varphi^+,\varphi^-,\Sigma)$.

Step 3. To prove (ii), let $g_{\Omega,\Sigma}$ be the Borel measurable selector of $G_{\Omega,\Sigma}$ which appears in (2), Λ be a Borel set (in the set of geodesics in $\bar\Omega$) satisfying

$$\mathrm{Graph}\,G_\Omega\setminus\mathrm{Graph}\,D_{\Omega,\Sigma}\subset\Lambda\subset\mathrm{Graph}\,G_\Omega,$$

and let $\mathcal{R}:=g^{-1}_{\Omega,\Sigma}(\Lambda)$. Note that for every Borel set $e\subset\bar\Omega$ one has

$$\begin{aligned}
\mu(e) &= \mu^0(e)+\mu^+(e)+\mu^-(e),\\
\tilde\mu(e) &= \tilde\mu^0(e)+\tilde\mu^+(e)+\tilde\mu^-(e), \text{ where}\\
\mu^0(e) &:= \int_{\mathcal{R}} \mathcal{H}^1(e\cap g_\Omega(x,y))\,d\gamma(x,y),\\
\tilde\mu^0(e) &:= \int_{\mathcal{R}} \mathcal{H}^1(e\cap g_\Omega(x,y))\,d\tilde\gamma(x,y),\\
\mu^\pm(e) &:= \int_{(\bar\Omega\times\bar\Omega)\setminus\mathcal{R}} \mathcal{H}^1(e\cap g_\Omega(x,k^\pm(x)))\,d\gamma(x,y),\\
\tilde\mu^\pm(e) &:= \int_{\mathrm{Graph}\,k^\pm} \mathcal{H}^1(e\cap g_\Omega(x,k^\pm(x)))\,d\gamma^\pm(x,y).
\end{aligned}$$

Here g_Ω is a Borel selector of the geodesic map G_Ω.

Since $\tilde{\gamma} = \gamma$ over $\mathcal{R}$, one has $\tilde{\mu}^0(e) = \mu^0(e)$. But

$$\begin{aligned}\mu^{\pm}(e) &= \int_{\bar{\Omega}\times\bar{\Omega}} \mathcal{H}^1(e \cap g_\Omega(x, k^{\pm}(x)))\, d\gamma_{\mathcal{R}^c}(x,y) = \\ &\int_{\bar{\Omega}} \mathcal{H}^1(e \cap g_\Omega(x, k^{\pm}(x)))\, d(\varphi_s^{\pm} + \tilde{\psi}^{\pm})(x) = \\ &\int_{\bar{\Omega}\setminus\Sigma} \mathcal{H}^1(e \cap g_\Omega(x, k^{\pm}(x))) d\varphi_s^{\pm}(x) = \tilde{\mu}^{\pm}(e),\end{aligned}$$

which completes the proof. □

We would like to emphasize that according to the above theorem there is, generally speaking, a family of reduced Monge-Kantorovich problems (i.e. without the Dirichlet condition) corresponding to a problem with Dirichlet condition, according to different choices of the set $\mathcal{R}$. Nevertheless, we will still be able to prove uniqueness of the transport density for the problem with Dirichlet condition in Euclidean geometry, when the given measures are absolutely continuous with respect to the Lebesgue measure.

4. Uniqueness

From now on we assume that Ω is an open bounded convex subset of $\mathbb{R}^n$, so that d_Ω coincides with the usual Euclidean distance. We denote also by $\mathcal{L}^n$ the Lebesgue measure in $\mathbb{R}^n$. In this case the following uniqueness result is valid.

Theorem 4.1. *Assume $\varphi^{\pm} \llcorner \bar{\Omega} \setminus \Sigma << \mathcal{L}^n$. Then the transport density for the Monge-Kantorovich problem $MK(\varphi^+, \varphi^-, \Sigma)$ is unique, namely, it is independent of the choice of optimal measure γ and of the choice of the geodesic selector $g_{\Omega,\Sigma}(x, y)$.*

In order to prove this result we need to introduce some auxiliary notions. We will denote by $\hat{G}_{\Omega,\Sigma}(x, y)$ the set of geodesic segments between $x \in \bar{\Omega}$ and $y \in \bar{\Omega}$ without endpoints x and y. Recall that we agreed to consider the elements of $G_\Omega(x, y)$, and hence those of $G_{\Omega,\Sigma}(x, y)$, to be closed geodesic segments, i.e. to include both endpoints x and y.

Definition 4.1. *Assume $u \in Lip_{1,\Sigma}$ to be a Kantorovich potential for the transport problem $MK(\varphi^+, \varphi^-, \Sigma)$. We call a geodesic transport ray the maximum geodesic segment $R \in \hat{G}_{\Omega,\Sigma}(l^+, l^-)$ not including endpoints l^+, l^- such that*

$$u(l^+) - u(l^-) = d_{\Omega,\Sigma}(l^+, l^-).$$

Furthermore, we call thick geodesic transport rays each of the following objects:

(i) *every geodesic transport ray R not piercing Σ (i.e. such that $R \cap \Sigma = \emptyset$);*
(ii) *the union of all geodesic transport rays piercing Σ (i.e. $\cup_{R\cap\Sigma\neq\emptyset} R$).*

The union T of all geodesic transport rays will further on be called transport set. We also denote by $\mathcal{E}$ the set of all ray ends.

Now we concentrate on the properties of thick geodesic transport rays. We prove first the following simple assertion.

Lemma 4.1. *For every pair of thick geodesic transport rays $\mathcal{R}_1$ and $\mathcal{R}_2$ one has that either $\mathcal{R}_1 = \mathcal{R}_2$ or $\mathcal{R}_1 \cap \mathcal{R}_2 = \emptyset$.*

Proof: If both $\mathcal{R}_1$ and $\mathcal{R}_2$ do not pierce Σ, then the announced property follows from lemma 10 of [3]. It remains to consider the case $\mathcal{R}_1 \cap \Sigma \neq \emptyset$ and $\mathcal{R}_2 \cap \Sigma = \emptyset$. Suppose there is an $x \in \mathcal{R}_1 \cap \mathcal{R}_2$. Without loss of generality we may assume $u(x) > 0$ (the case $u(x) < 0$ can be considered in an absolutely symmetric way). Let then l^+ stand for the upper end of $\mathcal{R}_2$. From

$$u(l^+) - u(x) = d_\Omega(l^+, x), \qquad\qquad u(x) - u(k^+(x)) = d_\Omega(x, k^+(x))$$

one derives

$$u(l^+) - u(k^+(x)) = d_\Omega(l^+, x) + d_\Omega(x, k^+(x)) \geq d_\Omega(l^+, k^+(x)),$$

while on the other hand since $u \in \mathrm{Lip}_{1,\Sigma}$, one has

$$u(l^+) - u(k^+(x)) \leq d_\Omega(l^+, k^+(x)).$$

Therefore, $d_\Omega(l^+, x) + d_\Omega(x, k^+(x)) = d_\Omega(l^+, k^+(x))$, and hence $k^+(x) \in \mathcal{R}_2$ contradicting the assumption $\mathcal{R}_2 \cap \Sigma = \emptyset$. Hence, in this case one has $\mathcal{R}_1 \cap \mathcal{R}_2 = \emptyset$. $\square$

In other words, the above lemma 4.1 states that the set of thick geodesic rays $\xi = \{\mathcal{R}_x\}_{x\in\bar{\Omega}}$, where $\mathcal{R}_x$ the unique thick geodesic ray passing through $x \in T$, defines a partition of T. The following auxiliary assertion on measurability of this partition is crucial for the proof of uniqueness of transport density.

Lemma 4.2. *The sets T and $\mathcal{E}$ are Borel sets satisfying $\mathcal{L}^n(\mathcal{E}) = 0$. The set of thick geodesic rays $\xi = \{\mathcal{R}_x\}_{x\in\bar{\Omega}}$ defines a Borel measurable partition of T.*

In order to prove this lemma we first recall the basic construction which goes back to [3]. If $u\colon \bar{\Omega} \to \mathbb{R}$ is a 1-Lipschitz function, then one can define geodesic transport rays relative to u as maximum geodesic intervals $g_\Omega(l^+, l^-)$ without endpoints such that $u(l^+) - u(l^-) = d_\Omega(l^+, l^-)$. For every $p \in R$ the set

$$\{x \in \Omega \,:\, u(x) = p \text{ and } \exists \nabla u(x) \neq 0\}$$

admits a Borel covering $\{S_{pi}\}_{i=1}^\infty$ such that there exist Lipschitz coordinates $U\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ and $V\colon \mathbb{R}^{n-1} \to \mathbb{R}^n$ satisfying $V(U(x)) = x$ for all $x \in S_{pi}$. For each triple $(p, i, j) \in \mathbb{Q}\times\mathbb{N}^2$ one can define the "ray clusters" T_{pij} as the union of geodesic transport rays R relative to u piercing S_{pi} in some point which is distant at least $1/j$ from both endpoints of R. Note that it is easy to prove that every geodesic transport ray R relative to u belongs to some of these clusters. Each cluster T_{pij} then admits coordinates $G := G_{pij}\colon T_{pij} \to \mathbb{R}^{n-1} \times \mathbb{R}$ (with Lipschitz inverse F), which are Lipschitz over

$$T^\sigma_{pij} := \left\{x \in T_{pij} \,:\, \min\{d_\Omega(z, l^+), d_\Omega(z, l^-)\} \geq \sigma > 0\right\},$$

where l^+ and l^- stand for the ends of the unique geodesic transport ray R_x passing through x. Moreover, one has

$$G(x) = (U(z), u(x) - u(z)), \text{ where } \{z\} := R_x \cap S_{pi}.$$

In particular, all the clusters T_{pij} are Borel measurable.

If $B \subset \bar{\Omega}$ is a Borel set, then we can define similarly the clusters $T_{pij}(B)$ as the union of geodesic transport rays R relative to u piercing $S_{pi} \cap B$ in some point which is distant at least $1/j$ from both endpoints of R. Much analogously one shows that every geodesic transport ray R relative to u intersecting B belongs to some of $T_{pij}(B)$ and that all $T_{pij}(B)$ are Borel sets.

Proof of lemma 4.2: Let $u \in \mathrm{Lip}_{1,\Sigma}$ be a Kantorovich potential of the problem $MK(\varphi^+, \varphi^-, \Sigma)$. Clearly, u is also a 1-Lipschitz function, since $d_{\Omega,\Sigma} \leq d_\Omega$. The set

T is Borel, since then

$$T = \bigcup_{(p,i,j)\in\mathbb{N}^2} T_{pij} \cup (\mathcal{E}_u \cap \Sigma),$$

where $\mathcal{E}_u$ stands for the set of endpoints of geodesic transport rays relative to u, while $\mathcal{E}_u$ as well as each of the ray clusters T_{pij} (relative to u) is a Borel set. One observes also that

$$\mathcal{E} = \mathcal{E}_u \setminus \Sigma.$$

Since $\mathcal{E}_u$ is a Borel set having Lebesgue measure zero by lemma 25 from [3], then so is $\mathcal{E}$.

Let T_1 stand for the thick geodesic transport ray intersecting Σ. It is easy to see that

$$T_1 = (\mathcal{S} \cap \mathcal{S}_0) \cup (\mathcal{E}_u \cap \Sigma), \text{ where } \mathcal{S}_0 := \bigcup_{p-1/j\leq 0\leq p+1/j} T_{pij}, \tag{6}$$

the sum being taken over $(p, i, j) \in \mathbb{Q} \times \mathbb{N}^2$, and $\mathcal{S}$ stands for the set of points admittng a unique projection on Σ. In fact, $\mathcal{S}_0$ is clearly is the union of alltransport rays relative to u without endpoints, which pierce the set $\{u = 0\}$ (maybe at one of the endpoints). Hence every $x \in \mathcal{S}_0$ admits a unique projection on $\{u = 0\}$ which coincides with the point where the geodesic transport ray $\bar{R}_x$ with endpoints pierces $\{u = 0\}$. But if $x \in \mathcal{S}_0 \cap \mathcal{S}$, then the respective projection point must belong to Σ, and hence $\bar{R}_x$ pierces Σ. This shows $\mathcal{S} \cap \mathcal{S}_0 \subset T_1 \setminus (\mathcal{E}_u \cap \Sigma)$, while the inverse inclusion is obvious. According to the above formula (6), the set T_1 is Borel since so are $\mathcal{S}$ (which can be viewed, for instance, as the transport set of the problem $MK(\mathcal{L}^n, 0, \Sigma)$), $\mathcal{E}_u$ and all the ray clusters T_{pij}. Here and below when using the transport rays and ray clusters T_{pij} we always refer to the ones relative to a Kantorovich potential of the problem $MK(\varphi^+, \varphi^-, \Sigma)$.

It suffices now to prove that $\xi' = \{\mathcal{R}_x\}_{x\in\bar{\Omega}} \setminus \{T_1\}$ is a Borel measurable partition of $T \setminus T_1$. This claim becomes immediate once one observes that ξ' is generated by the map π which sends $x \in T \setminus T_1$ into the geodesic transport ray $\mathcal{R}_x$ and shows that π is Borel measurable when viewed as a function acting from $T \setminus T_1$ into the metric space of S of segments of positive length without endpoints endowed with the distance defined by the formula

$$d(R_1, R_2) := d_\Omega(l_1^+, l_2^+) \vee d_\Omega(l_1^-, l_2^-) \tag{7}$$

whenever (l_1^+, l_1^-) and (l_2^+, l_2^-) are the pairs of upper and lower ray ends of R_1 and R_2 respectively.

To prove the latter measurability claim, consider the "set-valued version" Π of the map π, namely, the set-valued map Π: $T \setminus T_1 \multimap \bar{\Omega}$ defined by the formula $\Pi(x) := \mathcal{R}_x$. Then $\Pi_+^{-1}(e) := \{x : \Pi(x) \cap e \neq \emptyset\}$ is a Borel set whenever $e \subset \bar{\Omega}$ is Borel. In fact

$$\Pi_+^{-1}(e) = \bigcup_{(p,i,j)\in\mathbb{Q}\times\mathbb{N}^2} T_{pij}(e) \setminus T_1.$$

We observe now that

$$\Pi_+^{-1}(e) = \pi^{-1}(S_e), \text{ where } S_e := \{R \in S : R \cap e \neq \emptyset\}.$$

To conclude the proof, it is enough to show that the σ-algebra generated by all the subsets $S_e \subset S$ where e is a Borel subset of $\bar{\Omega}$, contains all the Borel sets of S. This is reduced to proving that for every open ball $B_\alpha(R) \subset S$, $B_\alpha(R) := \{R' \in S :$

$d(R', R) < \alpha\}$, there is a Borel set $e \subset \bar{\Omega}$ such that $B_\alpha(R) = S_e$. One observes now that the latter relationship holds with

$$e := \{x \,:\, x \in R' \in B_\alpha(R)\} \subset \bar{\Omega}$$

which is clearly a Borel set. □

At last, we need also the following lemma.

Lemma 4.3. *Let $\varphi^+(\bar{\Omega}) = \varphi^-(\bar{\Omega})$, γ' be any optimal transport plan of the problem $MK(\varphi^+, \varphi^-, \Sigma)$ and the measure μ defined by*

$$\mu(B) := \int_{\bar{\Omega}\times\bar{\Omega}} \mathcal{H}^1(g_{\Omega,\Sigma}(x,y) \cap B)\, d\gamma'(x,y)$$

for every Borel $B \subset \bar{\Omega}$ is a transport density corresponding to γ'. Then, setting $\lambda^\pm := \varphi^\pm - \varphi^+ \wedge \varphi^-$, there exists an optimal transport plan γ of the problem $MK(\varphi^+, \varphi^-, \Sigma)$

$$\pi^\pm_\# \gamma = \lambda^\pm \text{ on } \bar{\Omega} \text{ and } \mu(B) = \int_{\bar{\Omega}\times\bar{\Omega}} \mathcal{H}^1(g_{\Omega,\Sigma}(x,y) \cap B)\, d\gamma(x,y).$$

PROOF: We define the measure γ over $\bar{\Omega} \times \bar{\Omega}$ by

$$\gamma(e) := \gamma'(e) - \delta^+(\pi^+(e \cap \Delta) \cap (\bar{\Omega} \setminus \Sigma)) - (\delta^+ \otimes \delta^-)(e \cap (\Sigma \times \Sigma))/|\delta^-|(\Sigma),$$

where

$$\delta^\pm := \pi^\pm_\# \gamma' - \lambda^\pm$$

and $\Delta := \{(x,x) \,:\, x \in \bar{\Omega}\}$. Since γ' is an admissible transport plan, then $\delta^+ = \delta^-$ over $\bar{\Omega} \setminus \Sigma$, and therefore $|\delta^-|(\Sigma) = |\delta^+|(\Sigma)$. Now, a simple calculation for a Borel set $e \subset \bar{\Omega}$ shows

$$\begin{aligned} \gamma(e \times \bar{\Omega}) &= (\pi^+_\# \gamma')(e) - (\pi^+_\# \gamma')(e \setminus \Sigma) \\ &\quad + \lambda^+(e \setminus \Sigma) \\ &\quad - \lambda^+(e \cap \Sigma) \\ &= \lambda^+(e), \end{aligned}$$

and symmetrically, $\gamma(\bar{\Omega} \times e) = \lambda^-(e)$. Hence γ is still admissible for the problem $MK(\varphi^+, \varphi^-, \Sigma)$, while

$$\pi^\pm_\# \gamma = \lambda^\pm \text{ on } \bar{\Omega}.$$

It is enough now to observe that since the function $d_{\Omega,\Sigma}$ vanishes over the diagonal Δ and over $\Sigma \times \Sigma$, then $I_\Sigma(\gamma) = I_\Sigma(\gamma')$, that is, γ is still an optimal transport plan for $MK(\varphi^+, \varphi^-, \Sigma)$. In the same way, since the function

$$(x,y) \in \bar{\Omega} \times \bar{\Omega} \mapsto \mathcal{H}^1(g_{\Omega,\Sigma}(x,y) \cap B)$$

vanishes over Δ and over $\Sigma \times \Sigma$, then

$$\mu(B) = \int_{\bar{\Omega}\times\bar{\Omega}} \mathcal{H}^1(g_{\Omega,\Sigma}(x,y) \cap B)\, d\gamma(x,y),$$

and hence μ is a transport density with respect to γ. □

Now we are ready to prove the announced uniqueness theorem.

PROOF OF THEOREM 4.1:

Note that by redefining $\varphi^\pm$ over Σ we may assume without loss of generality that $\varphi^+(\bar{\Omega}) = \varphi^+(\bar{\Omega})$. Let γ' be an arbitrary optimal transport plan for $MK(\varphi^+, \varphi^-, \Sigma)$ and μ be a transport density calculated according to (2) with this particular transport plan γ' and some choice of a selector $g_{\Omega,\Sigma}$. Assume then γ to be an optimal transport plan for $MK(\varphi^+, \varphi^-, \Sigma)$ indicated in the statement of lemma 4.3, i.e. such that

$$\pi^{\pm}_{\#}\gamma = \lambda^{\pm} := \varphi^{\pm} - \varphi^+ \wedge \varphi^- \tag{8}$$

and μ is still a transport density calculated with the transport plan γ and the same choice of a selector $g_{\Omega,\Sigma}$.

We choose now a Kantorovich potential $u \in \mathrm{Lip}_{1,\Sigma}$ such that

$$|u(x) - u(y)| = d_{\Omega,\Sigma}(x, y) \qquad \hat{\gamma}\text{-a.e. over } \bar{\Omega} \times \bar{\Omega},$$

for *every* optimal transport plan $\hat{\gamma}$ of $MK(\varphi^+, \varphi^-, \Sigma)$ satisfying (8). Such a Kantorovich potential exists in view of corollary 2.1 from [1]. This choice of u determines the particular transport set T and the set of ray ends $\mathcal{E}$.

To prove the claim, note first that since $\mathcal{L}^n(\mathcal{E}) = 0$ in view of lemma 4.2, and $\varphi^{\pm} \llcorner \bar{\Omega} \setminus \Sigma << \mathcal{L}^n$, it is not restrictive to assume $\varphi^{\pm}$ to be concentrated on T. Hence, $\lambda^{\pm}$ are concentrated on T and $\lambda^+ \wedge \lambda^- = 0$. We note now that in view of lemma 4.2, the set $\eta = \{\mathcal{R}_x \times \mathcal{R}_x\}_{x\in\bar{\Omega}}$ defines a Borel measurable partition of $\bar{\Omega} \times \bar{\Omega}$ modulo a γ-nullset. The factor space $\bar{\Omega} \times \bar{\Omega}/\eta$ is naturally isomorphic to $\bar{\Omega}/\xi$ through a map i which sends $\mathcal{R}_x \times \mathcal{R}_x$ to $\mathcal{R}_x$. By the general version of the theorem on disintegration of measures due to V. Rokhlin [11], one has

$$\gamma = \int_{\bar{\Omega}/\xi} \gamma_{\mathcal{R}} \, d\nu(\mathcal{R}), \tag{9}$$

where $\nu := r_{\#}\gamma$, $r\colon \bar{\Omega} \times \bar{\Omega} \to \bar{\Omega}/\xi$ being defined γ-a.e. by setting

$$r(x, y) := \mathcal{R}_x \text{ whenever } \mathcal{R}_x = \mathcal{R}_y,$$

and $\{\gamma_{\mathcal{R}}\}_{\mathcal{R}\in\xi}$ be the family of disintegrated probability measures over $\bar{\Omega} \times \bar{\Omega}$, while each $\gamma_{\mathcal{R}}$ is concentrated on $\mathcal{R} \times \mathcal{R}$. Note that r is just the composition of i with the natural projection of $\bar{\Omega} \times \bar{\Omega}$ onto $\bar{\Omega} \times \bar{\Omega}/\eta$.

In view of (8) and (9) one has

$$\lambda^{\pm} = \int_{\bar{\Omega}/\xi} \lambda^{\pm}_{\mathcal{R}} \, d\nu(\mathcal{R}), \tag{10}$$

where $\lambda^{\pm}_{\mathcal{R}} := \pi^{\pm}_{\#}\gamma_{\mathcal{R}}$. For the Kantorovich potential $u \in \mathrm{Lip}_{1,\Sigma}$ one has

$$|u(x) - u(y)| = d_{\Omega,\Sigma}(x, y) \qquad \gamma\text{-a.e. over } \bar{\Omega} \times \bar{\Omega},$$

and hence the same relationship also holds $\gamma_{\mathcal{R}}$-a.e. in $\bar{\Omega}\times\bar{\Omega}$ for ν-a.e. $\mathcal{R} \in \bar{\Omega}/\xi$. Due to corollary 2.1 from [1] this implies that $\gamma_{\mathcal{R}}$ is an optimal plan for the transport problem $MK(\lambda^+_{\mathcal{R}}, \lambda^-_{\mathcal{R}}, \Sigma)$. Note that

$$\mu(B) = \int_{\bar{\Omega}\times\bar{\Omega}} \mathcal{H}^1(g_{\Omega,\Sigma}(x, y) \cap B) \, d\gamma(x, y) = \int_{\bar{\Omega}/\xi} \mu_{\mathcal{R}} \, d\nu(\mathcal{R}),$$

where the measures $\mu_{\mathcal{R}}$ are defined by

$$\mu_{\mathcal{R}}(B) := \int_{\bar{\Omega}\times\bar{\Omega}} \mathcal{H}^1(g_{\Omega,\Sigma}(x, y) \cap B) \, d\gamma_{\mathcal{R}}(x, y),$$

and hence they are the transport densities for the problems $MK(\lambda^+_{\mathcal{R}}, \lambda^-_{\mathcal{R}}, \Sigma)$. To prove the claim, it is enough thus to show that

(i) ν is independent of γ;

(ii) $\mu_{\mathcal{R}}$ are independent of the choice of $g_{\Omega,\Sigma}(x, y)$ as well as of the choice of γ.

To show (i), let $\pi\colon T \to \bar{\Omega}/\xi$ be the natural projection map $\pi(x) := \mathcal{R}_x$. Then γ-a.e. over $T \times T$ one has $r = \pi \circ \pi^+$ and hence

$$\nu = r_{\#}\gamma = \pi_{\#}(\pi^+_{\#}\gamma) = \pi_{\#}\lambda^+.$$

To show (ii), note first that since ν is already proven to be independent of γ, in view ν-a.e. uniqueness of $\lambda^{\pm}_{\mathcal{R}}$ in the representation (10), one has that $\lambda^{\pm}_{\mathcal{R}}$ is also independent of γ. Consider now two cases.

Case A. $\mathcal{R}$ is not piercing Σ, i.e. it is just a line segment without endpoints. Then (ii) follows from lemma 7.2 from [1].

Case B. $\mathcal{R} = T_1$ is the union of all geodesic transport rays piercing Σ. It is easy to observe then that

$$\mu_{\mathcal{R}} = \mu_{\mathcal{R}} \llcorner \mathcal{R}^+ + \mu_{\mathcal{R}} \llcorner \mathcal{R}^-,$$

where $\mathcal{R}^{\pm} := T_1 \cap \{\pm u > 0\}$, since $\mu_{\mathcal{R}}(\{u = 0\} = \mu_{\mathcal{R}}(\Sigma) = 0$. It remains therefore to show that each of the measures $\mu_{\mathcal{R}} \llcorner \mathcal{R}^{\pm}$ is independent of γ and of the choice of $g_{\Omega,\Sigma}(x, y)$.

We show the latter claim for $\mu_{\mathcal{R}} \llcorner \mathcal{R}^+$ (the case of $\mu_{\mathcal{R}} \llcorner \mathcal{R}^-$ is completely symmetric). Consider for this purpose a new Borel measure

$$\tilde{\gamma} := \gamma_{\mathcal{R}} \llcorner \mathcal{R}^+ \times \mathcal{R}^+ + (Id \times k^+)_{\#}(\pi^+_{\#}(\gamma_{\mathcal{R}} \llcorner \mathcal{R}^+ \times (T_1 \setminus \mathcal{R}^+)))$$

over $\mathcal{R}^+ \times \tilde{\Omega}$, where $\tilde{\Omega} := T_1 \cap \{u \geq 0\}$. A direct calculation gives

$$\pi^+_{\#}\tilde{\gamma} = \lambda^+_{\mathcal{R}} \llcorner \mathcal{R}^+, \qquad (\pi^+_{\#}\tilde{\gamma}) \llcorner \mathcal{R}^+ = \lambda^-_{\mathcal{R}} \llcorner \mathcal{R}^+,$$

while $|u(x) - u(y)| = d_{\Omega,\Sigma}(x, y)$ for $\tilde{\gamma}$-a.e. $(x, y) \in \tilde{\Omega} \times \tilde{\Omega}$. Then according to corollary 2.1 from [1], $\tilde{\gamma}$ is an optimal transport plan for $MK(\lambda^+_{\mathcal{R}} \llcorner \mathcal{R}^+, \lambda^-_{\mathcal{R}} \llcorner \mathcal{R}^+, \Sigma)$. Moreover, $\mu_{\mathcal{R}} \llcorner \mathcal{R}^{\pm}$ is easily seen to be a transport density of the latter problem. In fact, for a Borel set $B \subset \mathcal{R}^+$ one has

$$\begin{aligned}\mu_{\mathcal{R}}(B) = &\int_{T_1 \times T_1} \mathcal{H}^1(g_{\Omega,\Sigma}(x, y) \cap B)\, d\gamma_{\mathcal{R}}(x, y) = \\ &\int_{\mathcal{R}^+ \times \mathcal{R}^+} \mathcal{H}^1(g_{\Omega,\Sigma}(x, y) \cap B)\, d\gamma_{\mathcal{R}}(x, y) + \\ &\qquad \int_{\mathcal{R}^+ \times (T_1 \setminus \mathcal{R}^+)} \mathcal{H}^1(g_{\Omega,\Sigma}(x, k^+(x)) \cap B)\, d\gamma_{\mathcal{R}}(x, y) = \\ &\int_{\mathcal{R}^+ \times \tilde{\Omega}} \mathcal{H}^1(g_{\Omega,\Sigma}(x, y) \cap B)\, d\tilde{\gamma}(x, y)\end{aligned}$$

For $z \in \mathcal{R}^+$ set $R^+_z := R_z \cap \{u \geq 0\}$. Let also $\tilde{S}$ stand for the set of segments of positive length in $\bar{\Omega}$ with only the lower endpoint (and without the upper endpoint) endowed with the distance (7). One has then $R^+_z \in \tilde{S}$ for every $z \in \mathcal{R}^+$. Define the map $\tilde{r}\colon \mathcal{R}^+ \times \tilde{\Omega} \to \tilde{S}$ by the formula

$$\tilde{r}(x, y) := R^+_x \text{ whenever } y \in R^+_x,$$

The latter clearly defines $\tilde{r}$ for $\tilde{\gamma}$-a.e. on $(x, y) \in \mathcal{R}^+ \times \tilde{\Omega}$. By repeating substantially the arguments of the proof of lemma 4.2 this map can be easily shown to be Borel measurable. Then acting as above one can disintegrate

$$\tilde{\gamma} = \int_{\tilde{S}} \tilde{\gamma}_R \, d\tilde{\nu}(R), \tag{11}$$

where $\tilde{\nu} := \tilde{r}_{\#}\tilde{\gamma}$ and $\{\tilde{\gamma}_R\}_{R \in \tilde{S}}$ be the family of disintegrated probability measures over $\mathcal{R}^+ \times \tilde{\Omega}$, while each $\tilde{\gamma}_R$ is concentrated on $R \times R$. Again, analogously to the above, one has that $\tilde{\gamma}_R$ is an optimal plan for the problem $MK(\tilde{\lambda}_R^+, \tilde{\lambda}_R^-, \Sigma)$, where $\tilde{\lambda}_R^\pm := \pi_\#^\pm \tilde{\gamma}$, while the transport density

$$\mu_{\mathcal{R}} \llcorner \mathcal{R}^+ = \int_{\tilde{S}} \tilde{\mu}_{\mathcal{R}} \, d\tilde{\nu}(R),$$

where the measures $\tilde{\mu}_R$ are transport densities for the problems $MK(\tilde{\lambda}_R^+, \tilde{\lambda}_R^-, \Sigma)$. Analogously to (i), one shows then that $\tilde{\nu} = \tilde{\pi}_\#(\pi_\#^+ \tilde{\gamma}) = \tilde{\pi}_\#(\lambda_{\mathcal{R}}^+ \llcorner \mathcal{R}^+)$, where $\tilde{\pi}$: $\mathcal{R}^+ \to \tilde{S}$ is defined by $\tilde{\pi}(z) := R_z^+$. Therefore, $\tilde{\nu}$ is independent of γ. Then $\lambda_{\mathcal{R}}^\pm \llcorner (R \setminus \Sigma)$ are also independent of γ due to the uniqueness of the representation

$$\lambda_{\mathcal{R}}^\pm \llcorner \mathcal{R}^+ = \int_{\tilde{S}} \lambda_{\mathcal{R}}^\pm \llcorner (R \setminus \Sigma) \, d\tilde{\nu}(R).$$

But on the other hand, $\tilde{\mu}_R$ is also independent of γ and of the choice of $g_{\Omega,\Sigma}$ since it is a unique solution in the weak sense of the one-dimensional ODE

$$\tilde{\mu}_R' = \tilde{\lambda}_R^+ - \tilde{\lambda}_{\mathcal{R}}^-$$

over $R \setminus \Sigma$ which has given total mass

$$\mu_R(R) = MK(\tilde{\lambda}_R^+, \tilde{\lambda}_{\mathcal{R}}^-, \Sigma)$$

also independent of γ. $\square$

At last we show by means of the example below, that unlike the case $\Sigma = \emptyset$ no uniqueness of transport density is guaranteed, if only one of the measures φ^+ or φ^- is assumed to be absolutely continuous with respect to the Lebesgue measure.

Example 4.1. Let $\Sigma = \partial B_1(0)$ be the unit sphere with the center in the origin, $\varphi^+ := \delta_O$, where O stands for the origin, and φ^- be such that $\varphi^- << \mathcal{L}^n$ and $\operatorname{supp} \varphi^- \subset \mathbb{R}^n \setminus B_1(0)$. Then, clearly, for every $A \in \partial B_1(0)$ there is a transport density μ such that $\mu \llcorner B_1(0) = \mathcal{H}^1 \llcorner (OA)$, and hence the transport density is not unique.

5. Properties of transport density

5.1. **Dimension.** We recall the following notion [1].

Definition 5.1. *The Hausdorff dimension* $\dim \mu$ *of the Borel measure* μ *over* $\bar{\Omega}$ *is defined as*

$$\dim \mu = \sup\{k : \mu << \mathcal{H}^k\},$$

where $\mathcal{H}^k$ *stands for the* k*-dimensional Hausdorff measure.*

Another known characterization of the Hausdorff dimension of a measure is given by the formula

$$\dim \mu = \sup\{k : \Theta_k^*(\mu, x) < +\infty \text{ for } \mu - \text{ a.e. } x \in \bar{\Omega}\}, \tag{12}$$

where $\Theta_k^*(\mu, x)$ stands for the upper k-dimensional density of μ in the the point $x \in \bar{\Omega}$, namely,

$$\Theta_k^*(\mu, x) := \limsup_{r \to 0} \frac{\mu(B_r(x))}{\mathcal{H}^k(B_r(x))}.$$

In fact, if $\Theta_k^*(\mu, x) < +\infty$ for μ-a.e. $x \in \bar{\Omega}$, then setting $B_i := \{x : \Theta_k^*(\mu, x) \le i\}$, one has $\mu \llcorner B_i \le 2^k i \mathcal{H}^k \llcorner B_i$, and hence $\mathcal{H}^k(e) = 0$ for a Borel $e \subset \bar{\Omega}$ implies $\mu(e \cap B_i) = 0$, hence $\mu(e) = 0$, or, in other words, $\mu << \mathcal{H}^k$. Therefore, $k \le \dim \mu$. On the other hand, the assumption $\mu << \mathcal{H}^k$ implies, since always $\Theta_k^*(\mu, x) <$

$+\infty$ for $\mathcal{H}^k$-a.e. in $\bar{\Omega}$, that the same holds also μ-a.e. Thus, both definitions are equivalent.

We have the following result regarding the dimension of every transport density for the Monge-Kantorovich problem with Dirichlet condition.

Theorem 5.1. *Let μ be an arbitrary transport density for the Monge-Kantorovich problem $MK(\varphi^+, \varphi^-, \Sigma)$. Then one has*

$$\dim \mu \geq 1 \vee (\dim \varphi^+ \wedge \dim \varphi^-).$$

PROOF: According to theorem 3.1, one can represent the transport density μ as in (4), i.e.

$$\mu = \tilde{\mu}^0(e) + \tilde{\mu}^+(e) + \tilde{\mu}^-(e),$$

where $\tilde{\mu}^0(e)$ is a transport density for $MK(\varphi_r^+, \varphi_r^-)$, $\tilde{\mu}^+(e)$ is a transport density for $MK(\varphi_s^+, \psi^-)$ and $\tilde{\mu}^-(e)$ is a transport density for $MK(\psi^+, \varphi_s^-)$. In view of theorem 4.7 from [5] one has

$$\begin{aligned} \dim \tilde{\mu}^0 &\geq 1 \vee \dim \varphi_r^+ \vee \dim \varphi_r^-, \\ \dim \tilde{\mu}^\pm &\geq 1 \vee \dim \varphi_s^\pm \vee \dim \psi^\pm. \end{aligned}$$

Hence, clearly, $\dim \mu \geq 1$. To show the inequality $\dim \mu \geq \dim \varphi^+ \wedge \dim \varphi^-$, note that $\varphi^\pm = \varphi_r^\pm + \varphi_s^\pm$, and hence according to lemma 5.1 below

$$\dim \varphi^\pm = \dim \varphi_r^\pm \wedge \dim \varphi_s^\pm,$$

which completes the proof. □

Lemma 5.1. *Let μ, μ_1, μ_2 be finite Borel measures over $\bar{\Omega}$ satisfying $\mu = \mu_1 + \mu_2$. Then*

$$\dim \mu = \dim \mu_1 \wedge \dim \mu_2.$$

PROOF: To show $\dim \mu \leq \dim \mu_1$, suppose the contrary. Fix then a k satisfying $\dim \mu_1 < k < \dim \mu$. Then there is an $e \subset \bar{\Omega}$ with $\mu_1(e) > 0$ such that $\Theta_k^*(\mu_1, x) = +\infty$ for all $x \in e$. But $\Theta_k^*(\mu_1, x) < \Theta_k^*(\mu, x)$, hence for all $x \in e$ one has also $\Theta_k^*(\mu, x) = +\infty$, which means $\mu(e) = 0$ and therefore $\mu_1(e) = 0$ contradicting the assumption. In the same way one proves also $\dim \mu \leq \dim \mu_2$.

It remains to show that $\dim \mu \geq k$ whenever both $\dim \mu_1 \geq k$ and $\dim \mu_2 \geq k$. Again, suppose the contrary, i.e. $\dim \mu_1 \geq k$ and $\dim \mu_2 \geq k$ but $\dim \mu < k$. Then there is an $e \subset \bar{\Omega}$ with $\mu(e) > 0$ such that $\Theta_k^*(\mu, x) = +\infty$ for all $x \in e$. But

$$\Theta_k^*(\mu, x) \leq \Theta_k^*(\mu_1, x) + \Theta_k^*(\mu_2, x),$$

hence both $\Theta_k^*(\mu_1, x) = +\infty$ and $\Theta_k^*(\mu_2, x) = +\infty$ for all $x \in e$. The formula (12) implies then $\mu_1(e) = \mu_2(e) = 0$ and therefore also $\mu(e) = 0$, contradicting the assumption. □

Note that in the case without Dirichlet condition (i.e. when $\Sigma = \emptyset$), theorem 4.7 from [5] gives a stronger result, namely,

$$\dim \mu \geq 1 \vee \dim \varphi^+ \vee \dim \varphi^-.$$

However, the example below shows that theorem 5.1 is optimal in the general case when $\Sigma \neq \emptyset$.

Example 5.1. Assume $\varphi^+ := \mathcal{L}^2 \llcorner B_1(0)$, where $B_1(0) \subset \mathbb{R}^2$ is the ball with radius 1 centered at the origin, $\Sigma = [2, 3] \times \{0\}$ is the closed line segment and $\varphi^- := \delta_A$, where $A := (4, 0)$. Then, clearly, $\mu \llcorner \{x \geq 3\} = \mathcal{H}^1 \llcorner [3, 4] \times \{0\}$, hence $\dim \mu = 1$, while $\dim \varphi^+ = 2$.

5.2. **Summability.** The following result on the summability of the transport density is valid.

Theorem 5.2. *If $\varphi^\pm << \mathcal{L}^n$, then for the transport density μ for the problem $MK(\varphi^+, \varphi^-, \Sigma)$ one has $\mu << \mathcal{L}^n$, namely, it has the Radon-Nikodym density*

$$a := \frac{d\mu}{d\mathcal{L}^n} \in L^1(\Omega).$$

REMARK. Recall that in the case of the above theorem the transport density is unique according to theorem 4.1.

PROOF: In view of theorem 3.1 we represent μ as in (4). Note now that $\varphi^\pm << \mathcal{L}^n$ and $\varphi^\pm = \varphi_r^\pm + \varphi_s^\pm$ imply that both $\varphi_r^\pm << \mathcal{L}^n$ and $\varphi_s^\pm << \mathcal{L}^n$. Therefore, in view of theorem 4.11 from [5], one has $\tilde{\mu}^0 << \mathcal{L}^n$ and $\tilde{\mu}^\pm << \mathcal{L}^n$, which shows the claim. □

It is worth emphasizing that in view of example 5.1, unlike the case without Dirichlet condition (i.e. when $\Sigma = \emptyset$), in general the above result is not true when only φ^+ (or only φ^-) is absolutely continuous with respect to the Lebesgue measure.

Using the same idea, we can also show further summability results for the transport density. To abbreviate the notation, we will identify the Borel measures μ absolutely continuous with respect to the Lebesgue measure with the respective densities. Namely, we say that $\mu \in L^p(\Omega)$, if $\mu << \mathcal{L}^n$ and the respective density belongs to $L^p(\Omega)$.

Theorem 5.3. *Let $\varphi^\pm \in L^p(\Omega \setminus \Sigma)$, $1 \le p \le +\infty$. Then for the transport density μ for the problem $MK(\varphi^+, \varphi^-, \Sigma)$ one has*

$$a := \frac{d\mu}{d\mathcal{L}^n} \in L^q_{loc}(\Omega \setminus \Sigma)$$

for every $q < p$, if $p < +\infty$, and for $q = +\infty$, if $p = +\infty$. If, moreover, $\operatorname{supp}\varphi^+ \cap \operatorname{supp}\varphi^- = \emptyset$, then $a \in L^p_{loc}(\Omega \setminus \Sigma)$.

PROOF: We achieve the result in two steps.

Step 1. As in the proof of theorem 5.2, we represent μ according to (4). Since $\varphi^\pm \in L^p(\Omega \setminus \Sigma)$, $\varphi^\pm = \varphi_r^\pm + \varphi_s^\pm$, then both $\varphi_r^\pm \in L^p(\Omega \setminus \Sigma)$ and $\varphi_s^\pm \in L^p(\Omega \setminus \Sigma)$. Theorem 4.18 (for the case $p < +\infty$) or proposition 4.17 (for the case $p = +\infty$) from [5] implies $\tilde{\mu}^0 \in L^q(\Omega \setminus \Sigma)$ with q as indicated in the statement of the theorem being proven. If the supports of φ^+ and φ^- (and hence, those of φ_r^+ and φ_r^-) are disjoint, then one uses theorem 4.20 from the same paper instead to show that one can take $q = p$.

Step 2. We show now $\tilde{\mu}^\pm \in L^p_{loc}(\Omega \setminus \Sigma)$, which suffices to conclude the proof.

For $\varepsilon > 0$ we denote by Σ_ε the closed ε-neighborhood of Σ, i.e.

$$\Sigma_\varepsilon := \bigcup_{x \in \Sigma} \bar{B}_\varepsilon(x) = \{x \in \Omega \,:\, \operatorname{dist}_\Omega(x, \Sigma) \le \varepsilon\},$$

and let $\Sigma_\varepsilon^c := \bar{\Omega} \setminus \Sigma_\varepsilon$. We have to show $\tilde{\mu}^\pm \in L^p(\Omega \setminus \Sigma_\varepsilon)$ for every $\varepsilon > 0$. It is enough to consider $\tilde{\mu}^+$, since the arguments for $\tilde{\mu}^-$ are completely symmetric. Let γ^+ be an optimal measure for the problem $MK(\varphi_s^+, \psi^-)$ and set

$$\begin{array}{rcl} \gamma_\varepsilon^+ & := & \gamma^+ \llcorner (\Sigma_\varepsilon^c \times \bar{\Omega}), \\ \varphi_{s,\varepsilon}^+ & := & \pi^+_\# \gamma_\varepsilon^+ = \varphi_s^+ \llcorner \Sigma_\varepsilon^c, \\ \psi_\varepsilon^- & := & \pi^-_\# \gamma_\varepsilon^+ \end{array}$$

(clearly, ψ_ε^- is concentrated on Σ). The measure γ_ε^+ is optimal for the Monge-Kantorovich problem $MK(\varphi_{s,\varepsilon}^+, \psi_\varepsilon^-)$ in view of lemma 2.1.

In view of lemma 5.2 below it suffices now to prove that the transport density $\tilde\mu_{\varepsilon/2}^+$ of the Monge-Kantorovich problem $MK(\varphi_{s,\varepsilon/2}^+, \psi_{\varepsilon/2}^-)$ belongs to $L^p_{loc}(\mathbb{R}^n \setminus \Sigma)$ for every $\varepsilon > 0$. The latter follows from lemma 5.3 below, since

$$\operatorname{dist}_\Omega(\operatorname{supp}\varphi_{s,\varepsilon/2}^+, \operatorname{supp}\psi_{\varepsilon/2}^-) \geq \operatorname{dist}_\Omega(\operatorname{supp}\varphi_{s,\varepsilon/2}^+, \Sigma) > \varepsilon/2,$$

which concludes the proof. □

Lemma 5.2. *Let $\tilde\mu_{\varepsilon/2}^+$ stand for the transport density for the Monge-Kantorovich problem $MK(\varphi_{s,\varepsilon/2}^+, \psi_{\varepsilon/2}^-)$. Then*

$$\tilde\mu^+ \llcorner \Sigma_\varepsilon^c = \tilde\mu_{\varepsilon/2}^+ \llcorner \Sigma_\varepsilon^c.$$

PROOF: For a Borel set $e \subset \Sigma_\varepsilon^c$ we have

$$\begin{aligned}
\tilde\mu^+(e) &= \int_{\operatorname{Graph} K^+} \mathcal{H}^1(e \cap g_\Omega(x,y))\, d\gamma^+(x,y) = \\
&\int_{\operatorname{Graph} K^+ \cap (\Sigma_{\varepsilon/2} \times \bar\Omega)} \mathcal{H}^1(e \cap g_\Omega(x,y))\, d\gamma^+(x,y) + \\
&\int_{\operatorname{Graph} K^+ \cap (\Sigma_{\varepsilon/2}^c \times \bar\Omega)} \mathcal{H}^1(e \cap g_\Omega(x,y))\, d\gamma^+(x,y) = \\
&\int_{\operatorname{Graph} K^+ \cap (\Sigma_{\varepsilon/2}^c \times \bar\Omega)} \mathcal{H}^1(e \cap g_\Omega(x,y))\, d\gamma^+(x,y) = \\
&\int_{\operatorname{Graph} K^+} \mathcal{H}^1(e \cap g_\Omega(x,y))\, d\gamma_{\varepsilon/2}^+(x,y) = \tilde\mu_{\varepsilon/2}^+(e),
\end{aligned}$$

since $e \subset \Sigma_\varepsilon^c$. □

Lemma 5.3. *If $\varphi^+ \in L^p(\Omega)$ and $\operatorname{supp}\varphi^+ \cap \operatorname{supp}\varphi^- = \emptyset$. Then for the transport density μ stand of the Monge-Kantorovich problem $MK(\varphi^+, \varphi^-)$ one has $\mu \in L^p_{loc}(\mathbb{R}^n \setminus \operatorname{supp}\varphi^-)$.*

PROOF: The lemma is just a slight refinement of theorem 4.20 from [5]. □

We remark that one cannot expect in general $a \in L^q(\Omega \setminus \Sigma)$ (hence actually $a \in L^q(\Omega)$, since $a = 0$ a.e. in Σ in view of (3)) with q as indicated in the statement of the above theorem, even when the densities of $\varphi^\pm$ belong to $L^p(\Omega \setminus \Sigma)$ and have disjoint supports, as the following example shows.

Example 5.2. Let $D \subset \mathbb{R}^n$, $n \geq 2$ stand for the half annulus

$$D := (B_2(0) \setminus B_1(0)) \cap \{x_1 \leq 0\},$$

x_1 stands for the first coordinate. Assume Σ to be the line segment $\Sigma := [0, 10] \times \{0\}^{n-1}$, and take $\varphi^+ := \mathcal{L}^n \llcorner D$ and $\varphi^- := \mathcal{L}^n \llcorner B_1(12)$. Then it is easy to compute $\mu \llcorner \{x_1 \leq 0\}$. In fact, $\mu \llcorner \{x_1 \leq 0\} = a\mathcal{L}^n \llcorner \{x_1 \leq 0\}$, where

$$a(r) = \frac{1}{r^{n-1}} \int_1^2 \rho^{n-1}\, d\rho = \frac{C}{r^{n-1}},$$

where $C > 0$ is a constant and $r := ||x||$, $||\cdot||$ standing for the Euclidean norm. Therefore, although $\varphi^\pm \in L^\infty(\Omega)$, one only has $a \in L^q(\Omega)$ for $1 \leq q < 1 + 1/(n-1)$.

Nevertheless, we can claim more summability of μ, when Σ satisfies some extra regularity assumption.

For a set $D \subset \mathbb{R}^n$ we denote by ∂D its boundary and by $\operatorname{int} D$ its inner part.

Definition 5.2. *We say that the open set $D \subset \mathbb{R}^n$ satisfies the internal ball condition, if there is a $\rho > 0$ such that for every $x \in \partial D$ there is an open ball in D with a radius not less than ρ, whose boundary touches ∂D at x.*

The above condition admits an easy analytical interpretation when $\partial D \in C^2$, i.e. for every $y \in \partial D$ in a suitable coordinate system $(x_1, \dots, x_n)$ in $\mathbb{R}^n$ centered in y (i.e. with $y = 0$) one has

$$D \cap B_r(0) = \{x_n < \Psi(x')\} \cap B_r(0)$$

for sufficiently small $r > 0$, where $x' := (x_1, \dots, x_{n-1})$ and

$$\Psi(x') = \sum_{k=1}^{n-1} k_i x_i^2 + o(\|x'\|^2),$$

and k_i, $i = 1, \dots, n-1$ are the principal curvatures of ∂D at y. In this case the internal ball condition asserts that there is a the lower bound

$$k_i \geq -\frac{1}{\rho}, \qquad\qquad i = 1, \dots, n-1.$$

for the principal curvatures of ∂D at every point. It is well-known that $\partial D \in C^{1,1}$ is sufficient for the internal ball condition to hold, but, of course, not necessary (e.g. this condition also holds whenever D is a complement of some convex set in $\mathbb{R}^n$).

We are able now to claim the following result.

Theorem 5.4. *Assume that $D := \mathrm{int}\Sigma$ satisfies the internal ball condition. Let $\varphi^\pm \in L^p(\Omega \setminus \Sigma)$, $1 \leq p \leq +\infty$. Then for the transport density μ of the Monge-Kantorovich problem $MK(\varphi^+, \varphi^-, \Sigma)$ one has*

$$a := \frac{d\mu}{d\mathcal{L}^n} \in L^q(\Omega)$$

for every $q < p$, if $p < +\infty$, and for $q = +\infty$, if $p = +\infty$. If, moreover, $\mathrm{supp}\,\varphi^+ \cap \mathrm{supp}\,\varphi^- = \emptyset$, then $a \in L^p(\Omega)$.

PROOF: Keeping the notation of the proof of theorem 5.3, we note that it is enough to show that under the conditions of the statement being proven one has $\tilde{\mu}^\pm \in L^q(\Omega \setminus \Sigma)$. We will prove it for $\tilde{\mu}^+$, the analogous assertion for $\tilde{\mu}^-$ being completely symmetric.

For this purpose without loss of generality we consider φ_s^+ to be concentrated outside of Σ. Furhermore, since $\varphi_s^+ << \mathcal{L}^n$, then we may also assume φ_s^+ to be concentrated on a set Φ_s^+ of $x \in \bar{\Omega} \setminus \Sigma$ which admit a unique projection (i.e. the point of minimum distance) $k(x)$ to Σ (this set differs from $\Omega \setminus \Sigma$ by a set of nondifferentiability points of $\mathrm{dist}_\Omega(\cdot, \Sigma)$ which has Lebesgue measure zero since the distance function is Lipschitz continuous). Therefore according to (2), one has

$$\tilde{\mu}^+(B_r(x)) = \int_{\bar{\Omega}} \mathcal{H}^1(g_\Omega(z, k(z)) \cap B_r(x))\, d\varphi_s^+(z).$$

For $\varepsilon > 0$ let

$$D_{-\varepsilon} := \{x \in D \,:\, \mathrm{dist}_\Omega(x, \mathbb{R}^n \setminus D) \geq \varepsilon\}$$

and fix an $\varepsilon < \rho$, where ρ is the constant in the internal ball condition. Also for $x \in \Phi_s^+$ denote

$$k'(x) := k(x) + \varepsilon \frac{x - k(x)}{d_\Omega(x, k(x))}.$$

Define now the measure $\psi^{'-} := k'_{\#}\varphi_s^+$. According to lemma 5.4 below, $\psi^{'-}$ is supported on $\partial D_{-\varepsilon}$. Consider the problem $MK(\varphi_s^+, \psi^{'-})$. The respective transport density $\tilde{\mu}^{'+}$ satisfies

$$\tilde{\mu}^{'+}(B_r(x)) = \int_{\bar{\Omega}} \mathcal{H}^1(g_\Omega(z, k'(z)) \cap B_r(x))\, d\varphi_s^+(z).$$

For $x \in \bar{\Omega} \setminus \Sigma$ and sufficiently small $r > 0$ one obviously has then $\tilde{\mu}^+(B_r(x)) = \tilde{\mu}^{'+}(B_r(x))$, and hence

$$\tilde{a}^+(x) = \tilde{a}^{'+}(x) \qquad \text{for a.e. } x \in \bar{\Omega} \setminus \Sigma, \tag{13}$$

where $\tilde{a}^+$ and $\tilde{a}^{'+}$ are the densities of $\tilde{\mu}^+$ and $\tilde{\mu}^{'+}$ respectively with respect to the Lebesgue measure. On the other hand, Step 2 of the proof of theorem 5.3 shows $\tilde{a}^{'+} \in L^q_{loc}(\Omega \setminus \partial D_{-\varepsilon})$ and in particular $\tilde{a}^{'+} \in L^q(\Omega \setminus \Sigma)$, which together with (13) shows the claim. $\square$

Lemma 5.4. *For every $x \in \Phi_s^+$ one has $k'(x) \in \partial D_{-\varepsilon}$, and hence*

$$dist_\Omega(k'(x), \Phi_s^+) \geq \varepsilon.$$

PROOF: Due to the internal ball condition there is a ball $B_r(z) \subset D$ with $r \geq \rho$ touching ∂D at $k(x)$. Then all $y \neq z$ belonging to a line segment which joins z and $k(x)$ have the unique projection $k(x)$ on $\mathbb{R}^n \setminus D$. In fact, suppose the contrary, i.e. the existence of an y in this segment such that

$$d_\Omega(y, k(x)) \geq d_\Omega(y, y')$$

for some $y' \neq k(x)$, $y' \in \mathbb{R}^n \setminus D$. Denoting then by y_r the intersection of the segment $g_\Omega(y, y')$ with $\partial B_r(z)$, one would have in contradiction with the assumption

$$d_\Omega(y, y') \geq d_\Omega(y, y_r) > d_\Omega(y, k(x)),$$

since $y \neq z$ and $k(x)$ is then the unique projection of y on $\partial B_r(z)$.

Since by our choice $\varepsilon < \rho$, then in particular the unique projection of $k'(x)$ to $\mathbb{R}^n \setminus D$ is $k(x)$. But $d_\Omega(k'(x), k(x)) = \varepsilon$ by construction, hence $k'(x) \in \partial D_{-\varepsilon}$. $\square$

We remark that the internal ball condition is essential for the statement of the above theorem to hold, as the following example shows.

Example 5.3. Let $D \subset \mathbb{R}^2$ stand for the quarter of the annulus

$$D := (B_2(0) \setminus B_1(0)) \cap \{x_1 \leq -|x_2|\},$$

where x_1 and x_2 are the first and the second coordinates respectively, and let Σ be the square with vertices $(0,0)$, $(1,1)$, $(1,-1)$, $(2,0)$. Take now $\varphi^+ := \mathcal{L}^n \llcorner D$ and $\varphi^- := 0$. Then the computation identical to the one fulfilled in example 5.2 shows $a \in L^q(\Omega)$ only for $1 \leq q < 1 + 1/(n-1)$, while $\varphi^\pm \in L^\infty(\Omega)$. In fact, the inner part of Σ clearly does no satisfy the internal ball condition.

Another similar statement can also be proven for the case when Σ is a boundary of some set.

Theorem 5.5. *Assume that Σ is the $C^{1,1}$ boundary of an open set $D \subset \Omega$. Let $\varphi^\pm \in L^p(\Omega \setminus \Sigma)$, $1 \leq p \leq +\infty$. Then for the transport density μ of the Monge-Kantorovich problem $MK(\varphi^+, \varphi^-, \Sigma)$ one has*

$$a := \frac{d\mu}{d\mathcal{L}^n} \in L^q(\Omega)$$

for every $q < p$, if $p < +\infty$, and for $q = +\infty$, if $p = +\infty$. If, moreover, $\operatorname{supp}\varphi^+ \cap \operatorname{supp}\varphi^- = \emptyset$, then $a \in L^p(\Omega)$.

PROOF: Again, as in the proof of theorem 5.4, it is enough to show that the conditions of the statement being proven imply has $\tilde{\mu}^+ \in L^q(\Omega \setminus \Sigma)$ (the analogous assertion for $\tilde{\mu}^-$ is symmetric). Adopting the same notation as in the proof of theorem 5.4, we consider without loss of generality φ_s^+ to be concentrated over Φ_s^+.

Set $\varphi_{1,s}^+ := \varphi_s^+ \llcorner D$, $\varphi_{2,s}^+ := \varphi_s^+ \llcorner(\bar{\Omega} \setminus D)$ and $\psi_i^- := k_\# \varphi_{i,s}^+$, and consider the optimal transport problems $MK(\varphi_{i,s}^+, \psi_i^s)$, $i = 1, 2$. Since $\Sigma = \partial D \in C^{1,1}$, then both D and $\mathbb{R}^n \setminus D$ satisfy the internal ball condition, and we reiterate the arguments of the proof of theorem 5.4 to see that the respective transport densities $\tilde{\mu}_i^+ \in L^q(\Omega \setminus \Sigma)$, $i = 1, 2$, which shows the claim since $\tilde{\mu}^+ = \tilde{\mu}_1^+$ over D and $\tilde{\mu}^+ = \tilde{\mu}_2^+$ outside of $\bar{D}$. $\square$

Let us remark that in view of example 5.3 for the above result to hold the regularity (differentiability) of Σ is inevitable. In fact, in example 5.3 one could have taken Σ to be just the boundary of the square (which is locally Lipschitz) rather than the whole square. Moreover, the $C^{1,1}$ regularity condition on Σ in this case is sharp, i.e. it cannot be substituted even by $C^{1,\alpha}$ regularity requirement for $0 \leq \alpha < 1$, as the following example shows.

Example 5.4. Let $D \subset \mathbb{R}^2$ be a set which in a neighborhood B of the origin coincides with $\{y \leq f(x)\}$, where

$$f(x) := \frac{1}{1+\alpha} \int_0^x t^{1+\alpha} \sin 1/t \, dt,$$

$0 \leq \alpha < 1$, and suppose that outside of B it has smooth (say, C^2) boundary. Let $\Sigma := \partial D$ and note that $\Sigma \in C^{1,\alpha}$, but $\Sigma \notin C^{1,1}$. Assume $\varphi^+ := \mathcal{L}^n \llcorner B$ and $\varphi^- := 0$. We show now $a \notin L^\infty(B)$, although both $\varphi^\pm \in L^\infty(\Omega)$.

To calculate a, we pass from the coordinates (x, y) to the coordinates (t, r), where

$$r = d_D(x, y) := \operatorname{dist}((x, y), D) - \operatorname{dist}((x, y), \mathbb{R}^2 \setminus D)$$

is the signed distance function to D and t is the natural curvilinear coordinate of the projection of the point (x, y) on Σ. The respective transformation $(t, r) \to (x, y)$ is given by the formulae

$$\begin{cases} x = t - V(t, r) f'(t), \\ y = f(t) + V(t, r), \end{cases}$$

where $V(t, r) := r/(1 + f'^2(t))^{1/2}$. The Jacobian of this transformation is given by the expression

$$J(t, r) = (f'^2(t) + 1)^{1/2} |1 - K(t) r|,$$

where $K(t) := f''(t)/(1 + f'^2(t))^{3/2}$ is the curvature of Σ at the point with coordinate t. Consider such couples (t, r) that $K(t) > 0$ and $r < 0$. A direct calculation shows then

$$a(t, r) \geq \frac{1}{J(t, r)} \int_{C(t)}^r J(t, \rho) \, d\rho$$

in a neighborhood of origin, for some finite $C(t) < r < 0$ (in fact, we supposed that φ^+ has compact support). Suppose that there is a constant $C < 0$ such that $C(t) < C$. Then using the expression for $J(t, r)$ one estimates

$$a(t, r) \geq \frac{2 - K(t) r - C K(t)}{2(1 - K(t) r)} (r - C)$$

which clearly implies that a is not bounded in a neigborhood of $(0,0)$. In fact, it is enough to take a sequence $\{t_k\}_{k=1}^{\infty}$ such that $1/t_k \in (2\pi k + \pi/2, 2\pi k + \pi)$, hence $t_k \to 0$, and $K(t_k) \to +\infty$, and a sequence $\{r_k\}_{k=1}^{\infty}$ satisfying $K(t_k)r_k \to 0^-$ to see that $a(t_k, r_k) \to +\infty$.

5.3. **Regularity.** We claim the following result on higher regularity of the transport density for the problem with Dirichlet condition. Here and below we will refer to transport rays, ray clusters and the transport set relative to a Kantorovich potential $u\colon \bar{\Omega} \to \mathbb{R}$ of the problem $MK(\varphi^+, \varphi^-, \Sigma)$.

Theorem 5.6. *The following assertion is valid. Let $\varphi^{\pm} \in L^p_{loc}(\Omega \backslash \Sigma)$, $1 \leq p \leq +\infty$. Consider $a := \frac{d\mu}{d\mathcal{L}^n}$, where μ is the transport density of the Monge-Kantorovich problem $MK(\varphi^+, \varphi^-, \Sigma)$. Then for a.e. $x \in T$ the restriction a_{R_x} of a to the transport ray R_x passing through x is well defined and, moreover, $a_{R_x} \in W^{1,p}_{loc}(R_x)$.*

To prove the above theorem we need the following simple assertions.

Lemma 5.5. *Let $N \subset \mathbb{R}^n$ be a set with Lebesgue measure zero. Then for a.e. $x \in T$ one has $\mathcal{H}^1(R_x \cap N) = 0$.*

PROOF: Observe that $\mathcal{L}^n(G(T^{1/l}_{pij} \cap N)) = 0$ for every $(p,i,j) \in \mathbb{Q} \times \mathbb{N}^2$ and $l \in \mathbb{N}$, hence by Fubini theorem $\mathcal{H}^1(G(R_x \cap T^{1/l}_{pij}) \cap G(N)) = 0$ for a.e. $x \in T_{pij}$. Making the countable union over all $l \in \mathbb{N}$ we obtain $\mathcal{H}^1(G(R_x) \cap G(N)) = 0$ and hence $\mathcal{H}^1(R_x \cap N) = 0$ for a.e. $x \in T_{pij}$, which concludes the proof. □

We remark that according to the above lemma 5.5 for a.e. $x \in T$ one has that $\mathcal{H}^1$-a.e. $z \in R_x$ is a Lebesgue point of the density a.

Lemma 5.6. *Let $\rho \in L^p_{loc}(\Omega)$, $1 \leq p \leq +\infty$. Then for a.e. $x \in T$ one has $\rho \in L^p_{loc}(R_x)$.*

PROOF: Let $\rho \in L^p_{loc}(\Omega)$. The area formula gives

$$\int_{T_{pij}} \rho^p(x)\, dx = \int_{G(T_{pij})} \rho^p(F(x'))J_nF(x')\, dx',$$

where J_nF stands for the Jacobian of F. Since over $G(T^{\sigma}_{pij})$ one has $J_nF \geq c > 0$, then $\rho(F(\cdot)) \in L^p_{loc}(G(T_{pij}))$, and hence by Fubini theorem

$$\rho(F(z,\cdot)) \in L^p_{loc}(G(T_{pij}) \cap (\{z\} \times \mathbb{R}))$$

for $\mathcal{H}^{n-1}$-a.e. $z \in \mathbb{R}^{n-1}$. This means $\rho \in L^p_{loc}(R_z)$ for $\mathcal{H}^{n-1}$-a.e. $z \in S_{pi}$ which implies the first result. □

PROOF OF THEOREM 5.6:

Using theorem 3.1, we represent the transport density μ as in (4). Consider the densities a^0 and $a^{\pm}$ of $\tilde{\mu}^0$, $\tilde{\mu}^{\pm}$ respectively with respect to the Lebesgue measure. Clearly,

$$a = a^0 + a^+ + a^-.$$

Denote

$$f^{\pm} := \frac{d\varphi^{\pm}}{d\mathcal{L}^n}.$$

Let R_x be a transport ray relative to a Kantorovich potential $u\colon \bar{\Omega} \to \mathbb{R}$ of $MK(\varphi^+, \varphi^-, \Sigma)$, chosen in such a way that $\mathcal{H}^1$-a.e. $z \in R_x$ is a Lebesgue point of a^0, $a^{\pm}$, $f^{\pm}$. According to lemma 5.5 for a.e. $x \in T$ the transport ray R_x satisfies such a requirement.

Let $(p,i,j) \in \mathbb{Q} \times \mathbb{N}^2$ be such that $R_x \subset T_{pij}$. According to lemma 35 from [10], one has for a.e. $z \in T_{pij}$ the representation

$$a^0(z) = \frac{\hat{a}^0(G(z))}{J_n F(G(z))},$$

where $J_n F := |\det DF|$ is the Jacobian of F and the function $\hat{a}^0 \colon \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$\hat{a}^0(X, x_n) := \int_{x_n}^{+\infty} (f^+(F(X,\xi)) - f^-(F(X,\xi))) J_n F(X,\xi)\, d\xi.$$

Let $X \in \mathbb{R}^{n-1}$ be the projection of $G(R_x)$ on the plane $\{x_n = 0\}$. We may consider without loss of generality that

$$R_x = \{F(X, x_n) \,:\, x_n \in [0, l]\},$$

for some $l > 0$, and, moreover, that $F(X,l)$ and $F(X,0)$ be the upper and lower ray ends of R_x respectively. Minding that

$$F(X, x_n) = V(X) + x_n \nu(V(X)),$$

where $\nu = \nabla u$ is the unit direction of a transport ray, one has that $J_n F$ is a polynomial of x_n. Moreover, if $x_n \in [\sigma, l - \sigma]$ for a small $\sigma > 0$, then $J_n F > c$ for some $c > 0$. Assume now $f^\pm \in L^p(\Omega)$. Then $f^\pm \in L^p_{loc}(R_x)$ by lemma 5.6 and the straightforward computation shows

$$\frac{\partial a^0}{\partial x_n} \in L^p_{loc}(R_x),$$

hence $a^0 \in W^{1,p}_{loc}(R_x)$.

Now, using the notation introduced in Step 2 of the proof of theorem 5.3, we let further $a^\pm_\varepsilon$ stand for the densities of $\tilde{\mu}^\pm_\varepsilon$ with respect to Lebesgue measure. According to lemma 5.2, $a^\pm = a^\pm_{\varepsilon/2}$ a.e. over Σ^c_ε. Then, preserving the above introduced notation, one has

$$a^\pm_{\varepsilon/2}(z) = \frac{\hat{a}^\pm_{\varepsilon/2}(G(z))}{J_n F(G(z))},$$

for a.e. $z \in T_{pij} \cap \Sigma^c_\varepsilon$, where

$$\hat{a}^\pm_{\varepsilon/2}(X, x_n) := \pm \int_{x_n}^{\pm\infty} f^\pm(F(X,\xi)) J_n F(X,\xi)\, d\xi.$$

Therefore the arguments identical to those used above to establish the regularity of a^0, show that $a^\pm_{\varepsilon/2} \in W^{1,p}_{loc}(R_x)$, hence $a^\pm \in W^{1,p}_{loc}(R_x \setminus \Sigma)$ whenever $f^\pm \in L^p(\Omega)$. Combining all these results, we conclude the proof. □

5.4. **Behaviour at the ends of transport rays.** The result on vanishing of transport density at the ends of transport rays for Lipschitz continuous $\varphi^\pm$ was first proven and then extensively used in [6] in the proof of existence of optimal transport maps for Monge-Kantorovich problems without Dirichlet constraint via ODE approach. It was used as well in [7, 8, 9] for deriving the law of evolution of a sandpile shape.

Theorem 5.7. *Let $\varphi^+ \in L^\infty_{loc}(\Omega \setminus \Sigma)$ (resp. $\varphi^- \in L^\infty_{loc}(\Omega \setminus \Sigma)$). Consider $a := \frac{d\mu}{d\mathcal{L}^n}$, where μ is the transport density of the Monge-Kantorovich problem $MK(\varphi^+, \varphi^-, \Sigma)$. Then for a.e. $x \in T$ one has $a(z) \to 0$ when either $z \to l^+$ (resp. $z \to l^-$) while $z \in R_x$, where l^+ and l^- are the upper and lower ends, respectively, of the geodesic transport ray R_x passing through x.*

PROOF: We keep the notation of the proof of theorem 5.6. First, as in this proof, we represent $a = a^0 + a^+ + a^-$. The respective property fo a^0, as well as for the transport densities $a^\pm_\varepsilon$ for each $\varepsilon > 0$ follows directly from lemma 5.7 below. The same property holds also for $a^\pm$, since due to lemma 5.2, one has $a^\pm = a^\pm_{\varepsilon/2}$ a.e. over Σ^c_ε, which concludes the proof. □

Lemma 5.7. *Consider the problem $MK(\varphi^+, \varphi^-)$, where $\varphi^+ \in L^\infty(\bar{\Omega})$ (resp. $\varphi^- \in L^\infty(\bar{\Omega})$) and let $a := \frac{d\mu}{d\mathcal{L}^n}$, where μ is the transport density of $MK(\varphi^+, \varphi^-)$. Then for a.e. $x \in T$ one has $a(z) \to 0$ when either $z \to l^+$ (resp. $z \to l^-$) while $z \in R_x$, where l^+ and l^- are the upper and lower ends, respectively, of the geodesic transport ray R_x passing through x and T is the union of all geodesic transport rays of $MK(\varphi^+, \varphi^-)$.*

To prove the above lemma we need the following slightly refined version of lemma 4.6 from [5].

Lemma 5.8. *Let l^+, l^- be the upper and lower ends of the transport ray R_x respectively, $z \in R_x$ and $\varepsilon > 0$ be sufficiently small. If*

$$d_\Omega(l^+, L^-) \geq \delta \tag{14}$$

holds for some $\delta > 0$ and for all lower ray ends L^-, then the upper end of every transport ray R_y which intersects $B_\varepsilon(z)$ belongs to the cylinder with axis R_x and the crossection area $C\varepsilon^{n-1}/\delta^{n-1}$, where $C > 0$ is a constant depending only on the geometry of the problem.

PROOF OF LEMMA 5.7: It is enough to consider the restrictions of a to each of the ray clusters T_{pij}. These restrictions in fact are transport densities for the Monge-Kantorovich problem $MK(\varphi^+ \llcorner T_{pij}, \varphi^- \llcorner T_{pij})$. In the remaining part of the proof we will assume a ray cluster T_{pij} to be chosen and will be dealing with the respective restrictions without explicitly referring to the cluster.

Let $x \in T$ be such a point that both a and $f^\pm$ be continuous in the generalized (Lebesgue) sense $\mathcal{H}^1$-a.e. on R_x (a.e. $x \in T$ will suit for that purpose according to lemma 5.5). Suppose $\varphi^+ \in L^\infty(\bar{\Omega})$ (the remaining case $\varphi^- \in L^\infty(\bar{\Omega})$ is completely symmetric). For every $z \in T$ denote by $d(z)$ the distance from z to the upper end of R_z. Suppose now $z \in R_x$, $d(z) = \theta$. Since d is an upper semicontinuous function according to lemma 24 from [3], then for each sufficiently small $\varepsilon > 0$ one has $d(y) \leq 2\theta$ for all $y \in B_\varepsilon(z) \cap T$, where $B_\varepsilon(z) \subset \mathbb{R}^n$ stands for the ball with radius ε centered at z.

We also remark that there exists a $\delta > 0$ such that (14) holds for all lower ray ends L^-. In fact, since we are working within the ray cluster T_{pij}, we have $u(l^+) \leq p - 1/j$. If (14) were false, there would exist a sequence of lower ray ends $\{L^-_\nu\}$ such that $L^-_\nu \to l^+$. But $u(L^-_\nu) \geq p + 1/j$ by construction of the ray cluster, and hence, in view of continuity of u, one would have $u(l^+) \geq p + 1/j$ leading to a contradiction.

Applying lemma 5.8, one has that for $\varepsilon > 0$ sufficiently small all the upper ends of transport rays R_y which intersect $B_\varepsilon(z)$ necessarily belong to the piece $C_{\theta,\varepsilon}$ of the cylinder mentioned in the lemma, with the length 4θ and with z being the middle point of its axis. From (2) one derives

$$\mu(B_\varepsilon(z)) \leq 2\varepsilon \int_{C_{\theta,\varepsilon}\times\Omega} d\gamma = 2\varepsilon\varphi^+(C_{\theta,\varepsilon}).$$

Dividing the above relationship by ε^n and taking the limit as $\varepsilon \to 0$, we obtain $a(z) \leq C'\theta$ for some $C' > 0$, which shows the statement. $\square$

Unfortunately, nothing more can be said on behaviour of transport density at the ends of transport rays even if $\varphi^+ \in L^p(\Omega \setminus \Sigma)$ (resp. $\varphi^- \in L^p(\Omega \setminus \Sigma)$ when $1 \leq p < +\infty$. In general no vanishing phenomenon occurs in this case as the following example shows.

Example 5.5. Let $\Omega \subset \mathbb{R}^n$ stand for some big ball centered at the origin, $\Sigma = \emptyset$. Assume $\varphi^+ := |x|^\alpha \mathcal{L}^n \llcorner B_1(0)$ with $-n/p < \alpha < -1$. Then $\varphi^+ \in L^p(\Omega)$ for some $p < +\infty$. Let also

$$\varphi^- := \frac{n}{(n+\alpha)(L^n - l^n)}\mathcal{L}^n \llcorner \{x \in \mathbb{R}^n \, : \, l \leq |x| \leq L\},$$

where $l > 1$. Then one has $u(x) := -|x|$ and the mass transportation is radial. In particular, for $x \in B_1(0)$ one has

$$a(x) = \frac{1}{n+\alpha}|x|^{\alpha+1},$$

and hence $a(x) \not\to 0$ as $|x| \to 0$.

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