ANALYSIS OF A MODEL COUPLING VOLUME AND SURFACE PROCESSES IN THERMOVISCOELASTICITY

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Abstract. We focus on a highly nonlinear evolutionary abstract PDE system describing volume processes coupled with surfaces processes in thermoviscoelasticity, featuring the quasi-static momentum balance, the equation for the unidirectional evolution of an internal variable on the surface, and the equations for the temperature in the bulk domain and the temperature on the surface. A significant example of our system occurs in the modeling for the unidirectional evolution of the adhesion between a body and a rigid support, subject to thermal fluctuations and in contact with friction.

We investigate the related initial-boundary value problem, and in particular the issue of existence of *global-in-time* solutions, on an abstract level. This allows us to highlight the analytical features of the problem and, at the same time, to exploit the tight coupling between the various equations in order to deduce suitable estimates on (an approximation) of the problem.

Our existence result is proved by passing to the limit in a carefully tailored approximate problem, and by extending the obtained local-in-time solution by means of a refined prolongation argument.

1. Introduction

This paper tackles the analysis of a PDE system describing a class of models where volume and surface processes are coupled. Our main example, and the motivation for our study, stems from a specific PDE system modeling adhesive contact, with frictional effects, in thermoviscoelasticity.

A PDE system for contact with adhesion, friction, and thermal effects. Contact and delamination arise in many fields in solid mechanics: among others, we may mention here the application to (structural) adhesive materials in civil engineering, the investigation of earthquakes, and the study of layered composite structures within machine design and manufacturing. Indeed, the degradation of the adhesive substance between the various laminates leads to material failure. That is why, there is a rich literature on this kind of problems, both from the engineering and from the mathematical community: we refer to the monographs [15] and [21], and to the references in [4]–[9], for some survey.

In this paper, following up on the recent [9], we focus on a PDE system for frictional adhesion between a viscoelastic body, subject to thermal fluctuations, in contact with a *rigid support*. In [9] this system was derived, according to the laws of Thermomechanics, on the basis of the modeling approach proposed by M. Frémond (cf. [16, 17, 18]). Such approach has already been applied in the previous [4, 5, 6, 7] to the investigation of adhesive contact, both for an isothermal and for a temperature-dependent system, as well as in [8], for isothermal adhesive contact with frictional effects. The model we consider here, encompassing friction, adhesion, and thermal effects, was analyzed in [9] in the case where no *irreversibility* of the degradation of the adhesive substance is enforced. In the present contribution, we broaden our investigation by encompassing in the model this *unidirectionality* feature. As we are going to demonstrate in what follows, this extension of the model from [9] brings about substantial analytical difficulties.

Let us now get a closer look at the PDE system under investigation. In accord with Frémond's modeling ansatz, adhesion is described in terms of an internal variable χ which can be interpreted as a *surface damage paramater*: it accounts for the state of the bonds between the body and the support, on which some adhesive substance is present. The other state variables are the strain tensor (in a small-strain framework), related to

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the displacement vector $\mathbf{u}$, and the possibly different (absolute) temperatures ϑ and ϑ_s in the body and on the contact surface. We allow for two different temperatures in the bulk domain and on the contact surface because we are modeling a physical situation in which some adhesive substance may be present on the contact surface, with different thermomechanical properties in comparison to the material in the domain. The evolution of $(\vartheta, \vartheta_s, \mathbf{u}, \chi)$ is described by a rather complex PDE system, consisting of the quasi-static momentum balance (where the inertial contributions are neglected), a parabolic-type evolution equations for the temperature ϑ and ϑ_s, and a doubly nonlinear differential inclusion for χ. Denoting by $\Omega \subset \mathbb{R}^3$ the (sufficiently regular) domain occupied by the body and by $\Gamma_{\rm C}$ the part of the boundary $\partial\Omega$ on which the body may be in contact with the support, the system reads as follows

$$\partial_t \ln(\vartheta) - \mathrm{div}(\partial_t \mathbf{u}) - \mathrm{div}(\nabla g(\vartheta)) = h \qquad \text{in } \Omega \times (0,T), \tag{1.1a}$$

$$\nabla(g(\vartheta)) \cdot \mathbf{n} = 0 \qquad \text{in } (\partial\Omega \setminus \Gamma_{\rm C}) \times (0,T), \tag{1.1b}$$

$$\partial_{\mathbf{n}} \vartheta \in -k(\chi)(\vartheta - \vartheta_s) - \mathfrak{c}'(\vartheta - \vartheta_s)\partial I_{(-\infty,0]}(u_{\rm N})|\partial_t \mathbf{u}_{\rm T}| \qquad \text{in } \Gamma_{\rm C} \times (0,T), \tag{1.1c}$$

$$\partial_t \ln(\vartheta_s) - \partial_t \lambda(\chi) - \mathrm{div}(\nabla f(\vartheta_s)) \in k(\chi)(\vartheta - \vartheta_s) + \mathfrak{c}'(\vartheta - \vartheta_s)\partial I_{(-\infty,0]}(u_{\rm N})|\partial_t \mathbf{u}_{\rm T}| \quad \text{in } \Gamma_{\rm C} \times (0,T), \tag{1.1d}$$

$$\nabla(f(\vartheta_s)) \cdot \mathbf{n}_s = 0 \qquad \text{in } \partial\Gamma_{\rm C} \times (0,T), \tag{1.1e}$$

$$-\mathrm{div}\boldsymbol{\sigma} = \mathbf{f} \ \text{ with } \ \boldsymbol{\sigma} = K_e \varepsilon(\mathbf{u}) + K_v \varepsilon(\partial_t \mathbf{u}) + \vartheta \mathbf{1} \quad \text{in } \Omega \times (0,T), \tag{1.1f}$$

$$\mathbf{u} = \mathbf{0} \quad \text{in } \Gamma_{\rm D} \times (0,T), \quad \boldsymbol{\sigma}\mathbf{n} = \mathbf{g} \quad \text{in } \Gamma_{\rm N} \times (0,T), \tag{1.1g}$$

$$\sigma_{\rm N} \in -\chi u_{\rm N} - \partial I_{(-\infty,0]}(u_{\rm N}) \quad \text{in } \Gamma_{\rm C} \times (0,T), \tag{1.1h}$$

$$\boldsymbol{\sigma}_{\rm T} \in -\chi \mathbf{u}_{\rm T} - \mathfrak{c}(\vartheta - \vartheta_s)\partial I_{(-\infty,0]}(u_{\rm N})\mathbf{d}(\partial_t \mathbf{u}) \quad \text{in } \Gamma_{\rm C} \times (0,T), \tag{1.1i}$$

$$\partial_t \chi + \partial I_{(-\infty,0]}(\partial_t \chi) - \Delta\chi + \partial I_{[0,1]}(\chi) + \gamma'(\chi) \ni -\lambda'(\chi)(\vartheta_s - \vartheta_{\rm eq}) - \frac{1}{2}|\mathbf{u}|^2 \quad \text{in } \Gamma_{\rm C} \times (0,T), \tag{1.1j}$$

$$\partial_{\mathbf{n}_s} \chi = 0 \quad \text{in } \partial\Gamma_{\rm C} \times (0,T). \tag{1.1k}$$

In system (1.1), we have used the following notation: $\mathbf{n}$ is the outward unit normal to $\partial\Omega$, for which we suppose $\partial\Omega = \overline{\Gamma_{\rm D}} \cup \overline{\Gamma_{\rm N}} \cup \overline{\Gamma_{\rm C}}$, with $\Gamma_{\rm D}$ ($\Gamma_{\rm N}$, resp.) the Dirichlet (the Neumann) part of the boundary where zero displacement (a fixed traction) is prescribed. As for $\Gamma_{\rm C}$, we require that it is also sufficiently smooth (and denote by $\mathbf{n}_s$ the outward unit normal to $\partial\Gamma_{\rm C}$) and *flat* (cf. condition (2.1) below); for simplicity we shall write v, in place of $v_{|\Gamma_{\rm C}}$, for the trace on $\Gamma_{\rm C}$ of a function v defined in Ω. We also adopt the following convention: given a vector $\mathbf{v} \in \mathbb{R}^3$, we denote by $v_{\rm N}$ and $\mathbf{v}_{\rm T}$ its normal component and its tangential part, i.e. $v_{\rm N} := \mathbf{v} \cdot \mathbf{n}$ and $\mathbf{v}_{\rm T} := \mathbf{v} - v_{\rm N}\mathbf{n}$. Analogously, the normal component and the tangential part of the Cauchy stress tensor $\boldsymbol{\sigma}$ (while $\varepsilon(\mathbf{u})$ is the small-strain tensor), are denoted by $\sigma_{\rm N}$ and $\boldsymbol{\sigma}_{\rm T}$, with $\sigma_{\rm N} := \boldsymbol{\sigma}\mathbf{n} \cdot \mathbf{n}$ and $\boldsymbol{\sigma}_{\rm T} := \boldsymbol{\sigma}\mathbf{n} - \sigma_{\rm N}\mathbf{n}$. Finally, the multivalued operator $\partial I_C : \mathbb{R} \rightrightarrows \mathbb{R}$ (with C the interval $[0,1]$ or the half-line $(-\infty, 0]$) is the subdifferential (in the sense of convex analysis) of the indicator function of the convex set C.

While referring the reader to [9, Sec. 2] for the rigorous derivation of system (1.1), let us now briefly comment on the features, and the meaning, of the single equations. The momentum balance (1.1f), where the elasticity and viscosity tensors K_e and K_v are positive-definite and satisfy suitable symmetry conditions and $\mathbf{f}$ is a given volume force, is supplemented by the boundary conditions (1.1g) on $\Gamma_{\rm D}$ and $\Gamma_{\rm N}$ (with $\mathbf{g}$ given), and by (1.1h)–(1.1i) on the contact surface $\Gamma_{\rm C}$. Observe that (1.1h) can be recast in complementarity form as

$$u_{\rm N} \le 0, \quad \sigma_{\rm N} + \chi u_{\rm N} \le 0, \quad u_{\rm N}(\sigma_{\rm N} + \chi u_{\rm N}) = 0 \quad \text{in } \Gamma_{\rm C} \times (0,T). \tag{1.2}$$

For $\chi = 0$ (i.e. no adhesion), these conditions reduce to the classical Signorini conditions for unilateral contact. Instead, for $0 < \chi \le 1$ (1.1h) allows for $\sigma_{\rm N}$ positive, namely the action of the adhesive substance on $\Gamma_{\rm C}$ prevents separation when a tension is applied. In (1.1i), $\mathbf{d} : \mathbb{R}^3 \rightrightarrows \mathbb{R}^3$ is the subdifferential of the functional $\Psi : \mathbb{R}^3 \to [0, +\infty)$ given by $\Psi(\mathbf{v}) = |\mathbf{v}_{\rm T}|$, viz.

$$\mathbf{d}(\mathbf{v}) = \begin{cases} \frac{\mathbf{v}_{\rm T}}{|\mathbf{v}_{\rm T}|} & \text{if } \mathbf{v}_{\rm T} \neq \mathbf{0} \\ \{\mathbf{w}_{\rm T} \ : \ \mathbf{w} \in \overline{B}_1\} & \text{if } \mathbf{v}_{\rm T} = \mathbf{0}, \end{cases}$$

with $\overline{B}_1$ the closed unit ball in $\mathbb{R}^3$. Therefore, in view of (1.1h), condition (1.1i) rephrases as

$$\begin{aligned} &|\boldsymbol{\sigma}_{\mathrm{T}} + \chi \mathbf{u}_{\mathrm{T}}| \leq \mathfrak{c}(\vartheta - \vartheta_s)|\sigma_{\mathrm{N}} + \chi u_{\mathrm{N}}| && \text{in } \Gamma_{\mathrm{C}} \times (0,T), \\ &|\boldsymbol{\sigma}_{\mathrm{T}} + \chi \mathbf{u}_{\mathrm{T}}| < \mathfrak{c}(\vartheta - \vartheta_s)|\sigma_{\mathrm{N}} + \chi u_{\mathrm{N}}| \Longrightarrow \partial_t \mathbf{u}_{\mathrm{T}} = \mathbf{0} && \text{in } \Gamma_{\mathrm{C}} \times (0,T), \\ &|\boldsymbol{\sigma}_{\mathrm{T}} + \chi \mathbf{u}_{\mathrm{T}}| = \mathfrak{c}(\vartheta - \vartheta_s)|\sigma_{\mathrm{N}} + \chi u_{\mathrm{N}}| \Longrightarrow \exists\, \nu \geq 0 : \ \partial_t \mathbf{u}_{\mathrm{T}} = -\nu(\boldsymbol{\sigma}_{\mathrm{T}} + \chi \mathbf{u}_{\mathrm{T}}) && \text{in } \Gamma_{\mathrm{C}} \times (0,T), \end{aligned} \tag{1.3}$$

which generalize the *dry friction* Coulomb law, to the case when adhesion effects are taken into account. Note that the *positive* function $\mathfrak{c}$ in (1.3) has the meaning of a *friction coefficient*.

The temperature equations (1.1a) and (1.1d) (where g is related to the heat flux in the bulk, h is a given heat source, and k and λ are suitably smooth functions), feature the singular terms $\ln(\vartheta)$ and $\ln(\vartheta_s)$, which originate from deriving (1.1a) and (1.1d) from the *entropy-balance*, in place of the internal energy balance, equations in the bulk domain and on the contact surface. These terms ensure the strict positivity of ϑ and ϑ_s, which is a necessary property in view of the thermodynamical consistency of the model. Equation (1.1a) for ϑ is coupled to the quasi-static momentum balance though the term $-\operatorname{div}(\partial_t \mathbf{u})$, related to the presence of a thermal expansion contribution $\vartheta \mathbf{1}$ in the stress tensor $\boldsymbol{\sigma}$ (1.1f). The coupling to the equations for ϑ_s and χ occurs through the Robin type boundary condition (1.1c), where the frictional contribution $\mathfrak{c}'(\vartheta - \vartheta_s)\partial I_{(-\infty,0]}(u_{\mathrm{N}})|\partial_t \mathbf{u}_{\mathrm{T}}|$ features as a source of heat on the contact surface Γ_{C}. Accordingly, this term also appears on the right-hand side of the equation for ϑ_s, where the function f, in terms of which the heat flux on the contact surface is defined, will be chosen in a suitable way, cf. (1.7) below.

The evolution equation (1.1j) for the internal variable χ has a doubly nonlinear character, in that it features the subdifferential operators $\partial I_{(-\infty,0]},\ \partial I_{[0,1]} : \mathbb{R} \rightrightarrows \mathbb{R}$. The former acts on $\partial_t \chi$, thereby encompassing in the model the constraint $\partial_t \chi \leq 0$, i.e. that the degradation process of the adhesive on Γ_{C} is *irreversible*. The latter subdifferential operator enforces the constraint that χ takes values in the (physically admissible) interval $[0,1]$. The function γ' is a smooth (possibly) non-monotone perturbation of the monotone term $\partial I_{[0,1]}$; it derives from some (possibly) non-convex contribution to the free energy functional. Finally, ϑ_{eq} is a critical temperature.

Analytical difficulties. The analysis of system (1.1) presents difficulties of various type:

1) The coupling of *bulk* and *(contact) surface* equations requires sufficient regularity of the bulk variables ϑ and $\mathbf{u}$ for their traces on Γ_{C} to make sense.
2) On the other hand, the highly nonlinear character of the equations (due to the presence of several singular and multivalued operators), as well as the *mixed* boundary conditions for the bulk variables $\mathbf{u}$ and ϑ, do not allow for elliptic regularity estimates which could enhance the spatial regularity of $\mathbf{u}$, $\partial_t \mathbf{u}$, and ϑ. In particular, we are not in the position to get H^2-regularity for the bulk variables.
3) A further obstacle is the *singular* character of the temperature equations (1.6a) and (1.6c), due to the terms $\partial_t \ln(\vartheta)$ and $\partial_t \ln(\vartheta_s)$. Because of these terms, the basic energy estimates on system (1.1) leads to a very weak time-regularity of ϑ and ϑ_s. As we will see, it is possible to improve such regularity, for the sole ϑ_s, only through a series of enhanced estimates, which in turn rely on a precise form for the function f, cf. (1.7) below.
4) The *doubly nonlinear* character of equation (1.1j) for χ, with the two *unbounded* subdifferential terms $\partial I_{(\infty,0]}(\partial_t \chi)$ and $\partial I_{[0,1]}(\chi)$.
5) A major analytical problem is brought about by the coupling of unilateral contact and the dry friction Coulomb law in the model. This leads to the presence of the multivalued operator $\partial I_{(-\infty,0]}$ in the coupling terms between the equations for ϑ, ϑ_s, and $\mathbf{u}$, and to the product of two subdifferential terms in (1.1i).

Let us stress that contact problems with friction involve severe, and unresolved, difficulties even in the case without adhesion. That is why, as done in many other works (cf. the references in [8]), starting from the pioneering paper [14] by Duvaut, we have to regularize (1.1i) by resorting to a *nonlocal version* of the Coulomb law. More precisely, we shall replace the nonlinearity in (1.1i) involving friction by the term

$$\mathfrak{c}(\vartheta - \vartheta_s)|\mathcal{R}(\partial I_{(-\infty,0]}(u_{\mathrm{N}}))|\mathbf{d}(\partial_t \mathbf{u}), \tag{1.4}$$

and, correspondingly, the term $\mathfrak{c}'(\vartheta-\vartheta_s)\partial I_{(-\infty,0]}(u_{\rm N})|\partial_t \mathbf{u}_{\rm T}|$ in (1.1c) and (1.1d) by

$$\mathfrak{c}'(\vartheta-\vartheta_s)|\mathcal{R}(\partial I_{(-\infty,0]}(u_{\rm N}))||\partial_t \mathbf{u}_{\rm T}|. \tag{1.5}$$

In (1.4) and (1.5), $\mathcal{R}$ is a regularization operator with suitable properties, cf. Hypothesis 2.4. The regularized friction law resulting from the replacement (1.4) in (1.1i) can be interpreted as taking into account nonlocal interactions on the contact surface.

We now dwell on the difficulties attached to the doubly nonlinear character of (1.1j), which is due to the inclusion of *unidirectionality* in the model. In order to prove the existence of a solution χ to a reasonably strong formulation of (1.1j), featuring two selections $\zeta \in \partial I_{(\infty,0]}(\partial_t\chi)$ and $\xi \in \partial I_{[0,1]}(\chi)$, it is essential to estimate the terms $\partial I_{(\infty,0]}(\partial_t\chi)$ and $\partial I_{[0,1]}(\chi)$ (formally written as single-valued) *separately* in some suitable function space, in fact $L^2(0,T;L^2(\Gamma_{\rm C}))$. Starting from the paper [3] on the analysis of a model for irreversible phase transition, it has been well known that such an estimate can be achieved by testing (1.1j) by the (formally written) term $\partial_t(-\Delta\chi+\partial I_{[0,1]}(\chi))$ and employing monotonicity arguments. The related calculations also involve an integration by parts in time on the right-hand side of (1.1j), which in turn requires suitable time-regularity for ϑ_s.

That is why, in order to carry out the crucial estimate for handling of the terms $\partial I_{(\infty,0]}(\partial_t\chi)$ and $\partial I_{[0,1]}(\chi)$ in (1.1j), we will need to enhance the time-regularity of ϑ_s. This can be done through a series of estimates, which partially rely on choosing in (1.1d) a function f tailored to the logarithmic nonlinearity therein.

It seems to us that the structure of these estimates, and the technical reasons underlying our hypotheses on the various nonlinearities of the system, can be highlighted upon developing our analysis on an *abstract* version of system (1.1).

A generalization of system (1.1)**.** Hereafter, we shall address the following PDE system coupling a volume process in a domain $\Omega\subset\mathbb{R}^3$, with a surface process occurring on a portion $\Gamma_{\rm C}$ of the boundary of Ω, which fulfills $\partial\Omega=\overline{\Gamma_{\rm D}}\cup\overline{\Gamma_{\rm N}}\cup\overline{\Gamma_{\rm C}}$. The surface process is described by a suitable internal variable χ, and thermal effects, in the bulk and on the surface, are accounted for through the temperature variables ϑ and ϑ_s. Accordingly, we have

$$\partial_t(L(\vartheta))-{\rm div}(\partial_t\mathbf{u})-{\rm div}(\nabla g(\vartheta))=h \qquad \text{in } \Omega\times(0,T), \tag{1.6a}$$

$$\nabla g(\vartheta)\cdot{\rm n}\begin{cases}=0 & \text{in } (\partial\Omega\setminus\Gamma_{\rm C})\times(0,T),\\ \in -k(\chi)(\vartheta-\vartheta_s)-\mathfrak{c}'(\vartheta-\vartheta_s)\Psi(\partial_t\mathbf{u})|\mathcal{R}(\partial\Phi(\mathbf{u}))| & \text{in } \Gamma_{\rm C}\times(0,T),\end{cases} \tag{1.6b}$$

$$\partial_t(\ell(\vartheta_s))-\partial_t(\lambda(\chi))-{\rm div}(\nabla f(\vartheta_s))\in k(\chi)(\vartheta-\vartheta_s)+\mathfrak{c}'(\vartheta-\vartheta_s)\Psi(\partial_t\mathbf{u})|\mathcal{R}(\partial\Phi(\mathbf{u}))| \quad \text{in } \Gamma_{\rm C}\times(0,T), \tag{1.6c}$$

$$\nabla f(\vartheta_s)\cdot\mathbf{n}_s=0 \qquad \text{in } \partial\Gamma_{\rm C}\times(0,T), \tag{1.6d}$$

$$-{\rm div}\boldsymbol{\sigma}=\mathbf{f} \quad \text{with} \quad \boldsymbol{\sigma}=K_e\varepsilon(\mathbf{u})+K_v\varepsilon(\partial_t\mathbf{u})+\vartheta\mathbf{1} \quad \text{in } \Omega\times(0,T), \tag{1.6e}$$

$$\mathbf{u}=\mathbf{0} \quad \text{in } \Gamma_{\rm D}\times(0,T), \quad \boldsymbol{\sigma}\mathbf{n}=\mathbf{g} \quad \text{in } \Gamma_{\rm N}\times(0,T), \tag{1.6f}$$

$$\boldsymbol{\sigma}\mathbf{n}+\chi\mathbf{u}+\partial\Phi(\mathbf{u})+\mathfrak{c}(\vartheta-\vartheta_s)\partial\Psi(\partial_t\mathbf{u})|\mathcal{R}(\partial\Phi(\mathbf{u}))|\ni\mathbf{0} \quad \text{in } \Gamma_{\rm C}\times(0,T), \tag{1.6g}$$

$$\partial_t\chi+\partial\widehat{\rho}(\partial_t\chi)-\Delta\chi+\partial\widehat{\beta}(\chi)+\gamma'(\chi)\ni-\lambda'(\chi)(\vartheta_s-\vartheta_{\rm eq})-\frac{1}{2}|\mathbf{u}|^2 \quad \text{in } \Gamma_{\rm C}\times(0,T), \tag{1.6h}$$

$$\partial_{\mathbf{n}_s}\chi=0 \quad \text{in } \partial\Gamma_{\rm C}\times(0,T). \tag{1.6i}$$

Observe that the temperature equations (1.6a) and (1.6c) are a generalization of the "concrete" equations (1.1a) and (1.1d). The logarithms therein have been replaced by two (possibly different) maximal monotone (single-valued) operators L and ℓ fulfilling suitable conditions. The function g is a strictly increasing, bi-Lipschitz, and otherwise general. Instead, f depends on the choice of ℓ, as it is defined by

$$f(\vartheta_s):=\int_0^{\vartheta_s}\frac{1}{\ell'(r)}\,{\rm d}r\,. \tag{1.7}$$

Admissible choices are, for instance

$$(\ell(\vartheta_s)=\ln(\vartheta_s), f(\vartheta_s)=\vartheta_s^2), \qquad (\ell(\vartheta_s)=\vartheta_s, f(\vartheta_s)=\vartheta_s).$$

Nonetheless, in the second case the strict positivity of the temperature is no longer directly ensured by the form of ℓ like in the first case.

Note moreover that the terms $\partial I_{(-\infty,0]}(u_{\mathrm{N}})$ and $\mathbf{d}(\partial_t \mathbf{u})$ in system (1.1) have been replaced by the subdifferentials $\partial\Phi(\mathbf{u})$ and $\partial\Psi(\partial_t \mathbf{u})$ in (1.6g), and accordingly in (1.6b) and (1.6c) (with $\mathcal{R}$ the regularization operator used in the analysis of frictional problems). Here, Φ and Ψ are (possibly nonsmooth) positive, lower semicontinuous, and convex functionals, and in addition Ψ is positively homogeneous of degree 1, i.e. $\Psi(l\mathbf{v}) = l\Psi(\mathbf{v})$ for all $l \geq 0$ and $\mathbf{v} \in \mathbb{R}^3$. It turns out that the crucial requirement for tackling the simultaneous presence of the two terms $\partial\Phi(\mathbf{u})$ and $\partial\Psi(\partial_t \mathbf{u})$ in (1.6g) is that

$$\partial\Phi(\mathbf{u}) \text{ and } \partial\Psi(\mathbf{v}) \text{ are orthogonal for all } \mathbf{u},\, \mathbf{v} \in \mathbb{R}^3,$$

cf. Hypothesis 2.3, which is obviously fulfilled in the case of system (1.1).

Finally, in (1.6h) $\widehat{\rho}$ and $\widehat{\beta}$ are two convex and lower semicontinuous functionals, such that $\mathrm{dom}(\widehat{\beta}) \subset [0, +\infty)$ in such a way as to ensure the positivity of the internal variable χ, which is also crucial for the analysis of system (1.6) and guarantees the physical consistency of the phase variable.

An existence result for system (1.6)**.** The main result of this paper, Theorem 2.7, states the existence of solutions to the (Cauchy problem for a) variational formulation of (1.6), in an appropriate functional framework which reflects the energy estimates for this system. This variational formulation is given in Section 2, where all the hypotheses on the various nonlinearities of the system and on the problem data are collected.

The proof follows by passing to the limit in a carefully devised approximate system, where some of the multivalued subdifferential terms featured in (1.6) are replaced by their Yosida regularizations. In particular, the possibly singular functions L and ℓ are regularized, and in addition the viscous terms $\varepsilon\partial_t\vartheta$ and $\varepsilon\partial_t\vartheta_s$ are included into the left-hand sides of (1.6a) and (1.6c), thereby enhancing the time-regularity of the ϑ- and ϑ_s-components of the approximate solutions. That is why, this procedure requires a technically delicate construction of approximate initial data for ϑ and ϑ_s.

In Section 3 we develop the set-up of the approximate problem, we state its variational formulation and prove a *local-in-time* existence result for the approximate system via a Schauder fixed point argument. Then, we derive in Section 4 a series of a priori estimates on the approximate solutions. Relying on them, in Section 5 we perform the passage to the limit with the approximate solutions via refined compactness and lower semicontinuity arguments. We then obtain in Theorem 5.1 the (still local-in-time) existence of solutions to (1.6). In this passage to the limit we have to deal with a significant difficulty stemming from the coupling between thermal and frictional effects in the model. This is the dependence of the friction coefficient $\mathfrak{c}$ on the thermal gap $\vartheta - \vartheta_s$. To tackle the passage to the limit in the approximation of the terms $\mathfrak{c}'(\vartheta - \vartheta_s)|\mathcal{R}(\partial\Phi(\mathbf{u}))|\Psi(\partial_t \mathbf{u})$ in (1.6b) and (1.6c), and $\mathfrak{c}(\vartheta - \vartheta_s)|\mathcal{R}(\partial\Phi(\mathbf{u}))|\partial\Psi(\partial_t \mathbf{u})$ in (1.6g), it is essential to prove strong compactness for (the sequences approximating) ϑ and ϑ_s in suitable spaces. The key step for ϑ is to derive an estimate in $\mathrm{BV}(0,T;W^{1,3+\epsilon}(\Omega)')$ for all $\epsilon > 0$, which enables us to apply a suitable version of the Lions-Aubin compactness theorem generalized to BV spaces. As for ϑ_s, exploiting condition (1.7), we are indeed able to obtain an a priori bound for ϑ_s in $H^1(0,T;L^2(\Gamma_{\mathrm{C}}))$. As previously mentioned, this enhanced time-regularity estimate for ϑ_s plays a crucial role in the derivation of estimates for the terms $\partial\widehat{\rho}(\partial_t\chi)$ and $\partial\widehat{\beta}(\chi)$ in (1.6h).

Remark 1.1. This mismatch in the time-regularity properties of ϑ and ϑ_s, as well as the fact that (as we are dealing with possibly different thermal properties of the substances in the domain and on the contact surface) we can allow for different choices of the functions L and ℓ, highlights that the temperature equations (1.6a) and (1.6c) have a substantially different character. In fact, in (1.6a) the function g can be fairly general, whereas in (1.6c) the function f needs to be chosen of the form (1.7), in order to allow for the enhanced time-regularity estimate for ϑ_s. Such an estimate could not be carried out on equation (1.6a), due to the mixed boundary conditions (1.6b), which are also responsible for the low spatial regularity of ϑ.

The last step in our existence proof consists in the extension of the local-in-time solution to the Cauchy problem for system (1.1). This prolongation procedure follows the lines of an extension argument from [4].

Indeed, for technical reasons that shall be expounded at the beginning of Section 6, it is necessary to extend the local-in-time solution, whose existence is guaranteed by Theorem 5.1, along with its approximability properties. This makes the prolongation argument rather complex, that is why we have devoted to it the whole Sec. 6. Finally, in the Appendix we collect a series of auxiliary results, among which some lemmas addressing the approximation of the initial data for ϑ and ϑ_s and the properties of the functions regularizing the nonlinearities of the system.

2. Main result

Before stating the analytical problem we are solving and the corresponding existence result, we first set up the notation and the assumptions.

Throughout the paper we shall assume that

$$\begin{gathered}\Omega \text{ is a bounded Lipschitz domain in } \mathbb{R}^3, \text{ with}\\ \partial\Omega = \overline{\Gamma_{\rm D}} \cup \overline{\Gamma_{\rm N}} \cup \overline{\Gamma_{\rm C}}, \quad \Gamma_{\rm D}, \Gamma_{\rm N}, \Gamma_{\rm C}, \text{ open disjoint subsets in the relative topology of } \partial\Omega, \text{ such that}\\ \mathcal{H}^2(\Gamma_{\rm D}), \mathcal{H}^2(\Gamma_{\rm C}) > 0, \qquad \text{and } \Gamma_{\rm C} \subset \mathbb{R}^2 \text{ a sufficiently smooth } \textit{flat} \text{ surface.}\end{gathered} \tag{2.1}$$

More precisely, by *flat* we mean that $\Gamma_{\rm C}$ is a subset of a hyperplane of $\mathbb{R}^3$ and $\mathcal{H}^2(\Gamma_{\rm C}) = \mathcal{L}^2(\Gamma_{\rm C})$, $\mathcal{L}^d$ and $\mathcal{H}^d$ denoting the d-dimensional Lebesgue and Hausdorff measures, respectively. As for smoothness, we require that $\Gamma_{\rm C}$ has a C^2-boundary.

Notation 2.1. Given a Banach space X, we denote by $\langle\cdot,\cdot\rangle_X$ the duality pairing between its dual space X' and X itself, and by $\|\cdot\|_X$ the norm in X. In particular, we shall use the following short-hand notation for function spaces

$$\begin{gathered}H := L^2(\Omega), \quad V := H^1(\Omega), \quad \mathbf{H} := L^2(\Omega;\mathbb{R}^3), \quad \mathbf{V} := H^1(\Omega;\mathbb{R}^3),\\ H_{\Gamma_{\rm C}} := L^2(\Gamma_{\rm C}), \quad V_{\Gamma_{\rm C}} := H^1(\Gamma_{\rm C}), \quad Y_{\Gamma_{\rm C}} := H^{1/2}_{00,\Gamma_{\rm D}}(\Gamma_{\rm C}),\\ \mathbf{W} := \{\mathbf{v} \in \mathbf{V} \,:\, \mathbf{v} = \mathbf{0} \text{ a.e. on } \Gamma_{\rm D}\}, \quad \mathbf{H}_{\Gamma_{\rm C}} := L^2(\Gamma_{\rm C};\mathbb{R}^3), \quad \mathbf{Y}_{\Gamma_{\rm C}} := H^{1/2}_{00,\Gamma_{\rm D}}(\Gamma_{\rm C};\mathbb{R}^3),\end{gathered}$$

where we recall that

$$H^{1/2}_{00,\Gamma_{\rm D}}(\Gamma_{\rm C}) = \Big\{ w \in H^{1/2}(\Gamma_{\rm C}) \,:\, \exists\, \tilde{w} \in H^{1/2}(\Gamma) \text{ with } \tilde{w} = w \text{ in } \Gamma_{\rm C},\ \tilde{w} = 0 \text{ in } \Gamma_{\rm D} \Big\}$$

and $H^{1/2}_{00,\Gamma_{\rm D}}(\Gamma_{\rm C};\mathbb{R}^3)$ is analogously defined. We will also use the space $H^{1/2}_{00,\Gamma_{\rm D}}(\Gamma_{\rm N};\mathbb{R}^3)$. The space $\mathbf{W}$ is endowed with the natural norm induced by $\mathbf{V}$. We will make use of the operator

$$A : V_{\Gamma_{\rm C}} \to V'_{\Gamma_{\rm C}} \qquad \langle A\chi, w\rangle_{V_{\Gamma_{\rm C}}} := \int_{\Gamma_{\rm C}} \nabla\chi\cdot\nabla w \,\mathrm{d}x \quad \text{for all } \chi,\, w \in V_{\Gamma_{\rm C}} \tag{2.2}$$

and of the notation

$$m(w) := \frac{1}{\mathcal{L}^d(A)} \int_A w \,\mathrm{d}x \quad \text{for } w \in L^1(A). \tag{2.3}$$

Linear viscoelasticity. We are in the framework of linear viscoelasticity theory (see e.g. [9] for some more details). In particular, we prescribe that the fourth-order tensors K_e and K_v (denoting the elasticity and the viscosity tensor, respectively) are symmetric and positive definite. Moreover, we require that they are uniformly bounded, in such a way that the following bilinear symmetric forms $a, b : \mathbf{W} \times \mathbf{W} \to \mathbb{R}$, defined by

$$a(\mathbf{u},\mathbf{v}) := \int_\Omega \varepsilon(\mathbf{u}) K_e \varepsilon(\mathbf{v}) \,\mathrm{d}x \qquad\qquad b(\mathbf{u},\mathbf{v}) := \int_\Omega \varepsilon(\mathbf{u}) K_v \varepsilon(\mathbf{v}) \,\mathrm{d}x \quad \text{for all } \mathbf{u},\mathbf{v} \in \mathbf{W},$$

are continuous. In particular, we have

$$\exists\, \bar{C} > 0 : \ |a(\mathbf{u},\mathbf{v})| + |b(\mathbf{u},\mathbf{v})| \le \bar{C}\|\mathbf{u}\|_{\mathbf{W}}\|\mathbf{v}\|_{\mathbf{W}} \quad \text{for all } \mathbf{u},\mathbf{v} \in \mathbf{W}. \tag{2.4}$$

Moreover, since $\Gamma_{\rm D}$ has positive measure, by Korn's inequality we deduce that $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are $\mathbf{W}$-elliptic, i.e., there exist $C_a, C_b > 0$ such that

$$a(\mathbf{u},\mathbf{u}) \geq C_a \|\mathbf{u}\|^2_{\mathbf{W}}\,, \qquad b(\mathbf{u},\mathbf{u}) \geq C_b \|\mathbf{u}\|^2_{\mathbf{W}} \qquad \text{for all } \mathbf{u} \in \mathbf{W}. \tag{2.5}$$

2.2. **Assumptions.** In order to tackle the analysis of the PDE system (1.6), we require the following.

Hypothesis 2.1. *For the functions L and ℓ in* (1.6a) *and* (1.6c) *we assume that*

$$L : \mathrm{D}(L) \subset \mathbb{R} \to \mathbb{R} \text{ maximal monotone, with } \mathrm{D}(L) \text{ a (possibly unbounded) open interval} \tag{2.6a}$$

$$L \in \mathrm{C}^1(\mathrm{D}(L)) \text{ and } \frac{1}{L'} \in \mathrm{C}^{0,1}(\overline{\mathrm{D}(L)}). \tag{2.6b}$$

Moreover, denoting by J a primitive of L, we impose that the Fenchel-Moreau convex conjugate J^* of J (recall that its derivative coincides with the inverse function L^{-1}), fulfills the following coercivity *condition*

$$\exists\, C_1,\ C_2 > 0 \qquad \forall\, \vartheta \in \mathrm{D}(L)\ :\quad J^*(L(\vartheta)) \geq C_1|\vartheta| - C_2\,. \tag{2.6c}$$

Analogously, we assume for ℓ

$$\ell : \mathrm{D}(\ell) \subset \mathbb{R} \to \mathbb{R} \text{ maximal monotone, with } \mathrm{D}(\ell) \text{ a (possibly unbounded) open interval} \tag{2.7a}$$

$$\ell \in \mathrm{C}^1(\mathrm{D}(\ell)) \text{ and } \frac{1}{\ell'} \in \mathrm{C}^{0,1}(\overline{\mathrm{D}(\ell)}), \tag{2.7b}$$

as well as, again, the coercivity *condition*

$$\exists\, c_1,\ c_2 > 0 \qquad \forall\, \vartheta_s \in \mathrm{D}(\ell)\ :\ j^*(\ell(\vartheta_s)) \geq c_1|\vartheta_s| - c_2\,, \tag{2.7c}$$

where j^* is the Fenchel-Moreau convex conjugate of j, j denoting a primitive of ℓ.

A straightforward consequence of (2.6) and of (2.7) is that

$$\begin{cases} L'(x) > 0 & \text{for all } x \in \mathrm{D}(L), \\ \ell'(x) > 0 & \text{for all } x \in \mathrm{D}(\ell), \end{cases} \qquad \text{hence } L \text{ and } \ell \text{ are invertible.}$$

Furthermore, it is not restrictive to suppose that

$$0 \in \mathrm{D}(J) \quad \text{with } J(0) = 0 \tag{2.8}$$

(the latter relation is trivially obtained with a translation argument), and the same for j. Since $\overline{\mathrm{D}(\ell)} = \overline{\mathrm{D}(j)}$, this in particular implies that

$$0 \in \overline{\mathrm{D}(\ell)}\,, \tag{2.9}$$

which will be convenient for the definition of f and f_ε later on.

Example 2.2. An example for L in accord with conditions (2.6a)–(2.6c) is

$$L(\vartheta) = \ln(\vartheta) \qquad \forall\, \vartheta \in \mathrm{D}(L) = (0,+\infty)\,. \tag{2.10}$$

In this case, with, e.g. $J(\vartheta) = \vartheta(\ln(\vartheta) - 1)$ for all $\vartheta \in (0,+\infty)$ as primitive of L, we see that $J^*(w) = e^w = L^{-1}(w)$ for all $w \in \mathbb{R}$, and (2.6c) is satisfied. This choice of L is particularly meaningful from a modeling viewpoint, since it enforces that $\vartheta > 0$, in accord with its interpretation as the absolute temperature. Clearly, $\ell(\vartheta_s) = \ln(\vartheta_s)$ is also an admissible choice for ℓ.

As already mentioned in the introduction, let us point out that, in fact, for our analysis we do not need the positivity of ϑ and ϑ_s (namely, that $\mathrm{D}(L)$, $\mathrm{D}(\ell) \subset (0,+\infty)$). Hence, other admissible choices for L and ℓ are

$$L(\vartheta) = \vartheta, \qquad \ell(\vartheta_s) = \vartheta_s \qquad \text{with } \mathrm{D}(L) = \mathrm{D}(\ell) = \mathbb{R}. \tag{2.11}$$

Let us also stress that, in system (1.6) we can in principle combine two distinct choices for L and ℓ.

Hypothesis 2.2. *As far as the functions g and f are concerned, we impose that*

$$g \in \mathrm{C}^1(\mathbb{R}) \text{ and } \exists\, c_3, c_4 > 0 \ : \ \forall\, x \in \mathbb{R} \ \ c_3 \leq g'(x) \leq c_4. \tag{2.12}$$

As for f, as previously mentioned we require that it is is the primitive of $\frac{1}{\ell'}$; in view of (2.9)*, we set*

$$f(x) = \int_0^x \frac{1}{\ell'(s)}\, \mathrm{d}s\,, \ \forall\, x \in \overline{\mathrm{D}(\ell)}\,. \tag{2.13}$$

Example 2.3. Clearly, the function f depends on the choice for ℓ. For example,

$$\begin{cases} \ell(\vartheta_s) = \vartheta_s & \Rightarrow \quad f(\vartheta_s) = \vartheta_s, \\ \ell(\vartheta_s) = \ln(\vartheta_s) & \Rightarrow \quad f(\vartheta_s) = \vartheta_s^2, \end{cases}$$

Hypothesis 2.3 (The subdifferential operators in the momentum balance equation)**.** *We suppose that*

$$\Psi : \mathbb{R}^3 \to [0, +\infty) \text{ is convex, non-degenerate, and positively 1-homogeneous} \tag{2.14}$$

i.e. Ψ satisfies

$$\Psi(\mathbf{v}) > 0 \text{ if } \mathbf{v} \neq 0, \qquad \Psi(l\mathbf{v}) = l\Psi(\mathbf{v}) \text{ for all } l \geq 0 \text{ and } \mathbf{v} \in \mathbb{R}^3,$$

(in fact, under positive homogeneity of degree 1, sublinearity is equivalent to convexity). As for the function Φ, we assume that

$$\Phi : \mathbb{R}^3 \to [0, +\infty] \text{ is proper, convex and lower semicontinuous, with } \Phi(0) = 0 \tag{2.15}$$

and effective domain $\mathrm{dom}(\Phi)$*. We impose the following "compatibility" condition between the respective subdifferential operators $\partial\Psi : \mathbb{R}^3 \rightrightarrows \mathbb{R}^3$ and $\partial\Phi : \mathbb{R}^3 \rightrightarrows \mathbb{R}^3$:*

$$\forall\, \mathbf{u} \in \mathrm{dom}(\Phi) \text{ and } \mathbf{v} \in \mathbb{R}^3, \qquad \forall\, \boldsymbol{\eta} \in \partial\Phi(\mathbf{u}) \text{ and } \mathbf{z} \in \partial\Psi(\mathbf{v}) \ : \qquad \boldsymbol{\eta} \cdot \mathbf{z} = 0. \tag{2.16}$$

In the variational formulation of system (1.6) (cf. (2.36) ahead), in fact we are going to make use of the abstract realization of Φ as a functional on $\mathbf{Y}_{\Gamma_\mathrm{C}}$, viz.

$$\boldsymbol{\varphi} : \mathbf{Y}_{\Gamma_\mathrm{C}} \to [0, +\infty] \ \text{ defined by } \ \boldsymbol{\varphi}(\mathbf{u}) := \begin{cases} \int_{\Gamma_\mathrm{C}} \Phi(\mathbf{u})\, \mathrm{d}x & \text{if } \Phi(\mathbf{u}) \in L^1(\Gamma_\mathrm{C}), \\ +\infty & \text{otherwise} \end{cases} \qquad \text{for all } \mathbf{u} \in \mathbf{Y}_{\Gamma_\mathrm{C}}. \tag{2.17}$$

Since $\boldsymbol{\varphi} : \mathbf{Y}_{\Gamma_\mathrm{C}} \to [0, +\infty]$ is a proper, convex and lower semicontinuous functional on $\mathbf{Y}_{\Gamma_\mathrm{C}}$, its subdifferential

$$\partial\boldsymbol{\varphi} : \mathbf{Y}_{\Gamma_\mathrm{C}} \rightrightarrows \mathbf{Y}'_{\Gamma_\mathrm{C}} \text{ is a maximal monotone operator.}$$

With a slight abuse of notation, we will use the symbol $\boldsymbol{\eta}$ not only for the elements of $\partial\Phi$, but also for those of $\partial\boldsymbol{\varphi}$.

Instead, in formulation (2.36) we are going to stay with the "concrete" subdifferential operator $\partial\Psi : \mathbf{H}_{\Gamma_\mathrm{C}} \rightrightarrows \mathbf{H}_{\Gamma_\mathrm{C}}$, which with a slight abuse we denote in the same way as the operator $\partial\Psi : \mathbb{R}^3 \rightrightarrows \mathbb{R}^3$ inducing it. It follows from (2.14) (observe that $\mathrm{dom}(\Psi) = \mathbb{R}^3$), that $\partial\Psi : \mathbb{R}^3 \rightrightarrows \mathbb{R}^3$ satisfies

$$\exists\, C_\Psi > 0 \ \forall\, \mathbf{z} \in \partial\Psi(\mathbf{v}) \ : \qquad |\mathbf{z}| \leq C_\Psi. \tag{2.18}$$

Example 2.4. The prototypical example of functionals Φ and Ψ complying with Hypothesis (2.3) comes from the modeling of frictional contact. In this frame, we have

$$\begin{aligned} \Phi(\mathbf{u}) &:= I_{(-\infty,0]}(u_\mathrm{N}), \\ \Psi(\mathbf{v}) &:= |\mathbf{v}_\mathrm{T}|. \end{aligned} \tag{2.19}$$

Clearly, the orthogonality condition (2.16) is fulfilled in this case. Nonetheless, let us highlight that (2.16) allows for much more generality: for example, $\partial\Phi(\mathbf{u})$ and $\partial\Psi(\mathbf{v})$ might be of the form

$$\begin{cases} \boldsymbol{\eta} \in \partial\Phi(\mathbf{u}) & \text{with } \boldsymbol{\eta} = \eta\mathbf{w}_1(u) \quad \eta \in \mathbb{R}, \quad \mathbf{w}_1(\mathbf{u}) \in \mathbb{R}^3, \\ \mathbf{z} \in \partial\Psi(\mathbf{v}) & \text{with } \mathbf{z} = z\mathbf{w}_2(\mathbf{v}) \quad z \in \mathbb{R}, \quad \mathbf{w}_2(\mathbf{v}) \in \mathbb{R}^3, \end{cases}$$

with $\mathbf{w}_1(\mathbf{u})$ and $\mathbf{w}_2(\mathbf{v})$ depending on $\mathbf{u}$ and $\mathbf{v}$, respectively, and such that

$$\mathbf{w}_1(\mathbf{u}) \cdot \mathbf{w}_2(\mathbf{v}) = 0 \quad \text{for all } \mathbf{u},\, \mathbf{v} \in \mathbb{R}^3.$$

Hypothesis 2.4 (The regularizing operator $\mathcal{R}$)**.** *Following* [8, 9] *we require that there exists* $\nu > 0$ *such that*

$$\begin{aligned} &\mathcal{R} : L^2(0,T;\mathbf{Y}'_{\Gamma_{\mathrm{C}}}) \to L^\infty(0,T;L^{2+\nu}(\Gamma_{\mathrm{C}};\mathbb{R}^3)) \text{ is weakly-strongly continuous, viz.} \\ &\boldsymbol{\eta}_n \rightharpoonup \boldsymbol{\eta} \ \text{ in } L^2(0,T;\mathbf{Y}'_{\Gamma_{\mathrm{C}}}) \quad \Rightarrow \quad \mathcal{R}(\boldsymbol{\eta}_n) \to \mathcal{R}(\boldsymbol{\eta}) \ \text{ in } L^\infty(0,T;L^{2+\nu}(\Gamma_{\mathrm{C}};\mathbb{R}^3)) \end{aligned} \tag{2.20}$$

for all $(\boldsymbol{\eta}_n)$, $\boldsymbol{\eta} \in L^2(0,T;\mathbf{Y}'_{\Gamma_{\mathrm{C}}})$.

Observe that (2.20) implies that $\mathcal{R} : L^2(0,T;\mathbf{Y}'_{\Gamma_{\mathrm{C}}}) \to L^\infty(0,T;L^{2+\nu}(\Gamma_{\mathrm{C}};\mathbb{R}^3))$ is bounded. We refer to [9, Example 3.2] for the explicit construction of an operator $\mathcal{R}$ complying with (2.20).

Hypothesis 2.5 (The subdifferential operators in the equation for χ)**.** *We assume that* $\widehat{\beta}$ *in* (1.6h) *fulfills*

$$\widehat{\beta} : \mathbb{R} \to (-\infty, +\infty] \text{ is proper, convex and lower semicontinuous, with } \operatorname{dom}(\widehat{\beta}) \subset [0, +\infty). \tag{2.21}$$

In what follows, we use the notation $\beta := \partial\widehat{\beta}$.

We also require that

$$\widehat{\rho} : \mathbb{R} \to [0, +\infty] \text{ is proper, convex and lower semicontinuous, with } 0 \in \operatorname{dom}(\widehat{\rho}). \tag{2.22}$$

We use ρ *as a placeholder for* $\partial\widehat{\rho}$.

Observe that, with a translation we can always confine ourselves to the case in which $\widehat{\rho}(0) = 0 = \min_{x\in\mathbb{R}} \widehat{\rho}(x)$, therefore we may also suppose that

$$0 \in \rho(0). \tag{2.23}$$

The simplest examples for $\widehat{\beta}$ and $\widehat{\rho}$ are $\widehat{\beta}(\chi) = I_{[0,1]}(\chi)$ and $\widehat{\rho}(\chi_t) = I_{(-\infty,0]}(\chi_t)$.

Hypothesis 2.6 (The other nonlinearities)**.** *We assume that the functions* k *in* (1.6b)–(1.6c), $\mathfrak{c}$ *in* (1.6b), (1.6d), *and* (1.6g), λ *in* (1.6c) *and* (1.6h), *and* γ *in* (1.6h) *fulfill*

$$k : \mathbb{R} \to [0, +\infty) \ \text{ is Lipschitz continuous,} \tag{2.24}$$

$$\mathfrak{c} \in \mathrm{C}^1(\mathbb{R}), \qquad \exists\, c_5,\, c_6 > 0 \ \forall\, x \in \mathbb{R} : \ \mathfrak{c}(x) \geq c_5, \ \ |\mathfrak{c}'(x)| \leq c_5, \qquad \mathfrak{c}'(x)x \geq 0. \tag{2.25}$$

$$\lambda \in \mathrm{C}^2(\mathbb{R}) \text{ and } \exists\, c_7, c_8 > 0 \ \forall\, x \in \mathbb{R} : \ |\lambda'(x)| \leq c_7, \ |\lambda''(x)| \leq c_8, \tag{2.26}$$

$$\gamma \in \mathrm{C}^2(\mathbb{R}), \text{ with } \gamma' : \mathbb{R} \to \mathbb{R} \text{ Lipschitz continuous.} \tag{2.27}$$

Assumptions on the problem and on the initial data. We require that

$$h \in L^2(0,T;V') \cap L^1(0,T;H)\,, \tag{2.28a}$$

$$\mathbf{f} \in L^2(0,T;\mathbf{W}')\,, \tag{2.28b}$$

$$\mathbf{g} \in L^2(0,T;H^{1/2}_{00,\Gamma_{\mathrm{D}}}(\Gamma_{\mathrm{N}};\mathbb{R}^3)'). \tag{2.28c}$$

For later convenience, we remark that, thanks to (2.28b)–(2.28c) the function $\mathbf{F} : (0,T) \to \mathbf{W}'$ defined by

$$\langle \mathbf{F}(t), \mathbf{v}\rangle_{\mathbf{W}} := \langle \mathbf{f}(t), \mathbf{v}\rangle_{\mathbf{W}} + \langle \mathbf{g}(t), \mathbf{v}\rangle_{H^{1/2}_{00,\Gamma_{\mathrm{D}}}(\Gamma_{\mathrm{N}};\mathbb{R}^3)} \quad \text{for all } \mathbf{v} \in \mathbf{W} \text{ and almost all } t \in (0,T),$$

satisfies

$$\mathbf{F} \in L^2(0,T;\mathbf{W}')\,. \tag{2.29}$$

For the initial data we impose that

$$J^*(L(\vartheta_0)) \in L^1(\Omega) \quad \text{and} \quad L(\vartheta_0) \in H\,, \tag{2.30a}$$

$$\vartheta^0_s \in H_{\Gamma_{\mathrm{C}}}\,, \quad \ell(\vartheta^0_s) \in H_{\Gamma_{\mathrm{C}}}\,, \text{ and } \quad f(\vartheta^0_s) \in H^1(\Gamma_{\mathrm{C}})\,, \tag{2.30b}$$

$$\mathbf{u}_0 \in \mathbf{W} \text{ and } \mathbf{u}_0 \in \operatorname{dom}(\boldsymbol{\varphi})\,, \tag{2.30c}$$

$$\chi_0 \in H^2(\Gamma_{\mathrm{C}}), \ \partial_{\mathbf{n}_s}\chi_0 = 0 \text{ a.e. in } \partial\Gamma_{\mathrm{C}}, \quad \widehat{\beta}(\chi_0) \in L^1(\Gamma_{\mathrm{C}})\,. \tag{2.30d}$$

Concerning the initial data ϑ_0 and ϑ^0_s, we observe that the first of (2.30a) implies $\vartheta_0 \in L^1(\Omega)$, in view of (2.6c). Moreover, the enhanced regularity (2.30b) required of ϑ^0_s reflects that we shall obtain a higher temporal regularity for ϑ_s than for ϑ, see Theorem 2.7 ahead.

2.3. **Variational formulation of the problem and main result.** We are now in the position to detail the formulation for the initial-boundary value problem associated with system (1.6). Observe that, while the temperature equations (1.6a) and (1.6c) and the momentum equation (1.6e) need to be formulated in dual spaces, the equation (1.6h) for the internal variable χ can be given a.e. in $\Gamma_{\rm C} \times (0,T)$, due to the $H^2(\Gamma_{\rm C})$-regularity obtained for χ.

Problem 2.5. Given a quadruple of data $(\vartheta_0, \vartheta_s^0, \mathbf{u}_0, \chi_0)$ fulfilling (2.30), find $(\vartheta, \vartheta_s, \mathbf{u}, \chi, \boldsymbol{\eta}, \boldsymbol{\mu}, \xi, \zeta)$, with

$$\vartheta \in L^2(0,T;V) \cap L^\infty(0,T;L^1(\Omega)), \qquad g(\vartheta) \in L^2(0,T;V), \tag{2.31a}$$
$$L(\vartheta) \in L^\infty(0,T;H) \cap H^1(0,T;V')\,, \tag{2.31b}$$
$$\vartheta_s \in L^2(0,T;V_{\Gamma_{\rm C}}), \tag{2.31c}$$
$$f(\vartheta_s) \in L^2(0,T;V_{\Gamma_{\rm C}})\,, \tag{2.31d}$$
$$\ell(\vartheta_s) \in L^\infty(0,T;H_{\Gamma_{\rm C}}) \cap H^1(0,T;V'_{\Gamma_{\rm C}})\,, \tag{2.31e}$$
$$\mathbf{u} \in H^1(0,T;\mathbf{W})\,, \tag{2.31f}$$
$$\chi \in L^2(0,T;H^2(\Gamma_{\rm C})) \cap L^\infty(0,T;V_{\Gamma_{\rm C}}) \cap H^1(0,T;H_{\Gamma_{\rm C}})\,, \tag{2.31g}$$
$$\boldsymbol{\eta} \in L^2(0,T;\mathbf{Y}'_{\Gamma_{\rm C}})\,, \tag{2.31h}$$
$$\boldsymbol{\mu} \in L^2(0,T;\mathbf{H}_{\Gamma_{\rm C}})\,, \tag{2.31i}$$
$$\xi \in L^2(0,T;H_{\Gamma_{\rm C}})\,, \tag{2.31j}$$
$$\zeta \in L^2(0,T;H_{\Gamma_{\rm C}})\,, \tag{2.31k}$$

$$\vartheta(0) = \vartheta_0 \quad \text{a.e. in } \Omega\,, \tag{2.32}$$
$$\vartheta_s(0) = \vartheta_s^0 \quad \text{a.e. in } \Gamma_{\rm C}\,, \tag{2.33}$$
$$\mathbf{u}(0) = \mathbf{u}_0 \quad \text{a.e. in } \Omega\,, \tag{2.34}$$
$$\chi(0) = \chi_0 \quad \text{a.e. in } \Gamma_{\rm C}\,, \tag{2.35}$$

and satisfying

$$\begin{aligned}\langle \partial_t L(\vartheta), v\rangle_V &- \int_\Omega \operatorname{div}(\partial_t \mathbf{u})\, v \,\mathrm{d}x + \int_\Omega \nabla g(\vartheta)\, \nabla v \,\mathrm{d}x + \int_{\Gamma_{\rm C}} k(\chi)(\vartheta - \vartheta_s) v \,\mathrm{d}x \\ &+ \int_{\Gamma_{\rm C}} \mathfrak{c}'(\vartheta - \vartheta_s)|\mathcal{R}(\boldsymbol{\eta})|\Psi(\partial_t \mathbf{u}) v \,\mathrm{d}x = \langle h, v\rangle_V \quad \forall\, v \in V \ \text{ a.e. in } (0,T)\,,\end{aligned} \tag{2.36a}$$

$$\begin{aligned}&\langle \partial_t \ell(\vartheta_s), v\rangle_{V_{\Gamma_{\rm C}}} - \int_{\Gamma_{\rm C}} \partial_t \lambda(\chi)\, v \,\mathrm{d}x + \int_{\Gamma_{\rm C}} \nabla f(\vartheta_s)\, \nabla v \,\mathrm{d}x \\ &= \int_{\Gamma_{\rm C}} k(\chi)(\vartheta - \vartheta_s) v \,\mathrm{d}x + \int_{\Gamma_{\rm C}} \mathfrak{c}'(\vartheta - \vartheta_s)|\mathcal{R}(\boldsymbol{\eta})|\Psi(\partial_t \mathbf{u}) v \,\mathrm{d}x \quad \forall\, v \in V_{\Gamma_{\rm C}} \ \text{ a.e. in } (0,T)\,,\end{aligned} \tag{2.36b}$$

$$\begin{aligned}&b(\partial_t \mathbf{u}, \mathbf{v}) + a(\mathbf{u}, \mathbf{v}) + \int_\Omega \vartheta \operatorname{div}(\mathbf{v}) \,\mathrm{d}x \\ &+ \int_{\Gamma_{\rm C}} \chi \mathbf{u} \cdot \mathbf{v} \,\mathrm{d}x + \langle \eta, \mathbf{v}\rangle_{\mathbf{Y}_{\Gamma_{\rm C}}} + \int_{\Gamma_{\rm C}} \mathfrak{c}(\vartheta - \vartheta_s) \boldsymbol{\mu} \cdot \mathbf{v} \,\mathrm{d}x = \langle \mathbf{F}, \mathbf{v}\rangle_{\mathbf{W}} \quad \text{for all } \mathbf{v} \in \mathbf{W} \ \text{ a.e. in } (0,T)\,,\end{aligned} \tag{2.36c}$$

$$\boldsymbol{\eta} \in \partial \boldsymbol{\varphi}(\mathbf{u}) \quad \text{ in } \mathbf{Y}'_{\Gamma_{\rm C}}, \text{ a.e. in } (0,T), \tag{2.36d}$$
$$\boldsymbol{\mu} \in |\mathcal{R}(\boldsymbol{\eta})| \partial \Psi(\partial_t \mathbf{u}) \text{ a.e. in } \Gamma_{\rm C} \times (0,T), \tag{2.36e}$$
$$\partial_t \chi + \zeta + A\chi + \xi + \gamma'(\chi) = -\lambda'(\chi)\vartheta_s - \frac{1}{2}|\mathbf{u}|^2 \quad \text{a.e. in } \Gamma_{\rm C} \times (0,T), \tag{2.36f}$$
$$\xi \in \beta(\chi) \quad \text{a.e. in } \Gamma_{\rm C} \times (0,T), \tag{2.36g}$$
$$\zeta \in \rho(\partial_t \chi) \quad \text{a.e. in } \Gamma_{\rm C} \times (0,T). \tag{2.36h}$$

Definition 2.6. *We call a solution to Problem 2.5* energy solution *if, in addition, it satisfies the energy inequality*

$$\begin{aligned}
&\int_\Omega J^*(L(\vartheta(t)))\,\mathrm{d}x + \int_s^t\int_\Omega g'(\vartheta)|\nabla\vartheta|^2\,\mathrm{d}x\,\mathrm{d}r + \int_{\Gamma_\mathrm{C}} j^*(\ell(\vartheta_s(t)))\,\mathrm{d}x + \int_s^t\int_\Omega f'(\vartheta_s)|\nabla\vartheta_s|^2\,\mathrm{d}x\,\mathrm{d}r\\
&\quad+\int_s^t\int_{\Gamma_\mathrm{C}} k(\chi)(\vartheta-\vartheta_s)^2\,\mathrm{d}x\,\mathrm{d}r + \int_s^t b(\partial_t\mathbf{u},\partial_t\mathbf{u})\,\mathrm{d}r + \frac{1}{2}a(\mathbf{u}(t),\mathbf{u}(t)) + \frac{1}{2}\int_{\Gamma_\mathrm{C}}\chi(t)|\mathbf{u}(t)|^2\,\mathrm{d}x + \int_{\Gamma_\mathrm{C}}\Phi(\mathbf{u}(t))\,\mathrm{d}x\\
&\quad+\int_s^t\int_{\Gamma_\mathrm{C}}\mathfrak{c}(\vartheta-\vartheta_s)\Psi(\partial_t\mathbf{u})|\mathcal{R}(\mathrm{D}\Phi_\varepsilon)|\,\mathrm{d}x\,\mathrm{d}r + \int_s^t\int_{\Gamma_\mathrm{C}}\mathfrak{c}'(\vartheta-\vartheta_s)\Psi(\partial_t\mathbf{u})|\mathcal{R}(\mathrm{D}\Phi_\varepsilon)|(\vartheta-\vartheta_s)\,\mathrm{d}x\,\mathrm{d}r\\
&\quad+\int_s^t\int_{\Gamma_\mathrm{C}}|\partial_t\chi|^2\,\mathrm{d}x\,\mathrm{d}r + \int_s^t\int_{\Gamma_\mathrm{C}}\zeta\chi_t\,\mathrm{d}x\,\mathrm{d}r + \frac{1}{2}\int_{\Gamma_\mathrm{C}}|\nabla\chi(t)|^2\,\mathrm{d}x + \int_{\Gamma_\mathrm{C}}\widehat{\beta}(\chi(t))\,\mathrm{d}x + \int_{\Gamma_\mathrm{C}}\gamma(\chi(t))\,\mathrm{d}x\\
&\le \int_\Omega J^*(L(\vartheta(s)))\,\mathrm{d}x + \int_{\Gamma_\mathrm{C}} j^*(\ell(\vartheta_s(s)))\,\mathrm{d}x + \int_0^t \langle h,\vartheta\rangle_V\,\mathrm{d}r + \int_0^t\langle\mathbf{F},\partial_t\mathbf{u}\rangle_{\mathbf{W}}\,\mathrm{d}r + \frac{1}{2}a(\mathbf{u}(s),\mathbf{u}(s))\\
&\quad+\int_{\Gamma_\mathrm{C}}\Phi(\mathbf{u}(s))\,\mathrm{d}x + \frac{1}{2}\int_{\Gamma_\mathrm{C}}\chi(s)|\mathbf{u}(s)|^2\,\mathrm{d}x + \frac{1}{2}\int_{\Gamma_\mathrm{C}}|\nabla\chi(s)|^2\,\mathrm{d}x + \int_{\Gamma_\mathrm{C}}\widehat{\beta}_\varepsilon(\chi(s))\,\mathrm{d}x + \int_{\Gamma_\mathrm{C}}\gamma(\chi(s))\,\mathrm{d}x
\end{aligned}\tag{2.37}$$

for almost all $0\le s\le t\le T$*, and and for* $s=0$*.*

With our main result, Theorem 2.7, we state the existence of an *energy solution* to Problem 2.5, with the additional properties (2.38) below. Namely, ϑ has bounded variation, as a function of time, with values in some dual space: this in particular ensures that $t\mapsto\vartheta(t)$ is continuous, with values in that space, at almost all points $t_0\in(0,T)$. For ϑ_s we gain a better regularity, cf. (2.38b) and (2.38c), as a result of a further regularity estimate on the equation for ϑ_s. Such an estimate also implies (2.38d)–(2.38f).

Theorem 2.7. *Assume* (2.1) *and Hypotheses 2.1–2.6. Suppose that the data* $(h,\mathbf{f},\mathbf{g})$ *and* $(\vartheta_0,\vartheta_s^0,\mathbf{u}_0,\chi_0)$ *fulfill* (2.28) *and* (2.30)*.*

Then, Problem 2.5 admits an energy solution $(\vartheta,\vartheta_s,\mathbf{u},\chi,\boldsymbol{\eta},\boldsymbol{\mu},\xi,\zeta)$*, which in addition satisfies*

$$\vartheta\in\mathrm{BV}([0,T];W^{1,q}(\Omega)')\ \textit{for every } q>3,\tag{2.38a}$$

$$\vartheta_s\in H^1(0,T;H_{\Gamma_\mathrm{C}}),\tag{2.38b}$$

$$f(\vartheta_s)\in L^\infty(0,T;V_{\Gamma_\mathrm{C}}),\tag{2.38c}$$

$$\chi\in L^\infty(0,T;H^2(\Gamma_\mathrm{C}))\cap H^1(0,T;V_{\Gamma_\mathrm{C}})\cap W^{1,\infty}(0,T;H_{\Gamma_\mathrm{C}}),\tag{2.38d}$$

$$\xi\in L^\infty(0,T;H_{\Gamma_\mathrm{C}})\,,\tag{2.38e}$$

$$\zeta\in L^\infty(0,T;H_{\Gamma_\mathrm{C}})\,.\tag{2.38f}$$

Outline of the proof. We set up a suitable approximation of system (2.36) by regularizing some of the (maximal monotone) operators featured therein; we shall denote the regularization parameter with the symbol ε and accordingly refer to the approximate problem as (P_ε). In Section 3 we prove the existence of local-in-time solutions to Problem (P_ε) (cf. Proposition 3.3 ahead). Then, we show that the approximate solutions fulfill an energy identity, which serves as the basis for deriving a series of a priori estimates, *uniform* w.r.t. ε. Relying on them, in Section 5 we prove that, along a suitable subsequence, the approximate solutions converge to a *local-in-time* solution to Problem 2.5. Its extension to a *global-in-time* solution by means of a careful prolongation argument is the focus of Section 6. Some useful technical lemma we will use in the proofs are stated and proved in the Appendix.

3. Approximation

First, in Sec. 3.1 we introduce and explain our approximation of system (2.36), leading to Problem (P_ε); in the end, we state its variational formulation. The existence of a local-in-time solution to (P_ε) is proved

in Sec. 3.2 via the Schauder fixed point theorem. Most of the calculations for the (uniform w.r.t. ε) a priori estimates on the approximate solutions which we shall derive in Sec. 5 hinge on a series of technical results on the functions approximating the nonlinearities of the problem, which we have collected in the Appendix.

3.1. **The approximate problem.** To motivate the regularization procedures we are going to adopt, we discuss in advance some of the a priori estimates we shall perform on system (2.36) in Sec. 5. As we will see, the related calculations cannot be performed rigorously on system (2.36), and indeed necessitate of the Yosida-type regularizations by which we are going to replace some of the maximal monotone nonlinearities in system (2.36).
Outlook to the approximate problem. The basic *energy estimate* for system (2.36) (cf. the *First a priori estimate* in Sec. 4) follows by testing (2.36a) by ϑ, (2.36b) by ϑ_s, (2.36c) by $\partial_t \mathbf{u}$, (2.36f) by $\partial_t \chi$, adding the resulting relations, and integrating in time. The *formal* calculations

$$\begin{aligned}\int_0^t \langle \partial_t L(\vartheta), \vartheta \rangle_V \, \mathrm{d}r &= \int_0^t \left\langle \partial_t w, L^{-1}(w) \right\rangle_V \, \mathrm{d}r \\ &= \|J^*(L(\vartheta(t)))\|_{L^1(\Omega)} - \|J^*(L(\vartheta_0))\|_{L^1(\Omega)} \geq C_1 \|\vartheta(t)\|_{L^1(\Omega)} - C\end{aligned} \tag{3.1}$$

where we have used the auxiliary variable $w := L(\vartheta)$, the *formal identity*

$$\left\langle \partial_t w, L^{-1}(w) \right\rangle_V = \frac{\mathrm{d}}{\mathrm{d}t} \left(\int_\Omega J^*(w) \, \mathrm{d}x \right) \quad \text{a.e. in } (0,T),$$

and, finally, the coercivity condition (2.6c), lead to a bound for ϑ in $L^\infty(0,T;L^1(\Omega))$. The corresponding calculations on the level of (2.36b) yield an estimate for ϑ_s in $L^\infty(0,T;L^1(\Gamma_{\mathrm{C}}))$.

In order to make (3.1) rigorous, following [10, 7, 9]

(1) we replace L and ℓ in the equations (2.36a) and (2.36b) by the following approximating functions

$$\widetilde{L}_\varepsilon(r) := \varepsilon r + L_\varepsilon(r), \tag{3.2}$$

$$\widetilde{\ell}_\varepsilon(r) := \varepsilon r + \ell_\varepsilon(r), \tag{3.3}$$

where for $\varepsilon > 0$ L_ε and ℓ_ε denote the Yosida regularizations of L and ℓ, respectively, cf. (A.2) below.

Therefore, $\widetilde{L}_\varepsilon$ ($\widetilde{\ell}_\varepsilon$, respectively) is differentiable, strictly increasing and Lipschitz continuous, see also the upcoming Lemma A.4. Nonetheless, this procedure only partially serves to the purpose of rigorously justifying (3.1), as expounded in Remark 4.1 at the end of Section 4.

In accord with (2.13) and (3.3),

(2) we thus replace the function f in (2.36b) by

$$f_\varepsilon(x) = \int_0^x \frac{1}{\widetilde{\ell}'_\varepsilon(s)} \, \mathrm{d}s \,, \quad \forall x \in \mathbb{R}, \tag{3.4}$$

whose definition reflects the fact that $f(x) = \int_0^x \frac{1}{\ell'(s)} \, \mathrm{d}s$.

Combining the aforementioned *energy estimate* and a comparison argument in the momentum equation (2.36c) leads to the following estimate

$$\|\mathfrak{c}(\vartheta - \vartheta_s)\boldsymbol{\mu} + \boldsymbol{\eta}\|_{L^2(0,T;\mathbf{Y}'_{\Gamma_{\mathrm{C}}})} \leq C \tag{3.5}$$

with $\boldsymbol{\eta} \in \partial\boldsymbol{\varphi}(\mathbf{u})$ in $\mathbf{Y}'_{\Gamma_{\mathrm{C}}}$ a.e. in $(0,T)$ (cf. (2.36d)), and $\boldsymbol{\mu} \in |\mathcal{R}(\boldsymbol{\eta})|\partial\Psi(\partial_t \mathbf{u})$ a.e. in $\Gamma_{\mathrm{C}} \times (0,T)$ (cf. (2.36e)). From (3.5), it is crucial to conclude the *separate* estimates

$$\|\mathfrak{c}(\vartheta - \vartheta_s)\boldsymbol{\mu}\|_{L^2(0,T;\mathbf{Y}'_{\Gamma_{\mathrm{C}}})} + \|\boldsymbol{\eta}\|_{L^2(0,T;\mathbf{Y}'_{\Gamma_{\mathrm{C}}})} \leq C. \tag{3.6}$$

This follows from the orthogonality condition (2.16) only on a *formal* level, since (2.16) is not, in general, inherited by the abstract operator $\partial\boldsymbol{\varphi} : \mathbf{Y}_{\Gamma_{\mathrm{C}}} \rightrightarrows \mathbf{Y}'_{\Gamma_{\mathrm{C}}}$. In order to justify this argument, we need to suitably approximate $\partial\boldsymbol{\varphi} : \mathbf{Y}_{\Gamma_{\mathrm{C}}} \rightrightarrows \mathbf{Y}'_{\Gamma_{\mathrm{C}}}$ in such a way as to replace $\boldsymbol{\eta} \in \mathbf{Y}'_{\Gamma_{\mathrm{C}}}$ in (2.36c) with a term $\boldsymbol{\eta}_\varepsilon$ orthogonal to $\partial\Psi(\partial_t \mathbf{u})$. Along the lines of [8, 9], we

(3) approximate the function Φ 2.15, which defines the functional $\boldsymbol{\varphi}$ through (2.17), by its Yosida approximation $\Phi_\varepsilon : \mathbb{R}^3 \to [0, +\infty)$.

We recall that Φ_ε is convex, differentiable, and such that $\mathrm{D}\Phi_\varepsilon$ is the Yosida regularization of the subdifferential $\partial\Phi : \mathbb{R}^3 \rightrightarrows \mathbb{R}^3$. As such, it fulfills (cf. (A.3) below)

$$\mathrm{D}\Phi_\varepsilon(\mathbf{u}) \in \partial\Phi(\mathsf{r}_\varepsilon(\mathbf{u})), \tag{3.7}$$

where r_ε denotes the resolvent of the operator $\partial\Phi$. Therefore, in view of (2.16), any approximate solution $\mathbf{u}_\varepsilon$ satisfies

$$\mathrm{D}\Phi_\varepsilon(\mathbf{u}_\varepsilon) \cdot \mathbf{z} = 0 \qquad \text{for all } \mathbf{z} \in \partial\Psi(\partial_t \mathbf{u}_\varepsilon), \tag{3.8}$$

which will be crucial in order to deduce (3.6), cf. the *Third a priori estimate* in Sec. 4.

Finally, along on the lines of [3] we will also perform on the *doubly nonlinear* equation (2.36f) the test by (the formal quantity) $\partial_t(A\chi + \beta(\chi))$. Let us mention that such an estimate is by now classical for this kind of doubly nonlinear diffusive evolutionary differential inclusions. It allows one to estimate the terms $A\chi$ and $\xi \in \beta(\chi)$, *separately*, in $L^\infty(0,T;L^2(\Gamma_\mathrm{C}))$. Let us stress that this estimate requires ad hoc time-regularity for the right-hand side terms. Once the computations have been carried out, an estimate for $\zeta \in \rho(\partial_t\chi)$ in $L^2(0,T;L^2(\Gamma_\mathrm{C}))$ then follows from a comparison in (2.36f). In order to perform all the calculations in a rigorous way (cf. the *Seventh a priori estimate* in Sec. 4), it is necessary to

(4) replace ρ and β by their Yosida approximations ρ_ε and β_ε.

Furthermore, it will be convenient to use the functions $\mathcal{I}_\varepsilon, i_\varepsilon : \mathbb{R} \to \mathbb{R}$

$$\mathcal{I}_\varepsilon(x) := \int_0^x s\, \widetilde{L}'_\varepsilon(s)\, \mathrm{d}s \tag{3.9}$$

$$i_\varepsilon(x) := \int_0^x s\, \widetilde{\ell}'_\varepsilon(s)\, \mathrm{d}s \tag{3.10}$$

(cf. in particular the derivation of the approximate energy identity (3.52) later on), and the function $H_\varepsilon : \mathbb{R} \to \mathbb{R}$

$$H_\varepsilon(x) := \int_0^x \widetilde{\ell}'_\varepsilon(s)\, f_\varepsilon(s)\, \mathrm{d}s \quad \forall x \in \mathbb{R} \tag{3.11}$$

(cf. the derivation of *Sixth and Seventh a priori estimate* in Sec. 4).

Finally, we will supplement our approximate Problem (P_ε) by the initial data $(\vartheta_0^\varepsilon, \vartheta_s^{0,\varepsilon}, \mathbf{u}_0, \chi_0)$, where the family

$$(\vartheta_0^\varepsilon, \vartheta_s^{0,\varepsilon})_\varepsilon \in H \times H_{\Gamma_\mathrm{C}} \tag{3.12}$$

approximates the data $(\vartheta_0, \vartheta_s^0)$ from (2.30a)–(2.30b) in the sense that

$$(\vartheta_0^\varepsilon, \vartheta_s^{0,\varepsilon}) \to (\vartheta_0, \vartheta_s^0) \text{ in } L^1(\Omega) \times H_{\Gamma_\mathrm{C}} \text{ as } \varepsilon \downarrow 0\,, \tag{3.13}$$

$$\exists\, \bar{S}_0 > 0 \text{ depending on } \vartheta_0 \text{ and } \vartheta_s^0 : \quad \|L_\varepsilon(\vartheta_0^\varepsilon)\|_H + \|\ell_\varepsilon(\vartheta_s^{0,\varepsilon})\|_{H_{\Gamma_\mathrm{C}}} \le \bar{S}_0 \quad \text{for all}\, \varepsilon > 0\,, \tag{3.14}$$

$$\int_\Omega \mathcal{I}_\varepsilon(\vartheta_0^\varepsilon)\, \mathrm{d}x \to \int_\Omega J^*(L(\vartheta_0))\, \mathrm{d}x \qquad \text{as } \varepsilon \downarrow 0\,, \tag{3.15}$$

$$\int_{\Gamma_\mathrm{C}} i_\varepsilon(\vartheta_s^{0,\varepsilon})\, \mathrm{d}x \to \int_{\Gamma_\mathrm{C}} j^*(\ell(\vartheta_s^0))\, \mathrm{d}x \qquad \text{as } \varepsilon \downarrow 0\,. \tag{3.16}$$

Observe that (3.15) and (3.16) guarantee that

$$\exists\, \bar{S}_1 > 0 \text{ depending on } \vartheta_0 : \quad \forall\, \varepsilon \in (0,1) \quad \int_\Omega \mathcal{I}_\varepsilon(\vartheta_0^\varepsilon)\, \mathrm{d}x \le \bar{S}_1\,, \tag{3.17}$$

$$\exists\, \bar{S}_2 > 0 \text{ depending on } \vartheta_s^0 : \quad \forall\, \varepsilon \in (0,1) \quad \int_{\Gamma_\mathrm{C}} i_\varepsilon(\vartheta_s^{0,\varepsilon})\, \mathrm{d}x \le \bar{S}_2\,. \tag{3.18}$$

Finally, we also require that the family $(\vartheta_s^{0,\varepsilon})_\varepsilon$ fulfills

$$\exists\, \bar{S}_3 > 0 \text{ depending on } \vartheta_s^0 : \quad \forall\, \varepsilon \in (0,1) \quad \|H_\varepsilon(\vartheta_s^{0,\varepsilon})\|_{L^1(\Gamma_c)} \leq \bar{S}_3\,, \tag{3.19}$$

$$\exists\, \bar{S}_4 > 0 \text{ depending on } \vartheta_s^0 : \quad \forall\, \varepsilon \in (0,1) \qquad \|f_\varepsilon(\vartheta_s^{0,\varepsilon})\|_{V_{\Gamma_C}} \leq \bar{S}_4. \tag{3.20}$$

Observe that, since f_ε is bi-Lipschitz (cf. Lemma A.5 later on), (3.20) in fact implies that $\vartheta_s^{0,\varepsilon}$ is also in V_{Γ_C}. In the Appendix we state a series of Lemmas, in which we will construct sequences of initial data $(\vartheta_0^\varepsilon, \vartheta_s^{0,\varepsilon})_\varepsilon$ complying with the properties (3.12)–(3.20) and detail how the constants $\bar{S}_i\,,\ i = 1, \cdots, 4$ may depend on the data ϑ_0 and ϑ_s^0.

All in all, the variational formulation of the approximate problem reads:

Problem 3.1 (P_ε)**.** Let a quadruple of initial data $(\vartheta_0^\varepsilon, \vartheta_s^{0,\varepsilon}, \mathbf{u}_0, \chi_0)$ satisfy (2.30c)–(2.30d) and (3.12)–(3.20). Find a quintuple $(\vartheta, \vartheta_s, \mathbf{u}, \chi, \boldsymbol{\mu})$ fulfilling

$$\vartheta \in L^2(0,T;V) \cap \mathrm{C}^0([0,T];H), \tag{3.21}$$

$$\widetilde{L}_\varepsilon(\vartheta) \in L^2(0,T;V) \cap \mathrm{C}^0([0,T];H) \cap H^1(0,T;V'), \tag{3.22}$$

$$\vartheta_s \in L^\infty(0,T;V_{\Gamma_C}) \cap H^1(0,T;H_{\Gamma_C}), \tag{3.23}$$

$$f_\varepsilon(\vartheta_s) \in L^\infty(0,T;V_{\Gamma_C}), \tag{3.24}$$

$$\widetilde{\ell}_\varepsilon(\vartheta_s) \in L^\infty(0,T;V_{\Gamma_C}) \cap H^1(0,T;H_{\Gamma_C}), \tag{3.25}$$

and such that $(\mathbf{u}, \chi, \boldsymbol{\mu})$ comply with (2.31f), (2.38d), and (2.31i), satisfying the initial conditions

$$\vartheta(0) = \vartheta_0^\varepsilon \quad \text{a.e. in } \Omega\,, \tag{3.26}$$

$$\vartheta_s(0) = \vartheta_s^{0,\varepsilon} \quad \text{a.e. in } \Gamma_C\,, \tag{3.27}$$

as well as (2.34)–(2.35), and the equations

$$\begin{aligned}
&\Big\langle \partial_t \widetilde{L}_\varepsilon(\vartheta), v \Big\rangle_V - \int_\Omega \mathrm{div}(\partial_t \mathbf{u})\, v \,\mathrm{d}x + \int_\Omega \nabla g(\vartheta)\, \nabla v \,\mathrm{d}x \\
&+ \int_{\Gamma_C} k(\chi)(\vartheta - \vartheta_s) v \,\mathrm{d}x + \int_{\Gamma_C} \mathfrak{c}'(\vartheta - \vartheta_s)\Psi(\partial_t \mathbf{u})|\mathcal{R}(\mathrm{D}\Phi_\varepsilon(\mathbf{u}))| v \,\mathrm{d}x = \langle h, v\rangle_V \quad \forall\, v \in V \ \text{ a.e. in } (0,T)\,,
\end{aligned} \tag{3.28a}$$

$$\begin{aligned}
&\Big\langle \partial_t \widetilde{\ell}_\varepsilon(\vartheta_s), v \Big\rangle_{V_{\Gamma_C}} - \int_{\Gamma_C} \partial_t \lambda(\chi)\, v \,\mathrm{d}x + \int_{\Gamma_C} \nabla f_\varepsilon(\vartheta_s)\, \nabla v \,\mathrm{d}x \\
&= \int_{\Gamma_C} k(\chi)(\vartheta - \vartheta_s) v \,\mathrm{d}x + \int_{\Gamma_C} \mathfrak{c}'(\vartheta - \vartheta_s)\Psi(\partial_t \mathbf{u})|\mathcal{R}(\mathrm{D}\Phi_\varepsilon(\mathbf{u}))| v \,\mathrm{d}x \quad \forall\, v \in V_{\Gamma_C} \ \text{ a.e. in } (0,T)\,,
\end{aligned} \tag{3.28b}$$

$$\begin{aligned}
&b(\partial_t \mathbf{u}, \mathbf{v}) + a(\mathbf{u}, \mathbf{v}) + \int_\Omega \vartheta\, \mathrm{div}(\mathbf{v}) \,\mathrm{d}x + \int_{\Gamma_C} \chi \mathbf{u} \cdot \mathbf{v} \,\mathrm{d}x \\
&+ \int_{\Gamma_C} \mathrm{D}\Phi_\varepsilon(\mathbf{u}) \cdot \mathbf{v} \,\mathrm{d}x + \int_{\Gamma_C} \mathfrak{c}(\vartheta - \vartheta_s)\boldsymbol{\mu} \cdot \mathbf{v} \,\mathrm{d}x = \langle \mathbf{F}, \mathbf{v}\rangle_{\mathbf{W}} \quad \text{for all } \mathbf{v} \in \mathbf{W} \ \text{ a.e. in } (0,T)\,,
\end{aligned} \tag{3.28c}$$

$$\boldsymbol{\mu} = |\mathcal{R}(\mathrm{D}\Phi_\varepsilon(\mathbf{u}))|\mathbf{z} \quad \text{with} \quad \mathbf{z} \in \partial\Psi(\partial_t \mathbf{u}) \text{ a.e. in } \Gamma_C \times (0,T), \tag{3.28d}$$

$$\partial_t \chi + \rho_\varepsilon(\partial_t \chi) + A\chi + \beta_\varepsilon(\chi) + \gamma'(\chi) = -\lambda'(\chi)\vartheta_s - \frac{1}{2}|\mathbf{u}|^2 \quad \text{a.e. in } \Gamma_C \times (0,T)\,. \tag{3.28e}$$

Let us only briefly comment on the enhanced regularity properties (3.21)–(3.25). Since, for $\varepsilon > 0$ fixed, $\widetilde{L}_\varepsilon$ is bi-Lipschitz (cf. Lemma A.4), $\vartheta \in L^2(0,T;V)$ implies that $\widetilde{L}_\varepsilon(\vartheta)$ is in the same space. Therefore, by interpolation with $H^1(0,T;V')$ we conclude that $\widetilde{L}_\varepsilon(\vartheta) \in \mathrm{C}^0([0,T];H)$, whence $\vartheta \in \mathrm{C}^0([0,T];H)$. Analogous arguments apply to (3.23)–(3.25). Observe that ϑ_s also inherits the regularity of $f_\varepsilon(\vartheta_s)$, since f_ε is bi-Lipschitz as well, cf. Lemma A.5.

3.2. **Local existence of approximate solutions.** The main result of this section is the forthcoming Proposition 3.3, stating the existence of a local-in-time solution to Problem (P_ε). The latter features a structure very similar to the approximate problem for the PDE system analyzed in [9], cf. Problem 4.4 therein. Indeed, some of the arguments from [9] may be easily adapted to the present setting. Therefore, we only sketch the fixed point procedure yielding local existence. In particular, we only hint to the most relevant steps in the construction of the fixed point operator and in the proof of its continuity and compactness, referring to [9, Sect. 4.2] for all details.

Fixed point setup. In view of hypothesis (2.20) on the regularizing operator $\mathcal{R}$, we may choose $\delta \in (0,1)$ such that

$$\mathcal{R} : L^2(0,T;\mathbf{Y}'_{\Gamma_C}) \to L^\infty(0,T;L^{\frac{2}{1-\delta}}(\Gamma_C;\mathbb{R}^3)) \text{ is weakly-strongly continuous} \tag{3.29}$$

(and therefore bounded). For a fixed $\tau > 0$ and a fixed constant $M > 0$, we consider the set

$$\begin{aligned}\mathcal{Y}_\tau = \{&(\vartheta,\vartheta_s,\mathbf{u},\chi) \in L^2(0,\tau;H^{1-\delta}(\Omega)) \times L^2(0,\tau;H^{1-\delta}(\Gamma_C)) \times L^2(0,\tau;H^{1-\delta}(\Omega;\mathbb{R}^3)) \times L^2(0,\tau;H_{\Gamma_C}) : \\ &\|\vartheta\|_{L^2(0,\tau;H^{1-\delta}(\Omega))} + \|\vartheta_s\|_{L^2(0,\tau;H^{1-\delta}(\Gamma_C))} + \|\mathbf{u}\|_{L^2(0,\tau;H^{1-\delta}(\Omega;\mathbb{R}^3))} + \|\chi\|_{L^2(0,\tau;H_{\Gamma_C})} \le M\},\end{aligned} \tag{3.30}$$

with the topology induced by $L^2(0,\tau;H^{1-\delta}(\Omega)) \times L^2(0,\tau;H^{1-\delta}(\Gamma_C)) \times L^2(0,\tau;H^{1-\delta}(\Omega;\mathbb{R}^3)) \times L^2(0,\tau;H_{\Gamma_C})$. We are going to construct an operator $\mathcal{T}$ mapping $\mathcal{Y}_{\widehat{T}}$ into itself for a suitable time $0 \le \widehat{T} \le T$, depending on M, in such a way that any fixed point of $\mathcal{T}$ yields a solution to Problem (P_ε) on the interval $(0,\widehat{T})$.

Notation 3.2. In the following lines, we will denote by S_i, $i = 1,...,5$, positive constants depending on the problem data and on $M > 0$ in (3.30), but *independent* of $\varepsilon > 0$, and by $S_6(\varepsilon)$ a constant depending on the above quantities and on $\varepsilon > 0$ as well. Furthermore, with the symbols $\pi_i(A)$, $\pi_{i,j}(A), \dots$, we will denote the projection of a set A on its i-, or (i,j)-component.

Step 1: As a first step in the construction of $\mathcal{T}$, we fix $(\widehat{\vartheta},\widehat{\vartheta}_s,\widehat{\mathbf{u}},\widehat{\chi}) \in \mathcal{Y}_\tau$ and consider (the Cauchy problem for) the system (3.28c–3.28d), with $(\widehat{\vartheta},\widehat{\vartheta}_s,\widehat{\chi})$ in place of $(\vartheta,\vartheta_s,\chi)$, and $\mathfrak{c}(\widehat{\vartheta}-\widehat{\vartheta}_s)|\mathcal{R}(\mathrm{D}\Phi_\varepsilon(\widehat{\mathbf{u}}))|$ replacing $\mathfrak{c}(\vartheta - \vartheta_s)|\mathcal{R}(\mathrm{D}\Phi_\varepsilon(\mathbf{u}))$, that is

$$\begin{aligned}&b(\partial_t\mathbf{u},\mathbf{v}) + a(\mathbf{u},\mathbf{v}) + \int_\Omega \widehat{\vartheta}\,\mathrm{div}(\mathbf{v})\,\mathrm{d}x + \int_{\Gamma_C} \widehat{\chi}\mathbf{u}\cdot\mathbf{v}\,\mathrm{d}x \\ &+ \int_{\Gamma_C} \mathrm{D}\Phi_\varepsilon(\mathbf{u})\cdot\mathbf{v}\,\mathrm{d}x + \int_{\Gamma_C} \mathfrak{c}(\widehat{\vartheta}-\widehat{\vartheta}_s)\boldsymbol{\mu}\cdot\mathbf{v}\,\mathrm{d}x = \langle\mathbf{F},\mathbf{v}\rangle_{\mathbf{W}} \quad \text{for all } \mathbf{v}\in\mathbf{W} \text{ a.e. in } (0,T)\,, \\ &\boldsymbol{\mu} = |\mathcal{R}(\mathrm{D}\Phi_\varepsilon(\widehat{\mathbf{u}}))|\mathbf{z} \text{ with } \mathbf{z}\in\partial\Psi(\partial_t\mathbf{u}) \text{ a.e. in } \Gamma_C\times(0,T)\,.\end{aligned} \tag{3.31}$$

A well-posedness result for such a problem can be obtained easily adapting the arguments of the proof of [9, Lemma 4.6], observing that (3.31) has the very same structure of the corresponding momentum equation tackled in [9] (cf. Hypothesis 2.3 on the subdifferential operators). Then, there exists a constant $S_1 > 0$ and a unique pair $(\mathbf{u},\boldsymbol{\mu}) \in H^1(0,\tau;\mathbf{W}) \times L^\infty(0,\tau;L^{2+\nu}(\Gamma_C;\mathbb{R}^3))$ fulfilling the initial condition (2.34), equation (3.31) and the estimate

$$\|\mathbf{u}\|_{H^1(0,\tau;\mathbf{W})} + \|\mathbf{z}\|_{L^\infty(\Gamma_C\times(0,\tau))} + \|\boldsymbol{\mu}\|_{L^\infty(0,\tau;L^{2+\nu}(\Gamma_C;\mathbb{R}^3))} \le S_1. \tag{3.32}$$

For later convenience, let us detail the proof of the estimate for $\|\mathbf{u}\|_{H^1(0,\tau;\mathbf{W})}$; the estimate for $\mathbf{z}$ simply derives from (2.18), while the bound for $\boldsymbol{\mu}$ follows from the calculations for the forthcoming *Third a priori estimate*, cf. Sec. 4. We choose $\mathbf{v} = \partial_t\mathbf{u}$ in (3.31) and integrate in time over $(0,t)$. In particular, we exploit the ellipticity properties (2.5) and integrate by parts. It follows from the Hölder inequality

$$\begin{aligned}&C_b\int_0^t \|\partial_t\mathbf{u}\|^2_{\mathbf{W}}\,\mathrm{d}s + \frac{C_a}{2}\|\mathbf{u}(t)\|^2_{\mathbf{W}} + \int_{\Gamma_C}\Phi_\varepsilon(\mathbf{u}(t))\,\mathrm{d}x \\ &\le c(\|\mathbf{u}_0\|^2_{\mathbf{W}} + \boldsymbol{\varphi}(\mathbf{u}_0)) + \int_0^t \|\widehat{\vartheta}\|_H\|\mathrm{div}\ \partial_t\mathbf{u}\|_H\,\mathrm{d}s + \int_0^t \|\widehat{\chi}\|_{H_{\Gamma_C}}\|\mathbf{u}\|_{L^4(\Gamma_C)}\|\partial_t\mathbf{u}\|_{L^4(\Gamma_C)}\,\mathrm{d}s \\ &\qquad + \int_0^t \|\mathbf{F}\|_{\mathbf{W}'}\|\partial_t\mathbf{u}\|_{\mathbf{W}}\,\mathrm{d}s + c.\end{aligned} \tag{3.33}$$

Note that here we have exploited the fact that

$$\int_{\Gamma_{\rm C}} \mathfrak{c}(\widehat{\vartheta}-\widehat{\vartheta}_s)\boldsymbol{\mu}\cdot\partial_t\mathbf{u}\,{\rm d}x \geq 0 \tag{3.34}$$

in view of (2.25). Hence, the right-hand side of (3.33) can be handled by Young's inequality combined with trace theorems and Sobolev embeddings, which give

$$\begin{aligned}&\int_0^t \|\widehat{\vartheta}\|_H \|{\rm div}\,\partial_t\mathbf{u}\|_H\,{\rm d}s + \int_0^t \|\widehat{\chi}\|_{H_{\Gamma_{\rm C}}}\|\mathbf{u}\|_{L^4(\Gamma_{\rm C})}\|\partial_t\mathbf{u}\|_{L^4(\Gamma_{\rm C})}\,{\rm d}s + \int_0^t \|\mathbf{F}\|_{\mathbf{W}'}\|\partial_t\mathbf{u}\|_{\mathbf{W}}\,{\rm d}s + c \\ &\leq \frac{C_b}{2}\int_0^t \|\partial_t\mathbf{u}\|^2_{\mathbf{W}}\,{\rm d}s + c\left(\|\widehat{\theta}\|^2_{L^2(0,\tau;H)} + \|\mathbf{F}\|^2_{L^2(0,\tau;\mathbf{W}')} + \int_0^t \|\widehat{\chi}\|^2_{H_{\Gamma_{\rm C}}}\|\mathbf{u}\|^2_{\mathbf{W}}\,{\rm d}s\right).\end{aligned} \tag{3.35}$$

Exploiting the Gronwall lemma, recalling (2.29) for $\mathbf{F}$ and (3.30) (in particular that $\|\widehat{\chi}\|^2_{H_{\Gamma_{\rm C}}}$ belongs to $L^1(0,\tau)$), we infer that $\mathbf{u}$ is bounded in $H^1(0,\tau;\mathbf{W})$ by some constant S_1 depending on the data of the problem (and on M) but not on ε. As a consequence, we may define an operator

$$\mathcal{T}_1 : \mathcal{Y}_\tau \to \mathcal{U}_\tau := \{\mathbf{u}\in H^1(0,\tau;\mathbf{W}) \,:\, \|\mathbf{u}\|_{H^1(0,\tau;\mathbf{W})} \leq S_1\} \tag{3.36}$$

which maps every quadruple $(\widehat{\vartheta},\widehat{\vartheta}_s,\widehat{\mathbf{u}},\widehat{\chi}) \in \mathcal{Y}_\tau$ into the unique solution $\mathbf{u}$ of the Cauchy problem for (3.31) (with associated $\boldsymbol{\mu}\in|\mathcal{R}({\rm D}\Phi_\varepsilon(\widehat{\mathbf{u}}))|\partial\Psi(\partial_t\mathbf{u}_{\rm T})$).

Step 2: As a second step, we consider (the Cauchy problem for) (3.28e), with $\widehat{\vartheta}_s\in\pi_2(\mathcal{Y}_\tau)$ and $\mathbf{u}=\mathcal{T}_1(\widehat{\vartheta},\widehat{\vartheta}_s,\widehat{\mathbf{u}},\widehat{\chi})$ in $\mathcal{U}_\tau$ on the right-hand side, that is

$$\partial_t\chi + \rho_\varepsilon(\partial_t\chi) + A\chi + \beta_\varepsilon(\chi) + \gamma'(\chi) = -\lambda'(\chi)\widehat{\vartheta}_s - \frac{1}{2}|\mathbf{u}|^2 \quad \text{a.e. in } \Gamma_{\rm C}\times(0,T)\,. \tag{3.37}$$

Standard results in the theory of parabolic equations (recall the Lipschitz continuity of β_ε and ρ_ε, as ε is fixed) ensure that there exists a constant $S_2>0$ (depending on M via S_1), and a unique function $\chi\in L^2(0,\tau;H^2(\Gamma_{\rm C}))\cap L^\infty(0,\tau;V_{\Gamma_{\rm C}})\cap H^1(0,\tau;H_{\Gamma_{\rm C}})$, fulfilling the initial condition (2.35), equation (3.37) and

$$\|\chi\|_{L^2(0,\tau;H^2(\Gamma_{\rm C}))\cap L^\infty(0,\tau;V_{\Gamma_{\rm C}})\cap H^1(0,\tau;H_{\Gamma_{\rm C}})} \leq S_2. \tag{3.38}$$

It follows that we may define an operator

$$\begin{aligned}\mathcal{T}_2 : \pi_2(\mathcal{Y}_\tau)\times\mathcal{U}_\tau \to \mathcal{X}_\tau := \{&\chi\in L^2(0,\tau;H^2(\Gamma_{\rm C}))\cap L^\infty(0,\tau;H^1(\Gamma_{\rm C}))\cap H^1(0,\tau;H_{\Gamma_{\rm C}}) \,: \\ &\|\chi\|_{L^2(0,\tau;H^2(\Gamma_{\rm C}))\cap L^\infty(0,\tau;H^1(\Gamma_{\rm C}))\cap H^1(0,\tau;H_{\Gamma_{\rm C}})} \leq S_2\}\end{aligned} \tag{3.39}$$

mapping $(\widehat{\vartheta}_s,\mathbf{u})\in\pi_2(\mathcal{Y}_\tau)\times\mathcal{U}_\tau$ into the unique solution χ of the Cauchy problem for (3.37).

Step 3: Finally, we consider the Cauchy problem for the system (3.28a, 3.28b) with fixed $(\widehat{\vartheta},\widehat{\vartheta}_s)\in\pi_{1,2}(\mathcal{Y}_\tau)$ and $\mathbf{u}=\mathcal{T}_1(\widehat{\vartheta},\widehat{\vartheta}_s,\widehat{\mathbf{u}},\widehat{\chi})$, $\chi=\mathcal{T}_2(\widehat{\vartheta}_s,\mathcal{T}_1(\widehat{\vartheta},\widehat{\vartheta}_s,\widehat{\mathbf{u}},\widehat{\chi}))$ from the previous steps. In particular, we set

$$\widehat{\mathcal{F}} := k(\chi)(\widehat{\vartheta}-\widehat{\vartheta}_s) + \mathfrak{c}'(\widehat{\vartheta}-\widehat{\vartheta}_s)|\mathcal{R}({\rm D}\Phi_\varepsilon(\mathbf{u}))|\Psi(\mathbf{u}_t) \tag{3.40}$$

and plug it into the boundary integral on the left-hand side of (3.28a) and on the right-hand side of (3.28b). Observe that, due to (3.30), (3.32), (3.38), to the Lipschitz continuity of $\mathfrak{c}$ and k, as well as (2.25) and Sobolev embeddings, and trace theorems, there holds

$$\widehat{\mathcal{F}}\in L^2(0,\tau;L^{4/3+s}(\Gamma_{\rm C})) \quad \text{for some } s=s(\delta)>0. \tag{3.41}$$

Now we consider the Cauchy problem for the system

$$\left\langle \partial_t\widetilde{L}_\varepsilon(\vartheta), v\right\rangle_V - \int_\Omega {\rm div}(\partial_t\mathbf{u})\,v\,{\rm d}x + \int_\Omega \nabla g(\vartheta)\,\nabla v\,{\rm d}x + \int_{\Gamma_{\rm C}} \widehat{\mathcal{F}} v\,{\rm d}x = \langle h, v\rangle_V \quad \forall\, v\in V\,, \tag{3.42}$$

$$\left\langle \partial_t\widetilde{\ell}_\varepsilon(\vartheta_s), v\right\rangle_{V_{\Gamma_{\rm C}}} - \int_{\Gamma_{\rm C}} \partial_t\lambda(\chi)\,v\,{\rm d}x + \int_{\Gamma_{\rm C}} \nabla f_\varepsilon(\vartheta_s)\,\nabla v\,{\rm d}x = \int_{\Gamma_{\rm C}} \widehat{\mathcal{F}} v\,{\rm d}x \quad \forall\, v\in V_{\Gamma_{\rm C}}\,, \tag{3.43}$$

a.e. in $(0,T)$. Observe that system (3.42, 3.43) is decoupled, hence we will handle equations (3.42) and (3.43) separately.

The well-posedness for the Cauchy problem for the doubly nonlinear equation (3.42) follows from standard results (cf. e.g. [23, Chap. 3, Thm. 4,1]), taking into account that both $\widetilde{L}_\varepsilon$ and g are bi-Lipschitz. We only sketch here the uniqueness proof for (3.42). We subtract the equation for ϑ_2 from the equation for ϑ_1 and integrate on $(0,t)$, with $0 \le t \le \tau$. Hence, we choose $v = g(\vartheta_1) - g(\vartheta_2)$ as test function and integrate again in time. Using that $\widetilde{L}_\varepsilon(r) = \varepsilon r + L_\varepsilon(r)$ and that the functions L_ε and g are strictly increasing (cf. also (2.6a)–(2.6b), (2.12)), we obtain

$$\begin{aligned} c_3\,\varepsilon\|\vartheta_1-\vartheta_2\|^2_{L^2(0,\tau;H_{\Gamma_C})} &\le \varepsilon\int_0^t\int_\Omega(\vartheta_1-\vartheta_2)(g(\vartheta_1)-g(\vartheta_2))\,\mathrm{d}x\,\mathrm{d}s \\ &+\int_0^t\int_\Omega(L_\varepsilon(\vartheta_1)-L_\varepsilon(\vartheta_2))(g(\vartheta_1)-g(\vartheta_2))\,\mathrm{d}x\,\mathrm{d}s+\frac{1}{2}\int_\Omega|(1*\nabla(g(\vartheta_1)-g(\vartheta_2))(t)|^2\,\mathrm{d}x \le 0, \end{aligned} \tag{3.44}$$

whence the desired uniqueness. Next, we test (3.42) by ϑ and integrate on $(0,t)$, with $0 \le t \le \tau$. Arguing in a very similar way as in the derivation of the subsequent *First a priori estimate* in Sec. 4, we deduce that there exists a positive constant S_3 such that

$$\|\vartheta\|_{L^2(0,\tau;V)\cap L^\infty(0,\tau;L^1(\Omega))} \le S_3, \tag{3.45}$$

$$\varepsilon^{1/2}\|\vartheta\|_{L^\infty(0,\tau;H)} \le S_3 \tag{3.46}$$

so that also $g(\vartheta) \in L^2(0,\tau;V)$. Thus, recalling (3.36) as well, by comparison in (3.42), we get

$$\|\partial_t\widetilde{L}_\varepsilon(\vartheta)\|_{L^2(0,\tau;V')} \le S_3. \tag{3.47}$$

Analogously, we handle equation (3.43) taking into account the monotonicity, the bi-Lipschitz continuity of $\widetilde{\ell}_\varepsilon$ and f_ε and the coercivity of j^*_ε (cf. Lemma A.2 and Lemma A.4). In particular, in order to conclude suitable estimates for ϑ_s, we test (3.43) by ϑ_s and argue as in the derivation of the forthcoming *First a priori estimate*. We obtain

$$\|\vartheta_s\|_{L^\infty(0,\tau;L^1(\Gamma_C))} + \|\vartheta_s\|_{L^\infty(0,\tau;H_{\Gamma_C})} \le S_4 \tag{3.48}$$

for some constant $S_4 > 0$. Moreover, we test (3.43) by $\widetilde{\ell}_\varepsilon(\vartheta_s)$. Proceeding as in the upcoming *Second a priori estimate*, we deduce

$$\|\vartheta_s\|_{L^2(0,\tau;V_{\Gamma_C})} \le S_5, \tag{3.49}$$

for some constant $S_5 > 0$. Observe that the latter estimate holds uniformly w.r.t $\varepsilon > 0$ and this property will be crucial to deduce that the local-existence time does not depend on $\widehat{T} > 0$. Finally, by comparison in (3.43), taking into account that f'_ε is bounded by a constant depending on ε (cf. (3.4) and (A.18)), we find

$$\|\partial_t\widetilde{\ell}_\varepsilon(\vartheta_s)\|_{L^2(0,\tau;V'_{\Gamma_C})} \le S_6(\varepsilon), \tag{3.50}$$

for some constant $S_6(\varepsilon) > 0$ depending on $\varepsilon > 0$ as well.

Therefore we may define an operator

$$\begin{aligned} &\mathcal{T}_3 : \pi_{1,2}(\mathcal{Y}_\tau)\times\mathcal{U}_\tau\times\mathcal{X}_\tau \to \\ &\mathcal{W}_\tau := \{(\vartheta,\vartheta_s) \in (L^2(0,\tau;V)\cap L^\infty(0,\tau;H))\times(L^2(0,\tau;V_{\Gamma_C})\cap L^\infty(0,\tau;H_{\Gamma_C})) : \\ &\qquad \|\vartheta\|_{L^2(0,\tau;V)\cap L^\infty(0,\tau;L^1(\Omega))} + \varepsilon^{1/2}\|\vartheta\|_{L^\infty(0,\tau;H)} \le S_3, \\ &\qquad \|\vartheta_s\|_{L^2(0,\tau;V_{\Gamma_C})\cap L^\infty(0,\tau;L^1(\Gamma_C))} + \varepsilon^{1/2}\|\vartheta_s\|_{L^\infty(0,\tau;H_{\Gamma_C})} \le S_4\} \end{aligned} \tag{3.51}$$

mapping $(\widehat{\vartheta},\widehat{\vartheta}_s,\mathbf{u},\chi) \in \pi_{1,2}(\mathcal{Y}_\tau)\times\mathcal{U}_\tau\times\mathcal{X}_\tau$ into the unique solution (ϑ,ϑ_s) of the Cauchy problem for system (3.42)–(3.43)). We are now in the position to prove the existence of local-in-time solutions to Problem (P_ε), defined on some interval $[0,\widehat{T}]$ with $0 < \widehat{T} \le T$. We stress that $\widehat{T}$ in fact will not depend on the parameter $\varepsilon > 0$, and such a property will be crucial in the forthcoming passage to the limit procedure.

Proposition 3.3 (Local existence for Problem (P_ε))**.** *Assume* (2.1)*, Hypotheses 2.1–2.6, and conditions* (2.28) *on the data* h, $\mathbf{f}$, $\mathbf{g}$, (2.30c)–(2.30d) *on* $\mathbf{u}_0$, χ_0, *and* (3.12)–(3.20) *on* ϑ_0^ε, $\vartheta_s^{0,\varepsilon}$.

Then, there exists $\widehat{T} \in (0,T]$ *such that for every* $\varepsilon > 0$ *Problem* (P_ε) *admits a solution* $(\vartheta, \vartheta_s, \mathbf{u}, \chi, \boldsymbol{\mu})$ *on the interval* $(0,\widehat{T})$*, fulfilling the* (approximate) energy identity

$$
\begin{aligned}
&\int_\Omega \mathcal{I}_\varepsilon(\vartheta(t))\,\mathrm{d}x + \int_s^t \int_\Omega g'(\vartheta)|\nabla\vartheta|^2\,\mathrm{d}x\,\mathrm{d}r + \int_{\Gamma_C} i_\varepsilon(\vartheta_s(t))\,\mathrm{d}x + \int_s^t \int_\Omega \nabla f_\varepsilon(\vartheta_s)\nabla\vartheta_s\,\mathrm{d}x\,\mathrm{d}r \\
&\quad + \int_s^t \int_{\Gamma_C} k(\chi)(\vartheta-\vartheta_s)^2\,\mathrm{d}x\,\mathrm{d}r + \int_s^t b(\partial_t\mathbf{u},\partial_t\mathbf{u})\,\mathrm{d}r + \frac{1}{2}a(\mathbf{u}(t),\mathbf{u}(t)) + \frac{1}{2}\int_{\Gamma_C}\chi(t)|\mathbf{u}(t)|^2\,\mathrm{d}x + \int_{\Gamma_C}\Phi_\varepsilon(\mathbf{u}(t))\,\mathrm{d}x \\
&\quad + \int_s^t \int_{\Gamma_C} \mathfrak{c}(\vartheta-\vartheta_s)\Psi(\partial_t\mathbf{u})|\mathcal{R}(\mathrm{D}\Phi_\varepsilon(\mathbf{u}))|\,\mathrm{d}x\,\mathrm{d}r + \int_s^t \int_{\Gamma_C} \mathfrak{c}'(\vartheta-\vartheta_s)\Psi(\partial_t\mathbf{u})|\mathcal{R}(\mathrm{D}\Phi_\varepsilon(\mathbf{u}))|(\vartheta-\vartheta_s)\,\mathrm{d}x\,\mathrm{d}r \\
&\quad + \int_s^t \int_{\Gamma_C} |\partial_t\chi|^2\,\mathrm{d}x\,\mathrm{d}r + \int_s^t \int_{\Gamma_C} \rho_\varepsilon(\chi_t)\chi_t\,\mathrm{d}x\,\mathrm{d}r + \frac{1}{2}\int_{\Gamma_C}|\nabla\chi(t)|^2\,\mathrm{d}x + \int_{\Gamma_C}\widehat{\beta}_\varepsilon(\chi(t))\,\mathrm{d}x + \int_{\Gamma_C}\gamma(\chi(t))\,\mathrm{d}x \\
&= \int_\Omega \mathcal{I}_\varepsilon(\vartheta(s))\,\mathrm{d}x + \int_{\Gamma_C} i_\varepsilon(\vartheta_s(s))\,\mathrm{d}x + \int_s^t \langle h,\vartheta\rangle_V\,\mathrm{d}r + \int_s^t \langle \mathbf{F},\partial_t\mathbf{u}\rangle_{\mathbf{W}}\,\mathrm{d}r + \frac{1}{2}a(\mathbf{u}(s),\mathbf{u}(s)) \\
&\quad + \int_{\Gamma_C}\Phi_\varepsilon(\mathbf{u}(s))\,\mathrm{d}x + \frac{1}{2}\int_{\Gamma_C}\chi(s)|\mathbf{u}(s)|^2\,\mathrm{d}x + \frac{1}{2}\int_{\Gamma_C}|\nabla\chi(s)|^2\,\mathrm{d}x + \int_{\Gamma_C}\widehat{\beta}_\varepsilon(\chi(s))\,\mathrm{d}x + \int_{\Gamma_C}\gamma(\chi(s))\,\mathrm{d}x
\end{aligned}
\tag{3.52}
$$

for all $0 \le s \le t \le \widehat{T}$.

Proof. Let the operator $\mathcal{T} : \mathcal{Y}_\tau \to \mathcal{W}_\tau \times \mathcal{U}_\tau \times \mathcal{X}_\tau$ be defined by

$$
\mathcal{T}(\widehat{\vartheta},\widehat{\vartheta}_s,\widehat{\mathbf{u}},\widehat{\chi}) := (\vartheta,\vartheta_s,\mathbf{u},\chi) \quad \text{with} \quad \begin{cases} \mathbf{u} := \mathcal{T}_1(\widehat{\vartheta},\widehat{\vartheta}_s,\widehat{\mathbf{u}},\widehat{\chi}), \\ \chi := \mathcal{T}_2(\widehat{\vartheta}_s,\mathbf{u}), \\ (\vartheta,\vartheta_s) := \mathcal{T}_3(\widehat{\vartheta},\widehat{\vartheta}_s,\mathbf{u},\chi). \end{cases} \tag{3.53}
$$

Our aim is now to show that there exists $\widehat{T} \in (0,T]$ such that for every $\varepsilon > 0$

$$
\mathcal{T} \text{ maps } \mathcal{Y}_{\widehat{T}} \text{ into itself,} \tag{3.54}
$$

$$
\begin{gathered}
\mathcal{T} : \mathcal{Y}_{\widehat{T}} \to \mathcal{Y}_{\widehat{T}} \quad \text{is compact and continuous w.r.t. the topology of} \\
L^2(0,\tau;H^{1-\delta}(\Omega)) \times L^2(0,\tau;H^{1-\delta}(\Gamma_C)) \times L^2(0,\tau;H^{1-\delta}(\Omega;\mathbb{R}^3)) \times L^2(0,\tau;H_{\Gamma_C}).
\end{gathered} \tag{3.55}
$$

We will prove (3.54) and (3.55) following the lines of the proof of [9, Proposition 4.9].

We shall not repeat the arguments leading to (3.55), as they are completely analogous to those in [9]. We only observe that, in the proof of the continuity of the operator $\mathcal{T}_2$ from (3.39), providing the solution of the Cauchy problem for (3.37), the limit passage in the term $\rho_\varepsilon(\partial_t\chi)$ can be easily handled in the very same way as in the forthcoming Sec. 5, cf. (5.40) later on. Arguing as in [9, Proposition 4.9], compactness and continuity of the operator $\mathcal{T}$ (3.53) in the (ϑ,ϑ_s)-component can be proved by first deriving compactness for $(\widetilde{L}_\varepsilon(\vartheta),\widetilde{\ell}_\varepsilon(\vartheta_s))$ (exploiting also estimates (3.47) and (3.50)). Since, for $\varepsilon > 0$ fixed, $\widetilde{L}_\varepsilon$ and $\widetilde{\ell}_\varepsilon$ are bi-Lipschitz (cf. Lemma A.4), we can then infer compactness in (ϑ,ϑ_s).

In what follows we will detail the proof of (3.54), highlighting that the final time $\widehat{T}$ for which (3.54) holds is independent of ε. Let $(\widehat{\vartheta},\widehat{\vartheta}_s,\widehat{\mathbf{u}},\widehat{\chi}) \in \mathcal{Y}_\tau$ be fixed, and let $(\vartheta,\vartheta_s,\mathbf{u},\chi) := \mathcal{T}(\widehat{\vartheta},\widehat{\vartheta}_s,\widehat{\mathbf{u}}\,\widehat{\chi})$. We use the interpolation inequality

$$
\|\vartheta(t)\|_{H^{1-\delta}(\Omega)} \le c\|\vartheta(t)\|_{H^1(\Omega)}^{1-\delta}\|\vartheta(t)\|_{L^2(\Omega)}^{\delta} \qquad \text{for a.a. } t \in (0,\tau) \tag{3.56}
$$

(cf. e.g. [12, Cor. 3.2]). Now, a further interpolation between the spaces $L^2(0,\tau;V)$ and $L^\infty(0,\tau;L^1(\Omega))$ and estimate (3.45) also yield the bound $\|\vartheta\|_{L^{10/3}(0,\tau;L^2(\Omega))} \le \bar{C}S_3$ for some interpolation constant $\bar{C}$. Integrating

(3.56) in time and using Hölder's inequality we therefore have

$$\begin{aligned}\|\vartheta\|^2_{L^2(0,t;H^{1-\delta}(\Omega))} &\le c\int_0^t \|\vartheta(s)\|^{2(1-\delta)}_{H^1(\Omega)}\|\vartheta(s)\|^{2\delta}_{L^2(\Omega)}\,\mathrm{d}s \\ &\le c\|\vartheta\|^{2(1-\delta)}_{L^2(0,\tau;H^1(\Omega))}t^{(2\delta)/5}\|\vartheta\|^{2\delta}_{L^{10/3}(0,\tau;L^2(\Omega))} \le CS_3^2 t^{(2\delta)/5}\,.\end{aligned} \tag{3.57}$$

We use the analogues of (3.56) for ϑ_s and $\mathbf{u}$ to estimate the norms $\|\vartheta_s\|_{L^2(0,t;H^1(\Gamma_{\mathrm{C}}))}$ and $\|\mathbf{u}\|_{L^2(0,t;H^{1-\delta}(\Omega;\mathbb{R}^3))}$, in the same way as in (3.57). For χ we trivially have $\|\chi\|^2_{L^2(0,t;H_{\Gamma_{\mathrm{C}}})} \le t\|\chi\|^2_{L^\infty(0,t;H_{\Gamma_{\mathrm{C}}})} \le tS_2^2$. Combining all of these estimates, which hold uniformly w.r.t. ε, we infer that there exists a sufficiently small $\widehat{T}>0$ for which (3.54) holds.

Thus we conclude that the operator $\mathcal{T}$ admits a fixed point in $\mathcal{Y}_{\widehat{T}}$. Hence, there exists a solution $(\vartheta,\vartheta_s,\mathbf{u},\chi,\boldsymbol{\mu})$ to the Cauchy problem for system (3.28a)–(3.28e) on the interval $(0,\widehat{T})$, with the regularity (3.21), (3.22), (2.31f), (2.31i) and, in addition,

$$\vartheta_s \in L^2(0,\widehat{T};V_{\Gamma_{\mathrm{C}}})\cap L^\infty(0,\widehat{T};H_{\Gamma_{\mathrm{C}}}), \tag{3.58}$$

$$\widetilde{\ell}_\varepsilon(\vartheta_s) \in L^2(0,\widehat{T};V_{\Gamma_{\mathrm{C}}})\cap \mathrm{C}^0([0,\widehat{T}];H_{\Gamma_{\mathrm{C}}})\cap H^1(0,\widehat{T};V'_{\Gamma_{\mathrm{C}}}), \tag{3.59}$$

$$\chi \in L^2(0,\widehat{T};H^2(\Gamma_{\mathrm{C}}))\cap L^\infty(0,\widehat{T};V_{\Gamma_{\mathrm{C}}})\cap H^1(0,\widehat{T};H_{\Gamma_{\mathrm{C}}}). \tag{3.60}$$

Moreover, to obtain the enhanced regularity properties (3.23)–(3.25) and (2.38d), we have to perform a further priori estimate on the (ϑ_s,χ)-component of the solution. First, we readily deduce

$$f_\varepsilon(\vartheta_s) \in L^2(0,\widehat{T};V_{\Gamma_{\mathrm{C}}})\cap L^\infty(0,\widehat{T};L^1(\Gamma_{\mathrm{C}})), \tag{3.61}$$

by (3.58) and the bi-Lipschitz continuity of f_ε. Moreover, we test (3.28b) by $\partial_t f_\varepsilon(\vartheta_s)$, (3.28e) by $\partial_t(A\chi+\beta_\varepsilon(\chi))$, add the resulting relations, and integrate in $(0,t)$, $t\in(0,\widehat{T})$. Arguing as in the derivation of the upcoming *Seventh a priori estimate*, we obtain

$$\partial_t\vartheta_s \in L^2(0,\widehat{T};H_{\Gamma_{\mathrm{C}}}), \tag{3.62}$$

$$f_\varepsilon(\vartheta_s) \in L^\infty(0,\widehat{T};V_{\Gamma_{\mathrm{C}}}), \tag{3.63}$$

$$\partial_t\chi \in L^2(0,\widehat{T};V_{\Gamma_{\mathrm{C}}}) \text{ and } A(\chi)\in L^\infty(0,\widehat{T};H_{\Gamma_{\mathrm{C}}}), \tag{3.64}$$

and, by comparison in (3.28e),

$$\partial_t\chi \in L^\infty(0,\widehat{T};H_{\Gamma_{\mathrm{C}}}). \tag{3.65}$$

Then, also taking into account the bi-Lipschitz continuity of f_ε and $\widetilde{\ell}_\varepsilon$, we conclude (3.23)–(3.25) and (2.38d).

Proof of the energy identity (3.52)**.** We test (3.28a) by ϑ, (3.28b) by ϑ_s, (3.28c) by $\partial_t\mathbf{u}$, and (3.28e) by $\partial_t\chi$, add the resulting relations, and integrate on (s,t), $t\in(0,T]$. The thermal expansion term in (3.28c) cancels out with the one from (3.28a), and so does $-\int_s^t\int_{\Gamma_{\mathrm{C}}}\partial_t\lambda(\chi)\vartheta_s\,\mathrm{d}x\,\mathrm{d}r$ with the corresponding term from (3.28e). Integrating by parts in time $\int_s^t\int_{\Gamma_{\mathrm{C}}}\chi\mathbf{u}\cdot\mathbf{u}_t\,\mathrm{d}x\,\mathrm{d}r$ we also have a cancellation with the term $-\frac{1}{2}\int_s^t\int_{\Gamma_{\mathrm{C}}}\chi_t|\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}r$ from (3.28e). We also use the *formal* chain rule (cf. Remark 4.1)

$$\Big\langle\partial_t\widetilde{L}_\varepsilon(\vartheta),\vartheta\Big\rangle_V = \int_\Omega \widetilde{L}'_\varepsilon(\vartheta)\partial_t\vartheta\vartheta\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathcal{I}_\varepsilon(\vartheta)\,\mathrm{d}x \tag{3.66}$$

yielding

$$\int_s^t\Big\langle\partial_t\widetilde{L}_\varepsilon(\vartheta),\vartheta\Big\rangle_V\,\mathrm{d}r = \int_\Omega\mathcal{I}_\varepsilon(\vartheta(t))\,\mathrm{d}x - \int_\Omega\mathcal{I}_\varepsilon(\vartheta(s))\,\mathrm{d}x$$

and the same for the term $\int_s^t\Big\langle\partial_t\widetilde{\ell}_\varepsilon(\vartheta_s),\vartheta_s\Big\rangle_{H^1(\Gamma_{\mathrm{C}})}\,\mathrm{d}r$, giving $\int_{\Gamma_{\mathrm{C}}} i_\varepsilon(\vartheta_s(t))\,\mathrm{d}x - \int_{\Gamma_{\mathrm{C}}} i_\varepsilon(\vartheta_s(s))\,\mathrm{d}x$. Instead, the chain rule identity

$$\int_s^t\int_{\Gamma_{\mathrm{C}}}\mathrm{D}\Phi_\varepsilon(\mathbf{u})\cdot\mathbf{u}_t\,\mathrm{d}x\,\mathrm{d}r = \int_{\Gamma_{\mathrm{C}}}\Phi_\varepsilon(\mathbf{u}(t))\,\mathrm{d}x - \int_{\Gamma_{\mathrm{C}}}\Phi_\varepsilon(\mathbf{u}(s))\,\mathrm{d}x$$

holds rigorously. Furthermore, we exploit that $\mathbf{z}\cdot\partial_t\mathbf{u} = \Psi(\partial_t\mathbf{u})$ a.e. in $\Gamma_{\rm C}\times(0,\widehat{T})$ by the 1-homogeneity of the functional Ψ. This gives rise to the tenth term on the left-hand side of (3.52). Then, we conclude (3.52). $\square$

4. Uniform w.r.t. ε a priori estimates

In this section, we perform a series of priori estimates on the solutions to Problem 3.1 (i.e. Problem (P_ε)). From them, we derive bounds on suitable norms of the local solutions, which hold uniformly w.r.t. the parameter $\varepsilon\in(0,1)$. Exploiting them, we will pass to the limit with ε in Problem 3.1 (P_ε) and conclude in Theorem 5.1 the existence of a local-in-time solution to Problem 2.5.

Let us mention in advance that the forthcoming a priori estimates are not, however, global in time. This *local character* manifests itself already with the *First a priori estimate* (i.e. the *energy estimate*), which derives from the approximate energy identity (3.52). More precisely, the problem is to estimate the left-hand side term

$$\frac{1}{2}\int_{\Gamma_{\rm C}}\chi(t)|\mathbf{u}(t)|^2\,{\rm d}x \tag{4.1}$$

therein. Indeed, since in Problem (P_ε) the maximal monotone operator β has been replaced by its Yosida regularization β_ε, the approximate solution χ is no longer guaranteed to be positive a.e. in $\Gamma_{\rm C}\times(0,\widehat{T})$. Therefore the term in (4.1) is not, a priori, estimated from below by a constant. On the other hand, it cannot be moved to the right-hand side of (3.52) and absorbed into the left-hand side by Hölder and Young inequalities, or by use of the Gronwall lemma, mainly due to a lack of regularity of the terms in the left-hand side.

That is why, in order to estimate (4.1) we will resort to the following argument. It follows from the fixed point procedure in Sec. 3.2 that the local solution $(\vartheta,\vartheta_s,\mathbf{u},\chi)$ whose existence we have proved in Prop. 3.3 belongs to $\mathcal{Y}_{\widehat{T}}$ from (3.30), whence

$$\|\vartheta\|_{L^2(0,\widehat{T};H^{1-\delta}(\Omega))}+\|\vartheta_s\|_{L^2(0,\widehat{T};H^{1-\delta}(\Gamma_{\rm C}))}+\|\mathbf{u}\|_{L^2(0,\widehat{T};H^{1-\delta}(\Omega;\mathbb{R}^3))}+\|\chi\|_{L^2(0,\widehat{T};H_{\Gamma_{\rm C}})}\le M, \tag{4.2}$$

where M does not depend on ε (see (3.30)). In addition, as estimates (3.32) and (3.38) do not depend on ε, we infer

$$\|\mathbf{u}\|_{H^1(0,\widehat{T};\mathbf{W})}+\|\chi\|_{H^1(0,\widehat{T};H_{\Gamma_{\rm C}})}\le c. \tag{4.3}$$

Exploiting (4.3) as well as well-known embedding and trace results, we then have

$$\frac{1}{2}\int_{\Gamma_{\rm C}}\chi(t)|\mathbf{u}(t)|^2\,{\rm d}x\le c\|\chi\|_{L^\infty(0,\widehat{T};H_{\Gamma_{\rm C}})}\|\mathbf{u}\|^2_{L^\infty(0,\widehat{T};L^4(\Gamma_{\rm C}))}\le c\|\chi\|_{L^\infty(0,\widehat{T};H_{\Gamma_{\rm C}})}\|\mathbf{u}\|^2_{L^\infty(0,\widehat{T};\mathbf{W})}\le c, \tag{4.4}$$

where c depends here on S_1,S_2, on $\widehat{T}$, but not on $\varepsilon>0$.

Clearly, since the *First a priori estimate* is not global in time, neither of the subsequent estimates is. Nonetheless, for notational simplicity in the following calculations we shall write T in place of the local-existence time $\widehat{T}$. Moreover, we shall omit to indicate the dependence on ε for the approximate solutions. We also mention that with the symbols c and C we will denote possibly different positive constants, depending on the data of the problem, but not on ε.

First a priori estimate. We consider the approximate energy identity (3.52) on the interval $(0,t)$, with $t\in(0,T)$ (i.e. in $(0,\widehat{T})$). Thanks to Lemma A.2 (cf. (A.9b) and (A.9c)) we have

$$\int_\Omega \mathcal{I}_\varepsilon(\vartheta(t))\,{\rm d}x\ge\frac{\varepsilon}{2}\|\vartheta(t)\|_H^2+c\|\vartheta(t)\|_{L^1(\Omega)}-C,$$
$$\int_{\Gamma_{\rm C}} i_\varepsilon(\vartheta_s(t))\,{\rm d}x\ge\frac{\varepsilon}{2}\|\vartheta_s(t)\|^2_{H_{\Gamma_{\rm C}}}+c\|\vartheta_s(t)\|_{L^1(\Gamma_{\rm C})}-C,$$

while (3.17) and (3.18) ensure that

$$\int_\Omega \mathcal{I}_\varepsilon(\vartheta_0^\varepsilon)\,{\rm d}x\le C\,,\qquad \int_\Omega i_\varepsilon({\vartheta_s}^{0,\varepsilon})\,{\rm d}x\le C\,.$$

Since g' is bounded from below by a strictly positive constant (see (2.12)), and so is f'_ε (cf. Lemma A.4), we also have

$$\int_0^t \int_\Omega g'(\vartheta)|\nabla\vartheta|^2 \,\mathrm{d}x\,\mathrm{d}s \geq c_3 \int_0^t \int_\Omega |\nabla\vartheta|^2 \,\mathrm{d}x\,\mathrm{d}s,$$
$$\int_0^t \int_{\Gamma_\mathrm{C}} \nabla f_\varepsilon(\vartheta_s)\nabla\vartheta_s \,\mathrm{d}x\,\mathrm{d}r = \int_0^t \int_{\Gamma_\mathrm{C}} \frac{1}{\widetilde{\ell}'_\varepsilon(\vartheta_s)}|\nabla\vartheta_s|^2 \,\mathrm{d}x\,\mathrm{d}r \geq \frac{\varepsilon}{3}\int_0^t \int_{\Gamma_\mathrm{C}} |\nabla\vartheta_s|^2 \,\mathrm{d}x\,\mathrm{d}r.$$

We observe that the term $\int_0^t \int_{\Gamma_\mathrm{C}} k(\chi)(\vartheta-\vartheta_s)^2 \,\mathrm{d}x\,\mathrm{d}s$ is positive thanks to (2.24), and so are the ninth, the tenth and the eleventh terms on the left-hand side of (3.52) by (2.15) and (2.25), respectively , i.e.

$$\begin{aligned}&\int_{\Gamma_\mathrm{C}} \Phi_\varepsilon(\mathbf{u}(t))\,\mathrm{d}x + \int_0^t \int_{\Gamma_\mathrm{C}} \mathfrak{c}(\vartheta-\vartheta_s)\Psi(\partial_t\mathbf{u})|\mathcal{R}(\mathrm{D}\Phi_\varepsilon(\mathbf{u}))|\,\mathrm{d}x\,\mathrm{d}r \\ &+ \int_0^t \int_{\Gamma_\mathrm{C}} \mathfrak{c}'(\vartheta-\vartheta_s)\Psi(\partial_t\mathbf{u})|\mathcal{R}(\mathrm{D}\Phi_\varepsilon(\mathbf{u}))|(\vartheta-\vartheta_s)\,\mathrm{d}x\,\mathrm{d}r \geq 0.\end{aligned} \tag{4.5}$$

Since $\rho_\varepsilon(0)=0$, we also infer that

$$\int_0^t \int_{\Gamma_\mathrm{C}} \rho_\varepsilon(\chi_t)\chi_t \,\mathrm{d}x\,\mathrm{d}s \geq 0$$

by monotonicity of ρ_ε. Furthermore, we estimate (cf. (A.5), here $\mathfrak{r}_\varepsilon$ denotes the resolvent of β)

$$\int_{\Gamma_\mathrm{C}} \widehat{\beta}_\varepsilon(\chi(t))\,\mathrm{d}x \geq \int_{\Gamma_\mathrm{C}} \widehat{\beta}(\mathfrak{r}_\varepsilon(\chi(t)))\,\mathrm{d}x \geq -c\int_{\Gamma_\mathrm{C}} |\mathfrak{r}_\varepsilon(\chi(t))|\,\mathrm{d}x - C \geq -c\int_{\Gamma_\mathrm{C}} |\chi(t)|\,\mathrm{d}x - C$$

since $\mathfrak{r}_\varepsilon$ is a contraction, yielding

$$|\mathfrak{r}_\varepsilon(\chi(t))| \leq |\mathfrak{r}_\varepsilon(\chi(t)) - \mathfrak{r}_\varepsilon(x_0)| + |\mathfrak{r}_\varepsilon(x_0)| \leq |\chi(t)| + |\mathfrak{r}_\varepsilon(x_0)| + |x_0| \leq |\chi(t)| + c$$

(where x_0 is a point in $\mathrm{dom}(\widehat{\beta})$). Moreover, since γ' is Lipschitz by (2.27), γ has at most quadratic growth, therefore

$$\int_{\Gamma_\mathrm{C}} \gamma(\chi(t))\,\mathrm{d}x \geq -c\int_{\Gamma_\mathrm{C}} |\chi(t)|^2\,\mathrm{d}x - C.$$

Finally, we estimate

$$\int_0^t \langle \mathbf{F}, \mathbf{u}_t\rangle_{\mathbf{W}}\,\mathrm{d}s \leq \int_0^t \|\mathbf{F}\|_{\mathbf{W}'}\|\mathbf{u}_t\|_{\mathbf{W}}\,\mathrm{d}s,$$
$$\int_0^t \langle h, \vartheta\rangle_V\,\mathrm{d}s \leq c\int_0^t \|h\|_{V'}(\|\vartheta\|_{L^1(\Omega)} + \|\nabla\vartheta\|_H)\,\mathrm{d}s.$$

All in all, taking into account (2.5) we conclude

$$\begin{aligned}&\frac{\varepsilon}{2}\|\vartheta(t)\|_H^2 + c\|\vartheta(t)\|_{L^1(\Omega)} + c_3\int_0^t\int_\Omega |\nabla\vartheta|^2\,\mathrm{d}x\,\mathrm{d}s + \frac{\varepsilon}{2}\|\vartheta_s(t)\|_{H_{\Gamma_\mathrm{C}}}^2 + c\|\vartheta_s(t)\|_{L^1(\Gamma_\mathrm{C})} + \frac{\varepsilon}{3}\int_0^t\int_{\Gamma_\mathrm{C}} |\nabla\vartheta_s|^2\,\mathrm{d}x\,\mathrm{d}s \\ &\quad + \frac{C_a}{2}\|\mathbf{u}(t)\|_{\mathbf{W}}^2 + C_b\int_0^t \|\partial_t\mathbf{u}\|_{\mathbf{W}}^2\,\mathrm{d}s + \int_0^t \|\partial_t\chi\|_{H_{\Gamma_\mathrm{C}}}^2\,\mathrm{d}s + \frac{1}{2}\|\nabla\chi(t)\|_{\Gamma_\mathrm{C}}^2 \\ &\leq C_0 + c\left(\int_0^t \|\mathbf{F}\|_{\mathbf{W}'}^2\,\mathrm{d}s + \int_0^t \|h\|_{V'}^2\,\mathrm{d}s\right) + \frac{C_b}{4}\int_0^t \|\mathbf{u}_t\|_{\mathbf{W}}^2\,\mathrm{d}s + \frac{c_3}{4}\int_0^t\int_\Omega |\nabla\vartheta|^2\,\mathrm{d}x\,\mathrm{d}s \\ &\quad + c\int_0^t \|h\|_{V'}\|\vartheta\|_{L^1(\Omega)}\,\mathrm{d}s + C'\|\chi(t)\|_{H_{\Gamma_\mathrm{C}}}^2 + C\,.\end{aligned} \tag{4.6}$$

Here, the constant C_0 depends on the initial data, in view of (3.17)–(3.18)and of conditions (2.30c)–(2.30d) on $\mathbf{u}_0$ and χ_0, and we have also used (4.4) to estimate the term in (4.1).

Hence, applying the Gronwall lemma we conclude (in addition to (4.3))

$$\varepsilon^{1/2}\|\vartheta\|_{L^\infty(0,T;H)} + \|\vartheta\|_{L^2(0,T;V)\cap L^\infty(0,T;L^1(\Omega))} \leq c, \tag{4.7}$$

$$\|\vartheta_s\|_{L^\infty(0,T;L^1(\Gamma_C))} \leq c, \tag{4.8}$$

$$\|\chi\|_{L^\infty(0,T;V_{\Gamma_C})} \leq c. \tag{4.9}$$

Second a priori estimate. We test (3.28a) by $\widetilde{L}_\varepsilon(\vartheta)$ and (3.28b) by $\widetilde{\ell}_\varepsilon(\vartheta_s)$, add the resulting relations and integrate in time. In particular, we use that the term $\int_0^t\int_{\Gamma_C} \nabla f_\varepsilon(\vartheta_s)\,\nabla\widetilde{\ell}_\varepsilon(\vartheta_s)\,\mathrm{d}x\,\mathrm{d}r$ on the left-hand side of the resulting inequality fulfills

$$\int_0^t\int_{\Gamma_C} \nabla f_\varepsilon(\vartheta_s)\,\nabla\widetilde{\ell}_\varepsilon(\vartheta_s)\,\mathrm{d}x\,\mathrm{d}r = \int_0^t\int_{\Gamma_C} \frac{1}{\widetilde{\ell}'_\varepsilon(\vartheta_s)}\nabla\vartheta_s\,\widetilde{\ell}'_\varepsilon(\vartheta_s)\,\nabla\vartheta_s\,\mathrm{d}x\,\mathrm{d}r = \int_0^t\int_{\Gamma_C} |\nabla\vartheta_s|^2\,\mathrm{d}x\,\mathrm{d}r, \tag{4.10}$$

thanks to (3.4). Then, we observe that (see (A.16) and (2.12))

$$\int_0^t\int_\Omega \nabla g(\vartheta)\,\nabla\widetilde{L}_\varepsilon(\vartheta)\,\mathrm{d}x\,\mathrm{d}r \geq \varepsilon c_3\int_0^t\int_\Omega |\nabla\vartheta|^2\,\mathrm{d}x\,\mathrm{d}r \geq 0. \tag{4.11}$$

Hence, integrating by parts in time and exploiting (4.3), (4.7)–(4.9), using the Young inequality and the Gronwall lemma it is a standard matter to get

$$\|\widetilde{L}_\varepsilon(\vartheta)\|_{L^\infty(0,T;H)} + \|\widetilde{\ell}_\varepsilon(\vartheta_s)\|_{L^\infty(0,T;H_{\Gamma_C})} \leq c, \tag{4.12}$$

whence

$$\|L_\varepsilon(\vartheta)\|_{L^\infty(0,T;H)} + \|\ell_\varepsilon(\vartheta_s)\|_{L^\infty(0,T;H_{\Gamma_C})} \leq c. \tag{4.13}$$

Moreover, (4.8) and (4.10) yield

$$\|\vartheta_s\|_{L^2(0,T;V_{\Gamma_C})} \leq c. \tag{4.14}$$

Third a priori estimate. We are now in the position to estimate $\boldsymbol{\mu}$ and $\mathrm{D}\Phi_\varepsilon(\mathbf{u})$ in (3.28c). Indeed, by a comparison in the equation, and the previous a priori estimates, there holds that

$$\|\mathrm{D}\Phi_\varepsilon(\mathbf{u}) + \mathfrak{c}(\vartheta-\vartheta_s)\boldsymbol{\mu}\|_{L^2(0,T;\mathbf{Y}'_{\Gamma_C})} \leq c.$$

Then, after observing that the two addenda are orthogonal thanks to assumption (2.16) combined with (3.7), it is straightforward to deduce that each of them is bounded in $L^2(0,T;\mathbf{Y}'_{\Gamma_C})$ arguing in the same way as in [8, Sec. 4] In particular, we get

$$\|\mathrm{D}\Phi_\varepsilon(\mathbf{u})\|_{L^2(0,T;\mathbf{Y}'_{\Gamma_C})} \leq c. \tag{4.15}$$

Now, taking into account that $\boldsymbol{\mu} = |\mathcal{R}(\mathrm{D}\Phi_\varepsilon(\mathbf{u}))|\mathbf{z}$, and the fact that $\|\mathbf{z}\|_{L^\infty(\Gamma_C\times(0,T))} \leq C$ by (2.18), and exploiting that $\mathcal{R} : L^2(0,T;\mathbf{Y}'_{\Gamma_C}) \to L^\infty(0,T;L^{2+\nu}(\Gamma_C;\mathbb{R}^3))$ is bounded by (2.20), we ultimately conclude

$$\|\boldsymbol{\mu}\|_{L^\infty(0,T;L^{2+\nu}(\Gamma_C;\mathbb{R}^3))} \leq c. \tag{4.16}$$

Fourth a priori estimate. We perform a comparison argument in (3.28a).We take into account the previously proved estimates (4.3), (4.7), (4.14), (4.15), and combine them with (2.14) and (2.24)–(2.25). In particular, we infer (cf. (3.40)–(3.41)) that $k(\chi)(\vartheta-\vartheta_s)+\mathfrak{c}'(\vartheta-\vartheta_s)|\mathcal{R}(\mathrm{D}\Phi_\varepsilon(\mathbf{u}))|\Psi(\mathbf{u}_t)$ is bounded in $L^2(0,T;L^{4/3+s}(\Gamma_C))$ for some $s>0$. Then, also taking into account (2.28a), we conclude that

$$\|\partial_t\widetilde{L}_\varepsilon(\vartheta)\|_{L^2(0,T;V')} \leq c. \tag{4.17}$$

Fifth a priori estimate - BV estimate for ϑ. Now, we formally test (3.28a) by $\frac{1}{\widetilde{L}'_\varepsilon(\vartheta)}v$, with $v \in W^{1,q}(\Omega)$, $q > 3$. Observe that this guarantees $v \in L^\infty(\Omega)$ and that its trace is in $L^\infty(\Gamma_{\rm C})$. The formal identity

$$\left\langle \partial_t \widetilde{L}_\varepsilon(\vartheta), \frac{1}{\widetilde{L}'_\varepsilon(\vartheta)} v \right\rangle_V = \int_\Omega \partial_t \vartheta v \,\mathrm{d}x \tag{4.18}$$

holds. Next, we exploit the Lipschitz continuity of $\frac{1}{\widetilde{L}'_\varepsilon}$ (cf. Lemma A.4) to deduce that

$$\left\| \frac{1}{\widetilde{L}'_\varepsilon(\vartheta)} \right\|_H \le c\|\vartheta\|_H + C, \quad \left\| \nabla \frac{1}{\widetilde{L}'_\varepsilon(\vartheta)} \right\|_H \le c\|\nabla\vartheta\|_H, \quad \text{whence} \quad \left\| \frac{1}{\widetilde{L}'_\varepsilon(\vartheta)} \right\|_V \le c\|\vartheta\|_V + C\,. \tag{4.19}$$

Thus, by (4.7)

$$\left\| \frac{1}{\widetilde{L}'_\varepsilon(\vartheta)} \right\|_{L^2(0,T;V)} \le c.$$

Therefore, also taking into account (2.12), we estimate

$$\left| \int_\Omega \mathrm{div}(\mathbf{u}_t) \frac{1}{\widetilde{L}'_\varepsilon(\vartheta)} v \,\mathrm{d}x \right| \le c\|\mathbf{u}_t\|_{\mathbf{W}} \left\| \frac{1}{\widetilde{L}'_\varepsilon(\vartheta)} \right\|_V \|v\|_{L^3(\Omega)} \le c\|\mathbf{u}_t\|_{\mathbf{W}}(\|\vartheta\|_V + 1)\|v\|_{L^3(\Omega)} \doteq f_1 \in L^1(0,T), \tag{4.20}$$

$$\begin{aligned} &\left| \int_\Omega \nabla g(\vartheta)\, \nabla \left(\frac{1}{\widetilde{L}'_\varepsilon(\vartheta)} v \right) \mathrm{d}x \right| \\ &\le c\|\nabla\vartheta\|_H \left\| \nabla \frac{1}{\widetilde{L}'_\varepsilon(\vartheta)} \right\|_H \|v\|_{L^\infty(\Omega)} + c\|\nabla\vartheta\|_H \left\| \frac{1}{\widetilde{L}'_\varepsilon(\vartheta)} \right\|_{L^6(\Omega)} \|\nabla v\|_{L^3(\Omega)} \doteq f_2 \in L^1(0,T), \end{aligned} \tag{4.21}$$

$$\left| \int_{\Gamma_{\rm C}} k(\chi)(\vartheta - \vartheta_s) \frac{1}{\widetilde{L}'_\varepsilon(\vartheta)} v \,\mathrm{d}x \right| \le \|k(\chi)(\vartheta - \vartheta_s)\|_{H_{\Gamma_{\rm C}}} \left\| \frac{1}{\widetilde{L}'_\varepsilon(\vartheta)} \right\|_{L^4(\Gamma_{\rm C})} \|v\|_{L^4(\Gamma_{\rm C})} \doteq f_3 \in L^1(0,T), \tag{4.22}$$

$$\begin{aligned} &\left| \int_{\Gamma_{\rm C}} \mathfrak{c}'(\vartheta - \vartheta_s) \Psi(\partial_t \mathbf{u}) |\mathcal{R}(\mathrm{D}\Phi_\varepsilon(\mathbf{u}))| \frac{1}{\widetilde{L}'_\varepsilon(\vartheta)} v \,\mathrm{d}x \right| \\ &\le \|\mathfrak{c}'(\vartheta - \vartheta_s) \Psi(\partial_t \mathbf{u}) |\mathcal{R}(\mathrm{D}\Phi_\varepsilon(\mathbf{u}))|\|_{L^{4/3}(\Gamma_{\rm C})} \left\| \frac{1}{\widetilde{L}'_\varepsilon(\vartheta)} \right\|_{L^4(\Gamma_{\rm C})} \|v\|_{L^\infty(\Gamma_{\rm C})} \doteq f_4 \in L^1(0,T), \end{aligned} \tag{4.23}$$

$$\left| \left\langle h, \frac{1}{\widetilde{L}'_\varepsilon(\vartheta)} v \right\rangle_V \right| \le c\|h\|_{V'} \left(\left\| \frac{1}{\widetilde{L}'_\varepsilon(\vartheta)} \right\|_V \|v\|_{W^{1,q}(\Omega)} \right) \doteq f_5 \in L^1(0,T), \tag{4.24}$$

where (4.20)–(4.24) follow from the previously proved estimates. Thus we conclude that

$$\exists\, \mathcal{F} \in L^1(0,T) \quad \text{for a.a. } t \in (0,T)\ \forall\, v \in W^{1,q}(\Omega)\ : \quad \left| \int_\Omega \partial_t \vartheta v \,\mathrm{d}x \right| \le \mathcal{F}(t)\|v\|_{W^{1,q}(\Omega)}$$

and this implies

$$\|\partial_t \vartheta\|_{L^1(0,T;W^{1,q}(\Omega)')} \le c. \tag{4.25}$$

Sixth a priori estimate. We test (3.28b) by $f_\varepsilon(\vartheta_s)$ and we integrate in time. Taking into account the definition (3.11) of H_ε, we have the formal identity

$$\int_0^t \langle \partial_t \ell_\varepsilon(\vartheta_s), f_\varepsilon(\vartheta_s) \rangle_{V_{\Gamma_{\rm C}}} \,\mathrm{d}r = \int_{\Gamma_{\rm C}} H_\varepsilon(\vartheta_s(t)) \,\mathrm{d}x - \int_{\Gamma_{\rm C}} H_\varepsilon(\vartheta_s^{0,\varepsilon}) \,\mathrm{d}x. \tag{4.26}$$

Thus, estimate (A.22) for H_ε (see Lemma A.5) as well as estimate (3.19) for $\int_{\Gamma_{\rm C}} H_\varepsilon(\vartheta_s^{0,\varepsilon}) \,\mathrm{d}x$ yield

$$\|f_\varepsilon(\vartheta_s)(t)\|_{L^1(\Gamma_{\rm C})} + \|\nabla f_\varepsilon(\vartheta_s)\|^2_{L^2(0,t;H_{\Gamma_{\rm C}})} \le C + I_1 + I_2 + I_3\,, \tag{4.27}$$

where (here we use Hölder's inequality and, in particular, (4.3), (4.7), (4.14))

$$\begin{aligned} I_1 = \int_0^t \int_{\Gamma_C} \lambda'(\chi)\partial_t\chi f_\varepsilon(\vartheta_s)\,\mathrm{d}x\,\mathrm{d}r &\le c\int_0^t \|\partial_t\chi\|_{H_{\Gamma_C}}\|f_\varepsilon(\vartheta_s)\|_{V_{\Gamma_C}}\,\mathrm{d}r \\ &\le c\int_0^t \|\partial_t\chi\|_{H_{\Gamma_C}}\left(\|f_\varepsilon(\vartheta_s)-m(f_\varepsilon(\vartheta_s))\|_{V_{\Gamma_C}}+\|m(f_\varepsilon(\vartheta_s))\|_{V_{\Gamma_C}}\right)\mathrm{d}r \\ &\le c\int_0^t \|\partial_t\chi\|_{H_{\Gamma_C}}\left(\|\nabla f_\varepsilon(\vartheta_s)\|_{H_{\Gamma_C}}+\|f_\varepsilon(\vartheta_s)\|_{L^1(\Gamma_C)}\right)\mathrm{d}r \\ &\le \delta\|\nabla f_\varepsilon(\vartheta_s)\|^2_{H_{\Gamma_C}} + c\int_0^t \|\partial_t\chi\|_{H_{\Gamma_C}}\|f_\varepsilon(\vartheta_s)\|_{L^1(\Gamma_C)}\,\mathrm{d}r + c\,, \end{aligned} \tag{4.28}$$

$$\begin{aligned} I_2 = \int_0^t \int_{\Gamma_C} k(\chi)(\vartheta-\vartheta_s)f_\varepsilon(\vartheta_s)\,\mathrm{d}x\,\mathrm{d}r &\le c\int_0^t (\|\chi\|_{V_{\Gamma_C}}+1)\|\vartheta-\vartheta_s\|_{H_{\Gamma_C}}\|f_\varepsilon(\vartheta_s)\|_{V_{\Gamma_C}}\,\mathrm{d}r \\ &\le c\int_0^t \left(\|\vartheta\|_{H_{\Gamma_C}}+\|\vartheta_s\|_{H_{\Gamma_C}}\right)\|f_\varepsilon(\vartheta_s)\|_{V_{\Gamma_C}}\,\mathrm{d}r \\ &\le \delta\|\nabla f_\varepsilon(\vartheta_s)\|^2_{H_{\Gamma_C}} + c\int_0^t \|\left(\|\vartheta\|_{H_{\Gamma_C}}+\|\vartheta_s\|_{H_{\Gamma_C}}\right)\|f_\varepsilon(\vartheta_s)\|_{L^1(\Gamma_C)}\,\mathrm{d}r + c\,, \end{aligned} \tag{4.29}$$

$$\begin{aligned} I_3 = \int_0^t \int_{\Gamma_C} \mathfrak{c}'(\vartheta-\vartheta_s)|\mathcal{R}(\mathrm{D}\Phi_\varepsilon(\mathbf{u}))||\Psi(\mathbf{u}_t)|f_\varepsilon(\vartheta_s)\,\mathrm{d}x\,\mathrm{d}r &\le c\int_0^t \|\mathcal{R}(\mathrm{D}\Phi_\varepsilon(\mathbf{u}))\|_{H_{\Gamma_C}}\|\mathbf{u}_t\|_{L^4(\Gamma_C)}\|f_\varepsilon(\vartheta_s)\|_{V_{\Gamma_C}}\,\mathrm{d}r \\ &\le c\int_0^t \|\mathbf{u}_t\|_{L^4(\Gamma_C)}\|f_\varepsilon(\vartheta_s)\|_{V_{\Gamma_C}}\,\mathrm{d}r \\ &\le \delta\|\nabla f_\varepsilon(\vartheta_s)\|^2_{H_{\Gamma_C}} + c\int_0^t \|\mathbf{u}_t\|\mathbf{w}\|f_\varepsilon(\vartheta_s)\|_{L^1(\Gamma_C)}\,\mathrm{d}r + c\,, \end{aligned} \tag{4.30}$$

for some sufficiently small $\delta > 0$; note that in (4.28) and (4.29) we have used that λ and k are Lipschitz, while (4.30) follows from the fact that $|\Psi(\mathbf{u}_t)| \le C|\mathbf{u}_t|$ thanks to (2.18). We apply the Gronwall lemma and we conclude

$$\|f(\vartheta_s)\|_{L^\infty(0,T;L^1(\Gamma_C))} + \|f(\vartheta_s)\|_{L^2(0,T;V_{\Gamma_C})} \le c. \tag{4.31}$$

Seventh estimate. We test (3.28b) by $\partial_t f_\varepsilon(\vartheta_s)$ (observe that, since f_ε is bi-Lipschitz, $\vartheta_s \in H^1(0,T;H_{\Gamma_C})$ implies $f_\varepsilon(\vartheta_s) \in H^1(0,T;H_{\Gamma_C})$), (3.28e) by $\partial_t(A\chi+\beta_\varepsilon(\chi))$, add the resulting relations, and integrate in time. In particular, we (formally) have

$$\int_0^t \int_{\Gamma_C} \partial_t(\widetilde{\ell}_\varepsilon(\vartheta_s))\,\partial_t(f_\varepsilon(\vartheta_s))\,\mathrm{d}x\,\mathrm{d}r = \int_0^t \int_{\Gamma_C} \widetilde{\ell}'_\varepsilon(\vartheta_s)\partial_t\vartheta_s\,\frac{1}{\widetilde{\ell}'_\varepsilon(\vartheta_s)}\partial_t\vartheta_s\,\mathrm{d}x\,\mathrm{d}r = \int_0^t \int_{\Gamma_C} |\partial_t\vartheta_s|^2\,\mathrm{d}x\,\mathrm{d}r\,.$$

Thus, taking into account the monotonicity of β_ε and ρ_ε, we obtain

$$\|\partial_t\vartheta_s\|^2_{L^2(0,t;H_{\Gamma_C})} + \frac{1}{2}\|\nabla f_\varepsilon(\vartheta_s(t))\|^2_{H_{\Gamma_C}} + \|\partial_t\chi\|^2_{L^2(0,t;V_{\Gamma_C})} + \frac{1}{2}\|A(\chi(t))+\xi(t)\|^2_{H_{\Gamma_C}} \le C + I_4 + I_5 + I_6 + I_7 + I_8, \tag{4.32}$$

where we have used the place-holder $\xi := \beta_\varepsilon(\chi)$. Now, systematically using estimate (A.21), we have (cf. also (2.24)–(2.27)),

$$
\begin{aligned}
I_4 = \int_0^t \int_{\Gamma_C} \lambda'(\chi)\partial_t\chi\partial_t f_\varepsilon(\vartheta_s)\,\mathrm{d}x\,\mathrm{d}r &\le \delta\int_0^t \|\partial_t\vartheta_s\|^2_{H_{\Gamma_C}}\,\mathrm{d}r + c\int_0^t\int_{\Gamma_C} |\partial_t\chi|^2|f'_\varepsilon(\vartheta_s)|^2\,\mathrm{d}x\,\mathrm{d}r\\
&\le \delta\int_0^t \|\partial_t\vartheta_s\|^2_{H_{\Gamma_C}}\,\mathrm{d}r + c\int_0^t \|\partial_t\chi\|_{H_{\Gamma_C}}\|\partial_t\chi\|_{L^4(\Gamma_C)}(\|f_\varepsilon(\vartheta_s)\|_{V_{\Gamma_C}}+c)\,\mathrm{d}r\\
&\le \delta\int_0^t \|\partial_t\vartheta_s\|^2_{H_{\Gamma_C}}\,\mathrm{d}r + \delta\int_0^t \|\partial_t\chi\|^2_{V_{\Gamma_C}}\,\mathrm{d}r\\
&\qquad + c\int_0^t \|\partial_t\chi\|^2_{H_{\Gamma_C}}\left(\|f_\varepsilon(\vartheta_s)\|^2_{V_{\Gamma_C}}+1\right)\,\mathrm{d}r\,,\\
I_5 = \int_0^t \int_{\Gamma_C} k(\chi)(\vartheta-\vartheta_s)\partial_t f_\varepsilon(\vartheta_s)\,\mathrm{d}x\,\mathrm{d}r &\le \delta\int_0^t \|\partial_t\vartheta_s\|^2_{H_{\Gamma_C}}\,\mathrm{d}r + c\int_0^t\int_{\Gamma_C}(\chi^2+1)(\vartheta^2+\vartheta_s^2)|f'_\varepsilon(\vartheta_s)|^2\,\mathrm{d}x\,\mathrm{d}r\\
&\le \delta\int_0^t \|\partial_t\vartheta_s\|^2_{H_{\Gamma_C}}\,\mathrm{d}r\\
&\qquad + c\int_0^t \left(\|\vartheta\|^2_{L^4(\Gamma_C)}+\|\vartheta_s\|^2_{L^4(\Gamma_C)}\right)\left(\|f_\varepsilon(\vartheta_s)\|_{V_{\Gamma_C}}+1\right)\,\mathrm{d}r\,,\\
I_6 = \int_0^t \int_{\Gamma_C} \mathfrak{c}'(\vartheta-\vartheta_s)|\mathcal{R}(\boldsymbol{\eta})|\Psi(\mathbf{u}_t)\partial_t f_\varepsilon(\vartheta_s)\,\mathrm{d}x\,\mathrm{d}r&\\
\le \delta\int_0^t \|\partial_t\vartheta_s\|^2_{H_{\Gamma_C}}\,\mathrm{d}r + c\int_0^t \|\mathcal{R}(\boldsymbol{\eta})\|^2_{L^{2+\nu}(\Gamma_C;\mathbb{R}^3)}\|\mathbf{u}_t\|^2_{L^4(\Gamma_C;\mathbb{R}^3)}\left(\|f_\varepsilon(\vartheta_s)\|_{V_{\Gamma_C}}+1\right)\,\mathrm{d}r&\\
\le \delta\int_0^t \|\partial_t\vartheta_s\|^2_{H_{\Gamma_C}}\,\mathrm{d}r + c\int_0^t \|\mathbf{u}_t\|^2_{\mathbf{W}}\left(\|f_\varepsilon(\vartheta_s)\|_{V_{\Gamma_C}}+1\right)\,\mathrm{d}r\,,&\\
I_7 = -\int_0^t \int_{\Gamma_C} (\gamma'(\chi)+\lambda'(\chi)\vartheta_s)\,\partial_t(A\chi+\xi)\,\mathrm{d}x\,\mathrm{d}r&\\
= \int_0^t \int_{\Gamma_C} (\gamma''(\chi)+\lambda''(\chi)\vartheta_s)\partial_t\chi(A\chi+\xi)\,\mathrm{d}x\,\mathrm{d}r + \int_0^t\int_{\Gamma_C}\lambda'(\chi)\partial_t\vartheta_s(A\chi+\xi)\,\mathrm{d}x\,\mathrm{d}r&\\
\quad - \int_{\Gamma_C}(\gamma'(\chi(t))+\lambda'(\chi(t))\vartheta_s(t))(A\chi(t)+\xi(t))\,\mathrm{d}x + \int_{\Gamma_C}(\gamma'(\chi_0)+\lambda'(\chi_0)\vartheta_s^0)(A\chi_0+\xi(0))\,\mathrm{d}x&\\
\le c\int_0^t \|\partial_t\chi\|_{V_{\Gamma_C}}(\|\vartheta_s\|_{V_{\Gamma_C}}+1)\|A\chi+\xi\|_{H_{\Gamma_C}}\,\mathrm{d}r + c\int_0^t \|\partial_t\vartheta_s\|_{H_{\Gamma_C}}\|A\chi+\xi\|_{H_{\Gamma_C}}\,\mathrm{d}r&\\
+ c(\|\chi(t)\|_{H_{\Gamma_C}}+\|\vartheta_s(t)\|_{H_{\Gamma_C}}+1)\|A\chi(t)+\xi(t)\|_{H_{\Gamma_C}} + c&\\
\le \frac{1}{4}\|A\chi(t)+\xi(t)\|^2_{H_{\Gamma_C}} + \delta\int_0^t \|\partial_t\vartheta_s\|^2_{H_{\Gamma_C}}\,\mathrm{d}r + \delta\int_0^t \|\partial_t\chi\|^2_{V_{\Gamma_C}}\,\mathrm{d}r + c\int_0^t\left(\|\vartheta_s\|^2_{V_{\Gamma_C}}+1\right)\|A\chi+\xi\|^2_{H_{\Gamma_C}}\,\mathrm{d}r&\\
+ c\|\vartheta_s\|^2_{L^\infty(0,T;H_{\Gamma_C})} + c\|\chi\|^2_{L^\infty(0,T;H_{\Gamma_C})} + c\,,&
\end{aligned}
$$

for some small $\delta > 0$, and

$$
\begin{aligned}
I_8 &= -\frac{1}{2}\int_0^t\int_{\Gamma_C}|\mathbf{u}|^2\partial_t(A\chi+\xi)\,\mathrm{d}x\,\mathrm{d}r\\
&= \int_0^t\int_{\Gamma_C}\mathbf{u}\mathbf{u}_t(A\chi+\xi)\,\mathrm{d}x\,\mathrm{d}r - \frac{1}{2}\int_{\Gamma_C}|\mathbf{u}(t)|^2(A\chi(t)+\xi(t))\,\mathrm{d}x + \frac{1}{2}\int_{\Gamma_C}|\mathbf{u}_0|^2(A\chi_0+\xi(0))\,\mathrm{d}x\\
&\le c\|\mathbf{u}\|^2_{L^\infty(0,T;\mathbf{W})}\int_0^t\|\mathbf{u}_t\|^2_{\mathbf{W}}\,\mathrm{d}r + \frac{1}{2}\int_0^t\|A\chi+\xi\|^2_{H_{\Gamma_C}}\,\mathrm{d}r + \frac{1}{8}\|A\chi(t)+\xi(t)\|^2_{H_{\Gamma_C}} + c\|\mathbf{u}\|^4_{L^\infty(0,T;\mathbf{W})} + c.
\end{aligned}
$$

We plug the above estimates for $I_4, \dots, I_8$ into (4.32), exploit the previously obtained bounds and apply the Gronwall Lemma. In this way, we conclude that

$$\|\vartheta_s\|_{H^1(0,T;H_{\Gamma_C})} + \|f_\varepsilon(\vartheta_s)\|_{L^\infty(0,T;V_{\Gamma_C})} + \|\chi\|_{H^1(0,T;V_{\Gamma_C})} + \|A\chi + \xi\|_{L^\infty(0,T;H_{\Gamma_C})} \leq c. \tag{4.33}$$

We deduce moreover

$$\|\chi\|_{L^\infty(0,T;H^2(\Gamma_C))} + \|\xi\|_{L^\infty(0,T;H_{\Gamma_C})} \leq c, \tag{4.34}$$

yielding (by a comparison in (3.28e))

$$\|\rho_\varepsilon(\partial_t \chi)\|_{L^\infty(0,T;H_{\Gamma_C})} \leq c. \tag{4.35}$$

Remark 4.1 (A fully rigorous derivation of the a priori estimates)**.** As already mentioned, the *First* and *Fifth estimates* are not yet rigorously justified in the framework of the approximate Problem 3.1. This is due to a lack of regularity for the term $\partial_t \widetilde{L}_\varepsilon(\vartheta_\varepsilon)$, which is only in $L^2(0,\widehat{T};V')$.

In order fully justify them, we should add a further viscosity contribution to the equation for ϑ, modulated by a second parameter $\nu > 0$, hence perform a double passage to the limit, first as $\nu \downarrow 0$ and secondly as $\varepsilon \downarrow 0$.

In the present paper we have chosen not to explore this, to avoid overburdening the analysis. We refer the reader to [6], where this procedure has been carried out in detail.

5. Local existence for Problem 2.5

In this section, we pass to the limit as $\varepsilon \downarrow 0$ in Problem 3.1 and deduce from the local existence result in Proposition 3.3 the following

Theorem 5.1. *Assume* (2.1) *and Hypotheses 2.1–2.6. Suppose that the data* $(h, \mathbf{f}, \mathbf{g})$ *and* $(\vartheta_0, \vartheta_s^0, \mathbf{u}_0, \chi_0)$ *fulfill* (2.28) *and* (2.30)*.*

Then, there exists $\widehat{T} > 0$ *such that Problem 2.5 admits an* energy solution $(\vartheta, \vartheta_s, \mathbf{u}, \chi, \boldsymbol{\eta}, \boldsymbol{\mu}, \xi, \zeta)$ *on* $(0,\widehat{T})$ *(in the sense of Definition 2.6), having in addition the regularity properties* (2.38) *on* $(0,\widehat{T})$*.*

Proof. Let $(\vartheta_\varepsilon, \vartheta_{s,\varepsilon}, \mathbf{u}_\varepsilon, \chi_\varepsilon, \boldsymbol{\mu}_\varepsilon)_\varepsilon$ be a family of solutions to Problem (P_ε) with $\mathbf{z}_\varepsilon$ as in (3.28d); in what follows, we shall use the place-holders $\boldsymbol{\eta}_\varepsilon := \mathrm{D}\Phi_\varepsilon(\mathbf{u}_\varepsilon)$ and $\xi_\varepsilon = \beta_\varepsilon(\chi_\varepsilon)$. Relying on the (uniform w.r.t. ε) a priori estimates (4.3), (4.7)–(4.9), (4.13)–(4.14), (4.16)–(4.17), (4.31), (4.33)–(4.35), by weak and weak* compactness arguments we deduce that, along a suitable subsequence (which we do not relabel) the following weak and weak* convergences hold as $\varepsilon \searrow 0$

$$\mathbf{u}_\varepsilon \rightharpoonup \mathbf{u} \quad \text{in } H^1(0,\widehat{T};\mathbf{W}), \tag{5.1}$$

$$\chi_\varepsilon \rightharpoonup^* \chi \quad \text{in } L^\infty(0,\widehat{T};H^2(\Gamma_C)) \cap H^1(0,\widehat{T};V_{\Gamma_C}), \tag{5.2}$$

$$\vartheta_\varepsilon \rightharpoonup \vartheta \quad \text{in } L^2(0,\widehat{T};V), \tag{5.3}$$

$$\vartheta_{s,\varepsilon} \rightharpoonup \vartheta_s \quad \text{in } L^2(0,\widehat{T};V_{\Gamma_C}), \tag{5.4}$$

$$\xi_\varepsilon \rightharpoonup^* \xi \quad \text{in } L^\infty(0,\widehat{T};H_{\Gamma_C}), \tag{5.5}$$

$$\boldsymbol{\eta}_\varepsilon \rightharpoonup \boldsymbol{\eta} \quad \text{in } L^2(0,\widehat{T};\mathbf{Y}'_{\Gamma_C}), \tag{5.6}$$

$$\boldsymbol{\mu}_\varepsilon \rightharpoonup^* \boldsymbol{\mu} \quad \text{in } L^\infty(0,\widehat{T};L^{2+\nu}(\Gamma_C;\mathbb{R}^3)), \tag{5.7}$$

$$\mathbf{z}_\varepsilon \rightharpoonup^* \mathbf{z} \quad \text{in } L^\infty(\Gamma_C \times (0,\widehat{T})), \tag{5.8}$$

$$\widetilde{L}_\varepsilon(\vartheta_\varepsilon) \rightharpoonup^* \omega \quad \text{in } L^\infty(0,\widehat{T};H) \cap H^1(0,\widehat{T};V'), \tag{5.9}$$

$$\widetilde{\ell}_\varepsilon(\vartheta_{s,\varepsilon}) \rightharpoonup^* \omega_s \quad \text{in } L^\infty(0,\widehat{T};H_{\Gamma_C}) \cap H^1(0,\widehat{T};V'_{\Gamma_C}), \tag{5.10}$$

$$f_\varepsilon(\vartheta_{s,\varepsilon}) \rightharpoonup \alpha \quad \text{in } L^2(0,\widehat{T};V_{\Gamma_C}), \tag{5.11}$$

$$\rho_\varepsilon(\partial_t \chi) \rightharpoonup^* \zeta \quad \text{in } L^\infty(0,\widehat{T};H_{\Gamma_C}). \tag{5.12}$$

The compactness results from [20] (cf. Thm. 4 and Cor. 5) yield in addition the following strong convergences

$$\mathbf{u}_\varepsilon \to \mathbf{u} \quad \text{in } \mathrm{C}^0([0,\widehat{T}]; H^{1-\delta}(\Omega;\mathbb{R}^3)) \quad \text{for all } \delta \in (0,1), \tag{5.13}$$

$$\chi_\varepsilon \to \chi \quad \text{in } \mathrm{C}^0([0,\widehat{T}]; H^{2-\delta}(\Gamma_{\mathrm{C}})) \quad \text{for all } \delta \in (0,2), \tag{5.14}$$

$$\vartheta_{s,\varepsilon} \to \vartheta_s \quad \text{in } L^2(0,\widehat{T}; L^\delta(\Gamma_{\mathrm{C}})) \text{ for all } \delta \in [1,+\infty), \tag{5.15}$$

$$\widetilde{L}_\varepsilon(\vartheta_\varepsilon) \to \omega \quad \text{in } \mathrm{C}^0([0,\widehat{T}]; V'), \tag{5.16}$$

$$\widetilde{\ell}_\varepsilon(\vartheta_{s,\varepsilon}) \to \omega_s \quad \text{in } \mathrm{C}^0([0,\widehat{T}]; V'_{\Gamma_{\mathrm{C}}}). \tag{5.17}$$

Note that by virtue of (5.6) and Hypothesis 2.4 on $\mathcal{R}$, it follows that

$$\mathcal{R}(\boldsymbol{\eta}_\varepsilon) \to \mathcal{R}(\boldsymbol{\eta}) \quad \text{in } L^\infty(0,\widehat{T}; L^{2+\nu}(\Gamma_{\mathrm{C}};\mathbb{R}^3)). \tag{5.18}$$

Hence, combining (5.7) and (5.8) (with (5.18)) we end up with

$$\boldsymbol{\mu} = |\mathcal{R}(\boldsymbol{\eta})|\mathbf{z}. \tag{5.19}$$

Finally, we combine estimate (4.7) with (4.25) and resort to the BV-version of the Aubin-Lions compactnes theorem (cf. again [20, Cor. 4]), to deduce that

$$\vartheta_\varepsilon \to \vartheta \quad \text{in } L^2(0,\widehat{T}; H^{1-\delta}(\Omega)) \quad \text{for all } \delta \in (0,1), \tag{5.20}$$

$$\vartheta_\varepsilon \to \vartheta \quad \text{in } L^2(0,\widehat{T}; L^{4-\delta}(\Gamma_{\mathrm{C}})) \quad \text{for all } \delta \in (0,4) \tag{5.21}$$

the latter convergence ensuing from (5.20) via trace and embedding theorems. Let us also point out that the strong convergences (5.13), (5.14), (5.15), and (5.20) imply (for suitable subsequences) a.e. convergence. Now, by (5.15) and and (2.25) we can identify the limit of $\mathfrak{c}(\vartheta_\varepsilon - \vartheta_{s,\varepsilon})$ and $\mathfrak{c}'(\vartheta_\varepsilon - \vartheta_{s,\varepsilon})$ in $L^2(0,\widehat{T}; L^\delta(\Gamma_{\mathrm{C}}))$ for any $\delta \in [1,4)$ and $L^q(\Gamma_{\mathrm{C}} \times (0,\widehat{T}))$ for all $q \in [1,+\infty)$, respectively (the latter convergence is guaranteed by a generalization of the Lebesgue theorem and by (2.12)).

Passage to the limit in the momentum equation. We can pass to the limit in (3.28c) as $\varepsilon \to 0$ by virtue of the previous convergences and conclude that the sextuple $(\vartheta, \vartheta_s, \mathbf{u}, \chi, \boldsymbol{\mu}, \boldsymbol{\eta})$ satisfies (2.36c) on $(0,\widehat{T})$, with $\boldsymbol{\mu} = |\mathcal{R}(\boldsymbol{\eta})|\mathbf{z}$ (cf. (5.19)). It remains to identify $\mathbf{z}$ as an element in $\partial\Psi(\mathbf{u}_t)$ (thus obtaining (2.36e) for $\boldsymbol{\mu}$), and $\boldsymbol{\eta}$ (cf. (5.6)) as an element in $\partial\boldsymbol{\varphi}(\mathbf{u})$.

For the latter task, we proceed as in [9] and test (3.28c) by $\mathbf{u}_\varepsilon$. By the previous convergences, lower semicontinuity arguments, and the fact that the limiting sextuple $(\vartheta, \vartheta_s, \mathbf{u}, \chi, \boldsymbol{\mu}, \boldsymbol{\eta})$ fulfills (2.36c), it is straightforward to check that

$$\limsup_{\varepsilon \searrow 0} \int_0^t \int_{\Gamma_{\mathrm{C}}} \boldsymbol{\eta}_\varepsilon \cdot \mathbf{u}_\varepsilon \,\mathrm{d}x\,\mathrm{d}r \le \int_0^t \langle \eta, \mathbf{u} \rangle_{\mathbf{Y}_{\Gamma_{\mathrm{C}}}} \,\mathrm{d}r \tag{5.22}$$

for all $t \in (0,\widehat{T})$. We use (5.22) to deduce that for all $v \in \mathbf{Y}_{\Gamma_{\mathrm{C}}}$ there holds

$$\begin{aligned}
\int_0^t \langle \eta, \mathbf{v} - \mathbf{u} \rangle_{\mathbf{Y}_{\Gamma_{\mathrm{C}}}} \,\mathrm{d}r \le \liminf_{\varepsilon \to 0} \int_0^t \int_{\Gamma_{\mathrm{C}}} \boldsymbol{\eta}_\varepsilon \cdot (\mathbf{v} - \mathbf{u}_\varepsilon) \,\mathrm{d}x\,\mathrm{d}r &\le \liminf_{\varepsilon \to 0} \int_0^t \int_{\Gamma_{\mathrm{C}}} (\Phi_\varepsilon(\mathbf{v}) - \Phi_\varepsilon(\mathbf{u}_\varepsilon)) \,\mathrm{d}x\,\mathrm{d}r \\
&\le \int_0^t \int_{\Gamma_{\mathrm{C}}} (\Phi(\mathbf{v}) - \Phi(\mathbf{u})) \,\mathrm{d}x\,\mathrm{d}r \\
&= \int_0^t (\boldsymbol{\varphi}(\mathbf{v}) - \boldsymbol{\varphi}(\mathbf{u})) \,\mathrm{d}r
\end{aligned}$$

where the third inequality is due to the fact that Φ_ε Mosco converges to Φ. We have thus shown that

$$\boldsymbol{\eta} \in \boldsymbol{\varphi}(\mathbf{u}) \quad \text{in } \mathbf{Y}'_{\Gamma_{\mathrm{C}}} \text{ a.e. in } (0,\widehat{T}). \tag{5.23}$$

Instead of directly proving that $\mathbf{z} \in \partial\Psi(\mathbf{u}_t)$ a.e. in $\Gamma_{\mathrm{C}} \times (0,\widehat{T})$, we will show that

$$\mathcal{J}_{(\vartheta,\vartheta_s,\boldsymbol{\eta})}(\mathbf{w}) - \mathcal{J}_{(\vartheta,\vartheta_s,\boldsymbol{\eta})}(\mathbf{u}_t) \ge \int_0^{\widehat{T}} \int_{\Gamma_{\mathrm{C}}} \mathfrak{c}(\vartheta - \vartheta_s)|\mathcal{R}(\boldsymbol{\eta})|\mathbf{z} \cdot (\mathbf{w} - \mathbf{u}_t) \,\mathrm{d}x\,\mathrm{d}t \tag{5.24}$$

for all $\mathbf{w} \in L^2(0,\widehat{T};L^4(\Gamma_{\rm C};\mathbb{R}^3))$, where the functional $\mathcal{J}_{(\vartheta,\vartheta_s,\boldsymbol{\eta})} : L^2(0,\widehat{T};L^4(\Gamma_{\rm C};\mathbb{R}^3)) \to [0,+\infty)$ is defined for all $\mathbf{v} \in L^2(0,\widehat{T};L^4(\Gamma_{\rm C};\mathbb{R}^3))$ by

$$\mathcal{J}_{(\vartheta,\vartheta_s,\boldsymbol{\eta})}(\mathbf{v}) := \int_0^{\widehat{T}} \int_{\Gamma_{\rm C}} \mathfrak{c}(\vartheta(x,t)-\vartheta_s(x,t))|\mathcal{R}(\boldsymbol{\eta})(x,t)|\Psi(\mathbf{v}(x,t))\,\mathrm{d}x\,\mathrm{d}t. \tag{5.25}$$

Clearly, $\mathcal{J}_{(\vartheta,\vartheta_s,\boldsymbol{\eta})}$ is a convex and lower semicontinuous functional on $L^2(0,\widehat{T};L^4(\Gamma_{\rm C};\mathbb{R}^3))$. It can be easily verified that the subdifferential $\partial\mathcal{J}_{(\vartheta,\vartheta_s,\boldsymbol{\eta})} : L^2(0,\widehat{T};L^4(\Gamma_{\rm C};\mathbb{R}^3)) \rightrightarrows L^2(0,\widehat{T};L^{4/3}(\Gamma_{\rm C};\mathbb{R}^3))$ of $\mathcal{J}_{(\vartheta,\vartheta_s,\boldsymbol{\eta})}$ is given at every $\mathbf{v} \in L^2(0,\widehat{T};L^4(\Gamma_{\rm C};\mathbb{R}^3))$ by

$$\mathbf{h} \in \partial\mathcal{J}_{(\vartheta,\vartheta_s,\boldsymbol{\eta})}(\mathbf{v}) \;\Leftrightarrow\; \begin{cases} \mathbf{h} \in L^2(0,\widehat{T};L^{4/3}(\Gamma_{\rm C})), \\ \mathbf{h}(x,t) \in \mathfrak{c}(\vartheta(x,t)-\vartheta_s(x,t))|\mathcal{R}(\boldsymbol{\eta})(x,t)|\partial\Psi(\mathbf{v}(x,t)) \end{cases} \tag{5.26}$$

for almost all $(x,t) \in \Gamma_{\rm C} \times (0,\widehat{T})$. Therefore, (5.24) will yield

$$\mathfrak{c}(\vartheta-\vartheta_s)|\mathcal{R}(\boldsymbol{\eta})|\mathbf{z} \in \mathfrak{c}(\vartheta(x,t)-\vartheta_s(x,t))|\mathcal{R}(\boldsymbol{\eta})(x,t)|\partial\Psi(\mathbf{u}_t(x,t))$$

whence (2.36e) for $\boldsymbol{\mu} = |\mathcal{R}(\boldsymbol{\eta})|\mathbf{z}$, also in view of (2.25). In order to show (5.24), we first observe that

$$\limsup_{\varepsilon\to 0} \int_0^{\widehat{T}} \int_{\Gamma_{\rm C}} \mathfrak{c}(\vartheta_\varepsilon-\vartheta_{s,\varepsilon})|\mathcal{R}(\boldsymbol{\eta}_\varepsilon)|\mathbf{z}_\varepsilon\cdot\partial_t\mathbf{u}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t \le \int_0^{\widehat{T}} \int_{\Gamma_{\rm C}} \mathfrak{c}(\vartheta-\vartheta_s)|\mathcal{R}(\boldsymbol{\eta})|\mathbf{z}\cdot\mathbf{u}_t\,\mathrm{d}x\,\mathrm{d}t, \tag{5.27}$$

which can be checked by testing (3.28c) by $\partial_t\mathbf{u}_\varepsilon$ and passing to the limit via the above convergences and lower semicontinuity arguments, and again the Mosco convergence of Φ_ε to Φ. Therefore, by (5.27), the previously obtained weak-strong convergence we have

$$\begin{aligned} \int_0^{\widehat{T}} \int_{\Gamma_{\rm C}} \mathfrak{c}(\vartheta-\vartheta_s)|\mathcal{R}(\boldsymbol{\eta})|\mathbf{z}\cdot(\mathbf{w}-\mathbf{u}_t)\,\mathrm{d}x\,\mathrm{d}t &\le \liminf_{\varepsilon\to 0} \int_0^{\widehat{T}} \int_{\Gamma_{\rm C}} \mathfrak{c}(\vartheta_\varepsilon-\vartheta_{s,\varepsilon})|\mathcal{R}(\boldsymbol{\eta}_\varepsilon)|\mathbf{z}_\varepsilon\cdot(\mathbf{w}-\partial_t\mathbf{u}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \\ &\le \liminf_{\varepsilon\to 0} \int_0^{\widehat{T}} \int_{\Gamma_{\rm C}} \mathfrak{c}(\vartheta_\varepsilon-\vartheta_{s,\varepsilon})|\mathcal{R}(\boldsymbol{\eta}_\varepsilon)|(\Psi(\mathbf{w})-\Psi(\partial_t\mathbf{u}_\varepsilon))\,\mathrm{d}x\,\mathrm{d}t \\ &\le \int_0^{\widehat{T}} \int_{\Gamma_{\rm C}} \mathfrak{c}(\vartheta-\vartheta_s)|\mathcal{R}(\boldsymbol{\eta})|(\Psi(\mathbf{w})-\Psi(\mathbf{u}_t))\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{5.28}$$

Then, (5.24) ensues. Furthermore, arguing as in the derivation of (5.28), a.e. in $\Gamma_{\rm C} \times (0,\widehat{T})$, we deduce

$$\liminf_{\varepsilon\to 0} \int_0^{\widehat{T}} \int_{\Gamma_{\rm C}} \mathfrak{c}(\vartheta_\varepsilon-\vartheta_{s,\varepsilon})|\mathcal{R}(\boldsymbol{\eta}_\varepsilon)|\mathbf{z}_\varepsilon\cdot\partial_t\mathbf{u}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t \ge \int_0^{\widehat{T}} \int_{\Gamma_{\rm C}} \mathfrak{c}(\vartheta-\vartheta_s)|\mathcal{R}(\boldsymbol{\eta})|\mathbf{z}\cdot\mathbf{u}_t\,\mathrm{d}x\,\mathrm{d}t\,. \tag{5.29}$$

Ultimately, from (5.27) and (5.29), taking into account that $\mathbf{z}_\varepsilon\cdot\partial_t\mathbf{u}_\varepsilon = \Psi(\partial_t\mathbf{u}_\varepsilon)$ and the same for $\mathbf{z}$ and $\mathbf{u}_t$, we conclude

$$\lim_{\varepsilon\to 0} \int_0^{\widehat{T}} \int_{\Gamma_{\rm C}} \mathfrak{c}(\vartheta_\varepsilon-\vartheta_{s,\varepsilon})|\mathcal{R}(\boldsymbol{\eta}_\varepsilon)|\Psi(\partial_t\mathbf{u}_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t = \int_0^{\widehat{T}} \int_{\Gamma_{\rm C}} \mathfrak{c}(\vartheta-\vartheta_s)|\mathcal{R}(\boldsymbol{\eta})|\Psi(\partial_t\mathbf{u})\,\mathrm{d}x\,\mathrm{d}t\,. \tag{5.30}$$

We can develop a lim sup argument completely analogous to the one leading to (5.22) and conclude that

$$\limsup_{\varepsilon\to 0} \int_0^{\widehat{T}} b(\partial_t\mathbf{u}_\varepsilon,\partial_t\mathbf{u}_\varepsilon)\,\mathrm{d}t \le \int_0^{\widehat{T}} b(\partial_t\mathbf{u},\partial_t\mathbf{u})\,\mathrm{d}t \tag{5.31}$$

so that it follows

$$\lim_{\varepsilon\to 0} \int_0^{\widehat{T}} b(\partial_t\mathbf{u}_\varepsilon,\partial_t\mathbf{u}_\varepsilon)\,\mathrm{d}t = \int_0^{\widehat{T}} b(\partial_t\mathbf{u},\partial_t\mathbf{u})\,\mathrm{d}t.$$

This gives, by the $\mathbf{W}$-ellipticity of b (cf. (2.5)), the following strong convergence

$$\partial_t\mathbf{u}_\varepsilon \to \partial_t\mathbf{u} \quad \text{in } L^2(0,\widehat{T};\mathbf{W}). \tag{5.32}$$

Passage to the limit in the equation for ϑ. To pass to the limit in (3.28a) we combine the bi-Lipschitz continuity (2.12) of g with convergence (5.20) to conclude that

$$g'(\vartheta_\varepsilon) \to g'(\vartheta) \quad \text{in } L^q(\Omega \times (0,\widehat{T})) \quad \text{for all } q \in (1,\infty). \tag{5.33}$$

Taking into account that $(\nabla g(\vartheta_\varepsilon))_\varepsilon$ is bounded in $L^2(0,\widehat{T};H)$ (by (4.7) and (2.12)), we therefore conclude

$$\nabla g(\vartheta_\varepsilon) \rightharpoonup \nabla g(\vartheta) \quad \text{in } L^2(0,\widehat{T};H). \tag{5.34}$$

It follows from convergence (5.14) for χ_ε, (5.15) for $\vartheta_{s,\varepsilon}$, (5.21) for ϑ_ε, and (2.24) on k, that

$$k(\chi_\varepsilon)(\vartheta_\varepsilon - \vartheta_{s,\varepsilon}) \to k(\chi)(\vartheta - \vartheta_s) \quad \text{in } L^2(0,\widehat{T};H_{\Gamma_{\mathrm{C}}})\,. \tag{5.35}$$

Relying on the strong convergence (5.32) of $\mathbf{u}_\varepsilon$ and on the previously proved convergences for ϑ_ε, $\vartheta_{s,\varepsilon}$, and $\boldsymbol{\eta}_\varepsilon$, we show for the frictional contribution that

$$\lim_{\varepsilon\to 0}\int_0^{\widehat{T}}\int_{\Gamma_{\mathrm{C}}} \mathfrak{c}'(\vartheta_\varepsilon - \vartheta_{s,\varepsilon})|\mathcal{R}(\boldsymbol{\eta}_\varepsilon)|\Psi(\partial_t \mathbf{u}_\varepsilon)v\,\mathrm{d}x\,\mathrm{d}t = \int_0^{\widehat{T}}\int_{\Gamma_{\mathrm{C}}} \mathfrak{c}'(\vartheta - \vartheta_s)|\mathcal{R}(\boldsymbol{\eta})|\Psi(\mathbf{u}_t)v\,\mathrm{d}x\,\mathrm{d}t \quad \text{for all } v \in V.$$

Ultimately, on account of convergence (5.9) for $\widetilde{L}_\varepsilon(\vartheta_\varepsilon)$ we conclude that the functions $(\omega, \mathbf{u}, \chi, \vartheta, \vartheta_s, \boldsymbol{\eta})$ fulfill the weak formulation (2.36a) of the equation for ϑ. It then remains to prove that

$$\omega = L(\vartheta) \quad \text{a.e. in } \Omega \times (0,\widehat{T}). \tag{5.36}$$

This follows from the lim sup inequality

$$\limsup_{\varepsilon\searrow 0}\int_0^T\int_\Omega \widetilde{L}_\varepsilon(\vartheta_\varepsilon)\vartheta_\varepsilon\,\mathrm{d}x\,\mathrm{d}t \le \int_0^T\int_\Omega \omega\vartheta\,\mathrm{d}x\,\mathrm{d}t\,, \tag{5.37}$$

(which in turn ensues from combining the weak convergence (5.9) of $\widetilde{L}_\varepsilon(\vartheta_\varepsilon)$ and the strong (5.20) of ϑ_ε), taking into account that $\widetilde{L}_\varepsilon$ converges in the sense of graphs to L. In this way we conclude that $(\mathbf{u}, \chi, \vartheta, \vartheta_s, \boldsymbol{\eta})$ fulfill (2.36a).

Passage to the limit in the equation for ϑ_s. It proceeds exactly along the same lines as the limit passage to (2.36a) (cf. also the proof of [9, Thm. 1]). Let us only comment on the proof of

$$\nabla f_\varepsilon(\vartheta_{s,\varepsilon}) \rightharpoonup \nabla f(\vartheta_s) \quad \text{in } L^2(0,\widehat{T};H_{\Gamma_{\mathrm{C}}}). \tag{5.38}$$

Indeed, in view of convergence (5.15) for $\vartheta_{s,\varepsilon}$ and of Lemma (A.6), we easily conclude, e.g., that $f_\varepsilon(\vartheta_{s,\varepsilon}) \to f(\vartheta_s)$ in $L^1(\Gamma_{\mathrm{C}} \times (0,\widehat{T}))$, and thus a.e.. This is enough to identify the weak limit of $f_\varepsilon(\vartheta_{s,\varepsilon})$ in (5.11) and conclude (5.38).

Passage to the limit in the equation for χ. Finally, we pass to the limit in (3.28e) by virtue of convergences (5.2), (5.5), (5.12)–(5.14), and (5.15), also taking into account the properties (2.26) of λ. By the strong-weak closedness of the graph of (the maximal monotone operator induced by β on $H_{\Gamma_{\mathrm{C}}}$), we can directly identify ξ as an element of $\beta(\chi)$ a.e. in $\Gamma_{\mathrm{C}} \times (0,\widehat{T})$. It remains to show that

$$\zeta \in \rho(\chi_t) \quad \text{a.e. in } \Gamma_{\mathrm{C}} \times (0,\widehat{T}). \tag{5.39}$$

To this aim, we test (3.28e) by $\partial_t\chi_\varepsilon$ and prove that

$$\limsup_{\varepsilon\searrow 0}\int_0^t\int_\Omega \rho_\varepsilon(\partial_t\chi_\varepsilon)\partial_t\chi_\varepsilon\,\mathrm{d}x\,\mathrm{d}r \le \int_0^t\int_\Omega \zeta\partial_t\chi\,\mathrm{d}x\,\mathrm{d}r. \tag{5.40}$$

Therefore, (5.39) follows. As a byproduct, with the same arguments as in the previous lines we have

$$\lim_{\varepsilon\searrow 0}\int_0^t\int_\Omega \rho_\varepsilon(\partial_t\chi_\varepsilon)\partial_t\chi_\varepsilon\,\mathrm{d}x\,\mathrm{d}r = \int_0^t\int_\Omega \zeta\partial_t\chi\,\mathrm{d}x\,\mathrm{d}r. \tag{5.41}$$

Proof of the energy inequality (2.37)**.** We take the limit as $\varepsilon \to 0$ in the approximate energy inequality (3.52): let us only justify the passage to the limit in some of the terms on the left- and on the right-hand side. First of all, combining convergence (5.20) for (ϑ_ε) with the upcoming Lemma A.3 we obtain that for almost all $t \in (0, \widehat{T})$

$$\lim_{\varepsilon\to 0} \int_\Omega \mathcal{I}_\varepsilon(\vartheta_\varepsilon(t))\,\mathrm{d}x = \int_\Omega J^*(L(\vartheta(t)))\,\mathrm{d}x\,.$$

The convergence for $t = 0$ follows from condition (3.15).

Analogously, we pass to the limit in the term $\int_{\Gamma_\mathrm{C}} i_\varepsilon(\vartheta_{s,\varepsilon}(r))\,\mathrm{d}x$ for $r = t$, s and for $r = 0$. The weak convergence (5.3) of $\nabla\vartheta_\varepsilon$ and the strong convergence (5.33) of $g'(\vartheta_\varepsilon)$ allow us to conclude, via the Ioffe theorem [19], that

$$\liminf_{\varepsilon\to 0} \int_s^t \int_\Omega g'(\vartheta_\varepsilon)|\nabla\vartheta_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}r \geq \int_s^t \int_\Omega g'(\vartheta)|\nabla\vartheta|^2\,\mathrm{d}x\,\mathrm{d}r.$$

To take the limit in the term $\int_s^t \int_\Omega f'_\varepsilon(\vartheta_{s,\varepsilon})|\nabla\vartheta_{s,\varepsilon}|^2\,\mathrm{d}x\,\mathrm{d}r = \int_s^t \int_\Omega \nabla(f_\varepsilon(\vartheta_{s,\varepsilon}))\cdot\nabla\vartheta_{s,\varepsilon}\,\mathrm{d}x\,\mathrm{d}r$ we proceed in a completely analogous way, taking into account Lemma A.6. The passage to the limit in the fifth term on the left-hand side of (3.52) results from (5.35), and for the term $\int_s^t \int_{\Gamma_\mathrm{C}} \mathfrak{c}(\vartheta_\varepsilon - \vartheta_{s,\varepsilon})\Psi(\partial_t \mathbf{u}_\varepsilon)|\mathcal{R}(\boldsymbol{\eta}_\varepsilon)|\,\mathrm{d}x\,\mathrm{d}r$ it follows from (5.30). We use the strong convergences (5.15) and (5.20) for $\vartheta_{s,\varepsilon}$ and ϑ_ε combined with the properties (2.25) of $\mathfrak{c}$, (5.18) for $\mathcal{R}(\boldsymbol{\eta}_\varepsilon)$, the fact that $\Psi(\mathbf{u}_t) \to \Psi(\mathbf{u})$ in $L^2(0,T;\mathbf{H}_{\Gamma_\mathrm{C}})$, and the Ioffe theorem, to infer that

$$\liminf_{\varepsilon\to 0} \int_s^t \int_{\Gamma_\mathrm{C}} \mathfrak{c}'(\vartheta_\varepsilon - \vartheta_{s,\varepsilon})\Psi(\partial_t \mathbf{u}_\varepsilon)|\mathcal{R}(\boldsymbol{\eta}_\varepsilon)|\,\mathrm{d}x\,\mathrm{d}r \geq \int_s^t \int_{\Gamma_\mathrm{C}} \mathfrak{c}'(\vartheta - \vartheta_s)\Psi(\mathbf{u}_t)|\mathcal{R}(\boldsymbol{\eta})|\,\mathrm{d}x\,\mathrm{d}r.$$

Finally, observe that for almost all $t \in (0,T)$

$$\lim_{\varepsilon\to 0} \int_{\Gamma_\mathrm{C}} \widehat{\beta}_\varepsilon(\chi_\varepsilon(t))\,\mathrm{d}x = \int_{\Gamma_\mathrm{C}} \widehat{\beta}(\chi(t))\,\mathrm{d}x.$$

This can be checked by observing that, on the one hand, by Mosco convergence of $\widehat{\beta}_\varepsilon$ to $\widehat{\beta}$,

$$\liminf_{\varepsilon\to 0} \int_{\Gamma_\mathrm{C}} \widehat{\beta}_\varepsilon(\chi_\varepsilon(t))\,\mathrm{d}x \geq \int_{\Gamma_\mathrm{C}} \widehat{\beta}(\chi(t))\,\mathrm{d}x. \tag{5.42}$$

For the lim sup inequality we use that $\widehat{\beta}_\varepsilon(\chi_\varepsilon(t)) \leq \widehat{\beta}(\chi(t)) + \beta_\varepsilon(\chi_\varepsilon(t))(\chi_\varepsilon(t) - \chi(t))$ a.e. in Γ_c, and combine convergences (5.5) and (5.14). For $t = 0$ we have $\widehat{\beta}_\varepsilon(\chi_0) \to \widehat{\beta}(\chi_0)$ in $L^1(\Gamma_\mathrm{C})$ by the dominated convergence theorem.

All the remaining terms on the left- and on the right-hand side of (3.52) can be dealt with exploiting the previously proved convergences. This concludes the proof (2.37) on the interval (s,t) for almost all s, $t \in (0,\widehat{T})$ and for $s = 0$. □

6. Extension to a global-in-time solution and proof of Theorem 2.7

In this Section we show that the local solution to Problem 2.5 found in Theorem 5.1 (hereafter, we shall denote it by $(\widehat{\vartheta}, \widehat{\vartheta}_s, \widehat{\mathbf{u}}, \widehat{\chi}, \widehat{\boldsymbol{\eta}}, \widehat{\mu}, \widehat{\xi}, \widehat{\zeta})$), actually extends from the interval $(0,\widehat{T})$ to the whole $(0,T)$.

To this aim, we first of all observe that the "energy estimates" (cf. the *First estimate*) derived from the energy inequality (2.37) have a *global-in-time* character. Nonetheless, we cannot derive from such global bounds the other estimates (i.e. the *Second–Seventh estimates*), and therefore we cannot directly extend the local solution $(\widehat{\vartheta}, \widehat{\vartheta}_s, \widehat{\mathbf{u}}, \widehat{\chi})$, along with $(\widehat{\boldsymbol{\eta}}, \widehat{\mu}, \widehat{\xi}, \widehat{\zeta})$, to the whole interval $(0,T)$. The reason for this is that, as expounded in Sec. 3.1 and shown in Sec. 4, these estimates involve calculations which are only formal on the level of the limit problem. Thus, we need to perform them on the regularized system from Problem (P_ε). However, the energy estimates have only a local character for the approximate problem, since the term $\int_{\Gamma_\mathrm{C}} \chi(t)|\mathbf{u}(t)|^2\,\mathrm{d}x$ is estimated locally in time, cf. the discussion at the beginning of Sec. 4 and (4.2).

Therefore, along the lines of the prolongation argument from [4], we shall proceed in the following way. We will extend the local solution $(\widehat{\vartheta}, \widehat{\vartheta}_s, \widehat{\mathbf{u}}, \widehat{\chi})$ together with its *approximability properties* (cf. the notion of *approximable solution* in Definition 6.1 below). In this way, the approximate solutions shall "inherit" the

global-in-time energy estimates from $(\widehat{\vartheta}, \widehat{\vartheta}_s, \widehat{\mathbf{u}}, \widehat{\chi})$ (cf. (6.11) and (6.12)). Building on this, we will be able to perform rigorously all the estimates necessary for the extension procedure on the approximate level, and use them to conclude that $(\widehat{\vartheta}, \widehat{\vartheta}_s, \widehat{\mathbf{u}}, \widehat{\chi}, \widehat{\boldsymbol{\eta}}, \widehat{\mu}, \widehat{\xi}, \widehat{\zeta})$ is defined on the whole interval $(0, T)$. More precisely, we will consider the maximal extension of our (approximable) solution $(\widehat{\vartheta}, \widehat{\vartheta}_s, \widehat{\mathbf{u}}, \widehat{\chi}, \widehat{\boldsymbol{\eta}}, \widehat{\mu}, \widehat{\xi}, \widehat{\zeta})$ and show that it is defined $(0, T)$ with a standard contradiction argument (cf. Step 4 below). In doing so, we will meet with the technical difficulty that the (ϑ, ϑ_s)-components of our solution need not be continuous w.r.t. time and therefore we will not be in the position to extend them by continuity. Indeed, in accord with the notion of *approximable* solution, we will argue on the level of the approximate solutions and rely on their time-regularity to carry out this procedure rigorously.

In what follows, $(\vartheta_\varepsilon, \vartheta_{s,\varepsilon}, \mathbf{u}_\varepsilon, \chi_\varepsilon)_\varepsilon$ (with associated $\boldsymbol{\mu}_\varepsilon$) will be the family of solutions to Problem (P_ε) which converge, along a not-relabeled subsequence (cf. (5.1)–(5.21)), to the local solution $(\widehat{\vartheta}, \widehat{\vartheta}_s, \widehat{\mathbf{u}}, \widehat{\chi})$ from Theorem 5.1. For simplicity, hereafter we shall omit the functions $(\widehat{\boldsymbol{\eta}}, \widehat{\mu}, \widehat{\xi}, \widehat{\zeta})$ and refer to the quadruple $(\widehat{\vartheta}, \widehat{\vartheta}_s, \widehat{\mathbf{u}}, \widehat{\chi})$ as "the" local solution to our problem. Accordingly, we will give the definition of approximable solution only in terms of the $(\vartheta, \vartheta_s, \mathbf{u}, \chi)$-components. We are now in the position to introduce the notion of "approximable solution".

Definition 6.1. *Let $\tau \in (0, T]$. We say that a quadruple $(\vartheta_\tau, \vartheta_{s,\tau}, \mathbf{u}_\tau, \chi_\tau)$ is an* approximable solution *on $(0, \tau)$ to Problem 2.5 if the following conditions are verified*

- *it is an energy solution on $(0, \tau)$;*
- *there exists a subsequence $\varepsilon_n \downarrow 0$ such that the related solutions of problem (P_{ε_n}) on $(0, \tau)$ (here the dependence on τ is omitted in the notation) fulfill as $n \to \infty$*

$$\|\mathbf{u}_{\varepsilon_n} - \mathbf{u}\|_{\mathrm{C}^0([0,\tau]; H^{1-\delta}(\Omega))} \to 0 \qquad \textit{for all } \delta \in (0, 1], \tag{6.1}$$

$$\|\chi_{\varepsilon_n} - \chi\|_{\mathrm{C}^0([0,\tau]; H^{1-\delta}(\Gamma_{\mathrm{C}}))} \to 0 \qquad \textit{for all } \delta \in (0, 1], \tag{6.2}$$

$$\|\vartheta_{\varepsilon_n} - \vartheta\|_{L^2(0,\tau; L^2(\Omega))} \to 0, \tag{6.3}$$

$$\|\vartheta_{s\varepsilon_n} - \vartheta_s\|_{L^2(0,\tau; L^2(\Gamma_{\mathrm{C}}))} \to 0. \tag{6.4}$$

Note in particular that, by virtue of the above definition and convergences (5.13), (5.14), (5.20), (5.21), for $\tau \geq \widehat{T}$ the quadruple $(\vartheta_\tau, \vartheta_{s,\tau}, \mathbf{u}_\tau, \chi_\tau)$ is a proper extension on $(0, \tau)$ of the local solution $(\widehat{\vartheta}, \widehat{\vartheta}_s, \widehat{\mathbf{u}}, \widehat{\chi})$. More precisely, we have

$$(\mathbf{u}_\tau, \chi_\tau) = (\widehat{\mathbf{u}}, \widehat{\chi}) \quad \text{for all } t \in [0, \widehat{T}], \qquad (\vartheta_\tau, \vartheta_{s,\tau}) = (\widehat{\vartheta}, \widehat{\vartheta}_s) \quad \text{for a.a. } t \in (0, \widehat{T}). \tag{6.5}$$

We introduce the set

$$\mathcal{T} := \{\tau \in (0, T] \text{ such that there exists an approximable solution on } (0, \tau)\}.$$

It follows from the passage to the limit argument in Theorem 5.1 that $\mathcal{T}$ is not an empty set, as at least $\widehat{T} \in \mathcal{T}$. As a consequence, letting

$$T^* = \sup \mathcal{T}$$

we have $0 < T^* \leq T$. Hence, proving that the local solution $(\widehat{\vartheta}, \widehat{\vartheta}_s, \widehat{\mathbf{u}}, \widehat{\chi})$ extends to the whole $(0, T)$ reduces to showing that it extends to an approximable solution on $(0, T^*)$, and that $T^* = T$. To this aim let us outline the sketch of the proof. First, we prove that the "energy estimates" for an approximable solution hold with a constant independent of τ (Step 1). Then, we deduce that an approximable solution extends to $(0, T^*)$ (Steps 2 and 3). Finally, we show that $T = T^*$ by a contradiction argument (Step 4).

Step 1. Let us prove the following Lemma, stating global estimates on the energy solutions to Problem 2.5. Observe that the constant C below does not depend on τ.

Lemma 6.2. *Assume* (2.1) *and Hypotheses 2.1–2.6. Suppose that the data $(h, \mathbf{f}, \mathbf{g})$ and $(\vartheta_0, \vartheta_s^0, \mathbf{u}_0, \chi_0)$ fulfill* (2.28) *and* (2.30).

Then, there exists a constant $C > 0$ depending on the data of the problem such that for any $\tau > 0$ and for any energy solution $(\vartheta, \vartheta_s, \mathbf{u}, \chi)$ to Problem 2.5 on $(0, \tau)$, there holds

$$\|\mathbf{u}\|_{H^1(0,\tau;\mathbf{W})} + \|\chi\|_{L^\infty(0,\tau;H^1(\Gamma_{\rm C})\cap H^1(0,\tau;L^2(\Gamma_{\rm C}))} + \|\vartheta\|_{L^2(0,\tau;V)\cap L^\infty(0,\tau;L^1(\Omega))} + \|\vartheta_s\|_{L^\infty(0,\tau;L^1(\Gamma_{\rm C}))} \leq C. \quad (6.6)$$

The proof directly follows from the energy inequality (2.37), written on $(0, \tau)$: we develop the very same calculations as in the derivation of the *First estimate*, after observing that here $\int_{\Gamma_{\rm C}} \chi(t)|\mathbf{u}(t)|^2 \, {\rm d}x \geq 0$, as $\chi \in \mathrm{dom}\, \widehat{\beta}$, and thus $\chi \geq 0$, a.e. on $\Gamma_{\rm C}$, e.g. due to (5.42).

Step 2. Let $(\vartheta_\tau, \vartheta_{s,\tau}, \mathbf{u}_\tau, \chi_\tau)_\tau$ be a family of approximable solutions depending on τ, with $\tau \in \mathcal{T}$. In view of the regularity required of an approximable solution, we have that $(\mathbf{u}_\tau, \chi_\tau) \in {\rm C}^0([0,\tau];\mathbf{W}) \times {\rm C}^0([0,\tau]; H^1(\Gamma_{\rm C}))$. Hence we can consider the extension to $(0, T^*)$ by continuity of these functions. More precisely, we define

$$\widetilde{\mathbf{u}}_\tau(t) := \begin{cases} \mathbf{u}_\tau(t) & \text{if } t \in [0,\tau], \\ \mathbf{u}_\tau(\tau) & \text{if } t \in (\tau, T^*], \end{cases} \qquad \widetilde{\chi}_\tau(t) := \begin{cases} \chi_\tau(t) & \text{if } t \in [0,\tau], \\ \chi_\tau(\tau) & \text{if } t \in (\tau, T^*]. \end{cases}$$

Due to (6.6) (where the constant C does not depend on τ) there holds (independently of τ)

$$\|\widetilde{\mathbf{u}}_\tau\|_{H^1(0,T^*;\mathbf{W})} + \|\widetilde{\chi}_\tau\|_{H^1(0,T^*;L^2(\Gamma_{\rm C}))\cap L^\infty(0,T^*;H^1(\Gamma_{\rm C}))} \leq C. \quad (6.7)$$

Thus, after fixing a sequence $\tau_m \uparrow T^*$, by (weak, weak*, and strong) compactness results we can conclude that there exists a pair

$$(\mathbf{u}^*, \chi^*) \in H^1(0,T^*;\mathbf{W}) \times H^1(0,T^*;L^2(\Gamma_{\rm C})) \cap L^\infty(0,T^*;H^1(\Gamma_{\rm C}))$$

such that, at least along some not relabeled subsequence,

$$\|\widetilde{\mathbf{u}}_{\tau_m} - \mathbf{u}^*\|_{{\rm C}^0([0,T^*];H^{1-\delta}(\Omega;\mathbb{R}^3))} + \|\widetilde{\chi}_{\tau_m} - \chi^*\|_{{\rm C}^0([0,T^*];H^{1-\delta}(\Gamma_{\rm C}))} \to 0. \quad (6.8)$$

Hence, by construction of $\widetilde{\mathbf{u}}_\tau$ and $\widetilde{\chi}_\tau$, we can infer that $(\mathbf{u}^*(t), \chi^*(t)) = (\widehat{\mathbf{u}}(t), \widehat{\chi}(t))$ for all $t \in [0, \widehat{T}]$ (see (6.5)).

Step 3. Now, we will prove that there exists $(\vartheta^*, \vartheta_s^*)$ such that $(\vartheta^*, \vartheta_s^*, \mathbf{u}^*, \chi^*)$ is an approximable solution to Problem 2.5 on $(0, T^*)$. To this aim, let $\tau_m \uparrow T^*$ and (6.8) hold. By definition of approximable solution (see Def. 6.1), for any $m \in \mathbb{N}$ there exists a (sub)sequence $(\varepsilon_n^{\tau_m})_n$ such that $\varepsilon_n^{\tau_m} \downarrow 0$ and the corresponding approximating sequence of solutions to Problem $(P_{\varepsilon_n^{\tau_m}})$ on $(0, \tau_m)$ satisfies

$$\|\mathbf{u}_{\varepsilon_n^{\tau_m}}^{\tau_m} - \mathbf{u}_{\tau_m}\|_{{\rm C}^0([0,\tau_m];H^{1-\delta}(\Omega;\mathbb{R}^3))} + \|\chi_{\varepsilon_n^{\tau_m}}^{\tau_m} - \chi_{\tau_m}\|_{{\rm C}^0([0,\tau_m];H^{1-\delta}(\Gamma_{\rm C}))} \to 0. \quad (6.9)$$

Thus, by diagonalization, we find a further subsequence, which we will denote by ε_m, and, correspondingly, a sequence $(\vartheta_{\varepsilon_m}^{\tau_m}, \vartheta_{s,\varepsilon_m}^{\tau_m}, \mathbf{u}_{\varepsilon_m}^{\tau_m}, \chi_{\varepsilon_m}^{\tau_m})$ of solutions to Problem (P_{ε_m}) on $(0, \tau_m)$, such that for every $m \in \mathbb{N}$

$$\|\mathbf{u}_{\varepsilon_m}^{\tau_m} - \mathbf{u}_{\tau_m}\|_{{\rm C}^0([0,\tau_m];H^{1-\delta}(\Omega;\mathbb{R}^3))} + \|\chi_{\varepsilon_m}^{\tau_m} - \chi_{\tau_m}\|_{{\rm C}^0([0,\tau_m];H^{1-\delta}(\Gamma_{\rm C}))} \leq \frac{1}{m}. \quad (6.10)$$

Ultimately, we have for some $m^* \in \mathbb{N}$ that

$$\|\mathbf{u}_{\varepsilon_m}^{\tau_m} - \mathbf{u}^*\|_{{\rm C}^0([0,\tau_m];H^{1-\delta}(\Omega;\mathbb{R}^3))} + \|\chi_{\varepsilon_m}^{\tau_m} - \chi^*\|_{{\rm C}^0([0,\tau_m];H^{1-\delta}(\Gamma_{\rm C}))} \leq \frac{2}{m} \quad \forall m \geq m^*. \quad (6.11)$$

We exploit (6.6) and (6.11), combined with trace theorems, to deduce

$$\int_{\Gamma_{\rm C}} \chi_{\varepsilon_m}^{\tau_m}(t)|\mathbf{u}_{\varepsilon_m}^{\tau_m}(t)|^2 \, {\rm d}x \leq c \quad (6.12)$$

independently of τ_m. Now, we use (6.12) in the approximate energy identity (3.52) and get the analogue of estimates (6.6) for the approximate solutions, with a constant independent of τ_m. As a consequence we can perform the same a priori estimates as in Section 4, and we can now conclude that they hold *globally in time*. In particular, we get that (4.3), (4.7)–(4.9), (4.13)–(4.14), (4.16)–(4.17), (4.31), (4.33)–(4.35) hold on $[0, \tau_m]$, uniformly w.r.t. $m \in \mathbb{N}$.

We now extend the (ϑ, ϑ_s)-components of the solution (along with $(\boldsymbol{\eta}, \boldsymbol{\mu}, \xi, \zeta)$), on the interval $(0, T^*)$, together with their approximability properties. To this aim, we proceed with a diagonalization argument, which

we sketch here for the sake of completeness, referring to [4] for all details. Let us take $T_k := T^*(2^k-1)/2^k$, $k \geq 1$. For $k=1$ and for m sufficiently large $m \geq \bar{m}_1$ we have $[0,T_1] \subset [0,\tau_m]$. As a consequence we can apply compactness arguments analogous to the ones in the previous sections. In particular, we show that for some suitable subsequence $(m_j^1)_j$ with $m_j^1 \to \infty$ and $m_j^1 \geq \bar{m}_1$ for all $j \in \mathbb{N}$, the analogues of (5.1)–(5.17) hold on $[0,T_1]$. In particular, we deduce

$$\vartheta_{\varepsilon_{m_j^1}}^{\tau_{m_j^1}} \to \vartheta^* \quad \text{in } L^2(0,T_1;H), \tag{6.13}$$

$$\vartheta_{s,\varepsilon_{m_j^1}}^{\tau_{m_j^1}} \to \vartheta_s^* \quad \text{in } L^2(0,T_1;L^2(\Gamma_{\mathrm{C}})), \tag{6.14}$$

(as well as the existence of a limit quadruple $(\boldsymbol{\eta}^*, \boldsymbol{\mu}^*, \xi^*, \zeta^*)$). As a consequence it is straightforward to observe that $\vartheta^*, \vartheta_s^*$ can be identified (a.e.) with $\widehat{\vartheta}, \widehat{\vartheta}_s$ on $(0,\widehat{T})$ (see the second of (6.5)). The aforementioned convergences allow us to apply a similar passage to the limit procedure in the approximate problem, and conclude that $(\vartheta^*, \vartheta_s^*, \mathbf{u}^*, \chi^*)$ is a solution on $(0,T_1)$ to Problem 2.5 (we omit details as they follow the already detailed argument in the proof Theorem 5.1). We can now proceed repeating the argument for T_k with $k=2$, and extending the above convergences to the interval $[0,T_2]$ along a subsequence $(m_j^2)_j$ larger than some $\bar{m}_2 \geq \bar{m}_1$. Iterating this construction for any $k \in \mathbb{N}$ (cf. [4, pag. 1061] for details), we get that the limit functions $(\mathbf{u}^*, \chi^*, \vartheta^*, \vartheta_s^*)$ solve the limit Problem 2.5 on the interval $(0,T^*)$ (along with some $(\boldsymbol{\eta}^*, \boldsymbol{\mu}^*, \xi^*, \zeta^*)$), and conclude indeed that $(\mathbf{u}^*, \chi^*, \vartheta^*, \vartheta_s^*)$ is an approximable solution on $(0,T^*)$.

Step 4. We now prove that $T = T^*$, hence that $(\vartheta^*, \vartheta_s^*, \mathbf{u}^*, \chi^*)$ is an approximable solution to Problem 2.5 on the whole $(0,T)$. We proceed by contradiction, assuming $T^* < T$ and show that actually $(\vartheta^*, \vartheta_s^*, \mathbf{u}^*, \chi^*)$ can be extended to some approximable solution $(\widetilde{\vartheta}, \widetilde{\vartheta}_s, \widetilde{\mathbf{u}}, \widetilde{\chi})$ on $(0,T^*+\delta)$, with $\delta > 0$. Indeed, let us consider the sequence $(\vartheta_{\varepsilon_m}^{\tau_m}, \vartheta_{s,\varepsilon_m}^{\tau_m}, \mathbf{u}_{\varepsilon_m}^{\tau_m}, \chi_{\varepsilon_m}^{\tau_m})$ of the approximate solutions solving Problem (P_{ε_m}) on $[0,\tau_m]$ constructed above. We can extend it to some larger interval $[0,\tau_m+\delta]$ with $\delta > 0$ by applying our local existence result, Proposition 3.3, to Problem (P_{ε_m}), supplemented with the initial data $(\vartheta_{\varepsilon_m}^{\tau_m}(\tau_m), \vartheta_{s,\varepsilon_m}^{\tau_m}(\tau_m), \mathbf{u}_{\varepsilon_m}^{\tau_m}(\tau_m), \chi_{\varepsilon_m}^{\tau_m}(\tau_m))$ (observe that $\vartheta_{\varepsilon_m}^{\tau_m}(\tau_m)$ and $\vartheta_{s,\varepsilon_m}^{\tau_m}(\tau_m)$ are well defined by the time-regularity (3.21) and (3.23) of the approximate solutions). Since the local existence time does δ does not depend on m, we conclude that there exists a local solution to (P_{ε_m}) with the aforementioned initial data on on $[\tau_m, \tau_m+\delta]$. Now, we have that $T^*+\delta/2 < \tau_m+\delta$ for m sufficiently large (as $\tau_m \to T^*$), therefore $(\vartheta_{\varepsilon_m}^{\tau_m}, \vartheta_{s,\varepsilon_m}^{\tau_m}, \mathbf{u}_{\varepsilon_m}^{\tau_m}, \chi_{\varepsilon_m}^{\tau_m})$ turns out to be extended on the interval $(0,T^*+\delta/2) \subset (0,\tau_m+\delta)$. Proceeding as in Step 3, by compactness and passage to the limit procedures, we can prove that Problem 2.5 admits an approximable solution $(\widetilde{\vartheta}, \widetilde{\vartheta}_s, \widetilde{\mathbf{u}}, \widetilde{\chi})$ on $(0,T^*+\delta/2)$, against the definition T^*. Therefore, $T^* = T$, which concludes the proof of Theorem 2.7. ∎

Appendix A

We develop here a series of technical results, collecting useful properties and estimates for the functions $\widetilde{L}_\varepsilon$, $\widetilde{\ell}_\varepsilon$, f_ε, and related quantities, which play a crucial role in deriving a priori estimates for Problem (P_ε). We also detail the construction of a family of approximate initial data complying with the properties (3.12)–(3.20).

In what follows, we shall rely on some definitions and results from the theory of maximal monotone operators, for which we refer to the classical monographs [2, 11]. Preliminarily, we fix some notation.

Notation A.1. Hereafter, for fixed $\varepsilon > 0$ we will denote by

$$R_\varepsilon := (\mathrm{Id} + \varepsilon L)^{-1} : \mathbb{R} \to \mathbb{R}, \qquad r_\varepsilon := (\mathrm{Id} + \varepsilon \ell)^{-1} : \mathbb{R} \to \mathbb{R} \tag{A.1}$$

the resolvent operators associated with L and ℓ, respectively. We recall that R_ε and r_ε are contractions, and that the Yosida regularizations of L and ℓ are defined, respectively, by

$$L_\varepsilon := \frac{\mathrm{Id} - R_\varepsilon}{\varepsilon} : \mathbb{R} \to \mathbb{R}\,, \qquad \ell_\varepsilon := \frac{\mathrm{Id} - r_\varepsilon}{\varepsilon} : \mathbb{R} \to \mathbb{R} \tag{A.2}$$

and fulfill

$$L_\varepsilon(x) = L(R_\varepsilon(x)), \qquad \ell_\varepsilon(x) = \ell(r_\varepsilon(x)) \qquad \forall\, x \in \mathbb{R}\,. \tag{A.3}$$

We also introduce the Yosida approximations J_ε and j_ε of the primitives J and j of L and ℓ, respectively defined for all $\varepsilon > 0$ by

$$J_\varepsilon(x) := \min_{y\in\mathbb{R}} \left\{ \frac{|y-x|^2}{2\varepsilon} + J(y) \right\}, \qquad j_\varepsilon(x) := \min_{y\in\mathbb{R}} \left\{ \frac{|y-x|^2}{2\varepsilon} + j(y) \right\} \quad \forall\, x \in \mathbb{R}\,. \tag{A.4}$$

By [11, Prop. 2.11], for all $\varepsilon > 0$ the functions J_ε and j_ε are convex and differentiable on $\mathbb{R}$, with $J_\varepsilon'(x) = L_\varepsilon(x)$ and $j_\varepsilon'(x) = \ell_\varepsilon(x)$ for all $x \in \mathbb{R}$, and the following identities hold (see [11, Prop. 2.11])

$$J_\varepsilon(x) = \frac{\varepsilon}{2}|L_\varepsilon(x)|^2 + J(R_\varepsilon(x)) \quad \forall\, x \in \mathbb{R}, \qquad j_\varepsilon(x) = \frac{\varepsilon}{2}|\ell_\varepsilon(x)|^2 + j(r_\varepsilon(x)) \quad \forall\, x \in \mathbb{R}\,. \tag{A.5}$$

Moreover, the Fenchel-Moreau convex conjugates $J_\varepsilon^* := (J_\varepsilon)^*$ of J_ε and $j_\varepsilon^* := (j_\varepsilon)^*$ of j_ε (which in general differ from the Yosida approximations of J^* and j^*, respectively), fulfill

$$L_\varepsilon^{-1} \equiv J_\varepsilon^{*\prime}, \qquad \ell_\varepsilon^{-1} \equiv j_\varepsilon^{*\prime} \tag{A.6}$$

$$J_\varepsilon(x) + J_\varepsilon^*(L_\varepsilon(x)) = xL_\varepsilon(x), \qquad j_\varepsilon(x) + j_\varepsilon^*(\ell_\varepsilon(x)) = x\ell_\varepsilon(x) \qquad \forall\, x \in \mathbb{R}\,, \tag{A.7}$$

$$J_\varepsilon^*(y) = J^*(y) + \frac{\varepsilon}{2}|y|^2 \qquad \forall\, y \in \mathbb{R}\,, \qquad j_\varepsilon^*(y) = j^*(y) + \frac{\varepsilon}{2}|y|^2 \qquad \forall\, y \in \mathbb{R}\,, \tag{A.8}$$

(we refer to, e.g., [1, Prop. 3.3, p. 266] for the proof of the latter relation).

In the first lemma we prove that the coercivity properties (2.6c) and (2.7c) transfer to the approximate level and that, consequently, the functions $\mathcal{I}_\varepsilon, i_\varepsilon : \mathbb{R} \to \mathbb{R}$ (see (3.9)–(3.10)) satisfy suitable growth conditions.

Lemma A.2. *Assume* (2.6a) *and* (2.6c) *on* L *and* (2.7a) *and* (2.7c) *on* ℓ. *Then,*

$$\mathcal{I}_\varepsilon(x) = \frac{\varepsilon}{2}x^2 + J_\varepsilon^*(L_\varepsilon(x)) + J_\varepsilon(0) \quad \textit{and} \quad i_\varepsilon(x) = \frac{\varepsilon}{2}x^2 + j_\varepsilon^*(\ell_\varepsilon(x)) + j_\varepsilon(0) \qquad \textit{for all } x \in \mathbb{R}\,. \tag{A.9a}$$

As a consequence,

$$\exists\, C_1^*,\, C_2^* > 0 \;\; \forall\, \varepsilon \in (0,1),\; \forall\, x \in \mathbb{R} : \quad \mathcal{I}_\varepsilon(x) \geq \frac{\varepsilon}{2}x^2 + C_1^*|x| - C_2^*\,, \tag{A.9b}$$

$$\exists\, c_1^*,\, c_2^* > 0 \;\; \forall\, \varepsilon \in (0,1),\; \forall\, x \in \mathbb{R} : \quad i_\varepsilon(x) \geq \frac{\varepsilon}{2}x^2 + c_1^*|x| - c_2^*\,. \tag{A.9c}$$

Proof. We develop the proof of the first of (A.9a), and of (A.9b), only, since the arguments for the second of (A.9a) and for (A.9c) are identical. Integrating by parts, we have

$$\begin{aligned} \mathcal{I}_\varepsilon(x) = x\widetilde{L}_\varepsilon(x) - \int_0^x \widetilde{L}_\varepsilon(s)\,\mathrm{d}s = \frac{\varepsilon}{2}x^2 + xL_\varepsilon(x) - \int_0^x L_\varepsilon(s)\,\mathrm{d}s &= \frac{\varepsilon}{2}x^2 + xL_\varepsilon(x) - J_\varepsilon(x) + J_\varepsilon(0) \\ &= \frac{\varepsilon}{2}x^2 + J_\varepsilon^*(L_\varepsilon(x)) + J_\varepsilon(0) \end{aligned} \tag{A.10}$$

where the second identity follows from the fact that $\widetilde{L}_\varepsilon(x) = \varepsilon x + L_\varepsilon(x)$, the third one from the fact that J_ε is a primitive of L_ε, and the last one from (A.7). Next, we show that

$$\exists\, C_1^*,\, C_2^* > 0 \;\; \forall\, \varepsilon \in (0,1),\; \forall\, x \in \mathbb{R} : \quad J_\varepsilon^*(L_\varepsilon(x)) \geq C_1^*|x| - C_2^*\,.$$

Indeed, we use (A.8), (A.3), and (2.6c) to infer that

$$J_\varepsilon^*(L_\varepsilon(x)) = J^*(L_\varepsilon(x)) + \frac{\varepsilon}{2}|L_\varepsilon(x)|^2 \geq \frac{\varepsilon}{2}|L_\varepsilon(x)|^2 + C_1|R_\varepsilon(x)| - C_2 \qquad \forall\, x \in \mathbb{R}\,. \tag{A.11}$$

On the other hand, due to the definition (A.2) of L_ε,

$$C_1|R_\varepsilon(x) - x| = C_1|\varepsilon L_\varepsilon(x)| \leq C_1^2 + \frac{\varepsilon}{4}|L_\varepsilon(x)|^2 \qquad \forall\, x \in \mathbb{R}\,,\; \varepsilon \in (0,1)\,. \tag{A.12}$$

Therefore, collecting (A.11)–(A.12) we conclude that for every $\varepsilon \in (0,1)$

$$J_\varepsilon^*(L_\varepsilon(x)) \geq C_1|x| - C_1^2 - C_2 \qquad \forall\, x \in \mathbb{R}\,.$$

Finally, in view of (A.5) we have

$$J_\varepsilon(0) \geq J(R_\varepsilon(0)) \geq -C|R_\varepsilon(0)| - C' \geq -C$$

for a constant independent of ε: this follows from the fact that the convex function J is bounded from below by a linear function, and from $R_\varepsilon(0) \to 0$ as $\varepsilon \to 0$, since $0 \in \overline{D(L)}$. □

Our next result is crucial for the passage to the limit as $\varepsilon \to 0$, in particular to obtain the energy inequality (2.37).

Lemma A.3. *Assume* (2.6a) *and* (2.6c) *on* L. *Let* $(\theta_\varepsilon)_\varepsilon \subset H$ *fulfill*

$$\theta_\varepsilon \to \theta \quad \text{in } H, \qquad \sup_\varepsilon \|L_\varepsilon(\theta_\varepsilon)\|_H \le C. \tag{A.13}$$

Then,

$$\lim_{\varepsilon\to 0} \int_\Omega \mathcal{I}_\varepsilon(\theta_\varepsilon(x))\,\mathrm{d}x = \int_\Omega J^*(L(\theta(x)))\,\mathrm{d}x\,. \tag{A.14}$$

Under conditions (2.7a) *and* (2.7c) *on* ℓ, *the analogue of* (A.14) *holds for* $(i_\varepsilon)_\varepsilon$.

Proof. It follows from the second of (A.13) that, for every sequence $(\varepsilon_n) \downarrow 0$ there exist a (not relabeled) subsequence (θ_{ε_n}) and $\omega \in H$ such that $L_{\varepsilon_n}(\theta_{\varepsilon_n}) \rightharpoonup \omega$ in H. Therefore

$$\limsup_{n\to\infty} \int_\Omega L_{\varepsilon_n}(\theta_{\varepsilon_n})\theta_{\varepsilon_n}\,\mathrm{d}x \le \int_\Omega \omega\theta\,\mathrm{d}x,$$

which yields $\omega = L(\theta)$ thanks to [2, Lemma 1.3, p. 42]. Since the limit does not depend on the subsequence, we ultimately conclude that

$$L_\varepsilon(\theta_\varepsilon) \rightharpoonup L(\vartheta) \qquad \text{in } H \text{ as } \varepsilon \to 0. \tag{A.15}$$

On the one hand,

$$\begin{aligned}
\liminf_{\varepsilon\to 0} \int_\Omega \mathcal{I}_\varepsilon(\theta_\varepsilon(x))\,\mathrm{d}x &= \liminf_{\varepsilon\to 0} \int_\Omega \Big(\frac{\varepsilon}{2}|\theta_\varepsilon(x)|^2 + J_\varepsilon^*(L_\varepsilon(\theta_\varepsilon(x))) + J_\varepsilon(0)\Big)\,\mathrm{d}x \\
&= \liminf_{\varepsilon\to 0} \int_\Omega J_\varepsilon^*(L_\varepsilon(\theta_\varepsilon(x)))\,\mathrm{d}x \\
&\ge \liminf_{\varepsilon\to 0} \int_\Omega J^*(L_\varepsilon(\theta_\varepsilon(x)))\,\mathrm{d}x \ge \int_\Omega J^*(L(\theta(x)))\,\mathrm{d}x
\end{aligned}$$

where the first identity follows from (A.9a), the second one from (A.13) and the fact that $J_\varepsilon(0) \to J(0) = 0$ due to (2.8), the third inequality is due to (A.8), and the last one to the weak convergence (A.15) combined with the Ioffe theorem, cf. [19] as well as [22, Thm. 21].

On the other hand, from (A.7) we infer that

$$\begin{aligned}
\limsup_{\varepsilon\to 0} \int_\Omega \mathcal{I}_\varepsilon(\theta_\varepsilon(x))\,\mathrm{d}x &= \limsup_{\varepsilon\to 0} \int_\Omega J_\varepsilon^*(L_\varepsilon(\theta_\varepsilon(x)))\,\mathrm{d}x = \limsup_{\varepsilon\to 0} \int_\Omega (\theta_\varepsilon(x)L_\varepsilon(\theta_\varepsilon(x)) - J_\varepsilon(\theta_\varepsilon(x)))\,\mathrm{d}x \\
&\le \int_\Omega (\theta(x)L(\theta(x)) - J(\theta(x)))\,\mathrm{d}x = \int_\Omega J^*(L(\theta(x)))\,\mathrm{d}x
\end{aligned}$$

where the second equality follows from (A.7), the third one from (A.13) and (A.15), combined with the fact that the (integral functional associated with) J_ε Mosco-converges, as $\varepsilon \downarrow 0$, to (the integral functional associated with) J, and the last identity follows from elementary convex analysis. This concludes the proof of (A.14). □

With our next result we investigate the Lipschitz continuity of $\widetilde{L}_\varepsilon$ (of $\widetilde{\ell}_\varepsilon$, respectively) and of $\frac{1}{\widetilde{L}_\varepsilon}$ (of $\frac{1}{\widetilde{\ell}_\varepsilon}$, respectively). The latter will play a crucial role in the proof of Lemma A.5 below.

Lemma A.4. *The functions $\widetilde{L}_\varepsilon : \mathbb{R} \to \mathbb{R}$ and $\widetilde{\ell}_\varepsilon : \mathbb{R} \to \mathbb{R}$ satisfy*

$$\varepsilon < \widetilde{L}'_\varepsilon(x) \leq \varepsilon + \frac{2}{\varepsilon} \quad \textit{for all } x \in \mathbb{R}\,, \tag{A.16}$$

$$\exists\, C_L > 0 \;\forall\, x, y \in \mathbb{R} : \qquad \left| \frac{1}{\widetilde{L}'_\varepsilon(x)} - \frac{1}{\widetilde{L}'_\varepsilon(y)} \right| \leq C_L |x - y|\,, \tag{A.17}$$

$$\varepsilon < \widetilde{\ell}'_\varepsilon(x) \leq \varepsilon + \frac{2}{\varepsilon} \quad \textit{for all } x \in \mathbb{R}\,, \tag{A.18}$$

$$\exists\, C_\ell > 0 \;\forall\, x, y \in \mathbb{R} : \qquad \left| \frac{1}{\widetilde{\ell}'_\varepsilon(x)} - \frac{1}{\widetilde{\ell}'_\varepsilon(y)} \right| \leq C_l |x - y|\,. \tag{A.19}$$

Proof. We detail only the proof of (A.16) and (A.17), since the one for (A.18) and (A.19) is completely analogous. The left-hand side inequality in (A.16) directly follows from the definition (3.2) of $\widetilde{L}_\varepsilon$. Plugging (A.2) into the definition (3.2) of $\widetilde{L}_\varepsilon$ and using that R_ε is a contraction, we immediately deduce the right-hand side inequality in (A.16). Observe that, by the first of (A.16) the function $x \mapsto \frac{1}{\widetilde{L}'_\varepsilon(x)}$ is well-defined on $\mathbb{R}$. In order to show that it is itself Lipschitz continuous , we use the formula

$$L'_\varepsilon(x) = \frac{L'(R_\varepsilon(x))}{1 + \varepsilon L'(R_\varepsilon(x))} \qquad \text{for all } x \in \mathbb{R}\,. \tag{A.20}$$

Therefore, for every x, $y \in \mathbb{R}$

$$\begin{aligned}
\left| \frac{1}{\widetilde{L}'_\varepsilon(x)} - \frac{1}{\widetilde{L}'_\varepsilon(y)} \right| &= \left| \frac{1 + \varepsilon L'(R_\varepsilon(x))}{\varepsilon + \varepsilon^2 L'(R_\varepsilon(x)) + L'(R_\varepsilon(x))} - \frac{1 + \varepsilon L'(R_\varepsilon(y))}{\varepsilon + \varepsilon^2 L'(R_\varepsilon(y)) + L'(R_\varepsilon(y))} \right| \\
&= \left| \frac{L'(R_\varepsilon(x)) - L'(R_\varepsilon(y))}{(\varepsilon + \varepsilon^2 L'(R_\varepsilon(x)) + L'(R_\varepsilon(x)))\,(\varepsilon + \varepsilon^2 L'(R_\varepsilon(y)) + L'(R_\varepsilon(y)))} \right| \\
&\leq \left| \frac{1}{L'(R_\varepsilon(x))} - \frac{1}{L'(R_\varepsilon(y))} \right| \leq C_L |R_\varepsilon(x) - R_\varepsilon(y)|
\end{aligned}$$

where the last inequality follows from (2.6b) (C_L denoting the Lipschitz constant of $\frac{1}{L'}$). Thus (A.17) ensues, taking into account that R_ε is a contraction. □

Finally, we address the properties of the functions f_ε (3.4) and $H_\varepsilon(x)$ (3.11).

Lemma A.5. *Under conditions* (2.7) *on ℓ, the function $f_\varepsilon : \mathbb{R} \to \mathbb{R}$ is strictly increasing, with $f_\varepsilon(0) = 0$ and $f_\varepsilon(x) > 0$ if and only if $x > 0$, bi-Lipschitz, and satisfies*

$$\exists\, \bar{c}_1,\, \bar{c}_2 > 0 \;\forall\, x \in \mathbb{R} : \qquad (f'_\varepsilon(x))^2 \leq \bar{c}_1 |f_\varepsilon(x)| + \bar{c}_2\,. \tag{A.21}$$

The function $H_\varepsilon : \mathbb{R} \to \mathbb{R}$ is strictly increasing on $(0, +\infty)$ and strictly decreasing on $(-\infty, 0)$ (hence 0 is its absolute minimum), and it satisfies

$$\exists\, \bar{c}_1^*,\, \bar{c}_2^* > 0 \;\forall\, x \in \mathbb{R} : \qquad H_\varepsilon(x) \geq \bar{c}_1^* |f_\varepsilon(x)| - \bar{c}_2^* |\widetilde{\ell}_\varepsilon(x) - \widetilde{\ell}_\varepsilon(0)| \quad \textit{for all } x \in \mathbb{R}\,. \tag{A.22}$$

Before developing the proof, we preliminarily observe that it is not restrictive to suppose, in addition to (2.7), that

$$\exists\, x_0 \in \mathrm{D}(\ell) \;:\; \ell(x_0) = 0 \tag{A.23}$$

Indeed, let x_0 be a fixed point in $\mathrm{D}(\ell)$ with $\ell(x_0) = f_0$, and set $\ell_{f_0}(x) := \ell(x) - f_0$. Then ℓ_{f_0} clearly fulfills (A.23) and still complies with (2.7a)–(2.7b). Since the conjugate $j^*_{f_0}$ of a(ny) primitive of ℓ_{f_0} is given by $j^*_{f_0}(w) = j^*(w + f_0)$, it is immediate to check that, if ℓ and j^* fulfill (2.7c), so do ℓ_{f_0} and $j^*_{f_0}$. We will use (A.23) to prove estimate (A.27) below.

Proof. The properties of f_ε trivially follow from its definition (3.4), also in view of Lemma A.4. Owing to the Lipschitz continuity of $\frac{1}{\widetilde{\ell}'_\varepsilon}$ (cf. (A.19)), we can deduce that

$$-C_l \le \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\widetilde{\ell}'_\varepsilon(x)}\right) \le C_l \quad \text{for a.a. } x \in \mathbb{R}\,, \tag{A.24}$$

whence

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\widetilde{\ell}'_\varepsilon(x)}\right)^2 \le 2C_l \frac{1}{\widetilde{\ell}'_\varepsilon(x)} \quad \text{for a.a. } x \in \mathbb{R}\,. \tag{A.25}$$

Integrating between 0 and $x \in \mathbb{R}^+$ and taking into account definition (3.4), we find

$$\left(f'_\varepsilon(x)\right)^2 \le 2C_l f_\varepsilon(x) + \frac{1}{\left(\widetilde{\ell}'_\varepsilon(0)\right)^2} \quad \text{for all } x > 0\,. \tag{A.26}$$

In order to estimate $\dfrac{1}{\widetilde{\ell}'_\varepsilon(0)}$ we observe that, given x_0 as in (A.23), we have $r_\varepsilon(x_0) = x_0$. Then, from (A.19) and (A.20) written for ℓ, we deduce

$$\frac{1}{\widetilde{\ell}'_\varepsilon(0)} \le \left|\frac{1}{\widetilde{\ell}'_\varepsilon(0)} - \frac{1}{\widetilde{\ell}'_\varepsilon(x_0)}\right| + \left|\frac{1}{\widetilde{\ell}'_\varepsilon(x_0)}\right| \le C_l|x_0| + \frac{1+\ell'(x_0)}{\ell'(x_0)}\,. \tag{A.27}$$

Thus (A.21) is proved for all $x > 0$. In order to complete the proof for $x < 0$, we multiply the left-hand side inequality in (A.24) by $\dfrac{1}{\widetilde{\ell}'_\varepsilon(x)}$ and we integrate from 0 to $x < 0$. In view of (3.4) and (A.27), we deduce

$$(f'_\varepsilon(x))^2 \le -2C_l f_\varepsilon(x) + \frac{1}{\left(\widetilde{\ell}'_\varepsilon(0)\right)^2} \le 2C_l|f_\varepsilon(x)| + C \quad \text{for all } x < 0\,. \tag{A.28}$$

Finally, in order to prove (A.22), we again distinguish the case $x > 0$ and $x < 0$. For $x > 0$, we multiply (A.26) by $\widetilde{\ell}'_\varepsilon(x)$ and we integrate on $(0, x)$. Recalling the definition (3.4) of f_ε as well as estimate (A.27), we readily obtain (A.22). For $x < 0$, we develop calculations analogous to (A.28). □

We also use the following result to pass to the limit as $\varepsilon \downarrow 0$ in the term $f_\varepsilon(\vartheta_{s,\varepsilon})$.

Lemma A.6. *Under conditions* (2.7) *on* ℓ*, for every* $x \in \overline{\mathrm{D}(\ell)}$ *and every* $(x_\varepsilon) \subset \mathbb{R}$ *with* $x_\varepsilon \to x$ *there holds* $f_\varepsilon(x_\varepsilon) \to f(x)$ *and* $f'_\varepsilon(x_\varepsilon) \to f'(x)$ *as* $\varepsilon \to 0$.

Proof. We use

$$|f_\varepsilon(x_\varepsilon) - f(x)| \le |f_\varepsilon(x_\varepsilon) - f_\varepsilon(x)| + |f_\varepsilon(x) - f(x)| \doteq \Delta^1_\varepsilon + \Delta^2_\varepsilon.$$

First,

$$\Delta^1_\varepsilon \le \left|\int_x^{x_\varepsilon}\left(\frac{1}{\widetilde{\ell}'_\varepsilon(s)} - \frac{1}{\widetilde{\ell}'_\varepsilon(0)}\right)\mathrm{d}s\right| + \frac{1}{\widetilde{\ell}'_\varepsilon(0)}|x_\varepsilon - x| \le C_l \max(|x|, |x_\varepsilon|)|x_\varepsilon - x| + C|x_\varepsilon - x| \to 0$$

where the second inequality follows from (A.19) and (A.27). Second, in order to prove that $\Delta^2_\varepsilon \to 0$, we observe that

$$\frac{1}{\widetilde{\ell}'_\varepsilon(s)} \to \frac{1}{\ell'(s)} \quad \text{for all } s \in \overline{\mathrm{D}(\ell)}, \tag{A.29}$$

while, using (A.20) it is not difficult to check that

$$0 < \frac{1}{\widetilde{\ell}'_\varepsilon(s)} \le \frac{1}{\ell'(r_\varepsilon(s))} + \varepsilon \quad \text{for all } s \in \mathbb{R}\,.$$

Since for every $x \in \overline{\mathrm{D}(\ell)}$ we have that $\frac{1}{\ell'(r_\varepsilon)} \to \frac{1}{\ell'}$ in $L^1(0,x)$ by dominated convergence, an extended version of the Lebesgue theorem yields that also $\frac{1}{\widetilde{\ell'_\varepsilon}} \to \frac{1}{\ell'}$ in $L^1(0,x)$. This concludes the proof that $f_\varepsilon(x) \to f(x)$, whence $f_\varepsilon(x_\varepsilon) \to f(x)$.

In order to check the last assertion, it is sufficient to observe that

$$\left| \frac{1}{\widetilde{\ell'_\varepsilon}(x_\varepsilon)} - \frac{1}{\ell'(x)} \right| \leq \left| \frac{1}{\widetilde{\ell'_\varepsilon}(x_\varepsilon)} - \frac{1}{\widetilde{\ell'_\varepsilon}(x)} \right| + \left| \frac{1}{\widetilde{\ell'_\varepsilon}(x)} - \frac{1}{\ell'(x)} \right|$$

and use that $\frac{1}{\widetilde{\ell'_\varepsilon}}$ is Lipschitz (cf. (A.19)) to estimate the first term, and (A.29) for the second summand. □

The forthcoming Lemmas address the construction of a family of initial data $(\vartheta_0^\varepsilon, \vartheta_s^{0,\varepsilon})_\varepsilon$ fulfilling properties (3.12)–(3.20). In Lemma A.7 we exhibit an example of sequence $(\vartheta_0^\varepsilon, \vartheta_s^{0,\varepsilon})_\varepsilon$ complying with the first set of properties, i.e. (3.12)–(3.18). Since the construction developed in the proof of Lemma A.7 does not guarantee the other requirements, we tackle them in two different results. Namely, in Lemma A.8 we detail how the family $(\vartheta_s^{0,\varepsilon})_\varepsilon$ chosen in the Lemma A.7 satisfies the additional properties (3.19)–(3.20) in the case of a special class of functions ℓ (and function f) in (2.36b). Finally, in Lemma A.9, a family of data $(\vartheta_s^{0,\varepsilon})_\varepsilon$ complying with the whole set of properties (3.12)–(3.20) is exhibited in the case of the (physically relevant) choice $\ell(\vartheta_s) = \ln(\vartheta_s)$ (and $f(\vartheta_s) = (\vartheta_s)^2$).

Lemma A.7. *Assume that the initial data ϑ_0 and ϑ_s^0 respectively comply with* (2.30a)–(2.30b)*. Then, there exist sequences $(\vartheta_0^\varepsilon)_\varepsilon$ and $(\vartheta_s^{0,\varepsilon})_\varepsilon$ fulfilling* (3.12)–(3.13) *and such that*

$$\exists\, \bar{S}_0 > 0 \quad \forall\, \varepsilon > 0 : \quad \|L_\varepsilon(\vartheta_0^\varepsilon)\|_H \leq \bar{S}_0(1 + \|L(\vartheta_0)\|_H)\,, \qquad \|\ell_\varepsilon(\vartheta_s^{0,\varepsilon})\|_{H_{\Gamma_C}} \leq \|\ell(\vartheta_s^0)\|_{H_{\Gamma_C}}\,, \tag{A.30}$$

and convergences (3.15) *and* (3.16) *hold.*

Proof. We start by developing the construction of the sequence $(\vartheta_0^\varepsilon)_\varepsilon$. Denote by $\gamma = (J^*)'$ the inverse function L^{-1} and let γ_ε be its Yosida regularization and ϱ_ε its resolvent for any fixed $\varepsilon > 0$. We set

$$\vartheta_0^\varepsilon(x) := R_\varepsilon^{-1}\left(\gamma_\varepsilon(w_0(x))\right) \qquad \text{for a.a. } x \in \Omega\,, \tag{A.31}$$

where $w_0(x) := L(\vartheta_0(x))$. In view of (A.3), we have

$$L_\varepsilon\left(\vartheta_0^\varepsilon(x)\right) = L\left(R_\varepsilon(\vartheta_0^\varepsilon(x))\right) = L(\gamma_\varepsilon(w_0(x))) \qquad \text{for a.a. } x \in \Omega\,.$$

Again by (A.3), $\gamma_\varepsilon(w_0(x)) = \gamma(\varrho_\varepsilon(w_0(x))$ and hence it holds

$$L_\varepsilon\left(\vartheta_0^\varepsilon(x)\right) = \varrho_\varepsilon(w_0(x)) \qquad \text{for a.a. } x \in \Omega\,. \tag{A.32}$$

Moreover, recalling (A.31) and (A.1), we have

$$\vartheta_0^\varepsilon(x) = \gamma_\varepsilon(w_0(x)) + \varepsilon\varrho_\varepsilon(w_0(x)) \qquad \text{for a.a. } x \in \Omega\,. \tag{A.33}$$

By the Lipschitz continuity (with constant $1/\varepsilon$) of γ_ε and using that ϱ_ε is a contraction, in view of (2.30a) (which ensures that $w_0 \in H$), we readily deduce (3.12) and (A.30).

Next, in order to prove (3.13) and (3.15), we recall (A.9a) and (A.8) and (A.32). We find that

$$\begin{aligned} \mathfrak{I}_\varepsilon(\vartheta_0^\varepsilon(x)) = \frac{\varepsilon}{2}|\vartheta_0^\varepsilon(x)|^2 + J_\varepsilon^*(L_\varepsilon(\vartheta_0^\varepsilon(x))) + J_\varepsilon(0) &= \frac{\varepsilon}{2}|\vartheta_0^\varepsilon(x)|^2 + \frac{\varepsilon}{2}|L_\varepsilon(\vartheta_0^\varepsilon(x))|^2 + J^*(L_\varepsilon(\vartheta_0^\varepsilon(x))) + J_\varepsilon(0) \\ &= \frac{\varepsilon}{2}|\vartheta_0^\varepsilon(x)|^2 + \frac{\varepsilon}{2}|L_\varepsilon(\vartheta_0^\varepsilon(x))|^2 + J^*(\varrho_\varepsilon(w_0(x))) + J_\varepsilon(0) \end{aligned} \tag{A.34}$$

for a.a. $x \in \Omega$. On the other hand, writing (A.5) for J^*, we find

$$J^*(\varrho_\varepsilon(w_0(x))) = (J^*)_\varepsilon(w_0(x)) - \frac{\varepsilon}{2}|\gamma_\varepsilon(w_0(x))|^2 \qquad \text{for a.a. } x \in \Omega\,, \tag{A.35}$$

where $(J^*)_\varepsilon$ denotes the Yosida regularization of J^*. Now, we combine (A.33)–(A.35) with (2.8), the Lipschitz continuity (with constant $1/\varepsilon$) of γ_ε, and the fact that ϱ_ε is a contraction. Thus we deduce that

$$\mathcal{I}_\varepsilon(\vartheta_0^\varepsilon(x)) = \frac{\varepsilon}{2}|L_\varepsilon(\vartheta_0^\varepsilon(x))|^2 + (J^*)_\varepsilon(w_0(x)) + \frac{\varepsilon^3}{2}\varrho_\varepsilon^2(w_0(x)) + \varepsilon^2\varrho_\varepsilon(w_0(x))\,\gamma_\varepsilon(w_0(x)) + J_\varepsilon(0) \tag{A.36}$$

for a.a. $x \in \Omega$ and for all $\varepsilon \in (0,1)$. Then, to prove (3.13) we observe that

$$\varrho_\varepsilon(w_0(x)) \to w_0(x) \qquad \text{as } \varepsilon \downarrow 0 \ \text{ for a.a. } x \in \Omega \tag{A.37}$$

due to our assumption that $J^*(w_0) = J^*(L(\vartheta_0)) \in L^1(\Omega)$ which implies that $w_0(x) \in \overline{\mathrm{dom}(J^*)} = \overline{\mathrm{dom}(\gamma)}$ for a.a. $x \in \Omega$. Furthermore

$$\gamma_\varepsilon(L(\vartheta_0(x))) \to \vartheta_0(x) \qquad \text{as } \varepsilon \downarrow 0 \ \text{ for a.a. } x \in \Omega \tag{A.38}$$

and hence

$$\vartheta_0^\varepsilon(x) \to \vartheta_0(x) \qquad \text{as } \varepsilon \downarrow 0 \ \text{ for a.a. } x \in \Omega. \tag{A.39}$$

Then, combining (3.15), estimate (A.9b), (A.39), and applying the dominated convergence theorem, we conclude (3.13). Convergence (3.15) follows from

$$\liminf_{\varepsilon\downarrow 0} \int_\Omega \mathcal{I}_\varepsilon(\vartheta_0^\varepsilon)\,\mathrm{d}x \geq \int_\Omega J^*(L(\vartheta_0))\,\mathrm{d}x$$

(due to (A.34), (A.37), and the Fatou Lemma), combined with

$$\limsup_{\varepsilon\downarrow 0} \int_\Omega \mathcal{I}_\varepsilon(\vartheta_0^\varepsilon)\,\mathrm{d}x \leq \int_\Omega J^*(L(\vartheta_0))\,\mathrm{d}x\,,$$

which ensues from (A.36), noting that $J_\varepsilon(0) \to J(0) = 0$ and that the first, the second, and the fourth terms on the r.h.s. of (A.36) tend to zero by (A.32), (A.37), the fact that γ_ε is Lipschitz with constant $1/\varepsilon$, and the dominated convergence theorem.

Concerning the initial data for ϑ_s, in view of (2.30b), we can make the choice

$$\vartheta_s^{0,\varepsilon}(x) := \vartheta_s^0(x) \ \text{ for a.a. } x \in \Gamma_{\mathrm{C}}\,. \tag{A.40}$$

In this case, (3.12) and (3.13) are trivially verified. Moreover, (A.30) follows from the well-known properties of the Yosida regularization ℓ_ε. Now, we only have to prove (3.16). To this aim, we recall (A.7) and (A.9a), yielding for a.a. $x \in \Gamma_{\mathrm{C}}$ and for all $\varepsilon \in (0,1)$

$$\begin{aligned} i_\varepsilon(\vartheta_s^0(x)) = \frac{\varepsilon}{2}|\vartheta_s^0(x)|^2 + j_\varepsilon^*(\ell_\varepsilon(\vartheta_s^0(x))) + j_\varepsilon(0) &= \frac{\varepsilon}{2}|\vartheta_s^0(x)|^2 + \vartheta_s^0(x)\ell_\varepsilon(\vartheta_s^0(x)) - j_\varepsilon(\vartheta_s^0(x)) + j_\varepsilon(0) \\ &\leq c(1 + |\vartheta_s^0(x)|^2 + |\ell(\vartheta_s^0(x))|^2) - j_\varepsilon(\vartheta_s^0(x)) + j(0) \end{aligned} \tag{A.41}$$

for some positive constant c (independent of ε). Moreover, taking into account (A.5), and using the convexity of j, and the Lipschitz continuity of r_ε, we deduce

$$j_\varepsilon(\vartheta_s^0(x)) \geq j(r_\varepsilon(\vartheta_s^0(x))) \geq -c(1 + |\vartheta_s^0(x)|) \ \text{ for a.a. } x \in \Gamma_{\mathrm{C}}\,. \tag{A.42}$$

Thus, from (A.41) and (A.42), we can conclude

$$i_\varepsilon(\vartheta_s^0(x)) \leq c(1 + |\vartheta_s^0(x)|^2 + |\ell(\vartheta_s^0(x))|^2) \qquad \text{for a.a. } x \in \Gamma_{\mathrm{C}}\,.$$

Since $i_\varepsilon(\vartheta_s^0(x)) \to j^*(\ell(\vartheta_s^0(x)))$ for almost all $x \in \Gamma_{\mathrm{C}}$, (3.16) follows from the dominated convergence theorem. □

Lemma A.8. *Assume that the initial datum ϑ_s^0 complies with* (2.30b) *and that the function ℓ from* (2.7a) *is also bi-Lipschitz, i.e.*

$$\exists\, C_1\,, C_2 > 0 \,:\, C_1 \leq \ell'(x) \leq C_2 \ \ \forall\, x \in D(\ell)\,. \tag{A.43}$$

Then, there holds

$$\exists\, \bar{S}_1 > 0 \;\; \forall\, \varepsilon \in (0,1) \; : \quad \int_{\Gamma_{\rm C}} H_\varepsilon(\vartheta_0^s)\,{\rm d}x \le \bar{S}_1 \|\vartheta_0^s\|_H^2\,, \tag{A.44}$$

$$\exists\, \bar{S}_2 > 0 \;\; \forall\, \varepsilon \in (0,1) \; : \quad \|f_\varepsilon(\vartheta_s^0)\|_{V_{\Gamma_{\rm C}}} \le \bar{S}_2 \|f(\vartheta_s^0)\|_{V_{\Gamma_{\rm C}}}. \tag{A.45}$$

Observe that, since ℓ is bi-Lipschitz, f is also bi-Lipschitz. Therefore, (2.30b) automatically guarantees that $\vartheta_s^0 \in V_{\Gamma_{\rm C}}$.

Proof. First, we note that ℓ_ε inherits the bi-Lipschitz continuity of ℓ (cf. (A.20) written for ℓ). Then, from the definition (3.4) of f_ε it follows that there exists a positive constant C such that

$$|f_\varepsilon(x)| \le C|x| \;\; \forall\, x \in \mathbb{R}\,, \;\; \forall\, \varepsilon \in (0,1). \tag{A.46}$$

Thus, by (3.11) (i.e. the definition of H_ε), (A.43), and (A.46), we deduce (A.44). Moreover, observe that (A.43) implies the bi-Lipschitz continuity of f_ε, whence (A.45) easily follows. □

Lemma A.9. *Assume that the initial datum ϑ_s^0 complies with* (2.30b) *and that*

$$\ell(\vartheta_s) = \ln(\vartheta_s) \qquad \textit{(whence } f(\vartheta_s) = (\vartheta_s)^2\textit{)}.$$

Then, there exists a sequence $(\vartheta_s^{0,\varepsilon})_\varepsilon \subset H_{\Gamma_{\rm C}}$ such that

$$\vartheta_s^{0,\varepsilon} \to \vartheta_s^0 \;\; \textit{in } H_{\Gamma_{\rm C}} \;\; \textit{as } \varepsilon \downarrow 0\,, \tag{A.47}$$

$$\|\ln_\varepsilon(\vartheta_s^{0,\varepsilon})\|_{H_{\Gamma_{\rm C}}} \le \|\ln(\vartheta_s^0)\|_{H_{\Gamma_{\rm C}}} \qquad \textit{for all } \varepsilon > 0\,, \tag{A.48}$$

convergence (3.16) *holds, and moreover*

$$\exists\, \bar{S}_4 > 0 \;\; \forall\, \varepsilon \in (0,1) \; : \quad \int_{\Gamma_{\rm C}} H_\varepsilon(\vartheta_s^{0,\varepsilon})\,{\rm d}x \le \bar{S}_4(1 + \|\vartheta_0^s\|_H^2)\,, \tag{A.49}$$

$$\exists\, \bar{S}_5 > 0 \;\; \forall\, \varepsilon \in (0,1) \; : \quad \|f_\varepsilon(\vartheta_s^{0,\varepsilon})\|_{V_{\Gamma_{\rm C}}} \le \bar{S}_5(1 + \|f(\vartheta_s^0)\|_{V_{\Gamma_{\rm C}}}^2). \tag{A.50}$$

Proof. We will construct a sequence of data $(\vartheta_s^{0,\varepsilon})_\varepsilon$ satisfying (A.47)–(A.50) partially adapting the argument of [9, Example 4.3]. For $\varepsilon \in (0,1)$ we set

$$\vartheta_s^{0,\varepsilon} := \max\{\vartheta_0^s, \varepsilon^\alpha\} \qquad \text{for some } \alpha > 0 \tag{A.51}$$

to be chosen later. It is not difficult to check that $\vartheta_s^{0,\varepsilon}$ can be written as $\nu_\varepsilon(f(\vartheta_s^0))$ for a suitable Lipschitz function ν_ε. Therefore, $\vartheta_s^{0,\varepsilon}$ is also in $H^1(\Gamma_{\rm C})$. Observe that $\vartheta_s^{0,\varepsilon} > 0$ a.e. in $\Gamma_{\rm C}$ (since $\vartheta_0^s > 0$ a.e. in $\Gamma_{\rm C}$ thanks to the second of (2.30b) and the fact that $\ell(\vartheta_0^s) = \ln(\vartheta_0^s)$), that $\vartheta_s^{0,\varepsilon} \to \vartheta_0^s$ a.e. in $\Gamma_{\rm C}$, and moreover that $(\vartheta_s^{0,\varepsilon})_\varepsilon$ is uniformly integrable in $H_{\Gamma_{\rm C}}$. Thus, (A.47) follows. Furthermore, relying on the definition (A.51) and on well-known properties of the Yosida regularization $\ln_\varepsilon$, we find

$$|\ln_\varepsilon(\vartheta_s^{0,\varepsilon})| \le |\ln_\varepsilon(\vartheta_0^s)| \le |\ln(\vartheta_0^s)| \quad \text{a.e. in } \Gamma_{\rm C}, \tag{A.52}$$

whence (A.48). Next, arguing as in [9, Lemma 4.1], we can deduce that

$$i_\varepsilon(x) \le \frac{\varepsilon}{2}x^2 + 2x \qquad \text{for every } x \in [0,+\infty) \text{ and } \varepsilon \in (0,1),$$

which implies that

$$i_\varepsilon(\vartheta_s^{0,\varepsilon}(x)) \le C(1 + (\vartheta_s^0(x))^2) \qquad \text{for a.a. } x \in \Gamma_{\rm C}.$$

Then, (3.16) follows from the pointwise convergence of $i_\varepsilon(\vartheta_s^{0,\varepsilon})$ to $j^*(\ell(\vartheta_s^0))$, via the dominated convergence theorem.

Now we recall definitions (3.4), (3.11), and (A.20) written for ln (yielding in particular that $\ln_\varepsilon'$ is a decreasing function), and find

$$H_\varepsilon(x) = \int_0^x \widetilde{\ell}_\varepsilon'(s) \int_0^s \frac{1}{\widetilde{\ell}_\varepsilon'(\tau)}\,{\rm d}\tau\,{\rm d}s = \int_0^x \int_0^s \frac{\widetilde{\ell}_\varepsilon'(s)}{\widetilde{\ell}_\varepsilon'(\tau)}\,{\rm d}\tau\,{\rm d}s \le \int_0^x s\,{\rm d}s = \frac{x^2}{2} \quad \text{for all } x \ge 0\,, \tag{A.53}$$

whence (A.49).
In order to show (A.50), we first recall that there exists a positive constant C such that

$$\|f_\varepsilon(\vartheta_s^{0,\varepsilon})\|_{L^1(\Gamma_{\rm C})} \le C(1+\|\vartheta_s^0\|^2_{H_{\Gamma_{\rm C}}}) \tag{A.54}$$

again by definition (3.4) and (A.20) written for ln. Now, we have to prove that

$$\|\nabla f_\varepsilon(\vartheta_s^{0,\varepsilon})\|_{H_{\Gamma_{\rm C}}} \le C(1+\|\nabla f(\vartheta_s^0)\|_{H_{\Gamma_{\rm C}}}) \tag{A.55}$$

To this aim, we recall definition (3.4) and relation (A.20) written for $\ln_\varepsilon$, whence $\ln_\varepsilon'(x) = \frac{1}{\varepsilon + r_\varepsilon(x)}$. We have

$$\int_{\Gamma_{\rm C}} |\nabla f_\varepsilon(\vartheta_s^{0,\varepsilon})|^2 \,{\rm d}x \le \int_{\Gamma_{\rm C}} (\varepsilon + r_\varepsilon(\vartheta_s^{0,\varepsilon}))^2 |\nabla(\vartheta_s^{0,\varepsilon})|^2 \,{\rm d}x\,. \tag{A.56}$$

We first observe that

$$\varepsilon^2 \int_{\Gamma_{\rm C}} |\nabla \vartheta_s^{0,\varepsilon}|^2 \,{\rm d}x = \varepsilon^2 \int_{\{\vartheta_s^0 > \varepsilon^\alpha\}} |\nabla \vartheta_s^0|^2 \,{\rm d}x \le \varepsilon^{2-2\alpha} \int_{\Gamma_{\rm C}} |\vartheta_s^0|^2\, |\nabla \vartheta_s^0|^2 \,{\rm d}x = \frac{\varepsilon^{2-2\alpha}}{4} \|\nabla f(\vartheta_s^0)\|^2_{H_{\Gamma_{\rm C}}}\,. \tag{A.57}$$

Next, we evaluate the term

$$\begin{aligned} &\int_{\Gamma_{\rm C}} (r_\varepsilon(\vartheta_s^{0,\varepsilon}))^2 |\nabla(\vartheta_s^{0,\varepsilon})|^2 \,{\rm d}x \\ &= \int_{\{\vartheta_s^{0,\varepsilon} \ge 1\}} (r_\varepsilon(\vartheta_s^{0,\varepsilon}))^2 |\nabla(\vartheta_s^{0,\varepsilon})|^2 \,{\rm d}x + \int_{\{\vartheta_s^{0,\varepsilon} < 1\}} (r_\varepsilon(\vartheta_s^{0,\varepsilon}))^2 |\nabla(\vartheta_s^{0,\varepsilon})|^2 \,{\rm d}x\,. \end{aligned} \tag{A.58}$$

Noting that (cf. definition (A.3) written for ln)

$$r_\varepsilon(\vartheta_s^{0,\varepsilon}) \le \vartheta_s^{0,\varepsilon} \quad \text{on the set } \{x \in \Gamma_{\rm C} \,:\, \vartheta_s^{0,\varepsilon}(x) \ge 1\}\,, \tag{A.59}$$

we find

$$\begin{aligned} \int_{\{\vartheta_s^{0,\varepsilon} \ge 1\}} (r_\varepsilon(\vartheta_s^{0,\varepsilon}))^2 |\nabla(\vartheta_s^{0,\varepsilon})|^2 \,{\rm d}x &\le \int_{\{\vartheta_s^{0,\varepsilon} \ge 1\}} (\vartheta_s^{0,\varepsilon})^2 |\nabla(\vartheta_s^{0,\varepsilon})|^2 \,{\rm d}x \\ &\le \int_{\Gamma_{\rm C}} (\vartheta_s^0)^2 |\nabla(\vartheta_s^0)|^2 \,{\rm d}x = \frac{1}{4} \|\nabla f(\vartheta_s^0)\|^2_{H_{\Gamma_{\rm C}}}\,. \end{aligned} \tag{A.60}$$

In order to estimate the second term on the right-hand side of (A.58), we use the relation (cf. (A.3)) $r_\varepsilon(x) = x - \varepsilon \ln_\varepsilon(x)$ for all $x \in \mathbb{R}$, and we obtain

$$\int_{\{\vartheta_s^{0,\varepsilon} < 1\}} (r_\varepsilon(\vartheta_s^{0,\varepsilon}))^2 |\nabla(\vartheta_s^{0,\varepsilon})|^2 \,{\rm d}x \le c \int_{\{\vartheta_s^{0,\varepsilon} < 1\}} ((\vartheta_s^{0,\varepsilon})^2 + \varepsilon^2 \ln_\varepsilon^2(\vartheta_s^{0,\varepsilon})) |\nabla(\vartheta_s^{0,\varepsilon})|^2 \,{\rm d}x\,.$$

Again

$$\int_{\{\vartheta_s^{0,\varepsilon} < 1\}} (\vartheta_s^{0,\varepsilon})^2 |\nabla(\vartheta_s^{0,\varepsilon})|^2 \,{\rm d}x \le c \|\nabla f((\vartheta_s^0)\|^2_{H_{\Gamma_{\rm C}}} \tag{A.61}$$

Moreover, the inequalities $\varepsilon^\alpha \le \vartheta_s^{0,\varepsilon} < 1$ and well-known properties of the Yosida regularization $\ln_\varepsilon$ imply that $|\ln_\varepsilon(\vartheta_s^{0,\varepsilon})| \le |\ln_\varepsilon(\varepsilon^\alpha)| \le |\ln(\varepsilon^\alpha)|$, whence

$$\begin{aligned} \int_{\{\vartheta_s^{0,\varepsilon} < 1\}} \varepsilon^2 \ln_\varepsilon^2(\vartheta_s^{0,\varepsilon}) |\nabla(\vartheta_s^{0,\varepsilon})|^2 \,{\rm d}x &= \varepsilon^2 \int_{\{\vartheta_s^{0,\varepsilon} < 1\}} \frac{1}{(\vartheta_s^{0,\varepsilon})^2} (\vartheta_s^{0,\varepsilon})^2 \ln_\varepsilon^2(\vartheta_s^{0,\varepsilon}) |\nabla(\vartheta_s^{0,\varepsilon})|^2 \,{\rm d}x \\ &\le \varepsilon^{2-2\alpha} \ln^2(\varepsilon^\alpha) \int_{\{\vartheta_s^{0,\varepsilon} < 1\}} (\vartheta_s^{0,\varepsilon})^2 |\nabla(\vartheta_s^{0,\varepsilon})|^2 \,{\rm d}x \\ &\le c\varepsilon^{2-2\alpha} \ln^2(\varepsilon^\alpha) \|\nabla f(\vartheta_s^0)\|^2_{H_{\Gamma_{\rm C}}}\,. \end{aligned} \tag{A.62}$$

Observe that the right-hand side of (A.62) is bounded independently of ε, choosing $\alpha < 1$ and ε sufficiently small. Finally, collecting (A.57), (A.60), (A.61), and (A.62), we deduce (A.55). Thus (A.50) is also proved. □

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