REGULARITY OF FREE BOUNDARIES IN ANISOTROPIC CAPILLARITY PROBLEMS AND THE VALIDITY OF YOUNG'S LAW

G. DE PHILIPPIS AND F. MAGGI

Abstract. Local volume-constrained minimizers in anisotropic capillarity problems develop free boundaries on the walls of their containers. We prove the regularity of the free boundary outside a closed negligible set, showing in particular the validity of Young's law at almost every point of the free boundary. Our regularity results are not specific to capillarity problems, and actually apply to sets of finite perimeter (and thus to codimension one integer rectifiable currents) arising as minimizers in other variational problems with free boundaries.

1. Introduction

1.1. **Young's law in anisotropic capillarity problems.** According to the historical introduction to Finn's beautiful monograph [Fin86], Young [You05] introduced in 1805 the notion of mean curvature of a surface in the study of capillarity phenomena. Mean curvature was reintroduced the following year by Laplace, together with its analytic expression and its linearization (the Laplacian), the latter being recognized as inadequate to describe real liquids in equilibrium. In the same essay [You05], Young also formulates the equilibrium condition for the contact angle of a capillarity surface commonly known as *Young's law*. These ideas were later reformulated by Gauss [Gau30] through the principle of virtual work and the introduction of a suitable free energy. Gauss' free energy consists of four terms: a free surface energy, proportional to the area of the surface separating the fluid and the surrounding media (another fluid or gas) in the given solid container, a wetting energy, accounting for the adhesion between the fluid and the walls of the container, the gravitational energy; and, finally, a Lagrange multiplier taking into account the volume constraint on the region occupied by the liquid. Since then, a huge amount of interdisciplinary literature has been devoted to the study of qualitative and quantitative properties of local minimizers and stationary surfaces of Gauss' free energy.

A modern formulation of Gauss' model, including the case of possibly anisotropic surface tension densities, as well as that of general potential energy terms, and extending the setting of the problem to (the geometrically relevant case of) arbitrary ambient space dimension, is obtained as follows. Given $n \ge 2$, an open set Ω with Lipschitz boundary in $\mathbb{R}^n$ (the container), and a set $E \subset \Omega$ (the region of occupied by the liquid droplet) with $\partial E \cap \Omega$ a smooth hypersurface, one considers the free energy

$$\mathcal{F}(E) = \int_{\partial E \cap \Omega} \Phi(x, \nu_E)\, d\mathcal{H}^{n-1} + \int_{\partial E \cap \partial\Omega} \sigma(x)\, d\mathcal{H}^{n-1} + \int_E g(x)\, dx\,, \tag{1.1}$$

where $\mathcal{H}^k$ is the k-dimensional Hausdorff measure on $\mathbb{R}^n$, ν_Ω and ν_E denote the outer unit normals to Ω and E respectively. Here $\Phi : \Omega \times \mathbb{R}^n \to [0,\infty)$ is convex and positively one-homogeneous in the second variable and represents the (possibly anisotropic) surface tension density, $\sigma : \partial\Omega \to \mathbb{R}$ is the relative adhesion coefficient between the liquid and the boundary walls of the container and it satisfies

$$-\Phi(x, -\nu_\Omega(x)) \le \sigma(x) \le \Phi(x, \nu_\Omega(x))\,, \qquad \forall x \in \partial\Omega\,, \tag{1.2}$$

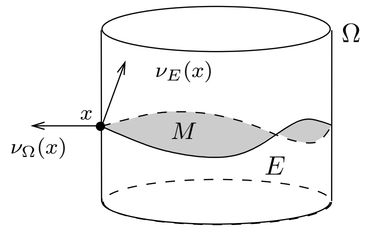

FIGURE 1.1. In this picture the region E occupied by the liquid lies at the bottom of the container Ω. The adhesion coefficient σ is integrated on the wetted surface $\partial E \cap \partial\Omega$, which consists of the bottom of the cylinder plus the lateral cylindrical surface below the free boundary $M \cap \partial\Omega$ of the surface $M = \mathrm{cl}(\Omega \cap \partial E)$. In particular, one expects $\partial_{\partial\Omega}(\partial E \cap \partial\Omega)$ (the topological boundary of the wetted surface relative to the boundary of the container) to coincide with $M \cap \partial\Omega$. The angle between $\nu_\Omega(x)$ and $\nu_E(x)$ at $x \in M \cap \partial\Omega$ is prescribed by Φ and σ through Young's law (1.5).

and $g : \Omega \to \mathbb{R}$ is a potential energy per unit mass. The classical capillarity problem is then obtained by taking $n = 3$, $\Phi = |\nu|$, and $g(x) = g_0\,\rho\, x_3$, where ρ is the constant density of the fluid and g_0 is the gravity of Earth.

If we are interested in global volume-constrained minimizers of $\mathcal{E}$, then we are led to consider the variational problem

$$\inf\Big\{\mathcal{F}(E) : |E| = m\Big\}\,, \qquad m > 0 \text{ fixed}\,, \tag{1.3}$$

where $|E|$ stands for the Lebesgue measure of E. Note that if we set $\sigma = 0$ and $g = 0$ in (1.1), then problem (1.3) reduces to the relative isoperimetric problem in Ω with respect to the Φ-perimeter, also known as the relative Wulff problem in Ω. Alternatively, one may consider local volume-constrained minimizers E of $\mathcal{F}$, or even stationary sets E for $\mathcal{F}$ with respect to volume-preserving flows. In all these cases, provided the objects involved are smooth enough, the equilibrium conditions (1.4) and (1.5) below are satisfied. Precisely, if $\partial\Omega$ and Φ are of class C^2, if g and σ are continuous, and if (denoting by cl topological closure in $\mathbb{R}^n$ and by $\partial_{\partial\Omega}$ topological boundary in the relative topology of $\partial\Omega$) the "capillarity surface"

$$M = \mathrm{cl}(\partial E \cap \Omega)\,,$$

is a class C^2 hypersurface with boundary $M \cap \partial\Omega = \partial_{\partial\Omega}(\partial E \cap \partial\Omega)$, then one has

$$\mathrm{div}_M\big[\nabla\Phi(x,\nu_E)\big] + \nabla_x\Phi(x,\nu_E)\cdot\nu_E(x) = -g(x) + \text{constant}\,, \qquad \forall\, x \in M \cap \Omega\,, \tag{1.4}$$

$$\nabla\Phi(x,\nu_E(x))\cdot\nu_\Omega(x) = \sigma(x)\,, \qquad \forall x \in M \cap \partial\Omega\,; \tag{1.5}$$

see Figure 1.1. Here, $\nabla_x\Phi$ and $\nabla\Phi$ denote the gradients of $\Phi(x,\nu)$ in the x and ν variables respectively, while div_M denote the tangential divergence with respect to M. In the isotropic case $\Phi(x,\nu) = |\nu|$, we thus find

$$\begin{aligned} &H_M(x) = -g(x) + \text{constant}\,, && \forall x \in M \cap \Omega\,,\\ &\nu_E(x)\cdot\nu_\Omega(x) = \sigma(x)\,, && \forall x \in M \cap \partial\Omega\,, \end{aligned} \tag{1.6}$$

where H_M is the scalar mean curvature of M with respect to the orientation induced by ν_E. In particular, the equilibrium condition (1.6) is Young's law. We also note that (1.5) implies that (1.2) is a necessary condition in order to have $M \cap \partial\Omega \neq \emptyset$. Indeed, since $\Phi(x,\nu) = \nabla\Phi(x,\nu)\cdot\nu$ for every x,ν, the convexity of Φ implies $(\nabla\Phi(x,\nu) - \nabla\Phi(x,\nu_0))\cdot\nu \geq 0$ for every x,ν,ν_0. By taking $\nu = \nu_\Omega(x)$ and $\nu_0 = \nu_E(x)$ one deduces the upper bound in (1.2) from (1.5); the lower bound is deduced by taking $\nu = -\nu_\Omega(x)$.

1.2. **Boundary regularity and validity of Young's law.** Mathematically speaking, the most elementary setting in which one can prove the existence of such capillarity surfaces is given by the theory of sets of finite perimeter developed by Caccioppoli and De Giorgi. In this framework, one can easily prove the existence of minimizers in (1.3) under natural assumptions on Ω, g and σ. (In particular, it is easy to see that the constraint (1.2) on the adhesion coefficient is, in general, a necessary condition to ensure the existence of minimizers; see [Mag12, Section 19.1] for various examples and remarks.)

When writing $\mathcal{F}(E)$ for E a set of finite perimeter one has to replace the topological boundary ∂E of E (that in the case of a generic set of finite perimeter could have positive *volume*!) with its *reduced boundary* $\partial^* E$. (See section 2.3 for the definition.) It is important to take into account that $\partial^* E$ is, in general, just a generalized hypersurface in the sense of Geometric Measure Theory, that is, $\partial^* E$ is just a countable union of compact subsets of C^1-hypersurfaces. Hence, existence theory only proves the existence of a "capillarity surfaces" M of the form

$$M = \mathrm{cl}(\partial^* E \cap \Omega)\,.$$

In other words, existence theory forces one to consider extremely rough hypersurfaces. Addressing the regularity issue is thus a fundamental task in order to understand the physical significance of the model itself and the validity of the equilibrium conditions (1.4) and (1.5), and, indeed, the problem has been considered by several authors. We now review the known results on this problem, that are mainly concerned with the case when E is a local volume-constrained minimizer of $\mathcal{F}$.

Interior regularity, that is, the regularity of $M \cap \Omega$, can be addressed in the framework developed by De Giorgi [DG60], Reifenberg [Rei60, Rei64a, Rei64b], and Almgren [Alm68]. Precisely, if we assume that Φ is a smooth, uniformly elliptic integrand (see Definition 1.1 below), and that g is a smooth function, then, by combining results from [Alm76, SSA77, Bom82], one can see that

$$M \cap \Omega = M^{\rm int}_{\rm reg} \cup M^{\rm int}_{\rm sing}\,,$$

where $M^{\rm int}_{\rm reg}$ is a smooth hypersurface, relatively open into $M \cap \Omega$, and where the singular set $M^{\rm int}_{\rm sing}$ is relatively closed, with $\mathcal{H}^{n-3}(M^{\rm int}_{\rm sing}) = 0$. Moreover, in the isotropic case $\Phi = |\nu|$, $M^{\rm int}_{\rm sing}$ is discrete if $n = 8$ and satisfies $\dim(M^{\rm int}_{\rm sing}) \le n - 8$ if $n \ge 9$, where dim stands for Hausdorff dimension. In particular, interior regularity ensures that the Euler–Lagrange equation (1.4) holds true in classical sense at every $x \in M^{\rm int}_{\rm reg}$. The picture for what concerns the regularity of the free-boundary $M \cap \partial\Omega$, and thus validity of Young's law (1.5), is however much more incomplete.

When Ω is the half-space $\{x_n > 0\}$, $\Phi = |\nu|$, σ is constant and $g = g(x_n)$ (this is the so-called sessile liquid drop problem when g is the gravity potential), then one can deduce the regularity of the free boundary by combining the interior regularity theory with the symmetry properties of minimizers, see [Gon77]. Although this kind of analysis was recently carried out in the anisotropic setting under suitable symmetry assumptions on $\Phi = \Phi(\nu)$, see [Bae14], it is clear that the approach itself is intrinsically limited to the case when Ω is a half-space, σ is a constant, and g is a function of the vertical variable x_n only.

Again in the case of the sessile liquid drops, Caffarelli and Friedman in [CF85] (see also [CM07]) study the regularity of the free boundary regularity when $2 \le n \le 7$ and σ is possibly non-constant and takes values in $(-1, 0)$. The non-positivity of σ, in combination with global minimality, implies that E is the subgraph of a function $u : \mathbb{R}^{n-1} \to [0, \infty)$. Since $\sigma \ne 0$, they can show that u is globally Lipschitz, and thus exploit the regularity theory for free boundaries of uniformly elliptic problems developed in [AC81, ACF84]. Note that it is (the a-posteriori validity of) Young's law $-\nu_E(x) \cdot e_n = \sigma(x)$ to show that the assumption $\sigma \ne 0$ is essential to this method: indeed, at a boundary point where $\sigma = 0$ one cannot expect u to be Lipschitz regular. We also point out that the proof in [CF85] highly relies on the analyticity of the minimizers in the interior, that is actually the reason for the restriction $2 \le n \le 7$ on the ambient space dimension, and a further obstruction to the extension to anisotropic problems.

In the case of generic containers Ω we are only aware of a sharp result by Taylor [Tay77] in dimension $n = 3$. Taylor fully addresses three-dimensional isotropic case $\Phi = |\nu|$ as a byproduct of the methods she developed in the study of Plateau's laws [Tay76]. Her result is fully satisfactory for what concerns local minimizers of Gauss' energy in physical space, but it does not extend to anisotropic surface energies (as it is based on monotonicity formulas and epiperimetric inequalities). Moreover, even in the isotropic case, her arguments seem to be somehow limited to the case $n = 3$ (although, of course, this is not really a limitation in the study of the capillarity problem).

The case $\Phi = |\nu|$ and $\sigma \equiv 0$ in arbitrary dimension is covered by the works of Grüter and Jost [GJ86] and of Grüter [Grü87a, Grü87b, Grü87c]. These results apply for instance to the regularity of free boundaries of minimizers of relative isoperimetric problems and of mass minimizing current in relative homology classes. Part of the theory also extends to case of stationary varifolds of arbitrary codimension, [GJ86]. The key idea here is to take advantage of the condition $\sigma \equiv 0$, together with the isotropy of the area functional, in order to apply the interior regularity theory after a local "reflection" of the minimizer across $\partial\Omega$.

1.3. **Main results.** Our main result, Theorem 1.10, is a general regularity theorem for free boundaries of local minimizers of anisotropic surface energies. One can deduce from Theorem 1.10 a regularity result for anisotropic capillarity surfaces, that works without artificial restrictions on the dimension or the geometry of the container, and that – in the anisotropic case – appears to be new even in dimension $n = 3$, see Theorem 1.2 below. Let us premise the following two definitions to the statements of these results:

Definition 1.1. [Elliptic integrands] Given an open set $\Omega \subset \mathbb{R}^n$, one says that Φ is an *elliptic integrand on* Ω if $\Phi : \mathrm{cl}(\Omega) \times \mathbb{R}^n \to [0, \infty]$ is lower semicontinuous, with $\Phi(x, \cdot)$ convex and positively one-homogeneous, i.e $\Phi(x, t\nu) = t\,\Phi(x, \nu)$ for every $t \ge 0$. If Φ is an elliptic integrand on Ω and E is a set of locally finite perimeter in Ω, then we set

$$\mathbf{\Phi}(E; G) = \int_{G\cap\partial^* E} \Phi(x, \nu_E(x))\, d\mathcal{H}^{n-1}(x) \in [0, \infty]\,,$$

for every Borel set $G \subset \Omega$. Given $\lambda \ge 1$ and $\ell \ge 0$, one says that Φ is a *regular elliptic integrand on* Ω *with ellipticity constant* λ *and Lipschitz constant* ℓ, and write

$$\Phi \in \boldsymbol{\mathcal{E}}(\Omega, \lambda, \ell)\,,$$

if Φ is an elliptic integrand on Ω, with $\Phi(x, \cdot) \in C^{2,1}(\mathbf{S}^{n-1})$ for every $x \in \mathrm{cl}(\Omega)$, and if the following properties hold true for every $x\,, y \in \mathrm{cl}(\Omega)$, $\nu, \nu' \in \mathbf{S}^{n-1}$, and $e \in \mathbb{R}^n$:

$$\frac{1}{\lambda} \le \Phi(x, \nu) \le \lambda\,, \tag{1.7}$$

$$|\Phi(x, \nu) - \Phi(y, \nu)| + |\nabla\Phi(x, \nu) - \nabla\Phi(y, \nu)| \le \ell\,|x - y|\,, \tag{1.8}$$

$$|\nabla\Phi(x, \nu)| + \|\nabla^2\Phi(x, \nu)\| + \frac{\|\nabla^2\Phi(x, \nu) - \nabla^2\Phi(x, \nu')\|}{|\nu - \nu'|} \le \lambda\,, \tag{1.9}$$

$$\nabla^2\Phi(x, \nu)e \cdot e \ge \frac{\big|e - (e\cdot\nu)\nu\big|^2}{\lambda}\,, \tag{1.10}$$

where $\nabla\Phi$ and $\nabla^2\Phi$ stand for the gradient and Hessian of Φ with respect to the ν-variable. Finally, any $\Phi \in \boldsymbol{\mathcal{E}}_*(\lambda) = \boldsymbol{\mathcal{E}}(\mathbb{R}^n, \lambda, 0)$ is said a *regular autonomous elliptic integrand.*

We now state our main regularity result concerning capillarity problems.

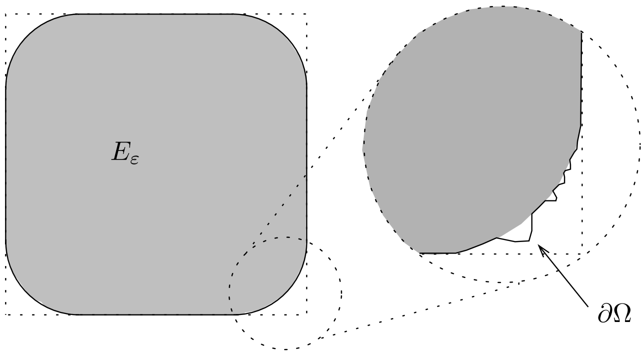

FIGURE 1.2. When the strict upper bound in (1.11) is an equality the conclusions of Theorem 1.10 can possibly fail.

Theorem 1.2. *If Ω is an open bounded set with $C^{1,1}$ boundary in $\mathbb{R}^n$, Φ is a regular elliptic integrand on Ω, $g \in L^\infty(\Omega)$, and $\sigma \in \mathrm{Lip}(\partial\Omega)$ satisfies*

$$-\Phi(x,-\nu_\Omega(x)) < \sigma(x) < \Phi(x,\nu_\Omega(x))\,, \qquad \forall x \in \partial\Omega\,, \tag{1.11}$$

then there exists a minimizer E in (1.3) such that E is an open set (possibly after a modification by a set of zero volume) and its trace $\partial E \cap \partial\Omega$ is a set of finite perimeter in $\partial\Omega$. Moreover if $M = \mathrm{cl}(\partial E \cap \Omega)$ then

$$\partial_{\partial\Omega}(\partial E \cap \partial\Omega) = M \cap \partial\Omega\,,$$

and there exists a closed set $\Sigma \subset M$, with $\mathcal{H}^{n-2}(\Sigma) = 0$ such that $M \setminus \Sigma$ is a $C^{1,1/2}$ hypersurface with boundary. In particular, Young's law (1.5) holds true at every $x \in (M \cap \partial\Omega) \setminus \Sigma$.

Remark 1.3. As proved in [SSA77] one has a better estimate on the singular set in the interior of Ω, namely $\mathcal{H}^{n-3}(\Sigma \cap \Omega) = 0$.

Corollary 1.4 (Isotropic case)**.** *Under the assumptions of Theorem 1.2, let $\Phi(x,\nu) = |\nu|$ for every $x \in \Omega$ and $\nu \in \mathbb{R}^n$. Then $\Sigma \cap \partial\Omega = \emptyset$ if $n = 3$, $\Sigma \cap \partial\Omega$ is a discrete set if $n = 4$, and $\mathcal{H}^s(\Sigma \cap \partial\Omega) = 0$ for every $s > n-4$ if $n \geq 5$.*

Remark 1.5. By the case $n = 3$ of Corollary 1.4 we obtain an alternative proof of Taylor's theorem [Tay77]. Notice also that, under the assumptions of Corollary 1.4, classical regularity for local minimizers of the perimeter gives that $\Sigma \cap \Omega = \emptyset$ if $n \leq 7$, $\Sigma \cap \Omega$ discrete if $n = 8$, and $\mathcal{H}^s(\Sigma \cap \Omega) = 0$ for every $s > n-8$ if $n \geq 9$.

Remark 1.6. Higher regularity of $\mathrm{cl}(\partial E \cap \Omega) \setminus \Sigma$ is obtained by combining Theorem 1.2 with elliptic regularity theory for non-parametric solutions of (1.4) and (1.5).

Remark 1.7. The strict inequality in (1.11) is somehow necessary. Indeed, according to (1.5) it predicts that M will intersect $\partial\Omega$ transversally. Moreover, if (1.11) fails, it is possible to construct examples of minimizers of (1.3) which do not satisfy the conclusion of Theorem 1.2. For example, if $Q = (0,1)^2 \subset \mathbb{R}^2$ is a unit square and $\varepsilon > 0$ is small enough, then the open set E_ε depicted in Figure 1.2 is a minimizer in

$$\inf\Big\{P(E) : E \subset Q\,, |E| = 1-\varepsilon\Big\}\,,$$

that is, in (1.3) with $\sigma = 1$ and $g = 0$ in the container Q. Let now Ω be an open set with smooth boundary such that $E_\varepsilon \subset \Omega \subset Q$ and $Q \cap \partial E_\varepsilon \cap \partial\Omega$ is a Cantor-type set contained in the circular arc $Q \cap \partial E_\varepsilon$. Then E_ε is a minimizer in

$$\inf\Big\{P(E) : E \subset \Omega\,, |E| = 1-\varepsilon\Big\}\,,$$

but E_ε does not satisfy the conclusion of Theorem 1.2.

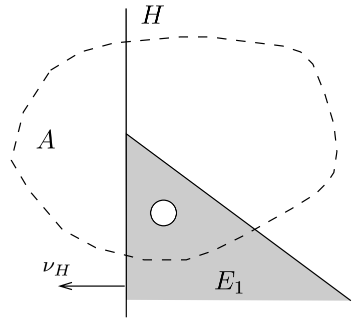

FIGURE 1.3. A (Λ, r_0)-minimizer of $(\mathbf{\Phi}, \sigma)$ in (A, H). Rougly speaking, inside balls $B_{x,r}$ of radius at most r_0 that are compactly contained in A, and up to a volume-type higher order perturbation, E is a minimizer of $F \mapsto \mathbf{\Phi}(F, H) + \int_{\partial F \cap \partial H} \sigma$ with respect to its own boundary data on $H \cap \partial B_{x,r}$, and with free boundary on $B_{x,r} \cap \partial H$. On balls $B_{x,r}$ that do not intersect ∂H, we just have a local almost-minimality condition.

Theorem 1.2 can be actually obtained as corollary of Theorem 1.10 below, which addresses the boundary regularity issue in the class of almost-minimizers introduced in the next definition.

Definition 1.8 (Almost-minimizers)**.** Let an open set A and an open half-space H in $\mathbb{R}^n$ be given (possibly $H = \mathbb{R}^n$), together with constants $r_0 \in (0, \infty]$ and $\Lambda \geq 0$, a regular elliptic integrand Φ on $A \cap H$, and a function $\sigma : A \cap \partial H \to \mathbb{R}$ with

$$-\Phi(x, -\nu_H) \leq \sigma(x) \leq \Phi(x, \nu_H) \qquad \forall\, x \in A \cap \partial H\,.$$

A set $E \subset H$ of locally finite perimeter in A is a (Λ, r_0)-*minimizer of* $(\mathbf{\Phi}, \sigma)$ *in* (A, H), if

$$\mathbf{\Phi}(E; H \cap W) + \int_{W \cap (\partial^* E \cap \partial H)} \sigma \, d\mathcal{H}^{n-1} \leq \mathbf{\Phi}(F; H \cap W) + \int_{W \cap (\partial^* F \cap \partial H)} \sigma \, d\mathcal{H}^{n-1} + \Lambda\, |E \Delta F|\,, \quad (1.12)$$

whenever $F \subset H$, $E\Delta F \subset\subset W$, and $W \subset\subset A$ is open, with $\mathrm{diam}(W) < 2r_0$; see Figure 1.3. When $\sigma = 0$, we simply say that E is a (Λ, r_0)-*minimizer of* $\mathbf{\Phi}$ *in* (A, H); when $\sigma = 0$, $\Lambda = 0$, and $r_0 = +\infty$, then we say that E is a *minimizer of* $\mathbf{\Phi}$ *in* (A, H).

Remark 1.9. Note that if $\mathrm{cl}(A) \subset H$ (as it happens, for example, in the limit case $H = \mathbb{R}^n$), then Definition 1.8 reduces to a local almost-minimality notion analogous to the ones considered in [Alm76, Bom82, Tam84, DS02] and [Mag12, Section 21].

Theorem 1.10. *If E is a (Λ, r_0)-minimizer of $(\mathbf{\Phi}, \sigma)$ in (A, H) for some $\sigma \in \mathrm{Lip}(A \cap \partial H)$ with*

$$-\Phi(x, -\nu_H) < \sigma(x) < \Phi(x, \nu_H)\,, \qquad \forall\, x \in A \cap \partial H\,, \quad (1.13)$$

then there is an open set $A' \subset A$ with $A \cap \partial H = A' \cap \partial H$ such that E is equivalent to an open set in A' and $\partial E \cap \partial H$ is a set of locally finite perimeter in $A' \cap \partial H$ (equivalently in $A \cap \partial H$). Moreover, if $M = \mathrm{cl}(\partial E \cap H)$ then

$$\partial_{\partial H}(\partial E \cap \partial H) \cap A = \partial_{\partial H}(\partial E \cap \partial H) \cap A' = M \cap \partial H$$

and there exists a relatively closed set $\Sigma \subset M \cap \partial H$ such that $\mathcal{H}^{n-2}(\Sigma) = 0$ and for every $x \in (M \cap \partial H) \setminus \Sigma$, M is a $C^{1,1/2}$ manifold with boundary in a neighborhood of x for which

$$\nabla\Phi(x, \nu_E(x)) \cdot \nu_H = \sigma(x) \qquad \forall x \in (M \cap \partial H) \setminus \Sigma\,.$$

Remark 1.11. The open set A' is just a countable union of small balls covering $A \cap \partial E$.

Since the class of regular elliptic integrands is invariant under $C^{1,1}$ diffeomorphisms (see the discussion in section 6), Theorem 1.10 applies to a wider class of variational problems than just (1.3). For instance, it applies to relative anisotropic isoperimetric problems in smooth domains, or in Riemannian and Finsler manifolds. Moreover, by arguing as in [Grü87b], Theorem 1.10 can be used to address the regularity of Φ-minimizing integer rectifiable codimension one currents in relative homology classes $\mathbf{H}_{n-1}(N,B)$ where N is a smooth n-dimensional manifold and $B\subset N$ is a smooth $(n-1)$-dimensional submanifold, see [Fed69, 4.4.1, 5.1.6] for definitions and terminology.

1.4. **Proof of Theorem 1.10 and organization of the paper.** We conclude this introduction with a few comments on our proofs, and with a brief description of the structure of the paper.

The core of the paper consists of sections 2–5, where we prove Theorem 1.10 in the $\sigma=0$ case. In section 2, after setting our notation and terminology, we prove several basic properties of almost-minimizers to be repeatedly used in subsequent arguments. Sections 3-4 are devoted to the proof of an "ε-regularity theorem" for almost-minimizers, Theorem 3.1. This theorem states the existence of a universal constant ε (i.e., depending only on the ambient space dimension and on the ellipticity constant of the integrand) with the following property: if around a free boundary point x, and for some $r>0$ sufficiently small, one has

$$\inf_{\nu\in\mathbf{S}^{n-1}}\frac{1}{r^{n-1}}\int_{H\cap B_{x,r}\cap\partial^*E}\frac{|\nu_E-\nu|^2}{2}\,d\mathcal{H}^{n-1}\le\varepsilon\,, \tag{1.14}$$

then $\mathrm{cl}(\partial E\cap H)$ is a $C^{1,1/2}$-manifold with boundary in a neighborhood of x. Here the consideration of the case $\sigma=0$, together with an appropriate choice of coordinates, allows us to "linearize" on a Neumann-type elliptic problem for which good estimates are known. (In other words, we develop the appropriate version of De Giorgi's harmonic approximation technique in our setting.) In section 5, Theorem 5.1, we estimate the size of the set where the ε-regularity theorem applies by exploiting some ideas introduced by Hardt in [Har77]. Note that when x is an interior point, De Giorgi's rectifiability theorem ensures that the set where (the appropriate version) of (1.14) holds true at some scale r is of full $\mathcal{H}^{n-1}$-measure in the boundary of E. However, as we expect the free boundary to be $(n-2)$-dimensional, and thus $\mathcal{H}^{n-1}$-negligible, we cannot deduce the existence of boundary points at which the ε-regularity theorem applies by De Giorgi's theorem only. We have instead to rely on ad hoc arguments based on minimality, and this is exactly the content of section 5.

In section 6 we begin by showing how to reduce the proof of Theorem 1.10 to the case when $\sigma=0$. This is achieved with the aid of the divergence theorem. Precisely, we show that if E is a (Λ,r_0)-minimizer of $(\mathbf{\Phi},\sigma)$ in (A,H) and $x\in A\cap\partial H$, then E is actually a (Λ_*,r_0)-minimizer of $(\mathbf{\Phi}_*,0)$ in (B_{x,r_*},H) for suitable constants Λ_* and r_*, and for a suitable regular elliptic integrand Φ_*. Having assumed strict inequalities in (1.13) plays a crucial role in showing that the new integrand Φ_* is still uniformly elliptic. Another interesting qualitative remark is that our method, even in the isotropic case, requires the consideration of anisotropic functionals in order to reduce to the case that $\sigma=0$. We finally conclude section 6 with the proofs of Theorem 1.10, Theorem 1.2 and Corollary 1.4.

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2. Almost-minimizers with free boundaries

In section 2.1 we fix our notation for sets in $\mathbb{R}^n$, while in section 2.2 we gather the basic facts concerning sets of finite perimeter. In section 2.3 we discuss some properties of the almost-minimizers introduced in Definition 1.8, while in section 2.4 we derive the anisotropic Young's law for half-spaces. Sections 2.5 and 2.6 contain classical density estimates and compactness properties of almost-minimizers. In section 2.7 we discuss some general properties of contact sets of almost-minimizers, prove a strong maximum principle, and set a useful normalization convention to be used in the rest of the paper. Finally, in section 2.8, we study the transformation of almost-minimizers under "shear-strained" deformations, a technical device that will be repeatedly applied in the proof of the ε-regularity theorem, Theorem 3.1.

2.1. **Basic notation.** *Norms and measures.* We denote by $v \cdot w$ the scalar product in $\mathbb{R}^n$ and by $|v| = (v \cdot v)^{1/2}$ the Euclidean norm. We set

$$\|L\| = \sup\{|Lx| : x \in \mathbb{R}^n\,, |x| < 1\}\,,$$

for the operator norm of a linear map $L : \mathbb{R}^n \to \mathbb{R}^n$. We denote by $\mathcal{H}^k$ the k-dimensional Hausdorff measure in $\mathbb{R}^n$ and set $\mathcal{H}^n(E) = |E|$ for every $E \subset \mathbb{R}^n$.
Reference cartesian decomposition. We denote by

$$\mathbf{p} : \mathbb{R}^n \to \mathbb{R}^{n-1} \quad \text{and} \qquad \mathbf{q} : \mathbb{R}^n \to \mathbb{R}$$

the orthogonal projections associated to the Cartesian decomposition of $\mathbb{R}^n$ as $\mathbb{R}^{n-1} \times \mathbb{R}$; correspondingly, $x = (\mathbf{p}x, \mathbf{q}x)$ for every $x \in \mathbb{R}^n$. We set

$$B = \{x \in \mathbb{R}^n : |x| < 1\}\,, \quad \mathbf{C} = \{x \in \mathbb{R}^n : |\mathbf{p}x| < 1\,, |\mathbf{q}x| < 1\}\,, \quad \mathbf{D} = \{z \in \mathbb{R}^{n-1} : |z| < 1\}\,,$$

so that $\mathbf{C} = \mathbf{D} \times (-1, 1)$. Sometimes we will identify $\mathbf{D}$ with the subset of $\mathbb{R}^n$ given by $\mathbf{D} \times \{0\}$. Even when doing so, $\partial\mathbf{D}$ denotes the boundary of $\mathbf{D}$ relative to $\mathbb{R}^{n-1}$, i.e. we always have

$$\partial\mathbf{D} = \Big\{z \in \mathbb{R}^{n-1} : |z| = 1\Big\}\,.$$

Given a vertical half-space $H = \{x_1 > b\} \subset \mathbb{R}^n$ ($b \in \mathbb{R}$), again with a slight abuse of notation we will set

$$\mathbf{D} \cap H = \Big\{z \in \mathbb{R}^{n-1} : |z| < 1\,, z_1 > b\Big\}\,,$$

as well as

$$\begin{aligned} H \cap \partial\mathbf{D} &= \Big\{z \in \mathbb{R}^{n-1} : |z| = 1\,, z_1 > b\Big\}\,, \\ \partial(\mathbf{D} \cap H) &= \Big(H \cap \partial\mathbf{D}\Big) \cup \Big\{z \in \mathbb{R}^{n-1} : |z| \le 1\,, z_1 = b\Big\}\,. \end{aligned}$$

Scaling maps. Given $E \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$ and $r > 0$, we set

$$E_{x,r} = x + r\,E\,, \qquad E^{x,r} = \frac{E - x}{r}\,.$$

In this way, for every $x \in \mathbb{R}^n$ and $r > 0$,

$$\begin{aligned} B_{x,r} &= \{y \in \mathbb{R}^n : |x - y| < r\} = B(x, r)\,, \\ \mathbf{C}_{x,r} &= \{y \in \mathbb{R}^n : |\mathbf{p}(y - x)| < r\,, |\mathbf{q}(y - x)| < r\} = \mathbf{C}(x, r)\,, \end{aligned}$$

and, similarly, for every $z \in \mathbb{R}^{n-1}$ and $r > 0$

$$\mathbf{D}_{z,r} = \{y \in \mathbb{R}^{n-1} : |y - z| < r\} = \mathbf{D}(z, r)\,.$$

In case $x\,, z = 0$ we simply write B_r, $\mathbf{C}_r$ and $\mathbf{D}_r$.
Convergence of sets. Let A be an open set in $\mathbb{R}^n$. Given a sequence of Lebesgue measurable sets $\{E_h\}_{h\in\mathbb{N}}$ in $\mathbb{R}^n$, we say that

$$E_h \to E \text{ in } L^1_{\rm loc}(A) \qquad \text{if } |(E_h \Delta E) \cap K| \to 0 \text{ as } h \to \infty \text{ for every } K \subset\subset A\,.$$

Given an open half-space $H \subset \mathbb{R}^n$ and a sequence of Borel sets $\{G_h\}_{h\in\mathbb{N}} \subset \partial H$, we say that

$$G_h \to G \text{ in } L^1_{\rm loc}(A \cap \partial H) \qquad \text{if } \mathcal{H}^{n-1}(K \cap (G_h \Delta G)) \to 0 \text{ as } h \to \infty \text{ for every } K \subset\subset A.$$

2.2. **Sets of finite perimeter.** Given a Lebesgue measurable set $E \subset \mathbb{R}^n$ and an open set $A \subset \mathbb{R}^n$, we say that E is of locally finite perimeter in A if there exists a $\mathbb{R}^n$-valued Radon measure μ_E (called the Gauss-Green measure of E) on A such that

$$\int_E \nabla\varphi(x)\,dx = \int_A \varphi\,d\mu_E\,, \qquad \forall \varphi \in C^1_c(A)\,,$$

and set $P(E;G) = |\mu_E|(G)$ for the perimeter of E relative to $G \subset A$. (Notice that $\mu_E = -D1_E$, where $D1_E$ denotes the distributional derivative of 1_E. In particular, if E is of locally finite perimeter in A and $|(E\Delta F)\cap A| = 0$, then F is of locally finite perimeter in A with $\mu_E = \mu_F$.) The well-known compactness theorem for sets of finite perimeter states that if $\{E_h\}_{h\in\mathbb{N}}$ is a sequence of sets of locally finite perimeter in A and $\{P(E_h;A_0)\}_{h\in\mathbb{N}}$ is bounded for every $A_0 \subset\subset A$, then there exists E of locally finite perimeter in A such that, up to extracting subsequences, $E_h \to E$ in $L^1_{\rm loc}(A)$; see, for instance, [Mag12, Corollary 12.27].

Support of μ_E and topological boundary. The support of μ_E can be characterized by

$$\mathrm{spt}\mu_E = \Big\{x \in A : 0 < |E \cap B(x,r)| < \omega_n\,r^n\,, \forall r > 0\Big\} \subset A \cap \partial E\,, \tag{2.1}$$

see [Mag12, Proposition 12.19].

Reduced and essential boundaries. If $E \subset \mathbb{R}^n$, $t \in [0,1]$,we set

$$E^{(t)} = \big\{x \in \mathbb{R}^n \text{ such that } |E \cap B_{x,r}| = t\,|B_{x,r}| + o(r^n) \text{ as } r \to 0^+\big\},$$

The *essential boundary* of E is defined as $\partial^{\rm e}E = \mathbb{R}^n \setminus (E^{(0)} \cup E^{(1)})$. If E is of locally finite perimeter in the open set A, then the *reduced boundary* $\partial^*E \subset A$ of E is the set of those $x \in \mathrm{spt}\mu_E$ such that

$$\lim_{r\to0^+} \frac{\mu_E(B_{x,r})}{|\mu_E|(B_{x,r})} \quad \text{exists and belongs to } \mathbf{S}^{n-1}\,.$$

This limit is denoted by $\nu_E(x)$, so that $\nu_E : \partial^*E \to \mathbf{S}^{n-1}$ is a Borel vector field. As it turns out,

$$\partial^*E \subset A \cap \partial^{\rm e}E \subset \mathrm{spt}\mu_E \subset A \cap \partial E\,, \qquad A \cap \mathrm{cl}(\partial^*E) = \mathrm{spt}\mu_E\,,$$

and each inclusion may be strict. Federer's criterion, see for instance [Mag12, Theorem 16.2], ensures that

$$\mathcal{H}^{n-1}((A \cap \partial^{\rm e}E) \setminus \partial^*E) = 0\,. \tag{2.2}$$

Moreover,

$$A =_{\mathcal{H}^{n-1}} \big(E^{(0)} \cup E^{(1)} \cup \partial^{\rm e}E\big) \cap A =_{\mathcal{H}^{n-1}} \big(E^{(0)} \cup E^{(1)} \cup \partial^*E\big) \cap A\,, \tag{2.3}$$

where the unions are $\mathcal{H}^{n-1}$ disjoints and we have introduced the notation $G =_{\mathcal{H}^{n-1}} F$ to mean $\mathcal{H}^{n-1}(G\Delta F) = 0$ (and, similarly, $G \subset_{\mathcal{H}^{n-1}} F$ means that $\mathcal{H}^{n-1}(F \setminus G) = 0$). We finally recall that De Giorgi's rectifiability theorem [Mag12, Theorem 15.5] asserts that, for every $x \in \partial^*E$,

$$E^{x,r} \to \{y \in \mathbb{R}^n : \nu_E(x)\cdot y \le 0\} \qquad \text{in } L^1_{\rm loc}(\mathbb{R}^n)\,,$$

and that $\mu_E = \nu_E\,\mathcal{H}^{n-1} \llcorner \partial^*E$ on Borel sets compactly contained in A where, given a Radon measure μ and a Borel set G, by $\mu \llcorner G$ we mean the measure given by $\mu \llcorner G(F) = \mu(G \cap F)$. In particular

$$\int_E \nabla\varphi(x)\,dx = \int_{\partial^*E} \varphi\,\nu_E\,d\mathcal{H}^{n-1}\,, \qquad \forall \varphi \in C^1_c(A)\,, \tag{2.4}$$

see for instance [Mag12, Section 15]. In particular $\mu_E(G) = 0$ if $\mathcal{H}^{n-1}(G) = 0$.

Gauss-Green measure and set operations. It is well-known that, if E and F are of locally finite perimeter in A, then $E \cap F$, $E \cup F$, $E \setminus F$ and $E\Delta F$ are sets of locally finite perimeter in A. Since the construction of competitors used in testing minimality inequalities often involves a combination of these set operations, being able to describe the corresponding behavior of Gauss–Green measures turns out to be extremely convenient. Recalling that $\nu_E(x) = \pm\nu_F(x)$

at $\mathcal{H}^{n-1}$-a.e. $x\in\partial^*E\cap\partial^*F$, setting $\{\nu_E=\nu_F\}$ for the sets of those $x\in\partial^*E\cap\partial^*F$ such that $\nu_E(x)=\nu_F(x)$, and defining similarly $\{\nu_E=-\nu_F\}$, one can prove that

$$\mu_{E\cap F} = \mu_E\llcorner(F^{(1)}\cap\partial^*E)+\mu_F\llcorner(E^{(1)}\cap\partial^*F)+\mu_E\llcorner\{\nu_E=\nu_F\}\,, \tag{2.5}$$

$$\mu_{E\cup F} = \mu_E\llcorner(F^{(0)}\cap\partial^*E)+\mu_F\llcorner(E^{(0)}\cap\partial^*F)+\mu_E\llcorner\{\nu_E=\nu_F\}\,, \tag{2.6}$$

$$\mu_{E\setminus F} = \mu_E\llcorner(F^{(0)}\cap\partial^*E)-\mu_F\llcorner(E^{(1)}\cap\partial^*F)+\mu_E\llcorner\{\nu_E=-\nu_F\}\,, \tag{2.7}$$

see [Mag12, Section 16.1]. Moreover, if $E\subset F$, then

$$\mu_E=\mu_E\llcorner F^{(1)}+\mu_F\llcorner\{\nu_E=\nu_F\}=\mu_E\llcorner F^{(1)}+\mu_F\llcorner(\partial^*E\cap\partial^*F)\,. \tag{2.8}$$

Reduced boundary and bi-Lipschitz transformations. If $f:\mathbb{R}^n\to\mathbb{R}^n$ is a Lipschitz diffeomorphism with $\det(\nabla f)>0$ on $\mathbb{R}^n$, then by the area formula it follows that $f(E)$ is a set of locally finite perimeter in $f(A)$, with $f(\partial^*E)=_{\mathcal{H}^{n-1}}\partial^*(f(E))$ and

$$\nu_{f(E)}(f(x))=\frac{\mathrm{cof}\,(\nabla f(x))\nu_E(x)}{|\mathrm{cof}\,(\nabla f(x))\nu_E(x)|}\,,\qquad\text{for }\mathcal{H}^{n-1}\text{-a.e. }x\in\partial^*f(E)\,.$$

(Recall that, if $L:\mathbb{R}^n\to\mathbb{R}^n$ is an invertible linear map, then

$$\mathrm{cof}\,L=(\det L)\,(L^{-1})^*\,,$$

where L^* denotes the adjoint map to L.) Moreover, one has

$$\int_{f(G\cap\partial^*E)}\Psi(y,\nu_{f(E)}(y))\,d\mathcal{H}^{n-1}(y)=\int_{G\cap\partial^*E}\Psi\Big(f(x),\mathrm{cof}\,(\nabla f(x))\,\nu_E(x)\Big)\,d\mathcal{H}^{n-1}(x)\,, \tag{2.9}$$

for every Borel measurable function $\Psi:A\times\mathbb{R}^n\to[0,\infty]$ which is one-homogeneous in the second variable and every $G\subset A$.

Traces of sets of finite perimeter. Let A be an open set in $\mathbb{R}^n$, let $H=\{x_n>0\}$, and let $E\subset H$ be a set of locally finite perimeter in A. Since $1_E\in BV(A'\cap H)$ for every open set $A'\subset\subset A$, by [Giu84, Lemma 2.4, Theorem 2.10] there exists a Borel set $\mathrm{Tr}_{\partial H}(E)\subset A\cap\partial H$ such that

$$\int_E\mathrm{div}\,T(x)dx=\int_{H\cap\partial^*E}T\cdot\nu_E\,d\mathcal{H}^{n-1}+\int_{\mathrm{Tr}_{\partial H}(E)}T\cdot\nu_H\,d\mathcal{H}^{n-1}\qquad\forall\,T\in C^1_c(A,\mathbb{R}^n)\,, \tag{2.10}$$

and with the property that, if $E_t=\{z\in\partial H:(z,t)\in E\cap A\}$ $(t>0)$, then

$$\lim_{t\to0^+}\mathcal{H}^{n-1}(K\cap(E_t\Delta\mathrm{Tr}_{\partial H}(E)))=0\,,\qquad\text{for every }K\subset\subset A. \tag{2.11}$$

On taking into account that, by (2.8),

$$\mu_E=\nu_E\,\mathcal{H}^{n-1}\llcorner(H\cap\partial^*E)-e_1\,\mathcal{H}^{n-1}\llcorner(\partial^*E\cap\partial H)\,, \tag{2.12}$$

by comparing (2.4), (2.12) and (2.10) we get

$$\mathrm{Tr}_{\partial H}(E)=_{\mathcal{H}^{n-1}}\partial^*E\cap\partial H\,. \tag{2.13}$$

We also notice that

$$\mathrm{Tr}_{\partial H}(H\setminus E)=_{\mathcal{H}^{n-1}}\partial H\setminus\mathrm{Tr}_{\partial H}(E)\,. \tag{2.14}$$

Finally, from [Giu84, Theorem 2.11], we have that if $\{E_h\}_{h\in\mathbb{N}}$ and E are sets of locally finite perimeter in A, then

$$\left\{\begin{array}{l}E_h\to E\text{ in }L^1_{\mathrm{loc}}(A)\,,\\ P(E_h;A'\cap H)\to P(E;A'\cap H)\text{ as }h\to\infty\\ \text{for every }A'\subset\subset A\,,\end{array}\right.\qquad\Rightarrow\qquad\mathrm{Tr}_{\partial H}(E_h)\to\mathrm{Tr}_{\partial H}(E)\text{ in }L^1_{\mathrm{loc}}(A\cap\partial H). \tag{2.15}$$

2.3. **Basic remarks on almost-minimizers.** Let us recall from Definition 1.8 that if A and H are an open set and an open half-space in $\mathbb{R}^n$, Φ is an elliptic integrand on $A\cap H$, $r_0 \in (0,\infty]$, and $\Lambda \geq 0$, then one says that E is a (Λ, r_0)-*minimizer of* $\boldsymbol{\Phi}$ *in* (A,H) provided $E \subset H$, E is a set of locally finite perimeter in A, and

$$\boldsymbol{\Phi}(E; W\cap H) \leq \boldsymbol{\Phi}(F; W\cap H) + \Lambda\,|E\Delta F| \tag{2.16}$$

whenever $F \subset H$ and $E\Delta F \subset\subset W$ for some open set $W \subset\subset A$ with $\mathrm{diam}(W) < 2r_0$. A $(0,\infty)$-minimizer is simply called minimizer.

The following two simple remarks concerning the behavior of almost minimizers with respect to the scaling and set complement will be frequently used in the sequel:

Remark 2.1 (Minimality and set complement)**.** If E is a (Λ, r_0)-minimizer of $\boldsymbol{\Phi}$ in (A,H), then $H\setminus E$ is a (Λ, r_0)-minimizer of $\widetilde{\boldsymbol{\Phi}}$ in (A,H), provided we set

$$\widetilde{\Phi}(x,\nu) = \Phi(x,-\nu)\,.$$

Of course, $\Phi \in \boldsymbol{\mathcal{E}}(A\cap H, \lambda, \ell)$ if and only if $\widetilde{\Phi} \in \boldsymbol{\mathcal{E}}(A\cap H,\lambda,\ell)$.

Remark 2.2 (Minimality and scaling)**.** Given $x \in \mathrm{cl}(A\cap H)$ and $r < r_0$ such that $B_{x,r} \subset\subset A$, one notices that E is a (Λ, r_0)-minimizer of $\boldsymbol{\Phi}$ in (A,H) if and only if $E^{x,r}$ is a $(\Lambda\, r, r_0/r)$-minimizer of $\boldsymbol{\Phi}^{x,r}$ in $(A^{x,r}, H^{x,r})$, where we have set

$$\Phi^{x,r}(y,\nu) = \Phi(x + r\,y, \nu)\,.$$

Notice that $\Phi \in \boldsymbol{\mathcal{E}}(A\cap H,\lambda,\ell)$ if and only if $\Phi^{x,r} \in \boldsymbol{\mathcal{E}}((A\cap H)^{x,r}, \lambda, r\,\ell)$.

It is sometimes convenient to consider sets which satisfy the minimality inequality (2.16) only with respect to inner or outer variations. Hence we also give the following definition, with A, H, Φ, r_0 and Λ as above.

Definition 2.3 (Sub/superminimizer)**.** One says that E is a (Λ, r_0)-*subminimizer of* $\boldsymbol{\Phi}$ *in* (A,H) if $E\subset H$, E is of locally finite perimeter in A, and inequality (2.16) holds true whenever $F \subset E$ and $E\setminus F \subset\subset W$ for some open set $W\subset\subset A$ with $\mathrm{diam}(W) < 2r_0$; and that E is a (Λ, r_0)-*superminimizer of* $\boldsymbol{\Phi}$ *in* (A,H) if inequality (2.16) holds true whenever $E\subset F\subset H$ and $F\setminus E \subset\subset W$ for some open set $W\subset\subset A$ with $\mathrm{diam}(W) < 2r_0$. In analogy with Definition 1.8, when E is a $(0,\infty)$-sub/superminimizer one simply says that E is a sub/superminimizer.

Remark 2.4. It is clear that a (Λ, r_0)-minimizer in (A,H) is both a (Λ, r_0)-superminimizer and a (Λ, r_0)-subminimizer. The converse is also true. Indeed, using (2.5) and (2.6), one easily verifies that for every sets $E, F\subset H$ of locally finite perimeter in A,

$$\boldsymbol{\Phi}(E\cap F, W\cap H) + \boldsymbol{\Phi}(E\cup F, W\cap H) \leq \boldsymbol{\Phi}(E, W\cap H) + \boldsymbol{\Phi}(F, W\cap H), \tag{2.17}$$

wherever $W\subset\subset A$. Hence, if E is a both a (Λ, r_0)-superminimizer and a (Λ, r_0)-subminimizer and $F\Delta E\subset\subset W\subset\subset A$, comparing E with $E\cup F$ and $E\cap F$ (which are immediately seen to be admissible) and using (2.17) we obtain

$$\begin{aligned} 2\,\boldsymbol{\Phi}(E, W\cap H) &\leq \boldsymbol{\Phi}(E\cap F, W\cap H) + \boldsymbol{\Phi}(E\cup F, W\cap H) + \Lambda\big(|E\setminus F| + |F\setminus E|\big)\\ &\leq \boldsymbol{\Phi}(E, W\cap H) + \boldsymbol{\Phi}(F, W\cap H) + \Lambda|E\Delta F|. \end{aligned}$$

The following " transfer of sub/superminimality property" will be useful in section 5.

Proposition 2.5. *Let A and be H be an open set and an open half-space in $\mathbb{R}^n$, let $\Phi \in \boldsymbol{\mathcal{E}}(A\cap H, \lambda,\ell)$, and let $E_1, E_2 \subset H$ be sets of locally finite perimeter in A with*

$$\partial^* E_2 \subset_{\mathcal{H}^{n-1}} \partial^* E_1\,. \tag{2.18}$$

If E_1 is a (Λ, r_0)-superminimizer of $\boldsymbol{\Phi}$ in (A,H) and $E_1\subset E_2$, then E_2 is a (Λ, r_0)-superminimizer of $\boldsymbol{\Phi}$ in (A,H); see Figure 2.1. Similarly, if E_1 is a (Λ, r_0)-subminimizer of $\boldsymbol{\Phi}$ in (A,H) and $E_2\subset E_1$, then E_2 is a (Λ, r_0)-subminimizer of $\boldsymbol{\Phi}$ in (A,H).

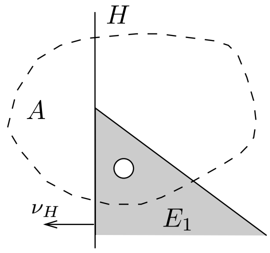

FIGURE 2.1. The situation in Proposition 2.5: if E_1 is a (Λ, r_0)-superminimizer of $\mathbf{\Phi}$ in (A, H) (which is the case for the set E_1 in the picture if Λ and r_0 are large and small enough respectively, and if the angle between the flat part of the boundary of E_1 and ∂H is in a suitable range, cf. with Proposition 2.6), then the set E_2 obtained by adding the interior of the missing disk to E_1 (larger set with smaller boundary) is still a superminimizer.

Proof. We give details only in the case of superminimizers, the case of subminimizers being entirely analogous. Let F be such that $E_2 \subset F \subset H$ and $F \setminus E_2 \subset\subset W$ for some open set $W \subset\subset A$ with $\mathrm{diam}(W) < 2r_0$. Setting $G^+ = G \cap H$ for every $G \subset \mathbb{R}^n$, we want to show that $\mathbf{\Phi}(E_2; W^+) \le \mathbf{\Phi}(F; W^+) + \Lambda\, |F \setminus E_2|$. By $E_2 \subset F$ and (2.8), this last inequality is equivalent to

$$\mathbf{\Phi}(E_2; F^{(1)} \cap W^+) \le \mathbf{\Phi}(F; E_2^{(0)} \cap W^+) + \Lambda\, |F \setminus E_2|\,. \tag{2.19}$$

To prove (2.19), we set

$$F_* = (F \setminus E_2) \cup E_1\,,$$

so that $E_1 \subset F_*$ with $F_* \setminus E_1 = F \setminus E_2 \subset\subset W$. By (Λ, r_0)-superminimality of E_1, we have

$$\mathbf{\Phi}(E_1; W^+) \le \mathbf{\Phi}(F_*; W^+) + \Lambda\, |F \setminus E_2|\,. \tag{2.20}$$

We now deduce (2.19) from (2.20) by repeatedly applying the formulas for Gauss-Green measures under set operations in conjunction with $E_1 \subset E_2$ and (2.18). We begin by noticing that, by (2.2), (2.8) and (2.18) we have

$$\mu_{E_1} = \mu_{E_2} + \mu_{E_1} \llcorner E_2^{(1)}\,, \qquad \mu_{E_2} = \mu_{E_2} \llcorner \partial^* F + \mu_{E_2} \llcorner F^{(1)}\,. \tag{2.21}$$

By (2.3) and (2.21) we find

$$\begin{aligned} \mathbf{\Phi}(E_1; W^+) &= \mathbf{\Phi}(E_2; W^+) + \mathbf{\Phi}(E_1; E_2^{(1)} \cap W^+) \\ &= \mathbf{\Phi}(E_2; F^{(1)} \cap W^+) + \mathbf{\Phi}(F; \partial^* E_2 \cap W^+) + \mathbf{\Phi}(E_1; E_2^{(1)} \cap W^+)\,. \end{aligned} \tag{2.22}$$

Since $\nu_{E_1} = -\nu_{F\setminus E_2}$ $\mathcal{H}^{n-1}$-a.e. on $\partial^* E_1 \cap \partial^*(F \setminus E_2)$ (due to the fact that $E_1 \subset E_2 \subset F$), by applying (2.6) to F_* we find that

$$\mathbf{\Phi}(F_*; W^+) = \mathbf{\Phi}(F \setminus E_2; E_1^{(0)} \cap W^+) + \mathbf{\Phi}(E_1; (F \setminus E_2)^{(0)} \cap W^+)\,. \tag{2.23}$$

We start noticing that

$$\mathbf{\Phi}(F \setminus E_2; E_1^{(0)} \cap W^+) = \mathbf{\Phi}(F; E_2^{(0)} \cap W^+)\,. \tag{2.24}$$

Indeed, by (2.21), (2.7) gives $\mu_{F \setminus E_2} = \mu_F \llcorner E_2^{(0)} - \mu_{E_2} \llcorner F^{(1)}$, so that we just need to show that $P(E_2; E_1^{(0)}) = 0$: but this is obvious, since $E_1 \subset E_2$ implies $E_2^{(0)} \subset E_1^{(0)}$, and thus, by (2.18),

$$\mathcal{H}^{n-1}(E_1^{(0)} \cap \partial^* E_2) \le \mathcal{H}^{n-1}(E_1^{(0)} \cap \partial^* E_1) = 0\,.$$

This proves (2.24). Next, we notice that $(F\setminus E_2)^{(0)} =_{\mathcal{H}^{n-1}} F^{(0)}\cup E_2^{(1)}\cup(\partial^*F\cap\partial^*E_2)$, with $\mathcal{H}^{n-1}(F^{(0)}\cap\partial^*E_1)=0$ by $F^{(0)}\subset E_1^{(0)}$ and (2.3), so that

$$
\begin{aligned}
\mathbf{\Phi}(E_1;(F\setminus E_2)^{(0)}\cap W^+) &= \mathbf{\Phi}(E_1;E_2^{(1)}\cap W^+)+\mathbf{\Phi}(E_1;\partial^*F\cap\partial^*E_2\cap W^+)\\
&= \mathbf{\Phi}(E_1;E_2^{(1)}\cap W^+)+\mathbf{\Phi}(F;\partial^*E_1\cap\partial^*E_2\cap W^+)\\
&= \mathbf{\Phi}(E_1;E_2^{(1)}\cap W^+)+\mathbf{\Phi}(F;\partial^*E_2\cap W^+)\,, \qquad (2.25)
\end{aligned}
$$

where in the last two identities we have first used that $\nu_F=\nu_{E_1}$ $\mathcal{H}^{n-1}$-a.e. on $\partial^*F\cap\partial^*E_1$, and then (2.18). By combining (2.20), (2.22), (2.23), (2.24), and (2.25) we thus find (2.19). □

2.4. **Anisotropic Young's law on half-spaces.** It is well known that, if A is an open set and Φ is an autonomous elliptic integrand, then

$$
\mathbf{\Phi}(\{x\cdot\nu<s\};A)\le\mathbf{\Phi}(F;A)\,, \qquad (2.26)
$$

whenever $\nu\in\mathbf{S}^{n-1}$, $s\in\mathbb{R}$, $\{x\cdot\nu<s\}=\{x\in\mathbb{R}^n:x\cdot\nu<s\}$ and $\{x\cdot\nu<s\}\Delta F\subset\subset A$. If, in addition, $\Phi\in\boldsymbol{\mathcal{E}}_*(\lambda)$ for some $\lambda>0$, then by Taylor's formula, (1.9) and (1.10), one can find positive constants κ_1 and κ_2 depending on λ only, such that

$$
\kappa_1\,\frac{|\nu_1-\nu_2|^2}{2}\le\Phi(\nu_2)-\Phi(\nu_1)-\nabla\Phi(\nu_1)\cdot(\nu_2-\nu_1)\le\kappa_2\,\frac{|\nu_1-\nu_2|^2}{2}\,, \qquad (2.27)
$$

for every $\nu_1,\nu_2\in\mathbf{S}^{n-1}$. Correspondingly, one can strengthen (2.26) into

$$
\kappa_1\int_{A\cap\partial^*F}\frac{|\nu_F-\nu|^2}{2}\le\mathbf{\Phi}(F;A)-\mathbf{\Phi}(\{x\cdot\nu<s\};A)\le\kappa_2\int_{A\cap\partial^*F}\frac{|\nu_F-\nu|^2}{2}\,, \qquad (2.28)
$$

which holds true whenever $\nu\in\mathbf{S}^{n-1}$, $s\in\mathbb{R}$, and $\{x\cdot\nu<s\}\Delta F\subset\subset A$. The following proposition provides similar assertions when a free boundary condition on a given hyperplane is considered.

Proposition 2.6 (Anisotropic Young's law)**.** *Let $H=\{x_1>0\}$, A be an open set, $\Phi\in\boldsymbol{\mathcal{E}}_*(\lambda)$ for some $\lambda>0$, $\nu\in\mathbf{S}^{n-1}\setminus\{\pm e_1\}$ and $c\in\mathbb{R}$ be such that the set*

$$
E=H\cap\{x\cdot\nu<c\}\,,
$$

satisfies $A\cap H\cap\partial E\ne\emptyset$, see Figure 2.2. Then, E is a superminimizer of $\mathbf{\Phi}$ in (A,H) if and only if

$$
\nabla\Phi(\nu)\cdot e_1\ge0\,; \qquad (2.29)
$$

similarly, E is a subminimizer of $\mathbf{\Phi}$ in (A,H) if and only if $\nabla\Phi(\nu)\cdot e_1\le0$. In particular, E is a minimizer of $\mathbf{\Phi}$ in (A,H) if and only if $\nabla\Phi(\nu)\cdot e_1=0$. Moreover, in this last case,

$$
\begin{aligned}
\kappa_1\int_{A\cap H\cap\partial^*F}\frac{|\nu_F-\nu|^2}{2}\,d\mathcal{H}^{n-1} &\le \mathbf{\Phi}(F;W\cap H)-\mathbf{\Phi}(E;W\cap H)\\
&\le \kappa_2\int_{A\cap H\cap\partial^*F}\frac{|\nu_F-\nu|^2}{2}\,d\mathcal{H}^{n-1}\,, \qquad (2.30)
\end{aligned}
$$

whenever $F\subset H$ with $E\Delta F\subset\subset W\subset\subset A$. Here, κ_1 and κ_2 are as in (2.27)*.*

Proof. Step one: We prove that (2.29) implies

$$
\mathbf{\Phi}(E;W\cap H)\le\mathbf{\Phi}(F;W\cap H)\,, \qquad (2.31)
$$

whenever $E\subset F\subset H$ with $F\setminus E\subset\subset W\subset\subset A$. Indeed, let $W'\subset\subset W$ be a set with smooth boundary such that $F\setminus E\subset\subset W'$ and

$$
\mathcal{H}^{n-1}(\partial W'\cap\partial^*E)=\mathcal{H}^{n-1}(\partial W'\cap\partial^*F)=0. \qquad (2.32)
$$

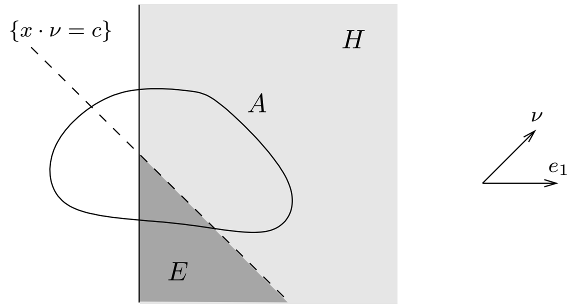

FIGURE 2.2. If $H = \{x_1 > 0\}$, then a half-space with outer unit normal ν is a superminimizer of $\mathbf{\Phi}$ on $(\mathbb{R}^n, H)$ if and only if $\nabla\Phi(\nu) \cdot e_1 \geq 0$.

By applying the divergence theorem to the constant vector field $\nabla\Phi(\nu)$ on the sets of finite perimeter $E \cap W'$ and $F \cap W'$, by taking into account (2.32), and by noticing that $\nu_E = -e_1$ $\mathcal{H}^{n-1}$-a.e. on $\partial^* E \cap \partial H$ (and that an analogous relation holds true for F), we obtain

$$\begin{aligned}\int_{\partial^* E\cap W'\cap H} \nabla\Phi(\nu)\cdot\nu_E\,d\mathcal{H}^{n-1} - (\nabla\Phi(\nu)\cdot e_1)\mathcal{H}^{n-1}(\partial^* E\cap\partial H\cap W') \\ = -\int_{E^{(1)}\cap\partial W'} \nabla\Phi(\nu)\cdot\nu_{W'}\,d\mathcal{H}^{n-1}\,, \\ \int_{\partial^* F\cap W'\cap H} \nabla\Phi(\nu)\cdot\nu_F\,d\mathcal{H}^{n-1} - (\nabla\Phi(\nu)\cdot e_1)\mathcal{H}^{n-1}(\partial^* F\cap\partial H\cap W') \\ = -\int_{F^{(1)}\cap\partial W'} \nabla\Phi(\nu)\cdot\nu_{W'}\,d\mathcal{H}^{n-1}\,.\end{aligned} \tag{2.33}$$

By $F \setminus E \subset\subset W'$, we have $E^{(1)} \cap \partial W' = F^{(1)} \cap \partial W'$; moreover, the inclusions $E \subset F \subset H$ and the definition of essential boundary imply that $\partial^* E \cap \partial H =_{\mathcal{H}^{n-1}} \partial^e E \cap \partial H \subset \partial^e F \cap \partial H =_{\mathcal{H}^{n-1}} \partial^* F \cap \partial H$: thus, by subtracting the two identities in (2.33), we find

$$\begin{aligned}\int_{\partial^* E\cap W'\cap H} \nabla\Phi(\nu)\cdot\nu_E\,d\mathcal{H}^{n-1} - \int_{\partial^* F\cap W'\cap H} \nabla\Phi(\nu)\cdot\nu_F\,d\mathcal{H}^{n-1} \\ = -(\nabla\Phi(\nu)\cdot e_1)\mathcal{H}^{n-1}\Big((\partial^* F\setminus\partial^* E)\cap\partial H\cap W'\Big)\,.\end{aligned} \tag{2.34}$$

Since $\nu_E = \nu$ on $\partial^* E \cap H$ and $\nabla\Phi(\nu)\cdot\nu = \Phi(\nu)$, the first integral on the left-hand side of (2.34) coincides with $\mathbf{\Phi}(E; W' \cap H)$. Therefore, (2.34) gives

$$\begin{aligned}\mathbf{\Phi}(E, W\cap H) + \int_{\partial^* F\cap H\cap W} \gamma_F\,d\mathcal{H}^{n-1} \\ = \mathbf{\Phi}(F, W\cap H) - (\nabla\Phi(\nu)\cdot e_1)\mathcal{H}^{n-1}\big((\partial^* F\setminus\partial^* E)\cap\partial H\cap W\big)\,,\end{aligned} \tag{2.35}$$

where we have defined $\gamma_F : \partial^* F \to \mathbb{R}$ by setting

$$\gamma_F = \Phi(\nu_F) - \nabla\Phi(\nu)\cdot\nu_F = \Phi(\nu_F) - \Phi(\nu) + \nabla\Phi(\nu)\cdot(\nu_F - \nu)\,.$$

By convexity of Φ, $\gamma_F \geq 0$ on $\partial^* F$, and thus (2.29) and (2.35) imply (2.31). The case of subminimizers is treated analogously, and then the characterization of minimizers follows by Remark 2.4. Moreover, in this last case, by exploiting (2.17) as in Remark 2.4, and by using (2.35), (2.6), and (2.5), we obtain

$$\mathbf{\Phi}(F, W\cap H) - \mathbf{\Phi}(E, W\cap H) = \int_{\partial^* F\cap H\cap W} \gamma_F\,d\mathcal{H}^{n-1}\,. \tag{2.36}$$

Since, by (2.27), $\kappa_1\,|\nu_F - \nu|^2 \leq 2\,\gamma_F(y) \leq \kappa_2\,|\nu_F - \nu|^2$ on $\partial^* F$, we see that (2.36) implies (2.30).

Step two: We now prove that (2.31) implies (2.29). Without loss of generality we shall assume that $0\in A$ and that $c=0$. In particular, by (2.31), there exists $r>0$ such that (2.31) holds true for every $E\subset F\subset H$ with $F\setminus E\subset\subset B_r$. To exploit this property, we pick $\zeta\in C^1_c(B_r)$, $\zeta\ge 0$, $e\in\mathbf{S}^{n-1}$ with

$$e\cdot e_1=0\qquad e\cdot\nu\ge 0,$$

and we define the maps $f_t(x)=x+t\,T(x)$ for $T=\zeta\,e\in C^1_c(B_r;\mathbb{R}^n)$, $t\ge 0$ and $x\in\mathbb{R}^n$. Clearly there exists $\varepsilon_0>0$ such that $\{f_t\}_{t\in[0,\varepsilon_0)}$ is a one-parameter family of diffeomorphisms on $\mathbb{R}^n$ such that, if we set $F_t=f_t(E)$, then $E\subset F_t\subset H$ with $F_t\setminus E\subset\subset B_r$. In particular by (2.31)

$$0\le \frac{d}{dt}\Big|_{t=0^+}\mathbf{\Phi}(f_t(E);B_r\cap H)\,. \tag{2.37}$$

By (2.9),

$$\begin{aligned}\mathbf{\Phi}(f_t(E);B_r\cap H)&=\int_{B_r\cap\partial^*E}\Phi(\mathrm{cof}\,(\nabla f_t)\nu_E)\,d\mathcal{H}^{n-1}\\ &=\int_{B_r\cap\partial^*E}\Phi(\nu_E)+t\Big(\Phi(\nu_E)\,\mathrm{div}\,T-\nabla\Phi(\nu_E)\cdot[(\nabla T)^*\nu_E]\Big)\,d\mathcal{H}^{n-1}+o(t)\,,\end{aligned}$$

where we have also used the fact that

$$\begin{aligned}\nabla f_t&=\mathrm{Id}\,+t\,\nabla T\,, & \mathrm{cof}\,(\nabla f_t)&=(J\,f_t)[(\nabla f_t)^{-1}\circ f_t]^*\,,\\ (\nabla f_t)^{-1}\circ f_t&=\mathrm{Id}\,-t\,\nabla T+O(t^2)\,, & Jf_t&=1+t\,\mathrm{div}\,T+O(t^2)\,.\end{aligned}\tag{2.38}$$

By (2.37), $\nabla T=e\otimes\nabla\zeta$, $\nu_E=\nu$, and $\Phi(\nu)=\nabla\Phi(\nu)\cdot\nu$, we thus find that

$$\begin{aligned}0&\le\int_{B_r\cap\partial^*E}\Phi(\nu)\,(e\cdot\nabla\zeta)-(e\cdot\nu)\,(\nabla\Phi(\nu)\cdot\nabla\zeta)\,d\mathcal{H}^{n-1}\\ &=\Big((\nabla\Phi(\nu)\cdot\nu)\,e-(e\cdot\nu)\nabla\Phi(\nu)\Big)\cdot\int_{B_r\cap\partial^*E}\nabla\zeta\,d\mathcal{H}^{n-1}\,.\end{aligned}\tag{2.39}$$

We now recall that $B_r\cap\partial^*E$ is the intersection with H of the $(n-1)$-dimensional disk in $\mathbb{R}^n$ of radius $r>0$, center at the origin, and perpendicular to ν, and that $\zeta=0$ on ∂B_r. Therefore, if we denote by ν_* its unit co-normal vector along $\{x\cdot\nu=0\}\cap\partial H$, then by divergence theorem

$$\int_{B_r\cap\partial^*E}\nabla\zeta\,d\mathcal{H}^{n-1}=\nu_*\int_{\{x\cdot\nu=0\}\cap\partial H}\zeta\,d\mathcal{H}^{n-2}\,. \tag{2.40}$$

By exploiting (2.39) and (2.40), and choosing $\zeta\in C_c(B_r)$ with $\int_{\{x\cdot\nu=0\}\cap H}\zeta\,d\mathcal{H}^{n-2}>0$, we find

$$\Big((\nabla\Phi(\nu)\cdot\nu)\,e-(e\cdot\nu)\nabla\Phi(\nu)\Big)\cdot\nu_*\ge 0\,,\qquad\forall\,e\in e_1^\perp\quad e\cdot\nu\ge 0\,. \tag{2.41}$$

Since $\nu\ne\pm e_1$ we can find α and $\beta<0$ such that $\nu_*=\alpha\,\nu+\beta\,e_1$. If we plug this identity into (2.41), as $\beta<0$, then we find

$$(e\cdot\nu)\,(\nabla\Phi(\nu)\cdot e_1)\ge 0\,,\qquad e\in e_1^\perp\quad e\cdot\nu\ge 0.$$

As $\nu\ne\pm e_1$, there exists $e\in e_1^\perp$ with $\nu\cdot e>0$. Thus $\nabla\Phi(\nu)\cdot e_1\ge 0$, as desired. □

We conclude this section on anisotropic Young's laws with an elementary technical lemma that shall be frequently used in the sequel. Given $\nu\in\mathbf{S}^{n-1}$ with $|\nu\cdot e_1|<1$, we shall set

$$\mathbf{e_1}(\nu)=\frac{\nu-(\nu\cdot e_1)\,e_1}{\sqrt{1-(\nu\cdot e_1)^2}}\,, \tag{2.42}$$

for the normalized projection of ν on $e_1^\perp$. In the light of Proposition 2.6, the following lemma says that if $\{x\cdot\nu<0\}\cap H$ is close to be a minimizer of $\mathbf{\Phi}\in\boldsymbol{\mathcal{E}}_*(\lambda)$ (in the sense that $\nabla\Phi(\nu)\cdot e_1$ is small), then there exists a minimizer of $\mathbf{\Phi}$ of the form $\{x\cdot\nu_0<0\}\cap H$ with ν_0 close to ν, and with the normalized projections of ν_0 and ν on $e_1^\perp$ being actually equal to each other.

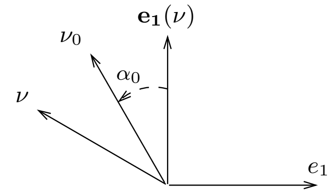

FIGURE 2.3. If we set $\nu(\alpha) = \cos\alpha\,\mathbf{e_1}(\nu) - \sin\alpha\, e_1$, then $\nu(\pi/2) = -e_1$, $\nu(-\pi/2) = e_1$, and $f(\alpha) = \nabla\Phi(\nu(\alpha)) \cdot e_1$ is strictly decreasing on $[-\pi/2, \pi/2]$ with $f(-\pi/2) \ge 1/\lambda$ and $f(\pi/2) \le -1/\lambda$. In particular, given $\Phi \in \boldsymbol{\mathcal{E}}_*(\lambda)$ and $e \in e_1^\perp \cap \mathbf{S}^{n-1}$, there exists a unique $\nu_0 \in \mathbf{S}^{n-1}$ such that $\mathbf{e_1}(\nu_0) = e$ and $\nabla\Phi(\nu_0) \cdot e_1 = 0$. If $\nu_0 = \nu(\alpha_0)$, then $H(\alpha) = \{x_1 > 0\} \cap \{x \cdot \nu(\alpha) < 0\}$ is a subminimizer of $\boldsymbol{\Phi}$ for every $\alpha \in [\alpha_0, \pi/2)$, and a superminimizer for every $\alpha \in (-\pi/2, \alpha_0]$.

Lemma 2.7. *For every $\lambda \ge 1$, there exist positive constants ε_0 and C_0, depending on λ only, with the following property. If $\Phi \in \boldsymbol{\mathcal{E}}_*(\lambda)$, $\nu \in \mathbf{S}^{n-1}$, and*

$$|\nabla\Phi(\nu) \cdot e_1| \le \varepsilon_0 \,, \tag{2.43}$$

then there exists $\nu_0 \in \mathbf{S}^{n-1}$ such that

$$\mathbf{e_1}(\nu_0) = \mathbf{e_1}(\nu)\,, \qquad \nabla\Phi(\nu_0) \cdot e_1 = 0\,, \qquad |\nu_0 - \nu| \le C_0\, |\nabla\Phi(\nu) \cdot e_1|\,. \tag{2.44}$$

Proof. We begin noticing that, for every $e \in \mathbf{S}^{n-1}$

$$|e \cdot e_1| \le \sqrt{1 - \frac{1}{\lambda^4}} + |\nabla\Phi(e) \cdot e_1|\,\lambda^3\,. \tag{2.45}$$

Indeed, if $\mathbf{j}$ denotes the projection of $\mathbb{R}^n$ onto $\nabla\Phi(e)^\perp$, then by $\Phi(e) = \nabla\Phi(e)\cdot e$, (1.7) and (1.9),

$$\begin{aligned} 1 &= \frac{|e \cdot \nabla\Phi(e)|^2}{|\nabla\Phi(e)|^2} + |\mathbf{j}\, e|^2 \ge \frac{1}{\lambda^4} + |\mathbf{j}\, e \cdot \mathbf{j}\, e_1|^2 = \frac{1}{\lambda^4} + |e \cdot \mathbf{j}\, e_1|^2 \\ &\ge \frac{1}{\lambda^4} + \Big(|e \cdot e_1| - \frac{(\nabla\Phi(e) \cdot e)\,(\nabla\Phi(e) \cdot e_1)}{|\nabla\Phi(e)|^2}\Big)^2\,, \end{aligned}$$

that leads to (2.45) by $\Phi(e) \le \lambda$ and $|\nabla\Phi(e)| \ge 1/\lambda$ (this last property follows by one homogeneity and (1.7)). We now notice that $\nu_0 \in \mathbf{S}^{n-1}$ is such that $\mathbf{e_1}(\nu_0) = \mathbf{e_1}(\nu)$ if and only if $\nu_0 = \cos\alpha_0\, \mathbf{e_1}(\nu) - \sin\alpha_0\, e_1$ for some $|\alpha_0| < \pi/2$. Let us thus set

$$f(\alpha) = \nabla\Phi(\cos\alpha\, \mathbf{e_1}(\nu) - \sin\alpha e_1) \cdot e_1 \qquad |\alpha| < \pi/2\,,$$

see Figure 2.3. By the one-homogeneity of Φ and by (1.7), we obtain

$$f(\pi/2) = \nabla\Phi(-e_1) \cdot e_1 = -\Phi(-e_1) \le -\frac{1}{\lambda}\,, \qquad f(-\pi/2) = \nabla\Phi(e_1) \cdot e_1 = \Phi(e_1) \ge \frac{1}{\lambda}\,,$$

so that there exists $\alpha_0 \in (-\pi/2, \pi/2)$ such that $f(\alpha_0) = 0$; correspondingly, ν_0 satisfies the first two identities in (2.44). We now notice that, by (2.45), by $\nabla\Phi(\nu_0) \cdot e_1 = 0$ and by (2.43)

$$|\nu_0 \cdot e_1| \le \sqrt{1 - \frac{1}{\lambda^4}}\,, \qquad |\nu \cdot e_1| \le \sqrt{1 - \frac{1}{\lambda^4}} + \varepsilon_0\,\lambda^3\,.$$

Hence, for every $\lambda \ge 1$ we can find $\eta(\lambda) \in (0,1)$ and $\varepsilon_0 = \varepsilon_0(\lambda)$ such that

$$\max\{|\nu_0 \cdot e_1|, |\nu \cdot e_1|\} \le 1 - \eta.$$

Correspondingly, for some $\tau(\lambda) < \pi/2$, we find that $|\alpha_0| \le \tau$ and, if $\alpha_1 \in (-\pi/2, \pi/2)$ is such that $\nu = \cos\alpha_1\, \mathbf{e_1}(\nu) - \sin\alpha_1\, e_1$, then $|\alpha_1| \le \tau$ too. Since, by zero-homogeneity of $\nabla\Phi$,

$f(\alpha)=\nabla\Phi(\mathbf{e_1}(\nu)-\tan\alpha\,e_1)\cdot e_1$ for every $|\alpha|<\pi/2$, by (1.10) we conclude that

$$f'(\alpha)=-\frac{e_1\cdot\nabla^2\Phi(\mathbf{e_1}(\nu)-\tan\alpha\,e_1)\,e_1}{\cos^2\alpha}\le-\frac{1}{\lambda\,\cos^2\alpha\,|\mathbf{e_1}(\nu)-\tan\alpha\,e_1|}=-\frac{1}{\lambda\,\cos\alpha}\,,\qquad(2.46)$$

for every $|\alpha|<\pi/2$. In particular, there exists $\bar\alpha$ between α_0 and α_1 such that

$$|\nabla\Phi(\nu)\cdot e_1|=|f(\alpha_1)|=|f(\alpha_0)-f(\alpha_1)|\ge\frac{|\alpha_0-\alpha_1|}{\lambda\,\cos\bar\alpha}\ge\frac{|\alpha_0-\alpha_1|}{\lambda\,\cos\tau}\ge\frac{|\alpha_0-\alpha_1|}{C(\lambda)}\,.$$

Since $|\nu-\nu_0|\le 2|\alpha_0-\alpha_1|$, the above equation concludes the proof of the lemma. □

2.5. **Density estimates.** Density estimates for almost-minimizers are proved by a classical argument. The only significant difference is that when deducing lower perimeter estimates from upper volume estimates, a whole family of relative isoperimetric inequalities has to be used in place of the sole relative isoperimetric inequality on a ball, see (2.51) below.

Lemma 2.8. *For every $\lambda\ge1$ there exist constants $c_1=c_1(n,\lambda)\in(0,1)$ and $C_1=C_1(n,\lambda)$ with the following property. If A is an open set, H is an open half-space, $\Phi\in\boldsymbol{\mathcal{E}}(A\cap H,\lambda,\ell)$, and E is a (Λ,r_0)-minimizer of $\mathbf{\Phi}$ in (A,H), then*

$$P(E;B_{x,r})=P(E;B_{x,r}\cap H)+P(E;B_{x,r}\cap\partial H)\le C_1\,r^{n-1}\,,\qquad(2.47)$$

for every $x\in A\cap\mathrm{cl}(H)$ and $r<\min\big\{r_0,\mathrm{dist}(x,\partial A),1/2\lambda\Lambda\big\}$, and

$$|E\cap B_{x,r}|\ge c_1|B_{x,r}\cap H|\,,\qquad\forall x\in\mathrm{cl}(H)\cap\mathrm{spt}\mu_E\,,\qquad(2.48)$$

$$|E\cap B_{x,r}|\le(1-c_1)|B_{x,r}\cap H|\qquad\forall x\in\mathrm{cl}(H)\cap\mathrm{spt}\mu_{H\setminus E}\,,\qquad(2.49)$$

$$P(E;B_{x,r}\cap H)\ge c_1\,r^{n-1}\,,\qquad\forall x\in A\cap\mathrm{cl}(H\cap\mathrm{spt}\mu_E)\,,\qquad(2.50)$$

for every $r<\big\{r_0,\mathrm{dist}(x,\partial A),1/2\lambda\Lambda\big\}$.

Recall that, in our notation, if $E\subset\mathbb{R}^n$ is of locally finite perimeter in the open set A, then $\mathrm{spt}\mu_E$ and ∂^*E are automatically defined as subsets of A.

Proof. Step one: Without loss of generality, we shall assume that $H=\{x_1>0\}$, and set $G^+=G\cap H$ for every $G\subset\mathbb{R}^n$. For $\sigma\in(1/2,1)$, we consider the relative isoperimetric problem in the truncated ball $B^+_{t\,e_1}$ $(t\ge0)$ with volume fraction σ, and set

$$\gamma(\sigma)=\inf_{t\ge0}\inf\Big\{\frac{P(F;B^+_{t\,e_1})}{|F|^{(n-1)/n}}:|F|\le\sigma|B^+_{t\,e_1}|\Big\}\,.\qquad(2.51)$$

Then $\gamma(\sigma)>0$ for every $\sigma\in(1/2,1)$

Step two: Given $x\in A$ we set $r_x=\min\{r_0,\mathrm{dist}(x,\partial A),1/2\lambda\Lambda\}$, and define

$$m_x(r)=|E\cap B_{x,r}|=|E\cap B_{x,r}\cap H|$$

for every $r\in(0,r_x)$, so that m_x is absolutely continuous on $(0,r_x)$ (and strictly positive if we also have $x\in\mathrm{spt}\,\mu_E$), with $m_x'(r)=\mathcal{H}^{n-1}(E\cap\partial B_{x,r})$ for a.e. $r\in(0,r_x)$. If $x\in A\cap\mathrm{cl}(H)$ and $r\in(0,r_x)$, then the set $F_r=E\setminus B_{x,r}\subset H$ satisfies $E\Delta F_r\subset\subset B_{x,r_*}\subset\subset A$ for some $r_*\in(r,r_x)$. Since $r_*<r_0$, the set F_r is admissible in (2.16), which gives

$$\mathbf{\Phi}(E;B^+_{x,r_*})\le\mathbf{\Phi}(F_r;B^+_{x,r_*})+\Lambda\,m_x(r)\,.$$

We combine this last inequality and (1.7) with the remark that, by (2.7),

$$\mathbf{\Phi}(F_r;B^+_{x,r_*})=\mathbf{\Phi}(F_r;B^+_{x,r_*}\setminus B^+_{x,r})+\int_{E\cap\partial B_{x,r}}\Phi(y,\nu_{B_{x,r}}(y))\,d\mathcal{H}^{n-1}(y)\,,\quad\text{for a.e. }r>0\,,$$

in order to get

$$P(E;B^+_{x,r})\le\lambda\mathbf{\Phi}(E;B^+_{x,r})\le\lambda^2\,m_x'(r)+\lambda\Lambda m_x(r)\,,\qquad\forall x\in A\cap\mathrm{cl}(H)\,,r<r_x\,.\qquad(2.52)$$

Since $m_x'(r)\le n\omega_n r^{n-1}$, $m_x(r)\le\omega_n\,r_x\,r^{n-1}$, and $2\lambda\Lambda\,r_x\le 1/\lambda$, this proves that

$$P(E;B^+_{x,r})\le C\,r^{n-1}\,,\qquad\forall x\in A\cap{\rm cl}(H)\,,r<r_x\,,\tag{2.53}$$

where $C=C(n,\lambda)$. Now, by the divergence theorem (see [Mag12, Proposition 19.22]), we have

$$P(E\cap B_{x,r};\partial H)\le P(E\cap B_{x,r};H)\,,\qquad\forall x\in\mathbb{R}^n\,,r>0\,.\tag{2.54}$$

At the same time, by (2.5) one has

$$P(E\cap B_{x,r};H)=P(E;B^+_{x,r})+m_x'(r)\qquad\text{for a.e. }r>0,$$

while $P(B_{x,r};\partial H)=0$ gives $P(E\cap B_{x,r};\partial H)=P(E;B_{x,r}\cap\partial H)$. By combining these facts with (2.53) and (2.54) we obtain (2.47). Moreover, for every $x\in\mathbb{R}^n$ and for a.e. $r>0$,

$$\begin{aligned}P(E\cap B_{x,r})&=P(E\cap B_{x,r};H)+P(E\cap B_{x,r};\partial H)\le 2\,P(E\cap B_{x,r};H)\\&=2\Big\{P(E;B^+_{x,r})+\mathcal{H}^{n-1}(E\cap\partial B_{x,r})\Big\}\,.\end{aligned}$$

This last inequality, together with (2.52) and the isoperimetric inequality, gives

$$n\omega_n^{1/n}m_x(r)^{(n-1)/n}\le 2\,(1+\lambda^2)\,m_x'(r)+2\,\lambda\,\Lambda\,m_x(r)\,,\tag{2.55}$$

for every $x\in A\cap{\rm cl}(H)$ and for a.e. $r<r_x$. Since $2\lambda\Lambda\,r_x\le 1$, for every $r<r_x$ we get

$$2\,\lambda\,\Lambda\,m_x(r)\le 2\,\lambda\,\Lambda\,m_x(r_x)^{1/n}\,m_x(r)^{(n-1)/n}\le 2\,\lambda\,\Lambda\,\omega_n^{1/n}r_x\,m_x(r)^{(n-1)/n}\le\omega_n^{1/n}m_x(r)^{(n-1)/n}\,,$$

so that (2.55) gives, for every $x\in A\cap{\rm cl}(H)$ and for a.e. $r<r_x$,

$$(n-1)\omega_n^{1/n}\,m_x(r)^{(n-1)/n}\le 2\,(1+\lambda^2)\,m_x'(r)\,.\tag{2.56}$$

If we now assume that $x\in{\rm spt}\,\mu_E$, then $m_x(r)>0$ for every $r>0$, and thus we can divide by $m_x(r)^{(n-1)/n}$ in (2.56). By integrating the resulting differential inequality, we get

$$|E\cap B_{x,r}|\ge\omega_n\Big(\frac{n-1}{2n(1+\lambda^2)}\Big)^n r^n\ge c(n,\lambda)|B_{x,r}\cap H|\,,\qquad\forall x\in{\rm cl}(H)\cap{\rm spt}\mu_E\,,r<r_x\,.$$

We have thus found a constant $c=c(n,\lambda)\in(0,1)$ such that (2.48) holds true with c in place of c_1 (the final value of c_1 will be smaller than this). We prove (2.49) (again, with c in place of c_1) by repeating the above argument with $H\setminus E$ in place of E; see Remark 2.1. Since $H\cap{\rm spt}\mu_E=H\cap{\rm spt}\mu_{H\setminus E}$, we notice that both (2.48) and (2.49) hold true for every $x\in A\cap{\rm cl}(H\cap{\rm spt}\mu_E)$ and $r<r_x$: correspondingly, by definition of $\gamma(\sigma)$, see (2.51), we find

$$\begin{aligned}P(E;B^+_{x,r})&=P(E\cap B^+_{x,r};B^+_{x,r})\ge\gamma(1-c)|E\cap B^+_{x,r}|^{(n-1)/n}\\&\ge\gamma(1-c)\big(|B^+_{x/r,1}|\,c\big)^{(n-1)/n}\,r^{n-1}\,,\end{aligned}$$

for every $x\in A\cap{\rm cl}(H\cap{\rm spt}\mu_E)$ and every $r<r_x$. Since $|B^+_{x/r,1}|\ge|B^+_{0,1}|=\omega_n/2$, this proves (2.50) with $c_1=\gamma(1-c)(c\,\omega_n/2)^{(n-1)/n}$. □

2.6. **Compactness theorem.** The density estimates of Lemma 2.8 lead to the following compactness theorem for almost-minimizers.

Theorem 2.9. *Let $\lambda\ge 1$, A an open set in $\mathbb{R}^n$ and H an open half-space in $\mathbb{R}^n$. For every $h\in\mathbb{N}$, let $\ell_h\ge 0$, $\Lambda_h\ge 0$, $r_h\in(0,\infty]$ be such that*

$$\lim_{h\to\infty}\ell_h=\ell<\infty\,,\qquad\lim_{h\to\infty}r_h=r_0>0\,,\qquad\lim_{h\to\infty}\Lambda_h=\Lambda_0<\infty\,.$$

If, for every $h\in\mathbb{N}$, $\Phi_h\in\boldsymbol{\mathcal{E}}(A\cap H,\lambda,\ell_h)$ and E_h is a (Λ_h,r_h)-minimizer of $\boldsymbol{\Phi}_h$ in (A,H), then there exist $\Phi\in\boldsymbol{\mathcal{E}}(A\cap H,\lambda,\ell)$ and a set E of locally finite perimeter in A such that, up to extracting a subsequence,

$$E_h\to E\ \text{ in }L^1_{\rm loc}(A)\,,\tag{2.57}$$

where E is a (Λ, r_0)-minimizer of $\mathbf{\Phi}$ in (A,H), and where

$$\mu_{E_h} \overset{*}{\rightharpoonup} \mu_E\,,\quad |\mu_{E_h}|\llcorner H \overset{*}{\rightharpoonup} |\mu_E|\llcorner H\,,\quad |\mu_{E_h}| \overset{*}{\rightharpoonup} |\mu_E|\,,\quad \textit{as Radon measures in } A\,. \tag{2.58}$$

Moreover,

$$\mathrm{Tr}_{\partial H}(E_h) \to \mathrm{Tr}_{\partial H}(E) \ \textit{in}\ L^1_{\mathrm{loc}}(A\cap\partial H)\,, \tag{2.59}$$

while

$$\textit{for every } x\in \mathrm{spt}\mu_E \textit{ there exists } x_h\in\mathrm{spt}\,\mu_{E_h} \textit{ such that } \lim_{h\to\infty} x_h = x\,, \tag{2.60}$$

and

$$\begin{cases} x_h\in \mathrm{cl}(H\cap \mathrm{spt}\,\mu_{E_h})\,, & \forall h\in\mathbb{N}\,,\\ \lim_{h\to\infty} x_h = x\,, \end{cases} \quad\Rightarrow\quad x\in\mathrm{cl}(H\cap\mathrm{spt}\,\mu_E)\,. \tag{2.61}$$

Proof. Step one: We assume without loss of generality that $H=\{x_1>0\}$, and set $G^+=G\cap H$ for every $G\subset\mathbb{R}^n$. Up to extracting subsequence, we may assume that

$$\frac{r_0}{2}<r_h<2\,r_0\,,\qquad \Lambda_h<2\Lambda_0\,,\qquad \forall h\in\mathbb{N}\,.$$

In particular, if $x\in A\cap\mathrm{cl}(H)$ and $s_x=\min\big\{r_0/2,\mathrm{dist}(x,\partial A),1/4\lambda\Lambda_0\big\}$, then we can apply the upper density estimate (2.47) to each set E_h at the point x and at any scale $r<s_x$, to find that

$$P(E_h;B_{x,r})\le C_1\,r^{n-1}\,,\qquad \forall x\in A\cap\mathrm{cl}(H)\,,r<s_x\,,$$

where $C_1=C_1(n,\lambda)$. Since $E\subset H$, a simple covering argument gives

$$\sup_{h\in\mathbb{N}} P(E_h;A_0)<\infty\,,\qquad \text{for every open set } A_0\subset\subset A.$$

Hence there exists a set E of locally finite perimeter in A such that, up to extracting subsequences, $E_h\to E$ in $L^1_{\mathrm{loc}}(A)$, and

$$\mu_{E_h}\overset{*}{\rightharpoonup}\mu_E\,,\qquad \text{as Radon measures in } A\,. \tag{2.62}$$

This proves (2.57) and the first part of (2.58),

Step two: By the Ascoli-Arzelá theorem, there exists $\Phi\in\boldsymbol{\mathcal{E}}(A^+,\lambda,\ell)$ such that, up to extracting a subsequence, $\Phi_h\to\Phi$ in $C^0(\mathrm{cl}(A^+)\times\mathbf{S}^{n-1})$ with $\Phi_h(x,\cdot)\to\Phi(x,\cdot)$ in $C^2(\mathbf{S}^{n-1})$ uniformly on $x\in\mathrm{cl}(A^+)$. By exploiting the uniform convergence of Φ_h to Φ on $\mathrm{cl}(A^+)\times\mathbf{S}^{n-1}$, together with the lower bound in (1.7), we see that for every $\varepsilon>0$ there exists h_ε such that if $h\ge h_\varepsilon$, then

$$(1+\varepsilon)\Phi(x,\nu)\ge\Phi_h(x,\nu)\ge(1-\varepsilon)\Phi(x,\nu)\,,\qquad \forall(x,\nu)\in\mathrm{cl}(A^+)\times\mathbf{S}^{n-1}\,. \tag{2.63}$$

By (2.62) and by Reshetnyak lower semicontinuity theorem [AFP00, Theorem 2.38], we thus find that, for every open set $U\subset A$,

$$\liminf_{h\to\infty}\mathbf{\Phi}_h(E_h;U)\ge\mathbf{\Phi}(E;U)\,. \tag{2.64}$$

Step three: In order to prove that E is a (Λ,r_0)-minimizer of $\mathbf{\Phi}$ in (A,H), we need to show that

$$\mathbf{\Phi}(E;W^+)\le\mathbf{\Phi}(F;W^+)+\Lambda\,|E\Delta F|\,, \tag{2.65}$$

whenever W is open and $F\subset H$ is such that

$$E\Delta F\subset\subset W\subset\subset A\,,\qquad \mathrm{diam}(W)<2r_0\,. \tag{2.66}$$

Indeed, let F and W satisfy (2.66). Clearly we can find an open set W' with

$$E\Delta F\subset\subset W\subset\subset W'\,,\qquad \mathcal{H}^{n-1}(\partial W'\cap(\partial^*E\cup\partial^*F))=0\,, \tag{2.67}$$

$$\lim_{h\to\infty}\mathcal{H}^{n-1}\Big((E\Delta E_h)\cap\partial W'\Big)=0\,, \tag{2.68}$$

and $\mathrm{diam}(W') < 2r_0$. Hence, there exists $h_* \in \mathbb{N}$ such that $\mathrm{diam}(W') < 2\,r_h$ for every $h \ge h_*$; in particular, we can find an open set $W'' \subset\subset A$, such that, if we set

$$F_h = (F \cap W') \cup (E_h \setminus W')\,, \tag{2.69}$$

then $F_h \subset H$, $E_h \Delta F_h \subset\subset W'' \subset\subset A$, and $\mathrm{diam}(W'') < 2r_h$ for every $h \ge h_*$. We can exploit the fact that E_h is a (Λ_h, r_h)-minimizer of $\mathbf{\Phi}$ in (A,H) to find

$$\mathbf{\Phi}_h(E_h;(W'')^+) \le \mathbf{\Phi}_h(F_h;(W'')^+) + \Lambda_h\,|E_h\Delta F_h|\,,$$

for every $h \ge h_*$. By taking into account (2.67), (2.5), (2.6) and (2.7) we find that

$$\mathbf{\Phi}_h(E_h;(W')^+) \le \mathbf{\Phi}_h(F;(W')^+) + \varepsilon_h + \Lambda_h\,\Big|(E_h\Delta F)\cap W'\Big|\,, \tag{2.70}$$

for every $h \ge h_*$, where we have applied (1.7) and set

$$\varepsilon_h = \lambda\,\mathcal{H}^{n-1}\Big((E\Delta E_h)\cap\partial W'\Big)\,,\qquad h\in\mathbb{N}\,.$$

By letting $h\to\infty$ in (2.70), by (2.68), and since $E_h \to E$ in $L^1_{\mathrm{loc}}(A)$, we conclude

$$\limsup_{h\to\infty}\mathbf{\Phi}_h(E_h;(W')^+) \le \mathbf{\Phi}(F;(W')^+) + \Lambda\,|E\Delta F|\,. \tag{2.71}$$

Since $(W')^+ \subset\subset A$, by (2.64) we get $\mathbf{\Phi}(E;(W')^+) \le \mathbf{\Phi}(F;(W')^+) + \Lambda\,|E\Delta F|$, that is exactly (2.65).

Step four: If we choose $F = E$ in the argument of step three, then the combination of (2.64) and (2.71) gives

$$\lim_{h\to\infty}\mathbf{\Phi}_h(E_h;B^+_{x,r}) = \mathbf{\Phi}(E;B^+_{x,r})\,,$$

for every $x \in A\cap\mathrm{cl}(H)$ and for a.e. $r < r_x = \min\{r_0, \mathrm{dist}(x,\partial A)\}$. By taking (2.63) into account, we thus find that

$$\lim_{h\to\infty}\mathbf{\Phi}(E_h;B^+_{x,r}) = \mathbf{\Phi}(E;B^+_{x,r})\,,$$

for every $x \in A\cap\mathrm{cl}(H)$ and for a.e. $r < r_x$. By (2.62) and by the strict convexity of $\Phi(x,\cdot)$ (in the sense of (1.10)), we can apply a classical result of Reshetnyak, see, e.g. [GMS98, Theorem 1, section 3.4], to find that

$$\lim_{h\to\infty}P(E_h;B^+_{x,r}) = P(E;B^+_{x,r})\,, \tag{2.72}$$

for every $x \in A\cap\mathrm{cl}(H)$ and for a.e. $r < r_x$. By (2.72) , (2.15) and a covering argument we deduce the validity of (2.59). Let now μ be a weak*-cluster point of the family of measures $|\mu_{E_h}|\llcorner H$. By (2.72) and by the Lebesgue-Besicovitch differentiation theorem, we find $\mu = |\mu_E|\llcorner H$, hence

$$|\mu_{E_h}|\llcorner H \overset{*}{\rightharpoonup} |\mu_E|\llcorner H\,,\qquad \text{as Radon measures in } A\,, \tag{2.73}$$

which proves the second statement in (2.58). We finally complete the proof of (2.58): given a compact set $K\subset A$, by $E\subset H$ we have $|\mu_E|(K) = |\mu_E|\llcorner H(K) + \mathcal{H}^{n-1}(K\cap\partial H\cap\partial^*E)$, where

$$|\mu_E|\llcorner H(K) \ge \limsup_{h\to\infty}|\mu_{E_h}|\llcorner H(K)\,,\quad \mathcal{H}^{n-1}(K\cap\partial H\cap\partial^*E) = \lim_{h\to\infty}\mathcal{H}^{n-1}(K\cap\partial H\cap\partial^*E_h)\,,$$

by (2.73) and by (2.59) respectively. This shows that $|\mu_E|(K) \ge \limsup_{h\to\infty}|\mu_{E_h}|(K)$ for every compact set $K\subset A$. This last fact, combined with (2.62), implies the last statement in (2.58), see for instance [Mag12, Proposition 4.26].

Step four: We finally prove (2.60) and (2.61). The validity of (2.60) is a standard consequence of (2.62): indeed, if (2.60) fails, then there exists $\varepsilon > 0$ such that, up to extracting subsequences, $B(x,\varepsilon)\cap\mathrm{spt}\mu_{E_h} = \emptyset$ for every $h\in\mathbb{N}$; but then, by (2.62) we get

$$|\mu_E|(B(x,\varepsilon)) \le \liminf_{h\to\infty}|\mu_{E_h}|(B(x,\varepsilon)) = 0,$$

against $x \in \mathrm{spt}\mu_E$. The validity of (2.61) follows, again via a standard argument, by the lower density estimate (2.50): indeed, if $x_h \in K \cap \mathrm{cl}(H \cap \mathrm{spt}\mu_{E_h})$, then by (2.50) we can find $r > 0$ such that $|\mu_{E_h}|(B_{x_h,s}) \ge c_1\, s^{n-1}$ for every $s < r$. In particular, since $B(x_h, r/2) \subset B(x,r)$ for h large enough, by (2.58) one gets

$$|\mu_E|(\mathrm{cl}(B_{x,r})) \ge \limsup_{h\to\infty} |\mu_{E_h}|(\mathrm{cl}(B_{x,r})) \ge c_1\, s^{n-1}\,, \qquad \forall s < \frac{r}{2}\,,$$

so that, necessarily, $x \in \mathrm{spt}\mu_E$, and (2.61) is proved. □

2.7. **Contact sets of almost-minimizers.** In this section we establish some "weak" regularity properties of the contact set $\mathrm{Tr}_{\partial H}(E)$ of a (Λ, r_0)-minimizer E. In Lemma 2.10 we show that $\mathrm{Tr}_{\partial H}(E)$ is of locally finite perimeter in $A\cap\partial H$. In Lemma 2.15 we prove lower density estimates for $\mathrm{Tr}_{\partial H}(E)$ in ∂H by means of a strong maximum principle discussed in Lemma 2.13. Finally, in Lemma 2.16, we set some normalization conventions on almost-minimizers to be used in the rest of the paper.

The following notation shall be used thorough this section. We decompose $\mathbb{R}^n$ as $\mathbb{R}\times\mathbb{R}^{n-1}$, denote by $\mathbf{h} : \mathbb{R}^n \to \mathbb{R}^{n-1}$ the corresponding projection of $\mathbb{R}^n$ onto $\mathbb{R}^{n-1}$, so that $x = (x_1, \mathbf{h}\,x)$ for every $x \in \mathbb{R}^n$, and define the *vertical* disk and the *vertical* cylinder centered at 0 as

$$\mathbf{D}^{\mathrm{v}}_r = \Big\{z \in \partial H : |z| < r\Big\}\,, \qquad \mathbf{C}^{\mathrm{v}}_r = \Big\{x \in \mathbb{R}^n : x_1 \in (0,r)\,, |\mathbf{h}\,x| < r\Big\} = (0,r)\times\mathbf{D}^{\mathrm{v}}_r\,. \tag{2.74}$$

With the usual abuse of notation (see section 2.1), we denote by $\partial\mathbf{D}^{\mathrm{v}}_r$ the boundary of $\mathbf{D}^{\mathrm{v}}_r$ inside ∂H, i.e. we set $\partial\mathbf{D}^{\mathrm{v}}_r = \big\{z \in \partial H : |z| = r\big\}$.

Lemma 2.10 (Contact sets are of locally finite perimeter)**.** *For every $\lambda \ge 1$ there exists a constant $C = C(n,\lambda)$ with the following property. If A is an open set, $H = \{x_1 > 0\}$, $\Phi \in \boldsymbol{\mathcal{E}}(A\cap H, \lambda, \ell)$, and E is a (Λ, r_0)-minimizer of $\mathbf{\Phi}$ in (A,H), then*

$$P(\mathrm{Tr}_{\partial H}(E); B_{x,r}\cap\partial H) \le C\, r^{n-2}\,,$$

for every $x \in \partial H$ and $r < \min\big\{r_0, \mathrm{dist}(x,\partial A), 1/2\lambda\Lambda, 1/\ell\big\}/4$. In particular $\mathrm{Tr}_{\partial H}(E)$ is a set of locally finite perimeter in $A\cap\partial H$.

Proof. Up to replace E and Φ with $E^{x,r}$ and $\Phi^{x,r}$ respectively, we can directly assume that $\Phi \in \boldsymbol{\mathcal{E}}(B_4\cap H, \lambda, \ell)$, $\ell \le 1$, E is a $(1/8\lambda, 4)$-minimizer of $\mathbf{\Phi}$ in (B_4, H), and prove that

$$P(\mathrm{Tr}_{\partial H}(E); \mathbf{D}^{\mathrm{v}}_1) \le C\,, \tag{2.75}$$

for a constant $C = C(n,\lambda)$. Given $s \in (0,1/2)$, let $\varphi_s \in C^1_c((0,2);[0,s])$ be such that $\varphi_s = s$ on $(0,1)$ and $|\varphi_s'| \le 3\,s$ on $(0,2)$, set

$$G_s = \Big\{x \in \mathbf{C}^{\mathrm{v}}_2 : x_1 \le \varphi_s(|\mathbf{h}\,x|)\Big\}\,,$$

and consider the bi-Lipschitz map $f_s : H\setminus G_s \to H$ defined as

$$f_s(x) = \left(1 - \Big(\frac{1-x_1}{1-\varphi_s(|\mathbf{h}\,x|)}\Big), \mathbf{h}\,x\right)\,, \qquad x \in [\mathbf{D}^{\mathrm{v}}_2\times(0,1)]\setminus G_s\,,$$
$$f_s(x) = x\,, \qquad x \in H\setminus[\mathbf{D}^{\mathrm{v}}_2\times(0,1)]\,,$$

see Figure 2.4. Notice that $f_s(\mathbf{C}^{\mathrm{v}}_2\setminus G_s) = \mathbf{C}^{\mathrm{v}}_2$, with

$$\sup_{x\in H\setminus G_s} |f_s(x) - x| + \|\nabla f_s(x) - \mathrm{Id}\,\| \le C\,s\,, \tag{2.76}$$

for a constant $C = C(n)$. If we set $E_s = f_s(E\setminus G_s)$, then E_s is a set of locally finite perimeter in B_4 (as $E\setminus G_s$ is), with $E_s \subset H$ and $E_s\Delta E \subset \mathbf{C}^{\mathrm{v}}_2$ with $\mathrm{diam}(\mathbf{C}^{\mathrm{v}}_2) = \sqrt{20} < 8$. We may thus exploit the fact that E is a $(1/8\lambda, 4)$-minimizer of $\mathbf{\Phi}$ in (B_4, H) to deduce that

$$\mathbf{\Phi}(E;\mathbf{C}^{\mathrm{v}}_2) \le \mathbf{\Phi}(f_s(E\setminus G_s);\mathbf{C}^{\mathrm{v}}_2) + \Lambda|(E\Delta f_s(E))\cap\mathbf{C}^{\mathrm{v}}_2|. \tag{2.77}$$

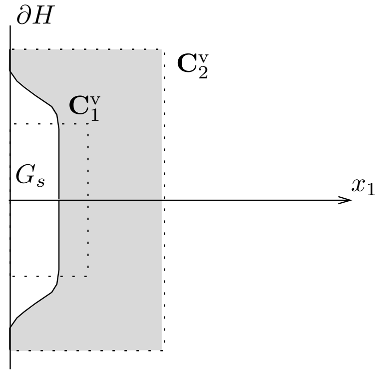

FIGURE 2.4. Given $z\in\mathbf{D}^{\rm v}_2$, the map f_s stretches each segment $[\varphi_s(|z|),1]\times\{z\}$ into $[0,1]\times\{z\}$, while keeping the point $(1,z)$ fixed.

By [Mag12, Lemma 17.9] and (2.76), we have

$$|(E\Delta f_s(E))\cap\mathbf{C}^{\rm v}_2|\le C\,P(E;B_3)\,s\,,\tag{2.78}$$

for some $C=C(n)$. Moreover, by (2.9) and $f_s(\mathbf{C}^{\rm v}_2)=\mathbf{C}^{\rm v}_2$ we find that

$$\begin{aligned}\mathbf{\Phi}(f_s(E\setminus G_s);H)&=\int_{\mathbf{C}^{\rm v}_2\cap\partial^*(E\setminus G_s)}\Phi\Big(f_s(x),\mathrm{cof}\,(\nabla f_s(x))\nu_{E\setminus G_s}(x)\Big)\,d\mathcal{H}^{n-1}(x)\\&\le\mathbf{\Phi}(E;\mathbf{C}^{\rm v}_2\setminus G_s)+C(n,\lambda)\,s\,P(E;\mathbf{C}^{\rm v}_2\setminus G_s)\,,\end{aligned}\tag{2.79}$$

where we have used (2.76), (1.8), (1.9) and the fact that $\ell\le 1$ (so that C depend on n and λ only). By (2.77), (2.78), (2.79) and (2.47) we thus find

$$\mathbf{\Phi}(E;G_s)\le CP(E,B_3)s\le C\,s$$

with $C=C(n,\lambda)$. Since $(0,s)\times\mathbf{D}^{\rm v}_1\subset G_s$, we conclude by (1.7) that

$$P(E;(0,s)\times\mathbf{D}^{\rm v}_1)\le C(n,\lambda)\,s\qquad\forall\,s\in(0,1/2)\,.$$

By the coarea formula for rectifiable set, see, e.g. [Mag12, Equation (18.25)], we find

$$\int_0^s P(E_t;\mathbf{D}^{\rm v}_1)\,dt\le P(E;(0,s)\times\mathbf{D}^{\rm v}_1)\le C(n,\lambda)\,s\,.$$

Hence, for every $s\in(0,1/2)$ we can find $t_s\in(0,s)$ such that $P(E_{t_s};\mathbf{D}^{\rm v}_1)\le C(n,\lambda)$. We deduce (2.75) by taking the limit as $s\to0^+$, thanks to (2.11) and the lower semicontinuity of perimeter:

$$P(\mathrm{Tr}_{\partial H}(E);\mathbf{D}^{\rm v}_1)\le\liminf_{s\to0^+}P(E_{t_s};\mathbf{D}^{\rm v}_1)\le C(n,\lambda)\,.\qquad\Box$$

We now prove a lower density estimates for the contact set. To this end, we shall need a strong maximum principle for local minimizers of regular autonomous elliptic integrands. (This should be compared with [SW89], where, however, even integrands are considered in order to deal with non-orientable surfaces.) We prove the strong maximum principle in Lemma 2.13, as a corollary of a comparison lemma, Lemma 2.12, illustrated in Figure 2.5. We premise to these results the following lemma, about the existence of Lipschitz solution of non-parametric problems. (Part two of the statement will be used in section 5.)

Lemma 2.11. *Let $\Phi\in\boldsymbol{\mathcal{E}}_*(\lambda)$, $\lambda>0$, and set $\Phi^\#(\xi)=\Phi(\xi,-1)$ for $\xi\in\mathbb{R}^{n-1}$.*

Part one: *If $\varphi\in C^{1,1}(\mathrm{cl}(\mathbf{D}_r))$, then there exists a unique $u\in C^2(\mathbf{D}_r)\cap\mathrm{Lip}(\mathrm{cl}(\mathbf{D}_r))$ such that*

$$\begin{cases}\mathrm{div}\,(\nabla_\xi\Phi^\#(\nabla u))=0\,, & in\ \mathbf{D}_r\,,\\ u=\varphi\,, & on\ \partial\mathbf{D}_r\,.\end{cases}\tag{2.80}$$

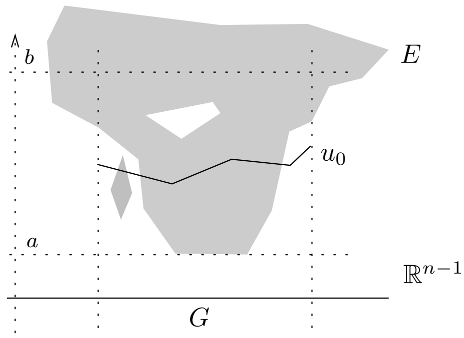

FIGURE 2.5. The situation in Lemma 2.12: the role of (2.83) is to ensure that, over the boundary of G, E lies above the graph of u_0. The figure tries to stress the fact that E does not need to be the epigraph of a function.

In addition, if $\varphi \ge 0$, $\varphi \not\equiv 0$, then $u(0) > 0$.

Part two: *Given $e \in \mathbb{R}^{n-1}$ with $|e| = 1$, let $H = \{z \in \mathbb{R}^{n-1} : z \cdot e \ge 0\}$. If $\varphi \in C^{1,1}(\mathrm{cl}(\mathbf{D}_r \cap H))$ with $\varphi = 0$ on $\mathbf{D}_r \cap \partial H$, then there exists a unique $u \in C^2(\mathbf{D}_r \cap \mathrm{cl}(H)) \cap \mathrm{Lip}(\mathrm{cl}(\mathbf{D}_r \cap H))$ with*

$$\begin{cases} \mathrm{div}\,(\nabla_\xi \Phi^\#(\nabla u)) = 0\,, & \text{in } \mathbf{D}_r \cap H\,,\\ u = \varphi\,, & \text{on } \partial(\mathbf{D}_r \cap H)\,. \end{cases} \tag{2.81}$$

In addition, if $\varphi \le M\,(e \cdot z)$ on $\partial(\mathbf{D}_r \cap H)$ for some $M \in \mathbb{R}$, then $|\nabla\varphi(0)| = |\partial_e \varphi(0)| < M$.

Proof. The pair $(\varphi, \mathbf{D}_r)$ satisfies the so called Bounded Slope Condition, see [Giu03, Theorem 1.1], hence existence of a solution to (2.80) follows by [Giu03, Theorem 1.2]. Uniqueness follows by noticing that for every two Lipschitz solutions to (2.80) the difference solves a uniformly linear elliptic equation, see for instance [GT98, Chapter 10]. The last statement in part one is just the strict maximum principle, [GT98, Theorem 3.5]. To prove the existence and uniqueness of solutions to (2.81), one can make an odd reflection around ∂H and then use part one. For what concerns the last statement of part two, note that $w = M z \cdot e - u(z) \ge 0$ on $\partial(\mathbf{D}_r \cap H)$, $w(0) = 0$ and that w solves the uniformly elliptic (linear) equation,

$$0 = \mathrm{div}\,(\nabla_\xi \Phi^\#(\nabla u)) = \sum_{i,j}^{n-1} \nabla^2_{\xi_i \xi_j} \Phi^\#(\nabla u) \partial_{ij} u = \sum_{i,j}^{n-1} A_{ij}\, \partial_{ij} w\,.$$

We conclude by Hopf's boundary lemma, [GT98, Lemma 3.4]. □

Lemma 2.12 (Comparison lemma). *If $\lambda \ge 1$, $\Phi \in \boldsymbol{\mathcal{E}}_*(\lambda)$, $G \subset \mathbb{R}^{n-1}$ is a bounded open set with Lipschitz boundary, $a < b$, $J = G \times (a, b)$, $E \subset \{x_n > a\}$ is a set of finite perimeter in an open neighborhood of J such that*

$$\mathbf{\Phi}(E; \mathrm{cl}(J)) \le \mathbf{\Phi}(F; \mathrm{cl}(J))\,, \qquad \text{whenever } F \subset E,\ E \setminus F \subset J\,, \tag{2.82}$$

and $u_0 \in C^2(G) \cap \mathrm{Lip}(\mathrm{cl}(G))$ with $a < u_0 < b$ on $\mathrm{cl}(G)$ and

$$E^{(1)} \cap [(\partial G) \times (a, b)] \subset \Big\{(z, t) \in (\partial G) \times (a, b) : t \ge u_0(z)\Big\}\,, \tag{2.83}$$

$$\mathrm{div}\,(\nabla_\xi \Phi^\#(\nabla u_0(z))) = 0\,, \qquad \forall z \in G\,, \tag{2.84}$$

where $\Phi^\#(\xi) = \Phi(\xi, -1)$ for $\xi \in \mathbb{R}^{n-1}$, then

$$E \cap J \subset_{\mathcal{H}^n} \Big\{(z, t) \in J : t \ge u_0(z)\Big\}\,. \tag{2.85}$$

Proof. Let us begin noticing that the lower bound in (2.27) can be written in the form

$$\kappa_1\,\frac{|\nu_1-\nu_2|^2}{2}\le\Big(\nabla\Phi(\nu_2)-\nabla\Phi(\nu_1)\Big)\cdot\nu_2\,,\qquad\forall\nu_1,\nu_2\in\mathbf{S}^{n-1}\,,\tag{2.86}$$

We now consider the open sets

$$F_+=\Big\{(z,t)\in J:t>u_0(z)\Big\}\,,\qquad F_-=\Big\{(z,t)\in J:t<u_0(z)\Big\}\,.$$

By (2.83) and since $a<u_0<b$ on $\mathrm{cl}(G)$, we can exploit (2.82) with $F=(E\setminus J)\cup(E\cap F_+)$ to find that

$$\int_{\partial^*E\cap[F_-\cup(G\times\{a\})]}\Phi(\nu_E)\,d\mathcal{H}^{n-1}\le\int_{\partial^*F_+\cap(E^{(1)}\cap J)}\Phi(\nu_{F_+})\,d\mathcal{H}^{n-1}\,.\tag{2.87}$$

Let us now set $T(x)=\nabla\Phi(\nabla u_0(\mathbf{p}x),-1)$ for $x\in G\times\mathbb{R}$, so that $T\in C^1(G\times\mathbb{R};\mathbb{R}^n)$ with $\mathrm{div}\,T(x)=0$ for every $x\in G\times\mathbb{R}$ thanks to (2.84). If we set $\widetilde{E}=(E\cap J)\setminus F_+$, then $\widetilde{E}\subset J$ and $\partial^*\widetilde{E}\cap[(\partial G)\times(a,b)]$ is $\mathcal{H}^{n-1}$-negligible thanks to (2.83). Hence we can apply the divergence theorem to T on $\widetilde{E}$ to find that

$$\int_{\partial^*F_+\cap(E^{(1)}\cap J)}T\cdot\nu_{F_+}\,d\mathcal{H}^{n-1}=\int_{\partial^*E\cap(F_-\cup(G\times\{a\}))}T\cdot\nu_E\,d\mathcal{H}^{n-1}\,.\tag{2.88}$$

Now, by zero-homogeneity of $\nabla\Phi$ and by

$$\nu_{F_+}(z,u_0(z))=\frac{(\nabla u_0(\mathbf{p}x),-1)}{\sqrt{1+|\nabla u_0(\mathbf{p}x)|^2}}\,,\qquad\forall x\in J\cap\partial F_+\,,\tag{2.89}$$

we find that, for $\mathcal{H}^{n-1}$-a.e. on $J\cap\partial F_+$,

$$T\cdot\nu_{F_+}=\nabla\Phi(\nu_{F_+})\cdot\nu_{F_+}=\Phi(\nu_{F_+})\,,\qquad\text{on }J\cap\partial F_+\,,\tag{2.90}$$

while, by (2.86),

$$T\cdot\nu_E=\Phi(\nu_E)-\Big(\nabla\Phi(\nu_E)-\nabla\Phi(\nabla u_0,-1)\Big)\cdot\nu_E\le\Phi(\nu_E)-\frac{\kappa_1}{2}\left|\nu_E-\frac{(\nabla u_0,-1)}{\sqrt{1+|\nabla u_0|^2}}\right|^2.\tag{2.91}$$

We may thus combine (2.87), (2.88), (2.90) and (2.91) to deduce

$$\int_{\partial^*E\cap(F_-\cup(G\times\{a\}))}\left|\nu_E-\frac{(\nabla u_0(\mathbf{p}(x)),-1)}{\sqrt{1+|\nabla u_0(\mathbf{p}x)|^2}}\right|^2d\,\mathcal{H}^{n-1}(x)=0,$$

so that

$$\nu_E=\frac{(\nabla u_0(\mathbf{p}(x)),-1)}{\sqrt{1+|\nabla u_0(\mathbf{p}x)|^2}}\qquad\mathcal{H}^{n-1}\text{-a.e. on }\partial^*E\cap(F_-\cup(G\times\{a\})).\tag{2.92}$$

Folllowing [DS94], we apply the divergence theorem on the set $\widetilde{E}$ to the vector field $S\in C^1(G\times\mathbb{R};\mathbb{R}^n)$ defined by $S(x)=\big(\mathbf{p}x,\mathbf{p}x\cdot\nabla u_0(\mathbf{p}x)\big)$ for every $x\in G\times\mathbb{R}$. Since $S\cdot\nu_{\widetilde{E}}=0$ $\mathcal{H}^{n-1}-a.e.$ on $\partial^*\widetilde{E}$ thanks to (2.5), (2.7), (2.83), (2.89) and (2.92), we conclude that

$$(n-1)|\widetilde{E}|=\int_{\widetilde{E}}\mathrm{div}\,S=\int_{\partial^*\widetilde{E}}S\cdot\nu_{\widetilde{E}}\,d\mathcal{H}^{n-1}=0\,,$$

that is, $|(E\cap J)\setminus F_+|=0$. This proves the lemma. □

Lemma 2.13 (Strong maximum principle)**.** *If $\lambda\ge1$, $H=\{x_1>0\}$, $\Phi\in\boldsymbol{\mathcal{E}}_*(\lambda)$, and $E\subset H$ is a set of locally finite perimeter in B such that*

$$\mathbf{\Phi}(E;W)\le\mathbf{\Phi}(F;W)\,,\qquad\textit{whenever }E\Delta F\subset\subset W\subset\subset B\,,\tag{2.93}$$

then either $\mathcal{H}^{n-1}(B\cap\mathrm{Tr}_{\partial H}(E))=\mathcal{H}^{n-1}(B\cap\partial H)$ or $0\notin\mathrm{spt}\mu_E$.

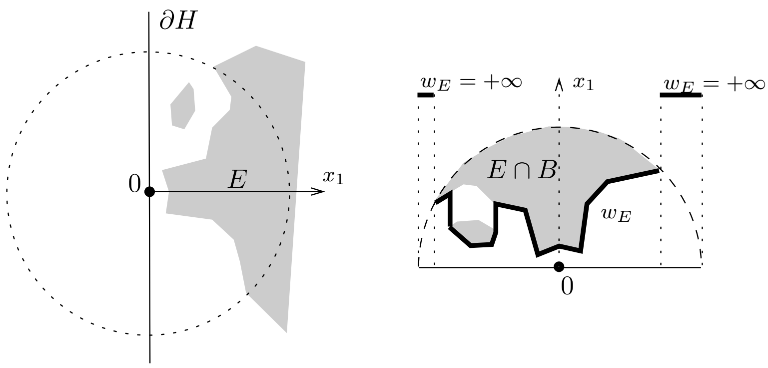

FIGURE 2.6. Inclusion (2.96). Notice that w_E may take the value $+\infty$.

Remark 2.14. We note that (2.93) is localized in W, not in $W \cap H$. Thus Lemma 2.13 says that the only minimizer E of $\mathbf{\Phi}$ in B which is contained in H and whose boundary touches ∂H at 0 is H itself (i.e., one must have $E \cap B = H \cap B$).

Proof of Lemma 2.13. Let us assume that

$$\mathcal{H}^{n-1}(B \cap \mathrm{Tr}_{\partial H}(E)) < \mathcal{H}^{n-1}(B \cap \partial H)\,. \tag{2.94}$$

Since (2.93) means that E is a minimizer of $\mathbf{\Phi}$ in $(B, \mathbb{R}^n)$, by Lemma 2.8 we find

$$c_1\,|B_{x,r}| < |E \cap B_{x,r}| < (1 - c_1)\,|B_{x,r}|$$

for every $x \in B \cap \mathrm{spt}\mu_E$ and $r < \mathrm{dist}(x, \partial B)$. In particular, $B \cap \mathrm{spt}\mu_E \subset B \cap \partial^{\mathrm{e}} E$, so that, by (2.2), $B \cap \mathrm{spt}\mu_E \subset_{\mathcal{H}^{n-1}} B \cap \partial^* E$. Thus, by (2.13) and (2.94) we find that

$$\mathcal{H}^{n-1}(B \cap \mathrm{spt}\mu_E) < \mathcal{H}^{n-1}(B \cap \partial H)\,. \tag{2.95}$$

We now define a function $w_E : \mathbf{D}^{\mathrm{v}}_1 \to [0, \infty]$ by setting

$$w_E(z) = \inf\Big\{t \in \mathbb{R} : (t, z) \in B \cap \mathrm{spt}\mu_E\Big\}\,, \qquad z \in \mathbf{D}^{\mathrm{v}}_1\,.$$

Since $\mathrm{spt}\mu_E$ is a closed subset of H, it turns out that w_E is non-negative and lower semicontinuous on $\mathbf{D}^{\mathrm{v}}_1$, with the property that

$$E \cap B \subset_{\mathcal{H}^n} \Big\{x \in \mathbb{R}^n : x_1 \geq w_E(\mathbf{h}\,x)\Big\}\,, \tag{2.96}$$

see Figure 2.6. By the coarea formula, (2.95) and (2.96), there exists $r_* \in (0, 1/2\sqrt{2})$ such that

$$\mathcal{H}^{n-2}(\mathrm{spt}\mu_E \cap \partial\mathbf{D}^{\mathrm{v}}_{r_*}) < \mathcal{H}^{n-2}(B \cap \partial\mathbf{D}^{\mathrm{v}}_{r_*})\,, \qquad \mathcal{H}^{n-1}(\partial^* E \cap \partial\mathbf{C}^{\mathrm{v}}_{r_*}) = 0\,, \tag{2.97}$$

and

$$x_1 \geq w_E(\mathbf{h}\,x)\,, \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in E^{(1)} \cap B \cap [\mathbb{R} \times \partial\mathbf{D}^{\mathrm{v}}_{r_*}]\,. \tag{2.98}$$

By definition of w_E and by (2.97), $w_E(z) > 0$ on a subset of $\partial\mathbf{D}^{\mathrm{v}}_{r_*}$ with positive $\mathcal{H}^{n-2}$-measure. Therefore, there exists $\varphi \in C^\infty(\partial\mathbf{D}^{\mathrm{v}}_{r_*})$ such that

$$\max_{\partial\mathbf{D}^{\mathrm{v}}_{r_*}} \varphi > 0\,, \qquad 0 \leq \varphi(z) \leq \min\Big\{w_E(z), \frac{r_*}{2}\Big\}\,, \qquad \forall\, z \in \partial\mathbf{D}^{\mathrm{v}}_{r_*}\,. \tag{2.99}$$

By Lemma 2.11, part one, there exists $u \in C^2(\mathbf{D}^{\mathrm{v}}_{r_*}) \cap \mathrm{Lip}(\mathrm{cl}(\mathbf{D}^{\mathrm{v}}_{r_*}))$ such that

$$\begin{cases} \mathrm{div}\,(\nabla_\xi \Phi^\#(\nabla u)) = 0\,, & \text{in } \mathbf{D}^{\mathrm{v}}_{r_*}\,, \\ u = \varphi\,, & \text{on } \partial\mathbf{D}^{\mathrm{v}}_{r_*}\,, \end{cases} \tag{2.100}$$

where $\Phi^\#(\xi) = \Phi(-1,\xi)$ for $\xi \in \mathbb{R}^{n-1}$; moreover, $u(0) > 0$ by (2.99). By (2.93), (2.98), (2.99), and (2.100), we can apply Lemma 2.12 to infer that

$$E \cap \mathbf{C}^{\rm v}_{r_*} \subset_{\mathcal{H}^n} \Big\{(z,t) \in \mathbf{C}^{\rm v}_{r_*} : t \geq u(z)\Big\}.$$

Since $u(0) > 0$, this last inclusion implies that $0 \notin {\rm spt}\mu_E$, and the lemma is proved. □

Lemma 2.15 (Lower density estimate for contact sets)**.** *For every $\lambda \geq 1$ there exist two positive constants $\varepsilon = \varepsilon(n,\lambda)$ and $c = c(n,\lambda)$ with the following property. If A is an open set, $H = \{x_1 > 0\}$, $\Phi \in \boldsymbol{\mathcal{E}}(A \cap H, \lambda, \ell)$, E is a (Λ, r_0)-minimizer of $\mathbf{\Phi}$ in (A,H), and $(\Lambda + \ell)r \leq \varepsilon$, then, for every $x \in \partial H \cap {\rm spt}\mu_E$ and $r < \min\{r_0, {\rm dist}(x,\partial A)\}/4$, we have*

$$\mathcal{H}^{n-1}(B_{x,r} \cap {\rm Tr}_{\partial H}(E)) \geq c\, r^{n-1}. \tag{2.101}$$

In particular, $\mathcal{H}^{n-1}({\rm spt}\mu_E \setminus \partial^* E) = 0$.

Proof. We start showing as (2.101) implies the last part of the statement. Indeed, (2.13) and (2.101) imply

$$\liminf_{r\to 0^+} \frac{\mathcal{H}^{n-1}(B_{x,r} \cap \partial^* E)}{r^{n-1}} > 0, \tag{2.102}$$

for every $x \in A \cap \partial H \cap {\rm spt}\mu_E$. Since (2.102) holds true at every $x \in A \cap {\rm cl}(H \cap {\rm spt}\mu_E)$ by (2.50), we conclude that (2.102) holds true at every $x \in {\rm spt}\mu_E$. By differentiation of Hausdorff measures (see, e.g., [Mag12, Corollary 6.5]), we conclude that $\mathcal{H}^{n-1}({\rm spt}\mu_E \setminus \partial^* E) = 0$.

We now prove (2.101). Up to consider $E^{x,r}$ and $\Phi^{x,r}$ in place of E and Φ we may reduce to prove the following statement: if $\Phi \in \boldsymbol{\mathcal{E}}(B_2 \cap H, \lambda, \ell)$, and E is a $(\Lambda, 4)$ minimizer of $\mathbf{\Phi}$ in (B_2, H) with $0 \in {\rm spt}\mu_E$ and $(\Lambda + \ell) \leq \varepsilon$, then

$$\mathcal{H}^{n-1}(B \cap {\rm Tr}_{\partial H}(E)) \geq c.$$

We argue by contradiction, and assume that for every $h \in \mathbb{N}$ there exist $\Phi_h \in \boldsymbol{\mathcal{E}}(B_2 \cap H, \lambda, \ell_h r_h)$ and E_h a $(\Lambda_h r_h, 4)$ minimizer of $\mathbf{\Phi}_h$ in (B_2, H), satisfying

$$0 \in \bigcap_{h\in\mathbb{N}} {\rm spt}\mu_{E_h}, \qquad \lim_{h\to\infty} (\Lambda_h + \ell_h) r_h = 0, \qquad \lim_{h\to\infty} \mathcal{H}^{n-1}(B \cap {\rm Tr}_{\partial H}(E_h)) = 0.$$

By Theorem 2.9, there exist $\Phi_\infty \in \boldsymbol{\mathcal{E}}_*(\lambda)$ and a $(0,4)$-minimizer E_∞ of $\mathbf{\Phi}_\infty$ in (B_2, H) such that, up to extracting a not relabeled subsequence, $E_h \to E_\infty$ in $L^1_{\rm loc}(B_2)$, and

$$\mathcal{H}^{n-1}(B \cap {\rm Tr}_{\partial H}(E_\infty)) = 0. \tag{2.103}$$

In fact, E_∞ is a minimizer of $\mathbf{\Phi}_\infty$ in $(B, \mathbb{R}^n)$, i.e.

$$\mathbf{\Phi}_\infty(E_\infty; B) \leq \mathbf{\Phi}_\infty(F; B), \tag{2.104}$$

whenever $E_\infty \Delta F \subset\subset B$ (note that this is a a stronger property than being a $(0,4)$-minimizer of $\mathbf{\Phi}_\infty$ in (B_2, H)).

To prove (2.104), pick F such that $E_\infty \Delta F \subset\subset B$: since $E_\infty \Delta (F \cap H) \subset\subset B$ and E_∞ is a $(0,4)$-minimizer $\mathbf{\Phi}_\infty$ in (B_2, H) we find

$$\mathbf{\Phi}_\infty(E_\infty; B \cap H) \leq \mathbf{\Phi}_\infty(F \cap H; B \cap H). \tag{2.105}$$

The left-hand sides of (2.104) and (2.105) coincide by (2.12), (2.13) and (2.103); the right-hand side of (2.105) is instead smaller than the right-hand side of (2.104) since $H \cap B \cap \partial^*(F \cap H) \subset B \cap \partial^* F$ with $\nu_{F\cap H} = \nu_F$ $\mathcal{H}^{n-1}$-a.e. on $H \cap B \cap \partial^*(F \cap H)$ by (2.5). This proves (2.104). Since $E_\infty \subset H$ and E_∞ satisfies (2.104) we can apply Lemma 2.13 to deduce that (2.103) implies $0 \notin {\rm spt}\mu_{E_\infty}$.

We now achieve a contradiction, and thus prove the lemma, by showing that $0 \in \mathrm{spt}\mu_{E_\infty}$. Indeed, let us set

$$\delta_h = \max\left\{\mathcal{H}^{n-1}(B \cap \mathrm{Tr}_{\partial H}(E_h))\,, \frac{1}{h}\right\} > 0\,, \qquad \varrho_h = \Big(\frac{2\delta_h}{\omega_{n-1}}\Big)^{1/(n-1)}\,, \qquad h \in \mathbb{N}\,.$$

(Notice that, up to take h large enough, we can assume $\varrho_h < 1$ for every $h \in \mathbb{N}$.) Since $\delta_h > 0$ and $0 \in \mathrm{spt}\mu_{E_h}$, we find $|E_h \cap B_{\varrho_h}| > 0$ for every $h \in \mathbb{N}$. Similarly, it must be $|(H \cap B_{\varrho_h}) \setminus E_h| > 0$ for every $h \in \mathbb{N}$, for otherwise, by the locality of the trace, we would have $B_{\varrho_h} \cap \partial H =_{\mathcal{H}^{n-1}} B_{\varrho_h} \cap \mathrm{Tr}_{\partial H}(E_h)$ for some $h \in \mathbb{N}$, and correspondingly

$$\delta_h \ge \mathcal{H}^{n-1}(B \cap \mathrm{Tr}_{\partial H}(E_h)) \ge \mathcal{H}^{n-1}(B_{\varrho_h} \cap \mathrm{Tr}_{\partial H}(E_h)) = \mathcal{H}^{n-1}(B_{\varrho_h} \cap \partial H) = \omega_{n-1}\varrho_h^{n-1} = 2\,\delta_h\,,$$

a contradiction to $\delta_h > 0$. This shows that $|E_h \cap B_{\varrho_h}|\,|(H \setminus E_h) \cap B_{\varrho_h}| > 0$ for every $h \in \mathbb{N}$. In particular, for every $h \in \mathbb{N}$ there exists $x_h \in \mathrm{cl}(H \cap \mathrm{spt}\mu_{E_h}) \cap B_{\varrho_h}$, and since $\varrho_h \to 0$ as $h \to \infty$, we conclude by (2.61) that $0 \in \mathrm{spt}\mu_{E_\infty}$. As already noticed, this completes the proof. ☐

We finally prove the following normalization lemma. Recall that, if E is a set of locally finite perimeter in A with $E \subset H$, then $\mathrm{spt}\mu_E$, $\partial^* E$ and $\mathrm{Tr}_{\partial H}(E)$ are defined as subsets of A. Moreover we are going to denote by $\partial_{\partial H}$ the topological boundary of subsets of ∂H, and by $\partial^*_{\partial H}$ the reduced boundary of sets of locally finite perimeter in ∂H.

Lemma 2.16 (Normalization)**.** *If A is an open set, $H = \{x_1 > 0\}$, $\Phi \in \boldsymbol{\mathcal{E}}(A \cap H, \lambda, \ell)$, and E is a (Λ, r_0)-minimizer of $\boldsymbol{\Phi}$ in (A, H), then, up to modify E on a set of measure zero,*

(i) *$E \cap A$ is open and $A \cap \partial E = \mathrm{spt}\mu_E$;*
(ii) *$\mathcal{H}^{n-1}((A \cap \partial E) \setminus \partial^* E) = 0$ and $\mathcal{H}^{n-1}\big((A \cap \partial E \cap \partial H)\Delta \mathrm{Tr}_{\partial H}(E)\big) = 0$;*
(iii) *$\partial E \cap \partial H$ is a set of locally finite perimeter in $A \cap \partial H$, and*

$$\mathcal{H}^{n-2}\Big(\big[\partial_{\partial H}(\partial E \cap \partial H) \setminus \partial^*_{\partial H}(\partial E \cap \partial H)\big] \cap A\Big) = 0\,. \tag{2.106}$$

Moreover,

$$\partial_{\partial H}\big(\partial E \cap \partial H\big) \cap A = \mathrm{cl}\big(\partial E \cap H\big) \cap \partial H \cap A\,. \tag{2.107}$$

Remark 2.17. In the sequel we will *always* assume that E is normalized in order to satisfy the conclusions of Lemma 2.16. In particular, ∂E shall be used in place of $\mathrm{spt}\mu_E$.

Proof of Lemma 2.16. Let us consider the set

$$\widetilde{E} = \big\{x \in A \cap H : |E \cap B_{x,r}| = |B_{x,r}| \text{ for some } r > 0\big\} \cup (E \setminus A)\,.$$

Obviously, $\widetilde{E} \cap A$ is open and, by (2.1),

$$A \cap \partial\widetilde{E} = \Big\{x \in A : 0 < |E \cap B_{x,r}| < |B_{x,r}| \quad \forall r > 0\Big\} = \mathrm{spt}\mu_E\,. \tag{2.108}$$

We now claim that $\widetilde{E}$ is equivalent to E. Clearly, $\widetilde{E} \cap A \subset E^{(1)} \cap A$. At the same time, if $x \in (E^{(1)} \cap A \cap H) \setminus \widetilde{E}$, then there exists $r_* > 0$ such that $0 < |E \cap B_{x,r}| < |B_{x,r}|$ for every $r < r_*$, that is, $x \in H \cap \mathrm{spt}\,\mu_E$: but then we cannot have $x \in E^{(1)}$ because of the density estimate (2.49). We have thus proved $H \cap A \cap (E^{(1)}\Delta\widetilde{E}) = \emptyset$, so that, by Lebesgue's density points theorem, $\widetilde{E}$ is equivalent to E. In particular, $\mu_E = \mu_{\widetilde{E}}$, and thus, by (2.108), we have

$$A \cap \partial\widetilde{E} = \mathrm{spt}\mu_{\widetilde{E}} \subset_{\mathcal{H}^{n-1}} \partial^*\widetilde{E}\,, \tag{2.109}$$

where the last inclusion follows by Lemma 2.15 (clearly, $\widetilde{E}$ is a (Λ, r_0)-minimizer of $\boldsymbol{\Phi}$ in (A, H)). By (2.109), $\mathcal{H}^{n-1}((A \cap \partial\widetilde{E})\Delta\partial^*\widetilde{E}) = 0$, and since $\mathcal{H}^{n-1}(\partial^*\widetilde{E}\Delta\mathrm{Tr}_{\partial H}(\widetilde{E})) = 0$ by (2.13), we have completed the proof of (ii). By (ii) and by Lemma 2.10, $\partial\widetilde{E}\cap\partial H$ is a set of locally finite perimeter in $\partial H\cap A$. Let us now prove (2.107). To this end we notice that, clearly,

$$\begin{aligned} A\cap\partial_{\partial H}\big(\partial\widetilde{E}\cap\partial H\big) &\subset \Big\{x\in A\cap\partial H:|E\cap B_{x,r}|\,|(H\setminus E)\cap B_{x,r}|>0\quad\forall r>0\Big\}\\ &\subset A\cap\partial H\cap\mathrm{cl}\big(H\cap\partial\widetilde{E}\big)\,, \end{aligned}\tag{2.110}$$

where the second inclusion follows as $A\cap\partial\widetilde{E}=\mathrm{spt}\mu_{\widetilde{E}}$. At the same time, since $H\cap\mathrm{spt}\mu_{\widetilde{E}}=H\cap\mathrm{spt}\mu_{H\setminus\widetilde{E}}$ and $A\cap\partial\widetilde{E}=\mathrm{spt}\mu_{\widetilde{E}}$, we have

$$A\cap\partial H\cap\mathrm{cl}\big(H\cap\partial\widetilde{E}\big)\subset\mathrm{spt}\mu_{\widetilde{E}}\cap\mathrm{spt}\mu_{H\setminus\widetilde{E}}\cap\partial H\,,$$

so that, by Remark 2.1, Lemma 2.15 (applied to both $\widetilde{E}$ and $H\setminus\widetilde{E}$) and (2.14), one has, for every $x\in A\cap\partial H\cap\mathrm{cl}(H\cap\partial\widetilde{E})$ and $r>0$ sufficiently small,

$$c\,\mathcal{H}^{n-1}(B_{x,r}\cap\partial H)\le\mathcal{H}^{n-1}(B_{x,r}\cap\partial\widetilde{E}\cap\partial H)\le(1-c)\mathcal{H}^{n-1}(B_{x,r}\cap\partial H)\,,\tag{2.111}$$

where $c=c(n,\lambda)$. This implies of course $A\cap\partial H\cap\mathrm{cl}(H\cap\partial\widetilde{E})\subset A\cap\partial_{\partial H}(\partial\widetilde{E}\cap\partial H)$, that, together with (2.110), implies (2.107). Finally, we notice that, by (2.107), the relative isoperimetric inequality in $\partial H\simeq\mathbb{R}^{n-1}$ and (2.111) give

$$\mathcal{H}^{n-2}\Big(\partial^*_{\partial H}(\partial\widetilde{E}\cap\partial H)\cap B_{x,r}\Big)\ge c(n,\lambda)\,r^{n-2}\,,\qquad\forall\,x\in\partial_{\partial H}(\partial\widetilde{E}\cap\partial H)\cap A\,,$$

which implies (2.106) by differentiation of Hausdorff measures, see [Mag12, Corollary 6.5]. □

2.8. **Ellipticity, minimality, and affine transformations.** In the proof of the ε-regularity theorem we will need to look at a given almost-minimizer from different directions at different scales. A very useful trick is then that of using affine transformations in order to always write things in the same system of coordinates. It is thus convenient, for the sake of clarity, to state separately how the considered class of elliptic functionals and almost-minimizers behave under these transformations.

Lemma 2.18. *If A is an open set in $\mathbb{R}^n$, H an open half-space, $\Phi\in\boldsymbol{\mathcal{E}}(A\cap H,\lambda,\ell)$, E is a (Λ,r_0)-minimizer of $\boldsymbol{\Phi}$ in (A,H), L is an invertible affine map on $\mathbb{R}^n$, and*

$$\Phi^L(x,\nu)=\Phi(L^{-1}x,(\mathrm{cof}\,\nabla L)^{-1}\nu)\,,\qquad(x,\nu)\in\mathrm{cl}(L(A\cap H))\times\mathbb{R}^n\,,\tag{2.112}$$

then $\Phi^L\in\boldsymbol{\mathcal{E}}(L(A\cap H),\widetilde{\lambda},\widetilde{\ell})$ and $L(E)$ is a $(\widetilde{\Lambda},\widetilde{r_0})$-minimizer of $\boldsymbol{\Phi}^L$ on $(L(A),L(H))$, where

$$\begin{aligned} \widetilde{\lambda}&=\lambda\,\max\Big\{\|\nabla L\|\|(\nabla L)^{-1}\|^2,\|\nabla L\|^2\|(\nabla L)^{-1}\|\Big\}^{n-1}\,,\\ \widetilde{\ell}&=\ell\,\|(\nabla L)^{-1}\|^n\,,\qquad\widetilde{\Lambda}=\frac{\Lambda}{|\det\nabla L|}\,,\qquad\widetilde{r_0}=\frac{r_0}{\|(\nabla L)^{-1}\|}\,. \end{aligned}$$

In particular, there exist positive constants $\varepsilon_*=\varepsilon_*(n)$ and $C_*=C_*(n)$, such that, if $\|\nabla L-\mathrm{Id}\,\|<\varepsilon_*$, then

$$\max\Big\{\frac{|\widetilde{\lambda}-\lambda|}{\lambda},\frac{|\widetilde{\ell}-\ell|}{\ell},\frac{|\widetilde{\Lambda}-\Lambda|}{\Lambda},\frac{|\widetilde{r_0}-r_0|}{r_0}\Big\}\le C_*\,\|\nabla L-\mathrm{Id}\,\|\,.\tag{2.113}$$

Remark 2.19. Definition (2.112) is conceived to give the identity

$$\boldsymbol{\Phi}^L(L(E);L(K))=\boldsymbol{\Phi}(E;K)\,,\tag{2.114}$$

for every E with locally finite perimeter in A and $K\subset\subset A$. Also, (2.113) has to be understood in the sense that if, say, $\Lambda=0$, then $\widetilde{\Lambda}=\Lambda$, and so on.

Proof of Lemma 2.18. Without loss of generality, in order to simplify the notation, we directly assume that L is linear, and write L in place of ∇L and L^{-1} in place of $(\nabla L)^{-1}$.

Step one: If $\sigma_{min} = \sigma_{min}(L)$ and $\sigma_{max} = \sigma_{max}(L)$ denote the square roots of the maximum and minimum eigenvalues of $L^* L$, then we have

$$\sigma_{min}|z| \le \min\{|Lz|, |L^* z|\} \le \max\{|Lz|, |L^* z|\} \le \sigma_{max}\,|z|\,, \qquad \forall z \in \mathbb{R}^n\,, \tag{2.115}$$

$$\|L\| = \sigma_{max}\,, \qquad \|L^{-1}\| = \sigma_{min}^{-1}\,, \qquad \sigma_{min}^n \le \det\, L \le \sigma_{max}^n\,. \tag{2.116}$$

On taking into account that $(\det\, L)\, L^{-1} = (\mathrm{cof}\, L)^*$, we find

$$(\mathrm{cof}\, L)^{-1} = (\det\, L)^{-1}\, L^*\,, \tag{2.117}$$

and thus, by (2.115) and (2.116),

$$\frac{|z|}{\sigma_{max}^{n-1}} \le |(\mathrm{cof}\, L)^{-1} z| \le \frac{|z|}{\sigma_{min}^{n-1}}\,, \qquad \forall z \in \mathbb{R}^n\,. \tag{2.118}$$

By (1.7), (2.112), and (2.118) we thus find

$$\frac{1}{\lambda\, \sigma_{max}^{n-1}} \le \Phi^L(x,\nu) \le \frac{\lambda}{\sigma_{min}^{n-1}}\,, \qquad \forall x \in \mathrm{cl}(L(A\cap H))\,, \nu \in \mathbf{S}^{n-1}\,.$$

Similarly, setting for the sake of brevity $M = (\mathrm{cof}\, L)^{-1}$, and by taking into account that for every $x \in \mathrm{cl}(L(A\cap H))$, $\nu \in \mathbf{S}^{n-1}$, and $z, w \in \mathbb{R}^n$, one has

$$\nabla\Phi^L(x,\nu)\cdot z = \nabla\Phi(L^{-1}x, M\nu)\cdot (Mz)\,,$$
$$\nabla^2\Phi^L(x,\nu) z\cdot w = \nabla^2\Phi(L^{-1}x, M\nu)(Mz)\cdot (Mw)\,,$$

we find that, for every $x, y \in \mathrm{cl}(L(A\cap H))$ and $\nu, \nu' \in \mathbf{S}^{n-1}$,

$$\begin{aligned}
|\nabla\Phi^L(x,\nu)| &\le \frac{\lambda}{\sigma_{min}^{n-1}}\,,\\
\|\nabla^2\Phi^L(x,\nu)\| &\le \lambda\Big(\frac{\sigma_{max}}{\sigma_{min}^2}\Big)^{n-1}\,,\\
\|\nabla^2\Phi^L(x,\nu) - \nabla^2\Phi^L(x,\nu')\| &\le \lambda\Big(\frac{\sigma_{max}}{\sigma_{min}^2}\Big)^{n-1} |\nu-\nu'|\,,\\
|\Phi^L(x,\nu) - \Phi^L(y,\nu)| &\le \frac{\ell}{\sigma_{min}^n}\,|x-y|\,,\\
|\nabla\Phi^L(x,\nu) - \nabla\Phi^L(y,\nu)| &\le \frac{\ell}{\sigma_{min}^n}\,|x-y|\,.
\end{aligned}$$

Finally if $\nu \in \mathbf{S}^{n-1}$ and $e \in \mathbb{R}^n$ then, by by (1.10) and the (-1) homogeneity of $\nabla^2\Phi$,

$$\begin{aligned}
\nabla^2\Phi^L(x,\nu)e\cdot e &\ge \frac{1}{\lambda\,|M\nu|}\Big|Me - \Big(Me\cdot\frac{M\nu}{|M\nu|}\Big)\frac{M\nu}{|M\nu|}\Big|^2 \ge \frac{\sigma_{min}^{n-1}}{\lambda}\Big|M\Big(e - \Big(Me\cdot\frac{M\nu}{|M\nu|^2}\Big)\nu\Big)\Big|^2\\
&\ge \frac{1}{\lambda}\Big(\frac{\sigma_{min}}{\sigma_{max}^2}\Big)^{n-1}\Big|e - \Big(Me\cdot\frac{M\nu}{|M\nu|^2}\Big)\nu\Big|^2 \ge \frac{1}{\lambda}\Big(\frac{\sigma_{min}}{\sigma_{max}^2}\Big)^{n-1}\big|e - (e\cdot\nu)\nu\big|^2\,,
\end{aligned}$$

where in the last inequality we have used that $t \mapsto |e - t\nu|^2$ is minimized by $t_* = e\cdot\nu$.

Step two: Clearly, $L(E) \subset L(H)$, and one easily checks from the distributional definition of relative perimeter that $L(E)$ has locally finite perimeter in $L(A)$. Let now $G \subset L(H)$ with $G\Delta L(E) \subset\subset V \subset\subset L(A)$ for some open set V with $\mathrm{diam}(V) < 2s_0$. If we set $W = L^{-1}(V)$ and $F = L^{-1}(G)$, then $F \subset H$, $F\Delta E \subset\subset W \subset\subset A$ and $\mathrm{diam}(W) < \|(\nabla L)^{-1}\|\mathrm{diam}(V) < \|(\nabla L)^{-1}\|\, s_0 < r_0$ (provided $s_0 = r_0/\|(\nabla L)^{-1}\|$), so that

$$\mathbf{\Phi}(E; W\cap H) \le \mathbf{\Phi}(F; W\cap H) + \Lambda\,|E\Delta F|\,.$$

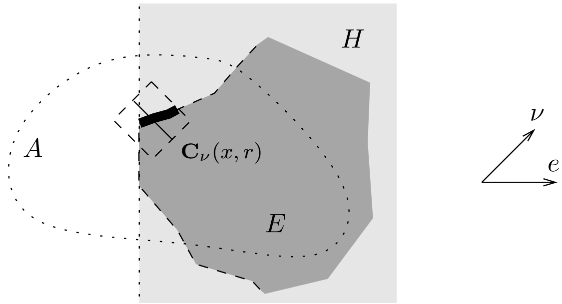

FIGURE 3.1. When computing $\mathbf{exc}_\nu^H(E,x,r)$, one considers only the part of the boundary of E that is interior to H (that is depicted as a bold line in this figure).

By (2.114) we find

$$\begin{aligned}\boldsymbol{\Phi}^L(L(E);V\cap L(H)) &\le \boldsymbol{\Phi}^L(G;V\cap L(H))+\Lambda\,|L^{-1}(L(E)\Delta G)|\\ &= \boldsymbol{\Phi}^L(G;V\cap L(H))+\Lambda\,|\det L|^{-1}\,|L(E)\Delta G|\,,\end{aligned}$$

and this concludes the proof. □

3. The ε-regularity theorem

This section is devoted to the proof of a boundary regularity criterion (Theorem 3.1) formulated in terms of the smallness of a quantity known as spherical excess. We thus begin by introducing the relevant notation and definitions needed in the formulation of this criterion. Given an open set A and an open half-space H in $\mathbb{R}^n$, we consider a set $E\subset H$ which is of locally finite perimeter in A, we fix $x\in A\cap\partial H$ and $r<\mathrm{dist}(x,\partial A)$, and then define the *spherical excess of E at the point x, at scale r, relative to H* as

$$\mathbf{exc}^H(E,x,r)=\inf\left\{\frac{1}{r^{n-1}}\int_{B_{x,r}\cap H\cap\partial^*E}\frac{|\nu_E-\nu|^2}{2}\,d\mathcal{H}^{n-1}:\nu\in\mathbf{S}^{n-1}\right\}.$$

Another useful notion of excess is that of cylindrical excess. Given $\nu\in\mathbf{S}^{n-1}$, we set $\mathbf{q}_\nu(y)=y\cdot\nu$, $\mathbf{p}_\nu(y)=y-(y\cdot\nu)\,\nu$ for every $y\in\mathbb{R}^n$, and let

$$\begin{aligned}\mathbf{C}_\nu(x,r)&=\Big\{y\in\mathbb{R}^n:|\mathbf{p}_\nu(y-x)|<r\,,|\mathbf{q}_\nu(y-x)|<r\Big\}\,,\\ \mathbf{D}_\nu(x,r)&=\Big\{y\in\mathbb{R}^n:|\mathbf{p}_\nu(y-x)|<r\,,|\mathbf{q}_\nu(y-x)|=0\Big\}\,.\end{aligned}$$

With this notation at hand, the *cylindrical excess of E at x, at scale r, in the direction ν, relative to H*, is defined as

$$\mathbf{exc}_\nu^H(E,x,r)=\frac{1}{r^{n-1}}\int_{H\cap\mathbf{C}_\nu(x,r)\cap\partial^*E}\frac{|\nu_E-\nu|^2}{2}\,d\mathcal{H}^{n-1}\,;$$

see Figure 3.1. If $\nu=e_n$, then we shall simply set $\mathbf{exc}_n^H$ in place of $\mathbf{exc}_{e_n}^H$. As usual, the definition is made so that the excess is invariant by scaling, precisely

$$\mathbf{exc}_\nu^H(E,x,r)=\mathbf{exc}_\nu^{H^{z,s}}\Big(E^{z,s},\frac{x-z}{s},\frac{r}{s}\Big)=\mathbf{exc}_\nu^{H^{x,r}}(E^{x,r},0,1)\,. \tag{3.1}$$

It is easily seen that, if $x\in\mathrm{cl}(H\cap\mathrm{spt}\mu_E)$ and $r>0$, then

$$\mathbf{exc}^H(E,x,r)=0\quad\text{iff}\quad\begin{cases}\text{there exist }\nu\in\mathbf{S}^{n-1}\text{ and }s\in\mathbb{R}\text{ such that}\\ E\cap H\cap B_{x,r}=\{x\cdot\nu<s\}\cap H\cap B_{x,r}\,,\end{cases} \tag{3.2}$$

$$\mathbf{exc}_\nu^H(E,x,r)=0\quad\text{iff}\quad\begin{cases}\text{there exists }s\in\mathbb{R}\text{ such that}\\ E\cap H\cap\mathbf{C}_\nu(x,r)=\{x\cdot\nu<s\}\cap H\cap\mathbf{C}_\nu(x,r)\,.\end{cases} \tag{3.3}$$

Finally, we recall that the normalized projection $\mathbf{e_1}(\nu)$ of $\nu \in \mathbf{S}^{n-1}$ (with $|\nu\cdot e_1|<1$) on $e_1^\perp$ was defined in (2.42) by $\mathbf{e_1}(\nu)=(\nu-(\nu\cdot e_1))/\sqrt{1-(\nu\cdot e_1)^2}$, so that

$$\nu^\perp=\Big\{x\in\mathbb{R}^n : x\cdot\nu=0\Big\}=\Big\{x\in\mathbb{R}^n : x\cdot\mathbf{e_1}(\nu)=-\frac{(\nu\cdot e_1)\,x_1}{\sqrt{1-(\nu\cdot e_1)^2}}\Big\}.$$

and that, the normalization of Lemma 2.16 being in force, we have $A\cap\partial E=\mathrm{spt}\mu_E$ on almost-minimizers.

Theorem 3.1 (ε-regularity theorem)**.** *For every $n\ge 2$ and $\lambda\ge 1$ there exist positive constants $\varepsilon_{\rm crit}=\varepsilon_{\rm crit}(n,\lambda)$, $\beta_1=\beta_1(n,\lambda)\le\beta_2=\beta_2(n,\lambda)$ and $C=C(n,\lambda)$ with the following properties. If $H=\{x_1>0\}$,*

$$\begin{aligned}&\Phi\in\boldsymbol{\mathcal{E}}(B_{4\,r}\cap H,\lambda,\ell)\,,\\ &E \textit{ is a } (\Lambda,r_0)\textit{-minimizer of } \mathbf{\Phi} \textit{ in } (B_{4\,r},H) \textit{ and } 0<2\,r\le r_0\,,\\ &0\in\mathrm{cl}(H\cap\partial E)\,,\\ &\mathbf{exc}^H(E,0,2\,r)+(\Lambda+\ell)\,r\le\varepsilon_{\rm crit}\,,\end{aligned}$$

then $M=\mathrm{cl}(\partial E\cap H)\cap B(0,\beta_1 r)$ is a $C^{1,1/2}$ manifold with boundary, with

$$M\cap\partial H=\partial_{\partial H}(\partial E\cap\partial H)\cap B(0,\beta_1 r)\,,$$

and such that the anisotropic Young's law holds true on $M\cap\partial H$, i.e.

$$\nabla\Phi\big(x,\nu_E(x)\big)\cdot e_1=0\,,\qquad\forall\,x\in M\cap\partial H\,.$$

More precisely, there exist $\nu\in\mathbf{S}^{n-1}$ with

$$\nabla\Phi(0,\nu)\cdot e_1=0\,,\qquad|\nu\cdot e_1|\le 1-\frac{1}{C}\,,$$

and $u\in C^{1,1/2}(\mathrm{cl}(\mathbf{D}_{\mathbf{e_1}(\nu)}(0,\beta_2\,r)\cap H))$ with

$$\sup_{x,y\in\mathbf{D}_{\mathbf{e_1}(\nu)}(0,\beta_2\,r)\cap H}\frac{|u(x)|}{r}+|\nabla u(x)|+r^{1/2}\,\frac{|\nabla u(x)-\nabla u(y)|}{|x-y|^{1/2}}\le C\,\sqrt{\varepsilon_{\rm crit}}\,,$$

such that, M is obtained, in a $\beta_2\,r$-neighborhood of $\nu^\perp$, as a perturbation of $\nu^\perp$ by u:

$$\begin{aligned}&\Big\{x\in H\cap\partial E:|\mathbf{p}_{\mathbf{e_1}(\nu)}x|<\beta_2\,r\,,\,-\beta_2 r-\frac{(\nu\cdot e_1)\,x_1}{\sqrt{1-(\nu\cdot e_1)^2}}<\mathbf{q}_{\mathbf{e_1}(\nu)}\,x<\beta_2 r-\frac{(\nu\cdot e_1)\,x_1}{\sqrt{1-(\nu\cdot e_1)^2}}\Big\}\\ &\qquad=\Big\{x\in H:|\mathbf{p}_{\mathbf{e_1}(\nu)}x|<\beta_2\,r\,,\,\mathbf{q}_{\mathbf{e_1}(\nu)}\,x=-\frac{(\nu\cdot e_1)\,x_1}{\sqrt{1-(\nu\cdot e_1)^2}}+u\big(\mathbf{p}_{\mathbf{e_1}(\nu)}\,x\big)\Big\}\,;\end{aligned}\tag{3.4}$$

see Figure 3.2.

Definition 3.2 (Boundary singular set)**.** If E is a (Λ,r_0)-minimizer of $\mathbf{\Phi}$ in (A,H) (normalized as in Lemma 2.16) and we set $M=A\cap\mathrm{cl}(H\cap\partial E)$, then the *boundary singular set* $\Sigma_A(E;\partial H)$ *of E* is defined as the subset of $M\cap\partial H$ such that

$$(M\cap\partial H)\setminus\Sigma_A(E,\partial H)=\left\{x\in M\cap\partial H:\begin{array}{l}\text{there exists } r_x>0 \text{ such that } M\cap B_{x,r_x}\\ \text{is a } C^{1,1/2} \text{ manifold with boundary}\end{array}\right\}.\tag{3.5}$$

Remark 3.3. By Theorem 3.1,

$$\Sigma_A(E;\partial H)=\Big\{x\in M\cap\partial H:\liminf_{r\to0^+}\mathbf{exc}^H(E,x,r)>0\Big\}\,.$$

This identity will be the starting point in section 5 to prove that $\mathcal{H}^{n-2}(\Sigma_A(E;\partial H))=0$.

The main step in the proof of Theorem 3.1 is proving the validity of the following lemma. (We will do this in section 4.)

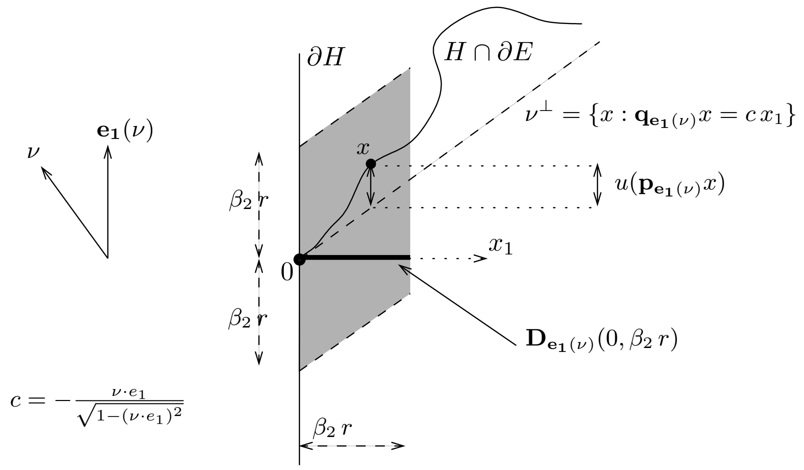

FIGURE 3.2. The situation in (3.4). The region (depicted in gray) where the graphicality of $H\cap\partial E$ is proved is obtained as a "shear-strained" deformation of $\mathbf{C}_{\mathbf{e_1}(\nu)}(0,\beta_2\,r)$, where the amount of vertical deformation depends on the coordinate x_1 only. Of course, the function u parameterizing $H\cap\partial E$ depends on the full set of variables $\mathbf{p}_{\mathbf{e_1}(\nu)}x$, not just on x_1.

Lemma 3.4. *For every $\lambda\ge 1$ there exist positive constants $\varepsilon_{\rm reg}=\varepsilon_{\rm reg}(n,\lambda)$ and $C=C(n,\lambda)$ with the following properties. If $H=\{x\cdot e_1>0\}$,*

$$
\begin{aligned}
&\Phi\in\boldsymbol{\mathcal{E}}(\mathbf{C}_{128\,r}\cap H,\lambda,\ell)\,,\\
&E \text{ is a } (\Lambda,r_0)\text{-minimizer of } \mathbf{\Phi} \text{ in } (\mathbf{C}_{128\,r},H) \text{ with } 0<64r\le r_0\,,\\
&0\in{\rm cl}(H\cap\partial E)\,,\\
&\big|\nabla\Phi(0,e_n)\cdot e_1\big|+\mathbf{exc}_n^H(E,0,64r)+(\Lambda+\ell)r<\varepsilon_{\rm reg}\,,
\end{aligned}
$$

then there exists a function $u\in C^{1,1/2}({\rm cl}(\mathbf{D}_r\cap H))$ such that

$$
\mathbf{C}_r^+\cap\partial E=\Big\{x\in H:|\mathbf{p}x|<r\,,\mathbf{q}x=u(\mathbf{p}x)\Big\}\,,
$$

$$
\sup_{z,y\in\mathbf{D}_r\cap H}\frac{|u(z)|}{r}+|\nabla u(z)|+r^{1/2}\frac{|\nabla u(z)-\nabla u(y)|}{|z-y|^{1/2}}\le C\,\sqrt{\varepsilon_{\rm reg}}\,, \tag{3.6}
$$

$$
\nabla\Phi\big((z,u(z)),(-\nabla u(z),1)\big)\cdot e_1=0\qquad\forall z\in\mathbf{D}_r\cap\partial H\,. \tag{3.7}
$$

We now devote the remaining part of this section to show how to deduce Theorem 3.1 from Lemma 3.4. The first step consists in showing that the smallness of the spherical excess implies the existence of a direction such that the cylindrical excess is small. Moreover, this direction is close to satisfy the anisotropic Young's law.

Lemma 3.5. *For every $\lambda\ge 1$ and $\tau>0$ there exists $\varepsilon_{\rm sc}=\varepsilon_{\rm sc}(n,\tau)>0$ with the following property. If $H=\{x_1>0\}$, $\Phi\in\boldsymbol{\mathcal{E}}(\mathbf{C}_{4r}\cap H,\lambda,\ell)$, and E is a (Λ,r_0)-minimizer of $\mathbf{\Phi}$ on $(\mathbf{C}_{4r},H)$ with $0\in{\rm cl}(H\cap\partial E)$, $2\,r<r_0$, and*

$$
\mathbf{exc}^H(E,0,2r)+(\Lambda+\ell)\,r<\varepsilon_{\rm sc}\,,
$$

then there exists $\nu\in\mathbf{S}^{n-1}$ with

$$
|\nabla\Phi(0,\nu)\cdot e_1|+\mathbf{exc}_\nu^H(E,0,r)+(\Lambda+\ell)\,r<\tau\,.
$$

Remark 3.6. The following continuity properties of the cylindrical excess are useful in the proof of Lemma 3.5 (as well as in other arguments): if $\{E_h\}_{h\in\mathbb{N}}$ and E are sets of locally finite perimeter in A, with $E_h \to E$ in $L^1_{\rm loc}(A)$ and

$$|\mu_{E_h}|\llcorner H \stackrel{*}{\rightharpoonup} |\mu_E|\llcorner H\,,\quad |\mu_{E_h}| \stackrel{*}{\rightharpoonup} |\mu_E|\,,\quad \text{as Radon measures in } A\,, \tag{3.8}$$

as $h\to\infty$, then

$$\mathbf{exc}_\nu^H(E,x,r) \le \liminf_{h\to\infty} \mathbf{exc}_\nu^H(E_h,x,r)\,,\qquad \text{whenever } \mathbf{C}_\nu(x,r)\subset\subset A\,, \tag{3.9}$$

$$\mathbf{exc}_\nu^H(E,x,r) = \lim_{h\to\infty} \mathbf{exc}_\nu^H(E_h,x,r)\,,\qquad \begin{array}{l}\text{whenever } \mathbf{C}_\nu(x,r)\subset\subset A\\ \text{with } P(E;\partial\mathbf{C}_\nu(x,r))=0\,.\end{array} \tag{3.10}$$

Indeed, (3.9) follows from (3.10) by monotonicity, while $|\nu_E-\nu|^2/2 = 1-(\nu_E\cdot\nu)$ gives

$$\mathbf{exc}_\nu^H(E,x,r) = \frac{|\mu_E|(\mathbf{C}_\nu(x,r)\cap H) - \nu\cdot\mu_E(\mathbf{C}_\nu(x,r)\cap H)}{r^{n-1}}\,,$$

by which (3.10) is immediately seen to be consequence of (3.8).

Proof of Lemma 3.5. By Remark 2.2 and (3.1), up to replace E and Φ with $E^{x_0,r}$ and $\Phi^{x_0,r}$ respectively, we can directly assume that $x_0=0$ and $r=1$. We then argue by contradiction, and assume the existence of $\tau_0>0$ and of sequences $\{\Phi_h\}_{h\in\mathbb{N}}\in\boldsymbol{\mathcal{E}}(\mathbf{C}_4\cap H,\lambda,\ell_h)$ and $\{E_h\}_{h\in\mathbb{N}}$ such that E_h is a $(\Lambda_h,2)$-minimizer of $\mathbf{\Phi}_h$ on $(\mathbf{C}_4,H)$ with $0\in{\rm cl}(H\cap\partial E_h)$ for every $h\in\mathbb{N}$ and

$$\lim_{h\to\infty} \mathbf{exc}^H(E_h,0,2)+\Lambda_h+\ell_h = 0\,, \tag{3.11}$$

$$\inf_{\nu\in\mathbf{S}^{n-1}} \Big\{|\nabla\Psi_h(0,\nu)\cdot e_1| + \mathbf{exc}_\nu^H(E_h,0,1)\Big\} \ge \tau_0\,. \tag{3.12}$$

By Theorem 2.9 and (3.11), there exist $\Phi_\infty\in\boldsymbol{\mathcal{E}}_*(\lambda)$, and a $(0,2)$-minimizer E_∞ of $\mathbf{\Phi}_\infty$ on $(\mathbf{C}_4,H)$ with $0\in{\rm cl}(H\cap\partial E_\infty)$, such that $\nabla\Phi_h(0,\nu)\to\nabla\Phi_\infty(\nu)$ uniformly on $\nu\in\mathbf{S}^{n-1}$, $E_h\to E_\infty$ in $L^1_{\rm loc}(\mathbf{C}_4)$, and $|\mu_{E_h}|$ and $|\mu_{E_h}|\llcorner H$ that converge, respectively, to $|\mu_{E_\infty}|$ and $|\mu_{E_\infty}|\llcorner H$, as Radon measures in $\mathbf{C}_4$ when $h\to\infty$. We can thus apply (3.9) and (3.11) to deduce that $\mathbf{exc}^H(E_\infty,0,2)=0$. By (3.2), there exist $\nu\in\mathbf{S}^{n-1}$ and $s\in\mathbb{R}$ such that

$$E_\infty\cap B_2\cap H = B_2\cap\{x\cdot\nu<s\}\cap H\,, \tag{3.13}$$

so that, in particular $\mathbf{exc}_\nu^H(E_\infty,0,1)=0$ and we can apply (3.10) to deduce that

$$\lim_{h\to\infty}\mathbf{exc}_\nu^H(E_h,0,1)=0\,.$$

By (3.12), we conclude that

$$|\nabla\Phi_\infty(\nu)\cdot e_1|\ge\tau_0\,.$$

However, $0\in{\rm cl}(H\cap\partial E_h)$ for every $h\in\mathbb{N}$ and (2.61) imply $0\in{\rm cl}(H\cap\partial E_\infty)$, so that $B_2\cap H\cap\partial E_\infty$ is non-empty, in particular $\nu\ne\pm e_1$. Thanks to (3.13) we can apply Proposition 2.6 to conclude that $\nabla\Phi_\infty(\nu)\cdot e_1=0$. We thus reach a contradiction and complete the proof of the lemma. □

The second tool we need to deduce Theorem 3.1 from Lemma 3.4 is the following lemma.

Lemma 3.7. *For every $\tau\in[0,1)$, there exists a positive constant $C=C(\tau)$ with the following property. If $H=\{x_1>0\}$, $\nu\in\mathbf{S}^{n-1}$ and $|\nu\cdot e_1|\le\tau<1$, then there exists a linear map $L:\mathbb{R}^n\to\mathbb{R}^n$ such that $L(H)=H$,*

$$L(\nu^\perp)=e_n^\perp\,,\qquad \textit{so that}\qquad e_n=\frac{({\rm cof}\,\nabla L)\nu}{|({\rm cof}\,\nabla L)\nu|}\,, \tag{3.14}$$

$$L^{-1}(\mathbf{C}\cap H)=\Big\{y\in H:|y-(y\cdot\mathbf{e_1}(\nu))\mathbf{e_1}(\nu)|<1\,,\qquad -1-\frac{(\nu\cdot e_1)\,y_1}{\sqrt{1-(\nu\cdot e_1)^2}}<y\cdot\mathbf{e_1}(\nu)<1-\frac{(\nu\cdot e_1)\,y_1}{\sqrt{1-(\nu\cdot e_1)^2}}\Big\}\,, \tag{3.15}$$

see Figure 3.3, and

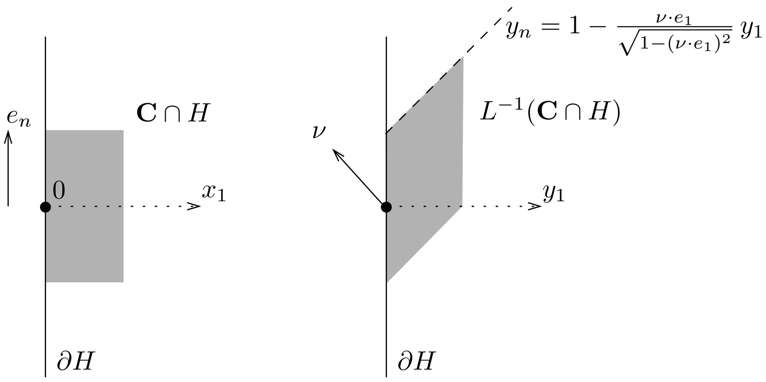

FIGURE 3.3. The image of $\mathbf{C}\cap H$ through L^{-1}. The picture refers to the situation when $\mathbf{e_1}(\nu)=e_n$. Notice that, in this case, the projection of $L^{-1}(\mathbf{C}\cap H)$ on $e_n^\perp$ is $\mathbf{D}\cap H$. In the general case, the projection of $L^{-1}(\mathbf{C}\cap H)$ over $\mathbf{e_1}(\nu)^\perp$ is $\mathbf{D}_{\mathbf{e_1}(\nu)}\cap H$.

$$\max\{\|\nabla L\|,\|(\nabla L)^{-1}\|\}\le C\,,\qquad \det\nabla L=1\,,\qquad \nabla\Phi^L(e_n)\cdot e_1=\nabla\Phi(\nu)\cdot e_1\,. \tag{3.16}$$

whenever Φ is an autonomous elliptic integrand and Φ^L is defined by $\Phi^L(e)=\Phi((\mathrm{cof}\ \nabla L)^{-1}e)$. Moreover,

$$\|\nabla L-\mathrm{Id}\,\|\le C\,|\nu-e_n|\,. \tag{3.17}$$

Proof. For some $|\alpha|\le\eta(\tau)<\pi/2$ we have

$$\nu=\cos\alpha\,\mathbf{e_1}(\nu)-\sin\alpha\,e_1\,,\qquad |\sin\,\alpha|=|\nu\cdot e_1|\le|\nu-e_n|\,. \tag{3.18}$$

We define a linear map Q by setting $Q=\mathrm{Id}$ if $\mathbf{e_1}(\nu)=e_n$, and by setting $Q=\mathrm{Id}$ on $e_n^\perp\cap\mathbf{e_1}(\nu)^\perp$ and Q to be the rotation taking $\mathbf{e_1}(\nu)$ into e_n on $\mathrm{Span}(e_n,\mathbf{e_1}(\nu))$ otherwise. Finally, we define a linear map $L:\mathbb{R}^n\to\mathbb{R}^n$ by setting

$$\nabla L=Q\circ\Big(\mathrm{Id}\,-\tan\alpha\,\mathbf{e_1}(\nu)\otimes e_1\Big)\,. \tag{3.19}$$

Trivially, $\det\nabla L=1$ and $\|\nabla L\|\le 1+|\tan\,\alpha|\le C(\tau)$. If $v\in\nu^\perp$, then $\mathbf{e_1}(\nu)\cdot v=\tan\,\alpha\,(e_1\cdot v)$ and thus

$$(Lv)\cdot e_n=\Big(v-\tan\alpha\,(e_1\cdot v)\,\mathbf{e_1}(\nu)\Big)\cdot Q^{-1}e_n=v\cdot\mathbf{e_1}(\nu)-\tan\,\alpha\,(e_1\cdot v)=0\,,$$

so that (3.14) holds true. By noticing that,

$$\nabla L^{-1}=\Big(\mathrm{Id}\,+\tan\alpha\,\mathbf{e_1}(\nu)\otimes e_1\Big)\circ Q^{-1}\,,$$

we also have $\|(\nabla L)^{-1}\|\le C(\tau)$. By definition of Φ^L, we see that

$$\nabla\Phi^L(e_n)\cdot e_1=\nabla\Phi((\mathrm{cof}\,\nabla L)^{-1}e_n)\cdot\Big((\mathrm{cof}\,\nabla L)^{-1}e_1\Big)\,. \tag{3.20}$$

Since $\nabla L^*=(\det\nabla L)\,(\mathrm{cof}\,\nabla L)^{-1}=(\mathrm{cof}\,\nabla L)^{-1}$ and $\nabla\Phi$ is zero-homogeneous, by (3.14) we find

$$\nabla\Phi^L(e_n)\cdot e_1=\nabla\Phi(\nu)\cdot(\nabla L^*e_1)=\nabla\Phi(\nu)\cdot e_1\,, \tag{3.21}$$

where we have used that $\nabla L^*e_1=e_1$, as it can be seen from (3.19). This completes the proof of (3.16), while the validity of (3.15) is easily checked. To finally prove (3.17), we notice that $\|Q-\mathrm{Id}\,\|\le C\,|\mathbf{e_1}(\nu)-e_n|\le C\,|\nu-e_n|$ for some constant C, so that for every $e\in\mathbf{S}^{n-1}$ one has

$$|\nabla L\,e-e|=|Qe-e-\tan\,a\,(e_1\cdot e)\,e_n|\le\|Q-\mathrm{Id}\,\|+|\tan\,\alpha|\le C\,|\nu-e_n|\,. \qquad\square$$

Finally, we estimate how cylindrical excess changes under transformation by affine maps.

Lemma 3.8. *For every $\eta \ge 1$ there exists a constant $C = C(n,\eta)$ with the following property. If H is an open half-space and $L : \mathbb{R}^n \to \mathbb{R}^n$ is an affine transformation, with $L(H) = H$ and*

$$\|\nabla L\|\,,\|\nabla L^{-1}\| \le \eta\,, \tag{3.22}$$

then for every set of finite perimeter E and every $\nu \in \mathbf{S}^{n-1}$,

$$\mathbf{exc}^H_{\hat\nu}\Big(L(E), Lx, \frac{r}{\sqrt{2}\,\eta}\Big) \le C\,\mathbf{exc}^H_\nu(E,x,r) \qquad \textit{where} \qquad \hat\nu = \frac{(\mathrm{cof}\,\nabla L)\nu}{|(\mathrm{cof}\,\nabla L)\nu|}\,. \tag{3.23}$$

Proof. By (3.1) we can assume that $x = 0$ and $r = 1$, as well as that L is linear. Correspondingly, we set L in place ∇L. In this way, by arguing as in (2.118), we have

$$\frac{|e|}{\|L^{-1}\|^{n-1}} \le \sigma_{min}^{n-1}|e| \le |\mathrm{cof}\,L\,e| \le \sigma_{max}^{n-1}|e| = \|L\|^{n-1}|e|\,, \qquad \forall e \in \mathbb{R}^n\,. \tag{3.24}$$

By (3.22), we find $\mathbf{C}_{\hat\nu}(0,1/\sqrt{2}\eta) \subset L(\mathbf{C})$, so that, if we set $M = \mathrm{cof}\ L$, then

$$\begin{aligned}\int_{\mathbf{C}_{\hat\nu}(0,1/\sqrt{2}\eta)\cap L(H)\cap L(\partial^* E)} |\hat\nu - \nu_{L(E)}|^2\,d\mathcal{H}^{n-1} &\le \int_{L(\mathbf{C}\cap H\cap\partial^* E)} |\hat\nu - \nu_{L(E)}|^2\,d\mathcal{H}^{n-1}\\ &\le \int_{\mathbf{C}\cap H\cap\partial^* E}\Big|\frac{M\nu}{|M\nu|} - \frac{M\nu_E}{|M\nu_E|}\Big|^2 |M\nu_E|\,d\mathcal{H}^{n-1}\,.\end{aligned}$$

We thus find (3.23) thanks to the fact that, by (3.24),

$$\Big|\frac{M\nu}{|M\nu|} - \frac{M\nu_E}{|M\nu_E|}\Big|^2 |M\nu_E| \le \Big(\frac{2|M\nu - M\nu_E|}{|M\nu_E|}\Big)^2 |M\nu_E| \le 4\,\eta^{3(n-1)}\,|\nu-\nu_E|^2\,. \qquad \square$$

We now combine Lemma 3.4 (to be proved in section 4) with Lemma 3.5, Lemma 3.7 and Lemma 3.8 to prove Theorem 3.1.

Proof of Theorem 3.1 (assuming Lemma 3.4). Correspondingly to $\lambda \ge 1$, we can find $\varepsilon_* = \varepsilon_*(\lambda)$ such that, if we set

$$\tau = \sqrt{1 - \frac{1}{\lambda^4}} + \varepsilon_*\,\lambda^4\,,$$

then $\tau \in (0,1)$. Let us now consider $\Phi \in \boldsymbol{\mathcal{E}}(B_{4r}\cap H,\lambda,\ell)$, E a (Λ, r_0)-minimizer of $\mathbf{\Phi}$ in (B_{4r}, H) with $0 < 2\,r \le r_0$, $0 \in \mathrm{cl}(H\cap\partial E)$, and

$$\mathbf{exc}^H(E,0,2\,r) + (\Lambda+\ell)\,r \le \varepsilon_{\rm crit}\,.$$

If $\varepsilon_{\rm crit}(n,\lambda) \le \varepsilon_{\rm sc}(n,\min\{\varepsilon_*(\lambda),\varepsilon_{\rm reg}(n,\lambda)\})$, then by Lemma 3.5 there exists $\nu \in \mathbf{S}^{n-1}$ such that

$$|\nabla\Phi(0,\nu)\cdot e_1| + \mathbf{exc}^H_\nu(E,0,r) + (\Lambda+\ell)\,r < \min\{\varepsilon_*(\lambda),\varepsilon_{\rm reg}(n,\lambda)\}\,.$$

By $|\nabla\Phi(0,\nu)| \ge 1/\lambda$, we have $|\nabla\Phi(0,\nu)\cdot e_1| \le \lambda\,\varepsilon_*\,|\nabla\Phi(0,\nu)|$, and thus, by (2.45),

$$|\nu\cdot e_1| \le \sqrt{1-\frac{1}{\lambda^4}} + \varepsilon_*\,\lambda^4 = \tau(\lambda) < 1\,. \tag{3.25}$$

By Lemma 3.7, there exists a linear map $L : \mathbb{R}^n \to \mathbb{R}^n$ such that $L(H) = H$ and (3.14) and (3.16) hold true. By Lemma 2.18 and (3.16), if we set

$$\Phi^L(x,\nu) = \Phi(L^{-1}x,(\mathrm{cof}\,\nabla L)^{-1}\nu)\,,$$

then $\Phi^L \in \boldsymbol{\mathcal{E}}(L(B_{4\,r})\cap H,\widetilde\lambda,\widetilde\ell)$ and $L(E)$ is a $(\Lambda,\widetilde{r_0})$-minimizer of $\mathbf{\Phi}^L$ on $(L(B_{4\,r}),H)$, where

$$\widetilde\lambda \le C\,\lambda\,,\qquad \widetilde\ell \le C\,\ell\,,\qquad r_0 \le C\,\widetilde{r_0}\,,\qquad B_{r/C}\subset L(B_{4r})\,,$$

for a constant $C = C(\lambda)$. Moreover, by Lemma 3.8 and (3.14), for some $\eta = \eta(\lambda) \ge 1$ and $C = C(n,\lambda)$, we have

$$\mathbf{exc}^H_n\Big(L(E),0,\frac{r}{\sqrt{2}\,\eta}\Big) \le C\,\mathbf{exc}^H_\nu(E,0,r) \qquad \text{since} \qquad e_n = \frac{(\mathrm{cof}\,\nabla L)\nu}{|(\mathrm{cof}\,\nabla L)\nu|}\,,$$

as well as, again by (3.16),

$$\nabla\Phi^L(0,e_n)\cdot e_1=\nabla\Phi(0,\nu)\cdot e_1\,.$$

Summarizing, there exist positive constants $C_*=C_*(n,\lambda)$ and $C_{**}=C_{**}(n,\lambda)$ such that $\Phi^L\in\boldsymbol{\mathcal{E}}(B_{r/C_*}\cap H,C_*\lambda,C_*\ell)$, $L(E)$ is a $(\Lambda,r_0/C_*)$-minimizer of $\mathbf{\Phi}^L$ on $(B_{r/C_*},H)$ with $0\in\mathrm{cl}(H\cap\partial L(E))$, and

$$|\nabla\Phi^L(0,e_n)\cdot e_1|+\mathbf{exc}_n^H\Big(L(E),0,\frac{r}{2C_*}\Big)+(\Lambda+C_*\,\ell)\,r\le C_{**}\,\varepsilon_{\rm crit}\,.$$

If $C_{**}\,\varepsilon_{\rm crit}\le\varepsilon_{\rm reg}(n,C_*\,\lambda)$, then by Lemma 3.4 there exists a function $u\in C^{1,1/2}(\mathrm{cl}(\mathbf{D}_{r/128\,C_*}\cap H))$ such that

$$\mathbf{C}_{r/128\,C_*}\cap H\cap\partial L(E)=\Big\{x\in H:|\mathbf{p}x|<r\,,\mathbf{q}x=u(\mathbf{p}x)\Big\}\,,\tag{3.26}$$

with

$$\sup_{z,y\in\mathbf{D}_{r/128\,C_*}\cap H}\frac{|u(z)|}{r}+|\nabla u(z)|+r^{1/2}\,\frac{|\nabla u(z)-\nabla u(y)|}{|z-y|^{1/2}}\le C\,\sqrt{\varepsilon_{\rm reg}}\,,$$
$$\nabla\Phi\big((z,u(z)),(-\nabla u(z),1)\big)\cdot e_1=0\,,\qquad\forall z\in\mathbf{D}_{r/128C_*}\cap\partial H\,.$$

for some $C=C(n,\lambda)$. By exploiting (3.15) and by applying L^{-1} to the identity (3.26) we complete the proof of Theorem 3.1. □

4. Proof of Lemma 3.4

In this section we prove Lemma 3.4. The argument is that commonly used in most proofs of ε-regularity criterions, and can be very roughly sketched as follows. Based on a *height bound* (Lemma 4.1), one shows that locally at points with small excess it is possible to cover a large portion of the boundary with the graph of a Lipschitz function u, that is close to solve the linearized Euler-Lagrange equation of a suitable non-parametric functional (Lemma 4.3). One then approximates u with a solution of the Euler-Lagrange equation it approximately solves, transfers to u the estimates that the exact solution enjoys by elliptic regularity theory, and then reads these estimates on the boundary of E (Lemma 4.6). An iteration of this scheme leads to prove that, at a sufficiently small scale, u covers *all* of the boundary of E, and that it is actually of class $C^{1,1/2}$ by a classical integral criterion for hölderianity due to Campanato. For ease of presentation, we dedicate a separate section to each step of this long argument. Recall that thorough the proof, the normalization conventions of Lemma 2.16 are in force. In particular, we always have $A\cap\partial E=\mathrm{spt}\mu_E$.

4.1. Height bound.

We start with the height bound.

Lemma 4.1 (Height bound)**.** *For every $\lambda\ge1$ and $\sigma\in(0,1/4)$ there exists a positive constant $\varepsilon_{\rm hb}=\varepsilon_{\rm hb}(n,\lambda,\sigma)$ with the following property. If $H=\{x_1>b\}$ for some $b\in\mathbb{R}$, $x_0\in\mathrm{cl}(H)$, and*

$$\Phi\in\boldsymbol{\mathcal{E}}(\mathbf{C}_{x_0,4\,r}\cap H,\lambda,\ell)\,,$$
$$E\textit{ is a }(\Lambda,r_0)\textit{-minimizer of }\mathbf{\Phi}\textit{ in }(\mathbf{C}_{x_0,4\,r},H)\textit{ with }0<r\le r_0\,,$$
$$x_0\in\mathrm{cl}(H\cap\partial E)\,,\tag{4.1}$$
$$(2\lambda\Lambda+\ell)\,r\le1\,,\tag{4.2}$$
$$\mathbf{exc}_n^H(E,x_0,2\,r)<\varepsilon_{\rm hb}\,,\tag{4.3}$$

then

$$\sup\left\{\frac{|\mathbf{q}(x-x_0)|}{r}:x\in\mathbf{C}_{x_0,r_0}\cap H\cap\partial E\right\}\le\sigma\,,\tag{4.4}$$

$$\Big|\Big\{x\in\mathbf{C}_{x_0,r}\cap H\cap E:\mathbf{q}(x-x_0)>\sigma\,r\Big\}\Big|=0\,,\tag{4.5}$$

$$\Big|\Big\{x\in(\mathbf{C}_{x_0,r}\cap H)\setminus E:\mathbf{q}(x-x_0)<-\sigma\,r\Big\}\Big|=0\,.\tag{4.6}$$

Moreover, the identity

$$\zeta(G)=P(E;\mathbf{p}^{-1}(G)\cap\mathbf{C}_{x_0,r}\cap H)-\mathcal{H}^{n-1}(G\cap H)\,,\qquad G\subset\mathbf{D}_{\mathbf{p}x_0,r}\,,$$

defines a finite Radon measure ζ *on* $\mathbf{D}_{\mathbf{p}x_0,r}$ *concentrated on* $H\cap\mathbf{D}_{\mathbf{p}x_0,r}$ *and such that*

$$\zeta(\mathbf{D}_{\mathbf{p}x_0,r})=r^{n-1}\,\mathbf{exc}_n^H(E,x_0,r)\,.\tag{4.7}$$

Proof of Lemma 4.1. The fact that ζ is a positive Radon measure and satisfies (4.7) follows by (4.4), (4.5) (4.6) and Lemma 4.2 below. We thus focus on the proof of these three properties. By (3.1) and by Remark 2.2, up to replace E and Φ with $E^{x_0,r}$ and $\Phi^{x_0,r}$ respectively, we may directly assume that $x_0=0$, $r=1$ and that $H=\{x_1>-t\}$ with $t\ge0$. Arguing by contradiction, we thus assume the existence of $\lambda\ge1$ and $\sigma\in(0,1/4)$ such that for every $h\in\mathbb{N}$ there exist a half-space $H_h=\{x_1>-t_h\}$ ($t_h\ge0$), and

$$\begin{aligned}&\Phi_h\in\boldsymbol{\mathcal{E}}(\mathbf{C}_4\cap H_h,\lambda,\ell_h)\,,\\&E_h\text{ a }(\Lambda_h,1)\text{-minimizer of }\boldsymbol{\Phi}_h\text{ in }(\mathbf{C}_4,H_h)\\&0\in\mathrm{cl}(H_h\cap\partial E_h)\,,\\&2\lambda\Lambda_h+\ell_h\le1\,,\end{aligned}$$

such that $\mathbf{exc}_n^{H_h}(E_h,0,2)\to0$ as $h\to\infty$, and

$$\text{either}\qquad\sup\Big\{|\mathbf{q}x|:x\in\mathbf{C}\cap H_h\cap\partial E_h\Big\}>\sigma\,,\tag{4.8}$$

$$\text{or}\qquad\Big|\Big\{x\in\mathbf{C}\cap H_h\cap\partial E_h:\mathbf{q}x>\sigma\Big\}\Big|>0\,,\tag{4.9}$$

$$\text{or}\qquad\Big|\Big\{x\in(\mathbf{C}\cap H_h)\setminus E_h:\mathbf{q}x<-\sigma\Big\}\Big|>0\,,\tag{4.10}$$

for infinitely many $h\in\mathbb{N}$.

Step one: We start showing that (4.8) cannot hold for infinitely many $h\in\mathbb{N}$. To this end, we set $H_0=\{x_1>0\}$, and notice that

$$H_0\subset H_h=H_0-t_h\,e_1\,,\qquad\forall h\in\mathbb{N}\,.$$

In order to apply the compactness theorem, Theorem 2.9, we need to get rid of the moving half-spaces H_h. Since $t_h\ge0$ for every $h\in\mathbb{N}$, up to extracting subsequences, we may assume that $t_h\to t_*\in[0,\infty]$ as $h\to\infty$. We then consider two cases separately:

Case one: We have $t_*\in[0,5]$. Set $F_h=E_h+t_h\,e_1$ and $\Psi_h(x,\nu)=\Phi_h(x-t_he_1,\nu)$ so that $\Psi_h\in\boldsymbol{\mathcal{E}}(\mathbf{C}(t_h\,e_1,4)\cap H,\lambda,1)$. Since $H_0=H_h+t_h\,e_1$, and, for h large enough, $\mathbf{C}(t_*e_1,3)\subset\subset\mathbf{C}(t_he_1,4)$, we find that F_h is a $(1/2\lambda,1)$-minimizer of $\boldsymbol{\Psi}_h$ on $(\mathbf{C}(t_*e_1,3),H)$ (recall that $2\lambda\Lambda_h\le1$), with $t_h\,e_1\in\mathrm{cl}(H\cap\partial F_h)$ and (up to extracting a subsequence)

$$\lim_{h\to\infty}\mathbf{exc}_n^H(F_h,t_h\,e_1,2)=0\,,$$

$$\sup\Big\{|\mathbf{q}x|:x\in\mathbf{C}(t_h\,e_1,1)\cap H\cap\partial F_h\Big\}\ge\sigma\,,\qquad\forall h\in\mathbb{N}\,.\tag{4.11}$$

By Theorem 2.9, we can find $\Psi_\infty \in \boldsymbol{\mathcal{E}}(\mathbf{C}(t_* e_1, 3) \cap H, \lambda, 1)$ and $F_\infty \subset H$ such that F_∞ is a $(1/2\lambda, 1)$-minimizer of $\boldsymbol{\Psi}_\infty$ on $(\mathbf{C}(t_* e_1, 3), H)$ with $t_* e_1 \in \mathrm{cl}(H \cap \partial F_\infty)$. By (3.9),

$$\begin{aligned}\mathbf{exc}_n^H(F_\infty, t_* e_1, 3/2) &\le \liminf_{h\to\infty} \mathbf{exc}_n^H(F_h, t_* e_1, 3/2)\\ &\le \Big(\frac{4}{3}\Big)^{n-1} \liminf_{h\to\infty} \mathbf{exc}_n^H\Big(F_h, t_h e_1, 2\Big) = 0\,.\end{aligned}$$

Thus, by (3.3),

$$F_\infty \cap \mathbf{C}(t_* e_1, 3/2) \cap H = \Big\{x \in \mathbf{C}(t_* e_1, 3/2) \cap H : \mathbf{q}x < 0\Big\}\,. \tag{4.12}$$

At the same time, by (4.11), for every $h \in \mathbb{N}$ we can find $z_h \in \mathbf{C}(t_h\, e_1, 1) \cap H \cap \partial F_h$ with $|\mathbf{q} z_h| > \sigma$. In particular, up to extracting subsequences and by (2.61), $z_h \to z_0$ for some $z_0 \in \mathrm{cl}(\mathbf{C}(t_*\, e_1, 1) \cap H \cap \partial F_\infty)$ with $|\mathbf{q} z_0| \ge \sigma$. By (4.12), it must be $P(F_\infty; B_{z_0,s}) = 0$ for s small enough: hence, $z_0 \notin \partial F_\infty$, contradiction.

Case two: We have $t_* > 5$. In this case the presence of H_h is not detected by the minimality condition of E_h in $\mathbf{C}_4$, so that E_h turns out to be a $(1/2\lambda, 1)$-minimizer of $\boldsymbol{\Phi}_h$ in $(\mathbf{C}_4, \mathbb{R}^n)$. This time we apply Theorem 2.9 with the degenerate half-space $\mathbb{R}^n$ and we find a contradiction with (4.8) by the same argument used in dealing with case one.

Step two: We now prove that neither (4.9) nor (4.10) can hold for infinitely many h. Once again, we argue by contradiction, and assume for example that (4.9) holds true for infinitely many values of h. Since we know by step one that

$$\mathbf{C} \cap H_h \cap \partial E_h \subset \mathbf{D} \times [-\sigma, \sigma]\,, \qquad \forall h \in \mathbb{N}\,,$$

this assumption, combined with basic properties of the distributional derivative, implies that

$$\mathbf{C} \cap H_h \cap E_h \cap \{\mathbf{q}x > \sigma\} = \mathbf{C} \cap H_h \cap \{\mathbf{q}x > \sigma\}\,, \qquad \forall h \in \mathbb{N}\,.$$

By exploiting the compactness theorem for almost-minimizers as in step one, we see however that E_h is converging to $\{\mathbf{q}x < 0\}$ (up to horizontal translations and inside of $\mathbf{C}$), thus reaching a contradiction. □

The following lemma can be proved by a simple application of the divergence theorem to vector fields of the form $\varphi(\mathbf{p}x)\, e_n$ for $\varphi \in C^1_c(\mathbf{D}_{x_0,r})$. We refer to [Mag12, Lemma 22.11], and leave the details to the reader.

Lemma 4.2. *If $H = \{x_1 > b\}$ for some $b \in \mathbb{R}$ and $E \subset H$ is a set of finite perimeter with*

$$|\mathbf{q}x| < \sigma_0\,, \qquad \forall x \in \mathbf{C} \cap H \cap \mathrm{spt}\mu_E\,,$$
$$\Big\{x \in \mathbf{C} \cap H : \mathbf{q}x < -\sigma_0\Big\} \subset E \cap \mathbf{C} \cap H \subset \Big\{x \in \mathbf{C} \cap H : \mathbf{q}x < \sigma_0\Big\}\,,$$

for some $\sigma_0 \in (0,1)$, then the set function

$$\zeta(G) = P(E; \mathbf{p}^{-1}(G) \cap \mathbf{C} \cap H) - \mathcal{H}^{n-1}(G \cap H)\,, \qquad G \subset \mathbf{D}\,,$$

defines a positive finite Radon measure on $\mathbf{D}$ concentrated on H with

$$\zeta(G) = \int_{\mathbf{p}^{-1}(G) \cap \mathbf{C} \cap H \cap \partial^* E} \frac{|\nu_E - e_n|^2}{2}\, d\mathcal{H}^{n-1}\,, \qquad \forall G \subset \mathbf{D}\,.$$

In particular, $\zeta(\mathbf{D}) = \mathbf{exc}_n^H(E, 0, 1)$.

4.2. **Lipschitz approximation.** The next key ingredient in the proof of Lemma 3.4 is the construction of a Lipschitz approximation of ∂E.

Lemma 4.3 (Lipschitz approximation)**.** *For every $\lambda \ge 1$ and $\sigma \in (0,1/4)$ there exist positive constants $\varepsilon_{\rm lip} = \varepsilon_{\rm lip}(n,\lambda,\sigma)$, $C_2 = C_2(n,\lambda)$, and $\delta_1 = \delta_1(n,\lambda)$ with the following property. If $H = \{x_1 > b\}$ for some $b \in \mathbb{R}$, $x_0 \in {\rm cl}(H)$, and*

$$\begin{aligned}
&\Phi \in \boldsymbol{\mathcal{E}}(\mathbf{C}_{x_0,16\,r} \cap H, \lambda, \ell)\,,\\
&E \text{ is a } (\Lambda, r_0)\text{-minimizer of } \mathbf{\Phi} \text{ in } (\mathbf{C}_{x_0,16\,r}, H) \text{ with } 0 < 4r \le r_0\,,\\
&x_0 \in {\rm cl}(H \cap {\rm spt}\,\partial E)\ \ ,\\
&(8\lambda\Lambda + 4\ell)\,r \le 1\,,\\
&\mathbf{exc}_n^H(E, x_0, 8r) < \varepsilon_{\rm lip}\,,
\end{aligned}$$

then there exists a Lipschitz function $u : \mathbb{R}^{n-1} \to \mathbb{R}$ such that, on setting,

$$\begin{aligned}
M &= \mathbf{C}_{x_0,r} \cap H \cap {\rm spt}\partial E\,,\\
M_0 &= \Big\{ y \in M : \sup_{0<s<4r} \mathbf{exc}_n^H(E, y, s) \le \delta_1 \Big\}\,,\\
\Gamma &= \Big\{ (z, u(z)) : z \in \mathbf{D}_{\mathbf{p}x_0,r} \cap H \Big\}\,,
\end{aligned}$$

we have

$$\sup_{\mathbb{R}^{n-1}} \frac{|u - \mathbf{q}x_0|}{r} \le \sigma\,, \qquad {\rm Lip}(u) \le 1\,, \qquad M_0 \subset M \cap \Gamma\,, \tag{4.13}$$

and

$$\frac{\mathcal{H}^{n-1}(M \Delta \Gamma)}{r^{n-1}} \le C_2\, \mathbf{exc}_n^H(E, x_0, 8\,r)\,, \tag{4.14}$$

$$\frac{1}{r^{n-1}} \int_{\mathbf{D}_{\mathbf{p}x_0,r} \cap H} |\nabla u|^2 \le C_2\, \mathbf{exc}_n^H(E, x_0, 8\,r)\,. \tag{4.15}$$

Finally, if

$$\text{either} \qquad x_0 \in \partial H \text{ and } \nabla\Phi(x_0, e_n) \cdot e_1 = 0\,, \tag{4.16}$$

$$\text{or} \qquad {\rm dist}(x_0, \partial H) > r\,, \tag{4.17}$$

then we also have that

$$\frac{1}{r^{n-1}} \int_{\mathbf{D}_{\mathbf{p}x_0,r} \cap H} \Big(\nabla^2 \Phi(x_0, e_n)(\nabla u, 0) \Big) \cdot (\nabla \varphi, 0) \le C_2\, \|\nabla \varphi\|_\infty \Big(\mathbf{exc}_n^H(E, x_0, 8\,r) + (\Lambda + \ell)\,r \Big)\,, \tag{4.18}$$

for every $\varphi \in C^1(\mathbf{D}_{\mathbf{p}x_0,r})$ with $\varphi = 0$ on $H \cap \partial \mathbf{D}_{\mathbf{p}x_0,r}$. (Notice that this implies $\varphi = 0$ on $\partial \mathbf{D}_{\mathbf{p}x_0,r}$ when (4.17) holds true.)

Proof. Step one: By (3.1) and Remark 2.2 we may directly assume that $x_0 = 0$, $r = 1$ and $H = \{x_1 > -t\}$ with $t \ge 0$. With $\varepsilon_{\rm hb}(n,\lambda,\sigma)$ defined as in Lemma 4.1, we shall assume that

$$\varepsilon_{\rm lip}(n,\lambda,\sigma) \le \varepsilon_{\rm hb}(n,\lambda,\sigma)\,. \tag{4.19}$$

Since E is a $(\Lambda, 4)$-minimizer of $\mathbf{\Phi}$ in $(\mathbf{C}_{16}, H)$, with $\Phi \in \boldsymbol{\mathcal{E}}(\mathbf{C}_{16} \cap H, \lambda, \ell)$, $0 \in {\rm cl}(H \cap \partial E)$, $\mathbf{exc}_n^H(E, 0, 8) < \varepsilon_{\rm lip}$, and $8\,\lambda\Lambda + 4\,\ell \le 1$, by (4.19) we can apply Lemma 4.1 to get that

$$\sup \Big\{ |\mathbf{q}y| : y \in \mathbf{C}_4 \cap H \cap \partial E \Big\} \le \sigma\,; \tag{4.20}$$

moreover,

$$\zeta(G) = P(E; \mathbf{p}^{-1}(G) \cap \mathbf{C}_4 \cap H) - \mathcal{H}^{n-1}(G \cap \mathbf{D}_4 \cap H)\,, \quad G \subset \mathbf{D}_4\,,$$

defines a positive finite Radon measure ζ on $\mathbf{D}_4\cap H$. We now notice that if $y\in M_0$, $x\in M$ and $s=\max\{|\mathbf{p}(x-y)|,|\mathbf{q}(x-y)|\}$, then $s<2$ and by definition of M_0, we have

$$\mathbf{C}_{y,4s}\subset\mathbf{C}_{16}\,,\qquad \mathbf{exc}_n^H(E,y,2s)\le\delta_1\,,\qquad (2\lambda\Lambda+\ell)s\le 1\,.$$

Up to assume that

$$\delta_1(n,\lambda)<\varepsilon_{\rm hb}\Big(n,\lambda,\frac{1}{8}\Big)\,,$$

and since, by construction, $x\in{\rm cl}(\mathbf{C}_{y,s}\cap\partial E)\cap H$, we can thus apply Lemma 4.1 at the point y at scale s to infer

$$|\mathbf{q}y-\mathbf{q}x|\le\frac{s}{8}\,,$$

which in turn implies

$$|\mathbf{q}x-\mathbf{q}y|\le\frac{|\mathbf{p}x-\mathbf{p}y|}{8}\,.$$

In particular, $\mathbf{p}$ is invertible on M_0, so that we can define a function $u:\mathbf{p}(M_0)\to\mathbb{R}$ with the property that $u(\mathbf{p}x)=\mathbf{q}x$ for every $x\in M_0$ (thus $|u(z)|<\sigma$ for every $z\in\mathbf{p}(M_0)$ by (4.20)), and

$$|u(\mathbf{p}y)-u(\mathbf{p}x)|\le\frac{|\mathbf{p}y-\mathbf{p}x|}{8}\,,\qquad\forall x,y\in M_0\,.$$

We may thus extend u as a Lipschitz function on $\mathbb{R}^{n-1}$ with the properties that

$$\sup_{\mathbb{R}^{n-1}}|u|\le\sigma\,,\qquad {\rm Lip}(u)\le\frac{1}{8}\,,\qquad M_0\subset\Gamma=\Big\{(z,u(z)):z\in\mathbf{D}\cap H\Big\}\,.$$

This proves (4.13). We now prove (4.14) by a standard application of Besicovitch's covering theorem (that we detail just for the sake of completeness). We start noticing that, if $x\in M\setminus M_0$, then there exists $s_x\in(0,4)$ such that

$$\delta_1\,s_x^{n-1}\le\int_{H\cap\mathbf{C}_{x,s_x}\cap\partial^*E}\frac{|\nu_E-e_n|^2}{2}\,d\mathcal{H}^{n-1}\,.$$

If $\xi(n)$ is the Besicovitch covering constant, then (see for instance [Mag12, Corollary 5.2]) we can find a countable disjoint family of balls $B(x_h,\sqrt{2}\,s_h)$ with $x_h\in M\setminus M_0$, $s_h=s_{x_h}\in(0,4)$,

$$\delta_1\,s_h^{n-1}\le\int_{H\cap\mathbf{C}_{x_h,s_h}\cap\partial^*E}\frac{|\nu_E-e_n|^2}{2}\,d\mathcal{H}^{n-1}\,,\qquad\forall h\in\mathbb{N}\,,$$

and

$$\mathcal{H}^{n-1}(M\setminus M_0)\le\xi(n)\sum_{h\in\mathbb{N}}\mathcal{H}^{n-1}\Big((M\setminus M_0)\cap B(x_h,\sqrt{2}\,s_h)\Big)\,.$$

By (2.47), $\mathcal{H}^{n-1}(M\cap B(x_h,\sqrt{2}\,s_h))\le C(n,\lambda)\,s_h^{n-1}$ for every $h\in\mathbb{N}$, so that, by combining these last three inequalities,

$$\begin{aligned}\mathcal{H}^{n-1}(M\setminus M_0)&\le\frac{C(n,\lambda)}{\delta_1}\sum_{h\in\mathbb{N}}\int_{H\cap\mathbf{C}_{x_h,s_h}\cap\partial^*E}\frac{|\nu_E-e_n|^2}{2}\,d\mathcal{H}^{n-1}\\&=\frac{C(n,\lambda)}{\delta_1}\int_{H\cap\mathbf{C}_8\cap\partial^*E}\frac{|\nu_E-e_n|^2}{2}\,d\mathcal{H}^{n-1}\,,\end{aligned}$$

where in the last identity we have used the fact that the cylinders $\mathbf{C}_{x_h,s_h}$ are disjoint (as they are contained in the disjoint balls $B(x_h,\sqrt{2}\,s_h)$), as well as the fact that their union is contained in $\mathbf{C}_8$. Recalling that $M\setminus\Gamma\subset M\setminus M_0$, we have proved that

$$\mathcal{H}^{n-1}(M\setminus\Gamma)\le C(n,\lambda)\,\mathbf{exc}_n^H(E,0,8)\,.$$

The proof of (4.14) is then completed by noticing that, since $\mathrm{Lip}(u)\le 1$,

$$\begin{aligned}\mathcal{H}^{n-1}(\Gamma\setminus M)&\le\sqrt{2}\mathcal{H}^{n-1}(\mathbf{p}(\Gamma\setminus M))\\&\le\sqrt{2}\mathcal{H}^{n-1}(M\cap\mathbf{p}^{-1}\mathbf{p}(\Gamma\setminus M))\le\sqrt{2}\mathcal{H}^{n-1}(M\setminus\Gamma)\,,\end{aligned}$$

where in the last inequality we have used that

$$0\le\zeta(\mathbf{p}(\Gamma\setminus M))=\mathcal{H}^{n-1}(M\cap\mathbf{p}^{-1}\mathbf{p}(\Gamma\setminus M))-\mathcal{H}^{n-1}(\mathbf{p}(\Gamma\setminus M)),$$

and that $M\cap\mathbf{p}^{-1}\mathbf{p}(\Gamma\setminus M)\subset M\setminus\Gamma$. We finally prove (4.15). We first notice that

$$\nu_E(x)=\pm\frac{(-\nabla u(\mathbf{p}x),1)}{\sqrt{1+|\nabla u(\mathbf{p}x)|^2}}\,,\qquad\text{for }\mathcal{H}^{n-1}\text{-a.e. }x\in M\cap\Gamma\,.\tag{4.21}$$

Since $|\nu_E-e_n|^2\ge|\mathbf{p}\nu_E|^2$ and $\mathrm{Lip}(u)\le 1$, by the area formula and by (4.21) we get

$$\begin{aligned}8^{n-1}\mathbf{exc}_n^H(E,0,8)&\ge\frac{1}{2}\int_{M\cap\Gamma}|\mathbf{p}\nu_E|^2=\frac{1}{2}\int_{M\cap\Gamma}\frac{|\nabla u(\mathbf{p}x)|^2}{1+|\nabla u(\mathbf{p}x)|^2}\,d\mathcal{H}^{n-1}\\&=\frac{1}{2}\int_{\mathbf{p}(M\cap\Gamma)}\frac{|\nabla u|^2}{\sqrt{1+|\nabla u|^2}}\ge\frac{1}{2\sqrt{2}}\int_{\mathbf{p}(M\cap\Gamma)}|\nabla u|^2\,,\end{aligned}$$

as well as $\int_{\mathbf{p}(M\Delta\Gamma)}|\nabla u|^2\le\mathcal{H}^{n-1}(\mathbf{p}(M\Delta\Gamma))\le\mathcal{H}^{n-1}(M\Delta\Gamma)$, so that (4.15) follows from (4.14). We now devote the next two steps of the proof to show the validity of (4.18).

Step two: We start showing that, setting $\Phi_0(\nu)=\Phi(0,\nu)$,

$$\Big|\int_{\mathbf{C}\cap H\cap\partial^*E}\nabla\Phi_0(\nu_E)\cdot(\nabla\varphi(\mathbf{p}x,0),0)\,(\nu_E\cdot e_n)\,d\mathcal{H}^{n-1}\Big|\le C\,\|\nabla\varphi\|_\infty\,(\Lambda+\ell)\,,\tag{4.22}$$

whenever $\varphi\in C^1(\mathbf{D}\cap H)$ with $\varphi=0$ on $\partial\mathbf{D}\cap H$. In showing this, we can assume without loss of generality that $\|\nabla\varphi\|_\infty=1$, and notice that, correspondingly, $\sup_{\mathbf{D}\cap H_*}|\varphi|\le 1$. Let us now fix $\alpha\in C^\infty_c([-1,1];[0,1])$ with $\alpha=1$ on $[0,1/2]$ and $|\alpha'|\le 3$, and consider the family of maps

$$f_t(x)=x+t\,\alpha(\mathbf{q}x)\,\varphi(\mathbf{p}x)\,e_n\,,\qquad x\in\mathbb{R}^n\,.$$

For t small enough, f_t is a diffeomorphisms of $\mathbb{R}^n$, with

$$f_t(E)\subset H,\qquad f_t(E)\Delta E\subset\subset\mathbf{C}_2\,,\qquad|f_t(E)\Delta E|\le C(n)\,|t|\,P(E;\mathbf{C}_2)\,,$$

(see [Mag12, Lemma 17.9] for the last inequality). Hence, by minimality of E and by (2.47),

$$\begin{aligned}\mathbf{\Phi}(E,H\cap\mathbf{C}_2)&\le\mathbf{\Phi}(f_t(E),H\cap\mathbf{C}_2)+\Lambda\,|f_t(E)\Delta E|\\&\le\mathbf{\Phi}(f_t(E),H\cap\mathbf{C}_2)+C(n,\lambda)\,\Lambda\,|t|\,.\end{aligned}\tag{4.23}$$

By (2.9), (1.8), $|f_t(x)-x|\le C\,|t|$, and, again, by (2.47), we find

$$\begin{aligned}\mathbf{\Phi}(f_t(E),H\cap\mathbf{C}_2)&=\int_{H\cap\mathbf{C}_2\cap\partial^*E}\Phi\big(f_t(x),(\mathrm{cof}\,\nabla f_t(x))\nu_E(x)\big)d\mathcal{H}^{n-1}\\&\le\int_{H\cap\mathbf{C}_2\cap\partial^*E}\Phi\big(x,(\mathrm{cof}\,\nabla f_t(x))\nu_E(x)\big)d\mathcal{H}^{n-1}+C(n,\lambda)\,\ell|t|\,.\end{aligned}\tag{4.24}$$

By (2.38), $(\mathrm{cof}\,\nabla f_t)\nu_E=\nu_E+t\,((\mathrm{div}\,T)\,\nu_E-(\nabla T)^*\nu_E)$, with $T(x)=\alpha(\mathbf{q}x)\,\varphi(\mathbf{p}x)\,e_n$. Since $\alpha(\mathbf{q}x)\equiv 1$ for x in a neighborhood of ∂^*E, we have

$$(\mathrm{cof}\,\nabla f_t)\nu_E=\nu_E-t\,(e_n\cdot\nu_E)\,(\nabla\varphi,0)\qquad\text{on }\partial^*E.$$

Thus, by (4.24),

$$\begin{aligned}\mathbf{\Phi}(f_t(E), H\cap\mathbf{C}_2) - \mathbf{\Phi}(E, H\cap\mathbf{C}_2) & \\ &\le \int_{H\cap\mathbf{C}_2\cap\partial^*E}\Big[\Phi(x,(\mathrm{cof}\,\nabla f_t)\nu_E) - \Phi(x,\nu_E)\Big]d\mathcal{H}^{n-1} + C(n,\lambda)\,\ell|t| \\ &\le -t\int_{H\cap\mathbf{C}_2\cap\partial^*E}\nabla\Phi(x,\nu_E)\cdot(\nabla\phi,0)(e_n\cdot\nu_E)d\mathcal{H}^{n-1} + C(n,\lambda)\Big(\ell\,|t| + t^2\Big) \\ &\le -t\int_{H\cap\mathbf{C}_2\cap\partial^*E}\nabla\Phi_0(\nu_E)\cdot(\nabla\varphi,0)\,(\nu_E\cdot e_n)\,d\mathcal{H}^{n-1} + C(n,\lambda)\Big(\ell\,|t| + t^2\Big)\,,\end{aligned} \tag{4.25}$$

where in the second inequality we have used (1.9) and that $P(E,\mathbf{C}_2)\le C(n,\lambda)$ (by (2.47)) while in the third one we have used (1.8) (and again that $P(E,\mathbf{C}_2)\le C(n,\lambda)$). We combine (4.23) and (4.25) to find that

$$\Big|\int_{H\cap\mathbf{C}_2\cap\partial^*E}\nabla\Phi_0(\nu_E)\cdot(\nabla\varphi,0)\,(\nu_E\cdot e_n)\,d\mathcal{H}^{n-1}\Big| \le C(n,\lambda)\,\Big(|t| + (\Lambda+\ell)\Big)\,.$$

We prove (4.22) by letting choosing $t\to 0$.

Step three: We conclude the proof of (4.18). We start by showing that

$$\Big|\int_{\mathbf{D}\cap H}\nabla\Phi_0((-\nabla u(z),1))\cdot(\nabla\varphi(z),0)\,dz\Big| \le C\,\|\nabla\varphi\|_\infty\,\Big(\mathbf{exc}_n^H(E,0,8) + (\Lambda+\ell)\Big)\,, \tag{4.26}$$

whenever $\varphi\in C^1(\mathbf{D}\cap H)$ with $\varphi=0$ on $H\cap\partial\mathbf{D}$. Indeed, let us set

$$\Gamma_1 = \Big\{x\in\Gamma : \nu_E(x) = \frac{(-\nabla u(\mathbf{p}x),1)}{\sqrt{1+|\nabla u(\mathbf{p}x)|^2}}\Big\}\subset\Gamma\,.$$

By (4.21), if $x\in\Gamma\setminus\Gamma_1$, then $|\nu_E - e_n|^2\ge 1$. Hence, by (4.14),

$$\mathcal{H}^{n-1}(M\Delta\Gamma_1)\le 8^{n-1}\mathbf{exc}_n^H(E,0,8) + \mathcal{H}^{n-1}(M\Delta\Gamma)\le C\,\mathbf{exc}_n^H(E,0,8)\,, \tag{4.27}$$

and thus, by (4.22),

$$\Big|\int_{H\cap\Gamma_1}\nabla\Phi_0(\nu_E)\cdot(\nabla\varphi(\mathbf{p}x),0)\,(\nu_E\cdot e_n)\,d\mathcal{H}^{n-1}\Big| \le C\,\|\nabla\varphi\|_\infty\,\Big(\mathbf{exc}_n^H(E,0,8) + (\Lambda+\ell)\Big)\,, \tag{4.28}$$

where $C=C(n,\lambda)$. By taking into account the definition of Γ_1 and the formula for the area of a graph of a Lipschitz function we thus find

$$\Big|\int_{H\cap\mathbf{p}(\Gamma_1)}\frac{\nabla\Phi_0((-\nabla u,1))\cdot(\nabla\varphi,0)}{\sqrt{1+|\nabla u|^2}}\Big| \le C\,\|\nabla\varphi\|_\infty\,\Big(\mathbf{exc}_n^H(E,0,8) + (\Lambda+\ell)\Big)\,.$$

By (1.9), for every $G\subset\mathbf{D}\cap H$, we have

$$\Big|\int_G\frac{\nabla\Phi_0((-\nabla u,1))\cdot(\nabla\varphi,0)}{\sqrt{1+|\nabla u|^2}} - \int_G\nabla\Phi_0((-\nabla u,1))\cdot(\nabla\varphi,0)\Big| \le C(n,\lambda)\,\|\nabla\varphi\|_\infty\int_G|\nabla u|^2\,,$$

and thus by (4.15) we find

$$\Big|\int_{H\cap\mathbf{p}(\Gamma_1)}\nabla\Phi_0((-\nabla u,1))\cdot(\nabla\varphi,0)\Big| \le C\,\|\nabla\varphi\|_\infty\,\Big(\mathbf{exc}_n^H(E,0,8) + (\Lambda+\ell)\Big)\,. \tag{4.29}$$

At the same time, by (1.9) and again by (4.27) we have

$$\begin{aligned}\Big|\int_{H\cap(\mathbf{D}\Delta\mathbf{p}(\Gamma_1))}\nabla\Phi_0((-\nabla u,1))\cdot(\nabla\varphi,0)\Big| &\le \lambda\,\|\nabla\varphi\|_\infty\,\mathcal{H}^{n-1}(M\Delta\Gamma_1) \\ &\le C\,\|\nabla\varphi\|_\infty\,\mathbf{exc}_n^H(E,0,8)\,.\end{aligned}$$

We combine this last inequality with (4.29) to prove (4.26). We now notice that, by (1.9),

$$|\nabla\Phi_0(-\nabla u,1)-\nabla\Phi_0(e_n)-\nabla^2\Phi_0(e_n)\cdot(-\nabla u,0)|\le C(\lambda)\,|\nabla u|^2\,.$$

This, together with (4.15) and (4.26), gives

$$\Big|\int_{\mathbf{D}\cap H}\Big(\nabla\Phi_0(e_n)+\nabla^2\Phi_0(e_n)(-\nabla u,0)\Big)\cdot(\nabla\varphi,0)\Big|\le C\,\|\nabla\varphi\|_\infty\,\Big(\mathbf{exc}_n^H(E,0,8)+(\Lambda+\ell)\Big)\,. \tag{4.30}$$

We finally notice that, by Gauss–Green theorem,

$$\int_{\mathbf{D}\cap H}\nabla\Phi_0(e_n)\cdot(\nabla\varphi,0)=\nabla\Phi_0(e_n)\cdot\int_{H\cap\partial\mathbf{D}}\varphi\,\nu_{\mathbf{D}}-(\nabla\Phi_0(e_n)\cdot e_1)\int_{\mathbf{D}\cap\partial H}\varphi\ ,$$

where the first term vanishes as $\varphi=0$ on $H\cap\partial\mathbf{D}$, and the second term vanishes as either (4.16) is in force (and thus $\nabla\Phi_0(e_n)\cdot e_1=0$ by assumption) or (4.17) holds true, and then one simply has $\mathbf{D}\cap\partial H=\emptyset$. This completes the proof of (4.18), thus of the lemma. □

4.3. **Caccioppoli inequality.** The third tool used in the proof of Lemma 3.4 is the *Caccioppoli inequality* of Lemma 4.4 below. This result, also known as reverse Poincaré inequality, is morally analogous to its well-known counterpart in elliptic regularity theory, and shall be used here to translate decay estimates for the flatness of almost-minimizers into decay estimates for their excess. Here, given an open set A and an open half-space H in $\mathbb{R}^n$, a set $E\subset H$ of locally finite perimeter in A, and $x\in A\cap\mathrm{cl}(H)$, $\nu\in\mathbf{S}^{n-1}$ and $r>0$ such that $\mathbf{C}_\nu(x,r)\subset\subset A$, we define the *flatness of E at x, at scale r, in the direction ν, relative to H* as

$$\mathbf{flat}_\nu^H(E,x,r)=\inf_{c\in\mathbb{R}}\frac{1}{r^{n-1}}\int_{H\cap\mathbf{C}_\nu(x,r)\cap\partial^*E}\frac{|(y-x)\cdot\nu-c|^2}{r^2}\,d\mathcal{H}^{n-1}\,.$$

As usual, we set $\mathbf{flat}_\nu^H(E,x,r)=\mathbf{flat}_n^H(E,x,r)$ when $\nu=e_n$, and notice that flatness enjoys analogous scaling properties to the one of excess, see (3.1).

Lemma 4.4 (Caccioppoli inequality). *For every $\lambda\ge1$ there exist positive constants $\varepsilon_{\rm Ca}=\varepsilon_{\rm Ca}(n,\lambda)$ and $C_3=C_3(n,\lambda)$ with the following property. If $H=\{x_1>b\}$ for some $b\in\mathbb{R}$, $x_0\in\mathrm{cl}(H)$, and*

$$\begin{aligned}
&\Phi\in\boldsymbol{\mathcal{E}}(\mathbf{C}_{x_0,16\,r}\cap H,\lambda,\ell)\,,\\
&E\text{ is a }(\Lambda,r_0)\text{-minimizer of }\mathbf{\Phi}\text{ in }(\mathbf{C}_{x_0,16\,r},H)\text{ with }0<8r\le r_0\,,\\
&x_0\in\mathrm{cl}(H\cap\partial E)\ \ ,\\
&(16\,\lambda\Lambda+8\ell)\,r\le1\,,\\
&\mathbf{exc}_n^H(E,x_0,8r)<\varepsilon_{\rm Ca}\,,
\end{aligned}$$

with

$$\text{either}\qquad x_0\in\partial H\text{ and }\nabla\Phi(x_0,e_n)\cdot e_1=0\,, \tag{4.31}$$

$$\text{or}\qquad \mathrm{dist}(x_0,\partial H)>16\,r\,, \tag{4.32}$$

then

$$\mathbf{exc}_n^H(E,x_0,r)\le C_3\,\Big(\mathbf{flat}_n^H(E,x_0,4r)+(\Lambda+\ell)\,r\Big)\,. \tag{4.33}$$

The proof is based on the construction of "interior" and "exterior" competitors in arbitrary cylinders, that is detailed in Lemma 4.5 below, and originates from [Alm68]; see also [Bom82, Section V] and [DS02, Section 4]. We shall need the following terminology and notation: first, we shall say that $E\subset\mathbb{R}^n$ is a polyhedron if E is open and ∂E is contained in finitely many hyperplanes (in this case, $\mathcal{H}^{n-2}(\partial E\setminus\partial^*E)=0$, and $\nu_E(x)$ agrees with the elementarily defined outer unit normal to E at every $x\in\partial^*E$); second, given $z\in\mathbb{R}^{n-1}$ and $r>0$, we shall set

$$\mathbf{K}_{z,r}=\Big\{x\in\mathbb{R}^n:|\mathbf{p}x-z|<r\,,|\mathbf{q}x|<1\Big\}\,. \tag{4.34}$$

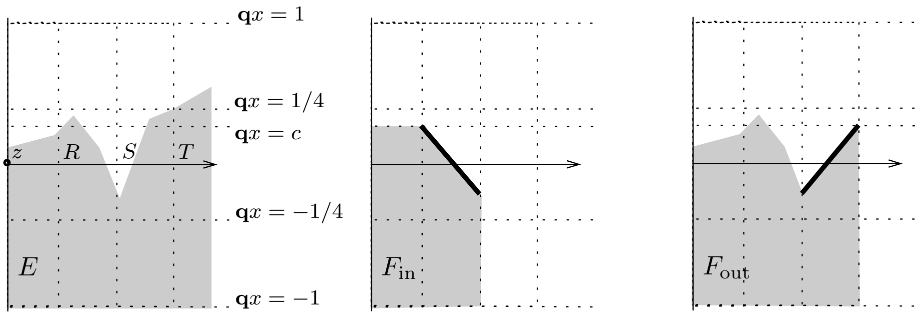

FIGURE 4.1. The construction of the the competitors $F_{\rm in}$ and $F_{\rm out}$ in Lemma 4.5. The picture is relative to the case when $z\in\partial H$. The bold lines represent the surfaces in the boundaries of $F_{\rm in}$ and $F_{\rm out}$ that are obtained by affine interpolation between $\{\mathbf{q}x=c\}\cap\partial\mathbf{K}_{z,R}$ and $\partial E\cap\partial\mathbf{K}_{z,S}$ in the case of $F_{\rm in}$, and between $\partial E\cap\partial\mathbf{K}_{z,S}$ and $\{\mathbf{q}x=c\}\cap\partial\mathbf{K}_{z,T}$ in the case of $F_{\rm out}$.

Referring to Figure 4.1 for an illustration of the considered construction, we now state and prove the following lemma.

Lemma 4.5. *If* $H=\{x_1>b\}$ *for some* $b\in\mathbb{R}$, $0<R<S<T$, $|c|<1/4$, $z\in\mathbb{R}^{n-1}\cap H$, $E\subset H$ *is a polyhedron, and*

$$|\nu_E(x)\cdot e_n|<1 \text{ for every } x\in H\cap\mathbf{K}_{z,T}\cap\partial^*E\,,\tag{4.35}$$

$$|\mathbf{q}x|<\frac{1}{4}\,,\qquad\forall x\in\mathbf{K}_{z,T}\cap H\cap\partial E\,,\tag{4.36}$$

$$\Big\{x\in\mathbf{K}_{z,T}\cap H:\mathbf{q}x<-\frac{1}{4}\Big\}\subset E\cap\mathbf{K}_{z,T}\subset\Big\{x\in\mathbf{K}_{z,T}\cap H:\mathbf{q}x<\frac{1}{4}\Big\}\,,\tag{4.37}$$

and if

$$\text{either } z\in\partial H \text{ or } \mathrm{dist}(z,\partial H)>T\,,\tag{4.38}$$

then there exist open sets of finite perimeter $F_{\rm in}$ *and* $F_{\rm out}$ *such that* $F_{\rm in}\subset\mathbf{K}_{z,S}\cap H$ *with*

$$H\cap\partial\mathbf{K}_{z,S}\cap\mathrm{cl}(F_{\rm in}) = H\cap\partial\mathbf{K}_{z,S}\cap E\,,\tag{4.39}$$

$$H\cap\mathbf{K}_{z,R}\cap F_{\rm in} = H\cap\mathbf{K}_{z,R}\cap\{\mathbf{q}x<c\}\,,\tag{4.40}$$

$$\Big\{x\in\mathbf{K}_{z,S}\cap H:\mathbf{q}x<-\frac{1}{4}\Big\}\subset F_{\rm in}\subset\Big\{x\in\mathbf{K}_{z,S}\cap H:\mathbf{q}x<\frac{1}{4}\Big\}\,,\tag{4.41}$$

and

$$\begin{aligned}P(F_{\rm in};\mathbf{K}_{z,S}\cap H) \;\le\;& \mathcal{H}^{n-1}(\mathbf{D}_{z,R}\cap H)\\&+\frac{S^{n-1}-R^{n-1}}{(n-1)S^{n-2}}\int_{H\cap\partial\mathbf{K}_{z,S}\cap\partial E}\sqrt{1+\Big(\frac{\mathbf{q}x-c}{S-R}\Big)^2}\,d\mathcal{H}^{n-2}\,,\end{aligned}\tag{4.42}$$

while $F_{\rm out}\subset\mathbf{K}_{z,T}\cap H$ *with*

$$H\cap\partial\mathbf{K}_{z,T}\cap\mathrm{cl}(F_{out}) = H\cap\partial\mathbf{K}_{z,T}\cap\{\mathbf{q}x<c\}\,,\tag{4.43}$$

$$H\cap\mathbf{K}_{z,S}\cap F_{\rm out} = H\cap\mathbf{K}_{z,S}\cap E\,,\tag{4.44}$$

$$\Big\{x\in\mathbf{K}_{z,T}\cap H:\mathbf{q}x<-\frac{1}{4}\Big\}\subset F_{\rm out}\subset\Big\{x\in\mathbf{K}_{z,T}\cap H:\mathbf{q}x<\frac{1}{4}\Big\}\,,\tag{4.45}$$

and

$$\begin{aligned}P(F_{\rm out};\mathbf{K}_{z,T}\cap H) \;\le\;& P(E;\mathbf{K}_{z,S}\cap H)\\&+\frac{T^{n-1}-S^{n-1}}{(n-1)\,S^{n-2}}\int_{H\cap\partial\mathbf{K}_{z,S}\cap\partial E}\sqrt{1+\Big(\frac{\mathbf{q}x-c}{T-S}\Big)^2}\,d\mathcal{H}^{n-2}\,.\end{aligned}\tag{4.46}$$

Proof. Without loss of generality we may assume that $z=0$, and write $\mathbf{K}_r$ in place of $\mathbf{K}_{0,r}$ for every $r>0$. Moreover, we shall set $G^+=G\cap H$ for every $G\subset\mathbb{R}^n$. By (4.35) there exists a partition (modulo $\mathcal{H}^{n-1}$) of $\mathbf{D}_T^+$ by finitely many open Lipschitz sets $\{\Omega_i\}_{i=1}^N$,

$$\mathbf{D}_T^+ =_{\mathcal{H}^{n-1}} \bigcup_{i=1}^N \Omega_i\,,\qquad \mathcal{H}^{n-1}(\Omega_i\cap\Omega_{i'})=0\quad\text{if } 1\le i<i'\le N\,,$$

and finitely many affine functions $\{f_{i,j}\}_{j=1}^{N(i)}$ and $\{g_{i,j}\}_{j=1}^{N(i)}$ with $f_{i,j}<g_{i,j}<f_{i,j+1}$ for every i and j, such that

$$\mathbf{K}_T^+\cap E =_{\mathcal{H}^n} \bigcup_{i=1}^N\bigcup_{j=1}^{N(i)}\Big\{x\in\Omega_i\times\mathbb{R}: f_{i,j}(\mathbf{p}x)<\mathbf{q}x<g_{i,j}(\mathbf{p}x)\Big\}\,. \tag{4.47}$$

By (4.36) and (4.37) we have

$$f_{i,1}=-1\,,\qquad g_{i,1}\ge-\frac14\,,\qquad g_{i,N(i)}\le\frac14\,,\qquad \forall i=1,\dots,N\,,$$

and, moreover,

$$\mathbf{K}_T^+\cap\partial E =_{\mathcal{H}^{n-1}} \bigcup_{i=1}^N\bigcup_{j=2}^{N(i)}\mathrm{graph}(f_{i,j},\Omega_i)\cup\bigcup_{i=1}^N\bigcup_{j=1}^{N(i)}\mathrm{graph}(g_{i,j},\Omega_i)\,.$$

Construction of $F_{\rm in}$. For every $i=1,\dots,N$, we can define an open set Σ_i such that

$$\Sigma_i =_{\mathcal{H}^{n-1}} \Big\{y\in(\mathbf{D}_S\setminus\mathbf{D}_R)^+ : S\,\hat y\in\Omega_i\cap(\partial\mathbf{D}_S)^+\Big\}\,,\quad\text{where}\quad \hat y=\frac{y}{|y|}\,.$$

Thanks to (4.38), $\{\Sigma_i\}_{i=1}^N$ is a partition modulo $\mathcal{H}^{n-1}$ of $(\mathbf{D}_S\setminus\mathbf{D}_R)^+$. We then define functions $f^*_{i,j}$ and $g^*_{i,j}$ on Σ_i by joining the values of $f_{i,j}$ and $g_{i,j}$ on $\Omega_i\cap(\partial\mathbf{D}_S)^+$ to the constant value $|c|<1/4$ via affine interpolation along radial directions: precisely, we set

$$f^*_{i,1}(y)=-1\,, \tag{4.48}$$

$$f^*_{i,j}(y)=\frac{|y|-R}{S-R}\,f_{i,j}(S\,\hat y)+\frac{S-|y|}{S-R}\,c\,,\qquad j=2,\dots,N(i)\,, \tag{4.49}$$

$$g^*_{i,j}(y)=\frac{|y|-R}{S-R}\,g_{i,j}(S\,\hat y)+\frac{S-|y|}{S-R}\,c\,,\qquad j=1,\dots,N(i)\,. \tag{4.50}$$

where $\hat y=y/|y|$. Finally, we define

$$F_{\rm in}=\Big(\mathbf{K}_R^+\cap\{\mathbf{q}x<c\}\Big)\cup\bigcup_{i=1}^N\bigcup_{j=1}^{N(i)}\Big\{x\in\Sigma_i\times\mathbb{R}: f^*_{i,j}(\mathbf{p}x)<\mathbf{q}x<g^*_{i,j}(\mathbf{p}x)\Big\}\,.$$

Trivially, $F_{\rm in}\subset\mathbf{K}_S^+$ is an open set of finite perimeter, and (4.39), (4.40) and (4.41) hold true. In order to prove (4.42), we start noticing that

$$P(F_{\rm in};\mathbf{K}_S^+)=\mathcal{H}^{n-1}(\mathbf{D}_R^+)+\sum_{i=1}^N\sum_{j=2}^{N(i)}\int_{\Sigma_i}\sqrt{1+|\nabla f^*_{i,j}|^2}+\sum_{i=1}^N\sum_{j=1}^{N(i)}\int_{\Sigma_i}\sqrt{1+|\nabla g^*_{i,j}|^2}\,.$$

Now, if $\varphi^S_{i,j}:(\partial\mathbf{D}_S)^+\to\mathbb{R}$ is defined as the restriction of $f_{i,j}$ to $(\partial\mathbf{D}_S)^+$, then for every $y\in\Sigma_i$,

$$\begin{aligned}\nabla f^*_{i,j}(y) &= \frac{|y|-R}{S-R}\,\frac{S}{|y|}\Big(\nabla f_{i,j}(S\hat y)-\Big(\nabla f_{i,j}(S\hat y)\cdot\hat y\Big)\hat y\Big)+\frac{f_{i,j}(S\,\hat y)-c}{S-R}\,\hat y\\ &= \frac{|y|-R}{S-R}\,\frac{S}{|y|}\,\nabla_\tau\varphi^S_{i,j}(S\hat y)+\frac{\varphi^S_{i,j}(S\,\hat y)-c}{S-R}\,\hat y\,,\end{aligned}$$

where ∇_τ is the tangential gradient along $(\partial\mathbf{D}_S)^+$. Since $\nabla_\tau\varphi^S_{i,j}(S\hat y)\cdot\hat y=0$ for every $y\in\Sigma_i$ and

$$0\le\frac{|y|-R}{S-R}\,\frac{S}{|y|}\le 1\qquad\text{for } R\le|y|\le S,$$

we obtain that

$$|\nabla f^*_{i,j}(y)|^2\le|\nabla_\tau\varphi^S_{i,j}(S\hat y)|^2+\Big(\frac{\varphi^S_{i,j}(S\hat y)-c}{S-R}\Big)^2\qquad\forall\,y\in\Sigma_i.$$

Hence by the co area formula and by the elementary inequality $\sqrt{1+a^2+b^2}\le\sqrt{1+a^2}\sqrt{1+b^2}$ we find

$$\begin{aligned}\int_{\Sigma_i}\sqrt{1+|\nabla f^*_{i,j}|^2}&=\int_R^S dr\int_{\Sigma_i\cap(\partial\mathbf{D}_r)^+}\sqrt{1+|\nabla f^*_{i,j}|^2}\,d\mathcal{H}^{n-2}\\
&\le\frac{1}{S^{n-2}}\int_R^S r^{n-2}\,dr\int_{\Sigma_i\cap(\partial\mathbf{D}_S)^+}\sqrt{1+|\nabla_\tau\varphi^S_{i,j}(S\hat y)|^2+\Big(\frac{\varphi^S_{i,j}(S\hat y)-c}{S-R}\Big)^2}\,d\mathcal{H}^{n-2}\\
&\le\frac{S^{n-1}-R^{n-1}}{(n-1)S^{n-2}}\int_{\Omega_i\cap(\partial\mathbf{D}_S)^+}\sqrt{1+|\nabla_\tau\varphi^S_{i,j}|^2}\sqrt{1+\Big(\frac{\varphi^S_{i,j}-c}{S-R}\Big)^2}\,d\mathcal{H}^{n-2}\\
&=\frac{S^{n-1}-R^{n-1}}{(n-1)S^{n-2}}\int_{\mathrm{graph}(f_{i,j})\cap(\partial\mathbf{K}_S)^+}\sqrt{1+\Big(\frac{x-c}{S-R}\Big)^2}\,d\mathcal{H}^{n-2}\,,\end{aligned}$$

where in the last step we have used the area formula. Since similar inequalities apply to $g^*_{i,j}$, by (4.47) we deduce the validity of (4.42).

Construction of $F_{\rm out}$. In this case we use affine interpolation along radial directions above the annulus $(\mathbf{D}_T\setminus\mathbf{D}_S)^+$. More precisely, this time setting $\Gamma_i=\{y\in(\mathbf{D}_T\setminus\mathbf{D}_S)^+:S\,\hat y\in\Omega_i\cap(\partial\mathbf{D}_S)^+\}$, we let, for every $y\in\Gamma_i$,

$$f^{**}_{i,1}(y)=-1\,,\tag{4.51}$$

$$f^{**}_{i,j}(y)=\frac{T-|y|}{T-S}\,f_{i,j}(S\,\hat y)+\frac{|y|-S}{T-S}\,c\,,\qquad j=2,\dots,N(i)\,,\tag{4.52}$$

$$g^{**}_{i,j}(y)=\frac{T-|y|}{T-S}\,g_{i,j}(S\,\hat y)+\frac{|y|-S}{T-S}\,c\,,\qquad j=1,\dots,N(i)\,,\tag{4.53}$$

and correspondingly define

$$F_{\rm out}=\Big(\mathbf{K}^+_S\cap E\Big)\cup\bigcup_{i=1}^N\bigcup_{j=1}^{N(i)}\Big\{x\in\Gamma_i\times\mathbb{R}:f^{**}_{i,j}(\mathbf{p}x)<\mathbf{q}x<g^{**}_{i,j}(\mathbf{p}x)\Big\}\,.$$

One checks the validity of (4.43), (4.44), (4.45) and (4.46) by arguing as above. □

Proof of Lemma 4.4. By (3.1) and Remark 2.2 we can assume that $x_0=0$ and $r=1$. By requiring that $\varepsilon_{\rm Ca}<\varepsilon_{\rm hb}(n,\lambda,1/8)$, we can apply Lemma 4.1 to find that

$$|\mathbf{q}x|<\frac18\,,\qquad\forall x\in\mathbf{C}_4\cap\partial E\,,\tag{4.54}$$

$$\Big\{x\in\mathbf{C}_4\cap H:\mathbf{q}x<-\frac18\Big\}\subset E\cap\mathbf{C}_4\cap H\subset\Big\{x\in\mathbf{C}_4\cap H:\mathbf{q}x<\frac18\Big\}\,,\tag{4.55}$$

and to have that

$$\zeta(G)=P(E;\mathbf{p}^{-1}(G)\cap\mathbf{C}_4\cap H))-\mathcal{H}^{n-1}(G\cap H)\,,\qquad G\subset\mathbf{D}_4\,,\tag{4.56}$$

defines a finite positive Radon measure on $\mathbf{D}_4$, concentrated on $\mathbf{D}_4 \cap H$, and such that

$$\zeta(\mathbf{D}_4) = 4^{n-1}\,\mathbf{exc}_n^H(E,0,4) \le 2^{n-1}\,\varepsilon_{\rm Ca}\,, \tag{4.57}$$
$$\zeta(G) = \int_{\mathbf{p}^{-1}(G)\cap\mathbf{C}_4\cap H\cap\partial^*E} \frac{|\nu_E - e_n|^2}{2}\,, \qquad \forall G \subset \mathbf{D}_4\,.$$

We now divide the argument in two steps, setting $G^+ = G \cap H$ for every $G \subset \mathbb{R}^n$.

Step one: We prove that for every $\xi \in (1,2)$ there exist positive constants $C_* = C_*(n,\lambda,\xi)$ and $\theta_* = \theta_*(\xi)$ such that if $z \in \mathbb{R}^{n-1}$, $s > 0$, $\mathbf{D}_{z,\xi\,s} \subset \mathbf{D}_4$ (i.e., $|z| + \xi\, s \le 4$), $|c| < 1/4$ and either $z \in \partial H$ or $\mathrm{dist}(z,\partial H) > \xi\, s$, then

$$P(E;\mathbf{K}^+_{z,s}) - \mathcal{H}^{n-1}(\mathbf{D}^+_{z,s}) \le C_*\Big\{\theta\Big(P(E;\mathbf{K}^+_{z,\xi\,s}) - \mathcal{H}^{n-1}(\mathbf{D}^+_{z,\xi\,s})\Big) + \frac{1}{\theta}\int_{\mathbf{K}^+_{z,\xi\,s}\cap\partial^*E} |\mathbf{q}x - c|^2\,d\mathcal{H}^{n-1}\Big\} + C_*(\Lambda + \ell)\,, \tag{4.58}$$

for every $\theta \in (0,\theta_*)$. This follows by testing the minimality of E against the competitors constructed in Lemma 4.5 and by exploiting the ellipticity of Φ to compare these competitors with half-spaces through Proposition 2.6. Precisely, with the end of exploiting Lemma 4.5, let us consider a sequence $\{E_k\}_{k\in\mathbb{N}}$ of open subsets of H with polyhedral boundaries such that

$$|\nu_{E_k}\cdot e_n| < 1 \text{ on } \partial E_k\,, \tag{4.59}$$
$$E_k \to E \text{ in } L^1_{\rm loc}(\mathbf{C}_{16}) \qquad\text{and}\qquad |\mu_{E_k}|\llcorner H \stackrel{*}{\rightharpoonup} |\mu_E|\llcorner H\,, \tag{4.60}$$
$$|\mathbf{q}x| < \frac{1}{4}\,, \qquad \forall x \in \mathbf{C}_4 \cap H \cap \partial E_k\,, \tag{4.61}$$
$$\Big\{x \in \mathbf{C}_4 \cap H : \mathbf{q}x < -\frac{1}{4}\Big\} \subset E_k \cap \mathbf{C}_4 \cap H \subset \Big\{x \in \mathbf{C}_4 \cap H : \mathbf{q}x < \frac{1}{4}\Big\}\,. \tag{4.62}$$

Note that the existence of a sequence $\{E_k\}_{k\in\mathbb{N}}$ satisfying the above properties can be obtained by trivial modifications of the classical polyhedral approximation of sets of finite perimeter, see e.g. [Mag12, Theorem 13.8]. In particular, since the normal of a polyhedron takes finitely many values, (4.59) can be achieved by performing arbitrary small rotations. Recalling that $\mathbf{D}_{\xi,s} \subset \mathbf{D}_4$, by (4.56),

$$P(E;\mathbf{K}^+_{z,\xi\,s}\setminus\mathbf{K}^+_{z,s}) - \mathcal{H}^{n-1}(\mathbf{D}^+_{z,\xi\,s}\setminus\mathbf{D}^+_{z,s}) = \zeta(\mathbf{D}_{z,\xi\,s}\setminus\mathbf{D}_{z,s}) \le \zeta(\mathbf{D}_{z,\xi\,s}) = P(E;\mathbf{K}^+_{z,\xi\,s}) - \mathcal{H}^{n-1}(\mathbf{D}^+_{z,\xi\,s})\,;$$

moreover, by (4.61), (4.62) and Lemma 4.2, an analogous inequality holds true with E_k in place of E. By a slicing argument based on coarea formula, there exists $\alpha \in (1,\xi)$ with

$$1 < \frac{2}{3} + \frac{\xi}{3} < \alpha < \frac{1}{3} + \frac{2\xi}{3} < \xi\,, \tag{4.63}$$

such that, up to extracting a not relabeled subsequence in k, one has

$$\mathcal{H}^{n-2}((\partial\mathbf{K}_{z,\alpha s})^+\cap\partial E_k) - \mathcal{H}^{n-2}((\partial\mathbf{D}_{\alpha s})^+) \le \frac{C}{s}\Big(P(E_k;\mathbf{K}^+_{z,\xi\,s}) - \mathcal{H}^{n-1}(\mathbf{D}^+_{z,\xi\,s})\Big)\,, \tag{4.64}$$
$$\int_{(\partial\mathbf{K}_{z,\alpha s})^+\cap\partial E_k} (\mathbf{q}x - c)^2\,d\mathcal{H}^{n-2} \le \frac{C}{s}\int_{\mathbf{K}^+_{z,\xi\,s}\cap\partial E_k} (\mathbf{q}x - c)^2\,d\mathcal{H}^{n-1}\,, \tag{4.65}$$

for every $k \in \mathbb{N}$ and for a suitable constant $C = C(\xi)$, and

$$\lim_{k\to\infty}\mathcal{H}^{n-1}\Big((E_k\Delta E^{(1)})\cap(\partial\mathbf{K}_{z,\alpha s})^+\Big) = 0\,, \tag{4.66}$$
$$P(E;(\partial\mathbf{K}_{\alpha\,s})^+) = 0\,. \tag{4.67}$$

Next, we choose any $\theta_*=\theta_*(\xi)$ such that

$$1<(1-\theta_*)\Big(\frac{2}{3}+\frac{\xi}{3}\Big)\,,\qquad\Big(\frac{1}{3}+\frac{2\xi}{3}\Big)<\xi(1-\theta_*)\,,$$

so to entail that,

$$s<(1-\theta)\alpha\,s<\alpha\,s<\frac{\alpha s}{(1-\theta)}<\xi s\,,\qquad\forall\,\theta\in(0,\theta_*]\,.$$

Finally, we set

$$R=(1-\theta)\,\alpha s\,,\qquad S=\alpha\,s\,,\qquad T=\frac{\alpha s}{1-\theta},\tag{4.68}$$

so that $s\le R<S<T<\xi s$.

Since we are assuming that either $z\in\partial H$ or $\mathrm{dist}(z,\partial H)>\xi\,s$, by (4.59), (4.61), and (4.62) we can apply Lemma 4.5 to find sequences of sets $\{F^k_{\rm in}\}_{k\in\mathbb{N}}$ and $\{F^k_{\rm out}\}_{k\in\mathbb{N}}$ corresponding to the values of R, S and T defined in (4.68). Let us now define

$$\widetilde F_k=(F^k_{\rm in}\cap\mathbf{K}^+_{z,S})\cup(E\setminus\mathbf{K}^+_{z,S})\,.$$

By exploiting the minimality of E, by (2.5), (2.6), and (2.7) (that shall be repeatedly used in the sequel), and by taking into account (4.39) and (4.40), we thus find that for every $k\in\mathbb{N}$,

$$\mathbf{\Phi}(E;\mathbf{K}^+_{z,S})\le\mathbf{\Phi}(F^k_{\rm in};\mathbf{K}^+_{z,S})+\lambda\,\mathcal{H}^{n-1}\Big((E^{(1)}\Delta E_k)\cap(\partial\mathbf{K}_{z,S})^+\Big)+\Lambda\,|E\Delta F^k_{\rm in}|\,,\tag{4.69}$$

where (1.7) was also taken into account. Since $|E\Delta F^k_{\rm in}|\le|K^+_T|\le|\mathbf{C}_4|$, and, by (4.60), (4.67), and Reshetnyak continuity theorem [GMS98, Theorem 1, section 3.4] $\mathbf{\Phi}(E_k;\mathbf{K}^+_{z,S})\to\mathbf{\Phi}(E;\mathbf{K}^+_{z,S})$ as $k\to\infty$, we deduce from (4.66) and (4.69) that

$$\mathbf{\Phi}(E_k;\mathbf{K}^+_{z,S})\le\mathbf{\Phi}(F^k_{\rm in};\mathbf{K}^+_{z,S})+\varepsilon_k+C\Lambda\,,\tag{4.70}$$

where $C=C(n)$ and $\varepsilon_k\to0$ as $k\to\infty$. We now notice that by (4.44) we can apply Lemma 4.2 to $F^k_{\rm out}$ on the cylinder $\mathbf{K}^+_{z,T}$ to see that

$$\zeta_k(G)=P(F^k_{\rm out};\mathbf{p}^{-1}(G)\cap\mathbf{K}^+_{z,T})-\mathcal{H}^{n-1}(G\cap\mathbf{D}^+_{z,T})\,,\qquad G\subset\mathbf{D}_{z,T}\,,$$

defines a positive Radon measure on $\mathbf{D}_{z,T}$ with

$$\begin{aligned}P(E_k;\mathbf{K}^+_{z,S})-\mathcal{H}^{n-1}(\mathbf{D}^+_{z,S})&=P(F^k_{\rm out};\mathbf{K}^+_{z,S})-\mathcal{H}^{n-1}(\mathbf{D}^+_{z,S})=\zeta_k(\mathbf{D}^+_{z,S})\\&\le\zeta_k(\mathbf{D}^+_{z,T})=\int_{\mathbf{K}^+_{z,T}\cap\partial F^k_{\rm out}}\frac{|\nu_{F^k_{\rm out}}-e_n|^2}{2}\,d\mathcal{H}^{n-1}\\&\le\frac{1}{\kappa_1}\Big\{\mathbf{\Phi}_0(F^k_{\rm out};\mathbf{K}^+_{z,T})-\Phi_0(e_n)\,\mathcal{H}^{n-1}(\mathbf{D}^+_{z,T})\Big\}\,.\end{aligned}\tag{4.71}$$

where in the last inequality we have applied either (2.30) or (2.28) (depending on whether $z\in\partial H$ and thus $\nabla\Phi(0,e_n)\cdot e_1=0$, or $\mathrm{dist}(z,\partial H)>\xi\,s>T$) to the autonomous elliptic integrand $\Phi_0(\nu)=\Phi(0,\nu)$. By (4.44) and (4.67) we find (with obvious notations)

$$\mathbf{\Phi}_0(F^k_{\rm out};\mathbf{K}^+_{z,T})=\mathbf{\Phi}(E_k;\mathbf{K}^+_{z,S})+\mathbf{\Phi}(F^k_{\rm out};\mathbf{K}^+_{z,T}\setminus\mathbf{K}^+_{z,S})+(\mathbf{\Phi}_0-\mathbf{\Phi})(F^k_{\rm out};\mathbf{K}^+_{z,T})\,.\tag{4.72}$$

Thus, we may combine (4.60), (4.70), (4.71), and (4.72) to get

$$
\begin{aligned}
&P(E;\mathbf{K}^+_{z,S})-\mathcal{H}^{n-1}(\mathbf{D}^+_{z,S})\\
&\qquad\le P(E_k;\mathbf{K}^+_{z,S})-\mathcal{H}^{n-1}(\mathbf{D}^+_{z,S})+\varepsilon_k\\
&\qquad\le C\Big\{\mathbf{\Phi}_0(F^k_{\rm out};\mathbf{K}^+_{z,T})-\Phi_0(e_n)\,\mathcal{H}^{n-1}(\mathbf{D}^+_{z,T})\Big\}+\varepsilon_k\\
&\qquad\le C\Big\{\mathbf{\Phi}(E_k;\mathbf{K}^+_{z,S})+\mathbf{\Phi}(F^k_{\rm out};\mathbf{K}^+_{z,T}\setminus\mathbf{K}^+_{z,S})-\Phi_0(e_n)\,\mathcal{H}^{n-1}(\mathbf{D}^+_{z,T})\\
&\qquad\qquad+(\mathbf{\Phi}_0-\mathbf{\Phi})(F^k_{\rm out};\mathbf{K}^+_{z,T})+\varepsilon_k\Big\}\\
&\qquad\le C\Big\{\mathbf{\Phi}(F^k_{\rm in};\mathbf{K}^+_{z,S})+\mathbf{\Phi}(F^k_{\rm out};\mathbf{K}^+_{z,T}\setminus\mathbf{K}^+_{z,S})-\Phi_0(e_n)\,\mathcal{H}^{n-1}(\mathbf{D}^+_{z,T})\\
&\qquad\qquad+(\mathbf{\Phi}_0-\mathbf{\Phi})(F^k_{\rm out};\mathbf{K}^+_{z,T})+\Lambda+\varepsilon_k\Big\}\\
&\qquad= C\Big\{\mathbf{\Phi}(F^k;\mathbf{K}^+_{z,T})-\Phi_0(e_n)\,\mathcal{H}^{n-1}(\mathbf{D}^+_{z,T})+(\mathbf{\Phi}_0-\mathbf{\Phi})(F^k_{\rm out};\mathbf{K}^+_{z,T})+\Lambda+\varepsilon_k\Big\}\,.
\end{aligned}
\tag{4.73}
$$

where $\varepsilon_k\to 0$ as $k\to\infty$, $C=C(n,\lambda)$, and we have set

$$
F^k=(F^k_{\rm in}\cap\mathbf{K}^+_{z,S})\cup(F^k_{\rm out}\cap(\mathbf{K}^+_{z,T}\setminus\mathbf{K}^+_{z,S}))\,. \tag{4.74}
$$

(Notice that, thanks to (4.39), (4.43), and (4.67) one has $P(F^k;(\partial\mathbf{K}_{z,S})^+)=0$.) By applying (2.30) or (2.28) to the set F^k, we see that

$$
\begin{aligned}
&\mathbf{\Phi}(F^k;\mathbf{K}^+_{z,T})-\Phi_0(e_n)\,\mathcal{H}^{n-1}(\mathbf{D}^+_{z,T})\\
&\qquad=\mathbf{\Phi}_0(F^k;\mathbf{K}^+_{z,T})-\Phi_0(e_n)\,\mathcal{H}^{n-1}(\mathbf{D}^+_{z,T})+(\mathbf{\Phi}-\mathbf{\Phi}_0)(F^k;\mathbf{K}^+_{z,T})\\
&\qquad\le\kappa_2\Big(P(F^k,\mathbf{K}^+_{z,T})-\mathcal{H}^{n-1}(\mathbf{D}^+_{z,T})\Big)+(\mathbf{\Phi}-\mathbf{\Phi}_0)(F^k;\mathbf{K}^+_{z,T})\,.
\end{aligned}
\tag{4.75}
$$

By combining (4.73) and (4.75) and by recalling the definition of F^k, (4.74), we then obtain

$$
\begin{aligned}
&P(E;\mathbf{K}^+_{z,S})-\mathcal{H}^{n-1}(\mathbf{D}^+_{z,S})\\
&\qquad\le C\Big\{\mathbf{\Phi}(F^k;\mathbf{K}^+_{z,T})-\Phi_0(e_n)\,\mathcal{H}^{n-1}(\mathbf{D}^+_{z,T})+(\mathbf{\Phi}_0-\mathbf{\Phi})(F^k_{\rm out};\mathbf{K}^+_{z,T})+\Lambda+\varepsilon_k\Big\}\\
&\qquad\le C\Big\{\kappa_2\big(P(F^k,\mathbf{K}^+_{z,T})-\mathcal{H}^{n-1}(\mathbf{D}^+_{z,T})\big)\\
&\qquad\qquad+(\mathbf{\Phi}-\mathbf{\Phi}_0)(F^k_{\rm in};\mathbf{K}^+_{z,S})+(\mathbf{\Phi}_0-\mathbf{\Phi})(E_k;\mathbf{K}^+_{z,S})+\Lambda+\varepsilon_k\Big\}\,.
\end{aligned}
\tag{4.76}
$$

Let us now notice that since $\mathbf{K}_{z,S}\subset\mathbf{C}_4$, thanks to (1.8), (4.60) and the density estimates (2.47) we have

$$
\big|(\mathbf{\Phi}-\mathbf{\Phi}_0)(E_k;\mathbf{K}^+_{z,S})\big|\le C\ell P(E_k,\mathbf{C}^+_4)\le C\ell P(E,\mathbf{C}^+_4)+\varepsilon_k\le C\ell+\varepsilon_k. \tag{4.77}
$$

Moreover, since $\mathcal{H}^{n-1}(\mathbf{D}^+_{z,S})\le C(n)$ (by $S\le\xi s\le 4$),

$$
\big|(\mathbf{\Phi}-\mathbf{\Phi}_0)(F^k_{\rm in};\mathbf{K}^+_{z,S})\big|\le C\ell P(F^k_{\rm in},\mathbf{K}^+_{z,S})\le C\ell\big\{P(F^k_{\rm in},\mathbf{K}^+_{z,S})-\mathcal{H}^{n-1}(\mathbf{D}^+_{z,S})\big\}+C\ell\,. \tag{4.78}
$$

Finally , by (4.39), (4.43), and (4.67),

$$
\begin{aligned}
P(F^k,\mathbf{K}^+_{z,T})-\mathcal{H}^{n-1}(\mathbf{D}^+_{z,T})&=P(F^k_{\rm in},\mathbf{K}^+_{z,S})-\mathcal{H}^{n-1}(\mathbf{D}^+_{z,S})\\
&\quad+P(F^k_{\rm out},\mathbf{K}^+_{z,T}\setminus\mathbf{K}^+_{z,S})-\mathcal{H}^{n-1}(\mathbf{D}^+_{z,T}\setminus\mathbf{D}^+_{z,S})\,.
\end{aligned}
\tag{4.79}
$$

Hence, by combining (4.76), (4.77), (4.78) and (4.79) and taking into account that $\ell \le 1$ and that both terms in the right hand side of (4.79) are non-negative (by Lemma 4.2), we obtain:

$$
\begin{aligned}
&P(E;\mathbf{K}_{z,S}^+) - \mathcal{H}^{n-1}(\mathbf{D}_{z,S}^+)\\
&\qquad\le C\Big\{P(F_{\rm in}^k,\mathbf{K}_{z,S}^+) - \mathcal{H}^{n-1}(\mathbf{D}_{z,S}^+)\\
&\qquad\qquad + P(F_{\rm out}^k,\mathbf{K}_{z,T}^+\setminus\mathbf{K}_{z,S}^+) - \mathcal{H}^{n-1}(\mathbf{D}_{z,T}^+\setminus\mathbf{D}_{z,S}^+) + (\Lambda+\ell)+\varepsilon_k\Big\}.
\end{aligned}
\tag{4.80}
$$

We now observe that both in the case $z\in\partial H$ and in the case $\mathrm{dist}(z,\partial H) > \xi\, s$ we have

$$
\begin{aligned}
\mathcal{H}^{n-1}(\mathbf{D}_{z,S}^+\setminus\mathbf{D}_{z,R}^+) &= \frac{S^{n-1}-R^{n-1}}{(n-1)S^{n-2}}\mathcal{H}^{n-2}\big((\partial\mathbf{D}_{z,S})^+\big)\\
\mathcal{H}^{n-1}(\mathbf{D}_{z,T}^+\setminus\mathbf{D}_{z,S}^+) &= \frac{T^{n-1}-S^{n-1}}{(n-1)T^{n-2}}\mathcal{H}^{n-2}\big((\partial\mathbf{D}_{z,S})^+\big)\,.
\end{aligned}
\tag{4.81}
$$

By (4.42), by (4.81) and since $\sqrt{1+t^2}\le 1+t^2$, we find

$$
\begin{aligned}
&P(F_{\rm in}^k;\mathbf{K}_{z,S}^+) - \mathcal{H}^{n-1}(\mathbf{D}_{z,S}^+)\\
&\qquad\le \frac{S^{n-1}-R^{n-1}}{(n-1)S^{n-2}}\int_{(\partial\mathbf{K}_{z,S})^+\cap\partial E_k} 1+\Big(\frac{\mathbf{q}x-c}{S-R}\Big)^2\,d\mathcal{H}^{n-2} - \mathcal{H}^{n-1}(\mathbf{D}_{z,S}^+\setminus\mathbf{D}_{z,R}^+)\\
&\qquad= \frac{S^{n-1}-R^{n-1}}{(n-1)S^{n-2}}\Big\{\int_{(\partial\mathbf{K}_{z,S})^+\cap\partial E_k}\Big(\frac{\mathbf{q}x-c}{S-R}\Big)^2\,d\mathcal{H}^{n-2}\\
&\qquad\qquad + \mathcal{H}^{n-1}((\partial\mathbf{K}_{z,S})^+\cap\partial E_k) - \mathcal{H}^{n-2}((\partial\mathbf{D}_{z,S})^+)\Big\}\\
&\qquad\le \frac{C(\xi)}{s}\,\frac{S^{n-1}-R^{n-1}}{(n-1)S^{n-2}}\Big\{\int_{\mathbf{K}_{z,\xi\, s}^+\cap\partial E_k}\Big(\frac{\mathbf{q}x-c}{S-R}\Big)^2\,d\mathcal{H}^{n-1} + P(E_k;\mathbf{K}_{z,\xi\, s}^+) - \mathcal{H}^{n-1}(\mathbf{D}_{z,\xi\, s}^+)\Big\}\,,
\end{aligned}
$$

where in the last inequality we have used (4.64) and (4.65). Since $S=\alpha\, s$ and $R=(1-\theta)\alpha\, s$ we have that

$$
S-R=\alpha\theta s \qquad\text{and}\qquad \frac{S^{n-1}-R^{n-1}}{(n-1)S^{n-2}}\le C(n)\,\theta\,\alpha\, s\,.
$$

Thus, by also taking into account that $\alpha\in(1,2)$ by (4.63), we conclude that

$$
\begin{aligned}
&P(F_{\rm in}^k;\mathbf{K}_{z,S}^+) - \mathcal{H}^{n-1}(\mathbf{D}_{z,S}^+)\\
&\qquad\le C\left\{\frac{1}{\theta}\int_{\mathbf{K}_{z,\xi\, s}^+\cap\partial E_k}\Big(\frac{\mathbf{q}x-c}{s}\Big)^2 d\mathcal{H}^{n-1} + \theta\Big(P(E_k;\mathbf{K}_{z,\xi\, s}^+) - \mathcal{H}^{n-1}(\mathbf{D}_{z,\xi\, s}^+)\Big)\right\},
\end{aligned}
\tag{4.82}
$$

for some $C=C(n,\xi)$. By an entirely similar argument we exploit (4.46), (4.64), and (4.65) to show that

$$
\begin{aligned}
&P(F_{\rm out}^k;\mathbf{K}_{z,T}^+\setminus\mathbf{K}_{z,S}^+) - \mathcal{H}^{n-1}(\mathbf{D}_{z,T}^+\setminus\mathbf{D}_{z,S}^+)\\
&\qquad\le C\left\{\frac{1}{\theta}\int_{\mathbf{K}_{z,\xi\, s}^+\cap\partial E_k}\Big(\frac{\mathbf{q}x-c}{s}\Big)^2 d\mathcal{H}^{n-1} + \theta\left(P(E_k;\mathbf{K}_{z,\xi\, s}^+) - \mathcal{H}^{n-1}(\mathbf{D}_{\xi\, s}^+)\right)\right\},
\end{aligned}
\tag{4.83}
$$

for some $C = C(n,\xi)$. By combining (4.80), (4.82) and (4.83) and by letting $k \to \infty$, taking also into account (4.60), we finally get

$$\begin{aligned} &P(E;\mathbf{K}_{z,S}^+) - \mathcal{H}^{n-1}(\mathbf{D}_{z,S}^+) \\ &\qquad \le C\left\{\frac{1}{\theta}\int_{\mathbf{K}_{z,\xi\,s}^+\cap\partial E}\Big(\frac{\mathbf{q}x-c}{s}\Big)^2\,d\mathcal{H}^{n-1} + \theta\Big(P(E;\mathbf{K}_{z,\xi\,s}^+) - \mathcal{H}^{n-1}(\mathbf{D}_{z,\xi\,s}^+)\Big) + (\Lambda+\ell)\right\}, \end{aligned} \tag{4.84}$$

for some $C = C(n,\lambda,\xi)$. By (4.56) and since $S > s$, the left-hand side of (4.84) is an upper bound to $P(E;\mathbf{K}_{z,s}^+) - \mathcal{H}^{n-1}(\mathbf{D}_{z,s}^+)$, so that (4.84) implies (4.58).

Step two: We finally deduce (4.33) from the weaker inequality (4.58) through a covering argument, see [Sim96]. We start noticing that, as a consequence of (4.58), for every $\xi \in (1,2)$ there exist $C_* = C_*(n,\lambda,\xi)$ and $\theta_* = \theta_*(\xi)$ such that

$$s^2\,\zeta(\mathbf{D}_{z,s}) \le C_*\left\{\theta s^2\,\zeta(\mathbf{D}_{z,\xi\,s}) + \frac{h}{\theta} + (\Lambda+\ell)\right\}, \qquad \forall\theta\in(0,\theta_*], \tag{4.85}$$

whenever $z \in \mathbb{R}^{n-1}$, $s > 0$, $\mathbf{D}_{z,\xi\,s} \subset \mathbf{D}_4$ with either $z \in \partial H$ or $\mathrm{dist}(z,\partial H) > \xi\,s$, ζ is given by (4.56) and

$$h = \inf_{|c|<1/4}\int_{\mathbf{C}_4\cap H\cap\partial^* E}|\mathbf{q}x-c|^2\,d\mathcal{H}^{n-1}. \tag{4.86}$$

We now conclude the proof of the lemma under the assumption (4.31), that is $0 \in \partial H$ and $\nabla\Phi(0,e_n)\cdot e_1 = 0$ (recall that we have set $x_0 = 0$ and $r = 1$). A simpler, analogous argument covers the case when (4.32) holds true. We start by showing that if $\xi \in (1,2)$ is sufficiently close to 1, then there there exist $C = C(n,\lambda,\theta)$ such that

$$s^2\zeta(\mathbf{D}_{z,s}) \le C\Big\{\theta s^2\zeta(\mathbf{D}_{z,4s}) + \frac{h}{\theta} + (\Lambda+\ell)\Big\}, \tag{4.87}$$

for every $\theta \in (0,\theta_*]$ and $\mathbf{D}_{z,4s} \subset \mathbf{D}_4$. Indeed, let $z \in \mathbb{R}^{n-1}$ and $s > 0$ satisfy $\mathbf{D}_{z,4s} \subset \mathbf{D}_4$. If $\mathrm{dist}(z,\partial H) > \xi s$ then (4.87) follows immediately from (4.85) by the trivial inclusion $\mathbf{D}_{z,\xi\,s} \subset \mathbf{D}_{z,4\,s}$ (recall that $\xi < 2$). If, instead, $\mathrm{dist}(z,\partial H) \le \xi s$, then we consider the projection $\bar z$ of z on ∂H; since $\mathbf{D}_{z,s} \subset \mathbf{D}_{\bar z,(\xi+1)s}$, we have $\zeta(\mathbf{D}_{z,s}) \le \zeta(\mathbf{D}_{\bar z,(\xi+1)\,s})$; at the same time, since $\bar z \in \partial H$ with $\mathbf{D}_{\bar z,\xi(\xi+1)\,s} \subset \mathbf{D}_{z,\xi(\xi+2)s} \subset \mathbf{D}_{z,4s} \subset \mathbf{D}_4$, we can apply (4.58) at $\bar z$ at the scale $(\xi+1)\,s$, to conclude that

$$s^2\zeta(\mathbf{D}_{z,s}) \le s^2\zeta(\mathbf{D}_{\bar z,(\xi+1)\,s}) \le C\Big\{\theta s^2\zeta(\mathbf{D}_{\bar z,(\xi^2+\xi)\,s}) + \frac{h}{\theta} + (\Lambda+\ell)\Big\}, \qquad \forall\theta\in(0,\theta_*].$$

If we choose ξ such that $\xi(\xi+2) < 4$, then $\mathbf{D}_{\bar z,(\xi^2+\xi)\,s} \subset \mathbf{D}_{z,(\xi^2+2\xi)\,s} \subset \mathbf{D}_{z,4s}$ and we deduce the validity of (4.87). Let us now define

$$Q = \sup\left\{s^2\,\zeta(\mathbf{D}_{z,s}) : \mathbf{D}_{z,4\,s} \subset \mathbf{D}_4\right\},$$

so that $Q < \infty$ by (4.57). We now notice that, if $\mathbf{D}_{z,4\,s} \subset \mathbf{D}_2$, then there exists a family of points $\{z_k\}_{k=1}^{N(n)} \subset \mathbf{D}_{z,s}$ such that $\mathbf{D}_{z,s} \subset \bigcup_{k=1}^{N(n)}\mathbf{D}_{z_k,s/16}$. Since, trivially $\mathbf{D}_{z_k,s} \subset \mathbf{D}_4$, by applying (4.87) at each z_k at scale $s/16$ we find that

$$\begin{aligned} s^2\,\zeta(\mathbf{D}_{z,s}) &\le 256\sum_{k=1}^{N(n)}\Big(\frac{s}{16}\Big)^2\zeta(\mathbf{D}_{z_k,s/16}) \\ &\le C\sum_{k=1}^{N(n)}\left\{\theta\Big(\frac{s}{16}\Big)^2\zeta(\mathbf{D}_{z_k,s/4}) + \frac{h}{\theta} + (\Lambda+\ell)\right\} \\ &\le C\Big\{\theta Q + \frac{h}{\theta} + (\Lambda+\ell)\Big\}, \qquad \forall\theta\in(0,\theta_*], \end{aligned}$$

where $C = C(n,\lambda,\xi)$. In other words,

$$Q \le C\Big\{\theta Q + \frac{h}{\theta} + (\Lambda+\ell)\Big\}\,, \qquad \forall\theta\in(0,\theta_*]\,.$$

Keeping ξ fixed, we choose $\theta<\theta_*$ such that $C\,\theta\le 1/2$ in order to conclude that

$$Q \le 2C\,(h+(\Lambda+\ell)\,r)\,.$$

By recalling the definition of h (i.e., (4.86)), and by noticing that $\mathbf{D}$ is admissible in the definition of Q, we conclude that

$$\mathbf{exc}_n^H(E,0,1) = \zeta(\mathbf{D}) \le Q \le C\Big\{\inf_{|c|<1/4}\int_{\mathbf{C}_4\cap H\cap\partial^* E}|\mathbf{q}x-c|^2\,d\mathcal{H}^{n-1} + (\Lambda+\ell)\Big\}\,. \tag{4.88}$$

Finally, if $|c|\ge 1/4$, then by (4.54), (2.50) and since $\mathcal{H}^{n-1}((\mathbf{C}_{16}\cap\partial E)\setminus\partial^* E)=0$ by Lemma 2.16, we find

$$\int_{\mathbf{C}_4\cap H\cap\partial^* E}|\mathbf{q}x-c|^2\,d\mathcal{H}^{n-1} \ge \frac{c_1 16^{n-1}}{8^2}\,.$$

Hence, provided $\varepsilon_{\rm Ca} < c_1\,(16)^{n-1}/64$, we find

$$\mathbf{exc}_n^H(E,0,1) \le 8^{n-1}\,\mathbf{exc}_n^H(E,0,8) \le 8^{n-1}\int_{\mathbf{C}_4\cap H\cap\partial^* E}|\mathbf{q}x-c|^2\,d\mathcal{H}^{n-1}\,. \tag{4.89}$$

We combine (4.88) and (4.89) to deduce (4.33). □

4.4. **Tilt lemma.** We now combine the results from the previous three sections to obtain the key estimate in the proof of Lemma 3.4. Indeed, Lemma 3.4 will follow by an iterated application of the following lemma.

Lemma 4.6 (Tilt lemma)**.** *For every $\lambda\ge 1$ and $\beta\in(0,1/64)$ there exist positive constants $\varepsilon_{\rm tilt}=\varepsilon_{\rm tilt}(n,\lambda,\beta)$ and $C_4=C_4(n,\lambda)$ with the following properties. If $H=\{x_1>b\}$ for some $b\in\mathbb{R}$, $x_0\in{\rm cl}(H)$,*

$$\begin{aligned}
&\Phi\in\boldsymbol{\mathcal{E}}(\mathbf{C}_{x_0,16\,r}\cap H,\lambda,\ell)\,,\\
&E \text{ is a } (\Lambda,r_0)\text{-minimizer of } \mathbf{\Phi} \text{ in } (\mathbf{C}_{x_0,16\,r},H) \text{ with } 0<8r\le r_0\,,\\
&x_0\in{\rm cl}(H\cap\partial E)\ ,\\
&\mathbf{exc}_n^H(E,x_0,8r)+(\Lambda+\ell)r<\varepsilon_{\rm tilt}\,,
\end{aligned}$$

and

$$\text{either} \qquad x_0\in\partial H \text{ and } \nabla\Phi(x_0,e_n)\cdot e_1=0\,, \tag{4.90}$$

$$\text{or} \qquad {\rm dist}(x_0,\partial H)>r\,, \tag{4.91}$$

then there exists an affine map $L:\mathbb{R}^n\to\mathbb{R}^n$ with $Lx_0=x_0$ and $L(H)=H$, such that

$$\begin{aligned}
\|\nabla L-{\rm Id}\,\|^2 &\le C_4\,\Big(\mathbf{exc}_n^H(E,x_0,8r)+(\Lambda+\ell)\,r\Big)\,,\\
\mathbf{exc}_n^H(L(E),x_0,\beta\,r) &\le C_4\,\Big(\beta^2\,\mathbf{exc}_n^H(E,x_0,8r)+\beta\,(\Lambda+\ell)r\Big)\,.
\end{aligned}$$

Moreover, if we set as usual $\Phi^L(x,\nu)=\Phi(L^{-1}(x),({\rm cof}\,\nabla L)^{-1}\nu)$, then $\Phi^L\in\boldsymbol{\mathcal{E}}(\mathbf{C}_{x_0,2\beta\,r}\cap H,\widetilde{\lambda},\widetilde{\ell})$ and $L(E)$ is a $(\Lambda,\widetilde{r_0})$-minimizer of $\mathbf{\Phi}^L$ on $(\mathbf{C}_{x_0,2\beta r},H)$, where

$$\max\Big\{\frac{|\widetilde{\lambda}-\lambda|}{\lambda},\frac{|\widetilde{\ell}-\ell|}{\ell},\frac{|\widetilde{r_0}-r_0|}{r_0}\Big\}\le C_4\,\Big(\mathbf{exc}_n^H(E,x_0,8r)+(\Lambda+\ell)\,r\Big)^{1/2}$$

Finally,

$$\nabla\Phi^L(x_0,e_n)\cdot e_1=0\,, \qquad \textit{if (4.90) holds.}$$

We premise the following lemma, usually known as a A-harmonic approximation lemma, that in our setting just amounts to a remark in the theory of constant coefficients elliptic PDEs.

Lemma 4.7. *For every $\lambda \geq 1$ and $\tau > 0$ there exists a positive constant $\varepsilon_{\rm har} = \varepsilon_{\rm har}(\tau, \lambda)$ with the following property. If $H = \{x_1 > b\}$ for some $b \in \mathbb{R}$, $A \in \mathbf{Sym}(n)$ with $\lambda^{-1}\,{\rm Id} \leq A \leq \lambda\,{\rm Id}$ and $u \in W^{1,2}(\mathbf{D} \cap H)$ is such that*

$$\int_{\mathbf{D}\cap H} |\nabla u|^2 \leq 1\,, \qquad \int_{\mathbf{D}\cap H} (A\nabla u) \cdot \nabla\varphi \leq \varepsilon_{\rm har} \|\nabla\varphi\|_\infty\,,$$

for every $\varphi \in C^1(\mathbf{D})$ with $\varphi = 0$ on $H \cap \partial\mathbf{D}$, then there exists $v \in W^{1,2}(\mathbf{D} \cap H)$ such that

$$\int_{\mathbf{D}\cap H} |u - v|^2 \leq \tau\,, \quad \int_{\mathbf{D}\cap H} |\nabla v|^2 \leq 1\,, \quad \int_{\mathbf{D}\cap H} (A\nabla v) \cdot \nabla\varphi = 0\,,$$

for every $\varphi \in C^1(\mathbf{D} \cap H)$ with $\varphi = 0$ on $H \cap \partial\mathbf{D}$.

Proof of Lemma 4.7. By contradiction; see, for example, [DM09, Lemma 2.1]. □

Proof of Lemma 4.6. Step one: By (3.1) and Remark 2.2, we may reduce to prove the following statement (where $H = \{x_1 \geq -t\}$ for some $t \geq 0$, and $G^+ = G \cap H$ for every $G \subset \mathbb{R}^n$). If

$$\begin{aligned}
&\Phi \in \boldsymbol{\mathcal{E}}(\mathbf{C}_{16} \cap H, \lambda, \ell)\,,\\
&E \text{ is a } (\Lambda, r_0)\text{-minimizer of } \mathbf{\Phi} \text{ in } (\mathbf{C}_{16}, H) \text{ with } r_0 > 8\,,\\
&0 \in {\rm cl}(H \cap {\rm spt}\,\partial E)\ ,\\
&\mathbf{exc}_n^H(E, 0, 8) + (\Lambda + \ell) < \varepsilon_{\rm tilt}\,,
\end{aligned} \tag{4.92}$$

and

$$\begin{aligned}
&\text{either} && 0 \in \partial H \text{ and } \nabla\Phi(0, e_n) \cdot e_1 = 0\,, && \text{(boundary case)} && (4.93)\\
&\text{or} && {\rm dist}(0, \partial H) > 1\,, && \text{(interior case)} && (4.94)
\end{aligned}$$

then there exists a linear map $L : \mathbb{R}^n \to \mathbb{R}^n$ with $L(H) = H$ such that

$$\|\nabla L - {\rm Id}\,\|^2 \leq C_4 \Big(\mathbf{exc}_n^H(E, 0, 8) + (\Lambda + \ell)\Big)\,, \tag{4.95}$$

$$\mathbf{exc}_n^H(L(E), 0, \beta) \leq C_4 \Big(\beta^2\,\mathbf{exc}_n^H(E, 0, 8) + \beta\,(\Lambda + \ell)\Big)\,, \tag{4.96}$$

$$\Phi^L \in \boldsymbol{\mathcal{E}}(\mathbf{C}_{2\beta}^+, \widetilde{\lambda}, \widetilde{\ell})\,, \tag{4.97}$$

$$L(E) \text{ is a } (\Lambda, \widetilde{r_0})\text{-minimizer of } \mathbf{\Phi}^L \text{ on } (\mathbf{C}_{2\beta}, H)\,, \tag{4.98}$$

$$\max\Big\{\frac{|\widetilde{\lambda} - \lambda|}{\lambda}, \frac{|\widetilde{\ell} - \ell|}{\ell}, \frac{|\widetilde{r_0} - r_0|}{r_0}\Big\} \leq C_4 \Big(\mathbf{exc}_n^H(E, 0, 8) + (\Lambda + \ell)\Big)^{1/2}\,, \tag{4.99}$$

and

$$\nabla\Phi^L(0, e_n) \cdot e_1 = 0 \qquad \text{if (4.93) holds}\,. \tag{4.100}$$

Step two: Given $\sigma \in (0, 1/4)$, let us consider the constant $\varepsilon_{\rm lip} = \varepsilon_{\rm lip}(n, \lambda, \sigma)$ determined by Lemma 4.3. We shall work under the assumption that

$$\varepsilon_{\rm tilt} < \min\Big\{\varepsilon_{\rm lip}(n, \lambda, \sigma), \frac{1}{8\,\lambda}\Big\}\,. \tag{4.101}$$

Of course, this will be compatible with $\varepsilon_{\rm tilt} = \varepsilon_{\rm tilt}(n, \lambda, \beta)$ as we shall fix (later on in the argument) a definite (sufficiently small) value of σ depending on n, λ, and β only. This said, by (4.92), we can apply Lemma 4.3 to find a Lipschitz function $u : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfying

$$\begin{aligned}
&\sup_{\mathbb{R}^{n-1}} |u| \leq \sigma\,, && {\rm Lip}(u) \leq 1\,,\\
&\mathcal{H}^{n-1}(M \Delta \Gamma) \leq C_2\,\mathbf{exc}_n^H(E, 0, 8)\,, && \int_{\mathbf{D}^+} |\nabla u|^2 \leq C_2\,\mathbf{exc}_n^H(E, 0, 8)\,,
\end{aligned} \tag{4.102}$$

where $C_2 = C_2(n,\lambda)$ (note in particular that C_2 does not depend on σ), and

$$M = \mathbf{C}^+ \cap \partial E \subset \mathbf{D}^+ \times (-\sigma,\sigma)\,, \qquad \Gamma = \big\{(z,u(z)) : z \in \mathbf{D}^+\big\}\,. \tag{4.103}$$

Moreover, setting $\Phi_0(\nu) = \Phi(0,\nu)$, we also know that

$$\int_{\mathbf{D}^+} \Big(\nabla^2\Phi_0(e_n)(\nabla u,0)\Big)\cdot(\nabla\varphi,0) \le C_2\,\|\nabla\varphi\|_\infty \Big(\mathbf{exc}_n^H(E,0,8) + (\Lambda+\ell)\Big)\,, \tag{4.104}$$

for every $\varphi \in C^1(\mathbf{D})$ with $\varphi = 0$ on $(\partial\mathbf{D})^+$. (Notice that $(\partial\mathbf{D})^+$ is half $\partial\mathbf{D}$ in the boundary case, and it it actually coincides with the whole $\partial\mathbf{D}$ in the interior case.) Let us set

$$\chi = C_2\Big(\mathbf{exc}_n^H(E,0,8) + (\Lambda+\ell)\Big) \le C_2\,\varepsilon_{\rm tilt} \tag{4.105}$$

and let us define

$$u_0 = \frac{u}{\sqrt{\chi}}\,, \qquad A_{ij} = \nabla^2\Phi_0(e_n)e_i\cdot e_j \qquad i\,,j = 1\dots,n-1\,. \tag{4.106}$$

If we require $\varepsilon_{\rm tilt} < 1/C_2(n,\lambda)$, then $\chi < 1$, while $\lambda^{-1}{\rm Id}_{\,n-1} \le A \le \lambda{\rm Id}_{\,n-1}$ thanks to (1.9) and (1.10). Moreover, by (4.102) and (4.104),

$$\int_{\mathbf{D}^+}|\nabla u_0|^2 \le 1\,, \quad \int_{\mathbf{D}^+} A\nabla u_0\cdot\nabla\varphi \le \|\nabla\varphi\|_\infty\sqrt{\chi}\,, \tag{4.107}$$

for every $\varphi \in C^1(\mathbf{D})$ with $\varphi = 0$ on $(\partial\mathbf{D})^+$. Let us now introduce, in addition to σ, an additional parameter $\tau > 0$ to be fixed later on depending on n, λ, and β only. In this way it makes sense to require that $\varepsilon_{\rm tilt} \le \varepsilon_{\rm har}(n,\lambda,\tau)/C_2(n,\lambda)$. Correspondingly, (4.105) and (4.107) allows us to apply Lemma 4.7 to find $v_0 \in W^{1,2}(\mathbf{D}^+)$ with

$$\int_{\mathbf{D}^+}|\nabla v_0|^2 \le 1\,, \qquad \int_{\mathbf{D}^+}(A\nabla v_0)\cdot\nabla\varphi = 0\,, \tag{4.108}$$

for every $\varphi \in C^1(\mathbf{D}^+)$ with $\varphi = 0$ on $(\partial\mathbf{D})^+$, and

$$\int_{\mathbf{D}^+}|u_0 - v_0|^2 \le \tau\,. \tag{4.109}$$

By elliptic regularity, there exists a constant $C = C(n,\lambda) > 1$ such that if

$$w_0(z) = v_0(0) + \nabla v_0(0)\cdot z\,, \qquad z \in \mathbf{D}\,, \tag{4.110}$$

is the tangent map to v_0 at the origin, then we have

$$|\nabla w_0| \le C\,, \tag{4.111}$$

$$(A\nabla w_0)\cdot e_1 = 0\,, \qquad \text{in the boundary case (4.93)}\,, \tag{4.112}$$

as well as, for every $s \le 1/2$,

$$|w_0(0)|^2 \le \frac{C}{s^{n-1}}\int_{\mathbf{D}_s^+}|v_0|^2\,, \qquad \frac{1}{s^{n-1}}\int_{\mathbf{D}_s^+}\frac{|v_0 - w_0|^2}{s^2} \le C\,s^2\,. \tag{4.113}$$

Let us now set

$$v = \sqrt{\chi}v_0\,, \qquad w = \sqrt{\chi}\,w_0\,, \qquad \nu = \frac{(-\nabla w,1)}{\sqrt{1+|\nabla w|^2}}\,, \qquad c = \frac{w(0)}{\sqrt{1+|\nabla w|^2}}\,. \tag{4.114}$$

By (4.111), $|\nabla w| \le C\sqrt{\chi}$ and, provided $\varepsilon_{\rm tilt}$ is sufficiently small (with respect to a constant depending on n and λ only), we have

$$|\nu - e_n| \le C(n,\lambda)|\nabla w| \le C(n,\lambda)\sqrt{\chi}\,. \tag{4.115}$$

We now claim that, if $|\nu - e_n| \le 1/4$ and we are in the boundary case (4.93), then

$$|\nabla\Phi_0(\nu)\cdot e_1| \le C(n,\lambda)\,\chi\,, \tag{4.116}$$

Indeed, by zero-homogeneity of $\nabla\Phi_0$, one has $\nabla\Phi_0(\nu)=\nabla\Phi_0(-\sqrt{\chi}\,\nabla w_0,1)$, and then (4.93), (4.106), (4.112), and a Taylor expansion (recall (1.9)) imply

$$
\begin{aligned}
|\nabla\Phi_0(\nu)\cdot e_1| &\le \big|\nabla\Phi_0(e_n)\cdot e_1+\big(\nabla^2\Phi_0(e_n)(\sqrt{\chi}\,\nabla w_0,0)\big)\cdot e_1\big|+C|\sqrt{\chi}\,\nabla w_0|^2\\
&= C|\sqrt{\chi}\,\nabla w_0|^2\le C\,\chi\,,
\end{aligned}
$$

for $C=C(n,\lambda)$. Up to further decrease the value of $\varepsilon_{\rm tilt}$ depending on n and λ only, (4.116) enables us to apply Lemma 2.7 to deduce that, if we are in the boundary case (4.93), then there exists $\nu_0\in\mathbf{S}^{n-1}$ such that

$$
\nabla\Phi_0(\nu_0)\cdot e_1=0\,, \tag{4.117}
$$

$$
|\nu_0-\nu|<C(n,\lambda)\,\chi\,. \tag{4.118}
$$

In the interior case, (4.94), we simply set $\nu_0=\nu$, so that (4.118) holds true in both cases. We now notice that, by (4.109) and (4.113), if $s\le 1/2$, then, for $C=C(n,\lambda)$,

$$
\frac{1}{s^{n-1}}\int_{\mathbf{D}_s^+}\frac{|u-w|^2}{s^2}\le C\Big(\frac{1}{s^{n-1}}\int_{D_s^+}\frac{|v-w|^2}{s^2}+\frac{\tau\,\chi}{s^{n+1}}\Big)\le C\Big(s^2+\frac{\tau}{s^{n+1}}\Big)\,\chi\,. \tag{4.119}
$$

By taking into account the definition of c in (4.114), and thanks to (4.113) and (4.109), we find

$$
|c|^2\le\chi|w_0(0)|^2\le C\,\chi\int_{\mathbf{D}_{1/2}^+}|v_0|^2\le C\Big(\chi\int_{\mathbf{D}_{1/2}^+}|v_0-u_0|^2+\int_{\mathbf{D}_{1/2}^+}|u|^2\Big\}\le C(\chi+\sigma)\,, \tag{4.120}
$$

for $C=C(n,\lambda)$, and where we have also taken into account that $u=\sqrt{\chi}u_0$ and $|u|\le\sigma$, as well as that $\sigma,\tau<1$. Moreover, by (4.114) and (4.111), for some $C=C(n,\lambda)$ we find

$$
\sup_{x\in M\cup\Gamma}|x\cdot\nu|^2\le\sup_{x\in M\cup\Gamma}\big(|\mathbf{p}x|\,|\nabla w|+|\mathbf{q}x|\big)^2\le C(\sqrt{\chi}+\sigma)^2\le C(\chi+\sigma)\,, \tag{4.121}
$$

where we have used that $\sigma,\chi<1$. Finally, setting $\mathbf{K}_s^+=\mathbf{D}_s^+\times(-1,1)\subset\mathbf{C}$ and using that we have both $\tau<1$ and $\chi<1$,

$$
\begin{aligned}
\frac{1}{s^{n-1}}\int_{\mathbf{K}_s^+\cap\partial^*E}\frac{|x\cdot\nu_0-c|^2}{s^2} &\le \frac{2}{s^{n-1}}\int_{\mathbf{K}_s^+\cap\partial^*E}\frac{|x\cdot\nu-c|^2}{s^2}+\frac{4P(E;\mathbf{C}^+)|\nu-\nu_0|^2}{s^{n+1}}\\
\text{(by (2.47) and (4.118))}\qquad &\le \frac{2}{s^{n-1}}\int_{\mathbf{K}_s^+\cap\partial^*E}\frac{|x\cdot\nu-c|^2}{s^2}+C\,\frac{\chi^2}{s^{n+1}}\\
&\le \frac{2}{s^{n-1}}\int_{\mathbf{K}_s^+\cap\Gamma}\frac{|x\cdot\nu-c|^2}{s^2}+C\,\frac{\chi^2}{s^{n+1}}\\
&\quad+\frac{C}{s^{n+1}}\,\mathcal{H}^{n-1}(M\Delta\Gamma)\Big(|c|^2+\sup_{x\in M\cup\Gamma}|x\cdot\nu|^2\Big)\\
\text{(by (4.102), (4.120), and (4.121))}\qquad &\le \frac{2}{s^{n-1}}\int_{\mathbf{K}_s^+\cap\Gamma}\frac{|x\cdot\nu-c|^2}{s^2}+\frac{C}{s^{n+1}}\,\chi\,(\chi+\sigma)\\
\text{(by (4.114))}\qquad &= \frac{2}{s^{n-1}}\int_{\mathbf{D}_s^+}\frac{|u-w|^2\sqrt{1+|\nabla u|^2}}{s^2\sqrt{1+|\nabla w|^2}}+\frac{C}{s^{n+1}}\chi\,(\chi+\sigma)\\
\text{(since Lip}(u)\le 1)\qquad &\le \frac{2\sqrt{2}}{s^{n-1}}\int_{\mathbf{D}_s^+}\frac{|u-w|^2}{s^2}+\frac{C}{s^{n+1}}\chi\,(\chi+\sigma)\\
\text{(by (4.119))}\qquad &\le C\,\chi\Big(s^2+\frac{\chi+\sigma+\tau}{s^{n+1}}\Big)\,.
\end{aligned}
$$

where $C = C(n,\lambda)$. We plug the value $s = 32\beta \le 1/2$ into this estimate so that, recalling the definition of χ (4.105), we get

$$\frac{1}{\beta^{n-1}}\int_{\mathbf{K}^+_{32\,\beta}\cap\partial^* E}\frac{|x\cdot\nu_0-c|^2}{\beta^2}\le C(n,\lambda)\Big(\beta^2+\frac{\chi+\sigma+\tau}{\beta^{n+1}}\Big)\Big(\mathbf{exc}_n^H(E,0,8)+(\Lambda+\ell)\Big)\,.$$

If we first choose $\sigma=\tau=\beta^{n+3}$ and then $\varepsilon_{\rm tilt}<\beta^{n+3}$, then the above estimate gives

$$\frac{1}{\beta^{n-1}}\int_{\mathbf{K}^+_{32\beta}\cap\partial^* E}\frac{|x\cdot\nu_0-c|^2}{\beta^2}\le C(n,\lambda)\beta^2\Big(\mathbf{exc}_n(E,0,8)+(\Lambda+\ell)\Big)\,. \tag{4.122}$$

We notice that, by (4.115) and (4.118), one has

$$|\nu_0-e_n|\le C(n,\lambda)\,\sqrt{\chi}\,. \tag{4.123}$$

We now use Lemma 3.7 to construct the map L. More precisely, (4.123) ensures that $|\nu_0\cdot e_1|\le 1/2$, provided $\varepsilon_{\rm tilt}$ is small enough. Then we can apply Lemma 3.7 to ν_0 to construct a linear map $L:\mathbb{R}^n\to\mathbb{R}^n$ with $L(H)=H$ and

$$L(\nu_0^\perp)=e_n^\perp\,,\quad\text{so that}\quad e_n=\frac{(\mathrm{cof}\,\nabla L)\nu_0}{|(\mathrm{cof}\,\nabla L)\nu_0|}\,, \tag{4.124}$$

$$\nabla\Phi^L(0,e_n)\cdot e_1=\nabla\Phi_0(\nu_0)\cdot e_1\,, \tag{4.125}$$

$$\|\nabla L-\mathrm{Id}\,\|\le C(n,\lambda)\,\sqrt{\chi}\,,\qquad \det\,\nabla L=1\,. \tag{4.126}$$

Thanks to Lemma 2.18, $\Phi^L\in\boldsymbol{\mathcal{E}}(L(\mathbf{C}_{16})\cap H,\widetilde{\lambda},\widetilde{\ell})$, $L(E)$ is a $(\Lambda,\widetilde{r_0})$-minimizer of $\mathbf{\Phi}^L$ on $(L(\mathbf{C}_{16}),H)$ and

$$\max\Big\{\frac{|\widetilde{\lambda}-\lambda|}{\lambda},\frac{|\widetilde{\ell}-\ell|}{\ell},\frac{|\widetilde{r_0}-r_0|}{r_0}\Big\}\le C(n,\lambda)\,\sqrt{\chi}\,. \tag{4.127}$$

Note that $\widetilde{\Lambda}=\Lambda$ in the application of Lemma 2.18, since $\det\,\nabla L=1$. This proves (4.97), (4.98), and (4.99). (Indeed $\mathbf{C}_{2\beta}\subset L(\mathbf{C}_{16})$ as, trivially, $\mathbf{C}_{2\beta}\subset\mathbf{C}$, and as one can make L close enough to the identity to ensure $\mathbf{C}\subset L(\mathbf{C}_{16})$ by (4.126) and up to further tuning the value of $\varepsilon_{\rm tilt}$.) Also, (4.125) implies (4.100) when we are in the boundary case. We are thus left to prove (4.96). Up to further decrease $\varepsilon_{\rm tilt}$, by (4.123) we can entail the inclusion $\mathbf{C}_{4\beta}\subset L(\mathbf{K}_{32\beta})$, so that, if

$$\widetilde{c}=\frac{c}{|(\mathrm{cof}\,\nabla L)\nu_0|}\,,$$

then by the area formula we find that

$$\int_{\mathbf{C}^+_{4\beta}\cap L(\partial^*E)}|x\cdot e_n-\widetilde{c}|^2\,d\mathcal{H}^{n-1}\le\int_{\partial^*E\cap\mathbf{K}^+_{32\beta}}\Big|Lx\cdot\frac{(\mathrm{cof}\,\nabla L)\nu_0}{|(\mathrm{cof}\,\nabla L)\nu_0|}-\frac{c}{|\mathrm{cof}\,\nabla L\nu_0|}\Big|^2|(\mathrm{cof}\,\nabla L)\nu_E|\,d\mathcal{H}^{n-1}\,. \tag{4.128}$$

Now, $\det L=1$ so that $L^*(\mathrm{cof}\,\nabla L)=\mathrm{Id}$ (see (2.117)). Hence

$$Lx\cdot(\mathrm{cof}\,\nabla L)\nu_0=\,x\cdot\nu_0\,,\qquad\forall x\in\mathbb{R}^n\,,$$

and thus, taking also into account that, by (4.126),

$$\frac{|(\mathrm{cof}\,\nabla L)\nu_E|}{|(\mathrm{cof}\,\nabla L)\nu_0|^2}\le 1+C(n)\,\|\nabla L-\mathrm{Id}\,\|\le C(n,\lambda)\,,$$

we deduce from (4.128) that

$$\int_{\mathbf{C}^+_{4\beta}\cap L(\partial^*E)}|x\cdot e_n-\widetilde{c}|^2\,d\mathcal{H}^{n-1}\le C(n,\lambda)\int_{\partial^*E\cap\mathbf{K}^+_{32\beta}}|x\cdot\nu_0-c|^2\,d\mathcal{H}^{n-1}\,.$$

Hence (4.122) implies that

$$\mathbf{flat}_n^H(L(E),0,4\beta)\le C(n,\lambda)\,\beta^2\Big(\mathbf{exc}_n^H(E,0,8)+(\Lambda+\ell)\Big)\,. \tag{4.129}$$

We now want to apply Lemma 4.4 to $L(E)$. To this end, we start noticing that, up to decrease the value of $\varepsilon_{\rm tilt}$ in order to entail $L^{-1}(\mathbf{K}_{8\beta})\subset\mathbf{C}_8$, and setting $M=\mathrm{cof}\,\nabla L$ for the sake of brevity, we have

$$\begin{aligned}
2\,\mathbf{exc}_n^H(L(E),0,8\beta) &= \frac{1}{(8\beta)^{n-1}}\int_{\mathbf{K}_{8\beta}^+\cap\partial^* L(E)}|\nu_{L(E)}-e_n|^2\,d\mathcal{H}^{n-1}\\
&= \frac{1}{(8\beta)^{n-1}}\int_{[L^{-1}(\mathbf{K}_{8\beta})]^+\cap\partial^* E}\Big|\frac{M\nu_E}{|M\nu_E|}-\frac{M\nu_0}{|M\nu_0|}\Big|^2\,|M\nu_E|\,d\mathcal{H}^{n-1}\\
&\le \frac{1+C(n,\lambda)\,\|\nabla L-\mathrm{Id}\,\|}{(8\beta)^{n-1}}\int_{\mathbf{C}_8^+\cap\partial^* E}|\nu_E-\nu_0|^2\,d\mathcal{H}^{n-1}\\
\text{(by (4.126))}\qquad &\le \frac{C(n,\lambda)}{\beta^{n-1}}\Big(\int_{\mathbf{C}_8^+\cap\partial^* E}|\nu_E-e_n|^2\,d\mathcal{H}^{n-1}+P(E;\mathbf{C}_8^+)\,|\nu_0-e_n|^2\Big)\\
\text{(by (2.47) and (4.123))}\qquad &\le \frac{C(n,\lambda)}{\beta^{n-1}}\,\chi<\varepsilon_{\rm Ca}(n,2\lambda)\,,
\end{aligned}\tag{4.130}$$

provided $\varepsilon_{\rm tilt}$ is small enough. In the same way, we can deduce from (4.127) that

$$\widetilde{\lambda}\le 2\lambda\,,\qquad \widetilde{\ell}\le 2\ell\,,\qquad \widetilde{r}_0\ge r_0/2\,,\tag{4.131}$$

and $\mathbf{C}_{16\beta}\subset L(\mathbf{C}_{16})$, again provided $\varepsilon_{\rm tilt}$ is small enough. In particular, since $\beta<1/64$, $r_0>8$, and $(\Lambda+\ell)<1/8\lambda$ by (4.101), we find that

$$\begin{aligned}
&\Phi^L\in\boldsymbol{\mathcal{E}}(\mathbf{C}_{16\,\beta}^+,2\lambda,2\ell)\,,\\
&L(E)\text{ is a }(2\Lambda,r_0/2)\text{-minimizer of }\mathbf{\Phi}^L\text{ in }(\mathbf{C}_{16\,\beta},H)\text{ with }8\,\beta<r_0/2\,,\\
&16\,(2\lambda)\Lambda\,\beta+8(2\ell)\,\beta\le 1\,,
\end{aligned}$$

with $\mathbf{exc}_n^H(L(E),0,8\beta)<\varepsilon_{\rm Ca}$ (by (4.130)), $0\in\mathrm{cl}(H\cap\mathrm{spt}\partial E)\cap\partial H$ and, when we are in the boundary case, with $\nabla\Phi^L(0,e_n)\cdot e_1=0$ (by (4.100)). We can thus apply Lemma 4.4 to $L(E)$ at scale 16β to deduce that

$$\mathbf{exc}_n^H(L(E),0,\beta)\le C_3\Big(\mathbf{flat}_n^H(L(E),0,4\beta)+\beta(\Lambda+\ell)\Big)\,.$$

We combine this estimate with (4.129) to obtain

$$\mathbf{exc}_n^H(L(E),0,\beta)\le C(n,\lambda)\Big(\beta^2\,\mathbf{exc}_n^H(E,0,8)+\beta\,(\Lambda+\ell)\Big)\,,$$

that is (4.96). This completes the proof of the lemma. □

4.5. **Proof of Lemma 3.4.** By scaling, see (3.1) and Remark 2.2, we can directly set $r=1$. With the notation $G^+=G\cap H$ for $G\subset\mathbb{R}^n$, we thus want to prove that, setting $\Phi_0(\nu)=\Phi(0,\nu)$, if

$$\Phi\in\boldsymbol{\mathcal{E}}(\mathbf{C}_{128}^+,\lambda,\ell)\,,\tag{4.132}$$
$$E\text{ is a }(\Lambda,r_0)\text{-minimizer of }\mathbf{\Phi}\text{ in }(\mathbf{C}_{128},H)\text{ with }r_0\ge 64\,,$$
$$0\in\mathrm{cl}(H\cap\partial E)\,,$$
$$|\nabla\Phi_0(e_n)\cdot e_1|+\mathbf{exc}_n^H(E,0,64)+(\Lambda+\ell)<\varepsilon_{\rm reg}\,,\tag{4.133}$$

then there exists $u\in C^{1,1/2}(\mathrm{cl}(\mathbf{D}^+))$ such that

$$\sup_{z,y\in\mathbf{D}^+}|u(x)|+|\nabla u(x)|+\frac{|\nabla u(x)-\nabla u(y)|}{|x-y|^{1/2}}\le C(n,\lambda)\sqrt{\varepsilon_{\rm reg}}\,,\tag{4.134}$$
$$\mathbf{C}^+\cap\partial E=\Big\{x\in H:|\mathbf{p}x|<1\,,\mathbf{q}x=u(\mathbf{p}x)\Big\}\,,\tag{4.135}$$
$$\nabla\Phi\big((z,u(z)),(-\nabla u(z),1)\big)\cdot e_1=0\,,\qquad\forall z\in\mathbf{D}\cap\partial H\,.\tag{4.136}$$

We divide the proof into four steps.

Step one: We claim that for every $x \in \mathrm{cl}(H \cap \partial E) \cap \partial H \cap \mathbf{C}_{16}$ there exists an affine map $L : \mathbb{R}^n \to \mathbb{R}^n$ (depending on x) with $L(x) = x$, $L(H) = H$, and

$$\|\nabla L - \mathrm{Id}\,\|^2 \le C\,\varepsilon_{\rm reg}\,, \qquad \mathbf{exc}_n^H(L(E), x, \varrho) \le C\,\varepsilon_{\rm reg}\,\varrho\,, \qquad \forall \varrho \le 16\,, \tag{4.137}$$

where $C = C(n, \lambda)$. Firsy we notice that it suffices to prove this under the assumption that

$$\varepsilon(x) = |\nabla\Phi(x, e_n) \cdot e_1| + \mathbf{exc}_n^H(E, x, 32) + (\Lambda + \ell) \le \varepsilon_0\,, \tag{4.138}$$

for a suitably small positive constant $\varepsilon_0 = \varepsilon_0(n, \lambda)$. Indeed, by (4.132), (1.8), and (4.133), if $x \in \mathbf{C}_{16} \cap \partial H$, then

$$\begin{aligned} |\nabla\Phi(x, e_n) \cdot e_1| &\le |\nabla\Phi_0(e_n) \cdot e_1| + 32\ell \le 32\,\varepsilon_{\rm reg}\,, \\ \mathbf{exc}_n^H(E, x, 32) &\le 2^{n-1}\mathbf{exc}_n^H(E, 0, 64) \le 2^{n-1}\varepsilon_{\rm reg}\,, \end{aligned}$$

so that $\varepsilon(x) \le C(n)\,\varepsilon_{\rm reg}$ for every $x \in \mathbf{C}_{16} \cap \partial H$; in particular, we can ensure the validity of (4.138) at every $x \in \mathrm{cl}(H \cap \partial E) \cap \partial H \cap \mathbf{C}_{16}$ provided we pick $\varepsilon_{\rm reg}$ sufficiently small.

We now prove our claim, setting ε in place of $\varepsilon(x)$ for the sake of brevity. By exploiting the convergence of the geometric series, it will suffice to prove the following statement:

There exist positive constants $\varepsilon_*, \beta_* < 1$, K_1 and K_2 (depending on n and λ only) such that, if $\varepsilon \le \varepsilon_*$, then for every $k \in \mathbb{N}$ there exists an affine map $L_k : \mathbb{R}^n \to \mathbb{R}^n$ with $L_k(x) = x$, $L_k(H) = H$ and

$$\begin{cases} \|\nabla L_k - \nabla L_{k-1}\|^2 \le K_1 \beta_*^k \varepsilon\,, & \text{if } k \ge 1\,, \\ \|\nabla L_0 - \mathrm{Id}\,\|^2 \le K_1\,\varepsilon\,, \end{cases} \tag{4.139}$$

such that,

$$\Phi_k = \Phi^{L_k} \in \boldsymbol{\mathcal{E}}\big(\mathbf{C}^+_{x,2\beta_*^k}, \lambda_k, \ell_k\big)\,, \tag{4.140}$$

$$E_k = L_k(E) \text{ is a } (\Lambda, r_{0,k}) \text{ minimizer of } \mathbf{\Phi}_k \text{ in } (\mathbf{C}_{x,2\beta_*^k}, H)\,, \tag{4.141}$$

$$\nabla\Phi^{L_k}(x, e_n) \cdot e_1 = 0\,, \tag{4.142}$$

$$\mathbf{exc}_n^H(E_k, x, \beta_*^k) \le K_2 \beta_*^k \varepsilon\,. \tag{4.143}$$

where $\lambda_0 = 2\lambda$, $\ell_0 = 2\ell$, $r_{0,0} = r_0/2$, and

$$\max\Big\{\frac{|\lambda_k - \lambda_{k-1}|}{\lambda}, \frac{|\ell_k - \ell_{k-1}|}{\ell}, \frac{|r_{0,k} - r_{0,k-1}|}{r_{0,k-1}}\Big\} \le K_1\,\sqrt{\beta_*^k\,\varepsilon}, \qquad \forall k \ge 1\,. \tag{4.144}$$

We prove this statement by induction.

Base case: If ε_* is small enough, then by $|\nabla\Phi(x, e_n) \cdot e_1| < \varepsilon$ we can apply Lemma 2.7 to find $\nu_0 \in \mathbf{S}^{n-1}$ such that

$$|\nu_0 - e_n| \le C(n, \lambda)\,\varepsilon\,, \qquad \nabla\Phi(x, \nu_0) \cdot e_1 = 0\,. \tag{4.145}$$

Up to further decrease ε_* so to entail $|e_1 \cdot \nu_0| \le 1/2$, we can apply Lemma 3.7 to ν_0 to find an affine map $L_0 : \mathbb{R}^n \to \mathbb{R}^n$ with $L_0(H) = H$, $L_0(x) = x$, $\|\nabla L_0 - \mathrm{Id}\,\| \le C(n, \lambda)\,|e_n - \nu_0|$, and $\nabla\Phi^{L_0}(x, e_n) \cdot e_1 = \nabla\Phi(x, \nu_0) \cdot e_1$ (so that (4.139) and (4.142) hold true by (4.145)). The validity of (4.140) and (4.141) is easily checked thanks to Lemma 2.18 and (4.139), up to further decrease the value of ε_*. Finally, by exploiting (4.139) (with $k = 0$) and (4.145) as in the proof of (4.130), we see that, if ε_* is small enough (also to entail that $L_0^{-1}(\mathbf{C}_x) \subset \mathbf{C}_{x,64}$), then we have

$$\mathbf{exc}_n^H(L_0(E), x, 1) \le C(n, \lambda)\,(32)^{n-1}\mathbf{exc}_n^H(E, x, 32) \le C(n, \lambda)\,\varepsilon\,.$$

This proves the case $k = 0$ of our claim.

Choice of ε_*, β_*, K_1 *and* K_2: Since ε_*, β_*, K_1 and K_2 have to be chosen in a careful order, it seems useful to fix their choice before entering into the inductive step. We shall pick $\beta_* = \beta_*(n,\lambda)$ so that

$$\beta_* < \min\left\{\frac{1}{512}\,,\frac{1}{64\,C_4(n,3\lambda)}\right\}, \tag{4.146}$$

where $C_4(n,3\lambda)$ is defined by means of Lemma 4.4. By (4.146), it is possible to choose $K_2 = K_2(n,\lambda)$ so that

$$K_2 \ge \frac{3\,C_4(n,3\lambda)}{1-64\,C_4(n,3\lambda)\,\beta_*}\,. \tag{4.147}$$

Finally, we choose $K_1 = K_1(n,\lambda)$ so that

$$K_1 \ge \frac{3\,C_4(n,3\lambda)\,\sqrt{K_2+3}}{\sqrt{\beta_*}}\,, \tag{4.148}$$

and in such a way that the case $k=0$ of (4.139) and (4.144) holds true. Finally, ε_* shall be chosen to be small enough with respect to other constants determined by n, λ, β_*, K_1 and K_2.

Inductive step: Let us assume our claim holds true for $j \le k$ and let us prove its validity for $j = k+1$. To this end, we notice that, by exploiting (4.144), and provided ε_* is small enough, we can certainly ensure that

$$\lambda_k \le 3\lambda\,, \qquad \ell_k \le 3\ell\,, \qquad r_{0,k} \ge \frac{r_0}{3}\,. \tag{4.149}$$

Let us set $\beta = 8\,\beta_* \in (0,1/64)$, so that we can consider the constant $\varepsilon_{\rm tilt}(n,3\lambda,8\beta_*)$ determined by Lemma 4.6. By the inductive step on (4.143), by (4.149) and by definition of ε, we see that

$$\mathbf{exc}_n^H(E_k,x,\beta_*^k) + (\Lambda+\ell_k)\,\beta_*^k \le K_2\,\varepsilon + 3\,(\Lambda+\ell) \le (K_2+3)\,\varepsilon\,,$$

so that, by (4.140), (4.141), (4.142) and provided we assume that

$$(K_2+3)\varepsilon_* \le \varepsilon_{\rm tilt}(n,3\lambda,8\,\beta_*)\,,$$

we can apply Lemma 4.6 with x, Φ_k, E_k, $2\beta_*^k$, $\lambda_k \le 3\lambda$, ℓ_k and $r_{0,k}$ in place of x_0, Φ, E, $16r$, λ and r_0 respectively. (Notice that we have $r_{0,k} \ge 16\beta_*^k$ thanks to (4.149), $r_0 \ge 64$, and $\beta_* < 1/512$.) Hence, there exists an affine map $\widetilde L:\mathbb{R}^n\to\mathbb{R}^n$ with $\widetilde L(x)=x$, $\widetilde L(H)=H$, and constants $\widetilde\lambda\ge 1$, $\widetilde\ell \ge 0$, and $\widetilde r_0 > 0$ such that

$$\Phi_k^{\widetilde L} \in \boldsymbol{\mathcal{E}}(\mathbf{C}^+_{x,\beta\beta_*^k/4},\widetilde\lambda,\widetilde\ell)\,, \tag{4.150}$$

$$\widetilde L(E_k) \text{ is a } (\Lambda,\widetilde r_0)\text{-minimizer of } \mathbf{\Phi}_k^{\widetilde L} \text{ in } (\mathbf{C}_{x,\beta\beta_*^k/4},H)\,, \tag{4.151}$$

$$\nabla\Phi_k^{\widetilde L}(x,e_n)\cdot e_1 = 0\,, \tag{4.152}$$

$$\mathbf{exc}_n^H\left(\widetilde L(E_k),x,\beta\frac{\beta_*^k}{8}\right) \le C_4(n,3\lambda)\left(\beta^2\,\mathbf{exc}_n^H(E_k,x,\beta_*^k) + \beta\,\frac{\beta_*^k}{8}\,(\Lambda+\ell_k)\right), \tag{4.153}$$

$$\max\left\{\|\widetilde L - {\rm Id}\,\|\,,\frac{|\widetilde\lambda-\lambda_k|}{\lambda_k},\frac{|\widetilde\ell-\ell_k|}{\ell_k},\frac{|\widetilde r_0-(r_{0,k})|}{r_{0,k}}\right\} \le C_4(n,3\lambda)\left(\mathbf{exc}_n^H(E_k,x,\beta_*^k)+(\Lambda+\ell_k)\,\beta_*^k\right)^{1/2}. \tag{4.154}$$

We claim that by setting

$$L_{k+1} = \widetilde L\circ L_k\,, \qquad \lambda_{k+1}=\widetilde\lambda\,, \qquad \ell_{k+1}=\widetilde\ell\,, \qquad r_{0,k+1}=\widetilde r_0\,, \tag{4.155}$$

the proof of the inductive step is completed. First, by (4.155) and since $\beta\,\beta_*^k/4 = 2\beta_*^{k+1}$ and $\Phi_k^{\widetilde{L}} = \Phi^{L_{k+1}}$, we see that (4.150), (4.151) and (4.142) immediately imply (4.140), (4.141), and (4.152) with $k+1$ in place of k respectively. Next we notice that, by (4.153) and by $\beta = 8\,\beta_*$,

$$\begin{aligned}
\mathbf{exc}_n^H(L_{k+1}(E),x,\beta_*^{k+1}) &\le C_4(n,3\lambda)\Big(64\,\beta_*^2\,\mathbf{exc}_n^H(E_k,x,\beta_*^k)+\beta_*^{k+1}\,(\Lambda+\ell_k)\Big)\\
\text{(by (4.143) and by (4.149))}\qquad &\le C_4(n,3\lambda)\Big(64\,K_2\,\beta_*^{k+2}\,\varepsilon+3\,\beta_*^{k+1}\,(\Lambda+\ell)\Big)\\
&\le C_4(n,3\lambda)(64\,K_2\,\beta_*+3)\,\beta_*^{k+1}\,\varepsilon\,, \qquad (4.156)
\end{aligned}$$

where in the last inequality we have used $\Lambda+\ell<\varepsilon$. By the choice (4.147) of K_2, (4.156) implies the validity of (4.143) with $k+1$ in place of k. Similarly, we notice that, by (4.143), by (4.149) and by definition of ε

$$\begin{aligned}
C_4(n,3\lambda)\Big(\mathbf{exc}_n^H(E_k,x,\beta_*^k)+(\Lambda+\ell_k)\,\beta_*^k\Big)^{1/2} &\le C_4(n,3\lambda)\Big(K_2\,\beta_*^k\,\varepsilon+3\,\varepsilon\,\beta_*^k\Big)^{1/2}\\
&= \frac{C_4(n,3\lambda)\,\sqrt{K_2+3}}{\sqrt{\beta_*}}\,\sqrt{\beta_*^{k+1}\varepsilon}\,. \qquad (4.157)
\end{aligned}$$

By (4.154), (4.155), (4.157) and (4.148) we deduce that (4.144) holds true with $k+1$ in place of k. Finally, by exploiting the validity of (4.139) for $j\le k$, we see that

$$\|\nabla L_k\| \le 1+\|\nabla L_0-\mathrm{Id}\,\|+\sum_{j=0}^{k-1}\|\nabla L_{j+1}-\nabla L_j\| \le 1+\Big(1+\frac{\sqrt{\varepsilon_*}}{1-\sqrt{\beta_*}}\Big)\sqrt{K_1}\le 3\,,$$

provided ε_* is small enough. Hence, by (4.154) and (4.157), we find

$$\begin{aligned}
\|\nabla L_{k+1}-\nabla L_k\| &\le \|\nabla L_k\|\|\nabla\widetilde{L}-\mathrm{Id}\,\|\\
&\le \frac{3\,C_4(n,3\lambda)\,\sqrt{K_2+3}}{\sqrt{\beta_*}}\sqrt{\beta_*^{k+1}\varepsilon}\le K_1\sqrt{\beta_*^{k+1}\varepsilon}\,,
\end{aligned}$$

once again thanks to (4.148). This completes the proof of step one.

Step two: The argument used in step one, where now the interior case of Lemma 4.6 is used in place of the boundary case at each step of the iteration, allows us to prove the following statement: There exists $\varepsilon_{**}=\varepsilon_{**}(n,\lambda)$ such that, if $x\in\mathrm{cl}(H\cap\partial E)\cap H\cap\mathbf{C}_{16}$ with

$$\varepsilon=\mathbf{exc}_n^H\big(E,x,2\,\mathrm{dist}(x,\partial H)\big)+2\mathrm{dist}(x,\partial H)(\Lambda+\lambda)\le\varepsilon_{**}\,, \qquad (4.158)$$

then there exists an affine map $L:\mathbb{R}^n\to\mathbb{R}^n$ (depending on x) with $L(x)=x$, $L(H)=H$, and

$$\|\nabla L-\mathrm{Id}\,\|^2\le C(n,\lambda)\,\varepsilon\,,\qquad \mathbf{exc}_n^H(L(E),x,\varrho)\le C(n,\lambda)\,\varepsilon\,\varrho\,,\qquad \forall\,\varrho\le\mathrm{dist}(x,\partial H)\,. \qquad (4.159)$$

This statement is an "interior" analogous to the "boundary" statement proved in step one, with (4.138) playing the role of (4.158). The only difference is that (4.138) follows directly from (4.133), while (4.158) cannot be so immediately deduced from it. Showing the validity of (4.158) at every $x\in\mathbf{C}_{16}^+\cap\partial E$ is, essentially, the content of the next step of the proof.

Step three: We now prove that for every $x\in\mathrm{cl}(H\cap\partial E)\cap\mathbf{C}$ there exists an affine map L such that $L(H)=H$ and

$$\|\nabla L-\mathrm{Id}\,\|^2\le C(n,\lambda)\,\varepsilon_{\mathrm{reg}}\,, \qquad (4.160)$$

$$\mathbf{exc}_n^H(L(E),L(x),\varrho)\le C(n,\lambda)\,\varepsilon_{\mathrm{reg}}\,\varrho\,,\qquad \forall\,\varrho\le 8\,. \qquad (4.161)$$

We start with the following simple observation: if $\varepsilon_{\rm reg}$ is sufficiently small with respect to $\varepsilon_{\rm hb}(n,\lambda,1/32)$, then by applying Lemma 4.1 to E in $\mathbf{C}_8$ we have

$$\begin{aligned}\sup\Big\{|\mathbf{q}y|:y\in\mathbf{C}_4^+\cap\partial E\Big\}&\le\frac{1}{32}\,,\\ \Big|\Big\{y\in\mathbf{C}_4^+\cap E:\mathbf{q}y>\frac{1}{32}\Big\}\Big|&=0\,,\\ \Big|\Big\{y\in\big(\mathbf{C}_4^+\setminus E\big):\mathbf{q}<-\frac{1}{32}\Big\}\Big|&=0\,.\end{aligned}\tag{4.162}$$

From this it follows that for every $y\in\partial H\cap\mathbf{C}_2$ there exists a point $y'\in\partial H$ such that

$$y'\in{\rm cl}(\mathbf{C}_2\cap\partial E)\cap\partial H\qquad\text{and}\qquad\mathbf{p}y'=\mathbf{p}y\,.\tag{4.163}$$

Indeed, thanks to (4.162), for every $s\in(0,2)$

$$|\mathbf{K}_{\mathbf{p}y,s}\cap E|>0\qquad|\mathbf{K}_{\mathbf{p}y,s}\setminus E|>0,$$

where $\mathbf{K}_{\mathbf{p}y,s}$ is defined as in (4.34). This gives that ${\rm spt}\mu_E\cap\mathbf{K}_{\mathbf{p}y,s}\ne\emptyset$, thus (4.163), since $\partial E={\rm spt}\mu_E$ by Lemma 2.16. Let now $x\in{\rm cl}(H\cap\partial E)\cap\mathbf{C}$ and $\bar x\in\partial H\cap\mathbf{C}_2$ be such that

$$|\mathbf{p}x-\mathbf{p}\bar x|=|x-\bar x|={\rm dist}(x,\partial H)\,,\tag{4.164}$$

Let $x'\in{\rm cl}(\mathbf{C}_2\cap\partial E)\cap\partial H$ be the corresponding point satisfying (4.163). By step one, there exists an affine map $L_1:\mathbb{R}^n\to\mathbb{R}^n$ with $L_1(H)=H$, $L_1(x')=x'$, and

$$\|\nabla L_1-{\rm Id}\,\|^2\le C(n,\lambda)\,\varepsilon_{\rm reg}\,,\tag{4.165}$$

$$\mathbf{exc}_n^H(L_1(E),x',\varrho)\le C(n,\lambda)\,\varepsilon_{\rm reg}\,\varrho\,,\qquad\forall\varrho\le16\,.\tag{4.166}$$

Since $L_1(x')=x'$ and L_1 is affine, by (4.165),

$$|L_1(x)-x|=|\nabla L_1(x-x')-(x-x')|\le C(n,\lambda)\sqrt{\varepsilon_{\rm reg}}\,|x-x'|\tag{4.167}$$

and

$$\big(1-C\sqrt{\varepsilon_{\rm reg}}\big)|L_1(x)-x'|\le|x-x'|\le\big(1+C\sqrt{\varepsilon_{\rm reg}}\big)|L_1(x)-x'|.\tag{4.168}$$

In particular we can choose $\varepsilon_{\rm reg}$ sufficiently small to ensure that $L_1(x)\in\mathbf{C}_2$. We now claim that, provided $\varepsilon_{\rm reg}$ is sufficiently small,

$$|L_1(x)-x'|\le2\,{\rm dist}(L_1(x),\partial H)\,.\tag{4.169}$$

First notice that thanks to (4.167) and (4.168),

$$\begin{aligned}{\rm dist}(x,\partial H)&\le{\rm dist}(L_1(x),\partial H)+C\sqrt{\varepsilon_{\rm reg}}\,|x-x'|\\&\le{\rm dist}(L_1(x),\partial H)+C\sqrt{\varepsilon_{\rm reg}}\,|L_1(x)-x'|.\end{aligned}\tag{4.170}$$

Moreover, thanks to Lemma 2.7,

$$\begin{aligned}&\Phi^{L_1}\in\boldsymbol{\mathcal{E}}(\mathbf{C}_{127}^+,2\lambda,2\ell)\,,\\&L_1(E)\text{ is a }(\Lambda,r_0/2)\text{-minimizer of }\mathbf{\Phi}^{L_1}\text{ on }(\mathbf{C}_{127},H)\text{ with }r_0/32\ge32\,.\end{aligned}\tag{4.171}$$

By (4.166) and (4.167), if $\varepsilon_{\rm reg}$ is small enough with respect to $\varepsilon_{\rm hb}(n,2\lambda,1/8)$, we can thus apply Lemma 4.1 to $L_1(E)$ on the cylinder $\mathbf{C}(x',4\,|x'-L_1(x)|)$, to deduce that

$$|\mathbf{q}L_1(x)-\mathbf{q}x'|\le|L_1(x)-x'|/8.$$

By this, (4.167), (4.168), (4.164) and recalling that $\mathbf{p}x'=\mathbf{p}\bar x$, we obtain

$$\begin{aligned}\frac78|L_1(x)-x'|&\le|\mathbf{p}L_1(x)-\mathbf{p}x'|\le|\mathbf{p}L_1(x)-\mathbf{p}x|+|\mathbf{p}x-\mathbf{p}x'|\\&\le|L_1(x)-x|+|\mathbf{p}x-\mathbf{p}\bar x|\\&\le C\sqrt{\varepsilon_{\rm reg}}|L_1(x)-x'|+{\rm dist}(x,\partial H)\\&\le C\sqrt{\varepsilon_{\rm reg}}|L_1(x)-x'|+{\rm dist}(L(x),\partial H),\end{aligned}$$

where in the last inequality we have used (4.170). Choosing $\varepsilon_{\rm reg}$ suitably small we obtain (4.169). By (4.169), if $\varrho\ge 2\,{\rm dist}(L_1(x),\partial H)$, then

$$\mathbf{C}(L_1(x),\varrho)\subset\mathbf{C}\big(x',\varrho+|L_1(x)-x'|\big)\subset\mathbf{C}(x',2\varrho)\,,$$

and thus

$$\mathbf{exc}_n^H(L_1(E),L_1(x),\varrho)\le 2^{n-1}\,\mathbf{exc}_n^H(L_1(E),x',2\varrho)\,.$$

Hence, (4.166) implies

$$\mathbf{exc}_n^H(L_1(E),L_1(x),\varrho)\le C\varepsilon_{\rm reg}\,\varrho\,,\qquad\forall\varrho\in\big(2\,{\rm dist}(L_1(x),\partial H),8\big)\,,$$

for a constant C depending on n and λ only. Of course up to suitably increase the constant C we also have

$$\mathbf{exc}_n^H(L_1(E),L_1(x),\varrho)\le C\varepsilon_{\rm reg}\,\varrho\,,\qquad\forall\varrho\in\big({\rm dist}(L_1(x),\partial H),8\big)\,. \tag{4.172}$$

We thus find

$$\mathbf{exc}_n^H\big(L_1(E),L_1(x),{\rm dist}(L_1(x),\partial H)\big)\le C\varepsilon_{\rm reg}\,{\rm dist}(L_1(x),\partial H)\,. \tag{4.173}$$

Since, by (4.133),

$${\rm dist}(L_1(x),\partial H)\,(\Lambda+\ell)\le{\rm dist}(L_1(x),\partial H)\,\varepsilon_{\rm reg}\,,$$

by choosing $\varepsilon_{\rm reg}$ sufficiently small we can exploit (4.171) to apply step two to $L_1(E)$ at $L_1(x)$, and deduce the existence of an affine map $L_2:\mathbb{R}^n\to\mathbb{R}^n$ such that $L_2(L_1(x))=L_1(x)$, $L_2(H)=H$ and

$$\|\nabla L_2-{\rm Id}\,\|^2\le C\varepsilon_{\rm reg}\,{\rm dist}(L_1(x),\partial H)\,, \tag{4.174}$$

$$\mathbf{exc}_n^H(L_2(L_1(E)),L_1(x),\varrho)\le C\varepsilon_{\rm reg}\,{\rm dist}(L_1(x),\partial H)\,\varrho\,,\quad\forall\,\varrho\le{\rm dist}(L_1(x),\partial H)\,. \tag{4.175}$$

We now claim that the map $L=L_2\circ L_1$ satisfies (4.160) and (4.161). Indeed, clearly $L(H)=H$ while (4.160) follows from (4.165) and (4.174). Let us now prove that

$$\mathbf{exc}_n^H(L(E),L(x),\varrho)\le C\varepsilon_{\rm reg}\,\varrho\,,\quad\forall\,\varrho\in\big({\rm dist}(L_1(x),\partial H),8\big)\,. \tag{4.176}$$

For, let us set $M_2={\rm cof}\,\nabla L_2$, so that

$$\nu_{L(E)}=\frac{M_2\,\nu_{L_1(E)}}{|M_2\nu_{L_1(E)}|}\,,$$

and consider $\hat\nu\in\mathbf{S}^{n-1}$ such that

$$e_n=\frac{M_2\hat\nu}{|M_2\hat\nu|}\,.$$

Since $L_2(L_1(x))=L_1(x)$ we can choose $\varepsilon_{\rm reg}$ suitably small to ensure that $(L_2)^{-1}(\mathbf{C}(L_1(x),\varrho))\subset\mathbf{C}(L_1(x),2\varrho)$, hence we get (compare with (4.130))

$$\begin{aligned}2\,\mathbf{exc}_n^H(L(E),L(x),\varrho)&=\frac{1}{\varrho^{n-1}}\int_{L(\partial^*E)\cap\mathbf{C}(L_1(x),\varrho)\cap H}|\nu_{L(E)}-e_n|^2\\&\le\frac{1}{\varrho^{n-1}}\int_{L_1(\partial^*E)\cap\mathbf{C}(L_1(x),2\varrho)\cap H}\left|\frac{M_2\,\nu_{L_1(E)}}{|M_2\nu_{L_1(E)}|}-\frac{M_2\hat\nu}{|M_2\hat\nu|}\right|^2|M_2\nu_{L_1(E)}|\\&\le\frac{C}{\varrho^{n-1}}\int_{L_1(\partial^*E)\cap\mathbf{C}(L_1(x),2\varrho)\cap H}\big|\nu_{L_1(E)}-\hat\nu\big|^2\\&\le C\,\mathbf{exc}_n^H(L_1(E),L_1(x),2\varrho)+\frac{CP\big(L_1(E);\mathbf{C}(L_1(x),2\varrho)\cap H\big)}{\varrho^{n-1}}\,|\hat\nu-e_n|^2\\&\le C\varepsilon_{\rm reg}\varrho+C\varepsilon_{\rm reg}{\rm dist}(L_1(x),\partial H)\,.\end{aligned} \tag{4.177}$$

Where in the last inequality we have used (2.47), as well as the fact that

$$|\hat\nu - e_n|^2 = \left|\frac{M_2\hat\nu}{|M_2\hat\nu|} - \hat\nu\right|^2 \le C\varepsilon_{\rm reg}{\rm dist}(L_1(x),\partial H)\,,$$

since $\|M_2 - {\rm Id}\,\|^2 \le C\varepsilon_{\rm reg}{\rm dist}(L_1(x),\partial H)$ by (4.174). Since (4.177) immediately implies (4.176), and (4.176) together with (4.175) gives (4.161), the proof of this step is complete.

Step four: We finally prove (4.134), (4.135) and (4.136). For $x \in {\rm cl}(\partial E\cap H)\cap \mathbf{C}$ let us define

$$\nu(x) = \frac{{\rm cof}\,(\nabla L^{-1})\,e_n}{|{\rm cof}\,(\nabla L^{-1})\,e_n|}\,. \tag{4.178}$$

where L is the affine map appearing in (4.161) (which, of course, depends on x). In this way, provided $\varepsilon_{\rm reg}$ is sufficiently small to ensure that $\mathbf{C}(x,\varrho)\subset L^{-1}(\mathbf{C}(L(x),2\varrho))$, the same computations done in (4.177) give

$$\frac{1}{\varrho^{n-1}}\int_{\partial^* E\cap \mathbf{C}^+_{x,\varrho}}\frac{|\nu_E-\nu(x)|^2}{2} \le C\,\mathbf{exc}^H_n(L(E),L(x),2\varrho)\le C\varepsilon_{\rm reg}\varrho\,,\qquad \forall\,\varrho\le 4\,. \tag{4.179}$$

Moreover thanks to (4.160) and the definition of $\nu(x)$, (4.178), $|\nu(x)-e_n|^2\le C\varepsilon_{\rm reg}$. By exploiting the upper density estimates (2.47) we get

$$\begin{aligned}\frac{1}{\varrho^{n-1}}\int_{\partial^* E\cap \mathbf{C}^+_{x,\varrho}}\frac{|\nu_E-e_n|^2}{2} &\le \frac{1}{\varrho^{n-1}}\int_{\partial^* E\cap \mathbf{C}^+_{x,\varrho}}|\nu_E-\nu(x)|^2 + \frac{P(E;\mathbf{C}^+_{x,\varrho})|\nu(x)-e_n|^2}{\varrho^{n-1}}\\ &\le C\varepsilon_{\rm reg}\varrho + C\varepsilon_{\rm reg}\le C\varepsilon_{\rm reg}\,,\qquad \forall \varrho\le 4\,.\end{aligned} \tag{4.180}$$

Now, (4.179) and (4.180) imply (4.134) and (4.135) by a classical argument. For the sake of completeness we give below a sketch of the proof. First, if we choose $\varepsilon_{\rm reg}$ small enough, then we can apply Lemma 4.3 to find a Lipschitz function $u:\mathbb{R}^{n-1}\to\mathbb{R}$ such that, if we set

$$M_0 = \Big\{x\in \mathbf{C}^+\cap\partial E : \sup_{0<s<4}\mathbf{exc}^H_n(E,x,s)\le \delta_1(n,\lambda)\Big\}\,,$$

and $\Gamma = \{(z,u(z)) : z\in \mathbf{D}\}$, then $M_0\subset\Gamma$. By (4.180), up to further decrease the value of $\varepsilon_{\rm reg}$, we have that $M_0 = \mathbf{C}^+\cap\partial E\subset\Gamma$. This easily implies, see for instance [Mag12, Theorem 23.1], that

$$\mathbf{C}^+\cap\partial E = \Big\{x\in H : |\mathbf{p}x|<1\,,\mathbf{q}x = u(\mathbf{p}x)\Big\}\,,$$

and this proves (4.135). We now notice that (4.135) and $0\in\partial E$ imply $u(0)=0$, while (4.15) gives

$$\int_{\mathbf{D}}|\nabla u|^2\le C\varepsilon_{\rm reg}\,,$$

so that (4.134) will follow by interpolation up to bound (in terms of a constant depending on n and λ only) the $C^{0,1/2}$ semi-norm of ∇u on $\mathbf{D}^+$. To this end, let us set

$$v(x) = -\frac{\mathbf{p}\nu(x)}{\mathbf{q}\nu(x)}\,,\qquad x\in{\rm cl}(H\cap\partial E)\cap\mathbf{C}\,,$$

(which is well-defined since $|\nu(x)-e_n|<1$). Since the map $\psi(\xi) = (-\xi,1)/(1+|\xi|^2)^{1/2}$ ($\xi\in\mathbb{R}^{n-1}$) satisfies ${\rm Lip}(\psi)\le 1$, by (4.179) we get

$$\inf_{v\in\mathbb{R}^{n-1}}\frac{1}{\varrho^{n-1}}\int_{\mathbf{D}_{z,\varrho}}|\nabla u - v|^2\le\frac{1}{\varrho^{n-1}}\int_{\mathbf{D}_{z,\varrho}}|\nabla u - v((z,u(z)))|^2\le C\,\varepsilon_{\rm reg}\varrho\,,$$

for every $z \in \mathbf{D}^+$ and $\varrho \le 4$. By Campanato's criterion, see for instance [Giu03, Theorem 2.9], the $C^{0,1/2}$ semi-norm of ∇u on $\mathbf{D}^+$ is bounded by some $C = C(n,\lambda)$. Finally (4.136) can be obtained by a simple first variation argument since the map u satisfies

$$\int_{\mathbf{D}^+} \Phi\big(z, u(z), (-\nabla u(z), 1)\big)\, dz \le \int_{\mathbf{D}^+} \Phi\big(z, w(z), (-\nabla w(z), 1)\big)\, dz + C\Lambda \int_{\mathbf{D}^+} |w-u|$$

for every $w \in \mathrm{Lip}(\mathbf{D})$ such that $w = u$ on $(\partial\mathbf{D})^+$, where $C = C(n,\lambda)$. This completes the proof of Lemma 3.4.

5. On the size of the boundary singular set

In this section we estimate the size of the set where Theorem 3.1 does not apply. More precisely, let us recall from Remark 3.3 that, if E is a (Λ, r_0)-minimizer of $\mathbf{\Phi}$ in (A,H) for some $\Phi \in \boldsymbol{\mathcal{E}}(A\cap H, \lambda, \ell)$, then the boundary singular set $\Sigma_A(E;\partial H)$ (i.e., the set of those $x \in \partial_{\partial H}(\partial E \cap \partial H)\cap A$ such that $A \cap \mathrm{cl}(H \cap \partial E)$ is not a $C^{1,1/2}$ manifold with boundary at x) is characterized in the terms of the spherical excess of E at x as

$$\Sigma_A(E;\partial H) = \Big\{ x \in \partial_{\partial H}(\partial E \cap \partial H) \cap A : \liminf_{r\to 0^+} \mathbf{exc}^H(E,x,r) > 0 \Big\}. \tag{5.1}$$

This identity provides a particularly useful starting point in the study of $\Sigma_A(E;\partial H)$ undertaken in this section, and leading to the following result.

Theorem 5.1. *If A and H are an open set and an open half-space in $\mathbb{R}^n$, $\Phi \in \boldsymbol{\mathcal{E}}(A\cap H,\lambda,\ell)$, and E is a (Λ, r_0)-minimizer of $\mathbf{\Phi}$ on (A,H), then for every $x \in \partial^*_{\partial H}(\partial E\cap\partial H)$ we have*

$$\lim_{r\to 0} \mathbf{exc}^H(E,x,r) = 0\,.$$

In particular,

$$\Sigma_A(E;\partial H) = (\partial_{\partial H}(\partial E \cap \partial H)\cap A) \setminus \partial^*_{\partial H}(\partial E\cap \partial H)\,, \tag{5.2}$$

and thus $\mathcal{H}^{n-2}(\Sigma_A(E;\partial H)) = 0$.

The proof of Theorem 5.1 is based on the study of blow-ups of E at points $x_0 \in \partial_{\partial H}(\partial E\cap\partial H)$. We first show that such blow-ups always exist and are non-trivial (i.e., they are neither empty nor equal to H), and that, if $x \in \partial^*_{\partial H}(\partial E\cap\partial H)$, then there exists a half-space inside ∂H that is the trace on ∂H of every such blow-up; see Lemma 5.2. Then, we show that if a blow-up F of E has the same trace on ∂H as that left by a half-space, then $H \cap \partial F$ is actually contained into a "wedge" of universal amplitude; see Lemma 5.3. At this point we follow some ideas of Hardt [Har77] to show that this wedge property forces a blow-up G of F (at the origin) to coincide with a half-space also *inside* of H; see Lemma 5.4. Since G is also a blow-up of E at x, Theorem 3.1 now implies that $H\cap\partial E$ is a $C^{1,1/2}$ manifold with boundary locally at x, and thus that $x \notin \Sigma_A(E;\partial H)$. We premise to the proof of these lemmas the following useful definition:
Given a set E of locally finite perimeter in A and $x_0 \in A$, we denote by $\mathcal{B}_{x_0}(E)$ the family of blow-ups of E at x_0, that is

$$\mathcal{B}_{x_0}(E) = \left\{ F \subset \mathbb{R}^n : \begin{array}{l} \text{there exists } r_h \to 0 \text{ as } h\to\infty \text{ such that} \\ E^{x_0,r_h} \to F \text{ in } L^1_{\mathrm{loc}}(\mathbb{R}^n) \text{ as } h \to \infty \end{array} \right\}. \tag{5.3}$$

By a diagonal argument, one immediately checks that $\mathcal{B}_{x_0}(E)$ is closed in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, and that

$$\mathcal{B}_0(F) \subset \mathcal{B}_{x_0}(E)\,, \qquad \forall\, F \in \mathcal{B}_{x_0}(E)\,. \tag{5.4}$$

We now start to implement the strategy described above.

Lemma 5.2. *If A is an open set, $H = \{x_1 > 0\}$, $\Phi \in \boldsymbol{\mathcal{E}}(A\cap H,\lambda,\ell)$, and E is a (Λ, r_0)-minimizer of $\mathbf{\Phi}$ in (A,H), then for every $x_0 \in \partial_{\partial H}(\partial E\cap\partial H)\cap A$*

$$\mathcal{B}_{x_0}(E) \ne \emptyset\,, \qquad \emptyset, H \notin \mathcal{B}_{x_0}(E)\,, \tag{5.5}$$

and every $F \in \mathcal{B}_{x_0}(E)$ is a minimizer of $\mathbf{\Phi}_{x_0}$ in $(\mathbb{R}^n, H)$ for $\Phi_{x_0} = \Phi(x_0, \cdot) \in \boldsymbol{\mathcal{E}}_*(\lambda)$. Moreover, for every $x_0 \in \partial^*_{\partial H}(\partial E \cap \partial H)$ there exists $e_{x_0} \in \mathbf{S}^{n-1} \cap e_1^\perp$ such that

$$\mathcal{H}^{n-1}\Big((\partial F \cap \partial H)\Delta\big\{x \in \partial H : x \cdot e_{x_0} \le 0\big\}\Big) = 0\,, \qquad \forall\, F \in \mathcal{B}_{x_0}(E)\,. \tag{5.6}$$

Proof. Let $x_0 \in \partial_{\partial H}(\partial E \cap \partial H) \cap A$. Given $r_h \to 0$ as $h \to \infty$, by Remark 2.2 E^{x_0,r_h} is a $(\Lambda r_h, r_0/r_h)$-minimizer of $\mathbf{\Phi}^{x_0,r_h}$ in (A^{x_0,r_h}, H), with $\Phi^{x_0,r_h} \in \boldsymbol{\mathcal{E}}(A^{x_0,r_h} \cap H, \lambda, r_h\,\ell)$ (note that since $x_0 \in \partial H$, $H^{x_0,r} = H$). By Theorem 2.9, up to extracting a not relabeled subsequence, $E^{x_0,r_h} \to F$ in $L^1_{\rm loc}(\mathbb{R}^n)$ as $h \to \infty$, where F is a minimizer of $\mathbf{\Phi}_{x_0}$ on $(\mathbb{R}^n, H)$. Moreover, by (2.59), as $h \to \infty$,

$$\mathrm{Tr}_{\partial H}(E^{x_0,r_h}) \to \mathrm{Tr}_{\partial H}(F)\,, \qquad \text{in } L^1_{\rm loc}(\partial H)\,. \tag{5.7}$$

Since, by (2.107), $\partial_{\partial H}(\partial E \cap \partial H) \cap A = \mathrm{cl}(\partial E \cap H) \cap \partial H \cap A$, we can apply both (2.48) and (2.49) to E at x_0, to find that

$$c_1\,|B_1 \cap H| \le |E^{x_0,r_h} \cap H \cap B_1| \le (1 - c_1)\,|B_1 \cap H|\,,$$

where $c_1 = c_1(n, \lambda) \in (0,1)$. By letting $h \to \infty$ in these inequalities, we thus find that $|F|\,|H \setminus F| > 0$, and prove (5.5). Let us now assume that $x_0 \in \partial^*_{\partial H}(\partial E \cap \partial H)$. By De Giorgi's rectifiability theorem (applied to the set of finite perimeter $\partial E \cap \partial H$ at the point x_0), there exists $e_{x_0} \in \mathbf{S}^{n-1} \cap e_1^\perp$ such that

$$(\partial E \cap \partial H)^{x_0,r} \to \{x \in \partial H : x \cdot e_{x_0} \le 0\}\,, \qquad \text{in } L^1_{\rm loc}(\partial H) \text{ as } r \to 0^+\,. \tag{5.8}$$

Now, for every $r > 0$, $(\partial E \cap \partial H)^{x_0,r} = \partial(E^{x_0,r}) \cap \partial H$, where $\partial(E^{x_0,r}) \cap \partial H =_{\mathcal{H}^{n-1}} \mathrm{Tr}_{\partial H}(E^{x_0,r})$ and $\partial F \cap \partial H =_{\mathcal{H}^{n-1}} \mathrm{Tr}_{\partial H}(F)$ by statement (ii) in Lemma 2.16. Therefore (5.6) follows by (5.7) and (5.8). $\square$

We now prove a (universal) wedge property for global minimizers with half-spaces as traces.

Lemma 5.3 (Wedge property)**.** *For ever $\lambda \ge 1$ there exists a positive constant $L = L(n, \lambda)$ with the following property. If $H = \{x_1 > 0\}$, $\Phi \in \boldsymbol{\mathcal{E}}_*(\lambda)$, E is a minimizer of $\mathbf{\Phi}$ in $(\mathbb{R}^n, H)$ and, for some $e \in \mathbf{S}^{n-1} \cap e_1^\perp$,*

$$\mathcal{H}^{n-1}\Big((\partial E \cap \partial H)\Delta\big\{x \in \partial H : x \cdot e \le 0\big\}\Big) = 0\,,$$

then

$$\sup\left\{\frac{|x \cdot e|}{x \cdot e_1} : x \in \partial E \cap H\right\} \le L\,.$$

Proof. We argue by contradiction, and thus assume that for every $h \in \mathbb{N}$ there exist $\Phi_h \in \boldsymbol{\mathcal{E}}_*(\lambda)$ and a minimizer E_h of $\mathbf{\Phi}_h$ in $(\mathbb{R}^n, H)$ with

$$\mathcal{H}^{n-1}\Big((\partial E_h \cap \partial H)\Delta\big\{x \in \partial H : x \cdot e_n \le 0\big\}\Big) = 0\,, \tag{5.9}$$

and $x_h \in H \cap \partial E_h$ such that, up to a rotation (keeping e_1 fixed),

$$\lim_{h\to\infty} \frac{|x_h \cdot e_n|}{x_h \cdot e_1} = +\infty\,. \tag{5.10}$$

Up to translating each set along a suitable direction in $e_1^\perp \cap e_n^\perp$ (note that both (5.9) and (5.10) are unaffected by such an operation), we can assume that $x_h = (x_h \cdot e_1, 0, \dots, 0, x_h \cdot e_n)$. Furthermore, up to changing E_h with $H \setminus E_h$, and to reflect along $\{x_n = 0\}$, we can assume that $x_h \cdot e_n > 0$ for every $h \in \mathbb{N}$. We now look at the sets $F_h = E_h^{0, x_h \cdot e_n}$. By Remark 2.2, F_h is a minimizer of $\mathbf{\Phi}_h$ in $(\mathbb{R}^n, H)$, with

$$\mathcal{H}^{n-1}\Big((\partial F_h \cap \partial H)\Delta\big\{x \in \partial H : x \cdot e_n \le 0\big\}\Big) = 0\,, \tag{5.11}$$

$$\Big(\frac{x_h \cdot e_1}{x_h \cdot e_n}\,, 0\,, \dots\,, 0\,, 1\Big) \in \partial F_h\,, \tag{5.12}$$

thanks to $x_h \cdot e_n > 0$. By Theorem 2.9, up to extracting a not relabeled subsequence, $F_h \to F_\infty$ in $L^1_{\rm loc}(H)$ as $h \to \infty$, where F_∞ is a minimizer of $\mathbf{\Phi}_\infty$ in $(\mathbb{R}^n, H)$ and $\Phi_\infty \in \boldsymbol{\mathcal{E}}_*(\lambda)$. By (2.59) and (5.11), we have

$$\mathcal{H}^{n-1}\Big((\partial F_\infty \cap \partial H)\Delta\big\{x \in \partial H : x \cdot e_n \le 0\big\}\Big) = 0\,. \tag{5.13}$$

However, (2.60), (5.10) and (5.12) imply that $p = (0, \dots, 0, 1) \in \partial F_\infty \cap \partial H$, so that, by Lemma 2.15, $\mathcal{H}^{n-1}(\partial F_\infty \cap \partial H \cap B_{p,1/2}) > 0$. This is a contradiction to (5.13), and the lemma is proved. □

The following lemma, which is the analogous of [Har77, Lemma 4.5], shows that for every point in the reduced boundary of the trace of a minimizer it is possible to find a blow-up given by the intersection of H with a half-space.

Lemma 5.4. *If A is an open set, $H = \{x_1 > 0\}$, $\Phi \in \boldsymbol{\mathcal{E}}(A\cap H, \lambda, \ell)$, and E is a (Λ, r_0)-minimizer of $\mathbf{\Phi}$ in (A, H), then for every $x_0 \in \partial^*_{\partial H}(\partial E \cap \partial H)$ there exists $\nu \in \mathbf{S}^{n-1}$ with*

$$H \cap \{\nu \cdot x \le 0\} \in \mathcal{B}_{x_0}(E)\,, \qquad \nabla\Phi(x_0, \nu) \cdot e_1 = 0\,. \tag{5.14}$$

Proof. Without loss of generality we take $x_0 = 0$, and then set $\Phi_0(\nu) = \Phi(0, \nu)$, so that $\Phi_0 \in \boldsymbol{\mathcal{E}}_*(\lambda)$. As usual, we shall set $G^+ = G \cap H$ for every $G \subset \mathbb{R}^n$. By Lemma 5.2, $\mathcal{B}_0(E)$ is a non-empty family of minimizers of $\mathbf{\Phi}_0$ in $(\mathbb{R}^n, H)$. By Theorem 2.9, $\mathcal{B}_0(E)$ is also a compact subset of $L^1_{\rm loc}(\mathbb{R}^n)$. By (5.6), there exists a vector $e \in \mathbf{S}^{n-1} \cap e_1^\perp$ such that

$$\mathcal{H}^{n-1}\Big((\partial F \cap \partial H)\Delta\big\{x \in \partial H : x \cdot e \le 0\big\}\Big) = 0\,, \qquad \forall F \in \mathcal{B}_0(E)\,. \tag{5.15}$$

Up to a rotation, we can assume that $e = e_n$, so that Lemma 5.3 ensures that

$$\sup_{(\partial F)^+} \frac{|x \cdot e_n|}{x \cdot e_1} \le L\,, \qquad \forall F \in \mathcal{B}_0(E)\,,$$

where $L = L(n, \lambda)$. Let us now define $\beta_1 : \mathcal{B}_0(E) \to [-L, L]$ by setting

$$\beta_1(F) = \sup_{(\partial F)^+} \frac{x \cdot e_n}{x \cdot e_1}\,, \qquad F \in \mathcal{B}_0(E)\,. \tag{5.16}$$

We notice that β_1 is lower semicontinuous on $\mathcal{B}_0(E)$ with respect to the $L^1_{\rm loc}(\mathbb{R}^n)$ convergence. Indeed, if $F_h, F \subset \mathcal{B}_0(E)$ and $F_h \to F$ in $L^1_{\rm loc}(\mathbb{R}^n)$, then, by (2.60), for every $x \in H \cap \partial F$ there exist $x_h \in H \cap \partial F_h$, $h \in \mathbb{N}$, such that $x_h \to x$ as $h \to \infty$. Hence,

$$\frac{x \cdot e_n}{x \cdot e_1} = \lim_{h\to\infty} \frac{x_h \cdot e_n}{x_h \cdot e_1} \le \liminf_{h\to\infty} \beta_1(F_h)\,,$$

as claimed. Since β_1 is lower semicontinuous and $\mathcal{B}_0(E)$ is a compact subset of $L^1_{\rm loc}(\mathbb{R}^n)$, we can find $F_1 \in \mathcal{B}_0(E)$ such that

$$\beta_1(F_1) \le \beta_1(F)\,, \qquad \forall F \in \mathcal{B}_0(E)\,. \tag{5.17}$$

Correspondingly, we define $\alpha_1 \in (-\pi/2, \pi/2)$ so that $\tan\alpha_1 = \beta_1(F_1)$ and set

$$\nu_1 = \cos\alpha_1\, e_n - \sin\alpha_1\, e_1 \in \mathbf{S}^{n-1}\,, \qquad H_1 = \Big\{x \in H : x \cdot \nu_1 \le 0\Big\}\,.$$

We now claim that

$$(\partial H_1)^+ \subset (\partial F_1)^+\,. \tag{5.18}$$

To prove (5.18), we first take into account the definition of β_1 to find

$$(\partial F_1)^+ \subset H_1\,. \tag{5.19}$$

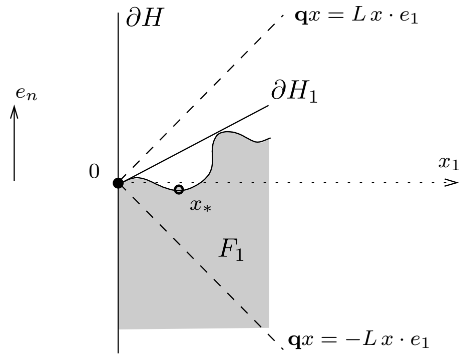

FIGURE 5.1. Failure of (5.18).

By (5.15) and (5.19), the upper semicontinuous function $w_{F_1} : \mathbb{R}^{n-1}_+ \to [-\infty, +\infty)$ defined by setting

$$w_{F_1}(z) = \sup\Big\{t \in \mathbb{R} : (z,t) \in \partial F_1\Big\}\,, \qquad z \in \mathbb{R}^{n-1}_+\,,$$

(here $\mathbb{R}^{n-1}_+ = \{z \in \mathbb{R}^{n-1} : z_1 > 0\}$) satisfies

$$F_1 \subset \Big\{x \in H : \mathbf{q}x \le w_{F_1}(\mathbf{p}x)\Big\}\,, \tag{5.20}$$

$$w_{F_1}(z) \le \beta_1(F_1)\, z\cdot e_1\,, \qquad \forall z \in \mathbb{R}^{n-1}_+\,. \tag{5.21}$$

Now, if (5.18) fails, see Figure 5.1, then there exists $x_* \in (\partial F_1)^+$ such that

$$w_{F_1}(\mathbf{p}x_*) < \beta_1(F_1)\, x_*\cdot e_1\,. \tag{5.22}$$

By (5.21) and (5.22), if we set $r_* = |\mathbf{p}x_*|$ and $z_* = \mathbf{p}x_*$, then we can find $\varphi \in C^{1,1}(\partial(\mathbf{D}^+_{r_*}))$ such that

$$w_{F_1}(z) \le \varphi(z) \le \beta_1(F_1)\, z\cdot e_1\,, \qquad \forall z \in \partial(\mathbf{D}_{r_*}\cap H) \tag{5.23}$$

$$\varphi(z_*) < \beta_1(F_1)\, z_*\cdot e_1\,. \tag{5.24}$$

In particular, $\varphi = 0$ on $\mathbf{D}_{r_*}\cap\partial H$. By part two of Lemma 2.11, there exists $u_0 \in C^{1,1}(\mathbf{D}^+_{r_*}) \cap \mathrm{Lip}(\mathrm{cl}(\mathbf{D}^+_{r_*}))$ such that, if we set $\Phi_0^\#(\xi) = \Phi_0(\xi,-1)$ for $\xi \in \mathbb{R}^{n-1}$, then

$$\begin{cases} \mathrm{div}\,(\nabla_\xi \Phi_0^\#(\nabla u)) = 0\,, & \text{in } \mathbf{D}^+_{r_*}\,,\\ u = \varphi\,, & \text{on } \partial(\mathbf{D}^+_{r_*})\,, \end{cases}$$

with

$$|\nabla u(0)| = |\nabla u(0)\cdot e_1| < \beta_1(F)\,. \tag{5.25}$$

By (5.20) and (5.23) we can apply Lemma 2.12 to infer that

$$F_1 \cap (\mathbf{D}^+_{r_*}\times\mathbb{R}) \subset_{\mathcal{H}^n} \Big\{(z,t) \in \mathbf{D}^+_{r_*}\times\mathbb{R} : t \le u(z)\Big\}\,. \tag{5.26}$$

If we now pick a sequence $\{s_h\}_{h\in\mathbb{N}}$ such that $s_h \to 0$ as $h\to\infty$ and $(F_1)^{0,s_h} \to \widetilde{F_1}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then, by (5.26) and $u(0) = 0$, we find that

$$\widetilde{F_1} \subset \Big\{(z,t) : t \le (\nabla u_0(0)\cdot e_1)\,(z\cdot e_1)\Big\}\,,$$

so that, thanks to (5.25), $\beta_1(\widetilde{F_1}) < \beta_1(F_1)$. Since $\widetilde{F_1} \in \mathcal{B}_0(F_1) \subset \mathcal{B}_0(E)$, this contradicts (5.17), and completes the proof of (5.18). By (5.15), (5.18), and (5.19),

$$\partial H_1 \subset_{\mathcal{H}^{n-1}} \partial F_1 \qquad \text{and} \qquad F_1 \subset H_1\,. \tag{5.27}$$

Since F_1 is a minimizer of $\mathbf{\Phi}_0$ on $(\mathbb{R}^n, H)$, by (5.27) and by Proposition 2.5 we find that H_1 is a superminimizer of $\mathbf{\Phi}_0$ on $(\mathbb{R}^n, H)$. Hence, by Proposition 2.6,

$$\nabla\Phi_0(\nu_1)\cdot e_1 \ge 0\,. \tag{5.28}$$

In order to prove the lemma, we now take a further blow-up of F_1 at 0. Precisely, we consider $\beta_2 : \mathcal{B}_0(F_1) \to [-L, L]$ to be defined as

$$\beta_2(F) = \inf_{(\partial F)^+} \frac{x\cdot e_n}{x\cdot e_1}\,, \qquad\qquad F\in\mathcal{B}_0(F_1)\,. \tag{5.29}$$

Since β_2 is upper semicontinuous and $\mathcal{B}_0(F_1)$ is a compact subset of $L^1_{\rm loc}(\mathbb{R}^n)$, we can find $F_2 \in \mathcal{B}_0(F_1)$ such that

$$\beta_2(F) \le \beta_2(F_2)\,, \qquad \forall F\in\mathcal{B}_0(F_1)\,. \tag{5.30}$$

If we now define $\alpha_2 \in (-\pi/2, \pi/2)$ so that $\tan\alpha_2 = \beta_2(F_2)$, and set

$$\nu_2 = \cos\alpha_2\, e_n - \sin\alpha_2\, e_1 \in \mathbf{S}^{n-1}\,, \qquad H_2 = \Big\{x\in H : x\cdot\nu_2 \le 0\Big\}\,,$$

then, by arguing as in the proof of (5.18) we find that $\partial H_2 \subset_{\mathcal{H}^{n-1}} \partial F_2$ and $H_2 \subset F_2$. By Proposition 2.5 H_2 is a subminimizer and hence Proposition 2.6 implies

$$\nabla\Phi_0(\nu_2)\cdot e_1 \le 0\,. \tag{5.31}$$

Note now that the second inclusion in (5.27) implies $F\subset H_1$ for every $F\in\mathcal{B}_0(F_1)$. In particular, $F_2\subset H_1$ and thus $\beta_2(F_2)\le\beta_1(F_2)=\beta_1(F_1)$, that is, $\alpha_2\le\alpha_1$. If we set, as in Lemma 2.7,

$$f(\alpha) = \nabla\Phi_0(\cos\alpha\, e_n - \sin\alpha\, e_1)\cdot e_1\,, \qquad \alpha\in[-\pi/2,\pi/2]\,,$$

then (5.28) and (5.31) give $f(\alpha_1)\ge 0$ and $f(\alpha_2)\le 0$, and since $f'(\alpha)<0$ by (2.46), we must conclude that $\alpha_1=\alpha_2$. In particular, $H_2=F_2\in\mathcal{B}_0(E)$ and $\nabla\Phi_0(\nu_2)\cdot e_1=0$, as required. □

Proof of Theorem 5.1. By (3.5) it is clear that if $x_0\in(A\cap\partial_{\partial H}(\partial H\cap\partial E))\setminus\Sigma_A(E;\partial H)$, then $x_0\in\partial^*_{\partial H}(\partial E\cap\partial H)$. This proves the inclusion $\supset$ in (5.2). To complete the proof of (5.2) we are going to show that if $x_0\in\partial^*_{\partial H}(\partial E\cap\partial H)$, then

$$\liminf_{r\to 0^+}\mathbf{exc}^H(E,x_0,r)=0\,. \tag{5.32}$$

To this end, we exploit Lemma 5.4 to find a sequence $\{r_h\}_{h\in\mathbb{N}}$ with $r_h\to 0$ as $h\to\infty$, such that

$$E^{x_0,r_h}\to F = H\cap\{\nu\cdot x\le 0\} \qquad \text{in } L^1(\mathbb{R}^n)\,,$$

as $h\to\infty$. By scale invariance of $\mathbf{exc}^H$ and by arguing as in Remark 3.6 we thus find

$$\lim_{h\to\infty}\mathbf{exc}^H(E,x_0,r_h)=\lim_{h\to\infty}\mathbf{exc}^H(E^{x_0,r_h},0,1)=\mathbf{exc}^H(H\cap\{\nu\cdot x\le 0\},0,1)=0\,,$$

that is (5.32). This proves (5.2), and then $\mathcal{H}^{n-2}(\Sigma_A(E,\partial H))=0$ follows by (2.106). □

6. Proofs of Theorems 1.2 and 1.10

In this section Theorem 1.2, Corollary 1.4, and Theorem 1.10 are finally deduced from Theorems 3.1 and 5.1. The key step is of course getting rid of the relative adhesion coefficient σ, as we do in the following lemma.

Lemma 6.1. *Given $\lambda\ge 1$ and $\Lambda,\ell,L\ge 0$, there exist constants $\ell_0,\Lambda_0\ge 0$ depending on n, λ, Λ, ℓ and L only, with the following property. If A is an open set in $\mathbb{R}^n$, $H=\{x_1>0\}$, $\Phi\in\boldsymbol{\mathcal{E}}(A\cap H,\lambda,\ell)$, $\sigma\in{\rm Lip}(A\cap\partial H)$ with ${\rm Lip}(\sigma)\le L$ and*

$$-\Phi(z,e_1)<\sigma(z)<\Phi(z,-e_1)\,, \qquad \forall\, z\in A\cap\partial H\,, \tag{6.1}$$

E is a (Λ,r_0)-minimizer of $(\mathbf{\Phi},\sigma)$ in (A,H) and $x_0\in A\cap\partial H$, then there exist $\lambda_*\ge 1$ and $\varrho_*>0$ and $\Psi\in\boldsymbol{\mathcal{E}}(B_{x_0,\varrho_*}\cap H,\lambda_*,\ell_0)$ such that E is a (Λ_0,r_0)-minimizer of $\mathbf{\Psi}$ in (B_{x_0,ϱ_*},H). Moreover, for every $x\in B_{x_0,\varrho}\cap\partial H$ and $\nu\in\mathbf{S}^{n-1}$ one has

$$\nabla\Psi(x,\nu)\cdot e_1=0\qquad\textit{if and only if}\qquad\nabla\Phi(x,\nu)\cdot(-e_1)=\sigma(x)\,. \tag{6.2}$$

Proof. Let us assume, without loss of generality, that $x_0=0$. We set $\Phi_0(\nu)=\Phi(0,\nu)$ and $G^+=G\cap H$ for every $G\subset\mathbb{R}^n$. We want to prove the existence of $\varrho_*>0$, $\lambda_*\ge1$, and $\Psi\in\boldsymbol{\mathcal{E}}(B^+_{\varrho_*},\lambda_*,\ell_0)$ such that

$$\mathbf{\Psi}(E;W^+)\le\mathbf{\Psi}(F;W^+)+\Lambda_0\,|E\Delta F|\,, \tag{6.3}$$

whenever $F\subset H$ with $E\Delta F\subset\subset W$ and W is an open set with $W\subset\subset B_{\varrho_*}$ and $\mathrm{diam}(W)<2\,r_0$. To this end, let us fix such a competitor F, with the requirement that $\varrho_*<\mathrm{dist}(0,\partial A)$, and, in correspondence to F, let W_0 be an open set with smooth boundary such that $E\Delta F\subset\subset W_0\subset\subset B_{\varrho_*}$, $\mathrm{diam}(W_0)<2\,r_0$, and

$$\mathcal{H}^{n-1}(\partial W_0\cap\partial^*E)=\mathcal{H}^{n-1}(\partial W_0\cap\partial^*F)=0\,. \tag{6.4}$$

Since E is a (Λ,r_0)-minimizer of $(\mathbf{\Phi},\sigma)$ in (A,H), we certainly have

$$\mathbf{\Phi}(E;W_0^+)+\int_{W_0\cap\partial H\cap\partial^*E}\sigma\,d\mathcal{H}^{n-1}\le\mathbf{\Phi}(F;W_0^+)+\int_{W_0\cap\partial H\cap\partial^*F}\sigma\,d\mathcal{H}^{n-1}+\Lambda\,|E\Delta F|\,. \tag{6.5}$$

Next we define a Lipschitz vector field $T:B_{\varrho_*}\to\mathbb{R}^n$ by setting, for every $x\in B_{\varrho_*}$,

$$T(x)=-\frac{\sigma(\mathbf{h}\,x)}{\Phi_0(e_1)}\,\nabla\Phi_0(e_1)\,,\qquad\text{if }\sigma(0)\le0\,,$$
$$T(x)=\frac{\sigma(\mathbf{h}\,x)}{\Phi_0(-e_1)}\,\nabla\Phi_0(-e_1)\,,\qquad\text{if }\sigma(0)>0\,,$$

where $\mathbf{h}:\mathbb{R}^n\to\partial H=\{x_1=0\}$ denotes the projection over ∂H. (The definition is well-posed since $0\in\partial H$, and thus $\mathbf{h}\,x\in A\cap\partial H$ whenever $x\in B_{\varrho_*}\subset A$.) Notice that, in both cases, since $\nabla\Phi_0(e)\cdot e=\Phi_0(e)$ for every $e\in\mathbf{S}^{n-1}$, one has

$$-T(x)\cdot e_1=\sigma(x)\,,\qquad\forall x\in B_{\varrho_*}\cap\partial H\,. \tag{6.6}$$

Since $E\subset H$, by (2.12), by (6.6) and by applying the divergence theorem to T over $E\cap W_0$,

$$\int_{E\cap W_0}\mathrm{div}\,T=\int_{W_0^+\cap\partial^*E}T\cdot\nu_E\,d\mathcal{H}^{n-1}+\int_{W_0\cap\partial H\cap\partial^*E}\sigma\,d\mathcal{H}^{n-1}+\int_{E^{(1)}\cap\partial W_0}T\cdot\nu_{W_0}\,d\mathcal{H}^{n-1}\,,$$

and an analogous relation holds true with F in place of E. By plugging these relations into (6.5), and taking also into account that $E^{(1)}\cap\partial W_0=F^{(1)}\cap\partial W_0$ since $E\Delta F\subset\subset W_0$, one finds

$$\begin{aligned}&\mathbf{\Phi}(E;W_0^+)-\int_{W_0\cap\partial H\cap\partial^*E}T\cdot\nu_E\,d\mathcal{H}^{n-1}\\ \le\ &\mathbf{\Phi}(F;W_0^+)-\int_{W_0\cap\partial H\cap\partial^*F}T\cdot\nu_F\,d\mathcal{H}^{n-1}+\Big(\Lambda+\sup_{B_{\varrho_*}}|\mathrm{div}\,T|\Big)\,|E\Delta F|\,.\end{aligned} \tag{6.7}$$

Thus, if we set

$$\Psi(x,\nu)=\Phi(x,\nu)-T(x)\cdot\nu\,,\qquad(x,\nu)\in B_{\varrho_*}\times\mathbb{R}^n\,,$$

then E is a (Λ_0,r_0)-minimizer of $\mathbf{\Psi}$ in (B_{ϱ_*},H), with $\Lambda_0=\Lambda+n\,L\,\lambda^2$ (as $|\nabla T|\le L\,\lambda^2$) provided we can check that $\Psi\in\boldsymbol{\mathcal{E}}(B^+_{\varrho_*},\lambda_*,\ell_0)$ for suitable values of λ_* and ℓ_0. A quick inspection of Definition 1.1 shows that indeed (1.9), (1.8), (1.10) and the upper bound in (1.7) hold true with suitable values of λ_* and ℓ_0 depending on λ, ℓ and L only. (In checking this, it is useful notice that $|\sigma|\le\lambda$ on $A\cap\partial H$ by (6.1).) One has to be more careful in the verification of the lower bound in (1.7), and indeed this is the place where the values of λ_* and ϱ_* has to be chosen in dependence of the positivity of

$$1+\frac{\sigma(0)}{\Phi_0(e_1)}\,,\qquad \text{if } \sigma(0)\le 0\,,$$

or in dependence of the positivity of

$$1-\frac{\sigma(0)}{\Phi_0(-e_1)}\,,\qquad \text{if } \sigma(0)> 0\,.$$

(Of course, both positivity properties descend from (6.1).) Precisely, let us first consider the case when $\sigma(0)\le 0$, and notice that by (1.7) and (1.9) (applied to Φ) one has

$$\Psi(x,\nu)\ge \Phi_0(\nu)+\frac{\sigma(0)}{\Phi_0(e_1)}\,\nabla\Phi_0(e_1)\cdot\nu-(\ell+L\lambda^2)\,\varrho_*\,,\tag{6.8}$$

for every $x\in B_{\varrho_*}$ and $\nu\in\mathbf{S}^{n-1}$. Let us now introduce a parameter $\tau_0>0$ and let us consider the following two cases:

(a) $\nabla\Phi_0(\nu)\cdot e_1\le\tau_0$,
(b) $\nabla\Phi_0(\nu)\cdot e_1\ge\tau_0$.

Case (a) We notice that, by (6.8), (1.7) (applied to Φ), and $|\sigma(0)|\le\lambda$, then

$$\Psi(x,\nu)\ge\frac{1}{\lambda}-\frac{|\sigma(0)|}{\Phi_0(e_1)}\,\tau_0-(\ell+L\lambda^2)\,\varrho_*\ge\frac{1}{\lambda}-\lambda^2\,\tau_0-(\ell+L\lambda^2)\,\varrho_*\ge\frac{1}{2\lambda}\,,\tag{6.9}$$

provided τ_0 and ϱ_* are small enough with respect to λ, L and ℓ.

Case (b) By convexity and one-homogeneity $\Phi_0(\nu)\ge\nabla\Phi_0(e_1)\cdot\nu$, hence (6.8) and the positivity of $1+(\sigma(0)/\Phi_0(e_1))$ implies that

$$\begin{aligned}\Psi(x,\nu)&\ge\Big(1+\frac{\sigma(0)}{\Phi_0(e_1)}\Big)\nabla\Phi_0(e_1)\cdot\nu-(L\lambda^2+\ell)\,\varrho_*\ge\Big(1+\frac{\sigma(0)}{\Phi_0(e_1)}\Big)\,\tau_0-(L\lambda^2+\ell)\,\varrho_*\\&\ge\Big(1+\frac{\sigma(0)}{\Phi_0(e_1)}\Big)\,\frac{\tau_0}{2}\,,\end{aligned}\tag{6.10}$$

provided ϱ_* is small enough depending on the size of $1+(\sigma(0)/\Phi_0(e_1))$, λ, L, ℓ and on the value of τ_0 chosen to ensure the validity of (6.9). By combining (6.9) and (6.10), we find that Ψ satisfies the lower bound in (1.7) for some value of λ_* depending on λ, L, ℓ, and the size of $1+(\sigma(0)/\Phi_0(e_1))$. In the case that $\sigma(0)>0$ one can check the validity for Ψ of the lower bound in (1.7) by an entirely analogous argument. This proves that $\Psi\in\boldsymbol{\mathcal{E}}(B^+_{\varrho_*},\lambda_*,\ell_0)$, while the validity of (6.2) is immediate from the definition of Ψ. □

Proof of Theorem 1.10. According to Lemma 6.1, we can cover $A\cap\partial H$ with countably many balls $\{B_h\}_{h\in\mathbb{N}}$ with the property that, for every $h\in\mathbb{N}$, $B_h\subset A$, E is a (Λ_0,r_0)-minimizer of $\boldsymbol{\Phi}_h$ in (B_h,H) for some $\Phi_h\in\boldsymbol{\mathcal{E}}(B_h\cap H,\lambda_h,\ell_0)$ such that, if $x\in B_h\cap\partial H$ and $\nu\in S^{n-1}$, then

$$\nabla\Phi_h(x,\nu)\cdot\nu_H=0\qquad\text{if and only if}\qquad\nabla\Phi(x,\nu)\cdot\nu_H=\sigma(x)\,.\tag{6.11}$$

Setting $M=\mathrm{cl}(H\cap\partial E)$, by Lemma 2.16, for every $h\in\mathbb{N}$,

$$\begin{aligned}&E\cap B_h\text{ is an open set}\,,\\&\partial E\cap\partial H\text{ is of locally finite perimeter in }B_h\cap\partial H\,,\\&B_h\cap\partial_{\partial H}(\partial E\cap\partial H)=B_h\cap M\cap\partial H\,.\end{aligned}$$

Let us define $A'=\cup_h B_h$ so that

$$A'\cap\partial H=A\cap\partial H\,,\qquad M\cap\partial H\cap A=M\cap\partial H\cap A'\qquad\text{and}\qquad\partial E\cap\partial H\cap A=\partial E\cap\partial H\cap A'\,.$$

Since B_h covers A' we see by the previous properties that $E\cap A'$ is (equivalent to) an open set, $\partial E\cap\partial H$ is of locally finite perimeter in $A'\cap\partial H=A\cap\partial H$, and $\partial_{\partial H}(\partial E\cap\partial H)=M\cap\partial H$. Moreover

$$\Sigma_{A'}(E;\partial H)\cap B_h=\Sigma_A(E;\partial H)\cap B_h=\Sigma_{B_h}(E;\partial H),$$

so that by Theorem 3.1 and Theorem 5.1, we find that $\mathcal{H}^{n-2}(\Sigma_A(E;\partial H))=0$, as well as that

$$\begin{aligned}&M\text{ is a }C^{1,1/2}\text{-manifold with boundary in a neighborhood of }x\\&\text{with }\nabla\Phi_h(x,\nu_E(x))\cdot\nu_H=0\,,\end{aligned}$$

for every $x\in\partial_{\partial H}(\partial E\cap\partial H)\setminus\Sigma_A(E;\partial H)$. This complete the proof of the theorem. □

Proof of Theorem 1.2. The existence of a minimizer E of (1.3) follows by applying the direct methods of the calculus of variation, see for instance [Mag12, Section 19.1] for the case $\Phi(x,\nu)=|\nu|$. By a"volume-fixing variation" argument, [Mag12, Example 21.3], we see that E satisfies the volume-constraint-free minimality property

$$\mathbf{\Phi}(E;\Omega)+\int_{\partial^*E\cap\partial\Omega}\sigma\,d\mathcal{H}^{n-1}\le\mathbf{\Phi}(F;\Omega)+\int_{\partial^*F\cap\partial\Omega}\sigma\,d\mathcal{H}^{n-1}+\Lambda\,|E\Delta F|\,,\tag{6.12}$$

whenever $F\subset\Omega$ and $\mathrm{diam}(E\Delta F)\le r_0$, where r_0 and Λ are constants depending on E, Ω, and $\|g\|_{L^\infty(\Omega)}$. Let us now fix $x_0\in\partial\Omega$: by assumption, there exist $r>0$, an open neighborhood A of the origin and a $C^{1,1}$-diffeomorphism f between $B_{x_0,r}\cap\Omega$ and $A\cap H$, and between $B_{x_0,r}\cap\partial\Omega$ and $A\cap\partial H$, where $H=\{x_1>0\}$. If we set $\Lambda^f=\Lambda\,\|\det\nabla f\|_{L^\infty(B(x_0,r))}$, and, for $x\in A\cap H$ and $\nu\in S^{n-1}$,

$$\begin{aligned}\Phi^f(x,\nu)&=\Phi(f^{-1}(x),\mathrm{cof}\,(\nabla f^{-1}(x))\,\nu)\,,\\ \sigma^f(x)&=\sigma(f^{-1}(x))\,\Big|\mathrm{cof}\,(\nabla f^{-1}(x))\,e_1\Big|\,,\end{aligned}$$

then we can find $r_*>0$, $\lambda_*\ge 1$, and $\ell_*>0$ such that, by (6.12), (2.9), and by arguing as in Lemma 2.18,

$$\mathbf{\Phi}^f(f(E);H)+\int_{\partial^*f(E)\cap\partial H}\sigma^f\,d\mathcal{H}^{n-1}\le\mathbf{\Phi}^f(G;H)+\int_{\partial^*G\cap\partial H}\sigma^f\,d\mathcal{H}^{n-1}+\Lambda^f|f(E)\Delta G|\,,$$

whenever $G\subset H$, $\mathrm{diam}(f(E)\Delta G)\le 2\,r_*$ and $f(E)\Delta G\subset\subset A$, with $\Phi^f\in\boldsymbol{\mathcal{E}}(A\cap H,\lambda_*,\ell_*)$. In particular, $f(E)$ is a (Λ^f,r_*)-minimizer of (Φ^f,σ^f) in (A,H), while

$$\nu_\Omega(f^{-1}(x))=\frac{\mathrm{cof}\,(\nabla f^{-1}(x))\,(-e_1)}{|\mathrm{cof}\,(\nabla f^{-1}(x))e_1|}\,,\qquad\forall x\in A\cap\partial H\,,$$

and (1.11) imply that

$$-\Phi^f(x,e_1)<\sigma^f(x)<\Phi^f(x,-e_1)\,,\qquad\forall\,x\in A\cap\partial H\,.$$

Hence, we can apply Theorem 1.10 to discuss the boundary regularity of $f(E)$ in A, and conclude the proof of the theorem by a covering argument and by a change of variables. □

Proof of Corollary 1.4. Step one: We start showing that $\Sigma\cap\partial\Omega=\emptyset$ if $n=3$. We argue by contradiction, and assume the existence of $x_0\in\Sigma\cap\partial\Omega$. Since Ω has boundary of class $C^{1,1}$, we can find $r>0$, an open neighborhood A of the origin, and a $C^{1,1}$ diffeomorphism f between $B_{x,r}\cap\Omega$ and $A\cap H$, and between $B_{x,r}\cap\partial\Omega$ and $A\cap\partial H$ such that $f(x_0)=0$ and $\nabla f(x_0)=\mathrm{Id}$. In particular, in the notation used in the proof of Theorem 1.2, we have

$$\Phi^f(0,\nu)=|\nu|\,,\qquad\sigma^f(0)=\sigma(x_0)\,.$$

By arguing as in Lemma 5.2 we thus see that every blow-up E_2 of $E_1=f(E)$ at 0 satisfies the minimality inequality

$$P(E_2;H)+\sigma(x_0)\,P(E_2;\partial H)\le P(F;H)+\sigma(x_0)\,P(F;\partial H)\,,$$

whenever $F \subset H$ and $E_2 \Delta F \subset\subset \mathbb{R}^n$. Given $r > 0$, if we plug into this inequality the cone-like comparison set F_r defined by

$$F_r = (E_2 \setminus B_r) \cup \Big\{ t\, x \in B_r : 0 \le t \le 1\,, x \in H \cap E_2^{(1)} \cap \partial B_r \Big\}\,,$$

then, by arguing for example as in [Mag12, Theorem 28.4], we find that the function

$$\alpha(r) = \frac{P(E_2; H \cap B_r) + \sigma(x_0) P(E_2; \partial H \cap B_r)}{r^2}\,, \qquad r > 0\,,$$

is increasing on $(0, \infty)$, with $\alpha(r) = \text{const}$ if and only if E_2 is a cone: in particular, every blow-up E_3 of E_2 at the origin is a cone. (Alternatively, we could have directly shown E_2 to be a cone by using almost-monotonicity formulas.) By interior regularity theory, $\partial E_3 \cap H$ is a smooth surface in $\mathbb{R}^3$ with zero mean curvature. Since this surface is also a cone, and $\partial E_3 \cap \partial B_1 \cap H$ must be a finite union of non-intersecting geodesics, we conclude that $\partial E_3 \cap H$ is a finite union of planes meeting along a common line $\gamma \subset \partial H$ with $0 \in \gamma$. Since $0 \in \Sigma(E_3; \partial H)$ and E_3 is a cone, it must be $\gamma \subset \Sigma(E_3; \partial H)$, and thus

$$\mathcal{H}^1(\Sigma(E_3; \partial H)) = +\infty\,.$$

However $\mathcal{H}^1(\Sigma(E_3; \partial H)) = 0$ by Theorem 1.10, and we have thus reached a contradiction.

Step two: By combining step one with the classical dimension reduction argument by Federer (see, [Sim83, Appendix A] or [Mag12, Sections 28.4-28.5]), one shows that $\Sigma(E; \partial\Omega)$ is discrete if $n = 4$, and that $\mathcal{H}^s(\Sigma(E; \partial\Omega)) = 0$ for every $s > n - 4$ if $n \ge 5$. □

References

[AC81] H. W. Alt and L. A. Caffarelli. Existence and regularity for a minimum problem with free boundary. *J. Reine Angew. Math.*, 325:105–144, 1981.

[ACF84] H.W. Alt, L. A. Caffarelli, and A. Friedman. A free boundary problem for quasilinear elliptic equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 11(1):1–44, 1984.

[AFP00] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[Alm68] F. J. Almgren. Existence and regularity almost everywhere of solutions to elliptic variational problems among surfaces of varying topological type and singularity structure. *Ann. Math.*, 87:321–391, 1968.

[Alm76] F. J. Almgren. Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints. *Mem. Amer. Math. Soc.*, 4(165):viii+199 pp, 1976.

[Bae14] E. Baer. Minimizers of anisotropic surface tensions under gravity: Higher dimensions via symmetrization. *to appear on Arch. Rat. Mech. Anal.*, 2014. doi 10.1007/s00205-014-0788-z.

[Bom82] E. Bombieri. Regularity theory for almost minimal currents. *Arch. Ration. Mech. Anal.*, 7(7):99–130, 1982.

[CF85] L. A. Caffarelli and A. Friedman. Regularity of the boundary of a capillary drop on an inhomogeneous plane and related variational problems. *Rev. Mat. Iberoamericana*, 1(1):61–84, 1985.

[CM07] L. A. Caffarelli and A. Mellet. Capillary drops: contact angle hysteresis and sticking drops. *Calc. Var. Partial Differential Equations*, 29(2):141–160, 2007.

[DG60] E. De Giorgi. *Frontiere orientate di misura minima.* Seminario di Matematica della Scuola Normale Superiore di Pisa. Editrice Tecnico Scientifica, Pisa, 1960.

[DM09] F. Duzaar and G. Mingione. Harmonic type approximation lemmas. *J. Math. Anal. Appl.*, 352(1):301–335, 2009.

[DS94] F. Duzaar and K. Steffen. Comparison principles for hypersurfaces of prescribed mean curvature. *J. Reine Angew. Math.*, 457:71–83, 1994.

[DS02] F. Duzaar and K. Steffen. Optimal interior and boundary regularity for almost minimizers to elliptic variational integrals. *J. Reine Angew. Math.*, 564:73–138, 2002.

[Fed69] H. Federer. *Geometric measure theory*, volume 153 of *Die Grundlehren der mathematischen Wissenschaften.* Springer-Verlag New York Inc., New York, 1969.

[Fin86] R. Finn. *Equilibrium Capillary Surfaces*, volume 284 of *Die Grundlehren der mathematischen Wissenschaften.* Springer-Verlag New York Inc., New York, 1986.

[Gau30] C. F. Gauss. Principia generalia theoriae figurae fluidorum. *Comment. Soc. Regiae Scient. Gottingensis Rec.*, 7, 1830.

[Giu84] E. Giusti. *Minimal surfaces and functions of bounded variation*, volume 80 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1984.

[Giu03] E. Giusti. *Direct Methods in the Calculus of Variations*, volume 2 of *Lecture Notes. Scuola Normale Superiore di Pisa (New Series)*. World Scientific Publishing Co. Inc., River Edge, NJ, 2003.

[GJ86] M. Grüter and J. Jost. Allard type regularity results for varifolds with free boundaries. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 13(1):129–169, 1986.

[GMS98] M. Giaquinta, G. Modica, and J. Soucek. *Cartesian currents in the Calculus of Variations. II. Variational integrals*, volume 38 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin, 1998.

[Gon77] E. Gonzalez. Regolarità per il problema della goccia appoggiata. *Rend. Sem. Mat. Univ. Padova*, 58:25–33, 1977.

[Grü87a] M. Grüter. Boundary regularity for solutions of a partitioning problem. *Arch. Rational Mech. Anal.*, 97(3):261–270, 1987.

[Grü87b] M. Grüter. Optimal regularity for codimension one minimal surfaces with a free boundary. *Manuscripta Math.*, 58(3):295–343, 1987.

[Grü87c] M. Grüter. Regularity results for minimizing currents with a free boundary. *J. Reine Angew. Math.*, 375/376:307–325, 1987.

[GT98] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Springer, Berlin; New York, 1998.

[Har77] R. Hardt. On boundary regularity for integral currents or flat chains modulo two minimizing the integral of an elliptic integrand. *Comm. Part. Diff. Equ.*, 2:1163–1232, 1977.

[Mag12] F. Maggi. *Sets of finite perimeter and geometric variational problems*, volume 135 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2012. An introduction to Geometric Measure Theory.

[Rei60] E. R. Reifenberg. Solution of the Plateau problem for m-dimensional surfaces of varying topological type. *Acta Math.*, 104:1–92, 1960.

[Rei64a] E. R. Reifenberg. An epiperimetric inequality related to the analyticity of minimal surfaces. *Ann. of Math.*, 80(2):1–14, 1964.

[Rei64b] E. R. Reifenberg. On the analyticity of minimal surfaces. *Ann. of Math.*, 80(2):15–21, 1964.

[Sim83] L. Simon. *Lectures on geometric measure theory*, volume 3 of *Proceedings of the Centre for Mathematical Analysis*. Australian National University, Centre for Mathematical Analysis, Canberra, 1983.

[Sim96] L. Simon. *Theorems on regularity and singularity of energy minimizing maps*. Birkhaüser-Verlag, Basel-Boston-Berlin, 1996.

[SSA77] R. Schoen, L. Simon, and F. J. Jr. Almgren. Regularity and singularity estimates on hypersurfaces minimizing parametric elliptic variational integrals. I, II. *Acta Math.*, 139(3-4):217–265, 1977.

[SW89] B. Solomon and B. White. A strong maximum principle for varifolds that are stationary with respect to even parametric elliptic functionals. *Indiana Univ. Math. J.*, 38(3):683–691, 1989.

[Tam84] I. Tamanini. *Regularity results for almost minimal oriented hypersurfaces in* $\mathbb{R}^N$. Quaderni del Dipartimento di Matematica dell'Università di Lecce. Universit di Lecce, 1984. available for download at http://cvgmt.sns.it/paper/1807/.

[Tay76] J. E. Taylor. The structure of singularities in soap-bubble-like and soap-film-like minimal surfaces. *Ann. of Math. (2)*, 103(3):489–539, 1976.

[Tay77] J. E. Taylor. Boundary regularity for solutions to various capillarity and free boundary problems. *Comm. Partial Differential Equations*, 2(4):323–357, 1977.

[You05] T. Young. An essay on the cohesion of fluids. *Philos. Trans. Roy. Soc. London*, 95:65–87, 1805.

Institute for Applied Mathematics, University of Bonn, Endenicher Allee 60, D-53115 Bonn, Germany

E-mail address: guido.de.philippis@hcm.uni-bonn.de

Department of Mathematics, The University of Texas at Austin, 2515 Speedway Stop C1200, Austin, Texas 78712-1202, USA

E-mail address: maggi@math.utexas.edu