Geometric characterizations of p-Poincaré inequalities in the metric setting *

Estibalitz Durand-Cartagena
Departamento de Matemática Aplicada, ETSI Industriales, UNED,
Juan del Rosal 12, Ciudad Universitaria, 28040 Madrid, Spain.
E-mail: `edurand@ind.uned.es`

Jesus A. Jaramillo †
Departamento de Análisis Matemático, Facultad de CC. Matemáticas,
Universidad Complutense de Madrid, 28040-Madrid, Spain.
E-mail: `jaram@mat.ucm.es`

Nageswari Shanmugalingam ‡
Department of Mathematical Sciences, University of Cincinnati,
P.O. Box 210025, Cincinnati, OH 45221-0025, U.S.A.
E-mail: `shanmun@uc.edu`

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Abstract. We prove that a locally complete metric space endowed with a doubling measure satisfies an ∞-Poincaré inequality if and only if given a null set, every two points can be joined by a quasiconvex curve which "almost avoids" that set. As an application, we characterize doubling measures on $\mathbb{R}$ satisfying an ∞-Poincaré inequality. For Ahlfors Q-regular spaces, we obtain a characterization of p-Poincaré inequality for $p > Q$ in terms of the p-modulus of quasiconvex curves connecting pairs of points in the space. A related characterization is given for the case $Q - 1 < p \leq Q$.

Key words p-Poincaré inequality, metric measure space, thick quasiconvexity, quasiconvexity, singular doubling measures in $\mathbb{R}$, $\mathrm{Lip} - \mathrm{lip}$ condition.

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1 Introduction

During the last decade, different theories have been proposed for developing a first order analysis on metric measure spaces, see for example [14], [17], [16], [7], and [30] for a sample. The common idea underpinning some of these non-linear theories is that, for a viable theory of first order calculus in this abstract setting, one needs plenty of curves well distributed along the space. One way of making this idea precise is to assume that the space supports a p-Poincaré inequality for some $1 \leq p < \infty$. This a priori analytical property involves the metric, the measure, and the (upper) gradients, and encodes geometric information about the space. The exponent p from the p-Poincaré inequality actually also plays a geometrical role. The bigger the exponent p, the weaker the p-Poincaré inequality, and hence less restriction on the geometry. The limiting case $p = \infty$ has been studied in [11] and has surprisingly different properties than the finite p-Poincaré inequality case. One of the key tools used to define the notion of a large family of curves is the p-modulus of a family of curves, an outer measure defined on the set of all rectifiable curves. The presence of a p-Poincaré inequality implies that the corresponding p-modulus of the collection of quasiconvex curves connecting two disjoint sets of positive measure has to be positive, that is, the space is p-thick quasiconvex. For $p = \infty$ this property turns out to be special in that ∞-Poincaré inequality is characterized in terms of ∞-thick quasiconvexity; see [11].

The three main theorems of this paper are Theorem 3.1, Theorem 5.1, and Theorem 5.3. In Theorem 3.1 we prove that a locally complete doubling metric space admits an ∞-Poincaré inequality if and only if one can find quasiconvex curves transversal to a given zero measure set, that is, given a zero measure set N and two points, one can find a quasiconvex curve γ connecting the two points such that $\mathscr{L}^1(\gamma^{-1}(\gamma \cap N)) = 0$. This purely geometric property is a very simple, but powerful tool, useful in different applications. Two such applications are studied in Section 4 of this paper. Furthermore, in Section 3, Theorem 3.1 is also used to answer two questions posed in [11] and [12]. Theorems 5.1 and 5.3 give analogous characterizations of p-Poincaré inequality for Ahlfors Q-regular spaces for $p > Q$ and $Q \geq p > Q - 1$ respectively.

An immediate consequence of Theorem 3.1 (Corollary 3.7) is that ∞-capacity of points is always positive when the space supports an ∞-Poincaré inequality. In particular every function in $N^{1,\infty}(X)$ is Lipschitz continuous. This solves an open problem posed in [11] and gives a complete understanding of the Newtonian function class for $p = \infty$.

It is worth mentioning that one can also apply Theorem 3.1 for p-finite type problems. It is known that complete metric spaces endowed with a doubling measure and supporting a p-Poincaré inequality for $1 \leq p \leq \infty$ are quasiconvex. As far as we know, completeness has been a crucial hypothesis for all the different proofs of this fact in the literature. As a byproduct of the main result, we can weaken the hypothesis of completeness to local completeness, see Remark 3.3.

It was proved by Buckley, Björn, and Keith in [6] that $(\mathbb{R}, |\cdot|, \mu)$, with μ doubling, will support a p-Poincaré inequality for some $1 \leq p < \infty$ if and only if $\mu \ll \mathscr{L}^1$ and the Radon-Nikodym derivative of μ with respect to $\mathscr{L}^1$ is a Muckenhoupt $\mathcal{A}_p$-weight. In contrast, we prove in Theorem 4.2 that to obtain an ∞-Poincaré inequality, it is both necessary and sufficient to have $\mathscr{L}^1 \ll \mu$. This completes the picture for $n = 1$. In higher dimensions it is not known so far whether doubling measures on $\mathbb{R}^n$ supporting a p-Poincaré inequality for some $1 \leq p < \infty$ must necessarily be absolutely continuous with respect to the Lebesgue measure $\mathscr{L}^n$. We will show that in higher dimensional Euclidean setting, if the measure μ satisfies $\mathscr{L}^n \ll \mu$, then

$(\mathbb{R}^n, |\cdot|, \mu)$ supports an ∞-Poincaré inequality. We do not know whether the converse is true. However, we use Theorem 3.1 to prove that certain singular measures μ on $\mathbb{R}^n$ cannot support an ∞-Poincaré inequality and hence cannot support a p-Poincaré inequality for any $p \geq 1$, see Example 4.5.

Cheeger [7] proved that doubling p-Poincaré spaces, $1 \leq p < \infty$, admit a measurable differentiable structure for which Rademacher's Theorem holds. Subsequently, Keith [20] obtained the same conclusion under a weaker hypothesis called the $\mathrm{Lip} - \mathrm{lip}$ condition, a condition that does not depend on p. Recently, Bate [4] and Gong [13] have proved independently that the $\mathrm{Lip} - \mathrm{lip}$ condition is not only a sufficient, but also a necessary condition for a Cheeger differentiable structure. In Example 4.7, we construct a complete doubling metric measure space supporting an ∞-Poincaré inequality but with no measurable differentiable structure. This in turn implies, by Theorem 4.6, that the space does not have the $\mathrm{Lip} - \mathrm{lip}$ condition. Therefore, without any extra-hypothesis, there is no relation between the $\mathrm{Lip} - \mathrm{lip}$ condition and the ∞-Poincaré inequality. This solves an open question posed in [12].

In the case $p < \infty$, the property of being p-thick quasiconvex is, in contrast to the $p = \infty$ case, too weak in order to characterize p-Poincaré inequalities, see [12]. The main reason is that one would need a more quantitative estimate for the p-moduli of curve families. Estimates of this nature that characterize p-Poincaré inequalities have been previously given in Heinonen–Koskela [17] (Loewner property), Keith [21] (Riesz measures), Bonk–Kleiner [5, Theorem 1.3], Semmes [29] (pencil of curves), and Maz'ya [24] (capacitary estimate); see also [1] for related results. In the particular case of graphs with polynomial volume growth, Coulhon–Koskela [9] obtains a characterization in terms of modulus of families of curves for all the range of exponents $1 \leq p < \infty$. In the spirit of [9], in Theorem 5.1 and Theorem 5.3 we give characterizations of p-Poincaré inequalities for the range of exponents $Q - 1 < p < \infty$. For the range $Q - 1 < p \leq Q$ this characterization is in terms of the p-modulus of curves connecting two continua and their diameter and relative distance. We believe this characterization is not true in general when $p < Q - 1$. The discrete setting considered in [9] is special in that the local dimension associated with a graph is 1, and hence locally the measure behaves as if $Q = 1$ in this case. Thus in [9] a lower bound is obtained for the p-modulus of curve families joining two continua, in terms of their relative separation, of a graph with polynomial growth of power Q and supporting a p-Poincaré inequality, even when $1 \leq p < Q - 1$. Such lower bound could fail in more general metric measure spaces, for then it is possible to have a 1-dimensional continuum of positive diameter but with zero p-capacity, as in the standard n-dimensional Euclidean setting for any $n \geq 3$.

2 Thick quasiconvex spaces: preliminaries

In this paper we will assume that $X = (X, d, \mu)$ is a metric measure space, that is, (X, d) is a metric space equipped with a Borel measure μ which is positive and finite on each ball, and that μ is doubling. Recall that μ is *doubling* if there is a constant C_μ such that, for each ball $B(x, r)$ in X,

$$\mu(B(x, 2r)) \leq C_\mu \, \mu(B(x, r)).$$

A *curve* in X is a continuous function $\gamma : I \to X$ for some compact interval $I \subset \mathbb{R}$. Such a curve is *rectifiable* if its length

$$\ell(\gamma) := \sup_{t_0<t_1<\cdots<t_n} \sum_{j=1}^{n} d(\gamma(t_{j-1}), \gamma(t_j))$$

is finite. In the above definition of length $\ell(\gamma)$, the supremum is taken over all finite subdivisions $t_0 < t_1 < \cdots < t_n$ of the interval I. A rectifiable curve γ can be re-parametrized so that it is arc-length parametrized, that is, $I = [0, \ell(\gamma)]$ and for each $s \in I$, with $I_s := \{t \in I : t \le s\}$, we have

$$\ell(\gamma|_{I_s}) = s.$$

Henceforth in the paper we will assume all rectifiable curves, unless otherwise indicated, to be arc-length parametrized as above. The integral of a Borel function $\rho : X \to [0, \infty]$ over an arc-length parametrized curve γ is defined as

$$\int_\gamma \rho\, ds := \int_0^{\ell(\gamma)} \rho(\gamma(t))\, dt.$$

The space X is said to be *quasiconvex* if there exists a constant $C \ge 1$ such that given two points $x, y \in X$, one can find a C-quasiconvex curve joining them, that is, a rectifiable curve γ such that $\ell(\gamma) \le C d(x, y)$.

Given $E \subset X$, let Γ_E^+ denote the family of curves γ in X such that $\mathscr{L}^1(\gamma^{-1}(\gamma \cap E)) > 0$, where $\mathscr{L}^1$ is the usual 1-dimensional Lebesgue measure on the line. We denote by Γ_E the family of curves γ such that $\gamma \cap E \neq \emptyset$.

Definition 2.1 Given a family Γ of curves in X, set $F(\Gamma)$ to be the family of all Borel measurable functions $\rho : X \to [0, \infty]$ such that

$$\int_\gamma \rho\, ds \ge 1 \quad \text{for all} \quad \gamma \in \Gamma.$$

We define the ∞-*modulus of* Γ by

$$\operatorname{Mod}_\infty(\Gamma) = \inf_{\rho \in F(\Gamma)} \|\rho\|_{L^\infty(X)},$$

and for $1 \le p < \infty$ the p-modulus of Γ is

$$\operatorname{Mod}_p(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_X \rho^p\, d\mu.$$

Note that if every curve in Γ is contained in a fixed ball B, then

$$\operatorname{Mod}_p(\Gamma)^{1/p} \le \mu(B)^{1/p} \operatorname{Mod}_\infty(\Gamma),$$

and therefore

$$\limsup_{p\to\infty} [\operatorname{Mod}_p(\Gamma)]^{1/p} \le \operatorname{Mod}_\infty(\Gamma).$$

We next recall a characterization of path families whose ∞-modulus is zero.

Lemma 2.2 [10, Lemma 5.7] *Let Γ be a family of curves in X. The following conditions are equivalent:*

(a) $\mathrm{Mod}_\infty \Gamma = 0$.

(b) *There is a Borel function $\rho \geq 0$ with $\|\rho\|_{L^\infty(X)} = 0$ such that $\int_\gamma \rho\, ds = +\infty$ for each $\gamma \in \Gamma$.*

Definition 2.3 For $1 \leq p \leq \infty$ we say that (X, d, μ) is a *p-thick quasiconvex* space if there exists $C \geq 1$ such that for all $x, y \in X$, all $0 < \varepsilon < \frac{1}{4} d(x, y)$, and all measurable sets $E \subset B(x, \varepsilon)$, $F \subset B(y, \varepsilon)$ satisfying $\mu(E)\mu(F) > 0$ we have

$$\mathrm{Mod}_p(\Gamma(E, F, C)) > 0,$$

where $\Gamma(E, F, C)$ denotes the collection of all curves $\gamma_{p,q}$ connecting $p \in E$ and $q \in F$ with $\ell(\gamma_{p,q}) \leq C d(p, q)$. Here we do not require quantitative control on the modulus of the curve family, but we do require a quantitative control over the length of the curves, the control being exercised by the constant C.

Remark 2.4 Every complete thick quasiconvex space X supporting a doubling measure is quasiconvex; see [11]. It was shown in [11] and [12] that if X supports a p-Poincaré inequality for some $1 \leq p \leq \infty$, then X is a p-thick quasiconvex space. It was also proved in [11] that ∞-thick quasiconvexity is also sufficient for the validity of an ∞-Poincaré inequality. However, the examples in [12] show that p-thick quasiconvexity is not sufficient for the validity of a p-Poincaré inequality when $1 \leq p < \infty$. The proof of Theorem 3.1 will also show that we can replace completeness of X with local completeness of X in the results mentioned above.

A non-negative Borel measurable function g on X is said to be a *p-weak upper gradient* of a function $u : X \to [-\infty, \infty]$ if there is a family Γ of non-constant curves with $\mathrm{Mod}_p(\Gamma) = 0$ such that whenever γ is a rectifiable curve in X with $\gamma \notin \Gamma$, we have

$$|u(y) - u(x)| \leq \int_\gamma g\, ds,$$

where x and y denote the end points of γ. The above inequality should also be interpreted to mean that $\int_\gamma g\, ds = \infty$ if at least one of $u(x)$, $u(y)$ is not finite; see [17]. We say that a p-weak upper gradient g is an *upper gradient* if the above inequality holds for *each* rectifiable curve γ on X.

Definition 2.5 We say that X supports a *p-Poincaré inequality*, $1 \leq p \leq \infty$, if there are constants $C > 0$, $\lambda \geq 1$ such that for each measurable function u on X, each p-weak upper gradient g of u, and each ball $B \subset X$ we have

$$\fint_B |u - u_B|\, d\mu \leq C \operatorname{rad}(B) \left(\fint_{\lambda B} g^p\, d\mu \right)^{1/p}.$$

Here λB denotes the ball concentric with B (with respect to the pre-determined center) but with radius λ-times the radius of B. When $p = \infty$, the term inside the parenthesis on the right-hand side of the above inequality should be interpreted to mean $\|g\|_{L^\infty(\lambda B)}$. For arbitrary $A \subset X$ with $0 < \mu(A) < \infty$ we write

$$u_A = \fint_A u := \frac{1}{\mu(A)} \int_A u\, d\mu.$$

It is well known that complete metric spaces endowed with a doubling measure and supporting a p-Poincaré inequality are quasiconvex (a property that does not depend on p); see for example [16], [7], and [22]. One in fact gains more information; X is p-thick quasiconvex, see [11] and [12].

For $1 \leq p \leq \infty$, let $\widetilde{N}^{1,p}(X)$ be the class of all p-integrable functions on X that have a p-weak upper gradient in $L^p(X)$. For $u \in \widetilde{N}^{1,p}(X)$ we define

$$\|u\|_{N^{1,p}} := \|u\|_{L^p(X)} + \inf_g \|g\|_{L^p(X)},$$

where the infimum is taken over all p-weak upper gradients g of u. Now, we define in $\widetilde{N}^{1,p}(X)$ an equivalence relation by $u_1 \sim u_2$ if and only if $\|u_1 - u_2\|_{N^{1,p}} = 0$. Then the corresponding Newtonian space is defined as the quotient $N^{1,p}(X) = \widetilde{N}^{1,p}(X)/\sim$ and it is equipped with the norm $\|u\|_{N^{1,p}(X)} := \|u\|_{N^{1,p}}$. It has been proved that $N^{1,p}(X)$ is a Banach space (see [30] for the case $1 \leq p < \infty$, and see [10] for the case $p = \infty$.)

From the results in [30] and [15] we know that when $1 \leq p < \infty$, given $u \in N^{1,p}(X)$ there is a unique p-weak upper gradient $g_u \in L^p(X)$ of u such that whenever $g \in L^p(X)$ is a p-weak upper gradient of u we have $g_u \leq g$ almost everywhere in X. Such g_u is called the *minimal p-weak upper gradient* of u. Given the non-locality of the norm of $L^\infty(X)$, such minimal weak upper gradients of functions in $N^{1,\infty}(X)$ are not readily verified to exist; however, using the approach of quasi-Banach function lattices, the paper [23] proved the existence of minimal p-weak upper gradients even for the case $p = \infty$.

The papers [2] and [3] together show that if the metric space X is metrically doubling and complete, then for $1 < p < \infty$ Lipschitz functions are dense in $N^{1,p}(X)$ and $N^{1,p}(X)$ is reflexive. If X supports a p-Poincaré inequality and the measure is doubling, then the above results hold even if X is not complete [30]. The case $p = \infty$ is slightly different; see [10], [12], and [11]. The results in [11] show that when X is complete and μ is doubling, X supports an ∞-Poincaré inequality if and only if for each $u \in N^{1,\infty}(X)$ there is a function $u_0 \in \mathrm{LIP}^\infty(X)$ such that $u = u_0$ μ-a.e. in X and the respective energy seminorms are comparable. Here $\mathrm{LIP}^\infty(X)$ denotes the space of all bounded Lipschitz functions on X endowed with the norm given by

$$\|u\|_{\mathrm{LIP}^\infty(X)} = \sup_{x \in X} |u(x)| + \sup_{x,y \in X; y \neq x} \frac{|u(y) - u(x)|}{d(x,y)},$$

where the second term forms the energy seminorm for $\mathrm{LIP}^\infty(X)$.

Associated with (locally) Lipschitz functions u on X there are two local "Lipschitz constant" functions that act like the (modulus of the) derivative of u:

$$\operatorname{Lip} u(x) := \limsup_{r \to 0^+} \sup_{0 < d(y,x) \leq r} \frac{|u(y) - u(x)|}{r},$$

and

$$\operatorname{lip} u(x) := \liminf_{r \to 0^+} \sup_{0 < d(y,x) \leq r} \frac{|u(y) - u(x)|}{r}.$$

It was shown in [7] that for complete metric spaces, $\operatorname{Lip} u$ and $\operatorname{lip} u$ are almost everywhere comparable to each other if μ is doubling and supports a p-Poincaré inequality for some $1 \leq p < \infty$. In Section 4 we will show that the above two "constant" functions are not necessarily related under ∞-Poincaré inequality, even if μ is doubling.

3 Thick quasiconvex spaces: the main theorem

In this section we state and prove the first of the three main theorems of this paper. The following theorem answers two open questions posed in [11] and [12]. See Corollary 3.7 and Example 4.7.

Theorem 3.1 *Suppose that X is a locally complete metric space supporting a doubling Borel measure μ which is nontrivial and finite on balls. Then the following conditions are equivalent:*

(a) *X supports an ∞-Poincaré inequality.*

(b) *X is ∞-thick quasiconvex.*

(c) *X is connected and $\mathrm{LIP}^\infty(X) = N^{1,\infty}(X)$ with comparable energy seminorms.*

(d) *X supports an ∞-Poincaré inequality for functions in $N^{1,\infty}(X)$.*

(e) *(X, d, μ) is a* very thick quasiconvex *space, that is, there exists $C \geq 1$ such that for all $x, y \in X$, with $d(x, y) > 0$ we have that*

$$\mathrm{Mod}_\infty(\Gamma(\{x\}, \{y\}, C)) > 0,$$

where $\Gamma(\{x\}, \{y\}, C)$ denotes the set of C-quasiconvex curves in X connecting x and y.

(f) *There is a constant $C \geq 1$ such that, for every null set N of X, and for every pair of points $x, y \in X$ there is a C-quasiconvex path γ in X connecting x to y with $\gamma \notin \Gamma_N^+$.*

Furthermore, under any of the above equivalent conditions, there is a constant $C \geq 1$ such that whenever $x, y \in X$ are distinct,

$$\frac{1}{d(x,y)} \geq \mathrm{Mod}_\infty(\Gamma(\{x\}, \{y\}, C)) \geq \frac{1}{C\, d(x,y)}.$$

Remark 3.2 The implication $(d) \implies (b)$ does not require the local completeness hypothesis. The equivalence of $(a), (b), (c)$, and (d) has already been established in Theorem 4.7 of [11]. We point out here that while [11] assumed X to be complete, the proof of Theorem 4.7 there did not require the completeness of X (indeed, we need the local completeness of X for the proof of $(b) \implies (f)$). Therefore, to prove the first part of the above theorem, it suffices to establish the equivalence of $(b), (e)$, and (f).

Proof. $(f) \implies (e)$ Assume (X, d, μ) is not a very thick quasiconvex space with respect to the constant C, where C is the constant from Condition (f). Then there exist $x, y \in X$ such that $\mathrm{Mod}_\infty(\Gamma(\{x\}, \{y\}, C)) = 0$. By Lemma 2.2 (b), there exists a non-negative Borel measurable function $g \in L^\infty(X)$ such that $\int_\gamma g\, ds = \infty$ for each $\gamma \in \Gamma(\{x\}, \{y\}, C)$ and $\|g\|_{L^\infty(X)} = 0$. Observe that $N = \{x \in X : g(x) > 0\}$ has zero measure. Then for each quasiconvex curve connecting x to y, $\mathscr{L}^1(\gamma^{-1}(\gamma \cap N)) > 0$. Hence $\Gamma(\{x\}, \{y\}, C) \subset \Gamma_N^+$, which then violates the hypothesis of (f). Therefore (e) holds true whenever (f) is true, with the constant associated with Condition (e) no larger than the constant associated with Condition (f).

$(e) \implies (f)$ Assume that (X, d, μ) is a very thick quasiconvex space. Let N be a zero measure set. Because $\mu(N) = 0$, we have $\mathrm{Mod}_\infty(\Gamma_N^+) = 0$ (since $\infty \cdot \chi_{N_0} \in F(\Gamma_N^+)$, where N_0 is a Borel set containing N such that $\mu(N_0)=0$). Therefore $\operatorname{Mod}_\infty(\Gamma(\{x\},\{y\},C)\setminus\Gamma_N^+)>0$ and hence we have condition (f), with the associated constant no more than the constant from Condition (e).

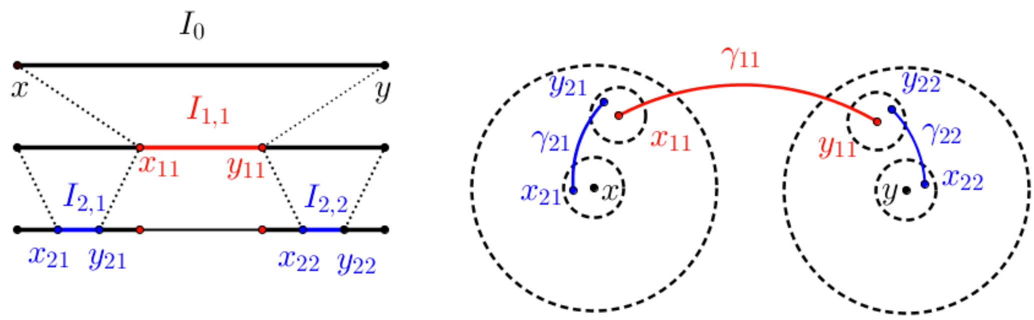
$(b)\implies(f)$ Fix $x,y\in X$, with $d(x,y)>0$. Since the space is locally complete, we can choose $0<\varepsilon_1\le\frac14 d(x,y)$ such that $\overline{B}(x,6C\varepsilon_1)$ and $\overline{B}(y,6C\varepsilon_1)$ are complete, where C is the constant associated to ∞-thick quasiconvexity. Let $N\subset X$ such that $\mu(N)=0$. Note that $\operatorname{Mod}_\infty(\Gamma_N^+)=0$, and since the space is ∞-thick quasiconvex, we have

$$\operatorname{Mod}_\infty(\Gamma(B(x,\varepsilon_1),B(y,\varepsilon_1),C)\setminus\Gamma_N^+)>0.$$

Thus there exist points $x_1\in B(x,\varepsilon_1)$, $y_1\in B(y,\varepsilon_1)$, and a curve γ_1 in X connecting x_1 and y_1, such that $\gamma_1\notin\Gamma_N^+$ and

$$\ell(\gamma_1)\le Cd(x_1,y_1)\le 2Cd(x,y).$$

It now suffices to be able to connect x_1 to x by a curve β_1 of length $\ell(\beta_1)\le C\,d(x_1,x)$, and connect y_1 to y by a curve β_2 of length $\ell(\beta_2)\le C\,d(y_1,y)$, such that $\mathscr{L}^1(\beta_1^{-1}(\beta_1\cap N)\cup\beta_2^{-1}(\beta_2\cap N))=0$. The concatenation of the three curves γ_1, β_1, and β_2 would then give the desired curve γ connecting x to y such that $\mathscr{L}^1(\gamma^{-1}(\gamma\cap N))=0$. The curves β_1 and β_2 are constructed in a manner similar to the construction of a Cantor set, as follows.



Let $I_0=[0,1]$, and for $k\in\mathbb{N}$ we inductively construct $I_{k,j}$, $j=1,\cdots,2^{k-1}$, as follows. Let $I_{1,1}=[1/3,2/3]$, and at step $k=2$, we set $I_{2,1}=[1/9,2/9]$, $I_{2,2}=[7/9,8/9]$ etc., so that for each $k\in\mathbb{N}$ the intervals $I_{k,j}$, $j=1,\cdots,2^{k-1}$, are of length 3^{-k}. We also consider the intervals $J_{k,j}$ which are the "gaps" at step $k-1$, that is, the complements of the interiors of intervals $I_{k,j}$. In this way at step $k=1$ we set $J_{2,1}=[0,1/3]$, $J_{2,2}=[2/3,1]$, and so on, so that for each $k\in\mathbb{N}$ the complement $[0,1]\setminus\bigcup_{i=1}^k\bigcup_{j=1}^{2^{k-1}}\operatorname{int}(I_{i,j})$ is the union of intervals $J_{k+1,j}$, $j=1,\cdots,2^k$. Note that the Cantor set is given by $\bigcap_{k\in\mathbb{N}}\bigcup_{j=1}^{2^k}J_{k+1,j}$.

With this notation, by reparametrizing we may think of γ_1 as a Lipschitz map $\gamma_1=\gamma_{1,1}:I_{1,1}\to X$ with Lipschitz constant at most $3C\,d(x_1,y_1)\le 6C\,d(x,y)$, connecting $x_1=x_{1,1}\in B(x,\varepsilon_1)$ to $y_1=y_{1,1}\in B(x,\varepsilon_1)$.

Now set $\varepsilon_2=\frac14\min\{\varepsilon_1,d(x,x_1),d(y,y_1)\}$. Then there exist points $x_{2,1}\in B(x,\varepsilon_2)$; $y_{2,1}\in B(x_{1,1},\varepsilon_2)$; $x_{2,2}\in B(y_{1,1},\varepsilon_2)$; and $y_{2,2}\in B(y,\varepsilon_2)$, and C-quasiconvex curves $\gamma_{2,1}:I_{2,1}\to X$ connecting $x_{2,1}$ to $y_{2,1}$ and $\gamma_{2,2}:I_{2,2}\to X$ connecting $x_{2,2}$ to $y_{2,2}$ with $\gamma_{2,1},\gamma_{2,2}\notin\Gamma_N^+$ such that

$$\ell(\gamma_{2,1})\le Cd(x_{2,1},y_{2,1})\le 2Cd(x,x_{1,1})\le\frac{C}{2}d(x,y)$$

and

$$\ell(\gamma_{2,2}) \le Cd(x_{2,2}, y_{2,2}) \le 2Cd(y, y_{1,1}) \le \frac{C}{2} d(x, y).$$

Now we can define $\tilde{\gamma} : I_{2,1} \cup I_{1,1} \cup I_{2,2} \cup \{0,1\} \to X$ by setting $\tilde{\gamma} = \gamma_{k,j}$ on each interval $I_{k,j}$ and $\tilde{\gamma}(0) = x$, $\tilde{\gamma}(1) = y$. It is not difficult to see that $\tilde{\gamma}$ is Lipschitz with constant $6Cd(x, y)$ on $I_{2,1} \cup I_{1,1} \cup I_{2,2} \cup \{0,1\}$. For example, if $s \in I_{2,1}$ and $t \in I_{1,1}$, noting that the gap between the two intervals is $J_{3,2}$ which has length 3^{-2} and so $|s - t| \ge 3^{-2}$, we have that

$$\begin{aligned} d(\tilde{\gamma}(s), \tilde{\gamma}(t)) &\le d(\gamma_{2,1}(s), y_{2,1}) + d(y_{2,1}, x_{1,1}) + d(x_{1,1}, \gamma_{1,1}(t)) \\ &\le 6Cd(x, y)|s - (2/9)| + \frac{1}{4} \cdot \frac{1}{4} d(x, y) + 6Cd(x, y)|(1/3) - t| \\ &\le 6Cd(x, y)|s - t|. \end{aligned}$$

We now iterate this process. Suppose we have already constructed step $k-1$, and we have the corresponding map $\tilde{\gamma} : \bigcup_{i=1}^{k-1} \bigcup_{j=1}^{2^{i-1}} I_{i,j} \cup \{0,1\} \to X$ which is $6Cd(x, y)$-Lipschitz. Consider $\varepsilon_k = \frac{1}{4} \min\{\varepsilon_{k-1}, \Delta\}$, where Δ is the minimum of all distances $d(\tilde{\gamma}(s_{k,j}), \tilde{\gamma}(t_{k,j}))$ where $s_{k,j}, t_{k,j}$ are the end points of the intervals $J_{k,j}$ that form the gap at step $k-1$, $j = 1, \cdots, 2^{k-1}$. We obtain as before for $j = 1, \cdots, 2^{k-1}$ points $x_{k,j}, y_{k,j}$ in X and a C-quasiconvex curve $\gamma_{k,j} : I_{k,j} \to X$ joining them, such that $\gamma_{k,j} \notin \Gamma_N^+$. In this way we extend $\tilde{\gamma}$ to a $6Cd(x, y)$-Lipschitz map on $\bigcup_{i=1}^{k} \bigcup_{j=1}^{2^{i-1}} I_{i,j} \cup \{0,1\}$.

Thus we can create a sequence of intervals $\{I_i\}_{i\in\mathbb{N}} := \{I_{k,j}\}_{\substack{k\in\mathbb{N} \\ j=1\cdots 2^{k-1}}}$ with each $I_i \subset I_0$, and a $6C\, d(x, y)$-Lipschitz continuous function

$$\tilde{\gamma} : \bigcup_{i\in\mathbb{N}} I_i \to Z = \gamma_1 \cup \overline{B}(x, \varepsilon_1) \cup \overline{B}(y, \varepsilon_1).$$

Since $(Z, d_{|Z})$ is complete there exists a $6C\, d(x, y)$-Lipschitz continuous extension $\gamma : I_0 \to Z$. Furthermore, we have that $\mathscr{L}^1(I_0 \setminus \bigcup_{i\in\mathbb{N}} I_i) = 0$, and from the construction we have $\gamma \notin \Gamma_N^+$ and

$$\ell(\gamma) = \sum_{i\in\mathbb{N}} \ell(\gamma|_{I_i}) \le 6Cd(x, y).$$

It follows that γ is a $6C$-quasiconvex curve connecting x to y, where C is the thick quasiconvexity constant from (b).

$(e) \Longrightarrow (b)$ is straightforward. This completes the proof of the first part of the theorem.

We next prove the second part of the theorem. To this end, we assume that Conditions (a)–(f) hold. Fixing $x_0, y_0 \in X$ such that $x_0 \ne y_0$, we denote the collection of all rectifiable curves in $B(x_0, 4Cd(x_0, y_0))$ connecting x_0 to y_0 by Γ_{x_0,y_0}. Let $g \in L^\infty(X)$ be a nonnegative Borel measurable function on X such that for all $\gamma \in \Gamma_{x_0,y_0}$, the integral $\int_\gamma g\, ds \ge 1$ and set $g_0 = g$ in $B(x_0, 2Cd(x_0, y_0))$ and $g_0 = g + 1/[2Cd(x_0, y_0)]$ on $X \setminus B(x_0, 2Cd(x_0, y_0))$. We then set

$$\tilde{u}(z) = \inf_{\gamma \text{ path connecting } z \text{ to } x_0} \int_\gamma g_0\, ds,$$

and consider $u = \min\{\tilde{u}, 2\}$. By the definition of u and by Condition (f), we can see that u is Lipschitz continuous on X. Indeed, if $z, w \in X$, then setting N to be the collection of all points $y \in X$ for which $g_0(y) > \|g_0\|_{L^\infty(X)}$ and noting that $\mu(N) = 0$, there must be a C-quasiconvex curve γ in X connecting z to w with $\mathscr{L}^1(\gamma^{-1}(\gamma \cap N)) = 0$. Hence by the fact that g_0 is an upper gradient of u, we have $|u(z) - u(w)| \le \int_\gamma g_0\, ds \le C\, \|g_0\|_{L^\infty(X)}\, d(z, w)$. From the definition of $\tilde{u}$ it follows that $u(x_0) = 0$, and by the choice of g and g_0, it also follows that $u(y_0) \ge 1$. Note that $\tilde{u}$ and hence u is measurable (see [18]) and that g_0 is an upper gradient of u; (see [11]) hence $u \in N^{1,\infty}(X)$.

Now for each $i \in \mathbb{Z}$ define $B_i = B(x_0, 2^{1-i} d(x_0, y_0))$ if $i \ge 0$, and $B_i = B(y_0, 2^{1+i} d(x_0, y_0))$ if $i \le -1$. We can choose the constant C in the above discussion to be large enough so that $C > 2\lambda$ where λ is the scaling constant related to the ∞-Poincaré inequality of Condition (a). So on the ball λB_i we know that $g_0 = g$. Since we know that x_0 and y_0 are Lebesgue points for u, we have that

$$\begin{aligned} 1 \le |u(x_0) - u(y_0)| \le \sum_{i \in \mathbb{Z}} \Big| \fint_{B_i} u d\mu - \fint_{B_{i+1}} u d\mu \Big| &\le C_\mu \sum_{i \in \mathbb{Z}} \fint_{B_i} \Big| u - \fint_{B_i} u d\mu \Big| d\mu \\ &\le C_\mu C d(x_0, y_0) \sum_{i \in \mathbb{Z}} 2^{-|i|} \|g_0\|_{L^\infty(\lambda B_i)} \\ &= C_\mu C d(x_0, y_0) \sum_{i \in \mathbb{Z}} 2^{-|i|} \|g\|_{L^\infty(\lambda B_i)} \\ &\le C d(x_0, y_0) \|g\|_{L^\infty(X)}. \end{aligned}$$

Hence

$$\|g\|_{L^\infty(X)} \ge \frac{1}{C\, d(x_0, y_0)}.$$

Taking the infimum over all such g we obtain the inequality

$$\mathrm{Mod}_\infty(\Gamma_{x_0, y_0}) \ge \frac{1}{C\, d(x_0, y_0)}.$$

For $m \ge 1$ we set $\Lambda(x_0, y_0, m) = \Gamma_{x_0, y_0} \setminus \Gamma(\{x_0\}, \{y_0\}, m)$. Each curve in $\Lambda(x_0, y_0, m)$ has length at least $m\, d(x_0, y_0)$, and so the function $\rho_m = [m\, d(x_0, y_0)]^{-1} \chi_{B(x_0, 4Cd(x_0, y_0))} \in F(\Lambda(x_0, y_0, m))$. It follows that

$$\mathrm{Mod}_\infty(\Lambda(x_0, y_0, m)) \le \frac{1}{m\, d(x_0, y_0)}.$$

So if $m = 2C$, then we have that

$$\mathrm{Mod}_\infty(\Gamma(\{x_0\}, \{y_0\}, 2C) \ge \mathrm{Mod}_\infty(\Gamma_{x_0, y_0}) - \mathrm{Mod}_\infty(\Lambda(x_0, y_0, 2C)) \ge \frac{1}{2C\, d(x_0, y_0)}.$$

For the upper bound, consider the constant function $g_1 = \frac{1}{d(x_0, y_0)}$. If γ is a rectifiable curve connecting x_0 to y_0, then the length of γ is at least $d(x_0, y_0)$, and hence $1 \le \int_\gamma g_1\, ds$. Therefore,

$$\mathrm{Mod}_\infty(\Gamma(\{x_0\}, \{y_0\}, C) \le \|g_1\|_{L^\infty(X)} = \frac{1}{d(x_0, y_0)}.$$

This completes the proof of Theorem 3.1. $\square$

Remark 3.3 Observe that ∞-thick quasiconvexity does not a priori imply quasiconvexity of the space. It only implies that given two points x, y in the space, we can find two points, one close to x and the other close to y, that can be connected by a quasiconvex curve. In general non-compact spaces this does not automatically give a quasiconvex curve connecting x and y themselves. However, a careful look at the proof of $(b) \Longrightarrow (f)$ of Theorem 3.1 reveals that a locally complete metric space (X, d) supporting a doubling Borel measure μ and a p-Poincaré inequality for some $1 \le p \le \infty$ is quasiconvex. Previous results required completeness of the space X, see [16], [7], [21], and [22].

Remark 3.4 It was proven in [11, Corollary 4.15] that the Sierpiński carpet endowed with the Euclidean distance and the s-dimensional Hausdorff measure with $s = \frac{\log 8}{\log 3}$ does not support an ∞-Poincaré inequality. Theorem 3.1 proves that for each $m \in \mathbb{N}$, there exists a null set N of the carpet X and a pair of points $x, y \in X$ such that every m-quasiconvex path γ in X connecting x to y belongs to $\gamma \in \Gamma_N^+$. This fact could help to understand the set of rectifiable curves in fractal type sets with no Poincaré inequalities.

Remark 3.5 Notice that Theorem 3.1 does not hold for $1 \le p < \infty$. In particular, the implication $(b) \Longrightarrow (e)$ is false. For example $(\mathbb{R}^n, |\cdot|, \mathscr{L}^n)$ has a 1-Poincaré inequality but the p-modulus of curves passing through a point is zero when $1 \le p \le n$.

Definition 3.6 The *p-capacity* of a set $E \subset X$ with respect to the space $N^{1,p}(X)$ is defined by

$$\mathrm{Cap}_p(E) = \inf_u \|u\|_{N^{1,p}(X)},$$

where the infimum is taken over all functions u in $N^{1,p}(X)$ such that $u_{|E} \ge 1$.

Corollary 3.7 *Under the hypothesis of Theorem 3.1, if X supports an ∞-Poincaré inequality, then $\mathrm{Mod}_\infty(\Gamma_{x_0}) > 0$ for each $x_0 \in X$, where Γ_{x_0} denotes the collection of all non-constant curves passing through the point x_0. In particular, $\mathrm{Cap}_\infty(\{x_0\}) > 0$ so each equivalence class $[u] \in N^{1,\infty}(X)$ has exactly one element in it. Thus every Newtonian function in $N^{1,\infty}(X)$ is Lipschitz continuous.*

Proof. Observe that for a set $F \subset X$ with $\mu(F) = 0$ we have $\mathrm{Cap}_\infty(F) = 0$ if and only if $\mathrm{Mod}_\infty(\Gamma_F) = 0$. Indeed, if $\mu(F) = 0$ and $\mathrm{Mod}_\infty(\Gamma_F) = 0$, then the function $u = \chi_F$ belongs to $N^{1,\infty}(X)$ with the constant function 0 as an ∞-weak upper gradient; in this case u is a test function for computing $\mathrm{Cap}_\infty(F)$, whence we obtain $\mathrm{Cap}_\infty(F) = 0$. For the converse, see [10, Lemma 5.17]. Since the measure of a singleton set in a quasiconvex doubling measure space is zero, the result follows. □

Remark 3.8 As the slit disc in the Euclidean plane shows, the converse of the above corollary does not hold.

Corollary 3.9 *Under the hypothesis of Theorem 3.1, if X supports an ∞-Poincaré inequality then there exists a constant $C \ge 1$ such that for each $u \in \mathrm{LIP}^\infty(X)$*

$$\sup_{x \in X} \mathrm{Lip}\, u(x) \le C \|\mathrm{Lip}\, u\|_{L^\infty(X)}.$$

Proof. Let $u \in \mathrm{LIP}^\infty(X)$ and $K = \|\operatorname{Lip} u\|_{L^\infty(X)} < \infty$. Then there exists a null set N such that $\operatorname{Lip} u(z) \le K$ for each $z \in X \setminus N$. Now, given $x, y \in X$, take γ in X connecting x and y, parametrized by the arc-length such that $\ell(\gamma) \le Cd(x,y)$ and $\mathscr{L}^1(\gamma^{-1}(\gamma \cap N)) = 0$. Then since $\operatorname{Lip} u(\gamma(t)) \le K$ for $\mathscr{L}^1$−a.e. $t \in [0, \ell(\gamma)]$, we have

$$|u(x) - u(y)| \le \int_\gamma \operatorname{Lip} u \, ds = \int_0^{\ell(\gamma)} \operatorname{Lip} u(\gamma(t)) dt \le K\ell(\gamma) \le KCd(x,y).$$

Therefore, $\sup_{x \in X} \operatorname{Lip} u(x) \le C\|\operatorname{Lip} u\|_{L^\infty(X)}$. □

4 Singular measures and Lip-lip condition

In this section we give some applications of Theorem 3.1 to the case of doubling measures on Euclidean spaces and, furthermore, we give a characterization of doubling measures on the real line that support an ∞-Poincaré inequality. We begin with the following simple Lemma.

Lemma 4.1 *If μ is a doubling measure on $\mathbb{R}^n$ and $\mathscr{L}^n \ll \mu$, then $(\mathbb{R}^n, |\cdot|, \mu)$ supports an ∞-Poincaré inequality.*

Proof. Recall that $(\mathbb{R}^n, |\cdot|, \mathscr{L}^n)$ supports an ∞-Poincaré inequality. Hence if $N \subset \mathbb{R}^n$ is such that $\mu(N) = 0$, then $\mathscr{L}^n(N) = 0$; now Condition (f) of Theorem 3.1 applied to $(\mathbb{R}^n, |\cdot|, \mathscr{L}^n)$ tells us that for each $x, y \in \mathbb{R}^n$ we can find a C-quasiconvex curve in $\mathbb{R}^n$ connecting x to y such that $\gamma \notin \Gamma_N^+$. Thus $(\mathbb{R}^n, |\cdot|, \mu)$ also satisfies Condition (f) of Theorem 3.1 and so supports an ∞-Poincaré inequality. □

In the case $n = 1$ we can also obtain a converse result.

Theorem 4.2 *Let μ be a doubling measure on $\mathbb{R}$. Then $(\mathbb{R}, |\cdot|, \mu)$ supports an ∞-Poincaré inequality if and only if $\mathscr{L}^1 \ll \mu$.*

Proof. Given the above lemma, it suffices to prove that if $\mathscr{L}^1 \not\ll \mu$ then $(\mathbb{R}, |\cdot|, \mu)$ does not support any ∞-Poincaré inequality. Suppose that there is a measurable set E in $\mathbb{R}$ with $\mathscr{L}^1(E) > 0$ and $\mu(E) = 0$. Choose two points $x, y \in \mathbb{R}$ such that $\mathscr{L}^1([x,y] \cap E) > 0$, and consider $N = [x,y] \cap E$. Now let $\gamma : [a,b] \to \mathbb{R}$ be an arc-length parametrized rectifiable curve connecting x to y. By connectedness, we have that

$$\gamma([a,b]) \supset [x,y] \supset N.$$

Thus by the arc-lengh parametrization, $\mathscr{L}^1(\gamma^{-1}(\gamma \cap N)) > 0$. Thus each curve γ connecting x and y belongs to Γ_N^+, and therefore by Theorem 3.1 we have that $(\mathbb{R}, |\cdot|, \mu)$ does not support any ∞-Poincaré inequality. □

Remark 4.3 A result in [6] tells us that when μ is doubling, then $(\mathbb{R}, |\cdot|, \mu)$ supports a p-Poincaré inequality for some $1 \le p < \infty$ if and only if $\mu \ll \mathscr{L}^1$ and the Radon-Nikodym derivative of μ with respect to $\mathscr{L}^1$ is a Muckenhoupt $\mathcal{A}_p$-weight. In contrast, Theorem 4.2 tells us that to obtain ∞-Poincaré inequality, it is both necessary and sufficient to have $\mathscr{L}^1 \ll \mu$. In particular, if ν is a singular doubling measure on $\mathbb{R}$, then $\mu = \mathscr{L}^1 + \nu$ would support an ∞-Poincaré inequality even though μ is not absolutely continuous with respect to $\mathscr{L}^1$. Recall that there are doubling measures on $\mathbb{R}$ that are mutually singular to $\mathscr{L}^1$, such as the Riesz measure constructed in [33] (see also [32] and [12]).

Remark 4.4 Unlike in the situation of Theorem 4.2, we do not know whether a doubling measure μ on $\mathbb{R}^n$, $n \geq 2$, supporting an ∞-Poincaré inequality, must necessarily satisfy $\mathscr{L}^n \ll \mu$. That is, the converse of the above lemma is not known when $n \geq 2$.

The next example illustrates another application of Theorem 3.1. Given the above remark, we cannot immediately claim that a singular measure μ on $\mathbb{R}^n$ cannot support an ∞-Poincaré inequality; we instead use Theorem 3.1.

Example 4.5 Let μ be given by $\mu = \mu_1 \times \nu$, where $\mu_1 \perp \mathscr{L}^1$ is a doubling measure on $\mathbb{R}$ and ν is an arbitrary doubling measure on $\mathbb{R}^{n-1}$. Then $(\mathbb{R}^n, |\cdot|, \mu)$ does not support ∞-Poincaré inequality. Indeed, since μ_1 is singular, there exists a set E such that $\mu_1(E_1) = 0$ while $\mathscr{L}^1(E_1) > 0$. Let $E = E_1 \times \mathbb{R}^{n-1}$ and notice that $\mu(E) = 0$. Choose two points $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ in $\mathbb{R}^n$ with $x_1 < y_1$ such that $\mathscr{L}^1(E_1 \cap [x_1, y_1]) > 0$ and a curve γ connecting x to y. Then $\mathcal{H}^1(E \cap \gamma) \geq \mathcal{H}^1(P_1(E \cap \gamma)) = \mathscr{L}^1(P_1(E \cap \gamma)) > 0$ where P_1 denotes the projection onto the first axis. We thus deduce that $\mathscr{L}^1(\gamma^{-1}(\gamma \cap E))$ is positive, and so $\gamma \in \Gamma_E^+$. Therefore $(\mathbb{R}^n, |\cdot|, \mu)$ does not support ∞-Poincaré inequality because μ violates Condition (f) of Theorem 3.1. We thank the anonymous referee for this improved version of our original example.

We conclude this section by considering the so-called $\mathrm{Lip} - \mathrm{lip}$ property of Keith [20]. In [7] Cheeger proved that doubling p-Poincaré spaces admit a (non-degenerate) differentiable structure for which Lipschitz functions are differentiable μ-a.e. in the sense that there exists a countable collection of pairs $\{(X_\alpha, \mathbf{x}_\alpha)\}$ of measurable sets $X_\alpha \subset X$ (*charts*) and Lipschitz maps

$$\mathbf{x}_\alpha = (x_\alpha^1, \ldots, x_\alpha^{N_\alpha}) : X \longrightarrow \mathbb{R}^{N_\alpha}$$

(*coordinates*), that satisfy the following conditions:

(*i*) $\mu\Big(X \setminus \bigcup_\alpha X_\alpha\Big) = 0$;

(*ii*) There exists $N \geq 1$ such that $N_\alpha \leq N$ for each $(X_\alpha, \mathbf{x}_\alpha)$;

(*iii*) If $u : X \to \mathbb{R}$ is Lipschitz, then for each $(X_\alpha, \mathbf{x}_\alpha)$ there exists a unique (up to a set of zero measure) measurable function $d^\alpha u : X_\alpha \longrightarrow \mathbb{R}^{N_\alpha}$ such that

$$\limsup_{\substack{y \to x \\ y \neq x}} \frac{|u(y) - u(x) - d^\alpha u(x) \cdot (\mathbf{x}_\alpha(y) - \mathbf{x}_\alpha(x))|}{d(y, x)} = 0 \tag{1}$$

for μ-a.e. $x \in X_\alpha$.

If the above holds, we say that (X, d, μ) supports a *measurable differentiable structure.*

Observe that the exponent p is present in the hypothesis of this result, but it has no role in the conclusions. Keith, in [20] weakened the hypotheses so as not to depend on p. He defined the

$\mathrm{Lip}-\mathrm{lip}$ condition as follows: a metric measure space X is said to satisfy a $\mathrm{Lip}-\mathrm{lip}$ *condition* if there exists a constant $K \geq 1$ such that whenever $u : X \longrightarrow \mathbb{R}$ is a Lipschitz function, we have

$$\mathrm{Lip}\, u(x) \leq K \,\mathrm{lip}\, u(x)$$

for μ-a.e. $x \in X$. The thesis [20, Section 1.4] conjectures that this condition can be understood as a version of Cheeger's theorem for $p = \infty$.

It is known that complete doubling metric measure spaces which admit a p-Poincaré inequality for any $1 \leq p < \infty$ satisfy the $\mathrm{Lip}-\mathrm{lip}$ condition as well. On the other hand, it is clear that the $\mathrm{Lip}-\mathrm{lip}$ condition does not imply the validity of a p-Poincaré inequality for any $1 \leq p \leq \infty$. A non-empty non-quasiconvex open set of $\mathbb{R}^n$ has the $\mathrm{Lip}-\mathrm{lip}$ condition with $K = 1$, but does not support any p-Poincaré inequality, $1 \leq p \leq \infty$.

Very recently it has been proved that the $\mathrm{Lip}-\mathrm{lip}$ condition is not only sufficient but also a necessary condition for the validity of a Rademacher theorem in the metric measure setting. The complete characterization is the following.

Theorem 4.6 ([4, Corollary 10.4], [13, Theorem 1.3]) *Let (X, d) be a complete metric space endowed with a Radon measure μ. Then (X, d, μ) supports a Cheeger differentiable structure if and only if the measure μ is pointwise doubling and if there exists a countable collection of measurable sets $\{Z_n\}$ with associated constants M_n such that $\mu\Big(X \setminus \bigcup_n Z_n\Big) = 0$ and for each $n \in \mathbb{N}$, the space (Z_n, d, μ) satisfies a* $\mathrm{Lip}-\mathrm{lip}$ *condition with constant M_n.*

Concerning the above Theorem, see also [28, Page 7]. In the next example we will construct a complete doubling metric measure space supporting an ∞-Poincaré inequality but with no measurable differentiable structure which in turn implies by Theorem 4.6 that the space does not satisfy the $\mathrm{Lip}-\mathrm{lip}$ condition. Therefore, without any extra-hypothesis, there is no relation between the $\mathrm{Lip}-\mathrm{lip}$ condition and the ∞-Poincaré inequality.

Example 4.7 Take any singular doubling measure with constant C in $\mathbb{R}$ denoted μ_s and define $\mu = \mu_s + \mathscr{L}^1$. Observe that μ is a doubling measure. Indeed,

$$\begin{aligned}\mu(B(x,2r)) \leq & \mu_s(B(x,2r)) + \mathscr{L}^1(B(x,2r)) \leq C\mu_s(B(x,r)) + 2\mathscr{L}^1(B(x,r)) \\ \leq & \max\{2, C\}(\mu_s(B(x,r)) + \mathscr{L}^1(B(x,r))).\end{aligned}$$

By Theorem 4.2, $(\mathbb{R}, |\cdot|, \mu)$ supports an ∞-Poincaré inequality. On the other hand, since $\mu_s \perp \mathscr{L}^1$, there exists a set N such that $\mu(N) > 0$ whereas $\mathscr{L}^1(N) = 0$. A classical result by Choquet [8] states that given a set $E \subset \mathbb{R}$, there exists a Lipschitz function $u_0 : \mathbb{R} \to \mathbb{R}$ which is non-differentiable at any point of $x \in E$ if and only if $\mathscr{L}^1(E) = 0$. Using this result we can construct a Lipschitz function u_0 that is Euclidean differentiable nowhere in N. Assume that $(\mathbb{R}, |\cdot|, \mu)$ has a measurable differentiable structure in the sense of Cheeger. For simplicity assume that $\mathbb{R}$ is decomposed in one single chart denoted by X_α (if there is more than one chart, one can merely focus on one of the charts, choose a point of density of that chart, and ignore the remaining part of $\mathbb{R}$ without difficulties in the following argument). Then, there exists a unique measurable function $du_0 : X_\alpha \longrightarrow \mathbb{R}^{N_\alpha}$ such that

$$\lim_{\substack{y \to x \\ y \neq x}} \frac{|u_0(y) - u_0(x) - du_0(x) \cdot (\mathbf{x}_\alpha(y) - \mathbf{x}_\alpha(x))|}{|y - x|} = 0 \tag{2}$$

for μ-a.e. $x \in X_\alpha$. In particular, by [27, Corollary 6.30] combined with [28, Lemma 4.1], or else by [13, Corollary 6.5], we know that we can choose the coordinate functions to be a certain collection of distance functions. More precisely, there exist points $x_1, x_2, ..., x_{N_\alpha} \in \mathbb{R}$ such that $\mathbf{x}_\alpha(x) = (|x_1 - x|, |x_2 - x|...., |x_{N_\alpha} - x|)$. Denote by Z the set of points where u_0 is non-differentiable with respect to the chart X_α. Observe that the function $\mathbf{x}_\alpha : \mathbb{R} \to \mathbb{R}^{N_\alpha}$ is Euclidean differentiable on $\mathbb{R} \setminus \{x_1, x_2, ..., x_{N_\alpha}\}$. Since μ being doubling cannot charge finite sets, we know that there is a point x_0 in $(X_\alpha \cap N) \setminus Z$ (that is not any of $x_1, x_2, ..., x_{N_\alpha}$), such that (2) holds for $x = x_0$, that is, u_0 is differentiable at x_0 with respect to the chart X_α. In particular u_0 is differentiable at x_0 with respect to the standard Euclidean coordinate functions, with Euclidean derivative given by

$$\sum_{i=1}^{N_\alpha} \alpha_i \frac{1}{|x_0 - x_i|}(x_0 - x_i),$$

where $(\alpha_1, \alpha_2, ..., \alpha_{N_\alpha})$ is the metric derivative of u_0 (with respect to the chart (X_α, x_α)) at x_0, yielding a contradiction.

5 A characterization of p-Poincaré inequality in Ahlfors Q-regular spaces for $p > Q - 1$

Poincaré and Sobolev inequalities for functions in the Sobolev classes have proven to be useful tools in the study of solutions to PDEs, and hence it is of interest to know what Euclidean domains, and more generally, metric measure spaces, support such inequalities. The first to study such inequalities and the associated embedding theorems was Sobolev, see [31]. Characterizations of such inequalities in terms of isoperimetric inequality and condenser inequalities, in the setting of Euclidean spaces and manifolds were given by Maz′ya [25], [26]; a nice exposition can be found in [24]. However, in this section we are concerned more with obtaining a characterization of p-Poincaré inequalities in terms of p-moduli of curve families. In the case that the metric measure space is complete and Ahlfors Q-regular, a geometric (Loewner property) characterization of Q-Poincaré inequality was first given in [17]. In this section, we focus on Ahlfors Q-regular metric measure spaces with $Q > 1$, and wish to characterize p-Poincaré inequality in terms of p-moduli of curve families that connect two sets, for the two cases $p > Q$ and $Q - 1 < p \leq Q$. Such a characterization for graphs was obtained by Coulhon and Koskela [9].

Given the characterization of ∞-Poincaré inequality from Theorem 3.1, it is natural to ask whether there is a similar characterization of p-Poincaré inequality for large enough p. Given the Morrey embedding theorem, we consider $p > Q$ with Q the Ahlfors regularity exponent of μ. Recall that a measure μ is *Ahlfors Q-regular* if there is a constant $C > 0$ such that whenever $x \in X$ and $0 < r < \operatorname{diam} X$, $r^Q/C \leq \mu(B(x, r)) \leq Cr^Q$.

A version of the following theorem holds if μ is known to be doubling, where Q is the logarithm of the doubling constant of μ. However, for the sake of simplicity we focus only on Ahlfors regular measures. Interested readers can easily modify the argument, but in this case the constant C depends not only on $\|g_u\|_{L^p(B(x,\tau d(x,y)))}$ but also on the choice of a compact subset $K \subset X$ that contains $B(x, \tau d(x, y))$.

Theorem 5.1 *Let X be a complete Ahlfors Q-regular space and $p > Q$. Then the following conditions are equivalent:*

(1) *X supports a p-Poincaré inequality.*

(2) *There are constants $C > 0, \tau \ge 1$ such that every $u \in N^{1,p}(X)$ is $(1-\frac{Q}{p})$-Hölder continuous and for all $x, y \in X$ we have*

$$|u(x) - u(y)| \le C \|g_u\|_{L^p(B(x,\tau d(x,y))} \, d(x,y)^{1-\frac{Q}{p}},$$

where g_u is the minimal p-weak upper gradient of u.

(3) *There is a constant $C \ge 1$ such that, for every pair of distinct points $x, y \in X$,*

$$\mathrm{Mod}_p(\Gamma(\{x\}, \{y\}, C)) \ge \frac{1}{C\, d(x,y)^{p-Q}},$$

where $\Gamma(\{x\}, \{y\}, C)$ denotes the family of C-quasiconvex curves connecting x to y.

Remark 5.2 Let Γ_{x_0} denote the collection of all non-constant rectifiable curves intersecting x_0. Condition (3) directly implies that $\mathrm{Mod}_p(\Gamma_{x_0}) > 0$ and therefore $\mathrm{Cap}_p(\{x_0\}) > 0$.

Note also that if X is not connected, then there are two non-empty disjoint open sets U, V such that $X = U \cup V$; and then for $x_0 \in U$ and $R > 0$, choosing a C-Lipschitz function η_R on X such that $\eta_R = 1$ on $B(x_0, R)$, $\eta_R = 0$ on $X \setminus B(x_0, R+1)$, and $0 \le \eta_R \le 1$ on X, we see that Condition (2) fails for large R for the functions $u_R = \eta_R \chi_U$. Thus Condition (2) also implies that X is connected.

Finally, observe that by considering the function $\rho(z) = d(x,y)^{-1}\chi_{B(x,2Cd(x,y))}(z)$ and noting that it is a test function for computing the p-modulus of $\Gamma(\{x\}, \{y\}, C)$, from Condition (3) above we also obtain

$$\frac{C}{d(x,y)^{p-Q}} \ge \mathrm{Mod}_p(\Gamma(\{x\}, \{y\}, C)) \ge \frac{1}{C\, d(x,y)^{p-Q}}.$$

This is comparable to the comparison of the ∞-modulus $\mathrm{Mod}_\infty(\Gamma(\{x\}, \{y\}, C))$ in terms of $d(x,y)^{-1}$ obtained in Theorem 3.1.

Proof. [Proof of Theorem 5.1.] That (1) $\Longrightarrow$ (2) follows from the Morrey embedding theorem, see for example [30, Theorem 5.1.] or [16, Theorem 5.1]. To show that (2) $\Longrightarrow$ (1), we suppose that (2) holds. Let $u \in N^{1,p}(X)$ with minimal p-weak upper gradient g_u, and let B be a ball in X. Then by Condition (2),

$$|u(x) - u(y)| \le C\, d(x,y)^{1-Q/p} \left(\int_{B(x,\tau d(x,y))} g_u^p \, d\mu \right)^{1/p}$$

whenever $x, y \in B$. Note that $B(x, \tau d(x,y)) \subset 3\tau B$ whenever $x, y \in B$. Therefore

$$|u(x) - u(y)| \le C\, d(x,y)^{1-Q/p} \left(\int_{3\tau B} g_u^p \, d\mu \right)^{1/p}.$$

Let R be the radius of B. Then by the fact that $p > Q$ and the Ahlfors Q-regularity of μ, for $x, y \in B$ we have

$$|u(x) - u(y)| \le C\, R \left(\frac{1}{\mu(B)} \int_{3\tau B} g_u^p \, d\mu \right)^{1/p}.$$

Integrating over x and y in B, we obtain

$$\fint_B |u - u_B| \, d\mu \le \fint_B \fint_B |u(x) - u(y)| \, d\mu(x) \, d\mu(y) \le C\, R \left(\fint_{3\tau B} g_u^p \, d\mu \right)^{1/p}.$$

That is, u, g_u satisfy the p-Poincaré inequality on B, with $\lambda = 3\tau$. Thus we have the desired Poincaré inequality for all functions in $N^{1,p}(X)$, and hence for all Lipschitz functions u and their corresponding natural upper gradients $\operatorname{Lip} u$. It follows from a result of Keith [21] that X supports a p-Poincaré inequality for *all* function-p-weak upper gradient pairs. This proves (1).

Let us prove now that (2) $\Longrightarrow$ (3). Fix $x_0, y_0 \in X$. We denote by Γ_{x_0,y_0} the collection of all rectifiable curves in X with end points x_0 and y_0. We wish to show that

$$\operatorname{Mod}_p(\Gamma_{x_0,y_0}) \ge \frac{1}{C\, d(x_0, y_0)^{p-Q}}.$$

This is clear if $\operatorname{Mod}_p(\Gamma_{x_0,y_0}) = +\infty$. Otherwise, consider a nonnegative Borel measurable function $g \in L^p(X)$ such that for all $\gamma \in \Gamma_{x_0,y_0}$, the integral $\int_\gamma g\, ds \ge 1$. We then set

$$\tilde{u}(z) = \inf_{\gamma \text{ path connecting } z \text{ to } x_0} \int_\gamma g\, ds.$$

By Corollary 1.10 of [18] and using the assumption that X is complete, we know that $\tilde{u}$ is measurable. Note that g is an upper gradient of $\min\{\tilde{u}, 2\}$, since if $x, y \in X$ and γ is a rectifiable curve connecting x to y and β is a rectifiable curve connecting x to x_0, then the concatenation $\beta + \gamma$ is a rectifiable curve connecting y to x_0.

Now let η a be Lipschitz function which satisfies the conditions $\eta = 1$ on $B(x_0, \tau d(x_0, y_0))$, $\eta = 0$ on $X \setminus B(x_0, 2\tau d(x_0, y_0))$, and $0 \le \eta \le 1$ on X, and consider $u = \eta \min\{\tilde{u}, 2\}$. Then it follows that $u(x_0) = 0$ and, by the choice of g, $u(y_0) \ge 1$. Notice that $u \in L^p(X)$, and, since $\operatorname{Lip}\eta$ is an upper gradient of η, it can be easily checked that $\tilde{g} = g + 2 \operatorname{Lip}\eta$ is an upper gradient of u. In particular, we have that $u \in N^{1,p}(X)$. Note that because η is constant on $B(x_0, \tau d(x_0, y_0))$, we have that g is an upper gradient of u in $B(x_0, \tau d(x_0, y_0))$. Therefore, by the hypothesis,

$$\begin{aligned} 1 \le |u(x_0) - u(y_0)| &\le C \|\tilde{g}\|_{L^p(B(x_0, \tau d(x_0,y_0)))}\, d(x_0, y_0)^{1-\frac{Q}{p}} \\ &= C \|g\|_{L^p(B(x_0, \tau d(x_0,y_0)))}\, d(x_0, y_0)^{1-\frac{Q}{p}}. \end{aligned}$$

Taking the infimum over all such g we obtain the estimate

$$\operatorname{Mod}_p(\Gamma_{x_0,y_0}) \ge \frac{1}{C\, d(x_0, y_0)^{p-Q}}.$$

Given a positive integer m, let $\Gamma(\{x_0\}, \{y_0\}, m)$ denote the collection of all rectifiable curves connecting x_0 to y_0 of length at most $m\, d(x_0, y_0)$. Set

$$\Lambda(x_0, y_0, m) = \Gamma_{x_0,y_0} \setminus \Gamma(\{x_0\}, \{y_0\}, m).$$

Then by the subadditivity property of the modulus, we have

$$\operatorname{Mod}_p(\Gamma(\{x_0\}, \{y_0\}, m)) + \operatorname{Mod}_p(\Lambda(x_0, y_0, m)) \ge \frac{1}{C\, d(x_0, y_0)^{p-Q}}.$$

On the other hand, since $\rho := [m\, d(x_0, y_0)]^{-1}\chi_{B(x_0, md(x_0,y_0))}$ is admissible for computing the p-modulus of $\Lambda(x_0, y_0, m)$, we have

$$\mathrm{Mod}_p(\Lambda(x_0, y_0, m)) \le \frac{\mu(B(x_0, md(x_0, y_0)))}{m^p\, d(x_0, y_0)^p} \le \frac{1}{C_1\, m^{p-Q}\, d(x_0, y_0)^{p-Q}},$$

where C_1 is a constant depending only on the Ahlfors Q-regularity constant. Hence, when

$$m > \left(\frac{2C}{C_1}\right)^{1/(p-Q)},$$

we must then have

$$\mathrm{Mod}_p(\Gamma(\{x_0\}, \{y_0\}, m)) \ge \frac{1}{2C\, d(x_0, y_0)^{p-Q}}.$$

Thus, choosing the quasiconvexity constant to be $1 + \left(\frac{2C}{C_1}\right)^{1/(p-Q)}$, we obtain the desired conclusion.

To complete the proof of the theorem, let us prove that (3) $\implies$ (2). Let $u \in N^{1,p}(X)$ and consider its minimal p-weak upper gradient g_u. Fix $x_0, y_0 \in X$ Lebesgue points for u with $u(x_0) \ne u(y_0)$. Recall that $\mu-$a.e. point is a Lebesgue point of every locally integrable function in X. Consider now the function $v = \dfrac{|u - u(x_0)|}{|u(y_0) - u(x_0)|}$ and observe that $v(x_0) = 0$, $v(y_0) = 1$. The function $g_v = \dfrac{g_u}{|u(y_0) - u(x_0)|}$ is the minimal p-weak upper gradient of v, and so for p-almost every rectifiable curve connecting x_0 and y_0 we have

$$1 = |v(x_0) - v(y_0)| \le \int_\gamma g_v\, ds.$$

Note that the curves in $\Gamma(\{x_0\}, \{y_0\}, C)$ stay inside $B(x_0, 2C\, d(x_0, y_0))$. In particular we obtain that $g_v\chi_{B(x_0,2Cd(x_0,y_0))}$ is an admissible function for computing the p-modulus of curves in $\Gamma(\{x_0\}, \{y_0\}, C)$ that satisfy the above inequality; note that the remaining curves in $\Gamma(\{x_0\}, \{y_0\}, C)$ form a family of p-modulus zero. Hence, by hypothesis we have

$$\int_{B(x_0,2C\, d(x_0,y_0))} \frac{g_u^p}{|u(y_0) - u(x_0)|^p} d\mu \ge \mathrm{Mod}_p(\Gamma(\{x_0\}, \{y_0\}, C)) \ge \frac{1}{C\, d(x_0, y_0)^{p-Q}},$$

and so

$$|u(y_0) - u(x_0)|^p \le Cd(x_0, y_0)^{p-Q} \|g_u\|^p_{L^p(B(x_0,2C\, d(x_0,y_0)))}.$$

Thus u is $(1 - \frac{Q}{p})$-Hölder continuous on its Lebesgue set, which is dense in X. Thus u admits a unique Hölder continuous extension to the whole space X. This extension defines the same element in $N^{1,p}(X)$ (notice that by Remark 5.2, points have positive capacity), and verifies the required inequality of Condition (2). □

We next focus on the case $Q - 1 < p \le Q$.

Recall first the definition of the *restricted maximal function* defined by

$$M_R(u)(x) = \sup_{0<r<R} \fint_{B(x,r)} |u| d\mu.$$

We also recall the notions of Hausdorff content and Hausdorff measure. Given $s > 0$, $0 < R \le \infty$, and $E \subset X$, the *s-dimensional Hausdorff R-content* (simply called the s-dimensional Hausdorff content when $R = \infty$) is the number

$$\mathcal{H}^s_R(E) = \inf\left\{ \sum_j \operatorname{diam}(B_j)^s \,:\, E \subset \bigcup_j B_j, \operatorname{diam}(B_j) < R \right\}.$$

The *s-dimensional Hausdorff measure* of E is the number

$$\mathcal{H}^s(E) = \lim_{R\to 0^+} \mathcal{H}^s_R(E).$$

Theorem 5.3 *Let X be a complete, Ahlfors $Q-$regular metric measure space. Then X supports a p-Poincaré inequality for some $Q - 1 < p \le Q$ if and only if X is path-connected and there exists a constant $C \ge 1$ such that for every two disjoint continua E, F in a ball B_R of radius $R > 0$, we have*

$$\operatorname{Mod}_p(\Gamma(E, F, C\,B_R)) \ge \frac{1}{CR^{1-Q+p}} \min\{\operatorname{diam}(E), \operatorname{diam}(F)\}, \tag{3}$$

where $\operatorname{Mod}_p(\Gamma(E, F, C\,B_R))$ denotes the modulus of rectifiable curves connecting E to F inside CB_R.

Recall that CB_R is the ball that is concentric with B_R but with radius CR.
Proof. We first prove that if either X supports a p-Poincaré inequality, or X is path-connected and satisfies the estimate (3), then X is quasiconvex. We already know that if X supports a p-Poincaré inequality, then X is quasiconvex because X is complete, see Theorem 3.1(e). So it suffices to show quasiconvexity under the hypothesis that X satisfies (3). Indeed, since X is path-connected, whenever $x \in X$ and $0 < r < \operatorname{diam}(X)/2$, the connected component $K(x, r)$ of $B(x, r)$ containing x must satisfy $\operatorname{diam}(K(x, r)) \ge r/2$. Fix $x, y \in X$ and set $E = K(x, d(x, y)/10)$, $F = K(y, d(x, y)/10)$. Then by (3), with $R = 10d(x, y)$,

$$\operatorname{Mod}_p(\Gamma(E, F, CB_R)) \ge \frac{d(x, y)^{Q-p}}{C}.$$

For $m > 0$ we let Γ_m denote the collection of curves in $\Gamma(E, F, CB_R)$ with length at most $mR = 10md(x, y)$. By the Ahlfors Q-regularity it follows that μ is doubling, and from the fact that $(mR)^{-1}\chi_{CB_R}$ is admissible for computing the modulus of $\Gamma(E, F, CB_R) \setminus \Gamma_m$, we see that

$$\operatorname{Mod}_p(\Gamma(E, F, CB_R) \setminus \Gamma_m) \le \frac{C_\mu\, d(x, y)^{Q-p}}{m^p}.$$

Thus if $m^p = 2CC_\mu + 1$, we have

$$\operatorname{Mod}_p(\Gamma_m) \ge \frac{d(x, y)^{Q-p}}{2C} > 0,$$

and so we can find a rectifiable curve γ with length at most $10md(x, y)$ connecting a point in $B(x, d(x, y)/10)$ to $B(y, d(x, y)/10)$. Let $\tau_x = \operatorname{dist}(\gamma, x)$. Then $\tau_x \le d(x, y)/10$. We set τ_y similarly. If $\tau_x = \tau_y = 0$ then we are done.

So suppose that $\tau_x > 0$. We now repeat the argument above with $E = \overline{K(x, \tau_x/10)}$ and F a connected component of $\gamma \cap [B(x, C\tau_x) \setminus B(x, \tau_x)]$ that connects $\overline{B}(x, \tau_x)$ to $X \setminus B(x, C\tau_x)$ if γ does not lie in $B(x, C\tau_x)$, and $F = \gamma$ otherwise. Thus we obtain a rectifiable curve γ_1 connecting γ to some point in $B(x, \tau_x/10)$ with length at most $10m\tau_x$, where m satisfies $m^p = 2CC_\mu + 1$ as before. If γ_1 passes through x we are done, for then we can take the concatenation of the two curves γ and γ_1 which would connect a point in $B(y, d(x, y)/10)$ to x (and a symmetric argument for y would then yield the desired quasiconvex curve). If not, then we again repeat the argument with $\tau_{x,2} = \operatorname{dist}(x, \gamma_1) < 10m\, d(x, y)/(10^2)$. By induction we obtain a curve connecting γ to x, with length at most $10m \sum_i \tau_{x,i} \le C_q\, d(x, y)$, as wanted.

If X supports a p-Poincaré inequality for some $Q - 1 < p \le Q$, it follows from [17, Theorem 5.9] that Condition (3) holds (use $s = 1$ and the fact that $\mathcal{H}^s_\infty(E) \ge \operatorname{diam}(E)$ when E is a continuum); see also the argument in [1]. Strictly speaking, Theorem 5.9 of [17] deals only with continuous functions u that satisfy $u \ge 1$ on E and $u \le 0$ on F, and obtains the lower bound estimate (3) for weak upper gradients of u, and hence [17, Theorem 5.9] shows that the continuous relative p-capacity of the condenser (E, F, CB_R) is bounded below by the estimate (3). However, the results of [19] show that when X is complete and the measure μ is doubling and supports a p-Poincaré inequality (as we have in our setting), the continuous relative p-capacity of the condenser is equal to the p-modulus of the family $\Gamma(E, F, CB_R)$. In the setting of (3), we know that g is admissible for computing the p-modulus of the collection of all curves in $\Gamma(E, F, CB_R)$ for which the upper gradient inequality between u and its upper gradient g holds. The collection of remaining curves from $\Gamma(E, F, CB_R)$ forms a zero p-modulus collection, and so we can use g to compute the p-modulus of $\Gamma(E, F, CB_R)$ itself.

For the converse, we model our proof along that of Lemma 5.17 of [17]. We suppose that Condition (3) holds. We fix a ball B in X, and let $x, y \in C_q^{-1}B$, where C_q is the quasiconvexity constant of X. Let u be a continuous function on X with upper gradient ρ, and let γ be a C_q-quasiconvex curve in X connecting x to y. Then $\gamma \subset 2B$. It will be sufficient to consider the case where $u(x) \ne u(y)$ so, by rescaling u if needed, we can assume that $|u(x) - u(y)| = 1$. Let $M > \max\{2, C_q\}$ be a large constant, and for each $j \in \mathbb{N}$, we set

$$A_j = B(x, M^{-3j}\tau) \setminus B(x, M^{-3j-2}\tau),$$

where $\tau > 0$ is chosen so that the sphere centered at x with radius τ intersects γ at its midpoint. Here, by midpoint we mean the point ζ on γ for which the length of the subcurve of γ with end points x and ζ equals the length of the subcurve of γ with end points ζ and y. It is easy to see that $0 < \tau \le C_q d(x, y)$. The goal is to show that we have a Hajłasz type inequality (4) for u, see below. For each j let γ_j denote a subcurve of γ lying in A_j and connecting the two spheres, centered at x, of radii $M^{-3j}\tau$ and $M^{-3j-2}\tau$. Let Γ_j denote the collection of curves in B connecting γ_j to γ_{j+1} in $B(x, M^{-3j+2}\tau)$. Now using (3) we obtain that there is a constant $C' > 0$ such that, for all j, we have

$$\operatorname{Mod}_p(\Gamma_j) \ge C' \frac{\tau^{Q-p}}{M^{3j(Q-p)}}.$$

Let

$$a_j = \inf_{\beta \in \Gamma_j} \int_\beta \rho\, ds.$$

Case 1: For μ-almost every $x, y \in C_q^{-1}B$ there is a choice of j for which

$$a_j^p\, M^{3jp} \ge 1$$

Since ρ/a_j is then admissible for computing $\mathrm{Mod}_p(\Gamma_j)$, we have that

$$M_{2\tau}(\rho^p)(x) \geq \fint_{B(x,M^{-3j+2}\tau)} \rho^p \, d\mu \geq \frac{1}{\mu(B(x, M^{-3j+2}\tau))} \, a_j^p \, \mathrm{Mod}_p(\Gamma_j) \geq c \, a_j^p \, \frac{M^{3jp}}{\tau^p} \geq \frac{c}{\tau^p},$$

for some constant c that depends only on the data of X and the choice of M. In this case, we have that (by recalling our choice of τ such that $0 < \tau \leq C_q d(x, y)$),

$$c^{1/p} \, |u(x) - u(y)| = c^{1/p} \leq (M_{2\tau}(\rho^p)(x))^{1/p} \, C_q d(x, y),$$

from which we obtain the Hajłasz type inequality

$$|u(x) - u(y)| \leq \frac{C_q}{c^{1/p}} \, d(x, y) \, \left[(M_{2\tau}(\rho^p)(x))^{1/p} + (M_{2\tau}(\rho^p)(y))^{1/p}\right]. \tag{4}$$

Note that if we have the above inequality for μ-almost all $x, y \in C_q^{-1}B$, then we obtain a p-Poincaré inequality for the function u and its upper gradient ρ, see [17, Lemma 5.15] or [16].

Case 2: There is a set of positive measure in $C_q^{-1}B$ for which no such choice of j exists. Fix x, y from that set. Then for each j we have that

$$a_j < \frac{1}{M^{3j}}$$

Then there is a curve $\beta_j \in \Gamma_j$, connecting γ_j to γ_{j+1}, so that

$$\int_{\beta_j} \rho \, ds < \frac{1}{M^{3j}}.$$

If β_j and β_{j+1} intersect for each j, then we can concatenate them to obtain (using a similar argument with x replaced with y) a rectifiable curve β connecting x to y such that

$$|u(x) - u(y)| = 1 \leq \int_\beta \rho \, ds \leq \sum_{j=1}^{\infty} M^{-3j} = \frac{1}{M^3 - 1},$$

which is not possible since $(M^3 - 1) \geq 1$. So there is some j for which β_j and β_{j+1} do not intersect. For such j we let Λ_j be the collection of all rectifiable curves in $B(x, M^{-3j+2}\tau)$ connecting β_j to β_{j+1}. We now set

$$b_j = \inf_{\alpha \in \Lambda_j} \int_\alpha \rho \, ds.$$

As in Case 1, if we have that for some choice of j,

$$b_j^p \, M^{3jp} \geq 1$$

then because ρ/b_j is admissible for computing $\mathrm{Mod}_p(\Gamma_j)$, we have the Hajłasz-type inequality (4) for u. So we assume that for each j,

$$b_j < \frac{1}{M^{3j}},$$

and hence we can choose a curve α_j connecting β_j and β_{j+1} so that

$$\int_{\alpha_j} \rho \, ds < \frac{1}{M^{3j}}.$$

Now we can concatenate β_j, α_j, and β_{j+1} for all such j, and concatenate β_j and β_{j+1} when they do intersect, to obtain a curve connecting x to y on which the path integral of ρ is smaller than 1, violating the upper gradient property of ρ again.

Therefore, in Case 2 there is some choice of j for which we have

$$b_j \geq \frac{1}{M^{3j}},$$

and so we have (4).

Combining Cases 1 and 2, we see that for each $x, y \in C_q^{-1}B$,

$$|u(x) - u(y)| \leq C\, d(x, y) \left[(M_{2\tau}(\rho^p)(x))^{1/p} + (M_{2\tau}(\rho^p)(y))^{1/p} \right].$$

It follows that X supports a p-Poincaré inequality for u and its upper gradient ρ.

The cases 1 and 2 together demonstrate that X supports a p-Poincaré inequality for all continuous functions and their upper gradients. It now follows by the results of [21] that X supports a p-Poincaré inequality for all function-upper gradient pairs (indeed, the results of [21] state that to verify p-Poincaré inequality for all functions and their upper gradients, it suffices to verify the p-Poincaré inequality for Lipschitz continuous functions and their upper gradients). $\square$

Remark 5.4 The case $p = Q$ corresponds to Loewner spaces of Heinonen–Koskela [17]. Recall that an Ahlfors Q-regular space is said to be a *Q-Loewner space* if there is a nonincreasing function $\psi : (0, \infty) \to (0, \infty)$ such that whenever $E, F \subset B \subset X$ (with B a ball in X) are disjoint nondegenerate continua, then

$$\mathrm{Mod}_Q(\Gamma(E, F, CB)) \geq \psi(\mathrm{dist}(E, F)/ \min\{\mathrm{diam}(E), \mathrm{diam}(F)\}).$$

We point out here that the Q-Loewner property characterization of Q-Poincaré inequality for Ahlfors Q-regular spaces is stronger than ours since we require a specific type of function ψ, namely $\psi(t) = 1/t$. For a related characterization of Q-Poincaré inequality, see [5].

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