CHARACTERIZATION OF ELLIPSOIDS AS K-DENSE SETS

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Abstract. Let $K \subset \mathbb{R}^N$ be any convex body containing the origin. A measurable set $G \subset \mathbb{R}^N$ with finite and positive Lebesgue measure is said to be K-dense if, for any fixed $r > 0$, the measure of $G \cap (x + rK)$ is constant when x varies on the boundary of G (here, $x + rK$ denotes a translation of a dilation of K). In [6], we proved for the case in which $N = 2$ that if G is K-dense, then both G and K must be homothetic to the same ellipse. Here, we completely characterize K-dense sets in $\mathbb{R}^N$: if G is K-dense, then both G and K must be homothetic to the same ellipsoid. Our proof, by building upon results obtained in [6], relies on an asymptotic formula for the measure of $G \cap (x + rK)$ for large values of the parameter r and a classical characterization of ellipsoids due to C.M. Petty [8].

1. Introduction

Let K be a convex body containing the origin of $\mathbb{R}^N$ and G be a measurable subset of $\mathbb{R}^N$ with finite and positive Lebesgue measure $V(G)$. We say that G is *K-dense* if there exists a function $c : (0, \infty) \to (0, \infty)$ such that

$$V(G \cap (x + r\,K)) = c(r) \quad \text{for} \quad x \in \partial G, \ r > 0. \tag{1.1}$$

Here, ∂G is the topological boundary of G and $x+r\,K$ denotes the translation by a vector x of a dilation of K by a factor $r > 0$.

Plane K-dense sets have been characterized in [1] and [6]. They cannot exist unless they are homothetic to K itself and, if this is the case, they must be ellipses (together with K). In this paper, we shall extend that characterization to general dimension by proving the following result.

Theorem 1.1. *Let $K \subset \mathbb{R}^N$ be a convex body and assume that there is a set $G \subset \mathbb{R}^N$ of finite positive measure such that* (1.1) *holds.*

Then, both K and G must be homothetic to the same ellipsoid.

The case $N = 2$ was first settled in [1] under some smoothness assumptions (∂K of class C^2 and ∂G of class C^4). It should also be noticed that the proof in [1] works even if condition (1.1) holds when r ranges in a sufficiently small interval $(0, r_0)$, since it only uses local information on ∂G.

In [6], we were able to remove such regularity assumptions. In fact, we showed that in the plane the occurrence of property (1.1) implies that both ∂G and ∂K are necessarily of class C^∞. Moreover, we gave an alternative proof of the characterization which is based on some local information on ∂G derived from (1.1) and classical affine inequalities for convex bodies.

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In [6], we also established some facts that hold in general dimension and will be useful in the remainder of this paper: let $K \subset \mathbb{R}^N$ be a convex body and assume that property (1.1) holds, then

(i) G is strictly convex;
(ii) ∂G is at least of class $C^{1,1}$;
(iii) if K is centrally symmetric (i.e. $-K = K$), then $K = G - G$ up to dilations, K is strictly convex and ∂K is at least of class $C^{1,1}$;
(iv) if ∂G is differentiable at x, then
$$V(G \cap (x + r\,K)) = V_0(x)\,r^N + o(r^N) \quad \text{as} \quad r \to 0^+;$$
(v) if ∂G is of class C^2 in a neighborhood of x, then
$$V(G \cap (x + r\,K)) = V_0(x)\,r^N + V_1(x)\,r^{N+1} + o(r^{N+1}) \quad \text{as} \quad r \to 0^+.$$

The coefficients $V_0(x)$ and $V_1(x)$ are explicitly computed; $G - G$ denotes the *Minkowski sum* of G and $-G$: $G - G = G + (-G) = \{x - y : x, y \in G\}$.

It will be useful to understand the mechanism of our proof in [6]. Since (1.1) holds, (ii) and (iv) imply that the function V_0 is constant on ∂G. By the explicit expression of $V_0(x)$ then one gets that

$$V(\{y \in K : y \cdot \nu(x) \ge 0\}) = \frac{1}{2}\,V(K) \quad \text{for every} \quad x \in \partial G, \tag{1.2}$$

where $\nu(x)$ denotes the exterior unit normal to ∂G at x. When $N = 2$, thanks to (i), it is not difficult to show that (1.2) implies that K is centrally symmetric — indeed, that is also true for $N \ge 3$, by a non-trivial result of Schneider [9]. Thus, (iii) comes into play and we can infer further regularity ($C^{2,1}$) for ∂G. Hence, (v) can be used: also the function V_1 must be constant on ∂G. This condition gives a pointwise constraint on the curvature of ∂G (see [6, (1.8)]) that — for $N = 2$ — ensures that $K = 2G$ up to homotheties and, with the help of Minkowski's inequality for mixed volumes and an inequality involving the *affine surface area* of ∂G, gives the desired conclusion.

Now, let us look at the case in which $N \ge 3$. Of course, (i) and (ii) still hold, if G is K-dense. Thus, the formula in (iv) still makes sense and hence, by the aforementioned result [9], K is centrally symmetric; consequently, (iii) holds, too. Therefore, also (v) makes sense and, even now, we can deduce that V_1 must be constant on ∂G. Unfortunately, the pointwise constraint on the principal curvatures [6, (1.8)] is no longer enough to deduce that $K = 2G$ and to conclude.

In this paper, we succeed in our purpose by changing stategy: we give up the asymptotic expansion for $r \to 0^+$ in (v) in favour of an expansion like

$$V(G \cap (x + r\,K)) = V(G) + W(x)\,(r_G - r)^{\frac{N+1}{2}} + o\left((r_G - r)^{\frac{N+1}{2}}\right) \tag{1.3}$$

as $r \to r_G^-$, where

$$r_G = \inf\{r > 0 : G \subseteq x + r\,K\}, \ x \in \partial G.$$

Notice that, if G is K-dense, then r_G is independent on $x \in \partial G$; since our problem is invariant with respect to dilations of K, throughout the paper, we shall assume that $r_G = 1$.

The computation of the coefficient $W(x)$ is carried out in Section 2 and involves the *support function* $h_K : \mathbb{S}^{N-1} \to \mathbb{R}$ of the convex body K with respect to the origin and the *shape operators* S_G and S_K of G and K, respectively. In fact, it turns out that for $x \in \partial G$

$$W(x) = -\frac{2\omega_{N-1}\, h_K(u)^{\frac{N+1}{2}}}{(N^2-1)\,\det[S_G(u) - S_K(u)]^{\frac{1}{2}}} \quad \text{with} \quad u = \nu(\overline{x}); \tag{1.4}$$

here, $\{\overline{x}\} = \partial G \cap (x + K)$,[1] $\nu(\overline{x})$ is the exterior unit normal to ∂G at $\overline{x}$, and ω_{N-1} denotes the surface measure of the unit sphere $\mathbb{S}^{N-2}$ of $\mathbb{R}^{N-1}$.

Properties (i) and (1.1) imply that the right-hand side of (1.4) must be constant as a function of $u \in \mathbb{S}^{N-1}$. A first consequence of this fact is that $K = 2G$ up to homotheties; a second consequence is that

$$\kappa_G(u) = c\, h_G(u)^{N+1} \quad \text{for every} \quad u \in \mathbb{S}^{N-1}, \tag{1.5}$$

for some positive constant c; here, κ_G denotes the *Gauss curvature* of ∂G at the (unique) point $x \in \partial G$ having normal equal to u.

The identity (1.5) is well-known in the theory of convex bodies: in fact, C.M. Petty proved in [8] that it characterizes G as an ellipsoid.

Section 2 contains all the details.

2. The proof of Theorem 1.1

Let $G \subset \mathbb{R}^N$ be a C^2 convex body. In a sufficiently small neighborhood of a point $x \in \partial G$, the set ∂G is the graph of a C^2-regular convex function over the tangent space to ∂G at x; we denote by S_G the Hessian of this function (the bilinear form associated is often called *shape operator*); it is well-known that its determinant κ_G is the *Gaussian curvature* of ∂G at that point. When G is strictly convex, without any ambiguity we can think of S_G as a function over the unit sphere, so that, for a given $u \in \mathbb{S}^{N-1}$, $S_G(u)$ denotes the shape operator at the only point $x \in \partial G$ with outward unit normal equal to u.

We know from [6] that, if G is K-dense, then ∂G is of class C^2 but, unfortunately, we can not assert that ∂K is of class C^2, even if we know that $K = G - G$ (see [2], for instance). Nevertheless, in [4] it is shown that, if G is strongly convex[2], then K has the same regularity as G; in particular the following result holds.

Theorem 2.1 (S. Krantz, H. Parks)**.** *If A is a strongly convex body with boundary of class C^∞ and B is a convex body with boundary of class C^2, then the Minkowski sum $A + B$ has boundary of class C^∞.*

Moreover, the shape operator of $A + B$ can be expressed by the following formula:

$$S_{A+B}(u) = \left[I + S_A(u)^{-1} S_B(u)\right]^{-1} S_B(u). \tag{2.1}$$

The proof of this theorem can be repeated step by step also in the case in which the C^∞-regularity of A is replaced by its C^2-regularity: one then gets that the boundary of $A + B$ is of class C^2 and that (2.1) holds, as well.

[1] It will be made clear in Section 2 that $\overline{x}$ is uniquely determined.

[2] That is $S_K(u) > 0$ for every $u \in \mathbb{S}^{N-1}$ — with which we mean that $S_K(u)$ is positive definite for every $u \in \mathbb{S}^{N-1}$.

Thus, our aim is now to show that K-dense bodies are strongly convex; then, by Theorem 2.1, we will gain the necessary regularity of K that gives a meaning to (2.1) with $A = G$ and $B = -G$.

In order to do this, for $x \in \partial G$ we shall study the asymptotic behavior of $V(G \setminus (x + rK))$ as $r \to 1^-$. As we shall see, if we want to express $V(G \setminus (x+rK))$ in terms of the shape operator of ∂G at some point $\overline{x} \in \partial G$, it is important to make sure that G shares with the boundary of $x + K$ only one point. We observe that this is not always the case: indeed, consider the Releaux triangle as the set G and let x denote one of its vertices; then, $K = G - G$ is a ball and $G \cap (x + K)$ is one of the arcs constituting the triangle's boundary; hence, so to speak, $G \setminus (x + rK)$ can not be localized around any point of ∂G.

Notice that such a G is strictly convex, but ∂G is not differentiable. Likewise, if we consider differentiable bodies which are not strictly convex, we can still provide an example of the same phenomenon: in fact, it is enough to set $G = B + Q$, where B is the unit ball and Q is the unit square.

The following lemma shows that we can get the desired result, if we assume that G is both differentiable and strictly convex.

Lemma 2.2. *Let G be a strictly convex body with differentiable boundary and set $K = G - G$, then for each $x \in \partial G$ the set $\partial(x + K) \cap G$ consists of only one point $\overline{x} \in \partial G$ characterized by $\nu_K(\overline{x} - x) = -\nu_G(x)$.*

Proof. Let $z \in \partial K \cap (G - x)$ and let $u = \nu_K(z)$. Clearly $z + x \in \partial G$ and, since the $G - x$ is contained in K and touches K at z from inside, then $\nu_G(z + x) = u$. Since $K = G - G$, we have

$$\begin{aligned} h_G(u) + h_G(-u) &= h_K(u) = \langle z, u\rangle = \langle z + x, u\rangle + \langle x, -u\rangle \\ &= h_G(u) + \langle x, -u\rangle. \end{aligned}$$

Thus, $h_G(-u) = \langle x, -u\rangle$, that is $\nu_G(x) = -u$. It is then enough to set $\overline{x} = z + x$.

Now, suppose that there exists another point z' such that $z' \in \partial K \cap (G-x)$ and set $u' = \nu_K(z')$; by the same argument, we get that $\nu_G(x) = -u'$, and hence $u = u'$. Since K is strictly convex (being G so), we finally find $z = z'$. □

The following lemma is helpful to prove that a K-dense set is positively curved.

Lemma 2.3. *Let G be a strictly convex body with boundary of class C^2 and let $K = G - G$. For $x \in \partial G$ and $\overline{x} \in \partial G$ such that $u = \nu_G(x) = -\nu_G(\overline{x})$, It holds:*

(i) *if $\kappa_G(u) = 0$, then*

$$\liminf_{r\to 1^-} \frac{V(G \setminus (x + rK))}{(1 - r)^{\frac{N+1}{2}}} = +\infty;$$

(ii) *if $\kappa_G(u) > 0$, then*

$$\limsup_{r\to 1^-} \frac{V(G \setminus (x + rK))}{(1 - r)^{\frac{N+1}{2}}} \le \frac{2\omega_{N-1}}{N^2 - 1}\, \kappa_G(u)\, h_K(u)^{\frac{N+1}{2}} (1 + \Lambda)^{\frac{N-1}{2}},$$

where Λ is the maximal principal curvature of ∂G at x.

Proof. First, notice that, by the above lemma, our choice of x and $\overline{x}$ ensures that $\{\overline{x}\} = \partial(x+K) \cap G$. Without loss of generality, we can always assume that $\overline{x} = 0$ and that $u = (0, 0, \dots, -1)$; then, in a neighborhood of $\overline{x}$, ∂G can be parametrized by

$$y_N = \langle S_G(u)\, y, y\rangle + o(|y|^2) \quad \text{as} \quad |y| \to 0, \tag{2.2}$$

where $y = (y_1, \dots, y_{N-1})$ ranges in the tangent space to ∂G at $\overline{x}$.

(i) Set $\varepsilon = 1 - r$. Let ε_n be an infinitesimal sequence of positive numbers on which the limit in (i) is attained and, to simplify notations, set $G_n = G \setminus (x + (1-\varepsilon_n)K)$; (2.2) suggests that, by possibly extracting a subsequence from ε_n, we can fit in G_n the set E_n bounded by the paraboloid

$$y_N = \langle S_G(u)\, y, y\rangle + \frac{1}{n}\, |y|^2$$

and the hyperplane $\overline{x} + \varepsilon_n\, h_K(u)\, u + u^\perp$ supporting the set $x + (1-\varepsilon_n)K$ at the point whose outer unit normal coincides with u. In our coordinates,

$$E_n = \{(y, y_N) : \langle S_G(u)\, y, y\rangle + \frac{1}{n}\, |y|^2 < y_N < \varepsilon_n\, h_K(u)\}$$

and $E_n \subseteq G_n$.

Thus, by Fubini's theorem and some calculations, we get:

$$V(G_n) \geq V(E_n) = \int_0^{\varepsilon_n\, h_K(u)} \mathcal{H}^{N-1}\left(\{y : \langle [S_G(u) + 1/n\, I]\, y, y\rangle \leq t\}\right)\, dt$$

$$= \frac{\omega_{N-1}}{(N-1)\det\left[S_G(u) + 1/n\, I\right]^{\frac{1}{2}}} \int_0^{\varepsilon_n\, h_K(u)} t^{\frac{N-1}{2}}\, dt = \frac{2\omega_{N-1}\, \varepsilon_n^{\frac{N+1}{2}}\, h_K(u)^{\frac{N+1}{2}}}{(N^2-1)\det\left[S_G(u) + 1/n\, I\right]^{\frac{1}{2}}}.$$

Therefore,

$$\liminf_{\varepsilon \to 0^+} \frac{V(G \setminus (x + (1-\varepsilon)K)}{\varepsilon^{\frac{N+1}{2}}} = \lim_{n\to\infty} \varepsilon_n^{-\frac{N+1}{2}}\, V(G_n)$$
$$\geq \lim_{n\to\infty} \frac{2\,\omega_{N-1}\, h_K(u)^{\frac{N+1}{2}}}{(N+1)\sqrt{\det\left(S_G(u) + 1/n\, I\right)}} = +\infty,$$

since $\det S_G(u) = \kappa_G(u) = 0$.

(ii) We shall obtain the desired inequality by observing that the domain $G \setminus (x + (1-\varepsilon)K)$ can be contained in the region $F_{\varepsilon,\delta}$ bounded by two paraboloids: one outside G and tangent to ∂G at $\overline{x}$, the other one tangent to the boundary of $x + (1-\varepsilon)K$ from inside. In order to show it, we assume as before that $\overline{x} = 0$ and $u = -e_N$ and, moreover, that $S_G(u) = I$ (this can be done since the affine tranformation $S_G(u)$ is invertible, being $\det S_G(u) = \kappa_G(u) > 0$): the desired formula will then be obtained by multiplying the right-hand side of (2.3) by the factor $\kappa_G(u)$.

We proceed to contruct $F_{\varepsilon,\delta}$. We choose any number $\lambda > 0$ such that $\lambda\, I > S_G(-u)$, that is such that $\lambda > \Lambda$. Since $\kappa_G(u) > 0$, Theorem 2.1 imply that ∂K is twice differentiable at u; moreover equation (2.1) turns into

$$S_K(u) < \frac{\lambda}{1+\lambda}\, I;$$

hence,

$$S_{(1-\varepsilon)K}(u) < \frac{\lambda}{(1+\lambda)(1-\varepsilon)}\, I.$$

For $\varepsilon > 0$ sufficiently small, we define $F_{\varepsilon,\delta}$ as

$$F_{\varepsilon,\delta} = \left\{ (y, y_N) : \delta\, |y|^2 \le y_N \le \varepsilon\, h_K(u) + \frac{\lambda}{(1+\lambda)(1-\varepsilon)}\, |y - \varepsilon x_*|^2 \right\},$$

where δ is chosen in the interval $(\frac{\lambda}{(1+\lambda)(1-\varepsilon)}, 1)$ and x_* is the projection of x on the tangent space to ∂G at $\overline{x}$; in this way,

$$G \setminus (x + (1-\varepsilon)K) \subset F_{\varepsilon,\delta}.$$

Indeed, equation (2.2) guarantees that the above inclusion holds, at least inside a small neighborhood of $\overline{x}$; however, by Lemma 2.2, we know that $G \setminus (x + (1-\varepsilon)K)$ is contained in a ball B_r around z whose radius $r = r(\varepsilon)$ tends to 0 as $\varepsilon \to 0$.

By using the rescaling $(y, y_N) = (\sqrt{\varepsilon}\, \xi, \varepsilon\, \xi_N)$, we obtain that $V(F_{\varepsilon,\delta}) = \varepsilon^{\frac{N+1}{2}}\, V(F'_{\varepsilon,\delta})$, where

$$F'_{\varepsilon,\delta} = \left\{ (\xi, \xi_N) : \delta\, |\xi|^2 \le \xi_N \le h_K(u) + \frac{\lambda}{(1+\lambda)(1-\varepsilon)}\, |\xi - \sqrt{\varepsilon}\, x_*|^2 \right\},$$

and it is easy to show that $V(F'_{\varepsilon,\delta}) \to V(F'_{0,\delta})$. By a straightforward computation of $V(F'_{0,\delta})$, we get that

$$\limsup_{\varepsilon \to 0} \frac{V(G \setminus (x + (1-\varepsilon)K))}{\varepsilon^{\frac{N+1}{2}}} \le \frac{2\omega_{N-1}}{N^2 - 1}\, \frac{h_K(u)^{\frac{N+1}{2}}}{(\delta - \frac{\lambda}{1+\lambda})^{\frac{N-1}{2}}},$$

and minimizing the right-hand side of this formula for $\lambda/(1+\lambda) < \delta < 1$ and $\lambda > \Lambda$ then gives:

$$\limsup_{\varepsilon \to 0} \frac{V(G \setminus (x + (1-\varepsilon)K))}{\varepsilon^{\frac{N+1}{2}}} \le \frac{2\omega_{N-1}}{N^2 - 1}\, h_K(u)^{\frac{N+1}{2}} (1+\Lambda)^{\frac{N-1}{2}}. \tag{2.3}$$

□

Corollary 2.4. *If G is K-dense, then ∂K is of class C^2 and every point of ∂K is a point of strong convexity. Moreover,*

$$S_K(u) = \left[I + S_G(u)^{-1} S_G(-u) \right]^{-1} S_G(-u). \tag{2.4}$$

Proof. Since G is K-dense, then the limits in items (i) and (ii) in Lemma 2.3 do not depend on the particular point $x \in \partial G$; in other words, they must be constant functions on ∂G. Since G is a convex body and ∂G is of class C^2, then κ_G is not identically zero; hence, the limit in item (ii) of Lemma 2.3 is a finite constant. As a consequence, item (i) of the same lemma implies that $\kappa_G > 0$ (and hence $S_G > 0$) on ∂G. Formula (2.4) is then a straightforward consequence of Theorem 2.1. □

Theorem 2.5. *Let G be a strongly convex body with boundary of class C^2 and set $K = G - G$. Chose x, u and $\overline{x}$ as in Lemma 2.3; then*

$$\lim_{r \to 1^-} \frac{V(G \setminus (x + rK))}{(1-r)^{\frac{N+1}{2}}} = \frac{2\omega_{N-1}\, h_K(u)^{\frac{N+1}{2}}}{(N^2 - 1) \det[S_G(u) - S_K(u)]^{\frac{1}{2}}}.$$

Proof. Again we set $\varepsilon = 1 - r$. We begin by showing that

$$\limsup_{\varepsilon\to 0^+} \frac{V(G\setminus(x+(1-\varepsilon)K))}{\varepsilon^{\frac{N+1}{2}}} \le \frac{2\omega_{N-1}\, h_K(u)^{\frac{N+1}{2}}}{(N^2-1)\det[S_G(u)-S_K(u)]^{\frac{1}{2}}}. \tag{2.5}$$

As in the proof of Lemma 2.3, without loss of generality, we can set $u = -e_N$ and $\overline{x} = 0$.

It is clear that $\varepsilon x \in \partial(x + (1-\varepsilon)K)$ and that u is the unit normal to $\partial(x+(1-\varepsilon)K)$ at that point; also, by a scaling argument, we know that

$$S_{x+(1-\varepsilon)K}(u) = \frac{S_K(u)}{1-\varepsilon}.$$

Notice that formula (2.4) implies that $S_G(u) > S_K(u)$; hence, we can chose $\overline{n}\in\mathbb{N}$ such that

$$\frac{S_G(u)-S_K(u)}{4} > \frac{I}{\overline{n}}. \tag{2.6}$$

In order to get an estimate from above for $V(G\setminus(x+(1-\varepsilon)K))$ we construct a set $C_{\varepsilon,n}$ containing $G\setminus(x+(1-\varepsilon)K)$. In fact, for $n > \overline{n}$ we set

$$C_{\varepsilon,n} = \big\{(y,y_N) : \langle \big(S_G(u)-n^{-1}I\big)\, y, y\rangle < y_N < \\ \varepsilon h_K(u) + \langle\big[(1-\varepsilon)^{-1}S_K(u)+n^{-1}I\big]\,(y-x_*),(y-x_*)\rangle\big\},$$

where x_* denotes the projection of x on $u^\perp$; $C_{\varepsilon,n}$ is the region bounded by two paraboloids, one touching ∂G at $\overline{x}$ from below, the other one touching $\partial(x+(1-\varepsilon)K)$ at εx from above and, for ε small enough, we have:

$$G\setminus(x+(1-\varepsilon)K)\subset C_{\varepsilon,n}.$$

Also, condition (2.6) guarantees that

$$S_G(u) - \frac{I}{n} > \frac{S_K(u)}{1-\varepsilon} + \frac{I}{n} > 0,$$

thus forcing $C_{\varepsilon,n}$ to be bounded.

The usual change of variables $(y, y_N) = (\sqrt{\varepsilon}\,\xi, \varepsilon\,\xi_N)$ gives that $V(C_{\varepsilon,n}) = \varepsilon^{\frac{N+1}{2}}\, V(C'_{\varepsilon,n})$, where

$$C'_{\varepsilon,n} = \big\{(\xi,\xi_N) : \langle\big[S_G(u)-n^{-1}I\big]\,\xi,\xi\rangle < \xi_N < h_K(u) + \\ \langle\big[(1-\varepsilon)^{-1}S_K(u)+n^{-1}I\big]\,(\xi-\sqrt{\varepsilon}x_*),(\xi-\sqrt{\varepsilon}x_*)\rangle\big\},$$

Since clearly $V(C'_{\varepsilon,n})\to V(C'_{0,n})$ as $\varepsilon\to 0$, a straightforward computation gives:

$$\limsup_{\varepsilon\to 0^+}\frac{V(G\setminus(\overline{z}+(1-\varepsilon)K))}{\varepsilon^{\frac{N+1}{2}}} \le V(C'_{0,n}) \tag{2.7}$$

$$= \frac{2\omega_{N-1}\, h_K(u)^{\frac{N+1}{2}}}{(N^2-1)\det\big[S_G(u)-S_K(u)-2/n\, I\big]^{\frac{1}{2}}}.$$

Since the right-hand side in (2.7) is independent on n, (2.5) follows at once by taking the limit for $n\to\infty$.

The converse inequality,

$$\liminf_{\varepsilon\to 0^+} \frac{V(G\setminus(\overline{z}+(1-\varepsilon)K))}{\varepsilon^{\frac{N+1}{2}}} \geq \frac{2\omega_{N-1}\, h_K(u)^{\frac{N+1}{2}}}{(N^2-1)\det\left[S_G(u)-S_K(u)-2/n\, I\right]^{\frac{1}{2}}},$$

is proved by using the same strategy used for (2.5): we choose $\overline{n}$ such that

$$S_G(u) > \frac{I}{\overline{n}}$$

and then we construct, for $n>\overline{n}$ and ε small, a set $D_{\varepsilon,n}\subseteq G\setminus(x+(1-\varepsilon)K)$:

$$D_{\varepsilon,n} = \big\{(y,y_N) : \langle\big(S_G(u)+n^{-1}I\big)\, y, y\rangle < y_N < \\ < \varepsilon h_K(u) + \langle\big[(1-\varepsilon)^{-1}S_K(u) - n^{-1}I\big]\,(y-\varepsilon x_*),(y-\varepsilon x_*)\rangle\big\}.$$

As before, the usual rescaling gives

$$\liminf_{\varepsilon\to 0^+} \frac{V(G\setminus(x+(1-\varepsilon)K))}{\varepsilon^{\frac{N+1}{2}}} \geq \frac{2\omega_{N-1}\, h_K(u)^{\frac{N+1}{2}}}{(N^2-1)\det\left[S_G(u)-S_K(u)+2/n\, I\right]^{\frac{1}{2}}}.$$

Again, we conclude by taking the limit for $n\to\infty$. □

Corollary 2.6. *Let G be a K-dense body, then* (1.3) *holds with the coefficient $W(x)$ given by* (1.4)*. In particular, the function defined by*

$$\frac{h_K(u)^{\frac{N+1}{2}}}{\det[S_G(u)-S_K(u)]^{\frac{1}{2}}}, \quad u\in\mathbb{S}^{N-1}, \tag{2.8}$$

is constant on $\mathbb{S}^{N-1}$.

Proof. Corollary 2.4 ensures that a K-dense body satisfies the assumptions of Theorem 2.5. Since $V(G\cap(x+r\,K)) = V(G)-V(G\setminus(x+r\,K))$, clearly

$$W(x) = \lim_{r\to 1^-} \frac{V(G\setminus(x+r\,K))}{(1-r)^{\frac{N+1}{2}}}$$

and $W(x)$ must be constant for $x\in\partial G$. Since G is strictly convex, (2.8) then follows from the suriectivity of the Gauss map. □

Now, we are going to show that if G is K-dense, then G and K must be equal up to homotheties.

Proposition 2.7. *Let G be a K-dense body, then $\kappa_G(u)=\kappa_G(-u)$.*

Proof. Let $u\in\mathbb{S}^{N-1}$ and $L=L_u$ be a linear map of $\mathbb{R}^N$ in itself, which leaves unchanged the unit vector u and whose restriction to $u^\perp$ equals $S_G(u)^{-\frac{1}{2}}$.

First, notice that, as an easy consequence of (1.1), the set LG is LK-dense, so that Corollary 2.6 holds for this set; in particular, (2.8) implies:

$$h_{LK}(-u)^{\frac{N+1}{2}}\{\det[S_{LG}(-u)-S_{LK}(-u)]\}^{-\frac{1}{2}} = \\ h_{LK}(u)^{\frac{N+1}{2}}\{\det[S_{LG}(u)-S_{LK}(u)]\}^{-\frac{1}{2}}. \tag{2.9}$$

Secondly, we know that K is centrally symmetric, and so must be LK; then, $S_{LK}(u)=S_{LK}(-u)$ and $h_{LK}(u)=h_{LK}(-u)$; (2.9) then becomes

$$\det[S_{LG}(-u)-S_{LK}(u)] = \det[S_{LG}(u)-S_{LK}(u)]. \tag{2.10}$$

As we shall see, this condition together with equation (2.1) is enough to prove that

$$\det[S_{LG}(u)] = \det[S_{LG}(-u)].$$

Indeed, by plugging (2.1) into (2.10) we get

$$\det\Big(S_{LG}(-u) - \big[I + S_{LG}(u)^{-1}S_{LG}(-u)\big]^{-1} S_{LG}(-u)\Big) \tag{2.11}$$
$$= \det\Big(S_{LG}(u) - \big[I + S_{LG}(u)^{-1}S_{LG}(-u)\big]^{-1} S_{LG}(-u)\Big);$$

furthermore, our chioice of the affine transformation L ensures that

$$S_{LG(u)} = I,$$

and

$$S_{LG}(-u) = S_G(u)^{-\frac{1}{2}} S_G(-u) S_G(u)^{-\frac{1}{2}}. \tag{2.12}$$

Equation (2.11) then turns into

$$\det\Big(S_{LG}(-u) - [I + S_{LG}(-u)]^{-1} S_{LG}(-u)\Big) \tag{2.13}$$
$$= \det\Big(I - [I + S_{LG}(-u)]^{-1} S_{LG}(-u)\Big);$$

by multiplying both sides of (2.13) by $\det[I + S_{LG}(-u)]$ and using Binet's identity, we get

$$\det[S_{LG}(-u)^2] = 1.$$

Finally, (2.12) gives that $\det[S_G(u)] = \det[S_G(-u)]$, that is $\kappa_G(u) = \kappa_G(-u)$. □

Corollary 2.8. *Let G be K-dense, then G is symmetric and $K = 2G$.*

Proof. The two bodies $G - G$ and $2G$ have the same Gaussian curvature as a function on $\mathbb{S}^{N-1}$; thus, they only differ by a translation. □

The following theorem and Petty's characterization of ellipsoids [8] complete the proof of Theorem 1.1.

Theorem 2.9. *Let G be a K-dense set. Then, for every $x \in \partial G$ it holds that*

$$\lim_{r\to 1^-} \frac{V(G \setminus (x + rK))}{(1-r)^{\frac{N+1}{2}}} = \frac{2\sqrt{2}\,\omega_{N-1}\, h_K(u)^{\frac{N+1}{2}}}{(N+1)\,\det[S_G(u)]^{\frac{1}{2}}} \quad \textit{with} \quad u = \nu(\overline{x})$$

and $\{\overline{x}\} = \partial G \cap (x + K)$.

In particular, there exists a positive constant c, depending only on N, such that

$$\kappa_G(u) = c\, h_G(u)^{N+1} \quad \textit{for every} \quad u \in \mathbb{S}^{N-1}.$$

Therefore, G must be an ellipsoid.

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