Power-law approximation under differential constraints

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Abstract

We study the Γ-convergence of the power-law functionals

$$F_p(V) = \Big(\int_\Omega f^p(x, V(x))dx\Big)^{1/p},$$

as p tends to $+\infty$, in the setting of constant-rank operator $\mathcal{A}$. We show that the Γ-limit is given by a supremal functional on $L^\infty(\Omega; \mathbb{M}^{d\times N}) \cap \mathrm{Ker}\mathcal{A}$ where $\mathbb{M}^{d\times N}$ is the space of $d \times N$ real matrices. We give an explicit representation formula for the supremand function. We provide some examples and as application of the Γ-convergence results we characterize the strength set in the context of electrical resistivity.

Keywords: supremal functionals, Γ-convergence, L^p-approximation, lower semicontinuity, $\mathcal{A}$-quasiconvexity.

Contents

1 Introduction

In this paper we study the asymptotic behaviour of family of integral functionals of the form

$$F_p(V) = \Big(\int_\Omega f^p(x, V(x))dx\Big)^{1/p} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set, $f : \Omega \times \mathbb{M}^{d\times N} \mapsto [0,\infty)$ is a Carathéodory function and $V \in L^\infty(\Omega;\mathbb{M}^{d\times N})$ is constrained to satisfy a system of first order linear partial differential equations:

$$\mathcal{A}V := \sum_{i=1}^{N} A^{(i)} \frac{\partial V}{\partial x_i} = 0. \tag{1.2}$$

Here $A^{(i)} : \mathbb{M}^{d\times N} \to \mathbb{R}^l$ are linear transformations for every $i = 1,\cdots,N$ and the operator $\mathcal{A}$ satisfies the so-called *constant-rank* property (see [21]).

This type of constraint arise naturally in the setting of continuum mechanics and electromagnestism: for example,

(a) in the case of solenoidal fields (divergence free fields) which are relevant to treat extreme resistivity: here

$$\mathcal{A}V = 0 \text{ if and only if } \operatorname{Div} V = 0$$

where $V : \Omega \to \mathbb{M}^{d\times N}$;

(b) in the context of effective conductivity (curl free fields) where

$$\mathcal{A}V = 0 \text{ if and only if } \operatorname{curl} V = 0;$$

(c) in the micromagnetics literature where the constraints are given by Maxwell's equations;

(d) in the case of higher gradients

(for further details see [18] Section 3).

In [18] Fonseca and Müller study the necessary and sufficient conditions for the (sequential) lower semicontinuity of integral functionals of the form

$$I(U,V) = \int_\Omega f(x,U(x),V(x))dx \tag{1.3}$$

when $U_n \to U$ in measure, $V_n \rightharpoonup V$ weakly in $L^p(\Omega,\mathbb{M}^{d\times N})$ (weakly* if $p = +\infty$) and $\mathcal{A}V_n \to 0$ in $W^{-1,p}(\Omega)$ ($\mathcal{A}V_n = 0$ if $p = +\infty$). In this framework they generalize the classical notion of quasiconvexity (see for example [2], [15]) and prove that a necessary and sufficient condition for the lower semicontinuity is $\mathcal{A}$-quasiconvexity of $f(x,U,\cdot)$. We recall that a function $f : \mathbb{M}^{d\times N} \to \mathbb{R}$ is $\mathcal{A}$-quasiconvex if

$$f(\Sigma) \le \int_Q f(\Sigma + V(x))\, dx$$

for every $V \in C^\infty(\mathbb{R}^N;\mathbb{M}^{d\times N})$ such that V is Q-periodic, $\mathcal{A}V = 0$ and $\int_Q V\, dx = 0$. Note that if $\mathcal{A} = \operatorname{curl}$ then $\mathcal{A}$-quasiconvexity coincides with the well-known notion of quasiconvexity due to Morrey in the case of the gradients.

So far the asymptotic behaviour of the family (F_p) has been studied in the curl-free case. In [20] Garroni, Nesi and Ponsiglione study the macroscopic behavior of two phases composite materials for the first failure dielectric breakdown. They consider the family of the *power-law functionals* $F_p : L^1(\Omega) \to [0,+\infty]$ given by

$$F_p(v) := \begin{cases} \left(\int_\Omega |\lambda(x)Dv|^p\, dx\right)^{1/p} & \text{if } v \in W^{1,p}(\Omega), \\ +\infty & \text{otherwise} \end{cases}$$

with $\lambda(x)$ piecewise-constant function (whose constant values represent the two phases) and prove that (F_p) Γ-converges with respect to the L^1-strong topology, as $p \to \infty$, to the so called *supremal functional*

$$F(v) := \begin{cases} \operatorname*{ess\,sup}_{x\in\Omega} |\lambda(x)Dv| & \text{if } v \in W^{1,\infty}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

In [14] Champion, De Pascale and Prinari consider $F_p : C(\overline{\Omega}, \mathbb{R}^d) \to [0,\infty]$ of the form

$$F_p(v) := \begin{cases} \left(\int_\Omega f^p(x, Dv(x))dx \right)^{1/p} & \text{if } v \in W^{1,p}(\Omega;\mathbb{R}^d), \\ +\infty & \text{otherwise} \end{cases}$$

where the function f satisfies a linear growth condition and the following generalized Jensen inequality: for every $x \in \Omega$

$$f\left(x, \int_{\mathbb{M}^{d\times N}} \Sigma d\nu_x(\Sigma)\right) \le \nu_x\text{-} \operatorname*{ess\,sup}_{\Sigma\in\mathbb{M}^{d\times N}} f(x,\Sigma) \tag{1.4}$$

whenever $(\nu_x)_{x\in\Omega}$ is a $W^{1,p}$-gradient Young measure for all $p \in (1,\infty)$. They prove that (F_p) Γ-converges with respect to the uniform convergence, as $p \to \infty$, to the functional

$$F(v) := \begin{cases} \operatorname*{ess\,sup}_{x\in\Omega} f(x, Dv(x)) & \text{if } v \in W^{1,\infty}(\Omega;\mathbb{R}^d), \\ +\infty & \text{otherwise.} \end{cases} \tag{1.5}$$

Notice that inequality (1.4) is satisfied, in particular, by the functions that are level convex in the second variable. We recall that a function $f : \mathbb{M}^{d\times N} \to \mathbb{R}$ is level convex if

$$f(\lambda\Sigma_1 + (1-\lambda)\Sigma_2) \le f(\Sigma_1) \vee f(\Sigma_2)$$

for every $\Sigma_1, \Sigma_2 \in \mathbb{M}^{d\times N}$ and $\lambda \in [0,1]$. In [24] Prinari removes the hypotheses (1.4) in the scalar case $d = 1$ and shows that (F_p) Γ-converges to the functional $\tilde{F}$ given by

$$\tilde{F}(v) := \begin{cases} \operatorname*{ess\,sup}_{x\in\Omega} f^{lc}(x, Du(x)) & \text{if } v \in W^{1,\infty}(\Omega), \\ +\infty & \text{otherwise} \end{cases}$$

where $f^{lc}(x,\cdot)$ is the greatest level convex function less or equal to $f(x,\cdot)$.

In [9] Bocea and Nesi study the L^p-approximation in the more general framework of $\mathcal{A}$-quasiconvexity. More precisely, they consider the power-law functionals $F_p : L^1(\Omega;\mathbb{M}^{d\times N}) \to [0,+\infty]$ defined by

$$F_p(V) := \begin{cases} \left(\int_\Omega f^p(x, V(x))dx \right)^{1/p} & \text{if } V \in L^p(\Omega;\mathbb{M}^{d\times N}) \cap \mathrm{Ker}\mathcal{A}, \\ +\infty & \text{otherwise,} \end{cases} \tag{1.6}$$

and, under the assumption that $f(x,\cdot)$ is $\mathcal{A}$-quasiconvex and satisfies standard growth conditions, they prove a liminf inequality with respect to the weak convergence in L^1 and in particular they show that (F_p) Γ-converges, with respect to the L^1-strong topology as $p \to \infty$, to the supremal functional

$$F(V) := \begin{cases} \operatorname*{ess\,sup}_{x\in\Omega} f(x, V(x)) & \text{if } V \in L^\infty(\Omega;\mathbb{M}^{d\times N}) \cap \mathrm{Ker}\mathcal{A}, \\ +\infty & \text{otherwise.} \end{cases} \tag{1.7}$$

However in the context of supremal functionals the $\mathcal{A}$-quasiconvexity is too restrictive since it is not necessary for the lower semicontinuity. For example, if $\mathcal{A} = \mathrm{curl}$, under suitable assumptions on f with respect to the variable x, in [8] Barron, Jensen and Wang prove that a supremal functional is weakly* lower semicontinuous on $W^{1,\infty}(\Omega,\mathbb{R}^d)$ if and only if $f(x,\cdot)$ is (strong) Morrey quasiconvex (see [8] Definition 2.1). We note that the curl-quasiconvexity, since it is equivalent to the quasiconvexity, only implies the Morrey quasiconvexity. In fact, if $d = 1$ or $N = 1$, we have that the curl-quasiconvexity coincides with the convexity while the Morrey quasiconvexity is equivalent only to the level convexity.

In this paper we generalize both results proved in [14] and [9] assuming milder assumption on f. More precisely, in Theorem 4.2 we consider the family $F_p : L^\infty(\Omega;\mathbb{M}^{d\times N}) \to [0,+\infty]$ given by

$$F_p(V) := \begin{cases} \left(\int_\Omega f^p(x, V(x))dx \right)^{1/p} & \text{if } V \in L^\infty(\Omega;\mathbb{M}^{d\times N}) \cap \mathrm{Ker}\mathcal{A}, \\ +\infty & \text{otherwise,} \end{cases} \tag{1.8}$$

and we prove that, for any Carathéodory function f satisfying linear growth condition, (F_p) Γ-converges, with respect to the L^∞-weak* topology as $p \to \infty$, to the functional

$$\tilde{F}(V) := \begin{cases} \operatorname*{ess\,sup}_{x\in\Omega} \tilde{f}(x, V(x)) & \text{if } V \in L^\infty(\Omega; \mathbb{M}^{d\times N}) \cap \mathrm{Ker}\mathcal{A}, \\ +\infty & \text{otherwise.} \end{cases} \tag{1.9}$$

In order to give an explicit formula of the supremand function $\tilde{f}$ we define the class of $\mathcal{A}$-∞ quasiconvex functions. We say that a function $f : \mathbb{M}^{d\times N} \mapsto [0, +\infty)$ is $\mathcal{A}$-∞ quasiconvex if for every $\Sigma \in \mathbb{M}^{d\times N}$

$$f(\Sigma) = \lim_{p\to\infty} \inf\Big\{ \big(\int_Q f^p(\Sigma + V(x))dx \big)^{1/p} : \, V \in L^\infty_\#(Q; \mathbb{M}^{d\times N}) \cap \mathrm{Ker}\,\mathcal{A}, \, \int_Q V\,dx = 0 \Big\}$$

where

$$L^\infty_\#(Q; \mathbb{M}^{d\times N}) = \{V \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N; \mathbb{M}^{d\times N}) : \, V \text{ is } Q\text{-periodic}\}\,.$$

We show that the function $\tilde{f}$ coincides with the $\mathcal{A}$-∞ quasiconvex envelope of f, that is, the greatest $\mathcal{A}$-∞ quasiconvex function below f. In particular, by Theorem 4.2 we have that the $\mathcal{A}$-∞ quasiconvexity is sufficient for the L^p-approximation of a supremal functional with respect to the L^∞- weak* topology. If f does not depend explicitly on x, in Theorem 4.4 we prove that the $\mathcal{A}$-∞ quasiconvexity is also necessary.

In general $\mathcal{A}$-quasiconvex functions are $\mathcal{A}$-∞-quasiconvex while the viceversa is not true (see Example 5.5). Therefore, in Theorem 4.1 we generalize the results obtained in [9] since we consider the same family of functionals (F_p) as in (1.6) with $f(x, \cdot)$ $\mathcal{A}$-∞ quasiconvex and satisfying standard growth conditions. Under these assumptions we prove that (F_p) Γ-converges, with respect to the L^1-strong topology as $p \to \infty$, to F as in (1.7).

Since the Γ-limit is always a lower semicontinuous functional (see Section 2.1), by Theorem 4.2 we can also conclude, under linear growth condition, that the notion of $\mathcal{A}$-∞ quasiconvexity provides a sufficient condition for the lower semicontinuity of the supremal functionals under differential constraint. In particular, the curl-∞ quasiconvex functions are a subclass of (strong) Morrey quasiconvex functions. Moreover, in the Example 5.9 we show that such inclusion is strict. In a forthcoming paper [4] we perform a deeper analysis in order to find necessary and sufficient conditions for the lower semicontinuity of supremal functional of the form (1.7).

In Section 6 we apply the results obtained to characterize via Γ-convergence the effective strength set K_{eff} in the context of electrical resistivity defined by

$$K_{\mathrm{eff}} := \{\bar{\sigma} := \int_\Omega \sigma(x)\,dx : \; \sigma(x) \in K(x) \text{ a.e. } x \in \Omega, \; \mathrm{div}\,\sigma = 0\}$$

where $\sigma : \Omega \mapsto \mathbb{R}^N$ is the current and

$$K(x) = \{\xi \in \mathbb{R}^N : \; f(x, \xi) \le 1\}\,.$$

In the context of (first failure) models of dielectric breakdown for composite made of two isotropic phases considered by Garroni, Nesi and Ponsiglione [20], the constraint $\sigma(x) \in K(x)$ is replaced by the condition that $\nabla u \in K(x)$ where ∇u is the electric field and $f(x, \xi) = \lambda(x)|\xi|$ with $\lambda(x)$ piecewise-constant function (whose constant values represent the two phases). Such a model is concerned with electrical conductivity and therefore the relevant fields are curl free. Here we want to model electrical resistivity then the right differential constraint is the divergence. We recall that in the context of plasticity the set $K(x)$ is called the yield set.

In [9] Bocea and Nesi characterize the set K_{eff} under the assumptions that f is a Carathéodory function, div-quasiconvex in the second variable and satisfying growth conditions (see Proposition 6.1 and 6.2 in [9]). Note that in their case, since $d = 1$, the div-quasiconvexity reduces to the convexity. In this paper we characterize the set K_{eff} under more general hypotheses; i.e., we assume that f is a Carathéodory function, div-∞ quasiconvex in the second variable and satisfying a growth condition from below. This is, in particular, equivalent to suppose that f is level convex in the second variable (see Proposition 5.4).

2 Notation and preliminaries

Let Ω be a bounded open subset of $\mathbb{R}^N$. We denote by $\mathcal{O}(\Omega)$ the family of open subsets of Ω. We write $\mathcal{L}^N(E)$ for the Lebesgue measure of $E \subset \mathbb{R}^N$. Let $\Sigma \in \mathbb{M}^{d\times N}$, where $\mathbb{M}^{d\times N}$ stands for the space of $d \times N$ real matrices, with a slight abuse of notation, we denote $|\Sigma| = \sum_{i=1}^d |\Sigma_i|$, where Σ_i is the i^{th} row of Σ and $|\Sigma_i|$ its Euclidean norm. We use ξ_i also to denote the i^{th} component of a vector ξ. Finally, if $V : \Omega \to \mathbb{M}^{d\times N}$ we define $\operatorname{Div} V : \Omega \mapsto \mathbb{R}^d$ such that

$$(\operatorname{Div} V)_i = \operatorname{div} V_i$$

for every $i = 1, \cdots, d$. Here, and in what follows, $\mathcal{A}$ is a constant-rank, first order linear partial differential operator defined on the space $L^p(\Omega, \mathbb{M}^{d\times N})$, $1 < p \le \infty$, by the formula

$$\mathcal{A}V := \sum_{i=1}^{N} A^{(i)} \frac{\partial V}{\partial x_i} \quad (\in W^{-1,p}(\Omega; \mathbb{R}^l))$$

where $A^{(i)} : \mathbb{M}^{d\times N} \to \mathbb{R}^l$ are linear transformations for every $i = 1, \cdots, N$. We recall that $\mathcal{A}$ satisfies the *constant-rank* property if there exists $r \in \mathbb{N}$ such that

$$\operatorname{rank} \mathbb{A}w = r \quad \text{ for all } w \in S^{N-1}$$

where

$$\mathbb{A}w = \sum_{i=1}^{N} A^{(i)} w_i, \quad w \in \mathbb{R}^N.$$

We define

$$L^p_\#(Q; \mathbb{M}^{d\times N}) := \{V \in L^p_{\mathrm{loc}}(\mathbb{R}^N; \mathbb{M}^{d\times N}) : V \text{ is } Q\text{-periodic}\},$$

for every $1 < p \le +\infty$, where Q denotes the unit cube in $\mathbb{R}^N$. Similarly, we denote by $C^\infty_\#$ the C^∞-functions that are Q-periodic.

2.1 Γ-convergence

We recall the sequential characterization of the Γ-limit when X is a metric space and when X is the dual of a separable Banach space that we will use in the sequel.

Proposition 2.1 ([16] Proposition 8.1) *Let X be a metric space and let $\varphi_n : X \to \mathbb{R} \cup \{\pm\infty\}$ for every $n \in \mathbb{N}$. Then (φ_n) Γ-converges to φ with respect to the strong topology of X (and we write $\Gamma(X)\text{-}\lim_{n\to\infty} \varphi_n = \varphi$) if and only if*

(i) *for every $x \in X$ and for every sequence (x_n) converging to x, it is*

$$\varphi(x) \le \liminf_{n\to\infty} \varphi_n(x_n);$$

(ii) *for every $x \in X$ there exists a sequence (x_n) converging to $x \in X$ such that*

$$\varphi(x) = \lim_{n\to\infty} \varphi_n(x_n).$$

We recall that the $\Gamma\text{-}\lim_{n\to\infty} \varphi_n$ is lower semicontinuous on X (see [16] Proposition 6.8).

Proposition 2.2 *Let X be the dual of a separable Banach space and let X be endowed with its weak* topology. Let $\varphi_n : X \to \mathbb{R} \cup \{\pm\infty\}$ for every $n \in \mathbb{N}$. Assume that there exists $\Phi : X \to \mathbb{R} \cup \{\pm\infty\}$ such that:*

$$\lim_{\|x\|_X \to +\infty} \Phi(x) = +\infty,$$

and $\varphi_n \ge \Phi$ for every $n \in \mathbb{N}$. Then (φ_n) Γ-converges to φ with respect to the weak topology of X (and we write $\Gamma(w^*\text{-}X)\text{-}\lim_{n\to\infty} \varphi_n = \varphi$) if and only if*

(i) *for every $x \in X$ and for every sequence (x_n) converging weakly* to $x \in X$, it is*

$$\varphi(x) \le \liminf_{n\to\infty} \varphi_n(x_n);$$

(ii) *for every $x \in X$ there exists a sequence (x_n) converging weakly* to $x \in X$ such that*

$$\varphi(x) = \lim_{n\to\infty} \varphi_n(x_n).$$

The proof of Proposition 2.2 easily follows the one of Proposition 8.10 in [16] with X endowed with its weak* topology.

Finally we recall also that the function $\varphi = \Gamma(w^*\text{-}X)\text{-}\lim_{n\to\infty}\varphi_n$ is weakly* lower semicontinuous on X (see [16] Proposition 6.8) and when $\varphi_n = \psi\ \forall n \in \mathbb{N}$ then φ coincides with the weakly* lower semicontinuous (l.s.c.) envelope of ψ, i.e.

$$\varphi(x) = \sup\big\{h(x):\ \forall\, h : X \to \mathbb{R}\cup\{\pm\infty\}\ \ w^* \text{ l.s.c.},\ h \le \psi \text{ on } X\big\} \tag{2.10}$$

(see Remark 4.5 in [16]).

We will say that a family (φ_p) Γ-converges to φ, with respect to the topology considered on X as $p \to \infty$, if (φ_{p_n}) Γ-converges to φ for all sequences (p_n) of positive numbers converging to ∞ as $n \to \infty$.

Finally we state the fundamental theorem of Γ-convergence.

Theorem 2.3 *Let (φ_n) be an equi-coercive sequence Γ-converging on X to the function φ with respect to the topology of X. Then we have the convergence of minima*

$$\min_X \varphi = \lim_{n\to\infty} \inf_X \varphi_n.$$

Moreover we have also the convergence of minimizers: if (x_n) is such that $\lim_{n\to\infty}\varphi_n(x_n) = \lim_{n\to\infty}\inf_X \varphi_n$ then, up to subsequences, $(x_n) \to x$ and x is a minimizer for φ.

For a comprehensive study of Γ-convergence we refer to the book of Dal Maso [16] (for a simplified introduction see [11]), while a detailed analysis of some of its applications to homogenization theory can be found in [12].

2.2 $\mathcal{A}$-quasiconvexity

In this section we recall the notion of $\mathcal{A}$-quasiconvexity and some related results that we will use in the sequel.

Definition 2.4 *Let $f : \mathbb{M}^{d\times N} \mapsto \mathbb{R}$ be a function. We say that f is* **$\mathcal{A}$-quasiconvex** *if*

$$f(\Sigma) \le \int_Q f(\Sigma + V(x))\,dx$$

for every $V \in C^\infty(\mathbb{R}^N;\mathbb{M}^{d\times N})$ such that V is Q-periodic, $\mathcal{A}V = 0$ and $\displaystyle\int_Q V\,dx = 0$.

In the next theorem we collect the results proved by Fonseca and Müller in [18] concerning the lower semicontinuity of integral functionals of the form

$$F(V) = \int_\Omega f(x, V(x))dx \tag{2.11}$$

in the context of the constant-rank operator $\mathcal{A}$.

Theorem 2.5 ([18] Theorems 3.6 - 3.7) *Let $1 \le p \le +\infty$ and suppose that $f : \Omega\times\mathbb{M}^{d\times N} \mapsto [0,\infty)$ is a Carathéodory function. Let $F : L^p(\Omega;\mathbb{M}^{d\times N}) \to [0,\infty)$ be the functional given by (2.11).*

1. *(sufficiency) Assume that* $\Sigma \mapsto f(x,\Sigma)$ *is* $\mathcal{A}$*-quasiconvex for a.e.* $x \in \Omega$*. If* $1 \le p < +\infty$*, then assume further that there exists a constant* $\beta > 0$ *such that*

$$0 \le f(x,\Sigma) \le \beta(1 + |\Sigma|^p)$$

for every $\Sigma \in \mathbb{M}^{d\times N}$ *and a.e.* $x \in \Omega$*. Then*

$$F(V) \le \liminf_{n\to\infty} F(V_n)$$

for every $V_n \rightharpoonup V$ *weakly in* $L^p(\Omega;\mathbb{M}^{d\times N})$ *(weakly* if* $p = +\infty$*) and* $\mathcal{A}V_n \to 0$ *in* $W^{-1,p}(\Omega;\mathbb{R}^l)$ *(*$\mathcal{A}V_n = 0$ *if* $p = +\infty$*).*

2. *(necessity) Assume that* $(f(\cdot,V_n(\cdot)))$ *is equi-integrable whenever* (V_n) *is a sequence bounded in* $L^\infty(\Omega;\mathbb{M}^{d\times N})$*. If*

$$F(V) \le \liminf_{n\to\infty} F(V_n)$$

for every $(V_n) \in C^\infty(\bar\Omega;\mathbb{M}^{d\times N})$ *such that* $V_n \rightharpoonup V$ *weakly* in* $L^\infty(\bar\Omega;\mathbb{M}^{d\times N})$ *and* $\mathcal{A}V_n = 0$ *then* $f(x,\cdot)$ *is* $\mathcal{A}$*-quasiconvex for a.e.* $x \in \Omega$*.*

In [13] Braides, Fonseca, and Leoni provide an integral representation formula for the relaxed energy of an integral functional of the form (2.11) in the case $p = +\infty$.

Theorem 2.6 ([13] Theorem 3.6) *Let* $f : \Omega \times \mathbb{M}^{d\times N} \mapsto [0,\infty)$ *be a Carathéodory function such that* $f \in L^\infty_{loc}(\Omega \times \mathbb{M}^{d\times N};[0,\infty))$*. Let* $F : L^\infty(\Omega;\mathbb{M}^{d\times N}) \cap Ker\mathcal{A} \to [0,\infty)$ *be the functional given by (2.11) and let* $\mathcal{F} : L^\infty(\Omega;\mathbb{M}^{d\times N}) \cap Ker\mathcal{A} \to [0,\infty)$ *be defined by*

$$\mathcal{F}(V) := \inf\left\{\liminf_{n\to\infty} F(V_n) \ : \ V_n \in L^\infty(\Omega;\mathbb{M}^{d\times N}) \cap Ker\mathcal{A}\,,\ V_n \rightharpoonup V \ \text{ weakly* in } L^\infty(\Omega;\mathbb{M}^{d\times N})\right\}. \tag{2.12}$$

Then we have

$$\mathcal{F}(V) = \int_\Omega \mathcal{Q}_\mathcal{A} f(x,V(x))\,dx,$$

for every $V \in L^\infty(\Omega;\mathbb{M}^{d\times N}) \cap Ker\mathcal{A}$*, where* $\mathcal{Q}_\mathcal{A} f(x,\cdot)$ *is the* $\mathcal{A}$*-quasiconvexification of* $f(x,\cdot)$*, namely*

$$\mathcal{Q}_\mathcal{A} f(x,\Sigma) := \inf\Big\{\int_Q f(x,\Sigma + V(y))\,dy \ : \ V \in C^\infty_\#(Q;\mathbb{M}^{d\times N}) \cap Ker\mathcal{A}\,,\ \int_Q V(y)\,dy = 0\Big\}. \tag{2.13}$$

Remark 2.7

1. If $f(x,\cdot)$ is upper semicontinuous for a.e. $x \in \Omega$ then $\mathcal{Q}_\mathcal{A} f$ is $\mathcal{A}$-quasiconvex (see [18] Proposition 3.4.)

2. If $f(x,\cdot)$ is upper semicontinuous and locally bounded from above, then, applying Fatou's lemma, it is easy to show that in the definition of $\mathcal{A}$-quasiconvexity and of $\mathcal{A}$-quasiconvexification the set of functions $C^\infty(\mathbb{R}^N;\mathbb{M}^{d\times N})$ may be replaced by $L^\infty(\mathbb{R}^N;\mathbb{M}^{d\times N})$. If, in addition, $|f(x,\Sigma)| \le \beta(1 + |\Sigma|^p)$ for some $\beta > 0$, for every $\Sigma \in \mathbb{M}^{d\times N}$ and for a.e. $x \in \Omega$, then $C^\infty(\mathbb{R}^N;\mathbb{M}^{d\times N})$ may be replaced by $L^p(\mathbb{R}^N;\mathbb{M}^{d\times N})$ (see [18] Remark 3.3(ii)).

3. If f satisfies also a coercivity condition then by Proposition 2.2 we have that $\mathcal{F}$, given by (2.12), is the $\Gamma(w^*\text{-}L^\infty)$-limit of the sequence $F_n \equiv F$, $\forall n$, and coincides with the weakly* lower semicontinuous envelope of F .

2.3 Young measures

In this section we recall briefly some results on the theory of Young measures (see e.g. [5], [10], [29]). If D is an open set (not necessarily bounded), we denote by $C_c(D;\mathbb{R}^k)$ the set of continuous functions with compact support in D, endowed with the supremum norm. The dual of the closure of $C_c(D;\mathbb{R}^k)$ may be identified with the set of $\mathbb{R}^k$-valued Radon measures with finite mass $\mathcal{M}(D;\mathbb{R}^k)$, through the duality

$$\langle \mu,\varphi\rangle := \int_D \varphi(y)\,d\mu(y)\,, \qquad \mu \in \mathcal{M}(D;\mathbb{R}^k)\,, \qquad \varphi \in C_c(D;\mathbb{R}^k)\,.$$

A map $\mu : \Omega \mapsto \mathcal{M}(D;\mathbb{R}^k)$ is said to be weak*-measurable if $x \mapsto \langle \mu_x,\varphi\rangle$ are measurable for all $\varphi \in C_c(D;\mathbb{R}^k)$.

Definition 2.8 *Let* (z_n) *be a bounded sequence in* $L^1(\Omega)$. *We say that* (z_n) *is equi-integrable if for all* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for every measurable* $E \subset \Omega$ *if* $\mathcal{L}^N(E) < \delta$ *then*

$$\sup_n \int_E |z_n(x)|\, dx < \varepsilon\,.$$

We say that (z_n) *is p-equi-integrable if* $(|z_n|^p)$ *is equi-integrable.*

Theorem 2.9 (Fundamental Theorem on Young Measures) *Let* $\Omega \subset \mathbb{R}^N$ *be a measurable set of finite measure and let* (V_n) *be a sequence of measurable functions,* $V_n : \Omega \mapsto \mathbb{M}^{d\times N}$. *Then there exists a subsequence* (V_{n_k}) *and a weak* measurable map* $\mu : \Omega \mapsto \mathcal{M}(\mathbb{M}^{d\times N})$ *such that the following hold:*

1. $\mu_x \geq 0$, $\|\mu_x\|_{\mathcal{M}(\mathbb{M}^{d\times N})} = \int_{\mathbb{M}^{d\times N}} d\mu_x \leq 1$ *for a.e.* $x \in \Omega$;
2. *if* $K \subset \mathbb{M}^{d\times N}$ *is a compact subset and* $\mathrm{dist}\,(V_{n_k}, K) \to 0$ *in measure, then*
$$\mathrm{supp}\,\mu_x \subset K \qquad \text{for a.e. } x \in \Omega\,;$$
3. $\|\mu_x\|_{\mathcal{M}(\mathbb{M}^{d\times N})} = 1$ *for a.e.* $x \in \Omega$ *if and only if*
$$\lim_{M\to\infty} \sup_k \mathcal{L}^N(\{|V_{n_k}| \geq M\}) = 0\,;$$
4. *if* (3) *holds then in* (2) *we may replace "if" with "if and only if";*
5. *if* $f : \Omega \times \mathbb{M}^{d\times N} \mapsto \mathbb{R}$ *is a Borel function bounded from below and* $f(x,\cdot)$ *is lower semicontinuous for a.e.* $x \in \Omega$, *then*
$$\liminf_{k\to\infty} \int_\Omega f(x, V_{n_k}(x))\, dx \geq \int_\Omega \bar{f}(x)\, dx\,,$$
where
$$\bar{f}(x) := \langle \mu_x, f(x,\cdot)\rangle = \int_{\mathbb{M}^{d\times N}} f(x,y)\, d\mu_x(y)\,;$$
6. *if* (3) *holds and if* $f : \Omega \times \mathbb{M}^{d\times N} \mapsto \mathbb{R}$ *is Carathéodory and bounded from below, then*
$$\lim_{k\to\infty} \int_\Omega f(x, V_{n_k}(x))\, dx = \int_\Omega \bar{f}(x)\, dx < +\infty$$
if and only if $(f(\cdot, V_{h_k}(\cdot)))$ *is equi-integrable. In this case*
$$f(\cdot, V_{n_k}(\cdot)) \rightharpoonup \bar{f} \qquad \textit{weakly in} \quad L^1(\Omega)\,.$$

The map $\mu : \Omega \mapsto \mathcal{M}(\mathbb{M}^{d\times N})$ as in Theorem 2.9 is called **Young measure** generated by the sequence (V_{n_k}).

Remark 2.10

1. We recall that if (V_n) is a bounded sequence in $L^1(\Omega; \mathbb{M}^{d\times N})$ and there exists a continuous function $g : [0; +\infty) \to [0; +\infty)$ such that
$$\lim_{t\to+\infty} \frac{g(t)}{t} = +\infty \text{ and } \sup_n \int_\Omega g(|V_n(x)|)\, dx < +\infty$$
then the sequence (V_n) is equi-integrable.
2. By Dunford-Pettits Theorem, if (V_n) is bounded in $L^q(\Omega; \mathbb{M}^{d\times N})$ for some $1 \leq q < +\infty$, then $(f(V_n))$ is equi-integrable whenever $f : \mathbb{M}^{d\times N} \to [0, +\infty)$ is a continuous function such that $\lim_{|\Sigma|\to\infty} \frac{f(\Sigma)}{|\Sigma|^q} = 0$. In particular, if f is a continuous function such that $|f(\Sigma)| \leq \beta(|\Sigma| + 1)$, then $\lim_{|\Sigma|\to\infty} \frac{f^p(\Sigma)}{|\Sigma|^q} = 0$ for every $q > p$; hence, for every (V_n) bounded in $L^\infty(\Omega; \mathbb{M}^{d\times N})$ we have that $(f^p(V_n))$ is equi-integrable for every p.

3. As a consequence of Theorem 2.9 (6), if (V_n) is equi-integrable then taking $f \equiv \mathrm{id}$ we obtain

$$V_{n_k} \rightharpoonup \bar{V} \qquad \text{weakly in } L^1(\Omega)\,, \qquad \bar{V}(x) := \langle \mu_x, \mathrm{id}\rangle\,.$$

(See [18] Remark 2.3 (ii)).

Definition 2.11 *Let μ be a Young measure. Then μ is said to be homogeneous if there is a Radon measure $\mu_0 \in \mathcal{M}(\mathbb{M}^{d\times N})$ such that $\mu_x = \mu_0$ for a.e. $x \in E$.*

We conclude this section by recalling the following proposition which will represent an important tool to prove the L^p-approximation Theorems 4.1.

Proposition 2.12 ([18] Proposition 3.8) *Let $1 \le p < \infty$ and let (V_n) be a p-equi-integrable sequence in $L^p(\Omega;\mathbb{M}^{d\times N})$ such that $\mathcal{A}V_n \to 0$ in $W^{-1,p}(\Omega)$ if $1 < p < \infty$, $\mathcal{A}V_n \to 0$ in $W^{-1,r}(\Omega)$ for some $r \in (1, \frac{N}{N-1})$ if $p = 1$, and (V_n) generates a Young measure μ. Let $V_n \rightharpoonup V$ weakly in $L^p(\Omega;\mathbb{M}^{d\times N})$. Then for a.e. $x_0 \in \Omega$ there exists a sequence $(\bar{W}_n) \subset L^p_{\#}(Q;\mathbb{M}^{d\times N}) \cap Ker\mathcal{A}$ that is p-equi-integrable, generates the homogeneous Young measure μ_{x_0} and satisfies*

$$\int_Q \bar{W}_n(y)dy = V(x_0)\,.$$

In particular for a.e. $x_0 \in \Omega$

$$f(V(x_0)) \le \langle \mu_{x_0}, f\rangle$$

for every continuous $\mathcal{A}$-quasiconvex function f such that

$$|f(\Sigma)| \le \beta(1 + |\Sigma|^p) \tag{2.14}$$

for some $\beta > 0$ and for all $\Sigma \in \mathbb{M}^{d\times N}$.

3 New sets of functions

In order to study the asymptotic behavior of the power-law functionals

$$F_p(V) := \left(\int_\Omega f^p(x, V(x))dx\right)^{1/p}, \qquad V \in L^p(\Omega;\mathbb{M}^{d\times N}) \cap \mathrm{Ker}\mathcal{A}$$

we introduce the classes of $\mathcal{A}$-weak and $\mathcal{A}$-∞ quasiconvex functions. In this section we analyse the main properties and their mutual connections.

3.1 $\mathcal{A}$-weak and $\mathcal{A}$-∞ quasiconvex functions

We start introducing the notion of $\mathcal{A}$-weak quasiconvexity. It seems the natural definition in the context of supremal functionals compared with the notion of $\mathcal{A}$-quasiconvexity (see Definition 2.4). However, we will see that the $\mathcal{A}$-weak quasiconvexity does not play the same role that the $\mathcal{A}$-quasiconvexity plays in the context of integral functionals.

Definition 3.1 *We say that a Borel function $f : \mathbb{M}^{d\times N} \mapsto \mathbb{R}$ is* **$\mathcal{A}$-weak quasiconvex** *if for all $\Sigma \in \mathbb{M}^{d\times N}$*

$$f(\Sigma) \le \operatorname*{ess\,sup}_{x\in Q} f(\Sigma + V(x))$$

for every $V \in L^\infty_{\#}(Q;\mathbb{M}^{d\times N}) \cap Ker\mathcal{A}$ with $\int_Q V\,dx = 0$.

Remark 3.2 1. Note that, by Remark 2.7(2), we have that every $\mathcal{A}$-quasiconvex function (upper semicontinuous and locally bounded from above) is $\mathcal{A}$-weak quasiconvex.

2. If f is curl-weak quasiconvex then f is weak Morrey quasiconvex; i.e.,

$$f(\Sigma) = \inf\Big\{\operatorname*{ess\,sup}_{x\in Q} f(\Sigma + D\varphi) : \varphi \in W_0^{1,\infty}(Q;\mathbb{R}^d)\Big\},$$

for all $\Sigma \in \mathbb{M}^{d\times N}$.

The weak Morrey quasiconvexity, introduced by Barron, Jensen, Wang in [8], is necessary for the lower semicontinuity of a supremal functional defined on $W^{1,\infty}(\Omega;\mathbb{R}^d)$. In the scalar case; i.e., $d = 1$ or $N = 1$, such condition is also sufficient and it coincides with the notion of level convexity. We recall that $f : \mathbb{R}^k \to \mathbb{R}$ is level convex if for every $t \in \mathbb{R}$ the level set $\big\{\xi \in \mathbb{R}^k : f(\xi) \le t\big\}$ is convex. It is an open problem to determine if the weak Morrey quasiconvexity is sufficient also in the vectorial case.

Since the aim of this paper is to prove that under suitable conditions on f the family (F_p) approximates via Γ-convergence a supremal functional (which is lower semicontinuous being a Γ-limit) we deduce that the $\mathcal{A}$-weak quasiconvexity may not be the right notion for f. Therefore we introduce the class of $\mathcal{A}$-∞ quasiconvex functions.

Definition 3.3 *We say that a Borel function $f : \mathbb{M}^{d\times N} \mapsto [0,+\infty)$ is $\mathcal{A}$-∞* **quasiconvex** *if for every $\Sigma \in \mathbb{M}^{d\times N}$*

$$f(\Sigma) = \lim_{p\to\infty} \inf\Big\{\big(\int_Q f^p(\Sigma + V(x))dx\big)^{1/p} : V \in L^\infty_\#(Q;\mathbb{M}^{d\times N}) \cap Ker\,\mathcal{A},\ \int_Q V\,dx = 0\Big\}. \tag{3.15}$$

In the sequel we will denote by

$$f_p(\Sigma) := \inf\Big\{\big(\int_Q f^p(\Sigma + V(x))dx\big)^{1/p} : V \in L^\infty_\#(Q;\mathbb{M}^{d\times N}) \cap \mathrm{Ker}\,\mathcal{A},\ \int_Q V\,dx = 0\Big\}. \tag{3.16}$$

Remark 3.4 Let f be an upper semicontinuous function. By Remark 2.7(2), if f satisfies also the growth conditions $|f(\Sigma)| \le \beta(1+|\Sigma|)$ then we may replace the space L^∞ with L^p in (3.15); i.e., f is $\mathcal{A}$-∞ quasiconvex if for every $\Sigma \in \mathbb{M}^{d\times N}$

$$f(\Sigma) = \lim_{p\to\infty} \inf\Big\{\big(\int_Q f^p(\Sigma + V(x))dx\big)^{1/p} : V \in L^p_\#(Q;\mathbb{M}^{d\times N}) \cap \mathrm{Ker}\,\mathcal{A},\ \int_Q V\,dx = 0\Big\}.$$

In the next Proposition 3.6 we study the connections between level convex, $\mathcal{A}$-quasiconvex, $\mathcal{A}$-weak quasiconvex and $\mathcal{A}$-∞ quasiconvex functions. In particular, we show that

$$f\ \mathcal{A}\text{-quasiconvex} \Longrightarrow f\ \mathcal{A}\text{-}\infty \text{ quasiconvex} \Longrightarrow f\ \mathcal{A}\text{-weak quasiconvex};$$

and

$$f \text{ level convex and lower semicontinuous} \Longrightarrow f\ \mathcal{A}\text{-weak quasiconvex}.$$

We first recall the Jensen inequality introduced by Barron, Jensen, and Liu in [7] for lower semicontinuous and level convex functions (see also [8] Theorem 1.2) that we use in the sequel.

Theorem 3.5 *Let $f : \mathbb{R}^k \to \mathbb{R}$ be a lower semicontinuous and level convex function, and let μ be a probability measure supported on Ω. Then for every function $u \in L^1_\mu(\Omega;\mathbb{R}^k)$, we have*

$$f\left(\int_\Omega u(x)d\mu(x)\right) \le \mu\text{-}\operatorname*{ess\,sup}_{x\in\Omega}(f\circ u)(x). \tag{3.17}$$

Proposition 3.6 *Let $f : \mathbb{M}^{d\times N} \mapsto [0,+\infty)$ be a Borel function.*

1. *If f^q is upper semicontinuous, locally bounded from above and $\mathcal{A}$-quasiconvex for some $q \ge 1$, then f is $\mathcal{A}$-∞ quasiconvex.*

2. *If f is $\mathcal{A}$-∞ quasiconvex then f is* $\mathcal{A}$-weak *quasiconvex.*

3. *If f is lower semicontinuous and level convex then f is $\mathcal{A}$-weak quasiconvex.*
4. *If f is continuous, level convex and there exist $\alpha > 0$ such that*
$$f(\Sigma) \geq \alpha|\Sigma| \qquad \text{for every } \Sigma \in \mathbb{M}^{d\times N}$$
then f is $\mathcal{A}$-∞ quasiconvex.

Proof.

1. By Definition 2.4 and Remark 2.7(2) for $q \geq 1$ we have that
$$f^q(\Sigma) = \inf\Big\{ \int_Q f^q(\Sigma + V(x))dx : V \in L^\infty_\#(Q;\mathbb{M}^{d\times N}) \cap \operatorname{Ker}\mathcal{A}, \int_Q V\,dx = 0\Big\}$$
for every $\Sigma \in \mathbb{M}^{d\times N}$; hence,
$$f(\Sigma) = \inf\Big\{ \big(\int_Q f^q(\Sigma + V(x))dx\big)^{1/q} : V \in L^\infty_\#(Q;\mathbb{M}^{d\times N}) \cap \operatorname{Ker}\mathcal{A}, \int_Q V\,dx = 0\Big\} = f_q(\Sigma).$$
For every $p \geq q$ and $V \in L^\infty_\#(Q;\mathbb{M}^{d\times N}) \cap \operatorname{Ker}\mathcal{A}$ with $\int_Q V\,dx = 0$, by Hölder's inequality, we have that
$$f(\Sigma) \leq \big(\int_Q f^q(\Sigma + V(x))dx\big)^{1/q} \leq \big(\int_Q f^p(\Sigma + V(x))dx\big)^{1/p}.$$
In particular
$$f(\Sigma) \leq \inf\Big\{ \big(\int_Q f^p(\Sigma + V(x))dx\big)^{1/p} : V \in L^\infty_\#(Q;\mathbb{M}^{d\times N}) \cap \operatorname{Ker}\mathcal{A}, \int_Q V\,dx = 0\Big\} \leq f(\Sigma).$$
This implies that
$$f_p(\Sigma) \equiv f(\Sigma)$$
for every $\Sigma \in \mathbb{M}^{d\times N}$ and $p \geq q$. Therefore we get in particular that $f(\Sigma) = \lim_{p\to\infty} f_p(\Sigma)$ for every $\Sigma \in \mathbb{M}^{d\times N}$, i.e. f is $\mathcal{A}$-∞ quasiconvex.
2. Let $\Sigma \in \mathbb{M}^{d\times N}$. Then
$$\begin{aligned} f(\Sigma) &= \lim_{p\to\infty} \inf\Big\{ \big(\int_Q f^p(\Sigma + V(x))dx\big)^{1/p} : V \in L^\infty_\#(Q;\mathbb{M}^{d\times N}) \cap \operatorname{Ker}\mathcal{A}, \int_Q V\,dx = 0\Big\} \\ &\leq \inf\Big\{ \operatorname*{ess\,sup}_{x\in Q} f(\Sigma + V(x)) : V \in L^\infty_\#(Q;\mathbb{M}^{d\times N}) \cap \operatorname{Ker}\mathcal{A}, \int_Q V\,dx = 0\Big\} \\ &\leq f(\Sigma) \end{aligned}$$
which concludes the proof.
3. Let $V \in L^\infty_\#(Q;\mathbb{M}^{d\times N}) \cap \operatorname{Ker}\mathcal{A}$ be such that $\int_Q V\,dx = 0$, applying Jensen's inequality (3.17), we have that
$$f(\Sigma) = f\Big(\fint_Q \Sigma + V(x)\,dx\Big) \leq \operatorname*{ess\,sup}_{x\in Q} f(\Sigma + V(x)).$$
Then f is $\mathcal{A}$-weak quasiconvex.
4. Let $(f^p)^{**}$ be the convex envelope of the function f^p. By Jensen's inequality we have that
$$(f^p)^{**}(\Sigma) \leq \int_Q (f^p)^{**}(\Sigma + V(x))\,dx \leq \int_Q f^p(\Sigma + V(x))\,dx$$
for every $V \in L^\infty_\#(Q;\mathbb{M}^{d\times N}) \cap \operatorname{Ker}\mathcal{A}$ such that $\int_Q V\,dx = 0$. It follows that
$$(f^p)^{**}(\Sigma) \leq \inf\Big\{ \int_Q f^p(\Sigma + V(x))dx : V \in L^\infty_\#(Q;\mathbb{M}^{d\times N}) \cap \operatorname{Ker}\mathcal{A}, \int_Q V\,dx = 0\Big\} = f_p^p(\Sigma); \quad (3.18)$$
hence, $((f^p)^{**})^{1/p} \leq f_p \leq f$. Moreover, since f is continuous, level convex and satisfies a linear growth condition, we have that $\lim_{p\to\infty}((f^p)^{**})^{1/p} = f$ (see e.g. [24] Remark 3.12). Hence, passing to the limit as $p \to \infty$ we get that in particular $f = \lim_{p\to\infty} f_p$; i.e., f is $\mathcal{A}$-∞ quasiconvex.

□

Remark 3.7 1. In Section 5.2 we will exhibit some counter-examples (see Examples 5.9 and 5.5) which show that if $d, N > 1$ then

$$f\ \mathcal{A}\text{-weak quasiconvex} \not\Longrightarrow f\ \mathcal{A}\text{-}\infty \text{ quasiconvex;}$$

$$f\ \mathcal{A}\text{-}\infty \text{ quasiconvex} \not\Longrightarrow f\ \mathcal{A}\text{-quasiconvex;}$$

respectively.

2. The coercivity assumption cannot be dropped in the statement of Proposition 3.6(4) (see Example 5.9).

3.2 $\mathcal{A}$-∞ quasiconvex envelope

For any function $f : \mathbb{M}^{d\times N} \to \mathbb{R}$ we define

$$\mathcal{Q}^{\infty}_{\mathcal{A}} f(\Sigma) := \sup\{h(\Sigma) :\ h \text{ is } \mathcal{A}\text{-}\infty \text{ quasiconvex and } h \leq f\} \tag{3.19}$$

the $\mathcal{A}$-∞ quasiconvex envelope of f. In the next proposition we prove, among others, that for any continuous function f, the $\mathcal{A}$-∞ quasiconvex envelope can be obtained as limit of the $\mathcal{A}$-quasiconvex functions f_p, as p tends to ∞.

Proposition 3.8 *Let* $f : \mathbb{M}^{d\times N} \to [0, +\infty)$ *be a continuous function and let* f_p *be defined as in* (3.16) *for every* $p > 1$*. Then*

1. f_p^p *is* $\mathcal{A}$*-quasiconvex;*
2. $\mathcal{Q}^{\infty}_{\mathcal{A}} f(\Sigma)$ *is* $\mathcal{A}$*-*∞ *quasiconvex;*
3. (f_p) *converges to* $\mathcal{Q}^{\infty}_{\mathcal{A}} f$*, as* $p \to \infty$*; i.e.,*

$$Q^{\infty}_{\mathcal{A}} f(\Sigma) = \lim_{p\to\infty} \inf\Big\{\big(\int_Q f^p(\Sigma+V(y))\,dy\big)^{1/p} :\ V \in L^{\infty}_{\#}(Q;\mathbb{M}^{d\times N})\cap Ker\,\mathcal{A},\ \int_Q V\,dy = 0\Big\}, \tag{3.20}$$

for every $\Sigma \in \mathbb{M}^{d\times N}$*.*

Proof. We recall that

$$f_p(\Sigma) := \inf\Big\{\big(\int_Q f^p(\Sigma + V(x))dx\big)^{1/p} :\ V \in L^{\infty}_{\#}(Q;\mathbb{M}^{d\times N}) \cap \mathrm{Ker}\,\mathcal{A},\ \int_Q V\,dx = 0\Big\}.$$

Note that (f_p) is an increasing sequence; hence, there exists the pointwise limit $\tilde{f} = \lim_{p\to\infty} f_p$ and $f_p \leq \tilde{f}$ for every p.

1. By Remark 2.7 (1) $\mathcal{Q}_{\mathcal{A}} f^p$ is $\mathcal{A}$-quasiconvex; hence, since $f_p^p(\Sigma) = \mathcal{Q}_{\mathcal{A}} f^p(\Sigma)$ we have that also f_p^p is $\mathcal{A}$-quasiconvex.
2. Fix $\Sigma \in \mathbb{M}^{d\times N}$ and $\epsilon > 0$. Let h_ε be a $\mathcal{A}$-∞ quasiconvex function such that $h_\varepsilon \leq f$ and such that

$$\mathcal{Q}^{\infty}_{\mathcal{A}} f(\Sigma) \leq h_\varepsilon(\Sigma) + \varepsilon.$$

Since $h_\varepsilon \leq \mathcal{Q}^{\infty}_{\mathcal{A}} f$ we have that $(h_\varepsilon)_p \leq (Q^{\infty}_{\mathcal{A}} f)_p$. This implies

$$\mathcal{Q}^{\infty}_{\mathcal{A}} f(\Sigma) \leq h_\varepsilon(\Sigma) + \varepsilon = \lim_{p\to\infty} (h_\varepsilon)_p(\Sigma) + \varepsilon \leq \lim_{p\to\infty} (Q^{\infty}_{\mathcal{A}} f)_p + \varepsilon$$

and, by the arbitrariness of $\varepsilon > 0$, we obtain that

$$\mathcal{Q}^{\infty}_{\mathcal{A}} f(\Sigma) \leq \lim_{p\to\infty} (Q^{\infty}_{\mathcal{A}} f)_p(\Sigma) \leq \mathcal{Q}^{\infty}_{\mathcal{A}} f(\Sigma).$$

3. Since by definition $\mathcal{Q}_{\mathcal{A}}^{\infty} f \leq f$, we get that for every $p > 1$ $(\mathcal{Q}_{\mathcal{A}}^{\infty} f)_p \leq f_p$. Since $\mathcal{Q}_{\mathcal{A}}^{\infty} f$ is $\mathcal{A}$-∞ quasiconvex, passing to the limit as $p \to \infty$, we obtain $\mathcal{Q}_{\mathcal{A}}^{\infty} f \leq \tilde{f}$. In order to show the converse inequality, we recall that by definition $f_p \leq f$ for every $p > 1$, passing to the limit as $p \to \infty$, we obtain that $\tilde{f} \leq f$. Therefore if we show that $\tilde{f}$ is $\mathcal{A}$-∞ quasiconvex, we get that $\tilde{f} \leq \mathcal{Q}_{\mathcal{A}}^{\infty} f$ which concludes the proof of step 3.

   We already know that f_p^p is $\mathcal{A}$-quasiconvex and $f_p \leq \tilde{f}$; hence,

$$
\begin{aligned}
f_p(\Sigma) &\leq \Big(\int_Q f_p^p(\Sigma + V(x))\, dx \Big)^{1/p} \\
&\leq \Big(\int_Q \tilde{f}^p(\Sigma + V(x))\, dx \Big)^{1/p} \\
&\leq \operatorname*{ess\,sup}_{x \in Q} \tilde{f}(\Sigma + V(x))
\end{aligned}
\tag{3.21}
$$

   for every $p > 1$ and $V \in L^{\infty}_{\#}(Q; \mathbb{M}^{d\times N}) \cap \operatorname{Ker} \mathcal{A}$, $\int_Q V\, dx = 0$. By (3.21), we have that

$$
\begin{aligned}
\tilde{f}(\Sigma) &= \lim_{p\to\infty} f_p(\Sigma) \\
&\leq \lim_{p\to\infty} \inf\Big\{ \big(\int_Q \tilde{f}^p(\Sigma + V(x))\, dx \big)^{1/p} : V \in L^{\infty}_{\#}(Q; \mathbb{M}^{d\times N}) \cap \operatorname{Ker} \mathcal{A},\ \int_Q V\, dx = 0 \Big\} \\
&\leq \inf\Big\{ \operatorname*{ess\,sup}_{x\in Q} \tilde{f}(\Sigma + V(x)) : V \in L^{\infty}_{\#}(Q; \mathbb{M}^{d\times N}) \cap \operatorname{Ker} \mathcal{A},\ \int_Q V\, dx = 0 \Big\} \\
&\leq \tilde{f}(\Sigma).
\end{aligned}
$$

   By Definition 3.3 we get that $\tilde{f}$ is $\mathcal{A}$-∞ quasiconvex.

□

Proposition 3.9 *Let $f : \mathbb{M}^{d\times N} \to [0, +\infty)$ be a continuous function satisfying weak growth condition: there exists $\alpha > 0$ such that*

$$
f(\Sigma) \geq \alpha |\Sigma| \qquad \text{for every } \Sigma \in \mathbb{M}^{d\times N}.
$$

Then for every $p > 1$, the function f_p given by (3.16) is continuous. In particular $Q_{\mathcal{A}}^{\infty} f$ is a lower semicontinuous function.

Proof. We start by proving that, for every $p > 1$, f_p is upper semicontinuous; i.e., for every sequence $(\Sigma_n) \in \mathbb{M}^{d\times N}$ converging to $\Sigma \in \mathbb{M}^{d\times N}$ we have that

$$
\limsup_{n\to\infty} f_p(\Sigma_n) \leq f_p(\Sigma)\,. \tag{3.22}
$$

Without loss of generality we may assume that $f_p(\Sigma) < +\infty$. Let $V_\varepsilon \in L^{\infty}_{\#}(Q; \mathbb{M}^{d\times N}) \cap \operatorname{Ker} \mathcal{A}$ be such that $\int_Q V_\varepsilon\, dx = 0$ and

$$
f_p(\Sigma) \geq \Big(\int_Q f^p(\Sigma + V_\varepsilon(x)) dx \Big)^{1/p} - \varepsilon.
$$

By definition of f_p we have that

$$
f_p(\Sigma_n) \leq \Big(\int_Q f^p(\Sigma_n + V(x)) dx \Big)^{1/p}
$$

for every $V \in L^{\infty}_{\#}(Q; \mathbb{M}^{d\times N}) \cap \operatorname{Ker} \mathcal{A}$, $\int_Q V\, dx = 0$. In particular, taking V_ε as test function we have

$$
\limsup_{n\to\infty} f_p(\Sigma_n) \leq \limsup_{n\to\infty} \Big(\int_Q f^p(\Sigma_n + V_\varepsilon(x)) dx \Big)^{1/p}.
$$

Since f is continuous and $V_\varepsilon \in L^\infty_\#(Q;\mathbb{M}^{d\times N}) \cap \operatorname{Ker}\mathcal{A}$, applying the Lebesgue theorem, we have that

$$\limsup_{n\to\infty} f_p(\Sigma_n) \le \Big(\int_Q f^p(\Sigma + V_\varepsilon(x))dx\Big)^{1/p} \le f_p(\Sigma) + \varepsilon\,.$$

By the arbitrariness of ε we get (3.22).

Let us deal with the lower semicontinuity of f_p; i.e., $f_p(\Sigma) \le \liminf_{n\to\infty} f_p(\Sigma_n)$. Since f is continuous and satisfies the weak growth condition, by Remark 2.7 (2)-(3) the functional

$$V \mapsto \int_Q f_p^p(V)\,dx = \int_Q (\mathcal{Q}_\mathcal{A} f^p)(V)\,dx$$

is weak* lower semicontinuous in L^∞. Hence, for every converging sequence $\Sigma_n \to \Sigma$ we have that

$$f_p^p(\Sigma) = \int_Q f_p^p(\Sigma)\,dx \le \liminf_{n\to\infty} \int_Q f_p^p(\Sigma_n)\,dx = \liminf_{n\to\infty} f_p^p(\Sigma_n)\,,$$

which concludes the proof of the lower semicontinuity of f_p. By (3.22) there follows that f_p is continuous. In particular, since $Q^\infty_\mathcal{A} f = \sup_p f_p$, we can conclude that $Q^\infty_\mathcal{A} f$ is lower semicontinuous. □

We can prove now that, under suitable growth conditions, the $\mathcal{A}$-∞ quasiconvex functions satisfy a Jensen inequality for a particular class of Young measures.

Proposition 3.10 *Let $V, (V_n) \subset L^1(\Omega, \mathbb{M}^{d\times N})$ be such that $\mathcal{A}V_n = 0$, $V_n \rightharpoonup V$ weakly in $L^q(\Omega;\mathbb{M}^{d\times N})$ for every $1 < q < \infty$. Assume that (V_n) generates a Young measure μ. Then there exists a negligible set $N \subset \Omega$ such that*

$$f(\int_{\mathbb{M}^{d\times N}} \Sigma d\mu_x(\Sigma)) \le \mu_x\text{-}\operatorname*{ess\,sup}_{\Sigma\in\mathbb{M}^{d\times N}} f(\Sigma)$$

for every $x \in \Omega \setminus N$ and for every continuous and $\mathcal{A}$-∞ quasiconvex function $f : \mathbb{M}^{d\times N} \to [0, +\infty)$ satisfying the following standard growth conditions: there exist $\alpha, \beta > 0$ such that

$$\alpha|\Sigma| \le f(\Sigma) \le \beta(|\Sigma| + 1) \qquad \textit{for every } \Sigma \in \mathbb{M}^{d\times N}.$$

Proof. Since the sequence $(V_n)_n$ is weakly converging in L^m, we have that (V_n) is m-equi-integrable. Moreover, by Propositions 3.9 and 3.8 we have that f_m^m is a continuous and $\mathcal{A}$-quasiconvex function satisfying (2.14) and $f_m \le f$ (where f_m is given by (3.16) with m in place of p).

By Proposition 2.12 applied to the subsequence $(V_n)_n$, with the function f_m^m in place of f and m in place of p, we have that for every $m \ge 1$ there exists a negligible set $E_m \subset \Omega$ (independent on f) such that

$$f_m^m(V(x)) \le \langle \mu_x, f_m^m\rangle \le \langle \mu_x, f^m\rangle = \int_{\mathbb{M}^{d\times N}} f^m(\Sigma)\,d\mu_x(\Sigma)$$

for every $x \in \Omega \setminus E_m$. In particular, we have that

$$f_m(V(x)) \le \Big(\int_{\mathbb{M}^{d\times N}} f^m(\Sigma)\,d\mu_x(\Sigma)\Big)^{1/m}$$

for every $m \ge 1$ and for every $x \in \Omega \setminus \bigcup_m E_m$. Hence, passing to the limit as $m \to \infty$, we obtain that

$$\lim_{m\to\infty} f_m(V(x)) \le \mu_x\text{-}\operatorname*{ess\,sup}_{\Sigma\in\mathbb{M}^{d\times N}} f(\Sigma)$$

for every $x \in \Omega \setminus \bigcup_m E_m$. Since f is $\mathcal{A}$-∞-quasiconvex, we can conclude that

$$f(\int_{\mathbb{M}^{d\times N}} \Sigma d\mu_x(\Sigma)) = f(V(x)) = \lim_{m\to\infty} f_m(V(x)) \le \mu_x\text{-}\operatorname*{ess\,sup}_{\Sigma\in\mathbb{M}^{d\times N}} f(\Sigma)$$

for a.e. $x \in \Omega$. □

4 The L^p-approximation theorems

In this section we study the L^p-approximation, via Γ-convergence, of supremal functional under differential constraint with respect to the L^1-strong topology (see Theorem 4.1) and to the L^∞- weak* topology (see Theorems 4.2). The results obtained generalize the Γ-convergence theorems proved by Bocea and Nesi in [9] and by Champion, De Pascale, and Prinari in [14]. We prove also that the $\mathcal{A}$-∞ quasiconvexity is a necessary and sufficient condition for the L^p-approximation with respect to the L^∞- weak* topology (see Theorem 4.4).

We start stating all theorems to easily compare the results obtained according to the different hypotheses and topologies considered. In Section 4.1 we collect the proofs of the theorems.

Theorem 4.1 *Let $f : \Omega \times \mathbb{M}^{d\times N} \to [0,+\infty)$ be a Carathéodory function such that $f(x,\cdot)$ is $\mathcal{A}$-∞ quasiconvex for a.e. $x \in \Omega$ and satisfying the standard growth conditions: there exist $\alpha, \beta > 0$ such that*

$$\alpha|\Sigma| \le f(x,\Sigma) \le \beta(|\Sigma|+1) \qquad \text{for a.e } x \in \Omega, \text{ for every } \Sigma \in \mathbb{M}^{d\times N}. \tag{4.23}$$

Let $F_p : L^1(\Omega;\mathbb{M}^{d\times N}) \to \mathbb{R}\cup\{+\infty\}$ be the functional defined by

$$F_p(V) := \begin{cases} \left(\int_\Omega f^p(x,V(x))dx\right)^{1/p} & \text{if } V \in L^p(\Omega;\mathbb{M}^{d\times N}) \cap Ker\mathcal{A}, \\ +\infty & \text{otherwise,} \end{cases} \tag{4.24}$$

and let $F : L^1(\Omega;\mathbb{M}^{d\times N}) \to \mathbb{R}\cup\{+\infty\}$ be the functional defined by

$$F(V) := \begin{cases} \operatorname*{ess\,sup}_\Omega f(x,V(x)) & \text{if } V \in L^\infty(\Omega;\mathbb{M}^{d\times N}) \cap Ker\mathcal{A}, \\ +\infty & \text{otherwise.} \end{cases} \tag{4.25}$$

Then,

(i) for every $V \in L^1(\Omega;\mathbb{M}^{d\times N})$ and $(V_p) \subset L^1(\Omega;\mathbb{M}^{d\times N})$ such that $V_p \rightharpoonup V$ weakly in $L^1(\Omega;\mathbb{M}^{d\times N})$, we have

$$F(V) \le \liminf_{p\to\infty} F_p(V_p);$$

(ii) for every $V \in L^1(\Omega;\mathbb{M}^{d\times N})$ there exists $(V_p) \subset L^1(\Omega;\mathbb{M}^{d\times N})$ such that $V_p \to V$ strongly in $L^1(\Omega;\mathbb{M}^{d\times N})$ and

$$\limsup_{p\to\infty} F_p(V_p) \le F(V).$$

In particular, (F_p) Γ- converges to F, as $p \to +\infty$, with respect to the L^1- strong convergence.

Theorem 4.2 *Let $f : \Omega \times \mathbb{M}^{d\times N} \to [0,+\infty)$ be a Carathéodory function satisfying the weak growth condition: there exists $\alpha > 0$ such that*

$$f(x,\Sigma) \ge \alpha|\Sigma| \qquad \text{for a.e } x \in \Omega, \text{ for every } \Sigma \in \mathbb{M}^{d\times N}. \tag{4.26}$$

Let $F_p : L^\infty(\Omega;\mathbb{M}^{d\times N}) \to \mathbb{R}\cup\{+\infty\}$ be the functional defined by

$$F_p(V) := \begin{cases} \left(\int_\Omega f^p(x,V(x))dx\right)^{1/p} & \text{if } V \in L^\infty(\Omega;\mathbb{M}^{d\times N}) \cap Ker\mathcal{A}, \\ +\infty & \text{otherwise,} \end{cases} \tag{4.27}$$

and let $\tilde{F} : L^\infty(\Omega;\mathbb{M}^{d\times N}) \to \mathbb{R}\cup\{+\infty\}$ be the functional defined by

$$\tilde{F}(V) := \begin{cases} \operatorname*{ess\,sup}_{x\in\Omega} \mathcal{Q}^\infty_\mathcal{A} f(x,V(x)) & \text{if } V \in L^\infty(\Omega;\mathbb{M}^{d\times N}) \cap Ker\,\mathcal{A}, \\ +\infty & \text{otherwise} \end{cases} \tag{4.28}$$

where $Q^\infty_\mathcal{A} f(x,\cdot)$ is the $\mathcal{A}$-∞ quasiconvex envelope of $f(x,\cdot)$. Then (F_p) Γ- converges to the functional $\tilde{F}$, as $p \to +\infty$, with respect to the L^∞- weak convergence.*

In particular, if $f(x,\cdot)$ is also $\mathcal{A}$-∞ quasiconvex for a.e. $x \in \Omega$, then (F_p) Γ- converges to the functional F given by (4.25) with respect to L^∞- weak convergence.*

Remark 4.3 Let $f : \Omega \times \mathbb{M}^{d\times N} \to [0, +\infty)$ be a Carathéodory function satisfying the standard growth conditions (4.23) and let F_p be given by (4.27). It is easy to show that F_p Γ- converges to the functional F given by (4.25) with respect to the L^∞-strong convergence without additional assumptions.

Note that Theorem 4.2 imply that the $\mathcal{A}$-∞ quasiconvexity is sufficient to get the L^p-approximation of the supremal functional F with respect to the L^∞-weak* convergence. The following theorem shows that if f does not depend explicitly on x then the $\mathcal{A}$-∞ quasiconvexity is also a necessary condition for the L^p-approximation of the supremal functional F.

Theorem 4.4 *Let $f : \mathbb{M}^{d\times N} \to [0, +\infty)$ be a continuous function satisfying the weak growth condition (4.26). Let $F_p, F : L^\infty(\Omega; \mathbb{M}^{d\times N}) \to \mathbb{R} \cup \{+\infty\}$ be given by*

$$F_p(V) := \begin{cases} \left(\int_\Omega f^p(V(x))dx\right)^{1/p} & \text{if } V \in L^\infty(\Omega; \mathbb{M}^{d\times N}) \cap Ker\,\mathcal{A}, \\ +\infty & \text{otherwise}, \end{cases} \tag{4.29}$$

and

$$F(V) := \begin{cases} \operatorname*{ess\,sup}_{x\in\Omega} f(V(x)) & \text{if } V \in L^\infty(\Omega; \mathbb{M}^{d\times N}) \cap Ker\,\mathcal{A}, \\ +\infty & \text{otherwise}, \end{cases} \tag{4.30}$$

respectively. Then the following statement are equivalent:

(i) f is $\mathcal{A}$-∞ quasiconvex function;

(ii) F_p Γ-converges to F, as $p \to \infty$, with respect to the L^∞- weak topology.*

4.1 Proofs of Theorems

We first prove the following lemma.

Lemma 4.5 *Let $f : \Omega \times \mathbb{M}^{d\times N} \to [0, +\infty)$ be a Carathéodory function. Then*

$$\liminf_{q\to\infty} \left(\int_\Omega \int_{\mathbb{M}^{d\times N}} f(x,\Sigma)^q d\mu_x(\Sigma)dx\right)^{1/q} = \operatorname*{ess\,sup}_{x\in\Omega} \big(\mu_x\text{-}\operatorname*{ess\,sup}_{\Sigma\in\mathbb{M}^{d\times N}} f(x,\Sigma)\big),$$

for every Young measure $(\mu_x)_{x\in\Omega}$.

Proof. The following inequality

$$\liminf_{q\to\infty} \left(\int_\Omega \int_{\mathbb{M}^{d\times N}} f(x,\Sigma)^q d\mu_x(\Sigma)dx\right)^{1/q} \le \operatorname*{ess\,sup}_{x\in\Omega} \big(\mu_x\text{-}\operatorname*{ess\,sup}_{\Sigma\in\mathbb{M}^{d\times N}} f(x,\Sigma)\big)$$

is straighforward. Let us prove the converse inequality. Without loss of generality we assume that

$$\liminf_{q\to\infty} \left(\int_\Omega \int_{\mathbb{M}^{d\times N}} f(x,\Sigma)^q d\mu_x(\Sigma)dx\right)^{1/q} < +\infty. \tag{4.31}$$

For every fixed $q > r$, by the convexity of $t \mapsto t^{q/r}$ on $[0, +\infty)$, we can apply the Jensen's inequality and we get that

$$\left(\int_\Omega \int_{\mathbb{M}^{d\times N}} f(x,\Sigma)^q d\mu_x(\Sigma)dx\right)^{1/q} \ge \left(\int_\Omega \left(\int_{\mathbb{M}^{d\times N}} f(x,\Sigma)^r d\mu_x(\Sigma)\right)^{q/r} dx\right)^{1/q}. \tag{4.32}$$

Passing to the limit as $q \to \infty$, by the convergence of the L^q-norm to the L^∞-norm, we have that

$$\lim_{q\to\infty} \left(\int_\Omega \left(\int_{\mathbb{M}^{d\times N}} f(x,\Sigma)^r d\mu_x(\Sigma)\right)^{q/r} dx\right)^{1/q} = \operatorname*{ess\,sup}_{x\in\Omega} \left(\int_{\mathbb{M}^{d\times N}} f(x,\Sigma)^r d\mu_x(\Sigma)\right)^{1/r}. \tag{4.33}$$

We now denote

$$g_r(x) := \left(\int_{\mathbb{M}^{d\times N}} f(x,\Sigma)^r d\mu_x(\Sigma) \right)^{1/r}.$$

Then (g_r) is an increasing positive sequence pointwise converging to the function

$$g(x) := \mu_x\text{- }\operatorname*{ess\,sup}_{\Sigma\in\mathbb{M}^{d\times N}} f(x,\Sigma)$$

as $r \to \infty$. Moreover, by (4.31)-(4.33), we have that $\sup_r ||g_r||_\infty < +\infty$. In particular, by Lebesgue Theorem, we have that $g_r \rightharpoonup g$ weakly* in L^∞. By (4.32), (4.33) and the weak* lower semicontinuity of the L^∞-norm, we have that

$$\liminf_{q\to\infty} \left(\int_\Omega \int_{\mathbb{M}^{d\times N}} f(x,\Sigma)^q d\mu_x(\Sigma) dx \right)^{1/q} \geq \operatorname*{ess\,sup}_{x\in\Omega} \big(\mu_x\text{- }\operatorname*{ess\,sup}_{\Sigma\in\mathbb{M}^{d\times N}} f(x,\Sigma)\big),$$

which concudes the proof. □

Proof of Theorem 4.1. The limsup inequality (ii) easily follows by the convergence of the L^p-norm to the L^∞-norm. We now deal with the liminf inequality (i). Let $(V_p) \in L^1(\Omega;\mathbb{M}^{d\times N})$ be a sequence L^1-weakly converging to $V \in L^1(\Omega;\mathbb{M}^{d\times N})$. Without loss of generality, we can assume that $M = \liminf_{p\to\infty} F_p(V_p) < +\infty$; hence, we have in particular that there exists $p_0 > 1$ such that

$$F_p(V_p) \leq M+1$$

for every $p \geq p_0$. This implies that $V_p \in L^p(\Omega;\mathbb{M}^{d\times N}) \cap \operatorname{Ker}\mathcal{A}$, for every $p \geq p_0$. By Hölder's inequality and (4.23) we have that

$$\begin{aligned} \Big(\int_\Omega |V_p(x)|^q dx\Big)^{1/q} &\leq \mathcal{L}^N(\Omega)^{1/q-1/p}\Big(\int_\Omega |V_p(x)|^p dx\Big)^{1/p} \\ &\leq \mathcal{L}^N(\Omega)^{1/q-1/p}\frac{1}{\alpha}F_p(V_p) \\ &\leq \mathcal{L}^N(\Omega)^{1/q}\frac{M+1}{\alpha} \end{aligned} \tag{4.34}$$

for every $p \geq q \geq p_0$. Then $(V_p)_{p\geq q}$ is bounded in $L^q(\Omega,\mathbb{M}^{d\times N})$ and it converges weakly in $L^q(\Omega;\mathbb{M}^{d\times N})$ to V for every $q \geq p_0$. We now prove that $V \in L^\infty(\Omega,\mathbb{M}^{d\times N})$. By (4.34) we have that

$$\Big(\int_\Omega |V(x)|^q dx\Big)^{1/q} \leq \liminf_{p\to\infty}\Big(\int_\Omega |V_p(x)|^q dx\Big)^{1/q} \leq \mathcal{L}^N(\Omega)^{1/q}\frac{M+1}{\alpha} \tag{4.35}$$

for every $q \geq p_0$. Moreover, for every $x_0 \in \Omega$, $r > 0$ and $q \geq p_0$, by Hölder's inequality, we have that

$$\begin{aligned} \fint_{B_r(x_0)} |V(x)|dx &\leq \Big(\fint_{B_r(x_0)} |V(x)|^q dx\Big)^{1/q} \\ &\leq \mathcal{L}^N(B_r(x_0))^{-1/q}\Big(\int_\Omega |V(x)|^q dx\Big)^{1/q}. \end{aligned}$$

Letting $q \to +\infty$ and using (4.35) we get

$$\fint_{B_r(x_0)} |V(x)|dx \leq \frac{M+1}{\alpha}$$

for every $r > 0$ and for every $x_0 \in \Omega$. In particular, if x_0 is a Lebesgue point of V, it follows that

$$|V(x_0)| \leq \lim_{r\to 0^+} \fint_{B_r(x_0)} |V(x)|dx \leq \frac{M+1}{\alpha}$$

which implies that $V \in L^\infty(\Omega;\mathbb{M}^{d\times N})$. In particular, $V \in L^\infty(\Omega,\mathbb{M}^{d\times N}) \cap \operatorname{Ker}\mathcal{A}$. In fact,

$$\langle \mathcal{A}V, \phi\rangle = \lim_{p\to+\infty} \langle \mathcal{A}V_p, \phi\rangle = 0, \qquad \forall \phi \in C_0^\infty(\Omega,\mathbb{M}^{d\times N}).$$

By the density of the $C_0^\infty(\Omega,\mathbb{M}^{d\times N})$ functions in $W_0^{1,1}(\Omega,\mathbb{M}^{d\times N})$ with respect to the strong convergence we get that also V satisfies the constraint $\mathcal{A}V = 0$.

Since (V_p) is L^1-weakly converging then (V_p) is also equi-integrable; hence, by Remark 2.10(3), we have that (V_p) generates a Young measure $(\mu_x)_{x\in\Omega}$ such that

$$V(x) = \int_{\mathbb{R}^{d\times N}} \Sigma \, d\mu_x(\Sigma)$$

for a.e. $x \in \Omega$. In particular, by Theorem 2.9(5) for any fixed $q > 1$, we have that

$$\begin{aligned} \liminf_{p\to\infty}\ F_q(V_p) &= \liminf_{p\to\infty} \left(\int_\Omega f^q(x, V_p(x))dx \right)^{1/q} \\ &\geq \left(\int_\Omega \int_{\mathbb{M}^{d\times N}} f^q(x,\Sigma) d\mu_x(\Sigma) dx \right)^{1/q}. \end{aligned}$$

Applying Lemma 4.5 we obtain

$$\liminf_{q\to\infty} \liminf_{p\to\infty}\ F_q(V_p) \geq \operatorname*{ess\,sup}_{x\in\Omega} \big(\mu_x\text{-} \operatorname*{ess\,sup}_{\Sigma\in\mathbb{M}^{d\times N}} f(x,\Sigma)\big). \tag{4.36}$$

Now, by assumption $f(y,\cdot)$ is $\mathcal{A}$-∞ quasiconvex for a.e. $y \in \Omega$; hence, we denote by

$$\Omega' := \{ y \in \Omega : \ f(y,\cdot) \text{ is } \mathcal{A}\text{-}\infty \text{ quasiconvex} \}.$$

Note that $\mathcal{L}^N(\Omega \setminus \Omega') = 0$. Since $(V_p)_{p\geq q}$ is bounded in $L^q(\Omega,\mathbb{M}^{d\times N})$, by Proposition 3.10 there exists a negligible set $N \subset \Omega$ such that

$$f(y, V(x)) \leq \mu_x\text{-} \operatorname*{ess\,sup}_{\Sigma\in\mathbb{M}^{d\times N}} f(x,\Sigma)$$

for every $y \in \Omega'$ and $x \in \Omega \setminus N$. In particular for every $x \in \Omega' \cap (\Omega \setminus N)$ we have that

$$f(x, V(x)) \leq \mu_x\text{-} \operatorname*{ess\,sup}_{\Sigma\in\mathbb{M}^{d\times N}} f(x,\Sigma).$$

Therefore

$$\operatorname*{ess\,sup}_{x\in\Omega} f(x,V(x)) \leq \operatorname*{ess\,sup}_{x\in\Omega} \big(\mu_x\text{-} \operatorname*{ess\,sup}_{\Sigma\in\mathbb{M}^{d\times N}} f(x,\Sigma)\big). \tag{4.37}$$

Finally, gathering (4.36) and (4.37), we infer

$$\begin{aligned} F(V) &= \operatorname*{ess\,sup}_{x\in\Omega} f(x,V(x)) \\ &\leq \operatorname*{ess\,sup}_{x\in\Omega} \big(\mu_x\text{-} \operatorname*{ess\,sup}_{\Sigma\in\mathbb{M}^{d\times N}} f(x,\Sigma)\big) \\ &\leq \liminf_{q\to\infty} \liminf_{p\to\infty}\ F_q(V_p) \\ &\leq \liminf_{q\to\infty} \liminf_{p\to\infty}\ \mathcal{L}^N(\Omega)^{1/q-1/p} F_p(V_p) \\ &= \liminf_{p\to\infty}\ F_p(V_p) \end{aligned} \tag{4.38}$$

which implies the liminf inequality. □

Remark 4.6 In the proof of Theorem 4.1 we deal with L^1-weakly convergent sequences (V_p) that are bounded in $L^q(\Omega;\mathbb{M}^{d\times N})$ for every $p_0 \leq q < +\infty$, for some $p_0 > 1$, and we prove that their limit functions are in $L^\infty(\Omega;\mathbb{M}^{d\times N})$. In view of Theorem 4.2 we want to observe that the following boundedness condition (see (4.34))

$$\sup_{1\leq q<+\infty} \sup_{p\geq q} ||V_p||_{L^q(\Omega,\mathbb{M}^{d\times N})} < +\infty\,,$$

does not imply that (V_p) is also bounded in $L^\infty(\Omega;\mathbb{M}^{d\times N})$ as the counter-example below shows. Therefore it does not give rise to a $L^\infty(\Omega;\mathbb{M}^{d\times N})$-weak* convergence to V.

Let us consider, the sequence $V_p(x) = \frac{1}{p}\log x$ where $x \in (0,1)$. For every $1 \leq q < +\infty$ we have that $(V_p) \subset L^q(\Omega,\mathbb{M}^{d\times N})$, $V_p \rightharpoonup 0$ weakly in $L^q(\Omega,\mathbb{M}^{d\times N})$ and it is not bounded in $L^\infty(\Omega,\mathbb{M}^{d\times N})$. Nevertheless, for every $1 \leq q < +\infty$, by Hölder's inequality, we have that

$$I_q := \Big(\int_0^1 |\log x|^q dx\Big)^{1/q} = \Big(\int_0^1 q|\log x|^{q-1} dx\Big)^{1/q} \leq q^{\frac{1}{q}} (I_q)^{\frac{q-1}{q}}$$

which implies that

$$I_q \leq q.$$

Therefore (V_p) satisfies the condition

$$\sup_{1\leq q<+\infty} \sup_{p\geq q} ||V_p||_{L^q(\Omega,\mathbb{M}^{d\times N})} \leq \sup_{q\geq 1} \frac{1}{q}\Big(\int_0^1 |\log x|^q dx\Big)^{1/q} \leq 1.$$

Proof of Theorem 4.2. Let us consider the sequence of functionals $(\mathcal{F}_p)$ given by

$$\mathcal{F}_p(V) := \begin{cases} \Big(\int_\Omega f_p^p(x,V(x))dx\Big)^{1/p} & \text{if } V \in L^\infty(\Omega,\mathbb{M}^{d\times N}) \cap \mathrm{Ker}\mathcal{A}, \\ +\infty & \text{otherwise.} \end{cases}$$

Since $f_p^p = \mathcal{Q}_\mathcal{A} f^p$, by Remark 2.7(3) we have that for every p the functional $\mathcal{F}_p$ is the lower semicontinuous envelope of the functional F_p on $L^\infty(\Omega,\mathbb{M}^{d\times N}) \cap \mathrm{Ker}\mathcal{A}$ with respect to the L^∞-weak* topology. Moreover $(\mathcal{F}_p)$ is non-decreasing. Therefore, by [16] Proposition 5.4, we have that

$$\Gamma(w^*\text{-}L^\infty)\text{-}\lim_{p\to\infty} F_p(V) = \lim_{p\to\infty} \mathcal{F}_p(V) = \sup_{p>1} \mathcal{F}_p(V). \tag{4.39}$$

We recall that (f_p) is an increasing sequence pointwise converging to $\mathcal{Q}_\mathcal{A}^\infty f$ (see Proposition 3.8(3)); hence, for every $p > 1$

$$\mathcal{F}_p(V) = \Big(\int_\Omega f_p^p(x,V(x))dx\Big)^{1/p} \leq \mathcal{L}^N(\Omega)^{\frac{1}{p}} \operatorname*{ess\,sup}_{x\in\Omega} \mathcal{Q}_\mathcal{A}^\infty f(x,V(x))$$

for every $V \in L^\infty(\Omega,\mathbb{M}^{d\times N}) \cap \mathrm{Ker}\mathcal{A}$. In particular,

$$\sup_{p>1} \mathcal{F}_p(V) \leq \lim_{p\to\infty} \mathcal{L}^N(\Omega)^{\frac{1}{p}} \tilde{F}(V) = \tilde{F}(V) \tag{4.40}$$

By (4.39) and (4.40) we get that

$$\Gamma(w^*\text{-}L^\infty)\text{-}\lim_{p\to\infty} F_p(V) \leq \tilde{F}(V)$$

for every $V \in L^\infty(\Omega,\mathbb{M}^{d\times N})$.

We now prove the converse inequality; i.e.,

$$\Gamma(w^*\text{-}L^\infty)\text{-}\lim_{p\to\infty} F_p(V) \geq \tilde{F}(V)$$

for every $V \in L^\infty(\Omega,\mathbb{M}^{d\times N})$. Let V be such that $\tilde{F}(V) < \infty$. Hence, $V \in L^\infty(\Omega,\mathbb{M}^{d\times N}) \cap \mathrm{Ker}\mathcal{A}$ and, by (4.39), it is sufficient to show that $\sup_{p>1} \mathcal{F}_p(V) \geq \tilde{F}(V)$.

For every fixed $\varepsilon > 0$ there exists a measurable set $B_\varepsilon \subset \Omega$ such that $\mathcal{L}^N(B_\varepsilon) > 0$ and

$$\operatorname*{ess\,sup}_{x\in\Omega} \mathcal{Q}^\infty_{\mathcal{A}} f(x, V(x)) \leq \mathcal{Q}^\infty_{\mathcal{A}} f(x, V(x)) + \varepsilon$$

for every $x \in B_\varepsilon$. This implies

$$\operatorname*{ess\,sup}_{x\in\Omega} \mathcal{Q}^\infty_{\mathcal{A}} f(x, V(x))\mathcal{L}^N(B_\varepsilon) \leq \int_{B_\varepsilon} \mathcal{Q}^\infty_{\mathcal{A}} f(x, V(x))dx + \varepsilon\mathcal{L}^N(B_\varepsilon).$$

By Proposition 3.8(3), Beppo Levi Theorem, and Hölder's inequality we obtain

$$\begin{aligned}\operatorname*{ess\,sup}_{x\in\Omega} Q^\infty_{\mathcal{A}} f(x, V(x))\mathcal{L}^N(B_\varepsilon) &\leq \lim_{p\to\infty} \int_{B_\varepsilon} f_p(x, V(x))dx + \varepsilon\mathcal{L}^N(B_\varepsilon)\\ &\leq \lim_{p\to\infty} \Big(\int_{B_\varepsilon} f_p^p(x, V(x))dx\Big)^{\frac{1}{p}} \mathcal{L}^N(B_\varepsilon)^{1-\frac{1}{p}} + \varepsilon\mathcal{L}^N(B_\varepsilon).\end{aligned}$$

This implies

$$\operatorname*{ess\,sup}_{x\in\Omega} \mathcal{Q}^\infty_{\mathcal{A}} f(x, V(x)) \quad\leq\quad \lim_{p\to\infty} \mathcal{F}_p(V)\mathcal{L}^N(B_\varepsilon)^{-\frac{1}{p}} + \varepsilon = \sup_{p>1} \mathcal{F}_p(V) + \varepsilon\,. \tag{4.41}$$

By (4.39), (4.41), and the arbitrariness of ε we have that

$$\Gamma(w^*\text{-}L^\infty)\text{-}\lim_{p\to\infty} F_p(V) \geq \operatorname*{ess\,sup}_{x\in\Omega} \mathcal{Q}^\infty_{\mathcal{A}} f(x, V(x)) = \tilde{F}(V)$$

for every $V \in L^\infty(\Omega; \mathbb{M}^{d\times N}) \cap \operatorname{Ker}\mathcal{A}$.
Let $V \in L^\infty(\Omega; \mathbb{M}^{d\times N})$ be such that $\tilde{F}(V) = +\infty$. In particular, we consider the non trivial case where $V \in L^\infty(\Omega, \mathbb{M}^{d\times N}) \cap \operatorname{Ker}\mathcal{A}$ and $\operatorname{ess\,sup}_{x\in\Omega} \mathcal{Q}^\infty_{\mathcal{A}} f(x, V(x)) = +\infty$. Hence, for every fixed $M > 0$ there exists a measurable set $B_M \subset \Omega$ such that $\mathcal{L}^N(B_M) > 0$ and for every $x \in B_M$

$$Q^\infty_{\mathcal{A}} f(x, V(x)) > M\,.$$

Let $\delta > 0$ be such that $\mathcal{L}^N(B_M) > \delta$, by Egoroff Theorem there exists E_δ such that $\mathcal{L}^N(E_\delta) < \delta$ and

$$\lim_{p\to\infty} \|f_p(\cdot, V(\cdot)) - \mathcal{Q}^\infty_{\mathcal{A}} f(\cdot, V(\cdot))\|_{L^\infty(\Omega\setminus E_\delta;\mathbb{M}^{d\times N})} = 0\,.$$

There follows that for every $\varepsilon > 0$ there exists p_ε such that for every $p > p_\varepsilon$

$$f_p(x, V(x)) - \mathcal{Q}^\infty_{\mathcal{A}} f(x, V(x)) > -\varepsilon$$

for every $x \in \Omega \setminus E_\delta$; hence, in particular

$$f_p(x, V(x)) > M - \varepsilon\,, \quad \forall x \in B_M \setminus E_\delta\,.$$

Then, we have

$$\mathcal{F}_p(V) = \left(\int_\Omega f_p^p(x, V(x))dx\right)^{1/p} \geq (M - \varepsilon)\mathcal{L}^N(B_M \setminus E_\delta)^{1/p}\,.$$

Passing to the limit as $p \to +\infty$ we get, by the arbitrariness of ε, that for every fixed $M > 0$

$$\sup_{p>1} \mathcal{F}_p(V) \geq M\,;$$

hence, also $\sup_{p>1} \mathcal{F}_p(V) = +\infty$; i.e., $\sup_{p>1} \mathcal{F}_p(V) = \tilde{F}(V)$. □

Proof of Theorem 4.4. (i) $\Longrightarrow$ (ii): follows by Theorem 4.2. (ii) $\Longrightarrow$ (i): by Theorem 4.2 we have that

$$\Gamma(w^*\text{-}L^\infty)\text{-}\lim_{p\to\infty} F_p(V) = \tilde{F}(V)$$

for every $V \in L^\infty(\Omega;\mathbb{M}^{d\times N})$, with $\tilde{F}$ given by (4.28). Therefore, by assumption we have that

$$\operatorname*{ess\,sup}_{x\in\Omega} \mathcal{Q}^\infty_{\mathcal{A}} f(V(x)) = F(V) = \operatorname*{ess\,sup}_{x\in\Omega} f(V(x))$$

for every $V \in L^\infty(\Omega;\mathbb{M}^{d\times N}) \cap \operatorname{Ker}\mathcal{A}$. In particular, we get

$$Q^\infty_{\mathcal{A}} f(\Sigma) = f(\Sigma)$$

for every $\Sigma \in \mathbb{M}^{d\times N}$. By Proposition 3.8(2), we can conclude that f is $\mathcal{A}$-∞ quasiconvex. □

5 Some remarks and examples

In this section we characterize the $\mathcal{A}$-∞ quasiconvex functions for some particular choice of the constant-rank operator $\mathcal{A}$ and of the dimension d and N.

5.1 $\mathcal{A}$-∞ quasiconvexity: some particular cases

We recall the following inequality characterizing the level convex function f:

$$f(t\Sigma_1 + (1-t)\Sigma_2) \le \max\{f(\Sigma_1)\,,\, f(\Sigma_2)\} \quad \forall t \in (0,1)\,,\ \Sigma^1 \neq \Sigma^2 \in \mathbb{M}^{d\times N}. \tag{5.42}$$

Proposition 5.1 *Let $f : \mathbb{M}^{d\times N} \mapsto [0,+\infty)$ be a Borel function.*

1. *If f is $\mathcal{A}$-weak quasiconvex function; i.e.,*

$$f(\Sigma) = \inf\Big\{\operatorname*{ess\,sup}_{x\in Q} f(\Sigma + V(x)) : V \in L^\infty_\#(Q;\mathbb{M}^{d\times N}) \cap Ker\,\mathcal{A},\ \int_Q V\,dx = 0\Big\}$$

   for every $\Sigma \in \mathbb{M}^{d\times N}$, then f satisfies (5.42) with $(\Sigma^1 - \Sigma^2) \in Ker\,\mathbb{A}(w)$ for every vector w of the canonical basis.

2. *If f is upper semicontinuous and $\mathcal{A}$-∞ quasiconvex then f satisfies (5.42) with $(\Sigma^1 - \Sigma^2) \in \Lambda$, where*

$$\Lambda := \bigcup_{w\in S^{N-1}} Ker\,\mathbb{A}(w)\,. \tag{5.43}$$

Proof.

1. Let $\Sigma^1, \Sigma^2 \in \mathbb{M}^{d\times N}$ and let $w \in \mathbb{R}^N$ be a vector of the canonical basis such that $(\Sigma^1 - \Sigma^2) \in \operatorname{Ker}\mathbb{A}(w)$. We define

$$V(x) = \begin{cases} (1-t)(\Sigma^1 - \Sigma^2)\,, & x \in A_1 \\ -t(\Sigma^1 - \Sigma^2)\,, & x \in A_2 \end{cases}$$

   where

$$A_1 = \{x \in \mathbb{R}^N : j < \langle x, w\rangle < j + t\,,\ j \in \mathbb{Z}\}\,,$$
$$A_2 = \{x \in \mathbb{R}^N : j + t < \langle x, w\rangle < j + 1\,,\ j \in \mathbb{Z}\}$$

   for fixed $t \in (0,1)$. Since $(\Sigma^1 - \Sigma^2) \in \operatorname{Ker}\mathbb{A}(w)$, we may easily check that $\mathcal{A}V = 0$ (see e.g. [3] Theorem 4.2 Step 3). Moreover, by construction $V \in L^\infty_\#(Q;\mathbb{M}^{d\times N})$ and satisfies $\int_Q V\,dx = 0$. Hence,

$$\begin{aligned} f(\Sigma) &\le \operatorname*{ess\,sup}_{x\in A_1\cup A_2} f(\Sigma + V(x)) \\ &= \max\{\operatorname*{ess\,sup}_{x\in A_1} f(\Sigma + (1-t)(\Sigma^1 - \Sigma^2))\,,\ \operatorname*{ess\,sup}_{x\in A_2} f(\Sigma - t(\Sigma^1 - \Sigma^2))\}\,. \end{aligned}$$

   In particular, for $\Sigma = t\Sigma^1 + (1-t)\Sigma^2$ we have that

$$f(t\Sigma^1 + (1-t)\Sigma^2) \le \max\{f(\Sigma^1)\,,\, f(\Sigma^2)\}\,.$$

2. Since f is upper semicontinuous, by [18] Proposition 3.4 we have that

$$\mathcal{Q}_{\mathcal{A}} f^p(\Sigma^1 + (1-t)\Sigma^2) \leq t\mathcal{Q}_{\mathcal{A}} f^p(\Sigma^1) + (1-t)\mathcal{Q}_{\mathcal{A}} f^p(\Sigma^2) \leq \max\{f^p(\Sigma^1)\,,\, f^p(\Sigma^2)\}\,.$$

for every $p \geq 1$ and $(\Sigma^1 - \Sigma^2) \in \Lambda$. By definition of f_p we have that

$$f_p(t\Sigma^1 + (1-t)\Sigma^2) \leq (\mathcal{Q}_{\mathcal{A}} f^p)^{1/p}(t\Sigma^1 + (1-t)\Sigma^2) \leq \max\{f(\Sigma^1)\,,\, f(\Sigma^2)\}$$

passing into the limit as $p \to +\infty$ we get that

$$f(t\Sigma^1 + (1-t)\Sigma^2) \leq \max\{f(\Sigma^1)\,,\, f(\Sigma^2)\}\,.$$

for every $\Sigma^1, \Sigma^2 \in \mathbb{M}^{d\times N}$ such that $(\Sigma^1 - \Sigma^2) \in \Lambda$.

□

Proposition 5.2 *1. ($\mathcal{A} = \operatorname{div}$, $d = 1$). Let $f : \mathbb{R}^N \mapsto [0, +\infty)$ be an upper semicontinuous and* div *-∞ quasiconvex function. Then f is level convex; i.e., f satisfies (5.42).*

2. ($\mathcal{A} = \operatorname{Div}$, $d \geq N > 1$). Let $f : \mathbb{M}^{d\times N} \mapsto [0, +\infty)$ be an upper semicontinuous and Div *-∞ quasiconvex function. Then f is rank-$(N-1)$ level convex; i.e., f satisfies (5.42) with $\operatorname{rank}(\Sigma^1 - \Sigma^2) \leq (N-1)$.*

3. ($\mathcal{A} = \operatorname{curl}$). Let $f : \mathbb{M}^{d\times N} \mapsto [0, +\infty)$ be an upper semicontinuous and curl*-weak quasiconvex function. Then f is rank-1 level convex; i.e., f satisfies (5.42) with $\operatorname{rank}(\Sigma^1 - \Sigma^2) \leq 1$. In particular, if either $d = 1$ or $N = 1$ then f is level convex.*

Proof.

1. If $d = 1$ and $\mathcal{A} = \operatorname{div}$ we can prove that

$$\operatorname{Ker} \mathbb{A}(w) = \{\xi \in \mathbb{R}^N : \langle \xi, w\rangle = 0\}$$

for every $w \in S^{N-1}$, which implies that $\Lambda = \mathbb{R}^N$. Note that, since for every $\xi^1 \neq \xi^2$ there always exists $w \in S^{N-1}$ such that $\langle \xi^1 - \xi^2, w\rangle = 0$, we have in particular that $(\xi^1 - \xi^2) \in \Lambda$. Hence, by Proposition 5.1(2), f satisfies (5.42) for every $\xi^1, \xi^2 \in \mathbb{R}^N$; i.e., f is level convex.

2. We recall that if $d > 1$, we define $\operatorname{Div} V : \Omega \mapsto \mathbb{R}^d$ such that

$$(\operatorname{Div} V)_i = \operatorname{div}(V)_i$$

for every $i = 1, \cdots, d$. Hence, assuming that $d \geq N > 1$ we can generalize the case $d = 1$ and prove that if $(\Sigma^1 - \Sigma^2) \in \Lambda$ then $\operatorname{rank}(\Sigma^1 - \Sigma^2) \leq (N-1)$. Hence, by Proposition 5.1(2) we have that f satisfies (5.42) for every $\Sigma^1 \neq \Sigma^2 \in \mathbb{M}^{d\times N}$ with $\operatorname{rank}(\Sigma^1 - \Sigma^2) \leq (N-1)$.

3. By Proposition 5.1(2) we have that f satisfies (5.42) for every $(\Sigma^1 - \Sigma^2) \in \Lambda$, where Λ is given by (5.43). By [18, Remark 3.3 (iii)] we have that

$$\operatorname{Ker} \mathbb{A}(w) = \{\xi \otimes w \in \mathbb{M}^{d\times N} : \ \xi \in \mathbb{R}^d, \ w \in S^{N-1}\}.$$

Therefore f is level convex along any rank-one directions; i.e., f is rank-1 level convex. It is easy to see that if either $d = 1$ or $N = 1$, then the rank-1 level convexity reduces to level convexity.

□

Remark 5.3 Higher gradients. Let $\mathcal{A}$ be the constant-rank operator defined by

$$\mathcal{A}V := \Big(\frac{\partial}{\partial x_i}V_{jk} - \frac{\partial}{\partial x_k}V_{ji}\Big)_{1\le i,j,k\le N}$$

for every $V \in C^\infty_\#(\mathbb{R}^N; E^d_2)$ with $E^d_2 :=$ {symmetric 2-linear maps $\mathbb{R}^N \mapsto \mathbb{R}^d$}. In this case

$$\Big\{V \in C^\infty_\#(Q; E^d_2) \cap \operatorname{Ker}\mathcal{A},\ \int_Q V\,dx = 0\Big\} = \{D^2 u:\ u \in C^\infty_\#(Q;\mathbb{R}^d)\}$$

and

$$\Lambda = \bigcup_{w\in S^{N-1}} \operatorname{Ker}\mathbb{A}(w) = \bigcup_{w\in S^{N-1}} \{\Sigma \in E^d_2:\ \Sigma = a\otimes w\otimes w \text{ for some } a \in \mathbb{R}^d\}$$

(see [18], Example 3.10 (d)). Hence, if f is upper semicontinuous and $\mathcal{A}$-∞ quasiconvex then, by Proposition 5.1(2), we have that f satisfies (5.42) for every $t \in (0,1)$ and $\Sigma^1 \neq \Sigma^2 \in \mathbb{M}^{d\times N}$ with $(\Sigma^1 - \Sigma^2) \in \Lambda$. In particular, if $d = 1$ we have that Λ is strictly included in the set of rank-1 matrices.

Proposition 5.4 *Let $f : \mathbb{M}^{d\times N} \to [0,+\infty)$ be a continuous function satisfying the weak growth condition (4.26).*

1. *If* $d = 1$ *then f is level convex* $\Longleftrightarrow$ div -∞ *quasiconvex.*
2. *If either* $d = 1$ *or* $N = 1$ *then f is level convex* $\Longleftrightarrow$ curl -*weak quasiconvex* $\Longleftrightarrow$ curl -∞ *quasiconvex.*

Proof. By Proposition 5.2 and Proposition 3.6(2)-(4) we get the thesis. □

5.2 Examples

In this section we discuss some examples which clarify the connection between the different classes of functions introduced in Section 3. More precisely, we start by constructing a $\mathcal{A}$-∞ quasiconvex function which is not $\mathcal{A}$-quasiconvex in the case $\mathcal{A} = \operatorname{curl}$. In particular such example allows us to conclude that the Γ-convergence result in Theorem 4.1 generalize the Theorem 3.2 in [9] proved by Bocea and Nesi. We recall that a function f is curl-quasiconvex if and only if f is quasiconvex.

Example 5.5 Let $f : \mathbb{M}^{d\times N} \to [0,+\infty)$ be the continuous function given by

$$f(\Sigma) := \begin{cases} |\Sigma| & \text{if } |\Sigma| \le 1 \\ 1 & \text{if } 1 \le |\Sigma| \le 2 \\ \frac{1}{2}|\Sigma| & \text{if } |\Sigma| \ge 2 \end{cases}$$

Then f is curl-∞ quasiconvex since it is level convex (see Proposition 3.6(4)) but it is not quasiconvex since it is not rank-1 convex.

We now recall the definition of polylevelconvex functions. Note that such functions have been referred in [8] as polyquasiconvex functions.

Definition 5.6 *A measurable function $g : \mathbb{M}^{d\times N} \to \mathbb{R}$ is called* **polylevelconvex** *if there exists a level convex function $f : \mathbb{R}^{c(N,d)} \to \mathbb{R}$ such that $g(\Sigma) = f(T(\Sigma))$ where $c(N,d)$ is given by*

$$c(N,d) = \sum_{s=1}^{\min(N,d)} \frac{d!N!}{(s!)^2(N-s)!(d-s)!}$$

and $T : \mathbb{M}^{d\times N} \to \mathbb{R}^{c(N,d)}$ is the map consisting of Σ and all of its $s\times s$ minors for $s \le min(N,d)$.

In the next proposition we prove that, under a suitable growth condition, the polylevelconvex functions are also curl-∞ quasiconvex.

Proposition 5.7 *Let $f : \mathbb{R}^{c(N,d)} \to [0, +\infty)$ be a continuous level convex function satisfying the weak growth condition* (4.26)*. Then the polylevelconvex function $g = f \circ T$ is* curl*-∞ quasiconvex.*

Proof. Let g_p and f_p be defined by (3.16) and let $(f^p)^{**}$ be the convex envelope of the function f^p. Reasoning as in the proof of Proposition 3.6(4) we have that

$$(f^p)^{**}(T(\Sigma)) \leq f^p(T(\Sigma)) = g^p(\Sigma)$$

and

$$f(T(\Sigma)) = \lim_{p\to\infty}((f^p)^{**})^{1/p}(T(\Sigma))$$

for every $\Sigma \in \mathbb{M}^{d\times N}$. Since the function $(f^p)^{**} \circ T$ is polyconvex it is in particular quasiconvex. Moreover, g_p^p is the quasiconvexification of g^p; hence,

$$(f^p)^{**}(T(\Sigma)) \leq g_p^p(\Sigma)$$

for every $\Sigma \in \mathbb{M}^{d\times N}$, which implies

$$((f^p)^{**})^{1/p}(T(\Sigma)) \leq g_p(\Sigma) \leq g(\Sigma)$$

for every $\Sigma \in \mathbb{M}^{d\times N}$. Then, passing to the limit as $p \to \infty$ in the previous inequality we get that

$$g(\Sigma) = f(T(\Sigma)) = \lim_{p\to\infty}((f^p)^{**})^{1/p}(T(\Sigma)) \leq \lim_{p\to\infty} g_p(\Sigma) \leq g(\Sigma)$$

for every $\Sigma \in \mathbb{M}^{d\times N}$; i.e., g is curl-∞ quasiconvex. □

In the following example we show that the class of $\mathcal{A}$-∞ quasiconvex functions strictly contains the class of the $\mathcal{A}$-quasiconvex functions and the class of the level convex functions.

Example 5.8 We consider the following family of functions $g_c : \mathbb{M}^{2\times 2} \to [0, +\infty)$ given by

$$g_c(\Sigma) = (\arctan \det \Sigma) \vee c|(\Sigma, \det \Sigma)|$$

where c is a positive constant and $(\Sigma, \det \Sigma)$ denotes the vector $(\Sigma_{11}, \Sigma_{12}, \Sigma_{21}, \Sigma_{22}, \det \Sigma)$ for every $\Sigma \in \mathbb{M}^{2\times 2}$.

Then

1. for every $c > 0$ the function g_c is curl-∞ quasiconvex. In fact, if we consider the level convex function $f_c : \mathbb{R}^5 \to [0, +\infty)$ given by

$$f_c(\xi) = (\arctan \xi_5) \vee c|\xi|,$$

we have that $g_c = f_c \circ T$. Therefore, by Definition 5.6, we have that g_c is polylevelconvex. By Proposition 5.7 we conclude that g_c is curl-∞ quasiconvex.

2. for every $0 < c < \arctan \frac{1}{4}$ the function g_c is not level convex. In fact, let $\Sigma^1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\Sigma^2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$; then for every $\lambda \in (0, \frac{1}{2}]$

$$g_c(\lambda\Sigma^1 + (1-\lambda)\Sigma^2) \geq \arctan \det(\lambda\Sigma^1 + (1-\lambda)\Sigma^2) = \arctan(\lambda(1-\lambda)) > 0$$

and

$$g_c(\Sigma^2) = g_c(\Sigma^1) = (\arctan \det \Sigma^1) \vee c|(1, 0, 0, 0, 0)| = 0 \vee c = c.$$

In particular when $0 < c < \arctan \frac{1}{4}$ we have that $g_c(\frac{1}{2}\Sigma^1 + \frac{1}{2}\Sigma^2) > c = g_c(\Sigma^1) \vee g_c(\Sigma^2)$ which implies that g_c is not level convex.

3. there exists $c_0 > 0$ such that the function g_c is not quasiconvex for every $0 < c < c_0$. At this end, we first note that the function $g_0(\Sigma) = (\arctan \det \Sigma) \vee 0$ is not quasiconvex. In fact, assume that g_0 is quasiconvex then it is also rank-1 convex. Since g_0 is bounded then g_0 is in particular constant (see [12] Exercise 4.2) but this is false.
Since g_0 is not quasiconvex there exists $\Sigma_0 \in \mathbb{M}^{2\times 2}$ and a Q-periodic function $V_0 \in C^\infty(\mathbb{R}^N; \mathbb{M}^{d\times N})$ satisfying $\operatorname{curl} V_0 = 0$, $\int_Q V_0(x)\,dx = 0$, and such that

$$g_0(\Sigma_0) > \int_Q g_0(\Sigma_0 + V_0(x))dx \geq 0.$$

Since $g_0 \leq g_c$, we have that

$$g_0(\Sigma) \leq g_c(\Sigma) \leq g_0(\Sigma) + c|(\Sigma, \det \Sigma)| \tag{5.44}$$

for every $\Sigma \in \mathbb{M}^{2\times 2}$. Therefore we get that

$$\begin{aligned} & \int_Q g_0(\Sigma_0 + V_0(x))dx \\ \leq & \int_Q g_c(\Sigma_0 + V_0(x))dx \\ \leq & \int_Q g_0(\Sigma_0 + V_0(x))dx + c\int_Q |(\Sigma_0 + V_0(x), \det(\Sigma_0 + V_0(x))|dx \end{aligned} \tag{5.45}$$

for every $c > 0$. By (5.44) and (5.45) it follows that

$$\lim_{c\to 0^+} g_c(\Sigma_0) - \int_Q g_c(\Sigma_0 + V_0(x))dx = g_0(\Sigma_0) - \int_Q g_0(\Sigma_0 + V_0(x))dx > 0$$

which implies that there exists $c_0 > 0$ such that

$$g_c(\Sigma_0) - \int_Q g_c(\Sigma_0 + V_0(x)dx > 0$$

for every $0 < c < c_0$; i.e., g_c is not quasiconvex for every $0 < c < c_0$.

Note that, for every $0 < c < \min\{c_0, \arctan \frac{1}{4}\}$, the function g_c is curl-∞ quasiconvex but it is neither quasiconvex nor level convex.

In Proposition 3.6(2) we prove that if f is coercive, continue and level convex, then f is $\mathcal{A}$-∞ quasiconvex. In the following example we show that if we drop the coercivity assumption this implication can be false. Moreover, Example 5.9 allows us to deduce that the curl-weak quasiconvex functions are not necessarily curl-∞ quasiconvex and that the class of curl-∞ quasiconvex functions is strictly contained in the class of (strong) Morrey quasiconvex functions.

Example 5.9 Let us consider the continuous function $f : \mathbb{R} \to [0, +\infty)$ given by

$$f(t) := \begin{cases} 0 & \text{if } t \leq 0 \\ t & \text{if } 0 \leq t \leq 1 \\ 1 & \text{if } t \geq 1 \end{cases}$$

Then

1. f is curl-weak quasiconvex and (strong) Morrey quasiconvex since f is level convex.
2. f is not curl-∞ quasiconvex. In fact, since we are in the scalar case, f^p_p coincides with the convex envelope $(f^p)^{**}$. Moreover f^p is bounded then $f^p_p = (f^p)^{**} = 0$. This implies that $Q^\infty_{\text{curl}} f = 0$; hence, $f(t) > Q^\infty_{\text{curl}} f(t)$, for every $t > 0$. Therefore f cannot be curl-∞ quasiconvex.

6 An application to the effective strenght for resistive materials

In this section we apply the results obtained to characterize via Γ-convergence the effective strength set $K_{\rm eff}$ in the context of electrical resistivity (how strongly a given material opposes the flow of electric current). More precisely, we consider

$$K_{\rm eff} = \{\xi \in \mathbb{R}^N \, : \exists\, \sigma \in L^\infty(Q;\mathbb{R}^N),\ \int_Q \sigma\, dx = 0,\ \operatorname{div}\sigma = 0,\ f(x, \xi + \sigma(x)) \le 1 \text{ a.e. } x \in Q\}. \tag{6.46}$$

Thanks to Theorem 4.2 we can characterize the set $K_{\rm eff}$ by assuming that f is a Carathéodory function, div-∞ quasiconvex in second variable and satisfying the weak growth condition (4.26). Note that, by Proposition 5.4, this is equivalent to supposing that f is level convex in the second variable and not necessarily convex as in [9].

Theorem 6.1 *Let $f : Q \times \mathbb{R}^N \mapsto [0, +\infty)$ be a Carathéodory function, level convex in the second variable and satisfying the weak growth condition (4.26). For any $\xi \in \mathbb{R}^N$ let*

$$j_p^{\rm eff}(\xi) := \inf\Big\{ \big(\int_Q f^p(x, \xi + \sigma(x)) dx \big)^{1/p} : \sigma \in L^\infty(Q;\mathbb{R}^N),\ \int_Q \sigma\, dx = 0,\ \operatorname{div}\sigma = 0 \Big\}.$$

Then, for any $\xi \in \mathbb{R}^N$, $j_p^{\rm eff}(\xi)$ converges to $j_\infty^{\rm eff}(\xi)$ given by

$$j_\infty^{\rm eff}(\xi) := \inf\Big\{ \operatorname*{ess\,sup}_{x\in Q} f(x, \xi + \sigma(x)) : \sigma \in L^\infty(Q;\mathbb{R}^N),\ \int_Q \sigma\, dx = 0,\ \operatorname{div}\sigma = 0 \Big\}.$$

Moreover, the set $K_{\rm eff}$ is described by

$$K_{\rm eff} = \{\xi \in \mathbb{R}^N : \ j_\infty^{\rm eff}(\xi) \le 1\}. \tag{6.47}$$

Proof. For a fixed $\xi \in \mathbb{R}^N$ we consider the functional $G_p : L^\infty(Q;\mathbb{R}^N) \to \mathbb{R} \cup \{+\infty\}$ given by

$$G_p(\sigma) := \begin{cases} \left(\int_Q f^p(x, \sigma(x) + \xi) dx \right)^{1/p} & \text{if } \sigma \in L^\infty(Q;\mathbb{R}^N),\ \operatorname{div}\sigma = 0,\ \int_Q \sigma\, dx = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

We state that

$$\Gamma(w^*\text{-}L^\infty)\text{-}\lim_{p\to\infty} G_p(\sigma) = \begin{cases} \operatorname*{ess\,sup}_{x\in Q} f(x, \sigma(x) + \xi) & \text{if } \sigma \in L^\infty(Q;\mathbb{R}^N),\ \operatorname{div}\sigma = 0,\ \int_Q \sigma\, dx = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

In fact, by Theorem 4.2 the Γ-liminf inequality is straightforward. The proof of the Γ-limsup inequality is an immediate consequence of the convergence of the L^p-norm to the L^∞-norm.

By Theorem 2.3, it follows that for any $\xi \in \mathbb{R}^N$

$$\lim_{p\to\infty} j_p^{\rm eff}(\xi) = j_\infty^{\rm eff}(\xi).$$

We now prove (6.47). Let $\xi \in K_{\rm eff}$, by (6.46), there exists $\sigma \in L^\infty(Q;\mathbb{R}^N)$ such that $\int_Q \sigma\, dx = 0$, $\operatorname{div}\sigma = 0$ and

$$f(x, \sigma(x) + \xi) \le 1$$

for a.e. $x \in Q$. This implies that

$$j_\infty^{\rm eff}(\xi) \le \operatorname*{ess\,sup}_{x\in Q} f(x, \sigma(x) + \xi) \le 1.$$

Conversely, let $\xi \in \mathbb{R}^N$ be such that $j_\infty^{\rm eff}(\xi) \leq 1$. By definition, there exists a sequence $\sigma_n \in L^\infty(Q;\mathbb{R}^N)$ such that $\int_Q \sigma_n\,dx = 0$, $\operatorname{div}\sigma_n = 0$ and

$$\lim_{n\to\infty} \operatorname*{ess\,sup}_{x\in Q} f(x,\xi+\sigma_n(x)) = j_\infty^{\rm eff}(\xi)$$

Thanks to the weak growth condition (4.26), there exists a subsequence of (σ_n) (not relabelled) such that $\sigma_n \rightharpoonup \sigma$ weakly* in $L^\infty(Q;\mathbb{R}^N)$ with $\int_Q \sigma\,dx = 0$, $\operatorname{div}\sigma = 0$. Since f is level convex, the functional $\operatorname{ess\,sup}_{x\in Q} f(x,\sigma(x)+\xi)$ is weakly* lower semicontinuous on $L^\infty(Q;\mathbb{R}^N)$. It follows that

$$\operatorname*{ess\,sup}_{x\in Q} f(x,\xi+\sigma(x)) \leq \liminf_{n\to\infty} \operatorname*{ess\,sup}_{x\in Q} f(x,\xi+\sigma_n(x)) = j_\infty^{\rm eff}(\xi) \leq 1$$

and therefore $\xi = \int_Q \xi + \sigma(x)\,dx \in K_{\rm eff}$. □

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