L^p-theory for some elliptic and parabolic problems with first order degeneracy at the boundary

Simona Fornaro, Giorgio Metafune, Diego Pallara

Dipartimento di Matematica "Ennio De Giorgi", Università di Lecce, C.P.193, 73100, Lecce, Italy.

Jan Prüss

FB Mathematik und Informatik, Martin-Luther-Universität Halle-Wittenberg, Theodor-Lieser-Str. 5, 60120-Halle, Germany.

Abstract

Let Ω be a smooth open bounded set in $\mathbf{R}^N$, let ϱ be the (smoothed in the interior) distance function from $\partial\Omega$, let (a_{ij}) be a uniformly elliptic matrix with continuous entries in Ω and A the associated second order elliptic operator. Under suitable conditions, we prove that the operator $L = -\varrho A + B$, with B a first order operator with continuous coefficients, with Dirichlet boundary conditions, generates an analytic semigroup in $L^p(\Omega)$, $1 < p < \infty$, and in $C(\overline{\Omega})$. In $L^p(\Omega)$ we also give a precise description of the domain.

Résumé

On considère un ouvert borné $\Omega \subset \mathbf{R}^N$ avec frontière régulière $\partial\Omega$, un opérateur uniformément elliptique $A = \sum_{ij} a_{ij} D_{ij}$ à coéfficients continus, et une fonction régulière ϱ qui coïncide avec la distance de $\partial\Omega$ dans un voisinage du bord. Sous des conditions qui lient les coéfficients (a_{ij}) à la géometrie de $\partial\Omega$, si B est n'importe quel opérateur du premier ordre à coéfficients continus, on prouve que $-\varrho A + B$ est le générateur d'un semi-groupe analytique dans $L^p(\Omega)$, $1 < p < \infty$, et dans $C(\overline{\Omega})$. Dans le cas de L^p, on décrit aussi d'un façon précise le domain du générateur.

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Email addresses: `simona.fornaro@unile.it`, `giorgio.metafune@unile.it`, `diego.pallara@unile.it` (Simona Fornaro, Giorgio Metafune, Diego Pallara), `jan.pruess@mathematik.uni-halle.de` (Jan Prüss).

1 Introduction

In this paper we study existence, uniqueness and regularity for elliptic and parabolic problems associated with a class of second order degenerate elliptic operators defined in smooth domains Ω of Euclidean spaces. We shall consider operators which are locally uniformly elliptic, i.e., nondegenerate in the interior, but whose ellipticity constant tends to zero when the point approaches the boundary. We confine ourselves to the case of first order complete degeneracy at the boundary, that is we assume that all the entries of the diffusion matrix tend to zero of order one with respect to the distance from the boundary $\partial\Omega$.

Let us comment briefly on the above hypotheses.

Complete degeneracy has been assumed having in mind the model case of the operator $(1-|x|^2)\Delta$ in the unit ball. This operator describes a Markov process without a drift term and with a diffusion part which vanishes at the boundary. The rate of degeneracy allows the random particle to reach the boundary in a finite time and therefore boundary conditions have to be imposed to the operator to describe the process. The same operator arises in some positive approximation problems, see [2, Section 6.3.9]. Since it seems to be very difficult to formulate a general theory which includes all possible types of boundary degeneracy (e.g. complete or only in some fixed directions), our choice has been determined by examples as above.

The rate of boundary degeneracy affects the results and the theory in a crucial way. For instance, the one-dimensional operator $(1-x^2)^\alpha D^2$ in the interval $[-1,1]$ describes a process which reaches the boundary if and only if $\alpha < 2$, see [16]. This implies that boundary conditions have to be imposed only when $\alpha < 2$. Surprisingly enough, the case $\alpha \geq 2$ can be handled in an easier way, whereas the case $\alpha < 2$ can be reduced to $\alpha = 1$ and treated via Bessel functions, see [23].

Let us present the main result of this paper. We consider the operator

$$A = -\varrho(\xi)\sum_{i,j=1}^{N+1} a_{ij}(\xi)D_{ij} + \sum_{i=1}^{N+1} b_i(\xi)D_i, \quad \xi \in \overline{\Omega}$$

in a smooth bounded domain $\Omega \subset \mathbf{R}^{N+1}$. The coefficients a_{ij}, b_i are continuous up to the boundary and the matrix a_{ij} is uniformly elliptic in $\overline{\Omega}$; ϱ is the (regularized in the interior) distance from the boundary and is responsible of the degeneracy. Under suitable conditions, we prove that $-A$, endowed with Dirichlet boundary conditions, generates an analytic semigroup in $L^p(\Omega)$ and $C(\overline{\Omega})$, and also estimates for $1 < p < \infty$ which provide an explicit description of the domain. These results give precise conditions for existence, uniqueness and regularity of elliptic and parabolic problems associated with A under

Dirichlet boundary conditions.

The study of second order degenerate differential operators, in connection with semigroups and Markov processes, started with the work of Feller who settled the one dimensional case, see [15]. The elliptic problem in several dimensions is treated in [24] following an approach due to Fichera. These results are used in [26], [27] to prove generation of semigroups, in the degenerate case. The approach of [24] and [26], [27] requires that the coefficients of the diffusion matrix can be smoothly extended to the whole space, keeping the non negativity of the associated quadratic form. This clearly excludes first order degeneracy. We also mention the papers of Baouendi and Goulaouic, [6], [7], Bolley and Camus [8], Višik and Grušin [29], Kohn and Nirenberg [19], as well as the treatises [28, Chapters 6, 7] and [25]. The approach in all these works, however, is confined to the elliptic problem in Hilbert spaces even though more general operators are allowed. Finally, let us mention that L^p estimates in the case of first order degeneracy of the tangential diffusion have been obtained in [18].

The paper is structured as follows. In Section 2 we analyse in detail a model problem in the halfspace $\mathbf{R}^{N+1}_+ = \{\xi = (x, y) \in \mathbf{R}^{N+1} : \ y > 0\}$, for $1 < p < \infty$. The differential operator under consideration is given by

$$L = -y\Delta + a \cdot \nabla_x + bD_y,$$

with constant first order coefficients. In this framework, we prove that if $b > -1/p$ then $-L$, endowed with domain $D_p^\circ = \Big\{u \in W_0^{1,p}(\mathbf{R}^{N+1}_+) \cap W^{2,p}_{\mathrm{loc}}(\mathbf{R}^{N+1}_+) : \sqrt{y}\,\nabla u, \ yD^2u \in L^p(\mathbf{R}^{N+1}_+)\Big\}$, generates an analytic semigroup of positive contractions in $L^p(\mathbf{R}^{N+1}_+)$. We note that the explicit description of the domain of the generator implies optimal elliptic regularity. This means that given a function $f \in L^p(\mathbf{R}^{N+1}_+)$, the solution of the equation $\lambda u + Lu = f$ has the best possible regularity. In particular, one cannot expect the p-summability of the second order derivatives, as a consequence of the degeneracy. However the *weighted* second order derivatives yD^2u actually belong to $L^p(\mathbf{R}^{N+1}_+)$. We remark that the assumption $b > -1/p$ is essential in order to characterize the domain of the generator, as it is shown in Example 2.11, but it is not necessary for the existence of a semigroup.

Section 3 deals with above described class of degenerate operators with variable coefficients in bounded regular domains Ω. The role of the coefficient y in the principal part of L is played by a regularized distance function from $\partial\Omega$, ϱ. By a standard technique based on local charts and freezing of the coefficients, we are able to recover the above result also in the present setting.

In Section 4, we study the case $p = \infty$, considering in the space $C(\overline{\Omega})$ the same class of operators introduced in the previous section. In order to prove their sectoriality (when endowed with a suitable domain), we employ the Masuda-

Stewart method, which is well-known for uniformly elliptic operators and relies on two main facts: a local version of the classical Morrey imbedding theorem for functions in $W^{1,p}(\Omega)$ with $p > N+1$, and the generation result in $L^p(\Omega)$. In our situation, the degeneracy at the boundary forces us to derive first a variant of the above imbedding theorem for functions in the weighted Sobolev space $\{u \in L^p(\Omega) \mid \sqrt{\varrho}\,\nabla u \in L^p(\Omega)\}$, which holds under the more restrictive assumption $p > 2(N+1)$. Once this is done, the generation in $L^p(\Omega)$ allows to complete the proof.

Finally, in Section 5, we apply the results of Section 4 to investigate regularity and asymptotic behaviour of a class of degenerate Feller semigroups. For simplicity, we just consider the operator $A = -m\Delta$, where m is a continuous function which can be estimated from above and below with a constant times the distance function. We impose Ventcel boundary conditions, which means, from a probabilistic point of view, that the diffusion process governed by $-A$ sticks forever at a point $x \in \partial\Omega$, whenever it reaches it. We can prove the analyticity of the semigroup and the exponential convergence to a limit projection. We also refer the reader to [23] and [10] for similar results in one dimension.

Notation The canonical basis of $\mathbf{R}^{N+1}$ is denoted by $\{e_1, \ldots, e_{N+1}\}$. We set $\mathbf{R}^{N+1}_+ = \{z = (x, y) : x \in \mathbf{R}^N, y > 0\}$. We use ∇_x, D^2_x for the gradient and the Hessian matrix with respect to the x-variables, respectively. Similarly, D_y and D^2_y denote first and second partial derivative with respect to y and the mixed derivative are denoted by $D_y\nabla_x$. We use D_{ij} to denote an arbitrary second order derivative, when we do not need to distinguish between x and y variables. Similarly, ∇u stands for the complete gradient of u, that is $\nabla u = (\nabla_x u, D_y u)$ and $D^2 u$ stands for the complete Hessian matrix of u. Similarly, we set $\Delta u = \Delta_x u + D^2_y u$.

If L is a closed operator in a Banach space X, we denote by $\sigma(L)$ and $\rho(L)$ the spectrum and the resolvent set of L. The resolvent operator is denoted by $(\lambda - L)^{-1}$.

2 A model problem in $L^p(\mathbf{R}^{N+1}_+)$

In this section we study the operator

$$L = -y\Big(\Delta_x + D^2_y\Big) + a \cdot \nabla_x + bD_y, \tag{2.1}$$

where $a \in \mathbf{R}^N$, $b \in \mathbf{R}$, in $L^p(\mathbf{R}^{N+1}_+)$, $1 < p < \infty$, with Dirichlet boundary conditions on $\mathbf{R}^N$.

We introduce the spaces

$$D_p = \Big\{ u \in L^p(\mathbf{R}^{N+1}_+) \cap W^{2,p}_{\rm loc}(\mathbf{R}^{N+1}_+) : \nabla u,\ \sqrt{y}\,\nabla u,\ yD^2u \in L^p(\mathbf{R}^{N+1}_+) \Big\}$$

and $D_p^\circ = \{u \in D_p : u(x,0)=0\}$. They are Banach spaces when endowed with their canonical norms. Moreover, we set $\mathcal{D} = \{u \in C_c^\infty(\mathbf{R}^{N+1}) : u(x,0)=0\}$. Here and in the sequel we prefer to deal with functions defined in the whole space, but we point out that sometimes only suitable restrictions will be used without mentioning. Notice also that the condition $D^2u \in L^p_{\rm loc}$ follows from $yD^2u \in L^p$. We have required the former in order to give a L^p meaning to the weak derivatives from the beginning.

If $0 < \varepsilon \le 1/2$, we define

$$S_\varepsilon = \{(x,y) : x \in \mathbf{R}^N, \varepsilon < y < \varepsilon^{-1}\}$$

and

$$\begin{aligned} D^\circ_{p,\varepsilon} &= W^{2,p}(S_\varepsilon) \cap W^{1,p}_0(S_\varepsilon), \\ \mathcal{D}_\varepsilon &= \{u \in C_c^\infty(\mathbf{R}^{N+1}) : u(x,\varepsilon) = u(x,\varepsilon^{-1}) = 0\}. \end{aligned}$$

To unify the notation, we use these spaces also with $\varepsilon = 0$ with the following agreements: $S_0 = \mathbf{R}^{N+1}_+$, $D^\circ_{p,0} = D^\circ_p$ and $\mathcal{D}_0 = \mathcal{D}$.

Clearly, $\mathcal{D}_\varepsilon$ is dense in $D^\circ_{p,\varepsilon}$ for $\varepsilon > 0$. A similar result also holds for D_p and D_p°.

Lemma 2.1 *$\mathcal{D}$ is dense in D_p° and $C_c^\infty(\mathbf{R}^{N+1})$ is dense in D_p.*

Proof. Let us first show that the functions in D_p° with compact support are dense in D_p°. Let $u \in D_p^\circ$ and let $\Phi \in C_c^\infty(\mathbf{R}^{N+1})$ be such that $\Phi = 1$ in $B_1(0)$, $\Phi = 0$ in $\mathbf{R}^{N+1} \setminus B_2(0)$ and $0 \le \Phi \le 1$ in $\mathbf{R}^{N+1}$. Set $\Phi_n(z) = \Phi(z/n)$, where $z = (x,y)$. Then $u_n = \Phi_n u \in D_p^\circ$ and has compact support in the closure of $\mathbf{R}^{N+1}_+$. By dominated convergence, $u_n \to u$ in $W^{1,p}(\mathbf{R}^{N+1}_+)$. Since $|\nabla \Phi_n| \le C/n$ in $B_{2n}(0) \setminus B_n(0)$ and $\nabla\Phi_n = 0$ elsewhere, $\sqrt{y}\, u \nabla \Phi_n \to 0$ in $L^p(\mathbf{R}^{N+1}_+)$. By dominated convergence, $\sqrt{y}\,\nabla u_n = \sqrt{y}\,u\nabla\Phi_n + \sqrt{y}\,\Phi_n \nabla u$ converges to $\sqrt{y}\,\nabla u$ in $L^p(\mathbf{R}^{N+1}_+)$. In a similar way, since $|D^2\Phi_n| \le C/n^2$ in $B_{2n}(0) \setminus B_n(0)$ and $D^2\Phi_n = 0$ elsewhere, we have that yD^2u_n tends to yD^2u in $L^p(\mathbf{R}^{N+1}_+)$ as $n \to \infty$.

Now, let $u \in D_p^\circ$ be such that ${\rm spt}u \subseteq \overline{B_R^+(0)}$, for some $R > 0$ and let us prove that $\mathcal{D}$ is dense in D_p°. Denote by $\tilde{u}$ the odd continuation of u with respect to y on $\mathbf{R}^{N+1}$. Then $\tilde{u} \in W^{1,p}(\mathbf{R}^{N+1})$ and has compact support in $\mathbf{R}^{N+1}$. Let ρ_ε be a standard family of mollifiers such that ρ is an even function in each variable. Then $\rho_\varepsilon * \tilde{u} \in C_c^\infty(\mathbf{R}^{N+1})$, $(\rho_\varepsilon * \tilde{u})(x,0) = 0$ and $\rho_\varepsilon * \tilde{u} \to \tilde{u}$ in $W^{1,p}(\mathbf{R}^{N+1})$ as $\varepsilon \to 0$. Since ${\rm spt}(\rho_\varepsilon * \tilde{u}) \subseteq B_{R+1}(0)$, we have also $\sqrt{y}\,\nabla(\rho_\varepsilon * \tilde{u}) \to \sqrt{y}\,\nabla\tilde{u}$ in

$L^p(\mathbf{R}^{N+1})$. Let $h = D_j\tilde{u}$ denote any first order derivative of $\tilde{u}$. If $i < N+1$, we have

$$
\begin{aligned}
yD_{ij}(\rho_\varepsilon * \tilde{u}) &= D_i(y(\rho_\varepsilon * h)) = D_i\Big(\rho_\varepsilon * (yh) + (y\rho_\varepsilon) * h\Big) \\
&= \rho_\varepsilon * (yD_i h) + (yD_i\rho_\varepsilon) * h.
\end{aligned}
$$

Concerning the first addend, we immediately have that $\rho_\varepsilon * (yD_i h) \to yD_{ij}\tilde{u}$ in $L^p(\mathbf{R}^{N+1})$. As far as the second term is concerned, a direct computation shows that $(yD_i\rho_\varepsilon) * h = (yD_i\rho)_\varepsilon * h$ and therefore it converges to $h\int_{\mathbf{R}^{N+1}} yD_i\rho(x,y)\,dx\,dy$, which is zero.
Thus, $yD_{ij}(\rho_\varepsilon * \tilde{u}) \to yD_{ij}\tilde{u}$ in $L^p(\mathbf{R}_+^{N+1})$.

If $i = N+1$, then

$$
\begin{aligned}
yD_y(\rho_\varepsilon * h) &= D_y\Big(y(\rho_\varepsilon * h)\Big) - \rho_\varepsilon * h = D_y(\rho_\varepsilon * (yh) + (y\rho_\varepsilon) * h) - \rho_\varepsilon * h \\
&= \rho_\varepsilon * (yD_y h + h) + (yD_y\rho_\varepsilon) * h.
\end{aligned}
$$

Now, $\rho_\varepsilon * (yD_y h + h) \to yD_y h + h$ in $L^p(\mathbf{R}^{N+1})$, and $(yD_y\rho_\varepsilon) * h = (yD_y\rho)_\varepsilon * h \to -h$, since $\int_{\mathbf{R}^{N+1}} yD_y\rho(x,y)dxdy = -\int_{\mathbf{R}^{N+1}} \rho(x,y)dxdy = -1$. The restrictions of $\rho_\varepsilon * \tilde{u}$ to $\mathbf{R}_+^{N+1}$ then converge to u in D_p° as $\varepsilon \to 0$.

As regards the second part of the statement, one can argue as before, just replacing the odd continuation of u with the even one. $\square$

Observe that the map $u \mapsto u/y$ is continuous from D_p° to $L^p(\mathbf{R}_+^{N+1})$. This follows from the classical Hardy inequality $\|u/y\|_p \le p/(p-1)\|D_y u\|_p$, which extends from $\mathcal{D}$ to D_p°, because of Lemma 2.1.

In the sequel we need the variant of Hardy inequality stated in the next lemma.

Lemma 2.2 *Let $0 \le \varepsilon \le 1/2$ and $u \in \mathcal{D}_\varepsilon$. Then*

$$
\|u/y\|_{L^p(S_\varepsilon)}^p \le \left(\frac{p}{p-1}\right)^2 \int_{S_\varepsilon} y^{2-p}|u|^{p-4}|\mathrm{Re}\,(\bar{u}D_y u)|^2.
$$

PROOF. Assume $\varepsilon > 0$. We first deal with the one dimensional case. Let $u \in \mathcal{D}_\varepsilon$ and let us define $w = u$ in $[\varepsilon, \varepsilon^{-1}]$ and zero elsewhere. Set $v = \mathrm{Re}\,(\bar{w}D_y w)$. Then $D_y|w|^p = p|w|^{p-2}v$ and

$$
\begin{aligned}
\int_0^\infty y^{-p}|w|^p\,dy &\le p\int_0^\infty y^{-p}\,dy\int_0^y |w(s)|^{p-2}|v(s)|\,ds\\
&= p\int_0^\infty y^{1-p}\,dy\int_0^1 |w(ty)|^{p-2}|v(ty)|\,dt\\
&= p\int_0^1 dt\int_0^\infty y^{1-p}|w(ty)|^{p-2}|v(ty)|\,dy\\
&= p\int_0^1 t^{p-2}\,dt\int_0^\infty s^{1-p}|w(s)|^{p-2}|v(s)|\,ds\\
&= \frac{p}{p-1}\int_0^\infty s^{1-p}|w(s)|^{p-2}|v(s)|\,ds.
\end{aligned}
$$

Since $w=0$ in $]-\infty,\varepsilon]$ and $[\varepsilon^{-1},+\infty[$ and $w=u$ in $[\varepsilon,\varepsilon^{-1}]$ this yields

$$
\begin{aligned}
&\int_\varepsilon^{\varepsilon^{-1}} y^{-p}|u|^p\,dy \le \frac{p}{p-1}\int_\varepsilon^{\varepsilon^{-1}} s^{1-p}|u(s)|^{p-2}|v(s)|\,ds\\
&\le \frac{p}{p-1}\Big(\int_\varepsilon^{\varepsilon^{-1}} s^{2-p}|u(s)|^{p-4}|v(s)|^2\,ds\Big)^{1/2}\Big(\int_\varepsilon^{\varepsilon^{-1}} s^{-p}|u|^p\,ds\Big)^{1/2}
\end{aligned}
$$

and therefore,

$$
\int_\varepsilon^{\varepsilon^{-1}} y^{-p}|u|^p\,dy \le \Big(\frac{p}{p-1}\Big)^2\int_\varepsilon^{\varepsilon^{-1}} y^{2-p}|u(y)|^{p-4}|\mathrm{Re}\,(\bar u D_y u)|^2\,dy.
$$

The multidimensional case easily follows from the one dimensional case, integrating with respect to $x\in\mathbf{R}^N$ the one dimensional inequality.

The case $\varepsilon=0$ can be handled similarly. $\square$

Some preliminary L^p-estimates for L are easy consequences of Calderón-Zygmund inequalities.

Lemma 2.3 *There exists $C=C(N,p)$ such that for every $u\in D^\circ_{p,\varepsilon}$, $0\le\varepsilon\le 1/2$, the following inequalities hold.*

$$
\begin{aligned}
\|yD_x^2u\|_{L^p(S_\varepsilon)} &\le C\Big(\|y\Delta u\|_{L^p(S_\varepsilon)}+\|D_yu\|_{L^p(S_\varepsilon)}\Big)\\
\|yD_y^2u\|_{L^p(S_\varepsilon)} &\le C\Big(\|y\Delta u\|_{L^p(S_\varepsilon)}+\|D_yu\|_{L^p(S_\varepsilon)}\Big)\\
\|yD_y\nabla_xu\|_{L^p(S_\varepsilon)} &\le C\Big(\|y\Delta u\|_{L^p(S_\varepsilon)}+\|D_yu\|_{L^p(S_\varepsilon)}+\|\nabla_xu\|_{L^p(S_\varepsilon)}\Big)
\end{aligned}
$$

Proof. First we consider the case $\varepsilon=0$. Let $u\in C_c^\infty(\mathbf{R}^{N+1})$ and consider $v=yu$. Since v vanishes on $\{y=0\}$, we may apply to it the Calderón-Zygmund estimates in $L^p(\mathbf{R}^{N+1}_+)$, see e.g. [17, Lemma 9.12]. Since $D_x^2(yu)=yD_x^2u$, $D_y^2(yu)=yD_y^2u+2D_yu$, $D_y\nabla_x(yu)=yD_y\nabla_xu+\nabla_xu$, the statement is proved for these u and, by density, for every $u\in D_p$ (not only in D_p°).

The case $\varepsilon > 0$ is similar. We consider $u \in \mathcal{D}_\varepsilon$ and observe that $v = yu$ vanishes on the boundary of S_ε. Therefore, we apply the Calderón-Zygmund estimates to v in $L^p(S_\varepsilon)$, which lead to $\|D^2 v\|_{L^p(S_\varepsilon)} \le C(\|\Delta v\|_{L^p(S_\varepsilon)} + \varepsilon^2 \|v\|_{L^p(S_\varepsilon)})$, where C is the constant related to a strip of width 1, hence it is independent of ε. As a consequence of Poincaré inequality, we have $\|u\|_{L^p(S_\varepsilon)} \le \varepsilon^{-1}\|D_y u\|_{L^p(S_\varepsilon)}$, $\|v\|_{L^p(S_\varepsilon)} \le \varepsilon^{-1}\|D_y v\|_{L^p(S_\varepsilon)} \le \varepsilon^{-1}(\|yD_y u\|_{L^p(S_\varepsilon)} + \|u\|_{L^p(S_\varepsilon)})$. Therefore

$$\begin{aligned}\|yD_x^2 u\|_{L^p(S_\varepsilon)} \le& C(\|y\Delta u\|_{L^p(S_\varepsilon)} + \|D_y u\|_{L^p(S_\varepsilon)} \\ &+ \varepsilon\|yD_y u\|_{L^p(S_\varepsilon)} + \varepsilon\|u\|_{L^p(S_\varepsilon)}) \\ \le& C(\|y\Delta u\|_{L^p(S_\varepsilon)} + \|D_y u\|_{L^p(S_\varepsilon)})\end{aligned}$$

The remaining estimates can be proved similarly. □

In the following we set $u^* = \bar{u}|u|^{p-2}$.

Proposition 2.4 *Let* $\mathrm{Re}\,\lambda \ge 0$, $u \in D^\circ_{p,\varepsilon}$, $0 \le \varepsilon \le 1/2$, *and* $f = \lambda u + Lu$. *Then*

$$\begin{aligned}(\mathrm{Re}\,\lambda)\|u\|_{L^p(S_\varepsilon)} &\le \|f\|_{L^p(S_\varepsilon)} \\ |\mathrm{Im}\,\lambda|\,\|u\|_{L^p(S_\varepsilon)} &\le \left(1 + \frac{|p-2|}{2\sqrt{p-1}}\right)\|f\|_{L^p(S_\varepsilon)} + \Big(|a| + |b+1|\Big)\|\nabla u\|_{L^p(S_\varepsilon)}.\end{aligned}$$

Proof. By density, we may assume that $u \in \mathcal{D}_\varepsilon$. Multiplying the equation $\lambda u + Lu = f$ by u^* and integrating by parts on S_ε, all boundary terms vanish and we have

$$\begin{aligned}\int_{S_\varepsilon} fu^* =& \lambda\|u\|^p_{L^p(S_\varepsilon)} + \int_{S_\varepsilon} y|u|^{p-4}\Big((p-1)|\mathrm{Re}\,(\bar{u}\nabla u)|^2 + |\mathrm{Im}\,(\bar{u}\nabla u)|^2\Big) \\ &+ i(p-2)\int_{S_\varepsilon} y|u|^{p-4}\Big(\mathrm{Re}\,(\bar{u}\nabla u)\mathrm{Im}\,(\bar{u}\nabla u)\Big) \\ &+ \int_{S_\varepsilon}(a\cdot\nabla_x u)u^* + (b+1)\int_{S_\varepsilon}(D_y u)u^*.\end{aligned}$$

Since the last two terms are purely imaginary, we deduce

$$\begin{aligned}\mathrm{Re}\int_{S_\varepsilon} fu^* =& (\mathrm{Re}\,\lambda)\|u\|^p_{L^p(S_\varepsilon)} \\ &+ \int_{S_\varepsilon} y|u|^{p-4}\Big((p-1)|\mathrm{Re}\,(\bar{u}\nabla u)|^2 + |\mathrm{Im}\,(\bar{u}\nabla u)|^2\Big) \\ \ge& (\mathrm{Re}\,\lambda)\|u\|^p_{L^p(S_\varepsilon)} \qquad (2.2)\end{aligned}$$

and

$$
\begin{aligned}
&|\mathrm{Im}\,\lambda|\,\|u\|^p_{L^p(S_\varepsilon)} \le \|f\|_{L^p(S_\varepsilon)}\|u\|^{p-1}_{L^p(S_\varepsilon)} \\
&+|p-2|\Big(\int_{S_\varepsilon} y|u|^{p-4}|\mathrm{Re}\,(\bar u\nabla u)|^2\Big)^{1/2}\Big(\int_{S_\varepsilon} y|u|^{p-4}|\mathrm{Im}\,(\bar u\nabla u)|^2\Big)^{1/2} \\
&+(|a|+|b+1|)\|\nabla u\|_{L^p(S_\varepsilon)}\|u\|^{p-1}_{L^p(S_\varepsilon)}.
\end{aligned}
$$

Using (2.2) we get

$$
\begin{aligned}
&\Big(\int_{S_\varepsilon} y|u|^{p-4}|\mathrm{Re}\,(\bar u\nabla u)|^2\Big)^{1/2}\Big(\int_{S_\varepsilon} y|u|^{p-4}|\mathrm{Im}\,(\bar u\nabla u)|^2\Big)^{1/2} \\
&\le \frac{1}{2\sqrt{p-1}}\Big((p-1)\int_{S_\varepsilon} y|u|^{p-4}|\mathrm{Re}\,(\bar u\nabla u)|^2+\int_{S_\varepsilon} y|u|^{p-4}|\mathrm{Im}\,(\bar u\nabla u)|^2\Big) \\
&\le \frac{1}{2\sqrt{p-1}}\|f\|_{L^p(S_\varepsilon)}\|u\|^{p-1}_{L^p(S_\varepsilon)}
\end{aligned}
$$

and hence

$$
|\mathrm{Im}\,\lambda|\,\|u\|_{L^p(S_\varepsilon)} \le \left(1+\frac{|p-2|}{2\sqrt{p-1}}\right)\|f\|_{L^p(S_\varepsilon)}+(|a|+|b+1|)\|\nabla u\|_{L^p(S_\varepsilon)}.
$$

□

Proposition 2.5 *Let* $\mathrm{Re}\,\lambda\ge 0$, $u\in D^\circ_{p,\varepsilon}$, $0\le\varepsilon\le 1/2$, *and* $f=\lambda u+Lu$. *Then*

$$
\begin{aligned}
\mathrm{Re}\int_{S_\varepsilon} y^{1-p}fu^* =& (\mathrm{Re}\,\lambda)\int_{S_\varepsilon} y^{1-p}|u|^p+(1-1/p)(b+2-p)\|u/y\|^p_{L^p(S_\varepsilon)} \\
&+\int_{S_\varepsilon} y^{2-p}|u|^{p-4}\Big((p-1)|\mathrm{Re}\,(\bar u\nabla u)|^2+|\mathrm{Im}\,(\bar u\nabla u)|^2\Big)
\end{aligned}
$$

Proof. Assume $u\in\mathcal{D}_\varepsilon$. Multiplying the equation $\lambda u+Lu=f$ by u^*y^{1-p} and integrating by parts we have

$$
\begin{aligned}
\int_{S_\varepsilon} y^{1-p}fu^* =&\lambda\int_{S_\varepsilon} y^{1-p}|u|^p \\
&+\int_{S_\varepsilon} y^{2-p}|u|^{p-4}\Big((p-1)|\mathrm{Re}\,(\bar u\nabla u)|^2+|\mathrm{Im}\,(\bar u\nabla u)|^2\Big) \\
&+i(p-2)\int_{S_\varepsilon} y^{2-p}|u|^{p-4}\Big(\mathrm{Re}\,(\bar u\nabla u)\mathrm{Im}\,(\bar u\nabla u)\Big) \\
&+(1-p)\int_{S_\varepsilon}(D_yu)y^{1-p}u^*+\int_{S_\varepsilon} y^{1-p}\Big(a\cdot\nabla_x u+(b+1)D_yu\Big)u^*.
\end{aligned}
$$

Observing that

$$
\mathrm{Re}\int_{S_\varepsilon}(D_yu)y^{1-p}u^*=\frac{1}{p}\int_{S_\varepsilon} y^{1-p}D_y|u|^p=(1-1/p)\|u/y\|^p_{L^p(S_\varepsilon)},
$$

and that $\int_{S_\varepsilon} y^{1-p}(a\cdot\nabla_x u)u^*$ is purely imaginary, the thesis follows taking the real part in the identity above. The general case can be handled by density. □

From Proposition 2.5 and Lemma 2.2 we obtain an estimate of $\|u/y\|_p$ in terms of $\|\lambda u + Lu\|_p$. Observe that the constant γ_p appearing in the statement of Proposition 2.6 below is positive if and only if $b > -1/p$.

Proposition 2.6 *Assume that $b > -1/p$, let $\operatorname{Re}\lambda \geq 0$, $u \in D^\circ_{p,\varepsilon}$ and $f = \lambda u + Lu$. Then*

$$\gamma_p\|u/y\|_{L^p(S_\varepsilon)} \leq \|f\|_{L^p(S_\varepsilon)}, \qquad \gamma_p = \frac{p-1}{p}(b+1/p).$$

PROOF. Using Lemma 2.2 and Proposition 2.5 we obtain for $u \in \mathcal{D}_\varepsilon$

$$\gamma_p\|u/y\|^p_{L^p(S_\varepsilon)} \leq \operatorname{Re}\int_{S_\varepsilon} y^{1-p} f u^* \leq \|f\|_{L^p(S_\varepsilon)}\|u/y\|^{p-1}_{L^p(S_\varepsilon)}.$$

By density one concludes the proof. □

To proceed further we need some interpolative inequalities.

Lemma 2.7 *There exist two constant $C > 0$, $\eta_0 > 0$ such that for every $u \in D_p$, $0 \leq \varepsilon \leq 1/2$ and $0 < \eta \leq \eta_0$ the following inequalities hold.*

(i) $\|D_y u\|_{L^p(S_\varepsilon)} \leq \eta\|yD^2_y u\|_{L^p(S_\varepsilon)} + (C/\eta)\|u/y\|_{L^p(S_\varepsilon)}$
(ii) $\|D_{x_i} u\|_{L^p(S_\varepsilon)} \leq \eta\|yD^2_{x_i} u\|_{L^p(S_\varepsilon)} + (C/\eta)\|u/y\|_{L^p(S_\varepsilon)}$
(iii) $\|\sqrt{y}\, D_y u\|_{L^p(S_\varepsilon)} \leq \eta\|yD^2_y u\|_{L^p(S_\varepsilon)} + (C/\eta)\|u\|_{L^p(S_\varepsilon)}$
(iv) $\|\sqrt{y}\, D_{x_i} u\|_{L^p(S_\varepsilon)} \leq \eta\|yD^2_{x_i} u\|_{L^p(S_\varepsilon)} + (C/\eta)\|u\|_{L^p(S_\varepsilon)}$.

PROOF. We deal only with the case $\varepsilon > 0$. The case $\varepsilon = 0$ can be proved letting $\varepsilon \to 0$. Set $S^1_\varepsilon = \{(x,y) : x \in \mathbf{R}^N, \varepsilon < y < (\varepsilon+\varepsilon^{-1})/2\}$, $S^2_\varepsilon = S_\varepsilon \setminus S^1_\varepsilon$ and let $u \in C^\infty_c(\mathbf{R}^{N+1})$. If $0 < \eta \leq \eta_0$, for some η_0 sufficiently small and independent of ε, then the points $(x,(1+\eta)y)$ and $(x,(1-\eta)y)$ belong to S_ε, whenever (x,y) belongs to S^1_ε, S^2_ε, respectively. Therefore, choosing $h = \pm\eta y$ in the Taylor formula

$$u(x,y+h) - u(x,y) = hD_y u(x,y) + h^2\int_0^1 (1-s)D^2_y u(x,y+sh)\,ds,$$

we find that

$$\begin{aligned} D_y u(x,y) = &\pm \frac{1}{\eta y}\Big(u(x,(1\pm\eta)y) - u(x,y)\Big) \\ &\mp \eta\int_0^1 (1-s)yD^2_y u(x,(1\pm\eta s)y)\,ds, \end{aligned}$$

in $S^1_\varepsilon, S^2_\varepsilon$, respectively. Observing that

$$\|y^{-1}u(x(1\pm\eta)y)\|_{L^p(S^i_\varepsilon)} \le C\|u/y\|_{L^p(S_\varepsilon)}$$
$$\|yD^2_y u(x,(1\pm\eta s)y)\|_{L^p(S^i_\varepsilon)} \le C\|yD^2_y u\|_{L^p(S_\varepsilon)}$$

$i=1,2$, with C independent of η, s, statement (i) easily follows.

As regards (ii), arguing as before, we obtain

$$\begin{aligned} D_{x_i}u(x,y) =& \frac{1}{\eta y}\Big(u(x+\eta y e_i, y) - u(x,y)\Big) \\ & - \eta\int_0^1 (1-s)yD^2_{x_i}u(x+s\eta y e_i, y)\,ds, \end{aligned}$$

where $\{e_1,\dots,e_N\}$ is the canonical basis of $\mathbf{R}^N$. Observing that the change of variables $(x,y)\mapsto(x+\eta y e_i, y)$ is measure-preserving and leaves $\mathbf{R}^{N+1}_+$ invariant, (ii) follows.

The proofs of (iii) and (iv) are similar, choosing $h=\sqrt{\eta y}$ and modifying, if necessary, the choice of η_0. By density, the result is proved for every $u \in D_p$. □

We can now prove that the operator (L, D°_p) is closed in $L^p(\mathbf{R}^{N+1}_+)$.

Proposition 2.8 *Assume that $b > -1/p$. Then there is a constant $C = C(N,p,a,b)$ such that for every $u \in D^\circ_{p,\varepsilon}$, $0\le\varepsilon\le 1/2$,*

$$\|u\|_{D^\circ_{p,\varepsilon}} \le C(\|Lu\|_{L^p(S_\varepsilon)} + \|u\|_{L^p(S_\varepsilon)}).$$

In particular, (L, D°_p) is closed in $L^p(\mathbf{R}^{N+1}_+)$.

Proof. Using Lemmas 2.3, 2.7 we obtain

$$\begin{aligned} \|yD^2u\|_{L^p(S_\varepsilon)} &\le C(\|y\Delta u\|_{L^p(S_\varepsilon)} + \|\nabla u\|_{L^p(S_\varepsilon)}) \\ &\le C(\|Lu\|_{L^p(S_\varepsilon)} + \|\nabla u\|_{L^p(S_\varepsilon)}) \\ &\le C\Big(\|Lu\|_{L^p(S_\varepsilon)} + \eta\|yD^2u\|_{L^p(S_\varepsilon)} + (1/\eta)\|u/y\|_{L^p(S_\varepsilon)}\Big). \end{aligned}$$

We remark that the constant C depends only on N,p,a,b. Taking η small enough and using Proposition 2.6 we obtain

$$\begin{aligned} \|yD^2u\|_{L^p(S_\varepsilon)} &\le C(\|Lu\|_{L^p(S_\varepsilon)} + \|u/y\|_{L^p(S_\varepsilon)}) \\ &\le C(\|Lu\|_{L^p(S_\varepsilon)} + \|u\|_{L^p(S_\varepsilon)}). \end{aligned} \tag{2.3}$$

The estimates for ∇u, $\sqrt{y}\,\nabla u$ now follow from Lemma 2.7 and Proposition 2.6. □

In the following two results we show that $(-L, D^\circ_p)$ generates an analytic semigroup in $L^p(\mathbf{R}^{N+1}_+)$.

Proposition 2.9 *Assume that $b > -1/p$. Then the operator $(-L, D_p^\circ)$ generates a semigroup of positive contractions $(T_p(t))_{t\geq 0}$ in $L^p(\mathbf{R}_+^{N+1})$. If, moreover, $1 < q < \infty$ and $b > -1/q$, then $T_p(t)f = T_q(t)f$ for every $f \in L^p(\mathbf{R}_+^{N+1}) \cap L^q(\mathbf{R}_+^{N+1})$.*

PROOF. Let $\lambda > 0$ and $f \in L^p(\mathbf{R}_+^{N+1})$ be given. Let us show that $u_\varepsilon \in D_{p,\varepsilon}^\circ$ exist, such that $(\lambda + L)u_\varepsilon = f$ in S_ε. In fact, Theorem 3.1.2 in [20] gives the result for λ large enough, and then the existence of u_ε for all $\lambda > 0$ follows from the dissipativity of $-L$, see Proposition 2.4. This same Proposition and Proposition 2.8 yield

$$\|u_\varepsilon\|_{L^p(S_\varepsilon)} \leq \lambda^{-1}\|f\|_{L^p(\mathbf{R}_+^{N+1})}, \quad \|u_\varepsilon\|_{D_{p,\varepsilon}^\circ} \leq C\|f\|_{L^p(\mathbf{R}_+^{N+1})}$$

for a suitable C independent of ε. By weak compactness, $u_{\varepsilon_n} \to u$ weakly in $W^{2,p}_{\rm loc}(\mathbf{R}_+^{N+1})$ for a suitable sequence $\varepsilon_n \to 0$. Then

$$\|u\|_{L^p(\mathbf{R}_+^{N+1})} \leq \lambda^{-1}\|f\|_{L^p(\mathbf{R}_+^{N+1})},$$
$$\|u\|_{D_p} \leq C\|f\|_{L^p(\mathbf{R}_+^{N+1})}$$

and $\lambda u + Lu = f$. It remains to show that u vanishes for $y = 0$.

Let $v_{\varepsilon_n}(x,y) = \eta(y)u_{\varepsilon_n}(x, y+\varepsilon_n)$, where η is a smooth function such that $\eta = 1$ in $[0,1]$, $\eta = 0$ in $[2,\infty[$. Then $v_{\varepsilon_n} \in W_0^{1,p}(\mathbf{R}_+^{N+1})$ and $\|\nabla v_{\varepsilon_n}\|_{L^p(\mathbf{R}_+^{N+1})} \leq C\|f\|_{L^p(\mathbf{R}_+^{N+1})}$. By weak compactness, $v_{\varepsilon_n} \to \eta u$ weakly in $W_0^{1,p}(\mathbf{R}_+^{N+1})$, hence $u \in W_0^{1,p}(\mathbf{R}_+^{N+1})$.

This shows that $(-L, D_p^\circ)$ generates a contraction semigroup $(T_p(t))_{t\geq 0}$ in $L^p(\mathbf{R}_+^{N+1})$. If f is positive then u_ε is positive and u too. Moreover, u_ε, hence u, do not depend on p. Therefore the resolvent of $-L$ is positive and p-independent and the proof is complete. □

Since $T_p(t)f = T_q(t)f$ for $f \in L^p(\mathbf{R}_+^{N+1}) \cap L^q(\mathbf{R}_+^{N+1})$, in the sequel we write simply $T(t)$.

From now on, until the end of the present section, we write $\|\cdot\|_p$ instead of $\|\cdot\|_{L^p(\mathbf{R}_+^{N+1})}$.

Theorem 2.10 *Assume that $b > -1/p$. Then the semigroup $(T(t))_{t\geq 0}$ is analytic in $L^p(\mathbf{R}_+^{N+1})$.*

PROOF. Let $u \in D_p^\circ$ and $\lambda = 1/\eta_0^2 + i\tau$, η_0 being given in Lemma 2.7, $f = \lambda u + Lu$. Using Lemma 2.7, equation (2.3) and Proposition 2.6, we get

$$\|\nabla u\|_p \leq \eta\|yD^2u\|_p + (C/\eta)\|u/y\|_p \leq C\Big(\eta\|Lu\|_p + 1/(\eta\gamma_p)\|\lambda u + Lu\|_p\Big)$$

and, taking $\eta = |\lambda|^{-1/2}$

$$\|\nabla u\|_p \le C|\lambda|^{1/2}\Big(\|f\|_p + \|u\|_p\Big).$$

Using Proposition 2.4 we obtain for $|\tau| \ge 1$,

$$|\tau|\|u\|_p \le C\Big(\|f\|_p + \|\nabla u\|_p\Big) \le C|\tau|^{1/2}\Big(\|f\|_p + \|u\|_p\Big)$$

and therefore $|\tau|^{1/2}\|u\|_p \le C\|f\|_p$ for large $|\tau|$. This implies that the norm of the resolvent operator $(1/\eta_0^2 + i\tau + L)^{-1}$ tends to zero as $|\tau| \to \infty$ and hence the resolvent set contains the half-lines $\{i\tau : |\tau| \ge M\}$ for a suitable constant $M > 0$.

Let $I_s : L^p(\mathbf{R}_+^{N+1}) \to L^p(\mathbf{R}_+^{N+1})$ be defined by $I_s u(x,y) = u(x/s, y/s)$. I_s is invertible with inverse $I_{s^{-1}}$, $I_s(D_p^\circ) = D_p^\circ$ and satisfies $I_s^{-1} L I_s = s^{-1} L$. This gives $\rho(L) = \rho(I_s^{-1} L I_s) = s^{-1}\rho(L)$, whence $\rho(L)$ contains $i\mathbf{R} \setminus \{0\}$. Moreover, if $i\lambda \ne 0$ and $\omega = \lambda/|\lambda| = \pm 1$, then

$$\|(i\lambda + L)^{-1}\| = |\lambda|^{-1}\|I_{|\lambda|}^{-1}(i\omega + L)^{-1} I_{|\lambda|}\| \le |\lambda|^{-1}\|(i\omega + L)^{-1}\|,$$

and the proof is complete. □

In the following example we show that the condition $b > -1/p$ is necessary in order that (L, D_p°) generates a semigroup in $L^p(\mathbf{R}_+^{N+1})$. For simplicity we work in $\mathbf{R}$ and, since the main problems come from the degeneracy at 0, we may work in the interval $[0,1]$, e.g. with a Neumann boundary condition at 1.

Example 2.11 Let $L = -yD_y^2 + bD_y$, where $0 < y < 1$. Assume that L endowed with the domain

$$D_p(L) = \Big\{u \in L^p(]0,1[) : D_y u,\ yD_y^2 u \in L^p(]0,1[),\ u(0) = D_y u(1) = 0\Big\}$$

has non-empty resolvent set in $L^p(]0,1[)$. Then L has compact resolvent and $0 \in \rho(L)$, since it is not an eigenvalue and therefore the equation $Lu = -1$ has a solution in $D_p(L)$. An explicit computation shows that

$$D_y u = \frac{1}{b}\Big(y^b - 1\Big)$$

and hence $D_y u \in L^p(]0,1[)$ if and only if $b > -1/p$.

We can now prove regularity results for the parabolic problem associated with L using purely functional analytic tools. Our approach requires the existence of the semigroup in a L^2 space and forces us to assume the condition $b > -1/2$. We recall that an analytic semigroup $(T(t))_{t \ge 0}$ on a Banach space X with generator $-B$ has *maximal regularity of type* L^q $(1 < q < \infty)$ if for each $f \in L^q([0,T], X)$ the function $t \mapsto u(t) = \int_0^t T(t-s) f(s)\, ds$ belongs to

$W^{1,q}([0,T],X) \cap L^q([0,T],D(B))$. This means that the mild solution of the evolution equation

$$u'(t) + Bu(t) = f(t), \quad t > 0, \qquad u(0) = 0,$$

is in fact a strong solution and has the best regularity one can expect. It is known that this property does not depend on $1 < q < \infty$ and $T > 0$. In recent years this concept has thoroughly been studied and applied in various directions, see e.g. [3], [14], [30], and the references therein. For our purposes we only need the following facts. Let $X = L^p(\mathbf{R}^{N+1}_+)$ for some $1 < p < \infty$. Then the operator $-B$ has maximal regularity of type L^q if its imaginary powers satisfy $\|B^{is}\| \le Me^{a|s|}$ for some $a \in [0, \pi/2)$ and all $s \in \mathbf{R}$ thanks to the Dore–Venni theorem, see e.g. [3, Theorem II.4.10.7]. If $-B$ generates a positive contraction semigroup on $L^p(\mathbf{R}^{N+1}_+)$, then $\|B^{is}\| \le M_\varepsilon \exp((\varepsilon + \pi/2)|s|)$ for each $\varepsilon > 0$ and $s \in \mathbf{R}$ because of the transference principle [12, Section 4], see [11, Theorem 5.8]. If, in addition, $p = 2$ and $-B$ is sectorial, then $\|B^{is}\| \le Me^{a|s|}$ for $a = \pi/2 - \phi$ and some $\phi \in (0, \pi/2]$, by a result due to McIntosh, [22]. If we combine these facts with the Riesz-Thorin interpolation theorem, Proposition 2.9 and Theorem 2.10 we obtain the following result.

Proposition 2.12 *Assume that that $b > \max\{-1/p\,, -1/2\}$. Then $(-L, D_p^\circ)$ has maximal regularity of type L^q on $L^p(\mathbf{R}^{N+1}_+)$.*

In order to deal with degenerate operators with variable coefficients, we prove the analogue of Theorem 2.10 for the operator

$$\hat{L} = -y \sum_{i,j=1}^{N+1} a_{ij} D_{ij} + a \cdot \nabla_x + bD_y, \qquad D_p(\hat{L}) = D_p^\circ.$$

Here we assume that $a_{ij} = a_{ji} \in \mathbf{R}$, $\sum_{i,j=1}^{N+1} a_{ij}\xi_i\xi_j \ge \alpha|\xi|^2$ for all $\xi \in \mathbf{R}^{N+1}$ with $\alpha > 0$. Set $M = \max |a_{ij}|$.

Lemma 2.13 *Let $c = a_{N+1\,N+1}$ and assume that $bc^{-1} > -1/p$. Then there exists a constant $C = C(N, p, a, bc^{-1}, M, \alpha)$ such that for every $\operatorname{Re}\lambda > 0$, the estimate $\|(\lambda + \hat{L})^{-1}\| \le C|\lambda|^{-1}$ holds.*

Moreover, for every $u \in D_p^\circ$, $\|u\|_{D_p} \le C(\|\hat{L}u\|_p + \|u\|_p)$.

Proof. Let Q_1 be a non-singular matrix such that $\sum_{i,j=1}^{N+1} a_{ij} D_{ij} u(z) = \Delta v(Q_1 z)$ whenever $u(z) = v(Q_1 z)$, $z = (x, y)$. Since the laplacian is rotation invariant, we may choose $Q = SQ_1$, $S^{-1} = S^*$, in such a way that it leaves $\mathbf{R}^{N+1}_+$ invariant and $\sum_{i,j=1}^{N+1} a_{ij} D_{ij} u(z) = \Delta v(Qz)$ whenever $u(z) = v(Qz)$. The invariance of $\mathbf{R}^{N+1}_+$ under Q implies $Q^* e_{N+1} = k e_{N+1}$ for some $k > 0$ and the identity $\sum_{i,j=1}^{N+1} a_{ij} D_{ij} u(z) = \Delta v(Qz)$ then yields $k^2 c = 1$.

The equation $\lambda u(z) + \hat{L}u(z) = f(z)$ is equivalent to

$$\lambda k v(z) - y\Delta v(z) + a_1 \cdot \nabla_x v(z) + bk^2 D_y v(z) = kf(Q^{-1}z),$$

for a suitable $a_1 \in \mathbf{R}^N$, and the thesis follows from Theorem 2.10 and Proposition 2.8. □

Corollary 2.14 *There exists a constant $C = C(N, p, a, bc^{-1}, M, \alpha, \eta_0)$, η_0 being given in Lemma 2.7, such that for all $u \in D_p^\circ$ and all $\lambda \in \mathbf{C}$ with* $\mathrm{Re}\,\lambda > 0$ *and $|\lambda| \geq 1/\eta_0^2$*

$$\|\sqrt{y}\,\nabla u\|_p \leq C|\lambda|^{-1/2}\|(\lambda + \hat{L})u\|_p.$$

Proof. It follows from Lemma 2.7 that there exists a constant $C > 0$ such that for every $u \in D_p^\circ$ and $0 < \eta \leq \eta_0$

$$\|\sqrt{y}\,\nabla u\|_p \leq \eta\|yD^2u\|_p + C/\eta\|u\|_p.$$

Taking Lemma 2.13 into account we get

$$\begin{aligned}\|\sqrt{y}\,\nabla u\|_p &\leq C\eta(\|\hat{L}u\|_p + \|u\|_p) + C/\eta\|u\|_p \\ &\leq C\left(\eta + \frac{1}{\eta|\lambda|}\right)\|(\lambda + \hat{L})u\|_p,\end{aligned}$$

where the constant C in the last step depends on the quantities listed in the statement. Choosing $\eta = |\lambda|^{-1/2}$, the thesis follows. □

3 General bounded domains

In the present section, we consider degenerate operators with variable coefficients in bounded domains. To be definite, let Ω be a bounded open subset of $\mathbf{R}^{N+1}$ with C^2 boundary and let ϱ be a function in $C^2(\overline{\Omega})$ such that $\varrho > 0$ in Ω, $\varrho = 0$ on $\partial\Omega$ and $\nabla\varrho(\xi) = \nu(\xi)$, for every $\xi \in \partial\Omega$. Here, $\nu(\xi)$ is the inward unitary normal vector to $\partial\Omega$ at ξ. Such a function ϱ can be constructed by extending the distance function from the boundary of Ω. We introduce the operator

$$A = -\varrho(\xi)\sum_{i,j=1}^{N+1} a_{ij}(\xi)D_{ij} + \sum_{i=1}^{N+1} b_i(\xi)D_i, \quad \xi \in \overline{\Omega} \tag{3.1}$$

under the following conditions on the coefficients.

(H1) a_{ij} are real continuous functions on $\overline{\Omega}$, $a_{ij} = a_{ji}$, and satisfy the ellipticity condition $\sum_{i,j=1}^{N+1} a_{ij}(\xi)\zeta_i\zeta_j \geq \alpha|\zeta|^2$ for every $\xi \in \overline{\Omega}$, $\zeta \in \mathbf{R}^{N+1}$ and some $\alpha > 0$.

(H2) b_i are real continuous functions on $\overline{\Omega}$.

(H3) $\min_{\xi\in\partial\Omega}\langle b(\xi),\nu(\xi)\rangle\langle a(\xi)\nu(\xi),\nu(\xi)\rangle^{-1}=\kappa>-1/p$, where $a(\xi)=\Big(a_{ij}(\xi)\Big)_{i,j}$.

Set $M=\max_{i,j}\{\|a_{ij}\|_\infty,\|b_i\|_\infty\}$. We endow A with the domain

$$D_p(A)=\Big\{u\in W^{2,p}_{\rm loc}(\Omega)\cap W^{1,p}_0(\Omega):\varrho D^2u\in L^p(\Omega)\Big\},$$

which is a Banach space with respect to the norm $\|u\|_{D_p(A)}=\|u\|_{W^{1,p}(\Omega)}+\|\varrho D^2u\|_{L^p(\Omega)}$ and is compactly embedded into $L^p(\Omega)$.

The main result of the section is stated in the next theorem.

Theorem 3.1 *Under assumptions (H1),(H2),(H3) the operator* $(-A,D_p(A))$ *generates an analytic semigroup in* $L^p(\Omega)$*. In particular, there exists* $\omega_p>0$*, such that*

$$\sup_{\mathrm{Re}\lambda\geq\omega_{\mathrm{p}}}\|\lambda(\lambda+A)^{-1}\|<+\infty.$$

In order to prove Theorem 3.1, we need some geometric preliminaries.

Let $\xi_0\in\partial\Omega$ be fixed. Without loss of generality, we can assume that at the point ξ_0 the ξ_{N+1} coordinate axis lies in the direction $\nu(\xi_0)$. By definition of a C^2 boundary, there exist an open neighbourhood $U=U_1\times U_2\subset\mathbf{R}^{N+1}$ of ξ_0, with $U_1\subset\mathbf{R}^N$ and $U_2\subset\mathbf{R}$ open and a function $F\in C^2(\overline{U_1})$, such that

$$\begin{aligned}U\cap\partial\Omega&=\{\xi=(\xi',\xi_{N+1})\in U\ :\ \xi_{N+1}=F(\xi')\},\\ U\cap\Omega&=\{\xi\in U\ :\ \xi_{N+1}>F(\xi')\}.\end{aligned}$$

The inward normal vector is given by

$$\nu(\xi)=(-D_{\xi_1}F(\xi'),\dots,-D_{\xi_N}F(\xi'),1)/(1+|DF(\xi')|^2)^{1/2},$$

for any $\xi\in U\cap\partial\Omega$. Setting

$$J(\xi)=z=(\xi',\xi_{N+1}-F(\xi')),\quad \xi\in U,$$

we obtain a C^2-diffeormorphism from U onto $\widetilde{U}=J(U)$ satisfying

$$\begin{aligned}J(U\cap\Omega)&=\widetilde{U}\cap\mathbf{R}^{N+1}_+,\\ J(U\cap\partial\Omega)&=\widetilde{U}\cap\partial\mathbf{R}^{N+1}_+.\end{aligned}$$

By compactness of $\partial\Omega$, all the derivatives of J and J^{-1} up to order 2 may be assumed to be bounded by a constant independent of ξ_0. To fix the notation, we set $H=J^{-1}$ and we suppose, for any $k=1,\dots,N+1$, that

$$\begin{aligned}&\|J_k\|_\infty+\|\nabla J_k\|_\infty+\|D^2J_k\|_\infty\leq L,\\ &\|H_k\|_\infty+\|\nabla H_k\|_\infty+\|D^2H_k\|_\infty\leq L.\end{aligned}$$

It is readily seen that the coordinate transformation J is *admissible* at ξ_0, i.e. the tangent space $T_{\partial\Omega,\xi_0}$ and the normal direction $\nu(\xi_0)$ at ξ_0 are mapped into the tangent space $T_{\partial\mathbf{R}^{N+1}_+,z_0}$ and the normal direction at $z_0 = J(\xi_0) = (x_0, 0)$. More precisely, we have $\mathrm{Jac}J(\xi_0) = I_{N+1}$.

Define $\phi(z) = \varrho(Hz)$, for $z \in \widetilde{U} \cap \mathbf{R}^{N+1}_+$. By using Taylor formula with respect to the last variable, if $z = (x, y)$ we find that

$$\begin{aligned}\phi(z) = \phi(x,y) &= \phi(x,0) + D_y\phi(x,0)\,y + \frac{1}{2}D_y^2\phi(x,t)\,y^2\\ &= y\left(D_y\phi(x,0) + \frac{1}{2}D_y^2\phi(x,t)\,y\right),\end{aligned}$$

where $t \in (0, y)$. Recalling that $\nabla\varrho(\xi) = \nu(\xi)$ for every $\xi \in \partial\Omega$, and using the explicit expressions of $\nu(\xi)$ and H, when $\xi \in U$, it follows that $D_y\phi(x, 0) = (1 + |DF(H(x, 0)')|^2)^{-1/2}$. Therefore

$$\phi(z) = yh(z), \tag{3.2}$$

where h is a continuous function which is bounded from above and below by positive constants, still independent of ξ_0, and $h(z_0) = 1$.

For a function $u : U \cap \Omega \to \mathbf{R}$, set $Tu = u \circ H$ on $\widetilde{U} \cap \mathbf{R}^{N+1}_+$. The boundedness of the derivatives of H and its inverse implies that T induces isomorphisms from $L^p(U\cap\Omega)$ onto $L^p(\widetilde{U}\cap\mathbf{R}^{N+1}_+)$ and from $W^{1,p}(U\cap\Omega)$ onto $W^{1,p}(\widetilde{U}\cap\mathbf{R}^{N+1}_+)$. Moreover, if $u \in W^{1,p}(U \cap \Omega)$, then $\varrho D^2u \in L^p(U \cap \Omega)$ iff $yD^2(Tu) \in L^p(\widetilde{U} \cap \mathbf{R}^{N+1}_+)$, with equivalence of the norms through constants independent of ξ_0.

The differential operator A is locally transformed into the operator $\tilde{A}$ given by

$$\tilde{A} = -\phi(z)\sum_{h,k=1}^{N+1}\alpha_{hk}(z)D_{z_hz_k} - \phi(z)\sum_{k=1}^{N+1}\beta_k(z)D_{z_k} + \sum_{k=1}^{N+1}\gamma_k(z)D_{z_k} \tag{3.3}$$

with

$$\begin{aligned}\alpha_{hk}(z) &= \sum_{i,j=1}^{N+1} a_{ij}(Hz)D_{\xi_j}J_h(Hz)D_{\xi_i}J_k(Hz),\\ \beta_k(z) &= \sum_{i,j=1}^{N+1} a_{ij}(Hz)D_{\xi_j\xi_i}J_k(Hz),\\ \gamma_k(z) &= \sum_{i=1}^{N+1} b_i(Hz)D_{\xi_i}J_k(Hz).\end{aligned} \tag{3.4}$$

In order to deal with the class of operators studied in the previous section, we freeze the coefficients of $\tilde{A}$ at the point z_0 as follows

$$\tilde{A}^0 = -y \sum_{h,k=1}^{N+1} \alpha_{hk}(z_0) D_{z_h z_k} + \sum_{k=1}^{N+1} \gamma_k(z_0) D_{z_k}. \tag{3.5}$$

Remark 3.2 Note that the coefficients $\alpha_{hk}(z_0)$ preserve the ellipticity condition with a constant independent of ξ_0. Moreover, since

$$\begin{aligned} \alpha_{N+1\,N+1}(z_0) &= \langle a(\xi_0)\nu(\xi_0), \nu(\xi_0)\rangle \\ \gamma_{N+1}(z_0) &= \langle b(\xi_0), \nu(\xi_0)\rangle, \end{aligned}$$

from assumption (H3) it follows that the operator $\tilde{A}^0$, defined by (3.5), satisfies the condition in Lemma 2.13 and the constant C in the statement may be chosen uniform in ξ_0.

For the sequel we need the following interpolative estimate.

Lemma 3.3 *There exist $\varepsilon_0, C > 0$ such that for every $0 < \varepsilon \le \varepsilon_0$ and every $u \in D_p(A)$ one has*

$$\|\sqrt{\varrho}\,\nabla u\|_{L^p(\Omega)} \le \varepsilon \|u\|_{D_p(A)} + \frac{C}{\varepsilon}\, \|u\|_{L^p(\Omega)}. \tag{3.6}$$

Proof. Let $\{U_0, U_1, \ldots, U_m\}$ be a finite covering of $\overline{\Omega}$ such that $U_0 \subset\subset \Omega$ and for every $i \ge 1$, U_i is a neighbourhood of some point $\xi_i \in \partial\Omega$ with the properties described above. Consider a partition of unity $\{\vartheta_i\}_{i=0}^m$ subordinate to this covering and set $\kappa = \max_{0\le i\le m} \{\|\nabla\vartheta_i\|_\infty, \|D^2\vartheta_i\|_\infty\}$. Fix $u \in D_p(A)$. Since $u_0 = \vartheta_0 u \in W^{2,p}(\Omega)$, by the classical interpolative estimate we get

$$\|\nabla u_0\|_{L^p(\Omega)} \le \delta \|u_0\|_{W^{2,p}(\Omega)} + C_1 \delta^{-1} \|u_0\|_{L^p(\Omega)},$$

for every $\delta > 0$ and some constant C_1. Since ϱ is bounded it follows that

$$\|\sqrt{\varrho}\,\nabla u_0\|_{L^p(\Omega)} \le \delta\Big(\|u\|_{D_p(A)} + \kappa \|u\|_{W^{1,p}(\Omega)}\Big) + C_2 \delta^{-1} \|u\|_{L^p(\Omega)}. \tag{3.7}$$

Now, let $i \ge 1$ and set $v_i(z) = (\vartheta_i u)(H_i z)$, where $H_i = J_i^{-1}$ and $J_i : U_i \to \widetilde{U}_i$ denotes the change of variables in U_i. Then $v_i \in D_p^\circ$ and has compact support in $\widetilde{U}_i$. From Lemma 2.7 it follows that $\|\sqrt{y}\,\nabla v_i\|_{L^p(\mathbf{R}_+^{N+1})} \le \eta \|y D^2 v_i\|_{L^p(\mathbf{R}_+^{N+1})} + (C/\eta)\|v_i\|_{L^p(\mathbf{R}_+^{N+1})}$, for every $0 < \eta \le \eta_0$. Therefore

$$\begin{aligned} \|\sqrt{\varrho}\,\nabla(\vartheta_i u)\|_{L^p(\Omega)} &\le C \|\sqrt{y}\,\nabla v_i\|_{L^p(\mathbf{R}_+^{N+1})} \\ &\le C(\eta \|u\|_{D_p(A)} + \eta^{-1} \|u\|_{L^p(\Omega)}), \end{aligned} \tag{3.8}$$

for a suitable constant $C > 0$, depending also on ϱ and κ. Finally, since $\|\vartheta_i\sqrt{\varrho}\,\nabla u\|_{L^p(\Omega)} \le \|\sqrt{\varrho}\,\nabla(\vartheta_i u)\|_{L^p(\Omega)} + \|u\sqrt{\varrho}\,\nabla\vartheta_i\|_{L^p(\Omega)}$, summing estimates (3.7) and (3.8) over $i = 0, \dots, m$ and using the arbitrariness of δ and η we obtain the statement. $\square$

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. For every $\xi_0 \in \partial\Omega$, let U_{ξ_0} be the open neighbourhood of ξ_0 and J_{ξ_0} the corresponding coordinate transformation described at the beginning of the section. Given $\varepsilon > 0$, choose a ball $B_{r(\xi_0)}(\xi_0) \subset U_{\xi_0}$ such that if $z \in J_{\xi_0}(B_{r(\xi_0)}(\xi_0)) \cap \mathbf{R}^{N+1}_+$, then for every $h, k = 1, \dots, N+1$

$$\begin{aligned} |h(z)\alpha_{hk}(z) - \alpha_{hk}(z_0)| &< \varepsilon, \\ |\phi(z)\beta_k(z)| &< \varepsilon, \\ |\gamma_k(z) - \gamma_k(z_0)| &< \varepsilon, \end{aligned} \tag{3.9}$$

where $z_0 = J_{\xi_0}(\xi_0)$, $\alpha_{hk}, \beta_k, \gamma_k$ are given in (3.4) and h, ϕ in (3.2). Set $\mathcal{F}_\varepsilon = \{B_{r(\xi)}(\xi) : \xi \in \partial\Omega\}$. By means of a suitable covering argument (see e.g. [4, Theorem 2.18]), recalling that $\partial\Omega$ is compact, we can extract a finite subcovering $\mathcal{F}'_\varepsilon = \{B_{r(\xi_i)}(\xi_i) : i = 1, \dots, m\}$ such that at most c_N among the balls of $\mathcal{F}'_\varepsilon$ overlap. Here c_N is a natural number which depends only on the dimension. Set $U_i = B_{r(\xi_i)}(\xi_i)$, $J_i = J_{\xi_i|B_{r(\xi_i)}(\xi_i)}$ and $\widetilde{U}_i = J_i(U_i)$, $z_i = J_i(\xi_i)$. Finally, let $U_0 \subset\subset \Omega$ be an open set with boundary of class C^2 such that $\{U_0, U_1, \dots, U_m\}$ is a covering of $\overline{\Omega}$.

To prove the statement it suffices to show that $(-A, D_p(A))$ is a sectorial operator in $L^p(\Omega)$. We split the proof in two steps.

Step 1. We first deal with the surjectivity of the operator $\lambda + A : D_p(A) \to L^p(\Omega)$. To be definite, we show that there exist $\omega'_p, C > 0$ such that for every $\operatorname{Re}\lambda \ge \omega'_p$ and $f \in L^p(\Omega)$ there is $u \in D_p(A)$ satisfying $\lambda u + Au = f$ and $|\lambda|\,\|u\|_{L^p(\Omega)} \le C\,\|f\|_{L^p(\Omega)}$.

Consider the open covering $\{U_0, U_1, \dots, U_m\}$ of $\overline{\Omega}$, as above, with ε to be determined. Let $H_i = J_i^{-1}$ and $T_i : L^p(U_i) \to L^p(\widetilde{U}_i)$, $T_i\varphi = \varphi \circ H_i$. Set $\Omega_i = U_i \cap \Omega$. Let $\{\eta_i^2\}_{i=0}^m$ be a partition of unity subordinate to such a covering, with $0 \le \eta_i \le 1$. To simplify the notation, in the constants appearing in the estimates below we make only the dependence on U_i explicit, whereas we omit the dependence on other quantities such as $N, p, \Omega, M, \kappa, \alpha, L$.

Let $f \in L^p(\Omega)$ be fixed. Since the operator A is nondegenerate in U_0, it is well-known that if $\operatorname{Re}\lambda \ge \lambda_0$, for a suitable $\lambda_0 \in \mathbf{R}$, then there exists a unique solution $u_0 \in W^{2,p}(U_0) \cap W^{1,p}_0(U_0)$ of the equation $\lambda u_0 + Au_0 = \eta_0 f$. Set

$R_0(\lambda)f = \eta_0 u_0$. Then $R_0(\lambda)f \in D_p(A)$ and

$$(\lambda + A)R_0(\lambda)f = \eta_0^2 f + [A, \eta_0]u_0 = \eta_0^2 f + E_0 f,$$

where the symbol $[\cdot\,,\cdot]$ denotes the commutator of two operators. It is easily seen that

$$\|E_0 f\|_{L^p(\Omega)} \le \frac{C_0}{|\lambda|^{1/2}} \|f\|_{L^p(U_0)}, \tag{3.10}$$

where the constant C_0 depends on U_0.

Now, fix $i \ge 1$. Denote by $\tilde{A}_i$, $\tilde{A}_i^0$ the operators obtained from $\tilde{A}$, $\tilde{A}^0$, defined in (3.3), (3.5), replacing J, H, z_0 with J_i, H_i, z_i, respectively. Taking Remark 3.2 into account, for every $\operatorname{Re}\lambda > 0$, there exists a unique solution $v_i \in D_p^\circ$ of $\lambda v_i + \tilde{A}_i^0 v_i = T_i(\eta_i f)$ in $\mathbf{R}_+^{N+1}$. Let $R_i(\lambda)f$ be the trivial extension to Ω of the function $T_i^{-1}\Big(T_i(\eta_i)v_i|_{\tilde{U}_i}\Big)$. Then, $R_i(\lambda)f \in D_p(A)$ and it has compact support contained in $\overline{\Omega}_i$. Since $A = T_i^{-1}\tilde{A}_i T_i$ in $L^p(\Omega_i)$, we easily get

$$\begin{aligned}(\lambda + A)R_i(\lambda)f &= T_i^{-1}(\lambda + \tilde{A}_i)\Big(T_i(\eta_i)v_i\Big) \\ &= \eta_i^2 f + B_i f + E_i f,\end{aligned}$$

where we have set

$$B_i f = \eta_i T_i^{-1}\Big((\tilde{A}_i - \tilde{A}_i^0)v_i\Big)$$

$$E_i f = T_i^{-1}\Big([\tilde{A}_i, T_i(\eta_i)]v_i\Big).$$

Now, we are going to estimate the L^p-norms of $B_i f$ and $E_i f$. Concerning $B_i f$, we observe that for every $z \in \widetilde{U}_i \cap \mathbf{R}_+^{N+1}$

$$\begin{aligned}(\tilde{A}_i - \tilde{A}_i^0)v_i(z) = &- \sum_{h,k=1}^{N+1} y\Big(h^i(z)\alpha_{hk}^i(z) - h^i(z_i)\alpha_{hk}^i(z_i)\Big)D_{z_h z_k} v_i(z) \\ &- \sum_{k=1}^{N+1} \phi^i(z)\beta_k^i(z)D_{z_k} v_i(z) + \sum_{k=1}^{N+1} (\gamma_k^i(z) - \gamma_k^i(z_i))D_{z_k} v_i(z),\end{aligned}$$

where the apex i means that the corresponding function refers to (U_i, J_i). Therefore

$$\|B_i f\|_{L^p(\Omega)} \le C\, \|(\tilde{A}_i - \tilde{A}_i^0)v_i\|_{L^p(\widetilde{U}_i \cap \mathbf{R}_+^{N+1})} \le C\varepsilon \|v_i\|_{D_p},$$

where, in the last step, we have used the fact that U_i has been constructed in such a way that (3.9) is satisfied. Applying Lemma 2.13 to the operator $\tilde{A}_i^0$ and recalling Remark 3.2, it turns out that

$$\begin{aligned}\|v_i\|_{D_p} &\le C\Big(\|\tilde{A}_i^0 v_i\|_{L^p(\mathbf{R}_+^{N+1})} + \|v_i\|_{L^p(\mathbf{R}_+^{N+1})}\Big) \\ &\le C\Big(\|T_i(\eta_i f)\|_{L^p(\mathbf{R}_+^{N+1})} + (|\lambda| + 1)|\lambda|^{-1}\|T_i(\eta_i f)\|_{L^p(\mathbf{R}_+^{N+1})}\Big) \\ &\le C\|f\|_{L^p(\Omega_i)}.\end{aligned}$$

Thus, we have established that

$$\|B_i f\|_{L^p(\Omega)} \le C\varepsilon \|f\|_{L^p(\Omega_i)}. \tag{3.11}$$

Concerning the norm of $E_i f$, we have

$$\begin{aligned}\|E_i f\|_{L^p(\Omega)} &\le C\|[\tilde{A}_i, T_i(\eta_i)]v_i\|_{L^p(\widetilde{U}_i\cap \mathbf{R}^{N+1}_+)}\\ &\le C_i\Big(\|\sqrt{y}\,\nabla v_i\|_{L^p(\mathbf{R}^{N+1}_+)} + \|v_i\|_{L^p(\mathbf{R}^{N+1}_+)}\Big).\end{aligned}$$

If $|\lambda| \ge 1/\eta_0^2$, then from Corollary 2.14 and Lemma 2.13 applied to $\tilde{A}_i^0$, it follows that

$$\|E_i f\|_{L^p(\Omega)} \le \frac{C_i}{|\lambda|^{1/2}}\|f\|_{L^p(\Omega_i)}. \tag{3.12}$$

Setting $R(\lambda)f = \sum_{i=0}^m R_i(\lambda)f$ and $S(\lambda)f = \sum_{i=1}^m \Big(B_i f + E_i f\Big) + E_0 f$ we find that

$$(\lambda + A)R(\lambda)f = f + S(\lambda)f. \tag{3.13}$$

Estimates (3.10), (3.11) and (3.12) imply that

$$\|S(\lambda)f\|_{L^p(\Omega)} \le \sum_{i=1}^m C\varepsilon\|f\|_{L^p(\Omega_i)} + \sum_{i=0}^m \frac{C_i}{|\lambda|^{1/2}}\|f\|_{L^p(\Omega_i)}.$$

Since at most c_N among the U_i's overlap, we get

$$\|S(\lambda)f\|_{L^p(\Omega)} \le c_N C\varepsilon\|f\|_{L^p(\Omega)} + \sum_{i=0}^m \frac{C_i}{|\lambda|^{1/2}}\|f\|_{L^p(\Omega_i)}.$$

Now, it is clear that we can choose $\varepsilon > 0$ sufficiently small and λ large enough to get $\|S(\lambda)\| \le 1/2$. This shows that there exists $\omega_p' \ge \max\{1/\eta_0^2, \lambda_0\} > 0$ such that for every $\operatorname{Re}\lambda \ge \omega_p'$, $I + S(\lambda) : L^p(\Omega) \to L^p(\Omega)$ is invertible and, denoted by $V(\lambda)$ its inverse, $\|V(\lambda)\| \le 2$. By (3.13), with $V(\lambda)f$ instead of f, we infer that $u = R(\lambda)V(\lambda)f$ is a function in $D_p(A)$ and solves the equation $(\lambda + A)u = f$. Moreover,

$$\|u\|_{L^p(\Omega)} \le \sum_{i=0}^m \|R_i(\lambda)V(\lambda)f\|_{L^p(\Omega)} \le \frac{C}{|\lambda|}\|V(\lambda)f\|_{L^p(\Omega)} \le \frac{2C}{|\lambda|}\|f\|_{L^p(\Omega)}.$$

Hence, the first step is done.

Step 2. Now, we study the injectivity of $\lambda + A$. According to the notation introduced in the first step, if $u \in D_p(A)$ and $\operatorname{Re}\lambda > \max\{0, \lambda_0\}$, we can write

$$\begin{aligned} R_i(\lambda)(\lambda + A)u &= \eta_i^2 u + F_i u + G_i u, \quad i \ge 1,\\ R_0(\lambda)(\lambda + A)u &= \eta_0^2 u + Hu \end{aligned}$$

where

$$F_i u = \eta_i T_i^{-1}\Big((\lambda+\tilde{A}_i^0)^{-1}(\tilde{A}_i-\tilde{A}_i^0)T_i(\eta_i u)\Big)$$
$$G_i u = \eta_i T_i^{-1}\Big((\lambda+\tilde{A}_i^0)^{-1}T_i\big([\eta_i, A]u\big)\Big),$$

and, if A_0 denotes the realization of A in $L^p(U_0)$ with Dirichlet boundary conditions

$$Hu = \eta_0(\lambda + A_0)^{-1}([A,\eta_0]u).$$

Summing over i, it turns out that

$$\sum_{i=0}^{m} R_i(\lambda)(\lambda+A)u = u + \sum_{i=1}^{m}(F_i u + G_i u) + Hu,$$

for every $u \in D_p(A)$. Let $u \in D_p(A)$ be such that $(\lambda + A)u = 0$. Then, the expression above implies that

$$u = -\sum_{i=1}^{m}\Big(F_i u + G_i u\Big) - Hu \tag{3.14}$$

We claim that $u = 0$. To prove this, we need to estimate the norms of u in $D_p(A)$ and in $L^p(\Omega)$. It is useful to set

$$\|\cdot\|_{D_{p,i}} = \|\cdot\|_{W^{1,p}(\Omega_i)} + \|\varrho D^2(\cdot)\|_{L^p(\Omega_i)},$$
$$\|\cdot\|_{p,i} = \|\cdot\|_{L^p(\Omega_i)}.$$

The easiest term to be estimated is Hu, since it involves a nondegenerate operator. To this aim, we observe that, as Hu is supported in U_0, its norm in $D_p(A)$ is equivalent to the $W^{2,p}$-norm, therefore the classical L^p estimates yield

$$\|Hu\|_{D_p(A)} \le C_0\|[A,\eta_0]u\|_{p,0}.$$

Since $[A,\eta_0]$ is a first-order operator, for every $\delta > 0$ there exists $C_\delta > 0$ such that

$$\|Hu\|_{D_p(A)} \le C_0\delta\|u\|_{D_{p,0}} + C_\delta\|u\|_{p,0}. \tag{3.15}$$

On the other hand

$$\|Hu\|_{L^p(\Omega)} \le \frac{C_0}{|\lambda|}\|u\|_{D_{p,0}}. \tag{3.16}$$

Here, C_0 denotes a suitable constant depending on η_0. Now, we estimate $F_i u$ and $G_i u$, for every $i \ge 1$. To keep the notation simpler, we set

$$f_i = (\tilde{A}_i - \tilde{A}_i^0)T_i(\eta_i u), \quad g_i = T_i[\eta_i, A]u$$

and we define

$$\varphi_i = T_i^{-1}(\lambda+\tilde{A}_i^0)^{-1}f_i, \quad \psi_i = T_i^{-1}(\lambda+\tilde{A}_i^0)^{-1}g_i.$$

As a consequence, we can write $F_i u = \eta_i \varphi_i$ and $G_i u = \eta_i \psi_i$. It is easily seen that

$$\|F_i u\|_{D_p(A)} \le \|\varphi_i\|_{D_{p,i}} + C_i(\|\varphi_i\|_{p,i} + \|\sqrt{\varrho}\,\nabla\varphi_i\|_{p,i}), \tag{3.17}$$

where C_i depends on $\|\nabla\eta_i\|_\infty, \|D^2\eta_i\|_\infty$ and Ω. Now, applying Remark 3.2 and recalling the choice of U_i we get

$$\|\varphi_i\|_{D_{p,i}} \le C\|(\lambda + \tilde{A}_i^0)^{-1} f_i\|_{D_p} \le C\|f_i\|_p \le C\varepsilon\|\eta_i u\|_{D_{p,i}}$$

and

$$\|\varphi_i\|_{p,i} \le \frac{C}{|\lambda|}\|f_i\|_p \le \frac{C}{|\lambda|}\varepsilon\|\eta_i u\|_{D_{p,i}}.$$

On the other hand, thanks to Corollary 2.14, if $|\lambda| \ge 1/\eta_0^2$ then

$$\|\sqrt{\varrho}\,\nabla\varphi_i\|_{p,i} \le C\|\sqrt{y}\,\nabla(\lambda + \tilde{A}_i^0)^{-1} f_i\|_p \le \frac{C}{|\lambda|^{1/2}}\|f_i\|_p \le \frac{C}{|\lambda|^{1/2}}\varepsilon\|\eta_i u\|_{D_{p,i}}. \tag{3.18}$$

As

$$\|\eta_i u\|_{D_{p,i}} \le \|u\|_{D_{p,i}} + C_i(\|u\|_{p,i} + \|\sqrt{\varrho}\,\nabla u\|_{p,i}),$$

we finally obtain

$$\begin{aligned}\|F_i u\|_{D_p(A)} &\le \left(C\varepsilon + \frac{C_i}{|\lambda|^{1/2}}\right)\|\eta_i u\|_{D_{p,i}}\\ &\le \left(C\varepsilon + \frac{C_i}{|\lambda|^{1/2}}\right)\|u\|_{D_{p,i}} + C_i(\|u\|_{p,i} + \|\sqrt{\varrho}\,\nabla u\|_{p,i}).\end{aligned} \tag{3.19}$$

For our purposes, we need to estimate the L^p norm of $F_i u$ independently. This is much easier; indeed, we immediately have

$$\|F_i u\|_{L^p(\Omega)} \le \frac{C}{|\lambda|}\|f_i\|_p \le \frac{C_i}{|\lambda|}\|u\|_{D_{p,i}}. \tag{3.20}$$

Next, we consider the term $G_i u$. Replacing φ_i, f_i with ψ_i, g_i, respectively, in (3.17)–(3.18) and observing that

$$\|g_i\|_p \le C_i(\|u\|_{p,i} + \|\sqrt{\varrho}\,\nabla u\|_{p,i}),$$

we infer

$$\|G_i u\|_{D_p(A)} \le C_i(\|u\|_{p,i} + \|\sqrt{\varrho}\,\nabla u\|_{p,i}), \tag{3.21}$$

and

$$\|G_i u\|_{L^p(\Omega)} \le \frac{C}{|\lambda|}\|g_i\|_p \le \frac{C_i}{|\lambda|}\|u\|_{D_{p,i}}. \tag{3.22}$$

Now, by (3.14), (3.15), (3.19) and (3.21) we derive

$$\begin{aligned}\|u\|_{D_p(A)} \le &\sum_{i=1}^m \left(C\varepsilon + \frac{C_i}{|\lambda|^{1/2}}\right)\|u\|_{D_{p,i}} + \sum_{i=1}^m C_i(\|u\|_{p,i} + \|\sqrt{\varrho}\,\nabla u\|_{p,i})\\ &+ C_0\delta\|u\|_{D_{p,0}} + C_\delta\|u\|_{p,0}.\end{aligned}$$

At this point, arguing as in the end of the first step, choose ε, δ sufficiently small and λ sufficiently large to obtain

$$\|u\|_{D_p(A)} \leq C(\|u\|_{L^p(\Omega)} + \|\sqrt{\varrho}\,\nabla u\|_{L^p(\Omega)}).$$

Using the interpolative estimate (3.6) we get

$$\|u\|_{D_p(A)} \leq C\|u\|_{L^p(\Omega)}.$$

Moreover, from (3.14), (3.16), (3.20) and (3.22) it follows that

$$\|u\|_{L^p(\Omega)} \leq \frac{C}{|\lambda|}\|u\|_{D_p(A)}.$$

Combining the last two estimates we obtain

$$\|u\|_{D_p(A)} \leq \frac{C}{|\lambda|}\|u\|_{D_p(A)},$$

which leads to a contradiction, for λ large, unless $u = 0$. Therefore, there exists $\omega''_p > 0$ such that $\lambda + A : D_p(A) \to L^p(\Omega)$ is injective for every $\mathrm{Re}\,\lambda \geq \omega''_p$. Hence, the second step is complete.

Now, we are immediately led to the conclusion. Indeed, from Steps 1,2 it follows that $\lambda + A$ is bijective from $D_p(A)$ onto $L^p(\Omega)$, for every $\mathrm{Re}\lambda \geq \omega_p = \max\{\omega'_p, \omega''_p\}$ and, in addition, $\sup_{\mathrm{Re}\lambda\geq\omega_p} \|\lambda(\lambda + A)^{-1}\| < +\infty$. □

As an immediate consequence of the result above we have the next corollary.

Corollary 3.4 *Denote by $(T_p(t))_{t\geq 0}$ the semigroup generated by $(-A, D_p(A))$. Assume that the constant κ given in (H3) satisfies $\kappa > -1/q$, with $1 < p < q < +\infty$. Then*

(i) *$T_p(t)f = T_q(t)f$, for every $f \in L^q(\Omega)$. Therefore, we may simply write $T(t)$ instead of $T_p(t)$. Moreover, $T(t)$ is compact;*
(ii) *the spectra and the spectral subspaces of $(A, D_p(A))$, $(A, D_q(A))$ coincide.*

Proof. The consistency of the semigroups $(T_p(t))_{t\geq 0}$ and $(T_q(t))_{t\geq 0}$ follows from that the consistency of the corresponding resolvents which is an immediate consequence of the inclusion $D_q(A) \subset D_p(A)$. The analyticity of $T(t)$ and the compactness of the resolvent operator $(\lambda + A)^{-1}$ yield the compactness of the semigroup. Thus (i) holds true. (ii) is well-known since the resolvents are compact. A proof can be found in [5, Proposition 2.6]. □

We conclude the section by proving some estimates that will be used to treat the case of continuous functions.

Corollary 3.5 *There exists a constant $C > 0$ such that for every $\mathrm{Re}\,\lambda \geq \Lambda_p = \max\{\omega_p, \varepsilon_0^{-2}, 1\}$, ε_0 being given in Lemma 3.3, and every $u \in D_p(A)$, it holds*

$$|\lambda|\|u\|_{L^p(\Omega)}+|\lambda|^{1/2}\|\sqrt{\varrho}\,\nabla u\|_{L^p(\Omega)}+\|\nabla u\|_{L^p(\Omega)}+\|\varrho D^2 u\|_{L^p(\Omega)} \leq C\|\lambda u+Au\|_{L^p(\Omega)}.$$

PROOF. Let $\mathrm{Re}\,\lambda \geq \Lambda_p$ be fixed. From Theorem 3.1 it follows that for every $u \in D_p(A)$ we have $|\lambda|\,\|u\|_{L^p(\Omega)} \leq c\|\lambda u+Au\|_{L^p(\Omega)}$, where $c = \sup_{\mathrm{Re}\lambda\geq\omega_p} \|\lambda(\lambda+A)^{-1}\|$.

Since $(A, D_p(A))$ is closed, $D_p(A)$ is complete with respect to the graph norm. On the other hand, $D_p(A)$ is complete also with respect to $\|\cdot\|_{D_p(A)}$ and $\|Au\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)} \leq c_1\|u\|_{D_p(A)}$, for every $u \in D_p(A)$. The open mapping theorem implies that the two norms are equivalent. In particular, there exists a constant $c_2 > 0$ such that

$$\|u\|_{D_p(A)} \leq c_2(\|Au\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}),\quad u \in D_p(A).$$

As $\|Au\|_{L^p(\Omega)} \leq (c+1)\|\lambda u + Au\|_{L^p(\Omega)}$, it follows that

$$\|\nabla u\|_{L^p(\Omega)} + \|\varrho D^2 u\|_{L^p(\Omega)} \leq c_2(2c+1)\|\lambda u + Au\|_{L^p(\Omega)}.$$

It remains to estimate $\|\sqrt{\varrho}\,\nabla u\|_{L^p(\Omega)}$. Since $|\lambda| \geq \max\{\varepsilon_0^{-2}, 1\}$, choosing $\varepsilon = |\lambda|^{-1/2}$ in (3.6) we get

$$\|\sqrt{\varrho}\,\nabla u\|_{L^p(\Omega)} \leq C\,|\lambda|^{-1/2}\|\lambda u + Au\|_{L^p(\Omega)},$$

for a suitable constant $C > 0$, as stated. □

4 Spaces of continuous functions

The aim of this section is to show that the operator $-A$, where A is defined in (3.1), endowed with the domain

$$D_0(A) = \left\{u \in C(\overline{\Omega}) \cap \bigcap_{1\leq p<\infty} W^{2,p}_{\mathrm{loc}}(\Omega)\,|\,\sqrt{\varrho}\,\nabla u,\ Au \in C(\overline{\Omega}),\ u_{|\partial\Omega} = 0\right\}$$

generates an analytic semigroup in $C(\overline{\Omega})$. In order to prove such a result, we adapt to our situation the Masuda-Stewart method, which is well-known in the case of uniformly elliptic operators (see [20, §3.1.2, 3.1.5]).

We start by proving the analogue of the classical Morrey imbedding theorem, *i.e.* if $p > 2(N+1)$, then the weighted Sobolev space of the functions $u \in L^p(\Omega)$

with $\sqrt{\varrho}\,\nabla u \in L^p(\Omega)$ is continuously imbedded into $C(\overline{\Omega})$. Actually, in the next lemma we provide a sharp local estimate which implies the above imbedding theorem and is the main ingredient in order to make the Masuda-Stewart method working in our setting. We observe that $C^1(\overline{\Omega})$ is dense in the space $\{u \in L^p(\Omega) \,|\, \sqrt{\varrho}\,\nabla u \in L^p(\Omega)\}$. This can be seen arguing as in the proof of Lemma 2.1 if $\Omega = \mathbf{R}^{N+1}_+$ and using local coordinates transformations, as in §3, in the case of a bounded open set.

As in the sequel we need the results of §3, we assume, since the beginning of the section, that Ω is a C^2 bounded open subset of $\mathbf{R}^{N+1}$, even though the imbedding result could be proved under weaker regularity assumptions on the domain. Here ϱ is the same function as in §3.

Lemma 4.1 *The following assertions hold.*

(i) *There exists a cone $\mathcal{C} = B_h(0) \cap \{\lambda y \,|\, \lambda > 0, y \in \Sigma\}$, where $h > 0$ and Σ is an open, non-empty set on the unitary sphere $\mathbf{S}^N = \{y \in \mathbf{R}^{N+1}, |y| = 1\}$, such that every $x \in \overline{\Omega}$ is the vertex of a cone $\mathcal{C}_x$ congruent to $\mathcal{C}$ and contained in Ω, with $\overline{\mathcal{C}_x} \cap \partial\Omega \subset \{x\}$. Moreover, there is a positive constant K with the property that*

$$\varrho(y) \geq K\Big(\varrho(x) + |x-y|\Big), \tag{4.1}$$

for every $x \in \overline{\Omega}$ and $y \in \mathcal{C}_x$.

(ii) *If $p > 2(N+1)$ and $\phi \in L^p(\Omega)$ with $\sqrt{\varrho}\,\nabla\phi \in L^p(\Omega)$ then $\phi \in C(\overline{\Omega})$. Moreover, there exists $C > 0$ such that for every ϕ as above and every $x \in \overline{\Omega}$ it holds*

$$|\phi(x)| \leq \begin{cases} C\,h^{-\frac{N+1}{p}}\Big(\|\phi\|_{L^p(\mathcal{C}_x)} + h^{\frac{1}{2}}\|\sqrt{\varrho}\,\nabla\phi\|_{L^p(\mathcal{C}_x)}\Big) \\ \qquad\qquad\qquad\qquad \text{if } \varrho(x) \leq h, \\ C\,h^{-\frac{N+1}{p}}\left(\|\phi\|_{L^p(\mathcal{C}_x)} + \dfrac{h}{\sqrt{\varrho(x)}}\|\sqrt{\varrho}\,\nabla\phi\|_{L^p(\mathcal{C}_x)}\right) \\ \qquad\qquad\qquad\qquad \text{if } \varrho(x) \geq h, \end{cases} \tag{4.2}$$

where $\mathcal{C}_x$ is defined in (i).

Proof. (i) As Ω is of class C^2, there exist $h > 0$, $\delta > 0$, with the following property: for every $x \in \overline{\Omega}$ there is a ball of radius h, B^x_h, contained in Ω, with $x \in \partial B^x_h$; if $\varrho(x) > \delta$ then $B^x_h \subset \{z \in \Omega \,|\, \varrho(z) > \delta\}$ and, if $\varrho(x) \leq \delta$ then x is the unique point on ∂B^x_h such that $\mathrm{dist}(\partial B^x_h, \partial\Omega) = \varrho(x)$.

Assume, first, that $\varrho(x) \leq \delta$ and let ξ be the unique point on $\partial\Omega$ of minimal distance from x. Without loss of generality, we suppose that x and the center of B^x_h, $\tilde{x}$, lie on the normal direction to $\partial\Omega$ at ξ. Let us consider the hyperplane π perpendicular to $\nu(\xi)$ through $\tilde{x}$. Let $\mathcal{K}$ be the set obtained by joining the points of $\pi \cap B^x_{h/2}$ to x, where $B^x_{h/2}$ is the ball having radius $h/2$ and center

$\tilde{x}$. We also denote by $\widetilde{\mathcal{K}}$ the set obtained by joining the points of $\pi \cap B_h^x$ to x. Set

$$\mathcal{C}_x = \mathcal{K} \cap B_h(x)$$

and let us prove that $\mathcal{C}_x$ fulfills (4.1), for a suitable value of $K > 0$, independent of x. For every $y \in \mathcal{C}_x$ let z be the intersection point between the boundary of $\widetilde{\mathcal{K}}$ and the normal direction to $\partial\Omega$ through y. Since $z \in B_h^x$, we have that $\varrho(z) \geq \operatorname{dist}(\partial B_h^x, \partial\Omega) = \varrho(x)$. On the other hand, $|y - z| \geq \sin\beta |y - x|$, where β is the difference of the opening angles of $\widetilde{\mathcal{K}}$ and $\mathcal{C}_x$. It follows that

$$\varrho(y) = |y - z| + \varrho(z) \geq \sin\beta |y - x| + \varrho(x) \geq \sin\beta \Big(|y - x| + \varrho(x)\Big).$$

If $\varrho(x) > \delta$, the situation is simpler. We consider the cone $\mathcal{C}_x$ obtained by joining the points of $\pi \cap B_{h/2}^x$ to x, where now π is the hyperplane through the center of B_h^x and perpendicular to the radial direction passing through x, intersecting with $B_h(x)$. Clearly, $\mathcal{C}_x$ is congruent to the previous one. Moreover, for every $y \in \mathcal{C}_x$ we have $|y - x| \leq h$ and then

$$\frac{\varrho(y)}{\varrho(x) + |y - x|} \geq \frac{\delta}{\|\varrho\|_\infty + h}.$$

Therefore, (4.1) is satisfied with $K = \min\left\{\frac{\delta}{\|\varrho\|_\infty + h}, \sin\beta\right\}$.

(ii) Assume first that $\phi \in C^1(\overline{\Omega})$. Then we have only to show estimate (4.2). The idea to prove it is similar to [1, Lemma 5.15]. Let $x \in \overline{\Omega}$ and let $\mathcal{C}_x$ be the cone given by (i). By introducing spherical coordinates r, ω with origin at x, we can describe $\mathcal{C}_x$ by $0 < r \leq h$, $\omega \in \Sigma \subset \mathbf{S}^N$ and we can write

$$\phi(x) = \phi(r\omega) - \int_0^r \frac{d}{dt}\phi(t\omega)dt.$$

It follows that

$$|\phi(x)| \leq |\phi(r\omega)| + \int_0^h |\nabla\phi(t\omega)|dt.$$

Multiplying by r^N and integrating r over $(0, h]$ and ω over Σ we obtain

$$\begin{aligned} |\mathcal{C}_x|\,|\phi(x)| &\leq \int_{\mathcal{C}_x} |\phi(y)|dy + \frac{h^{N+1}}{N+1}\int_{\mathcal{C}_x} \frac{|\nabla\phi(y)|}{|x - y|^N}\frac{\sqrt{\varrho(y)}}{\sqrt{\varrho(y)}}\,dy \\ &\leq |\mathcal{C}_x|^{1-\frac{1}{p}}\|\phi\|_{L^p(\mathcal{C}_x)} + \frac{h^{N+1}}{N+1}\|\sqrt{\varrho}\,\nabla\phi\|_{L^p(\mathcal{C}_x)}\, I^{1-\frac{1}{p}}, \end{aligned} \tag{4.3}$$

where we have set $I = \displaystyle\int_{\mathcal{C}_x} |x - y|^{-\frac{Np}{p-1}}\varrho(y)^{-\frac{p}{2(p-1)}}dy$. Now, we have to estimate I. By (4.1), we get

$$
\begin{aligned}
I &\le K^{-\frac{p}{2(p-1)}} \int_{\mathcal{C}_x} |x-y|^{-\frac{Np}{p-1}} \Big(\varrho(x)+|x-y|\Big)^{-\frac{p}{2(p-1)}} dy \\
&= K^{-\frac{p}{2(p-1)}} |\Sigma| \int_0^h r^{-\frac{N}{p-1}} \Big(\varrho(x)+r\Big)^{-\frac{p}{2(p-1)}} dr.
\end{aligned}
$$

We observe that since $p > 2N+2$ the right hand side is finite. If $\varrho(x)=0$, then $I \le CK^{-\frac{p}{2(p-1)}}|\Sigma| h^{\frac{p-2N-2}{2(p-1)}}$. If $\varrho(x) \neq 0$, by changing variable in the integral we obtain

$$
I \le K^{-\frac{p}{2(p-1)}} \, |\Sigma| \, \varrho(x)^{\frac{p-2N-2}{2(p-1)}} \int_0^{h/\varrho(x)} s^{-\frac{N}{p-1}} (1+s)^{-\frac{p}{2(p-1)}} ds. \tag{4.4}
$$

Now, it is convenient to estimate the last integral in two different ways, getting

$$
\varrho(x)^{\frac{p-2N-2}{2(p-1)}} \int_0^{h/\varrho(x)} s^{-\frac{N}{p-1}} (1+s)^{-\frac{p}{2(p-1)}} ds \le C \begin{cases} h^{1-\frac{N}{p-1}} \varrho(x)^{-\frac{p}{2(p-1)}} & \\ & \text{if } \frac{h}{\varrho(x)} \le 1, \\ h^{\frac{p-2N-2}{2(p-1)}} & \\ & \text{if } \frac{h}{\varrho(x)} > 1, \end{cases} \tag{4.5}
$$

The thesis now follows from (4.3), (4.4) and (4.5), since $|\mathcal{C}_x| = |\Sigma| \, \frac{h^{N+1}}{N+1}$.

If $\phi \in L^p(\Omega)$ and $\sqrt{\varrho}\,\nabla\phi \in L^p(\Omega)$ then there exists a sequence $\{\phi_n\} \subset C^1(\overline{\Omega})$ such that $\phi_n \to \phi$ and $\sqrt{\varrho}\,\nabla\phi_n \to \sqrt{\varrho}\,\nabla\phi$ in $L^p(\Omega)$. Estimate (4.2) implies that ϕ_n converges to ϕ uniformly in $\overline{\Omega}$. Therefore $\phi \in C(\overline{\Omega})$ and ϕ satisfies (4.2) as well. □

Theorem 4.2 *Assume that (H1), (H2) are satisfied and that (H3) holds with* $\kappa > -\frac{1}{2N+2}$. *Then, the operator* $(-A, D_0(A))$ *generates an analytic semigroup in* $C(\overline{\Omega})$.

PROOF. For every $x \in \overline{\Omega}$, let $\mathcal{C}_x$ be a cone with vertex x and height $h < 1$ having the properties of Lemma 4.1 (i). We can write $\mathcal{C}_x$ in the form $\mathcal{C}_x = x + \tau_x(\mathcal{C})$, which means that $\mathcal{C}_x$ is obtained from a fixed cone $\mathcal{C}$ with vertex at the origin and height h by a rotation τ_x and a translation. If $\lambda \in \mathbf{C} \setminus \{0\}$, define

$$
r(x) = \begin{cases} h^{-1}|\lambda|^{-1} & \text{if } \varrho(x) \le |\lambda|^{-1}, \\ \sqrt{\varrho(x)}h^{-1}|\lambda|^{-1/2} & \text{if } \varrho(x) \ge |\lambda|^{-1}, \end{cases}
$$

and set $\mathcal{B}_x = B_{r(x)}(x)$. Choosing $|\lambda|$ large enough, we have that $x + r(x)\tau_x(\mathcal{C}) \subset \mathcal{B}_x \cap \mathcal{C}_x$, for every $x \in \overline{\Omega}$. In particular, we may apply Lemma 4.1 with $\mathcal{C}_x$ replaced by $\mathcal{C}_x^* = x + r(x)\tau_x(\mathcal{C})$ and with the same constant K.

Since $\kappa > -1/(2N+2)$, there exists $p > 2N+2$ such that $\kappa > -1/p$. Take $f \in C(\overline{\Omega})$. Then $f \in L^p(\Omega)$ and from Theorem 3.1 we deduce that if $\mathrm{Re}\lambda \ge \omega_p$, there is a unique solution $u \in D_p(A)$ of the equation $\lambda u + Au = f$. We observe that Lemma 4.1 (ii) implies that $u, \sqrt{\varrho}\,\nabla u \in C(\overline{\Omega})$ and, clearly, $u_{|\partial\Omega} = 0$.

Let η_x be a smooth function satisfying $0 \le \eta_x \le 1$, $\eta_x \equiv 1$ in $\mathcal{B}_x$, $\eta_x \equiv 0$ outside $\mathcal{B}_x^\alpha = B_{(\alpha+1)r(x)}(x)$, where α is a positive parameter to be determined. Then

$$\alpha\, r(x)\|\nabla \eta_x\|_\infty + \alpha^2 r(x)^2 \|D^2\eta_x\|_\infty \le L,$$

with L independent of x and α. Set $v(y) = \eta_x(y)u(y)$. It is easily seen that $v \in D_p(A)$ and solves the equation

$$\lambda v + Av = \eta_x f - 2\varrho \sum_{i,j=1}^{N+1} a_{ij} D_i\eta_x D_j u + u \sum_{i=1}^{N+1} b_i D_i \eta_x - \varrho\, u \sum_{i,j=1}^{N+1} a_{ij} D_{ij}\eta_x.$$

If Re$\lambda \ge \Lambda_p$, we may apply Corollary 3.5 to v, getting

$$\begin{aligned}
&|\lambda|\|u\|_{L^p(\Omega\cap\mathcal{B}_x)} + |\lambda|^{1/2}\|\sqrt{\varrho}\,\nabla u\|_{L^p(\Omega\cap\mathcal{B}_x)} + \|\nabla u\|_{L^p(\Omega\cap\mathcal{B}_x)} + \|\varrho D^2 u\|_{L^p(\Omega\cap\mathcal{B}_x)} \\
\le& C\Bigg(\|f\|_{L^p(\Omega\cap\mathcal{B}_x^\alpha)} + \frac{1}{\alpha\, r(x)}\|\varrho\nabla u\|_{L^p(\Omega\cap\mathcal{B}_x^\alpha)} + \frac{1}{\alpha\, r(x)}\|u\|_{L^p(\Omega\cap\mathcal{B}_x^\alpha)} \\
&+ \frac{1}{\alpha^2 r(x)^2}\|\varrho\, u\|_{L^p(\Omega\cap\mathcal{B}_x^\alpha)}\Bigg) \\
\le& C(\alpha+1)^{\frac{N+1}{p}} r(x)^{\frac{N+1}{p}}\Bigg(\|f\|_{L^\infty(\Omega\cap\mathcal{B}_x^\alpha)} + \frac{1}{\alpha\, r(x)}\|\varrho\nabla u\|_{L^\infty(\Omega\cap\mathcal{B}_x^\alpha)} \\
&+ \frac{1}{\alpha\, r(x)}\|u\|_{L^\infty(\Omega\cap\mathcal{B}_x^\alpha)} + \frac{1}{\alpha^2 r(x)^2}\|\varrho\, u\|_{L^\infty(\Omega\cap\mathcal{B}_x^\alpha)}\Bigg)
\end{aligned} \tag{4.6}$$

with C independent of x and α. Applying (4.2) to the function u and to the cone $\mathcal{C}_x^*$ with vertex x and height $h\,r(x)$ we obtain

$$|u(x)| \le \begin{cases} C\,(h\,r(x))^{-\frac{N+1}{p}}\left(\|u\|_{L^p(\mathcal{C}_x^*)} + (h\,r(x))^{\frac{1}{2}}\|\sqrt{\varrho}\,\nabla u\|_{L^p(\mathcal{C}_x^*)}\right) \\ \qquad\qquad\qquad \text{if } \varrho(x) \le h\,r(x), \\ C\,(h\,r(x))^{-\frac{N+1}{p}}\left(\|u\|_{L^p(\mathcal{C}_x^*)} + \dfrac{h\,r(x)}{\sqrt{\varrho(x)}}\|\sqrt{\varrho}\,\nabla u\|_{L^p(\mathcal{C}_x^*)}\right) \\ \qquad\qquad\qquad \text{if } \varrho(x) \ge h\,r(x), \end{cases}$$

Taking the definition of $r(x)$ into account and recalling that $\mathcal{C}_x^* \subseteq \mathcal{B}_x \cap \Omega$, we find

$$|u(x)| \le C\, r(x)^{-\frac{N+1}{p}}\left(\|u\|_{L^p(\mathcal{B}_x\cap\Omega)} + |\lambda|^{-\frac{1}{2}}\|\sqrt{\varrho}\,\nabla u\|_{L^p(\mathcal{B}_x\cap\Omega)}\right)$$

with C independent of x and λ. The same argument applied to $\sqrt{\varrho}\,\nabla u$ shows that

$$|\sqrt{\varrho(x)}\,\nabla u(x)| \leq C\, r(x)^{-\frac{N+1}{p}}\Big(\|\sqrt{\varrho}\,\nabla u\|_{L^p(\mathcal{B}_x\cap\Omega)} + |\lambda|^{-\frac{1}{2}}\|\varrho D^2 u\|_{L^p(\mathcal{B}_x\cap\Omega)} + |\lambda|^{-\frac{1}{2}}\|\nabla u\|_{L^p(\mathcal{B}_x\cap\Omega)}\Big).$$

Therefore,

$$|\lambda|\,|u(x)| + |\lambda|^{1/2}|\sqrt{\varrho(x)}\,\nabla u(x)| \leq C\, r(x)^{-\frac{N+1}{p}} \times \Big(|\lambda|\|u\|_{L^p(\mathcal{B}_x\cap\Omega)} + |\lambda|^{1/2}\|\sqrt{\varrho}\,\nabla u\|_{L^p(\mathcal{B}_x\cap\Omega)} + \|\varrho D^2 u\|_{L^p(\mathcal{B}_x\cap\Omega)} + \|\nabla u\|_{L^p(\mathcal{B}_x\cap\Omega)}\Big)$$

so that, by (4.6),

$$|\lambda|\,|u(x)| + |\lambda|^{1/2}|\sqrt{\varrho(x)}\,\nabla u(x)| \leq C\,(\alpha+1)^{\frac{N+1}{p}}\Bigg(\|f\|_{L^\infty(\Omega\cap\mathcal{B}_x^\alpha)} + \frac{1}{\alpha\, r(x)}\|\varrho\nabla u\|_{L^\infty(\Omega\cap\mathcal{B}_x^\alpha)} + \frac{1}{\alpha\, r(x)}\|u\|_{L^\infty(\Omega\cap\mathcal{B}_x^\alpha)} + \frac{1}{\alpha^2 r(x)^2}\|\varrho u\|_{L^\infty(\Omega\cap\mathcal{B}_x^\alpha)}\Bigg).$$

Now, if $\varrho(x) \leq |\lambda|^{-1}$, then $r(x) = h^{-1}|\lambda|^{-1}$ and therefore, for every $y \in \mathcal{B}_x^\alpha \cap \Omega$, $\varrho(y) \leq \varrho(x) + (\alpha+1)r(x) \leq (\alpha+2)r(x)$. This leads to

$$\frac{1}{\alpha\, r(x)}\|\varrho\,\nabla u\|_{L^\infty(\Omega\cap\mathcal{B}_x^\alpha)} \leq \frac{\sqrt{\alpha+2}}{\alpha} h^{1/2}|\lambda|^{1/2}\|\sqrt{\varrho}\,\nabla u\|_{L^\infty(\Omega)} \tag{4.7}$$

$$\frac{1}{\alpha\, r(x)}\|u\|_{L^\infty(\Omega\cap\mathcal{B}_x^\alpha)} \leq \frac{1}{\alpha} h|\lambda|\,\|u\|_{L^\infty(\Omega)} \tag{4.8}$$

and

$$\frac{1}{\alpha^2 r(x)^2}\|\varrho\, u\|_{L^\infty(\Omega\cap\mathcal{B}_x^\alpha)} \leq \frac{\alpha+2}{\alpha^2} h|\lambda|\,\|u\|_{L^\infty(\Omega)}. \tag{4.9}$$

If $\varrho(x) \geq |\lambda|^{-1}$, then $r(x) = \sqrt{\varrho(x)}h^{-1}|\lambda|^{-1/2}$ and for any $y \in \mathcal{B}_x^\alpha \cap \Omega$ we have $\varrho(y) \leq \varrho(x) + (\alpha+1)r(x) \leq (\alpha+2)h|\lambda|r^2(x)$. This allows to obtain (4.7)–(4.9) in the present case, too. Summing up all the estimates we obtained so far, we find out that

$$|\lambda|\,|u(x)| + |\lambda|^{1/2}|\sqrt{\varrho(x)}\,\nabla u(x)| \leq C\,(\alpha+1)^{\frac{N+1}{p}}\Bigg(\|f\|_{L^\infty(\Omega)} + \frac{\sqrt{\alpha+2}}{\alpha}|\lambda|^{1/2}\|\sqrt{\varrho}\,\nabla u\|_{L^\infty(\Omega)} + \frac{1}{\alpha}|\lambda|\,\|u\|_{L^\infty(\Omega)} + \frac{\alpha+2}{\alpha^2}|\lambda|\,\|u\|_{L^\infty(\Omega)}\Bigg),$$

with C independent of x, α and λ. Taking first the supremum over $x \in \overline{\Omega}$ and then choosing α sufficiently large, we get

$$|\lambda|\,\|u\|_{L^\infty(\Omega)} + |\lambda|^{1/2}\|\sqrt{\varrho}\,\nabla u\|_{L^\infty(\Omega)} \leq C\,\|f\|_{L^\infty(\Omega)}.$$

Finally, let us note that, since $u \in D_p(A) \subset W^{2,p}_{\text{loc}}(\Omega)$ and $Au \in L^q(\Omega)$ for every $1 < q < \infty$ and A is nondegenerate in the interior, then $u \in W^{2,q}_{\text{loc}}(\Omega)$, by local elliptic regularity, see [17, Lemma 9.16].

Thus, we have established that there is ω_0 such that for every $\mathrm{Re}\lambda \geq \omega_0$ and $f \in C(\overline{\Omega})$ there exists a solution $u \in D_0(A)$ of $\lambda u + Au = f$, such that $\|u\|_{L^\infty(\Omega)} \leq C|\lambda|^{-1}\|f\|_{L^\infty(\Omega)}$. It remains to show that this is the unique solution. To this aim, it is sufficient to observe that from the maximum principle in [9] it follows that if $\lambda > 0$, then $\lambda + A$ is injective in $D_0(A)$. A simple argument based on connectedness now shows that $\{\mathrm{Re}\lambda \geq \omega_0\} \subset \rho(-A)$ and $\sup_{\mathrm{Re}\lambda \geq \omega_0} \|\lambda(\lambda + A)^{-1}\| < \infty$. □

Corollary 4.3 *If $p > 2N + 2$ is such that $\kappa > -1/p$, then $D_0(A) \subset D_p(A)$. Moreover, the semigroup $(T_0(t))_{t\geq 0}$ generated by $(-A, D_0(A))$ is positive, compact, contractive and coincides with the restriction of $(T_p(t))_{t\geq 0}$ to $C(\overline{\Omega})$.*

PROOF. From the proof of Theorem 4.2 it follows that the domain $D_0(A)$ is contained in $D_p(A)$ and it is continuously embedded into $W^{1,p}(\Omega)$, too. Therefore, the classical Sobolev embedding theorem leads to the compactness of the resolvent operator. As $(T_0(t))_{t\geq 0}$ is analytic, it is compact as well. The same proof shows that the resolvent operators of $(-A, D_0(A))$ and $(-A, D_p(A))$ coincide on $C(\overline{\Omega})$, hence the semigroups coincide. The positivity and contractivity of $(T_0(t))_{t\geq 0}$ (hence the positivity of $(T_p(t))_{t\geq 0}$) follow from Bony's maximum principle, [9]. □

Finally, we use the results of this section to investigate the solvability of the problem $\lambda u + Au = f$ with Dirichlet boundary conditions and for real values of λ. Observe that Theorem 3.1 implies solvability in $L^p(\Omega)$ for large λ whereas Corollary 4.3 yields solvability for $\lambda > 0$ in $C(\overline{\Omega})$.

Corollary 4.4 *Assume that $p > 2N + 2$, $\kappa > -1/p$. Then the resolvent sets of $(-A, D_p(A))$ and $(-A, D_0(A))$ contain a half-line $[-\delta, +\infty[$ for some $\delta > 0$.*

PROOF. Since the spectra of $(-A, D_p(A))$ and $(-A, D_0(A))$ coincide, see the proof of Corollary 3.4, we argue only for $(-A, D_0(A))$. Observe that $]0, +\infty[$ is contained in the resolvent sets of $(-A, D_0(A))$, by Corollary 4.3. Since its resolvent is compact it suffices to show that 0 is not an eigenvalue. Let $u \in D_0(A)$ be such that $Au = 0$. If u does not vanish, then it has an interior maximum or an interior minimum point in Ω, contradicting the strong maximum principle. Note that the strong maximum principle holds also for $W^{2,p}$ functions for $p > N + 1$, as it can be seen arguing as in [17, Lemma 3.4, Theorem 3.5] just using Bony maximum principle instead of the classical weak maximum principle. This concludes the proof. □

5 Ventcel boundary conditions in spaces of continuous functions

In this section we apply the previous results to investigate regularity and asymptotic behaviour of a class of degenerate Feller semigroups.

Let Ω be, as before, a bounded open subset of $\mathbf{R}^{N+1}$ with boundary of class C^2 and let $m \in C(\overline{\Omega})$. We assume that for every $x \in \overline{\Omega}$

$$c_1 \varrho(x) \leq m(x) \leq c_2 \varrho(x),$$

for some constants $c_1, c_2 > 0$. Here, ϱ denotes the function introduced at the beginning of §3.

For simplicity, we restrict ourselves to considering the operator

$$A = -m\Delta,$$

but we could replace Δ with any uniformly elliptic operator L with Hölder continuous coefficients. We impose that Au vanishes at $\partial\Omega$ for every u in the domain of A and this means, from a probabilistic point of view, that the diffusion process governed by $-A$ sticks forever at a point $x \in \partial\Omega$, whenever it reaches it. We refer to [2, Chapter 6] for a systematic study of these processes and their relationships with approximation theory and point out that we need smoothness of Ω rather than convexity. We can prove the analyticity of the semigroup and the exponential convergence to a limit projection. We also refer the reader to [23] and [10] for similar results in one dimension.

For every $f \in C(\overline{\Omega})$, let Pf be the unique solution in $C^2(\Omega) \cap C(\overline{\Omega})$ of

$$\begin{cases} Au = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

Note that the equation $Au = 0$ is equivalent to $\Delta u = 0$, because of the special form of the operator A. Therefore, problem (5.1) is solvable by means of the classical theory. Moreover, the maximum principle leads to $\|Pf\|_{C(\overline{\Omega})} = \|f\|_{C(\partial\Omega)} \leq \|f\|_{C(\overline{\Omega})}$, so that P is a continuous projection from $C(\overline{\Omega})$ onto

$$X = \{u \in C^2(\Omega) \cap C(\overline{\Omega}) \mid Au = 0\}$$

and $C(\overline{\Omega})$ is the direct sum of X and

$$C_0(\overline{\Omega}) = \{u \in C(\overline{\Omega}) \,|\, u_{|\partial\Omega} = 0\}.$$

More precisely, each $f \in C(\overline{\Omega})$ can be written, uniquely, in the form $f = (f - Pf) + Pf$, with $f - Pf \in C_0(\overline{\Omega})$ and $Pf \in X$. Since $C_0(\overline{\Omega})$ is a closed subspace of $C(\overline{\Omega})$, which is invariant under the semigroup $T_0(t)$, generated by $(-A, D_0(A))$ according to Theorem 4.2, the restriction of $(T_0(t))_{t\geq 0}$ to $C_0(\overline{\Omega})$, still denoted by $(T_0(t))_{t\geq 0}$, is an analytic semigroup, whose generator $-A_0$ is the part of $(-A, D_0(A))$ in $C_0(\overline{\Omega})$. Since $A = 0$ on X, it is easily seen that $-A$, endowed with the domain

$$D_V(A) = \left\{ u \in C(\overline{\Omega}) \cap \bigcap_{1\leq p<\infty} W^{2,p}_{\rm loc}(\Omega) \,|\, \sqrt{\varrho}\,\nabla u \in C(\overline{\Omega}),\ Au \in C_0(\overline{\Omega}) \right\}$$

generates the analytic semigroup $T_V(t)$ in $C(\overline{\Omega})$, given by the formula

$$T_V(t) = T_0(t)(I - P) + P, \qquad t \geq 0. \tag{5.2}$$

Thus, we have proved the following result.

Theorem 5.1 *The operator $(-A, D_V(A))$ generates an analytic semigroup in $C(\overline{\Omega})$.*

The asymptotic behaviour of $(T_V(t))_{t\geq 0}$ in $C(\overline{\Omega})$ is determined by that of $(T_0(t))_{t\geq 0}$ in $C_0(\overline{\Omega})$, hence by the spectral bound of its generator $-A_0$.

Proposition 5.2 *There exists $\delta > 0$ such that $\sigma(A_0) \subset [\delta, +\infty[$. Therefore for every $\varepsilon > 0$ one can find $C_\varepsilon > 0$ in such a way that*

$$\|T_V(t) - P\| \leq C_\varepsilon e^{-(\delta-\varepsilon)t}, \quad \forall\, t \geq 0. \tag{5.3}$$

Proof. Since A_0 has compact resolvent, we already know that its spectrum consists only of eigenvalues. Let $\lambda \in \sigma(A_0)$ and $u \in D(A_0) \setminus \{0\}$ be such that $A_0 u = \lambda u$. In particular, by Corollary 4.3, u belongs to $D_2(A)$ and therefore $u \in H^1_0(\Omega)$. Since u satisfies $\Delta u = f$, where $f = -\lambda u/m$ is in $L^2(\Omega)$ by (5.4), by elliptic regularity we infer that $u \in H^2(\Omega)$. Therefore, multiplying the equation $A_0 u = \lambda u$ by u/m and integrating by parts we get

$$\lambda \int_\Omega \frac{u^2}{m}\, dx = \int_\Omega |\nabla u|^2\, dx,$$

hence $\lambda \geq 0$. If $\lambda = 0$, then necessarily $u \equiv 0$, which is impossible. So, we have established that $\sigma(A_0) \subset [\delta, +\infty[$, for some $\delta > 0$. As the semigroup $(T_0(t))_{t\geq 0}$ is analytic, its spectral bound coincides with the growth bound and this leads to estimate (5.3), using (5.2). $\square$

Now, we claim that estimate (5.3) holds true also with $\varepsilon = 0$ and with δ given by the first eigenvalue of A_0. To show this, we study the spectrum of A_0 more carefully, introducing in the weighted space

$$H := L^2_{1/m}(\Omega) = \left\{ u : \Omega \to \mathbf{R} \text{ measurable } : \int_\Omega u^2 m^{-1} dx < +\infty \right\}$$

a selfadjoint operator B having the same spectral properties of A_0.

It is useful for the sequel to recall the classical Hardy inequality

$$\int_\Omega \frac{u^2}{m^2}\,dx \le C_{\mathcal{H}} \int_\Omega |\nabla u|^2\,dx, \tag{5.4}$$

which holds for every $u \in H_0^1(\Omega)$, (see e.g. [13, Theorem 1.5.6.]).

Set $V = H_0^1(\Omega)$. Then V and H are Hilbert spaces satisfying $V \hookrightarrow H \hookrightarrow L^2(\Omega)$ and V is dense in H. Indeed, by (5.4) we deduce that, for every $u \in V$, $u/m \in L^2(\Omega)$ and hence $u \in H$. Moreover, if $\phi \in H$, then there exists a sequence $(\phi_n) \subseteq V$ converging to $\phi/\sqrt{\varrho}$ in $L^2(\Omega)$. It follows from (5.4) that $\psi_n = \phi_n\sqrt{\varrho}$ belongs to V and converges to ϕ in H. This proves that V is dense in H. The inclusion $H \hookrightarrow L^2(\Omega)$ is obvious. Let us introduce the bilinear form

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v\,dx, \qquad u,v \in V.$$

As a is coercive, continuous and symmetric, there exists an accretive, selfadjoint operator B associated with a defined by

$$\begin{aligned} D(B) &= \{u \in V \mid \exists f \in H \,:\, a(u,v) = (f,v)_H,\ \forall\, v \in V\},\\ Bu &= f, \end{aligned}$$

where $(\cdot,\cdot)_H$ is the inner product of H. If $u \in D(B)$ and $f = Bu$, then

$$\int_\Omega \nabla u \cdot \nabla v\,dx = \int_\Omega f\,v\,\frac{dx}{m},$$

for every $v \in C_c^1(\Omega)$. Since $f/m \in L^2_{\mathrm{loc}}(\Omega)$, by the previous identity and elliptic regularity we infer that $u \in H^2_{\mathrm{loc}}(\Omega)$ and $\Delta u = -f/m$ in Ω. Hence,

$$Bu = -m\Delta u. \tag{5.5}$$

Lemma 5.3 *The operator $(B, D(B))$ has compact resolvent in H. Therefore its spectrum consists of an increasing sequence of nonnegative eigenvalues $\{\lambda_n\}$ diverging to $+\infty$.*

Proof. It suffices to show that V is compactly embedded into H. Let $\mathcal{F} \subset V$ be such that

$$\int_\Omega |\nabla u|^2\,dx \le C,$$

for every $u \in \mathcal{F}$. Given $\varepsilon > 0$, set $\Omega_\varepsilon = \{x \in \Omega \,|\, m(x) \le \varepsilon\}$. Using (5.4), we get

$$\int_{\Omega_\varepsilon} u^2\,\frac{dx}{m} \le \varepsilon \int_\Omega \frac{u^2}{m^2}\,dx \le \varepsilon\, C_{\mathcal{H}} \int_\Omega |\nabla u|^2\,dx \le \varepsilon\, C_{\mathcal{H}} C$$

for any $u \in \mathcal{F}$. On the other hand, the injection $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact and $L^2(\Omega)$ is continuously embedded into $L^2_{1/m}(\Omega \setminus \Omega_\varepsilon)$. Therefore, there are

$f_1, \ldots, f_r \in L^2_{1/m}(\Omega \setminus \Omega_\varepsilon)$ such that for every $u \in \mathcal{F}$ there exists $j \in \{1, \ldots, r\}$ with the property that $\|u - f_j\|_{L^2_{1/m}(\Omega\setminus\Omega_\varepsilon)} \leq \varepsilon$. By extending the f_i's to zero in Ω_ε, we determine new functions $\tilde{f}_i \in L^2_{1/m}(\Omega)$ such that for every $u \in \mathcal{F}$ there is $j \in \{1, \ldots, r\}$ in such a way that

$$\int_\Omega |u - \tilde{f}_j|^2 \frac{dx}{m} = \int_{\Omega\setminus\Omega_\varepsilon} |u - f_j|^2 \frac{dx}{m} + \int_{\Omega_\varepsilon} u^2 \frac{dx}{m} \leq \varepsilon^2 + \varepsilon\, C_{\mathcal{H}} C =: \varepsilon_1^2.$$

This means that

$$\mathcal{F} \subseteq \bigcup_{i=1}^{r} \{f \in H \mid \|f - \tilde{f}_i\|_H \leq \varepsilon_1\}$$

and the proof is complete. □

Our procedure requires the consistency of the resolvents of A_0 and B. We show this property in the next lemma.

Lemma 5.4 *If $\lambda \in \rho(A_0) \cap \rho(B)$ then $(\lambda - A_0)^{-1} = (\lambda - B)^{-1}$ on $C_0(\overline{\Omega}) \cap H$.*

Proof. Let us first note that $\rho(A_0) \cap \rho(B)$ is nonempty, as both $-A_0$ and $-B$ are generators. Let $\lambda \in \rho(A_0) \cap \rho(B)$ and $f \in C_0(\overline{\Omega}) \cap H$. Set $u = (\lambda - A_0)^{-1} f$. Then $u \in D(A_0)$ and $\lambda u - A_0 u = f$. In particular, $u \in H^1_0(\Omega)$, by Corollary 4.3. Moreover, if $\phi \in C^\infty_c(\Omega)$, multiplying the equation $\lambda u - A_0 u = f$ by ϕ/m and integrating over Ω we get

$$\lambda \int_\Omega u\, \phi \frac{dx}{m} + \int_\Omega \Delta u\, \phi\, dx = \int_\Omega f \phi \frac{dx}{m}.$$

Since $u \in H^2_{\mathrm{loc}}(\Omega)$, we may integrate by parts in the second integral obtaining

$$\lambda \int_\Omega u\, \phi \frac{dx}{m} - \int_\Omega \nabla u \cdot \nabla \phi\, dx = \int_\Omega f \phi \frac{dx}{m}, \tag{5.6}$$

for every $\phi \in C^\infty_c(\Omega)$. By density, (5.6) can be extended to every $\phi \in V$, so that $u \in D(B)$ and $Bu = -f + \lambda u$, as claimed. □

Proposition 5.5 *The operators A_0 and B have the same spectrum $\{\lambda_n\}$ and the first eigenvalue, λ_1, is positive. Moreover, for every $n \in \mathbf{N}$, λ_n is a simple pole both for $(\lambda - A_0)^{-1}$ and $(\lambda - B)^{-1}$ and the spectral subspaces of A_0 and B relative to λ_n coincide.*

Proof. From [5, Proposition 2.6] it follows that $\sigma(A_0) = \sigma(B)$. By Proposition 5.2 we have also that $\lambda_1 > 0$. Now, fix $n \in \mathbf{N}$ and consider the Laurent expansions of the resolvents at λ_n

$$(\lambda - A_0)^{-1} = \sum_{k=-\infty}^{+\infty} A_k (\lambda_n - \lambda)^k, \qquad (\lambda - B)^{-1} = \sum_{k=-\infty}^{+\infty} B_k (\lambda_n - \lambda)^k,$$

where

$$A_k = -\frac{1}{2\pi i}\int_\gamma \frac{(\lambda - A_0)^{-1}}{(\lambda_n - \lambda)^{k+1}}\,d\lambda, \qquad B_k = -\frac{1}{2\pi i}\int_\gamma \frac{(\lambda - B)^{-1}}{(\lambda_n - \lambda)^{k+1}}\,d\lambda$$

and γ is a small circle centered at λ_n, oriented counterclockwise. Note that A_{-1} and B_{-1} are, up to the sign, the spectral projections corresponding to λ_n. Since the resolvents coincide on $C_0(\overline{\Omega})\cap H$, A_k and B_k coincide on $C_0(\overline{\Omega})\cap H$, as well. By the density of $C_0(\overline{\Omega})\cap H$ both in $C_0(\overline{\Omega})$ and in H, we deduce that $A_k = 0$ if $k \le -2$, since $B_k = 0$ for $k \le -2$, B being self-adjoint. This shows that λ_n is a simple pole for $(\lambda - A_0)^{-1}$. Finally, as $A_{-1}(C_0(\overline{\Omega}))$ and $B_{-1}(H)$ are finite dimensional and $C_0(\overline{\Omega})\cap H$ is dense both in $C_0(\overline{\Omega})$ and in H, we deduce that $A_{-1}(C_0(\overline{\Omega})) = A_{-1}(C_0(\overline{\Omega})\cap H) = B_{-1}(C_0(\overline{\Omega})\cap H) = B_{-1}(H)$. □

We can now improve Proposition 5.2, recalling (5.2).

Theorem 5.6 *One has*

$$\lim_{t\to+\infty} e^{\lambda_1 t} T_0(t) = P_1,$$

in the norm topology of $C_0(\overline{\Omega})$, where P_1 is the spectral projection of A_0 corresponding to λ_1. As a consequence, estimate (5.3) is satisfied also when $\varepsilon = 0$ and $\delta = \lambda_1$.

Proof. By Proposition 5.5, λ_1 is a simple pole for the resolvent $(\lambda - A_0)^{-1}$, hence $P_1(C_0(\overline{\Omega})) = \ker(\lambda_1 - A_0)$. Moreover, by [20, Section A.2] the space $C_0(\overline{\Omega})$ can be decomposed as

$$C_0(\overline{\Omega}) = \ker(\lambda_1 - A_0) \oplus \mathrm{Rg}(\lambda_1 - A_0)$$

and the spectrum of the part of A_0 in $\mathrm{Rg}(\lambda_1 - A_0)$ is given by $\{\lambda_n : \lambda_n > \lambda_1\}$. According to this decomposition, we have

$$T_0(t) = e^{-\lambda_1 t} P_1 + T_0(t)(I - P_1),$$

hence

$$e^{\lambda_1 t} T_0(t) = P_1 + e^{\lambda_1 t} T_0(t)(I - P_1).$$

As $e^{\lambda_1 t} T_0(t)(I - P_1)$ tends to zero in the norm topology, exponentially, as $t \to +\infty$, the proof is complete.

Finally we show that the first eigenvalue λ_1 has similar properties as for uniformly elliptic operators.

Proposition 5.7 *The first eigenvalue λ_1 is simple (i.e., the corresponding eigenspace is one-dimensional). Moreover, each eigenfunction relative to λ_1 is either positive or negative in Ω.*

Proof. By Proposition 5.5, the spectral subspaces of A_0 and B relative to λ_1 coincide, therefore it suffices to prove the statement for the operator B. Let u

be an eigenfunction of B relative to λ_1 and assume that $\|u\|_H = 1$. Set $v = |u|$. Then $v \in H_0^1(\Omega)$ and $\|v\|_H = \|u\|_H = 1$. Furthermore, $a(v,v) = a(u,u) = \lambda_1$. The spectral theorem for self-adjoint operators with compact resolvent implies that $v \in D(B)$ and $Bv = \lambda_1 v$. Recalling (5.5), we find that $\Delta v = -\lambda_1 v/m$. By local elliptic regularity it turns out that $v \in W^{2,p}_{\mathrm{loc}}(\Omega)$ for all $p > 1$. As $\Delta v \le 0$, v cannot vanish inside Ω. Otherwise, by the strong maximum principle (see e.g. [17, Theorem 9.6]), v should be constant, which means $v \equiv 0$. This, of course, is impossible. Thus, $v > 0$ in Ω so that u is either positive or negative in Ω. Since every eigenfunction does not change sign, two of them cannot be orthogonal, hence the kernel of $\lambda_1 - B$ is one-dimensional. □

We end this section by considering a particular case where we can provide an estimate from above and below of the first eigenvalue λ_1.

Example 5.8 Let $\Omega = B_1(0)$ and $m(x) = 1 - |x|$. By the mini-max formula we have

$$\lambda_1 = \min_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\displaystyle\int_\Omega |\nabla u|^2\,dx}{\displaystyle\int_\Omega u^2 m^{-1}\,dx}. \tag{5.7}$$

Analogously, the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions is given by

$$\lambda_1^{-\Delta} = \min_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\displaystyle\int_\Omega |\nabla u|^2\,dx}{\displaystyle\int_\Omega u^2\,dx}.$$

By taking advantage of [21], we find that

$$\int_\Omega \frac{u^2}{m^2}\,dx \le 4 \int_\Omega |\nabla u|^2\,dx, \tag{5.8}$$

for every $u \in H_0^1(\Omega)$ and then

$$\begin{aligned}
\int_\Omega u^2 m^{-1}\,dx &\le \left(\int_\Omega \frac{u^2}{m^2}\,dx\right)^{\frac{1}{2}} \left(\int_\Omega u^2\,dx\right)^{\frac{1}{2}} \\
&\le 2\left(\int_\Omega |\nabla u|^2\,dx\right)^{\frac{1}{2}} \left(\int_\Omega u^2\,dx\right)^{\frac{1}{2}} \\
&\le 2(\lambda_1^{-\Delta})^{-1/2} \int_\Omega |\nabla u|^2\,dx.
\end{aligned}$$

Therefore $\lambda_1 \ge (\lambda_1^{-\Delta})^{1/2}/2$. In order to estimate $\lambda_1^{-\Delta}$, we observe that for every $u \in C_c^\infty(\Omega)$ and every $\varepsilon \in (0,1)$, integrating by parts we get

$$\gamma \int_{B_1(0)\setminus B_\varepsilon(0)} u^2 \Delta m\,dx = -2\gamma \int_{B_1(0)\setminus B_\varepsilon(0)} u\nabla u \cdot \nabla m\,dx - \gamma \int_{\partial B_\varepsilon(0)} u^2 \nabla m \cdot \frac{x}{\varepsilon}\,d\sigma,$$

for every positive constant γ. By the definition of m and Hölder inequality we have

$$\gamma \int_{B_1(0)\setminus B_\varepsilon(0)} u^2 \frac{N}{|x|}\, dx \le \int_{B_1(0)\setminus B_\varepsilon(0)} |\nabla u|^2\, dx \\ + \gamma^2 \int_{B_1(0)\setminus B_\varepsilon(0)} u^2\, dx + \gamma \int_{\partial B_\varepsilon(0)} u^2\, d\sigma,$$

and then

$$\int_{B_1(0)\setminus B_\varepsilon(0)} |\nabla u|^2\, dx \ge \gamma \int_{B_1(0)\setminus B_\varepsilon(0)} u^2 \left(N-\gamma\right)\, dx - \gamma \int_{\partial B_\varepsilon(0)} u^2\, d\sigma.$$

Letting ε go to 0 and taking the maximum over γ give

$$\int_{B_1(0)} |\nabla u|^2\, dx \ge \frac{N^2}{4} \int_{B_1(0)} u^2\, dx.$$

By density, the previous estimate still holds for every $u \in H_0^1(\Omega)$. Then $\lambda_1^{-\Delta} \ge N^2/4$ and consequently $\lambda_1 \ge N/4$. On the other hand, by plugging $u = m$ in (5.7), it easily follows that $\lambda_1 \le N+2$. Summing up, we have proved that

$$\frac{N}{4} \le \lambda_1 \le N+2.$$

Finally, we note that choosing $T = \gamma \frac{\nabla d}{d}$, where d is the distance from the boundary of Ω and γ a positive constant, integrating by parts in $\int_\Omega u^2 \mathrm{div} T$ and arguing as above, one can show (5.8) in the case of a convex smooth open set Ω.

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