A MULTIPHASE SHAPE OPTIMIZATION PROBLEM FOR EIGENVALUES: QUALITATIVE STUDY AND NUMERICAL RESULTS

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ABSTRACT. We consider the multiphase shape optimization problem

$$\min\Big\{\sum_{i=1}^h \lambda_1(\Omega_i)+c|\Omega_i|:\ \Omega_i \text{ open},\ \Omega_i\subset D,\ \Omega_i\cap\Omega_j=\emptyset\Big\},$$

where $c>0$ is a given constant and $D\subset\mathbb{R}^2$ is a bounded open set with Lipschitz boundary. We give some new results concerning the qualitative properties of the optimal sets and the regularity of the corresponding eigenfunctions. We also provide numerical results for the optimal partitions.

1. Introduction

In this paper we consider a multiphase shape optimization problem of the form

$$\min\Big\{F\big(\lambda_1(\Omega_1),\dots,\lambda_1(\Omega_h)\big)+c\sum_{i=1}^h|\Omega_i|:\ \Omega_i \text{ open},\ \Omega_i\subset D,\ \Omega_i\cap\Omega_j=\emptyset\Big\},\tag{1.1}$$

where $D\subset\mathbb{R}^2$ is an open set of finite measure, $F:\mathbb{R}^h\to\mathbb{R}$ is a given increasing in each variable Lipschitz continuous function and, for a generic open set $\Omega\subset\mathbb{R}^2$, $\lambda_1(\Omega)$ is the first eigenvalue of the Dirichlet Laplacian, which is variationally characterized as

$$\lambda_1(\Omega)=\min\Big\{\int_\Omega|\nabla u|^2\,dx:\ u\in H^1_0(\Omega),\int_\Omega u^2\,dx=1\Big\},\tag{1.2}$$

where $H^1_0(\Omega)$ is the Sobolev space on Ω. More precisely, we study the following model problem:

$$\min\Big\{\sum_{i=1}^h\lambda_1(\Omega_i)+c|\Omega_i|:\ \Omega_i \text{ open},\ \Omega_i\subset D,\ \Omega_i\cap\Omega_j=\emptyset\Big\};\tag{1.3}$$

The variational problem (1.3) is widely studied in the literature in the case $c=0$. We refer to the papers [11], [10], [14] and [3] for a theoretical and numerical analysis in this case. The other limit case appears when the constant $c>0$ is large enough. Indeed, we recall that the solution of the problem

$$\min\Big\{\lambda_1(\Omega)+c|\Omega|:\Omega \text{ open},\ \Omega\subset\mathbb{R}^2\Big\},\tag{1.4}$$

is a disk of radius $r_c=\left(\frac{\lambda_1(B_1)}{c\pi}\right)^{\frac14}$. It is straightforward to check that if $c>0$ is such that there are h disjoint disks of radius r_c that fit in the box D, then the solution of (1.3) is given by the h-uple of these disks. Finding the smallest real number $\overline{c}>0$, for which the above happens, reduces to solving the optimal packing problem

$$\max\Big\{r:\ \text{there exist } h \text{ disjoint balls } B_r(x_1),\dots,B_r(x_h) \text{ in } D\Big\}.\tag{1.5}$$

The multiphase problem (1.3), in variation of the parameter $c>0$, present an interpolation between the optimal partition problem (corresponding to the case $c=0$) and the optimal packing problem (1.5). The aim of this paper is to study the solutions of (1.3), providing some regularity and qualitative results, as well as some fine numerical results.

The paper is organized as follows. In Section 2 we recall the results concerning the existence of optimal configuration and we give the main technical tools concerning the eigenfunctions of the Dirichlet Laplacian, i.e. the Sobolev functions that realize the minimum in (1.2). In Section 3 prove that the eigenfunctions on the optimal sets are Lipschitz continuous on $\mathbb{R}^2$. In Section 4, we give some results concerning the qualitative behaviour of the optimal configurations. We recall a result from [7] which states that, for $c > 0$, there are no triple boundary points. We prove that there are no double boundary points on ∂D, provided that ∂D is locally a graph of a Lipschitz function. We also prove that for some optimal configurations the boundary of the set $\Omega = \Omega_1 \cup \cdots \cup \Omega_h$ may contain cusps. In Section 5 we present a numerical algorithm for calculating the minimizers of (1.3) as well as some numerical results for different values of h and c and we confirm numerically some of the theoretical results concerning the lack of triple points and the lack of double points on the boundary.

2. Preliminaries

2.1. **Eigenvalues and eigenfunctions.** Let $\Omega \subset \mathbb{R}^2$ be an open set. We denote with $H_0^1(\Omega)$ the Sobolev space obtained as the closure in $H^1(\mathbb{R}^2)$ of $C_c^\infty(\Omega)$, i.e. the smooth functions with compact support in Ω, with respect to the Sobolev norm

$$\|u\|_{H^1} := \left(\|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2\right)^{1/2} = \left(\int_{\mathbb{R}^2} |\nabla u|^2 + u^2\,dx\right)^{1/2}.$$

We note that $H_0^1(\Omega)$ can be characterized as

$$H_0^1(\Omega) = \Big\{u \in H^1(\mathbb{R}^2) :\ \mathrm{cap}\left(\{u \neq 0\} \setminus \Omega\right) = 0\Big\}, \tag{2.1}$$

where the capacity $\mathrm{cap}(E)$ of a measurable set $E \subset \mathbb{R}^2$ is defined as

$$\mathrm{cap}(E) = \min\Big\{\|u\|_{H^1}^2 :\ u \geq 1 \text{ in a neighbourhood of } E\Big\}^1.$$

The kth eigenvalue of the Dirichlet Laplacian can be defined through the min-max variational formulation

$$\lambda_k(\Omega) := \min_{S_k \subset H_0^1(\Omega)} \max_{u \in S_k} \frac{\int_\Omega |\nabla u|^2\,dx}{\int_\Omega u^2\,dx}, \tag{2.2}$$

where the maximum is over all non-zero functions $u \in S_k$ and the minimum is over all k dimensional subspaces S_k of $H_0^1(\Omega)$. There are functions $u_1, \dots, u_k, \dots$ in $H_0^1(\Omega)$, orthonormal in $L^2(\Omega)$, that solve the equation

$$-\Delta u_k = \lambda_k(\Omega) u_k, \qquad u_k \in H_0^1(\Omega),$$

in a weak sense in $H_0^1(\Omega)$. In particular, if $k = 1$, then the first eigenfunction u_1 of Ω is the solution of the minimization problem (1.2). Since $|u_1|$ is also a solution of (1.2), from now on we will always assume that u_1 is non-negative and normalized in L^2. Moreover, we have the following properties of u_1 on a generic open[2] set Ω of finite measure:

- u_1 is bounded and we have the estimate[3]

$$\|u_1\|_{L^\infty} \leq \frac{1}{\pi} \lambda_1(\Omega) |\Omega|^{1/2}. \tag{2.3}$$

- $u_1 \in H^1(\mathbb{R}^2)$, extended as zero outside Ω, satisfies the following inequality in sense of distributions:

$$\Delta u_1 + \lambda_1(\Omega) u_1 \geq 0 \qquad \text{in} \qquad \left[C_c^\infty(\mathbb{R}^2)\right]'. \tag{2.4}$$

[1] for more details see, for example, [13] or [15]

[2] The same properties hold for the first eigenfunction on quasi-open set of finite measure.

[3] We note that the infinity norm of u_1 can also be estimated in terms of $\lambda_1(\Omega)$ only as $\|u_1\|_{L^\infty} \leq C\lambda_1(\Omega)^{d/4}$. This estimate is more general and can be found in [12, Example 8.1.3].

- Every point $x_0 \in \mathbb{R}^2$ is a Lebesgue point for u_1. Pointwise defined as
$$u_1(x_0) := \lim_{r\to 0} \fint_{B_r(x_0)} u(x)\,dx,$$
u_1 is upper semi-continuous on $\mathbb{R}^2$.
- u_1 is almost subharmonic in sense that for every $x_0 \in \mathbb{R}^2$, we have
$$u_1(x_0) \le \|u_1\|_{L^\infty}\lambda_1(\Omega)r^2 + \fint_{B_r(x_0)} u_1(x)\,dx, \qquad \forall r > 0. \tag{2.5}$$

2.2. **Monotonicity formulas for eigenfunctions.** The monotonicity formula of Alt-Caffarelli-Friedman is an essential tool in the study of the behaviour of the eigenfunctions in the points of the common boundary of the optimal sets for (1.1). Since the eigenfunctions are not subharmonic, but satisfy (2.4), we will need another version of the monotonicity formula from [2]. We state here the following monotonicity theorem, which contains a refined version of the result in [11] and we will prove it in the Appendix A.

Theorem 2.1 (Two-phase monotonicity formula)**.** *Consider the unit ball $B_1 \subset \mathbb{R}^2$. Let $u^+, u^- \in H^1(B_1) \cap L^\infty(B_1)$ be two non-negative functions with disjoint supports, i.e. such that $\int_{B_1} u^+u^-\,dx = 0$, and let $\lambda_+, \lambda_- \ge 0$ be two real numbers such that*
$$\Delta u^+ + \lambda_+ u^+ \ge 0 \qquad \text{and} \qquad \Delta u^- + \lambda_- u^- \ge 0.$$
(a) Then there are constants $1/2 \ge r_0 > 0$ and $C > 0$, depending on d, λ_+ and λ_-, such that for every $r \in (0, r_0)$ we have
$$\left(\frac{1}{r^2}\int_{B_r} |\nabla u^+|^2\,dx\right)\left(\frac{1}{r^2}\int_{B_r} |\nabla u^-|^2\,dx\right) \le C\left(1 + \|u^+ + u^-\|^2_{L^\infty(B_{2r_0})}\right)^2. \tag{2.6}$$
(b) If, moreover, the set $\Omega := B_1 \cap \{u^+ = 0\} \cap \{u^- = 0\}$ has positive density in 0, i.e.
$$\liminf_{r\to 0} \frac{|\Omega \cap B_r|}{|B_r|} = c > 0,$$
then there is some $\varepsilon > 0$, depending on d, λ_+, λ_- and c such that
$$\left(\frac{1}{r^2}\int_{B_r} |\nabla u^+|^2\,dx\right)\left(\frac{1}{r^2}\int_{B_r} |\nabla u^-|^2\,dx\right) = o(r^\varepsilon). \tag{2.7}$$

We note that the estimate (2.6) follows by the more general result by Caffarelli, Jerison and Kenig (see [9] and also the note [17], where the continuity assumption was dropped). In order to obtain (2.7) we use the idea of Conti, Terracini and Verzini (see [11]), which works exclusively for eigenfunctions, but can be easily refined to obtain fine qualitative results as (2.7).

The three-phase version of Theorem 2.1 is the main tool that allows to exclude the presence of triple boundary points in the optimal configuration. The following three-phase monotonicity formula was proved for eigenfunctions in [11], while the general three-phase version of the Caffarelli-Jerison-Kenig result can be found in [7] (see also [17] for the detailed proof).

Theorem 2.2 (Three-phase monotonicity formula)**.** *Consider the unit ball $B_1 \subset \mathbb{R}^2$. Let $u_1, u_2, u_3 \in H^1(B_1) \cap L^\infty(B_1)$ be three non-negative functions with disjoint supports, i.e. such that $\int_{B_1} u_iu_j\,dx = 0$ for all $i \ne j$, and let $\lambda_1, \lambda_2, \lambda_3 \ge 0$ be real numbers such that*
$$\Delta u_i + \lambda_i u_i \ge 0, \qquad \forall i = 1, 2, 3.$$
Then there are constants $0 < r_0 \le 1/2$, $C > 0$ and $\varepsilon > 0$, depending on d, λ_1, λ_2 and λ_3, such that for every $r \in (0, r_0)$ we have
$$\prod_{i=1}^{3}\left(\frac{1}{r^2}\int_{B_r} |\nabla u_i|^2\,dx\right) \le Cr^\varepsilon\left(1 + \|u_1 + u_2 + u_3\|^2_{L^\infty(B_{2r_0})}\right)^3. \tag{2.8}$$

Remark 2.3. In [11] it was proved that one can take $\varepsilon = 3$.

2.3. **Existence of optimal configurations.** The shape optimization problems of the form (1.1) admit solutions for a very general cost functionals $\mathcal{F}(\Omega_1, \dots, \Omega_h)$. The general existence result in this direction is well known and is due to the classical Buttazzo-Dal Maso result from [8]. The price to pay for such a general result is that one has to relax the problem to a wider class of domains, which contains the open ones. Indeed, one notes that the capacitary definition of a Sobolev space (2.1) can be easily extended to generic measurable sets. In particular, it is well known (we refer, for example, to the books [15] and [5]) that it is sufficient to restrict the analysis to the class of *quasi-open* sets, i.e. the level sets of Sobolev functions. Since the definition of the first eigenvalue (1.2) is of purely variational character, one may also extend it to the quasi-open sets and then apply the theorem of Buttazzo and Dal Maso [8] to obtain existence for (1.1) in the family of quasi-open sets under the minimal assumptions of monotonicity and semi-continuity of the function F. Thus, the study of the problem of existence of a solution of (1.1) reduces to the analysis of the regularity of the optimal quasi-open sets.

Following the above idea, the existence of an open solution of (1.1) was proved in [7]. More precisely, the following existence result was proved in [7].

Theorem 2.4. *Let $F : \mathbb{R}^h \to \mathbb{R}$ be a locally Lipschitz function, increasing in each variable and let $c > 0$. Then, for every open set $D \subset \mathbb{R}^2$ of finite measure, there is a solution of the problem* (1.1)*. Moreover, every solution $(\Omega_1, \dots, \Omega_h)$ of* (1.1) *is such that:*

(a) the sets Ω_i are bounded and we have the estimate $diam(\Omega_i) \le C$, where $C > 0$ is a constant depending on c, $\lambda_1(\Omega_i)$ and $|\Omega_i|$;

(b) the sets Ω_i are of finite perimeter and we have the estimate

$$P(\Omega_i) \le c^{-1/2}\lambda_1(\Omega_i)|\Omega_i|^{1/2}; \tag{2.9}$$

(c) there is a lower bound on the eigenvalue $\lambda_1(\Omega_i)$ given by

$$\lambda_1(\Omega_i) \ge \big(4\pi c\big)^{1/2}; \tag{2.10}$$

(d) there are no triple boundary points, i.e. if $i, j, k \in \{1, \dots, h\}$ are three different indices, then the set $\partial\Omega_i \cap \partial\Omega_j \cap \partial\Omega_k$ is empty.

3. Lipschitz continuity of the eigenfunctions

In this section we prove that the first eigenfunctions on the optimal sets for (1.3) are Lipschitz continuous. To fix the notation, in the rest of this section we will denote with $(\Omega_1, \dots, \Omega_h)$ a generic solution of (1.3) and with $u_i \in H^1_0(\Omega_i)$ the first eigenfunction on Ω_i, i.e. u_i are non-negative function such that $\int_{\mathbb{R}^2} u_i^2\,dx = 1$ satisfying (2.3), (2.4) and the equation

$$-\Delta u_i = \lambda_1(\Omega_i)u_i, \qquad u_i \in H^1_0(\Omega),$$

weakly in $H^1_0(\Omega_i)$.

3.1. **Non-degeneracy of the eigenfunctions.** We first note that for every $\omega_i \subset \Omega_i$, the optimality of $(\Omega_1, \dots, \Omega_i, \dots, \Omega_h)$ tested against the h-uple of open sets $(\Omega_1, \dots, \omega_i, \dots, \Omega_h)$ gives the inequality

$$\lambda_1(\Omega_i) + c|\Omega_i| \le \lambda_1(\omega_i) + c|\omega_i|,$$

i.e. Ω_i is a *subsolution* for the functional $\lambda_1 + c|\cdot|$. Thus using the argument from the Alt-Caffarelli non-degeneracy lemma (see [1, Lemma 3.4] and also [7, Section 3]), we have the following result.

Lemma 3.1. *Suppose that* $(\Omega_1,\dots,\Omega_h)$ *is optimal for* (1.3). *Then there are constants* C_{nd} *and* $r_0>0$ *such that for all the first eigenfunctions* u_i*, every* $0<r\le r_0$ *and every* $x_0\in\mathbb{R}^2$ *we have the following implication*

$$\Big(\ B_{r/2}(x_0)\cap\Omega_i\neq\emptyset\ \Big)\ \Rightarrow\ \Big(\ \frac{1}{r}\fint_{B_r(x_0)}u_i\,dx\ge C_{nd}\ \Big).\tag{3.1}$$

Remark 3.2. Together with the estimate (2.5), Lemma 3.1 gives that there is $r_0>0$ such that

$$\|u_i\|_{L^\infty(B_{r/2}(x_0))}\le 5\fint_{B_r(x_0)}u_i\,dx,\qquad \forall r\le r_0\ \text{ such that }\ B_{r/2}(x_0)\cap\Omega_i\neq\emptyset.\tag{3.2}$$

On the common boundary of two optimal sets the non-degeneracy (3.1) of the mean $\fint_{B_r(x_0)}u_i\,dx$ gives a bound from below for the gradient $\fint_{B_r(x_0)}|\nabla u_i|^2\,dx$. This fact follows by the elementary lemma proved below.

Lemma 3.3. *Let* $R>0$*,* $B_R(x_0)\subset\mathbb{R}^2$ *and* $U\in H^1(B_R(x_0))$ *be a Sobolev function such that for almost every* $r\in(0,R)$ *the set* $\{U=0\}\cap\partial B_r(x_0)$ *is non-empty. Then we have*

$$\frac{1}{R}\fint_{B_R(x_0)}U\,d\mathcal{H}^1\le 2\left(\fint_{B_R(x_0)}|\nabla U|^2\,dx\right)^{1/2}.\tag{3.3}$$

Proof. Without loss of generality we suppose that $x_0=0$. We first note that for almost every $r\in(0,R)$ the restriction $U|_{\partial B_r}$ is Sobolev. If, moreover, $\{U=0\}\cap\partial B_r\neq\emptyset$, then we have

$$\int_{\partial B_r}U^2\,d\mathcal{H}^1\le 4r^2\int_{\partial B_r}|\nabla U|^2\,d\mathcal{H}^1.$$

Applying the Cauchy-Schwartz inequality and integrating for $r\in(0,R)$, we get

$$\Big(\frac{1}{R}\fint_{B_R}U\,dx\Big)^2\le\frac{1}{R^2}\fint_{B_R}U^2\,dx\le 4\fint_{B_R}|\nabla U|^2\,dx.$$

□

Corollary 3.4. *Suppose that* $(\Omega_1,\dots,\Omega_h)$ *is optimal for* (1.3). *Then there is a constant* $r_0>0$ *such that for every* $x_0\in\partial\Omega_i\cap\partial\Omega_j$*, for some* $i\neq j$ *we have*

$$\fint_{B_r(x_0)}|\nabla u_i|^2\,dx\ge 4C_{nd}^2,\forall r\in(0,r_0),\tag{3.4}$$

where $C_{nd}>0$ *is the non-degeneracy constant from Lemma* 3.1.

Proof. Since $x_0\in\partial\Omega_i\cap\partial\Omega_j$, we have that for every $r>0$ $\Omega_i\cap B_r(x_0)\neq\emptyset$ and $\Omega_j\cap B_r(x_0)\neq\emptyset$. In view of Lemma 3.1, it is sufficient to check that $\Omega_i\cap\partial B_r(x_0)\neq\emptyset$ and $\Omega_j\cap\partial B_r(x_0)\neq 0$, for almost every $r\in(0,r_0)$. Indeed, suppose that this is not the case and that $\Omega_i\cap\partial B_r(x_0)=\emptyset$. Since Ω_i is connected, we have that $\Omega_i\subset B_r(x_0)$, which gives $\lambda_1(\Omega_i)\ge\lambda_1(B_{r_0})$, which is impossible if we choose r_0 small enough. □

3.2. **Growth estimate of the eigenfunctions on the boundary.** We now prove the two key estimates of the growth of u_i close to the boundary $\partial\Omega_i$. We consider two kinds of estimates, one holds around the points, where two phases Ω_i and Ω_j are close to each other, and is reported in Lemma 3.5. The other estimate concerns the one-phase points, i.e. the points on one boundary, say $\partial\Omega_i$, which are far away from all other sets Ω_j.

Lemma 3.5. *Suppose that* $(\Omega_1,\dots,\Omega_h)$ *is optimal for* (1.3). *Then there are constants* C_2 *and* $r_0>0$ *such that if* $x_0\in\partial\Omega_i$ *is such that* $\Omega_j\cap B_r(x_0)\neq\emptyset$*, for some* $j\neq i$ *and* $r\le r_0$*, then*

$$\|u_i\|_{L^\infty(B_r(x_0))}\le C_2 r.\tag{3.5}$$

Proof. Without loss of generality we suppose that $0 = x_0 \in \partial\Omega_i$. Let now $0 < r \le r_0$ be such that $\Omega_j \cap B_r \neq \emptyset$. Choosing r_0 small enough we may apply Lemma 3.1 obtaining that

$$\fint_{B_{3r}} u_j\,dx \ge 3C_{nd}\,r.$$

Again by choosing r_0 small enough we may suppose that for every $r \in (0, r_0)$ we have $\partial B_{3r} \cap \Omega_i \neq 0$. Indeed, if this is not the case for some r, then the set Ω_i is entirely contained in B_{3r} and so $\lambda_1(\Omega_i) \ge \lambda_1(B_{3r}) \ge \lambda_1(B_{3r_0})$, contradicting the optimality of Ω_i. Thus, we may apply the estimate (3.3) for u_j obtaining

$$C_{nd}^2 \le \Big(\frac{1}{3r}\fint_{B_{3r}} u_j\,dx\Big)^2 \le 4\fint_{B_{3r}} |\nabla u_j|^2\,dx.$$

By the two-phase monotonicity formula applied for u_i and u_j, we get that there is a constant $C > 0$ such that

$$\frac{4C}{C_{nd}^2} \ge \fint_{B_{3r}} |\nabla u_i|^2\,dx.$$

Since $B_r \cap \Omega_j \neq \emptyset$, by choosing r_0 small enough an reasoning as above we may suppose that for every $\tilde r \in (r, 3r)$ $\partial B_{\tilde r} \cap \Omega_j \neq 0$. Thus, reasoning as in Lemma 3.3, we get that

$$4(3r)^2 \int_{B_{3r}\setminus B_{2r}} |\nabla u_i|^2\,dx \ge \int_{B_{3r}\setminus B_{2r}} u_i^2\,dx \ge \frac{1}{5\pi r^2}\Big(\int_{B_{3r}\setminus B_{2r}} u_i\,dx\Big)^2.$$

By the mean value formula, there is $R \in (2r, 3r)$ such that

$$\int_{\partial B_R} u_i\,dx \le \frac{1}{r}\int_{2r}^{3r}\Big(\int_{\partial B_s} u_i\,d\mathcal{H}^1\Big)\,ds \le 27r\Big(\int_{B_{3r}} |\nabla u_i|^2\,dx\Big)^{1/2} \tag{3.6}$$

We now note that by (2.4) the function $v(x) = u_i(x) - \lambda_1(\Omega_i)\|u_i\|_{L^\infty}(R^2 - |x|^2)$ is subharmonic. Then, for every $x \in B_r$, we use the Poisson formula

$$u_i(x) - \lambda_1(\Omega_i)\|u_i\|_{L^\infty}(3r)^2 \le \frac{R^2 - |x|^2}{2\pi R}\int_{\partial B_R} \frac{u_i(y)}{|y - x|^2}\,d\mathcal{H}^1(y) \le 9\fint_{\partial B_R} u_i\,d\mathcal{H}^1. \tag{3.7}$$

Using the non-degeneracy of u_i (Lemma 3.1) and combining the estimates from (3.6) and (3.7) we get

$$\|u_i\|_{L^\infty(B_r)} \le 3^6 r\Big(\int_{B_{3r}} |\nabla u_i|^2\,dx\Big)^{1/2} \le \frac{2\sqrt{C}3^6}{C_{nd}} r. \tag{3.8}$$

□

The following Lemma is similar to [1, Lemma] and can be found also in [4].

Lemma 3.6. *Suppose that $(\Omega_1, \dots, \Omega_h)$ is optimal for (1.3). Then there is are constants $C_1 > 0$ and $r_0 > 0$ such that if $x_0 \in \partial\Omega_i$ and $0 < r \le r_0$ are such that $\Omega_j \cap B_{2r}(x_0) = \emptyset$, for every $j \neq i$, then*

$$\|u_i\|_{L^\infty(B_r(x_0))} \le C_1 r. \tag{3.9}$$

Proof. Without loss of generality we may suppose that $x_0 = 0$. Since $\Omega_j \cap B_{2r} = \emptyset$, for every $j \neq i$, we may use the h-uple $(\Omega_1, \dots, \Omega_i \cap B_{2r}, \dots, \Omega_h)$ to test the optimality of $(\Omega_1, \dots, \Omega_i, \dots, \Omega_h)$. Thus we have

$$\begin{aligned}\int_{\mathbb{R}^2} |\nabla u_i|^2\,dx + c|\Omega_i| &= \lambda_1(\Omega_i) + c|\Omega_i| \le \lambda_1(\Omega_i \cup B_{2r}) + c|\Omega_i \cup B_r| \\ &\le \frac{\int_{\mathbb{R}^2} |\nabla \widetilde u_i|^2\,dx}{\int_{\mathbb{R}^2} \widetilde u_i^2\,dx} + c|\Omega_i \cup B_{2r}| \le \int_{\mathbb{R}^2} |\nabla \widetilde u_i|^2\,dx + c|\Omega_i \cup B_{2r}|,\end{aligned} \tag{3.10}$$

where we used the test function $\widetilde{u}_i \in H^1_0(\Omega_i \cap B_{2r})$ defined as $\widetilde{u}_i = v_i \mathbb{1}_{B_{2r}} + u_i \mathbb{1}_{B_{2r}^c}$ and $v_i \in H^1(B_{2r})$ is the solution of the obstacle problem

$$\min\Big\{ \int_{B_{2r}} |\nabla v|^2\,dx :\ v \in H^1(B_{2r}),\ v - u_i \in H^1_0(B_{2r}),\ v \geq u_i \Big\}. \tag{3.11}$$

By (3.10) an the fact that v_i is harmonic on the set $\{v_i > u_i\}$, we get

$$\int_{B_{2r}} |\nabla(u_i - v_i)|^2\,dx = \int_{B_{2r}} \big(|\nabla u_i|^2 - |\nabla v_i|^2\big)\,dx \leq c|B_{2r} \setminus \Omega_i|. \tag{3.12}$$

Now, reasoning as in [1, Lemma 3.2] (see also [16, Lemma 4.3.20] and [7]), there is a constant $C > 0$ such that

$$\big|\{u_i = 0\} \cap B_{2r}\big| \left(\frac{1}{2r} \fint_{\partial B_{2r}} u_i\,d\mathcal{H}^1\right)^2 \leq C \int_{B_{2r}} |\nabla(u_i - v_i)|^2\,dx. \tag{3.13}$$

Now we note that by the optimality of Ω_i, we have $\Omega_i = \{u_i > 0\}$ and $|B_{2r} \cap \{u_i = 0\}| > 0$ (if $|B_{2r} \cap \{u_i = 0\}| = 0$, then by the optimality $v_i = u_i$ in B_{2r}; thus u_i is superharmonic in B_{2r} and so $u_i > 0$ in B_{2r}, which contradicts the assumption $0 \in \partial\Omega_i$). Now (3.12) and (3.13) give

$$\frac{1}{2r} \fint_{\partial B_{2r}} u_i\,d\mathcal{H}^1 \leq \sqrt{C/c}. \tag{3.14}$$

Since the function $\Big\{x \mapsto \Big(u_i(x) - \lambda_1(\Omega_i)\|u_i\|_{L^\infty}(4r^2 - |x|^2)\Big)\Big\}$ is subharmonic, we can use the Poisson formula for every $x \in B_r$

$$u_i(x) - 4\lambda_1(\Omega_i)\|u_i\|_{L^\infty} r^2 \leq \frac{(2r)^2 - |x|^2}{4\pi r} \int_{\partial B_{2r}} \frac{u_i(y)}{|y - x|^2}\,d\mathcal{H}^1(y) \leq 4 \fint_{\partial B_{2r}} u_i\,d\mathcal{H}^1. \tag{3.15}$$

By the non-degeneracy of u_i (Lemma 3.1) and (3.15), we have that for r_0 small enough

$$\frac{\|u_i\|_{L^\infty(B_r)}}{r} \leq \frac{5}{2r} \fint_{\partial B_{2r}} u_i\,d\mathcal{H}^1 \leq 5\sqrt{C/c},$$

which gives the claim. □

We combine the estimates from Lemma 3.6 and Lemma 3.5, obtaining the following

Proposition 3.7. *Suppose that $(\Omega_1, \dots, \Omega_h)$ is optimal for (1.3). Then there are constants $r_0 > 0$ and $C_{12} > 0$ such that for every $i \in \{1, \dots, h\}$ we have*

$$\|u_i\|_{L^\infty(B_r(x_0))} \leq C_{12}\, r, \qquad \forall r \in (0, r_0). \tag{3.16}$$

3.3. **Lipschitz continuity of the eigenfunctions.** We now use the estimate from Proposition 3.7 to deduce the Lipschitz continuity of u_i. The argument is standard and we recall it briefly for the sake of completeness. It is based on the following classical lemma.

Lemma 3.8. *Suppose that $B_r \subset \mathbb{R}^2$, $f \in L^\infty(B_r)$ and $u \in H^1(B_r)$ satisfies the equation*

$$-\Delta u = f \quad \text{weakly in} \quad [H^1_0(B_r)]'.$$

Then there is a dimensional constant $C > 0$ such that the following estimate holds

$$\|\nabla u_i\|_{L^\infty(B_{r/2})} \leq C\left(\|f\|_{L^\infty(B_r)} + \frac{\|u\|_{L^\infty(B_r)}}{r}\right). \tag{3.17}$$

Theorem 3.9. *Let $D \subset \mathbb{R}^2$ be a bounded open set. Let $(\Omega_1, \dots, \Omega_h)$ be optimal for (1.3). Then the corresponding first eigenfunctions $u_1, \dots, u_h$ are locally Lipschitz continuous in D. If, moreover, D is such that the weak solution w_D of the problem*

$$-\Delta w_D = 1, \qquad w_D \in H^1_0(D),$$

is Lipschitz continuous on $\mathbb{R}^2$, then the first eigenfunctions $u_1, \dots, u_h$ are globally Lipschitz continuous on $\mathbb{R}^2$.

Proof. Let $r_0 > 0$ be the constant from Proposition 3.7 and fix $r_1 \le r_0/2$. Let $x_0 \in \Omega_i$ be such that $\mathrm{dist}(x_0, \partial D) \ge r_1$. If $r := \mathrm{dist}(x_0, \partial\Omega_i) \ge r_1$, then by (3.17), we have

$$|\nabla u_i(x_0)| \le C\left(\lambda_1(\Omega_i) + r_1^{-1}\right)\|u_i\|_{L^\infty}. \tag{3.18}$$

If $r := \mathrm{dist}(x_0, \partial\Omega_i) < r_1$, then we set $y_0 \in \partial\Omega_i$ to be such that $|x_0 - y_0| = \mathrm{dist}(x_0, \partial\Omega_i)$. Using Proposition 3.7 and again (3.17), we have

$$\begin{aligned} |\nabla u_i(x_0)| &\le C\left(\lambda_1(\Omega_i)\|u_i\|_{L^\infty} + \frac{\|u_i\|_{L^\infty(B_r(x_0))}}{r}\right) \\ &\le C\left(\lambda_1(\Omega_i)\|u_i\|_{L^\infty} + \frac{\|u_i\|_{L^\infty(B_{2r}(y_0))}}{r}\right) \le C\Big(\lambda_1(\Omega_i)\|u_i\|_{L^\infty} + 2C_{12}\Big), \end{aligned} \tag{3.19}$$

which gives the local Lipschitz continuity of u_i.

If the function w_D is Lipschitz continuous on $\mathbb{R}^d$, we consider for every point $x_0 \in \Omega_i$ two possibilities for $r := \mathrm{dist}(x_0, \partial\Omega_i)$: if $3r \ge \mathrm{dist}(x_0, \partial D)$, then the maximum principle $u_i \le \lambda_1(\Omega_i)\|u_i\|_{L^\infty} w_D$ and the gradient estimate (3.17) gives

$$\begin{aligned} |\nabla u_i(x_0)| &\le C\left(\lambda_1(\Omega_i)\|u_i\|_{L^\infty} + \frac{\|u_i\|_{L^\infty(B_r(x_0))}}{r}\right) \\ &\le C\lambda_1(\Omega_i)\|u_i\|_{L^\infty}\left(1 + \frac{\|\nabla w_D\|_{L^\infty}}{r}\Big(\mathrm{dist}(x_0, \partial D) + r\Big)\right) \\ &\le C\lambda_1(\Omega_i)\|u_i\|_{L^\infty}\Big(1 + 4\|\nabla w_D\|_{L^\infty}\Big). \end{aligned} \tag{3.20}$$

If $3r \le \mathrm{dist}(x_0, \partial D)$ and $r \le r_0/2$, then the gradient estimate (3.17) gives again (3.19). If $r \ge r_0/2$, then we have (3.18) with $r_1 = r_0/2$ and this concludes the proof. ☐

4. Qualitative properties of the optimal sets

4.1. **Lack of triple points.** The lack of triple points was proved in [7] in the more general case of partitions concerning general functionals depending on the spectrum of the Dirichlet Laplacian. We recall here the result for the problem (1.3), which is a simple consequence of the non-degeneracy of the gradient (Corollary 3.4) and the three-phase monotonicity formula (Theorem 2.2).

Proposition 4.1. *Let $D \subset \mathbb{R}^2$ be a bounded open set. Let $(\Omega_1, \dots, \Omega_h)$ be optimal for* (1.3)*. Then for any three distinct indices $i, j, k \in \{1, \dots, h\}$, we have that $\partial\Omega_i \cap \partial\Omega_j \cap \partial\Omega_k = \emptyset$.*

4.2. **Lack of two-phase points on the boundary of the box.** Our first numerical simulations showed the lack of double points (i.e. points on the boundary of two distinct sets) on the boundary of the box D. There is a quick argument that proves the above claim in the case when the boundary ∂D is smooth. Indeed, if this is the case and if $x_0 \in \partial D$, then there is a ball $B \subset D^c$ such that $x_0 \in \partial B$. Since the gradient of the first eigenfunction u on B satisfies the non-degeneracy inequality (3.4), we can deduce as in Section 4.1 that for any distinct $i, j \in \{1, \dots, h\}$ we have $x_0 \notin \partial\Omega_i \cap \partial\Omega_j$. If the boundary ∂D is only Lipschitz a more refined argument is needed.

Proposition 4.2. *Let $D \subset \mathbb{R}^2$ be a bounded open set with Lipschitz boundary ∂D. Let $(\Omega_1, \dots, \Omega_h)$ be optimal for* (1.3)*. Then for any pair of distinct indices $i, j \in \{1, \dots, h\}$, we have that $\partial\Omega_i \cap \partial\Omega_j \cap \partial D = \emptyset$.*

Proof. Suppose, by absurd, that there is a point $x_0 \in \partial\Omega_i \cap \partial\Omega_j \cap \partial D$. If u_i and u_j are the first eigenfunctions on Ω_i and Ω_j, by Corollary 3.4 we have

$$\fint_{B_r(x_0)} |\nabla u_i|^2\,dx \ge C_{nd} \qquad \text{and} \qquad \fint_{B_r(x_0)} |\nabla u_j|^2\,dx \ge C_{nd}, \tag{4.1}$$

for small enough $r > 0$ and some non-degeneracy constant $C_{nd} > 0$. Since ∂D is Lipschitz, we have the density estimate $\liminf_{r\to 0} \frac{|D^c \cap B_r(x_0)|}{|B_r|} > 0$ and so, we can apply Theorem 2.1 (B), obtaining a contradiction. □

4.3. **Remarks on the regularity of the free boundary.** Let again $D \subset \mathbb{R}^2$ be a bounded open set and let $(\Omega_1, \dots, \Omega_h)$ be a solution of (1.3). We also denote with E the set $\mathbb{R}^2 \setminus \left(\bigcup_{i=1}^h \Omega_i\right)$. The lack of triple boundary points (Section 4.1) allows to classify the boundary points in three categories:

- *One-phase points*, i.e. points $x_0 \in \partial\Omega_i$ such that $x_0 \notin \partial\Omega_j$, for $j \neq i$.
- *Internal two-phase points*, i.e. points $x_0 \in \partial\Omega_i \cap \partial\Omega_j$ such that $x_0 \notin \partial^M E$, i.e. $|B_r(x_0) \cap E| = 0$, for some $r > 0$.
- *Boundary two-phase points*, i.e. points $x_0 \in \partial\Omega_i \cap \partial\Omega_j$ such that $|B_r(x_0) \cap E| > 0$, for every $r > 0$.

Remark 4.3. The boundary of an optimal set Ω_i around a one-phase point $x_0 \in \partial\Omega_i$ is analytic. Indeed, there is a ball $B_r(x_0)$ such that $B_r(x_0) \cap \Omega_j = \emptyset$, for every $j \neq i$. Thus, Ω_i solves the problem

$$\min\Big\{\lambda_1(\Omega) + c|\Omega| : \ \Omega \text{ open}, \ \Omega\Delta\Omega_i \subset B_r(x_0)\Big\}.$$

Thus, applying the classical Alt-Caffarelli technique from [1], one can obtain that $\partial\Omega_i \cap B_r(x_0)$ is analytic. We refer to [4] for the proof of this fact.

Remark 4.4. The boundary of an optimal set Ω_i around an internal two-phase point $x_0 \in \partial\Omega_i \cap \partial\Omega_j$ is $C^{2,\alpha}$, for every $\alpha \in (0,1)$. Indeed, since there is a ball $B_r(x_0)$ such that $|B_r(x_0) \cap E| = 0$, we have that the pair (Ω_i, Ω_j) is a solution of the optimal partition problem

$$\min\Big\{\sum_{k=1}^{2} \lambda_1(\omega_k) : \ \omega_1, \omega_2 \subset D_{ij} \text{ open}, \ \omega_1 \cap \omega_2 = \emptyset\Big\}, \tag{4.2}$$

where $D_{ij} := \Omega_i \cup \Omega_j \cup B_r(x_0)$. Applying the regularity result from [10], we get that the free boundary $\partial\Omega_i \cap \partial\Omega_j \cap B_r(x_0)$ is $C^{2,\alpha}$, for every $\alpha \in (0,1)$.

Remark 4.5. If $x_0 \in \partial\Omega_i \cap \partial\Omega_j$ is a boundary two-phase point, then the set $\Omega_i \cap \Omega_j$ has a measure theoretic cusp in x_0, i.e. we have that

$$\liminf_{r\to 0} \frac{|E \cap B_r(x_0)|}{|B_r|} = 0.$$

Indeed, if this is not the case we can use the non-degeneracy of u_i and u_j (Corollary 3.4) and the improved monotonicity estimate (Theorem 2.1 (B)) to obtain a contradiction.

Unfortunately, we are not able to give a complete regularity result for the boundary $\partial\Omega_i$, the reason is that there is no available estimates even on the one-dimensional Hausdorff measure of the set of boundary two-phase points. Nevertheless, the numerical evidence we provide below suggests that the number of boundary value points is finite.

5. Numerical Results

5.1. **Approximation of the optimal sets.** In order to calculate numerically the shape and the position of the optimal sets, we will need a suitable approximation of the optimal sets with optimal potentials. We use the technique developed in [3], which is based on some fine Γ-convergence results. This technique applies also to multiphase problems involving higher eigenvalues, treated in [7]

$$\min\Big\{\sum_{i=1}^{h} \lambda_k(\Omega_i) + c|\Omega_i| : \ \Omega_i \subset D \text{ quasi-open}, \ \Omega_i \cap \Omega_j = \emptyset\Big\}. \tag{5.1}$$

where with $\lambda_k(\Omega)$ we denote the kth eigenvalue of the Dirichlet Laplacian on $\Omega \subset D$, variationally characterized as

$$\lambda_k(\Omega) := \min_{S_k \subset H_0^1(\Omega)} \max_{u \in S_k} \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega u^2 \, dx},$$

where the minimum is over all k-dimensional subspaces S_k of $H_0^1(\Omega)$.

For a given measurable function $\varphi : \Omega \in [0,1]$ and constant $C > 0$, we consider the spectrum of the operator $-\Delta + C(1-\varphi)$ on D, consisting on the eigenvalues with variational characterization

$$\lambda_k(\varphi, C) := \min_{S_k \subset H_0^1(\Omega)} \max_{u \in S_k} \frac{\int_\Omega |\nabla u|^2 + C(1-\varphi)u^2 \, dx}{\int_\Omega u^2 \, dx},$$

where the minimum is over all k-dimensional subspaces S_k of $H_0^1(D)$. The corresponding kth eigenfunction satisfies the equation

$$-\Delta u_k + C(1-\varphi) = \lambda_k(\varphi, C) u_k, \qquad u_k \in H_0^1(D), \quad \int_D u_k^2 \, dx = 1. \tag{5.2}$$

By the general existence theorem of Buttazzo and Dal Maso [8], there is a solution $\big(\phi_1^C, \dots, \phi_h^C\big)$ of the problem

$$\min \Big\{ \sum_{i=1}^h \Big(\lambda_k(\varphi_i, C) + c \int_D \varphi_i \, dx \Big) : \ \varphi_i : D \to [0,1] \text{ measurable}, \ \sum_{i=1}^h \varphi_i \le 1 \Big\}. \tag{5.3}$$

Moreover, by the approximation result [3, Theorem 2.4] we have that, for every $i = 1, \dots, h$,

$$\lim_{C \to +\infty} \lambda_k(\varphi_i^C, C) = \lambda_k(\Omega_i) \qquad \text{and} \qquad \lim_{C \to +\infty} \varphi_i^C = \mathbb{1}_{\Omega_i},$$

where the second limit is strong in $L^1(D)$ and the h-uple $(\Omega_1, \dots, \Omega_h)$ is optimal for (5.1).

5.2. **Algorithm for finding the optimal sets in the unit square.** In the numerical computations we perform we consider the box $D = (0,1) \times (0,1)$. In view of the results discussed in the preceding section, we represent each of the sets Ω_l by a function $\varphi_l : D \to [0,1]$. Each of these functions is then numerically approximated by its values on a regular even spaced grid of $N \times N$ points with spacing $h = 1/(N-1)$. For each Ω_l and its corresponding function φ_l we consider the discretization $(\varphi^l)_{i,j} = \varphi_{i,j}^l := \varphi_l\big(\frac{i}{N-1}, \frac{j}{N-1}\big)$ and the following finite difference approximation of the eigenvalue problem (5.2)

$$\frac{4U_{i,j}^l - U_{i+1,j}^l - U_{i-1,j}^l - U_{i,j+1}^l - U_{i,j-1}^l}{h^2} + C(1 - \varphi_{i,j}^l) U_{i,j}^l = \lambda_k(C, \varphi_l) U_{i,j}^l, \tag{5.4}$$

for every $1 \le i, j \le N-1$. Note that the above discrete formulation can be written as a matrix eigenvalue problem $A\tilde{U}^l = \lambda \tilde{U}^l$, where $\tilde{U}^l$ is a column vector, obtained as a concatenation of the columns of the matrix $(U_{i,j}^l)_{i,j=1}^N$. Thus, for every $l = 1, \dots, h$, the discretized matrix eigenvalue problem above gives us the values of $\lambda_k(\varphi_l, C)$.

We note that setting $\varphi_{h+1} := 1 - \sum_{i=1}^h \varphi_i$, one may write the multiphase problem (5.3) in the equivalent form

$$\min \Big\{ \sum_{i=1}^h \lambda_k(\varphi_i, C) - c \int_D \varphi_{h+1} \, dx : \ \varphi_i : D \to [0,1] \text{ measurable}, \ \sum_{i=1}^{h+1} \varphi_i = 1 \Big\}, \tag{5.5}$$

which is more suitable for the numerical implementation we perform and which approximates (5.1) reformulated as an optimal partition problem

$$\min \Big\{ \sum_{i=1}^h \lambda_k(\Omega_i) - c|\Omega_{h+1}| : \ \Omega_i \subset \mathbb{R}^d \text{ quasi-open}, \ \Omega_i \cap \Omega_j = \emptyset, \ \text{for } i,j = 1, \dots, h+1 \Big\}.$$

To finish the numerical cost computation for the above problem we use the discrete approximation of the volume of Ω_{h+1} given by

$$|\Omega_{h+1}| \simeq \frac{1}{N^2} \sum_{i,j=1}^{N^2} \varphi_{i,j}^{h+1}.$$

In order to use an optimization algorithm we need to compute the derivative of the eigenvalues $\lambda_k(\varphi_l, C)$ with respect to the discretization points of the grid. The precise expression of this derivative was given in [3] and has the form

$$\partial_{i,j}\lambda_k(\varphi_l, C) = -C(U_{i,j}^l)^2, \tag{5.6}$$

where U^l is the lth normalized eigenvector solution of the discrete equation (5.4). We give below a formal justification of formula (5.6) using a slightly different approach, while for the detailed proof we refer to [3].

Let φ and θ be two given functions on D. We consider the perturbation $A(t) := -\Delta + C(1-\varphi-t\theta)$ of the operator $A(0) := -\Delta + C(1-\varphi)$. Let $\lambda_k(t) := \lambda_k(\phi + t\theta, C)$ be the kth eigenfunction of $A(t)$ and $u_k(t)$ be the corresponding eigenfunction, normalized in $L^2(D)$ and satisfying the equation

$$A(t)u_k(t) = \lambda_k(t)u_k(t), \qquad u_k \in H_0^1(D).$$

Suppose that the functions $\lambda_k(t)$, $u_k(t)$ and $A(t)$, depending on the variable t are differentiable in a neighbourhood of $t = 0$. Taking the derivative of the above equation we get

$$A'(t)u_k(t) + A(t)u_k'(t) = \lambda_k'(t)u_k(t) + \lambda_k(t)u_k'(t).$$

Multiplying both sides by $u_k(t)$ and integrating on D for $t = 0$, we get

$$-C\int_D \theta u_k(0)^2\,dx + \int_D u_k(0)A(0)u_k'(0)\,dx = \lambda_k'(0)\int_{Dr} u_k(0)^2\,dx + \lambda_k(0)\int_D u_k'(0)u_k(0)\,dx.$$

Since $A(0)$ is self-adjoint, we obtain

$$\frac{d}{dt}\lambda_k(\varphi + t\theta, C)\Big|_{t=0} = -C\int_D \theta u_k^2\,dx.$$

Considering the discrete case of the above directional derivative formula for φ_l and $\theta = \delta_{i,j}$ we obtain (5.6).

Reasoning in a similar way we get that the directional derivative of $\int_D \varphi_{h+1}\,dx$ in the direction of θ is just $\int_D \theta\,dx$ and thus the discrete derivative of the volume is

$$\partial_{i,j}|\Omega_{h+1}| = 1/N^2.$$

In order to perform the optimization under the constraint $\sum_{l=1}^{h+1}\varphi_l = 1$ we will use the projection operator on the simplex

$$\mathbb{S}^h = \Big\{X = (X_1, .., X_{h+1}) \in [0,1]^{h+1} : \sum_{l=1}^{h+1} X_l = 1\Big\},$$

defined by

$$\Big(\Pi_{\mathbb{S}^h}\varphi^l\Big)_{i,j} = \frac{|\varphi_{i,j}^l|}{\sum_{l=1}^{h+1}|\varphi_{i,j}^l|}.$$

More details about the justification of the choice of the projection operator and the algorithm used can be found in [3].

Algorithm 1 General form of the projected gradient algorithm

Require: $k,\ c,\ h,\ \alpha,\ \alpha_{min},\ \alpha_{max},\ \omega,\ \varepsilon,\ p_{max}$

$p = 1$
repeat
    for $i = 1$ **to** h **do**
        Compute the eigenpair $(\lambda_k(\varphi^l), U_k(\varphi^l))$ of the operator $A(\varphi^l)$
        $\varphi^l_{\text{temp}} \leftarrow \varphi^l - \alpha \nabla_d \lambda_k(\varphi^l)$
    end for
    $\varphi^{h+1}_{\text{temp}} \leftarrow \varphi^{h+1} - \alpha \nabla_d |\Omega_{h+1}|$
    $\varphi^l_{\text{temp}} \leftarrow \Pi_{\mathbb{S}^h} \varphi^l_{\text{temp}},\ l = 1..h+1$
    Compute $J_p = \sum_{l=1}^{h+1} \lambda_k(\varphi^l) - c \int_D \varphi^{h+1}$
    if $J_p \leq J_{p-1}$ **then**
        $\varphi^l \leftarrow \varphi^l_{\text{temp}},\ l = 1..h+1$
        $\alpha \leftarrow \min((1+\omega)\alpha, \alpha_{max})$
    else
        $\alpha \leftarrow \max((1-\omega)\alpha, \alpha_{min})$
    end if
    $p \leftarrow p + 1$
until $p = p_{max}$ or $\sup_{i,j} \alpha |(\Pi_{\mathbb{S}^h} \varphi^l)_{i,j}| < \varepsilon$

5.3. **Numerical results.** In this section we present some numerical simulations that confirm some the theoretical results from Section 4 and the paper [7]. Most of the tests we made were in the case $k = 1$, but the algorithm works for higher eigenvalues as well. The main issue in the case of higher eigenvalues concerns the differentiability of the eigenvalues with respect to perturbations, which is well known to be closely related to their multiplicity. Nevertheless, we were able to obtain some interesting numerical results also in the case $k = 2$ and one example can be seen in Figure 2.

In all the cases the lack of triple junction points, proved in [7], is clearly observed, provided that the parameter $c > 0$ is large enough. The lack of double points on the boundary of the square proved in Proposition 4.2 can also be noticed on Figure 1. Another phenomenon that can be observed is that the sets Ω_i near the corner of the square D will not fill the corner. This is a fact that can be easily proved by adding a ball B (i.e. subsolution for the functional $\lambda_1 + c|\cdot|$) outside D, for which the corner of the square lies on the sphere ∂B. Now the claim can be deduced by the monotonicity Theorem 2.1 (B), as in Proposition 4.2.

In conclusion, we considered the periodic version of the problem (1.3) on the square $[0,1] \times [0,1]$ in attempt to simulate a "partition" of the whole space $\mathbb{R}^2$ (see Figure 2). For small enough constant $c > 0$ we obtain a configuration with touching hexagons with rounded corners, in support of the numerical results in [3]. We note that there is a critical value of the parameter $c > 0$, for which the optimal configuration is formed of pentagons with rounded corners. This phenomenon appears as a consequence of the fact that the empty space E grows larger, while the phases $(\Omega_i)_{i=1}^h$ tend to maintain a balanced distribution.

Appendix A. Proof of the two-phase monotonicity formula

The proof of Theorem 2.1 is based on Lemma A.2, which involves the auxiliary functions $\widetilde{u}^+$ and $\widetilde{u}^-$ constructed below. Let $\lambda := \max\{\lambda_+, \lambda_-\}$ and let $r_0 > 0$ be small enough such that there is a positive radially symmetric function $\varphi \in H^1(B_{r_0})$ satisfying

$$-\Delta\varphi = \lambda\varphi \ \text{ in } \ B_{r_0}, \qquad 0 < a \leq \varphi \leq b, \tag{A.1}$$

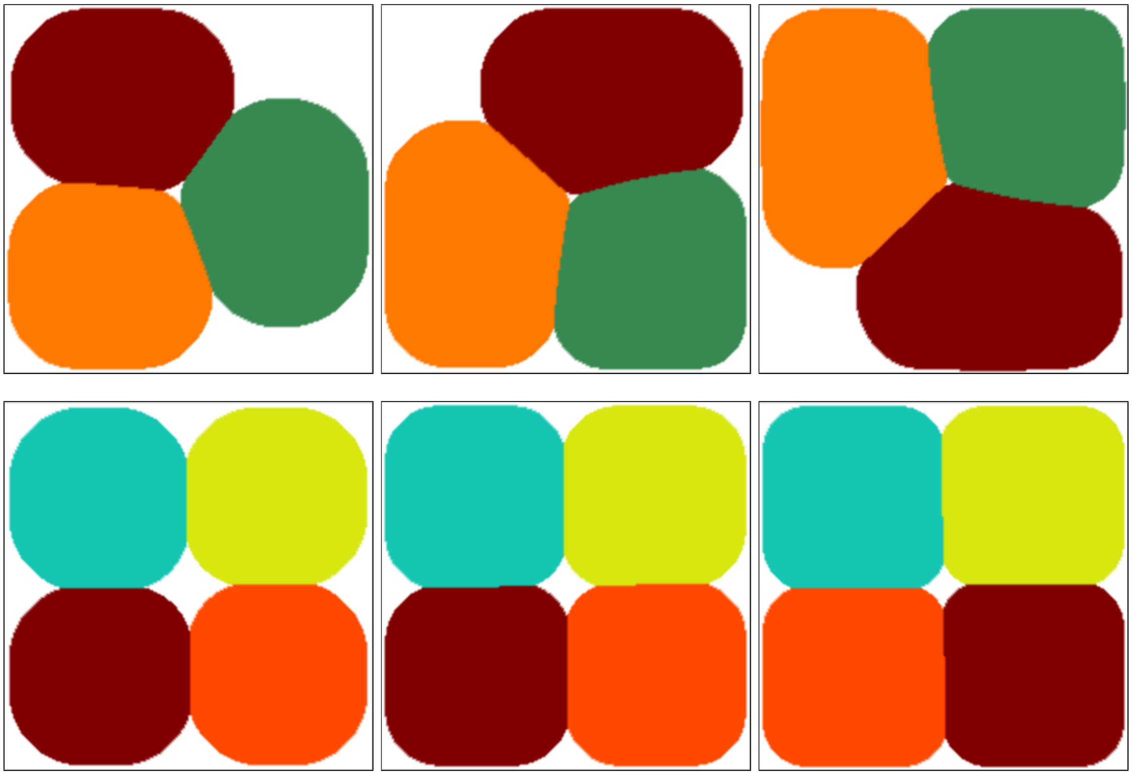

FIGURE 1. $k=1$, 200×200 non-periodic grid, 3 phases ($c=170,100,80$) and 4 phases ($c=250,150,100$)

FIGURE 2. $k=1$, 200×200 periodic grid, 8 phases, $c=500,580$ and $k=2$, 8 phases, $c=270$

for some constants $0<a\le b$ depending on d, λ and r_0. We now introduce the notation

$$U_1 := \frac{u^+}{\varphi} \qquad \text{and} \qquad U_2 := \frac{u^-}{\varphi}. \tag{A.2}$$

Remark A.1. A direct computation of the gradient and the Laplacian of U_i on B_{r_0} gives

$$\nabla U_1 = \varphi^{-1}\nabla u^+ - \varphi^2 u^+\nabla\varphi \qquad \text{and} \qquad \Delta U_1 = \varphi^{-1}\Delta u^+ - 2\varphi^{-2}(1+\varphi^{-1}u^+)\nabla u^+\cdot\nabla\varphi - \varphi^{-2}u^+\Delta\varphi.$$

We define the function $\Phi:[0,r_0]\to\mathbb{R}^+$ as

$$\Phi(r) := \left(\frac{1}{r^2}\int_{B_r}\varphi^2|\nabla U_1|^2\,dx\right)\left(\frac{1}{r^2}\int_{B_r}\varphi^2|\nabla U_2|^2\,dx\right) \tag{A.3}$$

Lemma A.2. *Consider the unit ball $B_1 \subset \mathbb{R}^2$. Let $u^+, u^- \in H^1(B_1) \cap L^\infty(B_1)$ be as in Theorem 2.1 and let $\Phi : [0, r_0] \to \mathbb{R}^+$ be given by (A.3). Then*

(a) Φ is decreasing on the interval $(0, r_0)$;

(b) If, moreover, the set $\Omega := B_1 \cap \{u^+ = 0\} \cap \{u^- = 0\}$ has positive density in 0, then there are constants $C > 0$ and $\varepsilon > 0$ such that

$$\frac{1}{r^\varepsilon}\Phi(r) \le \frac{C}{r_0^\varepsilon}\Phi(r_0).$$

Proof. We first estimate the derivative of Φ, using the notations $\nabla_n u$ and $\nabla_\tau u$ respectively for the normal and the tangential part of the gradient ∇u on the boundary of ∂B_r.

$$\begin{aligned}
\frac{\Phi'(r)}{\Phi(r)} &= -\frac{4}{r} + \sum_{i=1,2} \frac{\int_{\partial B_r} \varphi^2 |\nabla U_i|^2 \, d\mathcal{H}^1}{\int_{B_r} \varphi^2 |\nabla \widetilde{U}_i|^2 \, dx} \\
&\ge -\frac{4}{r} + \sum_{i=1,2} \frac{\int_{\partial B_r} \varphi^2 \big(|\nabla_\tau U_i|^2 + |\nabla_n U_i|^2\big) \, d\mathcal{H}^1}{\int_{\partial B_r} \varphi^2 U_i |\nabla_n U_i| \, d\mathcal{H}^1} && \text{(A.4)} \\
&\ge -\frac{4}{r} + \sum_{i=1,2} \frac{2\left(\int_{\partial B_r} \varphi^2 |\nabla_n U_i|^2 \, d\mathcal{H}^1\right)^{1/2} \left(\int_{\partial B_r} \varphi^2 |\nabla_\tau U_i|^2 \, d\mathcal{H}^1\right)^{1/2}}{\left(\int_{\partial B_r} \varphi^2 U_i^2 \, d\mathcal{H}^1\right)^{1/2} \left(\int_{\partial B_r} \varphi^2 |\nabla_n U_i|^2 \, d\mathcal{H}^1\right)^{1/2}} && \text{(A.5)} \\
&= -\frac{4}{r} + 2\sum_{i=1,2} \left(\frac{\int_{\partial B_r} |\nabla_\tau U_i|^2 \, d\mathcal{H}^1}{\int_{\partial B_r} U_i^2 \, d\mathcal{H}^1}\right)^{1/2} && \text{(A.6)} \\
&\ge -\frac{4}{r} + 2\sum_{i=1,2} \sqrt{\lambda_1(\partial B_r \cap \{U_i > 0\})} \\
&\ge -\frac{4}{r} + \sum_{i=1,2} \frac{2\pi}{\mathcal{H}^1(\partial B_r \cap \{U_i > 0\})}, && \text{(A.7)}
\end{aligned}$$

where (A.4) follows by integration by parts and the inequality $-\mathrm{div}(\phi^2 \nabla U_i) \ge 0$ obtained using Remark A.1; (A.5) is obtained by applying the mean quadratic-mean geometric inequality in the nominator and the Cauchy-Schwartz inequality in the denominator; (A.6) is due to the fact that φ is constant on ∂B_r; (A.7) follows by a standart symmetrization argument. Setting

$$\theta(r) := \frac{\mathcal{H}^1(\Omega \cap \partial B_r)}{\mathcal{H}^1(\partial B_r)},$$

and applying the mean arithmetic-mean harmonic inequality to (A.7), we get

$$\frac{\Phi'(r)}{\Phi(r)} \ge \frac{4}{r}\Big(-1 + \frac{1}{1-\theta(r)}\Big) \ge \frac{4\theta(r)}{r}, \tag{A.8}$$

which gives *(a)*. In order to prove (b), we note that for $r_0 > 0$ small enough we have the density estimate

$$|\Omega \cap B_r| \ge c|B_r|, \qquad \forall 0 < r \le r_0.$$

Using the fact that $\frac{\partial}{\partial r}|\Omega \cap B_r| = \mathcal{H}^1(\Omega \cap \partial B_r) = 2\pi r \theta(r)$ we get

$$\int_0^r 2\pi s(\theta(s) - c) \, ds \ge 0, \qquad \forall r \in (0, r_0). \tag{A.9}$$

As a consequence we have that

$$\int_{rc/2}^r 2\pi s \left(\theta(s) - \frac{c}{2}\right) \, ds \ge 0, \qquad \forall r \in (0, r_0). \tag{A.10}$$

Indeed, if this is not the case, then

$$0 \leq \int_0^r 2\pi s(\theta(s) - c)\,ds \leq \int_0^{cr/2} 2\pi s(1-c)\,ds - \int_{cr/2}^r 2\pi s \frac{c}{2}\,ds \leq -\pi r^2 c(1-c)^2,$$

which is in contradiction with (A.9). By (A.10), we get that there is a constant $c_0 > 0$ such that

$$\int_{rc/2}^r \theta(s)\,ds \geq c_0 r, \qquad \forall r < r_0. \tag{A.11}$$

By (A.8) we have

$$\begin{aligned} \log\left(r^{-\varepsilon}\Phi(r)\right) - \log\left(\left(rc/2\right)^{-\varepsilon}\Phi(rc/2)\right) &= \int_{rc/2}^r \left(-\frac{\varepsilon}{s} + \frac{\Phi'(s)}{\Phi(s)}\right) ds \\ &\geq \int_{rc/2}^r \frac{4}{s}\left(-\frac{\varepsilon}{4} + \theta(s)\right) ds \geq \varepsilon \log(c/2) + 4c_0, \end{aligned}$$

which is positive for $\varepsilon > 0$ small enough. Thus, we obtain that the sequence

$$a_n := r_n^{-\varepsilon}\Phi(r_n), \qquad \text{where} \quad r_n = (c/2)^n r_0,$$

is decreasing and so, by rescaling we obtain *(b)*. □

Proof of Theorem 2.1. We first note that as a consequence of Remark A.1, we have the estimates:

$$\begin{aligned} \int_{B_r} \frac{|\nabla u^\pm|^2}{|x|^{d-2}}\,dx &\leq 2\int_{B_r} \varphi^2 \frac{|\nabla \widetilde{u}^\pm|^2}{|x|^{d-2}}\,dx + 2\|\varphi^{-1}\nabla\varphi\|^2_{L^\infty(B_{r_0})} \int_{B_r} \frac{u^2}{|x|^{d-2}}\,dx, \\ \int_{B_r} \varphi^2 \frac{|\nabla \widetilde{u}^\pm|^2}{|x|^{d-2}}\,dx &\leq 2\int_{B_r} \frac{|\nabla u^\pm|^2}{|x|^{d-2}}\,dx + 2\|\varphi^{-1}\nabla\varphi\|^2_{L^\infty(B_{r_0})} \int_{B_r} \frac{u^2}{|x|^{d-2}}\,dx. \end{aligned} \tag{A.12}$$

Taking in consideration the inequality

$$\int_{B_{r_0}} \frac{|\nabla u^\pm|^2}{|x|^{d-2}}\,dx \leq C\left(1 + \int_{B_{2r_0}} |u^\pm|^2\,dx\right), \tag{A.13}$$

proved in [9], we obtain Theorem 2.1 *(a)* and *(b)* by Lemma A.2 and simple arithmetic. □

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