RIGIDITY OF ASYMPTOTICALLY CYLINDRICAL GRADIENT SHRINKING RICCI SOLITON

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Abstract. In this paper we prove that every asymptotically cylindrical gradient shrinking Ricci soliton is a cylinder.

1. Introduction and statement of the result

A gradient shrinking Ricci soliton is a complete Riemannian manifold (M^n, g) satisfying

$$\mathrm{Ric} + \nabla^2 f \;=\; \lambda\, g\,,$$

for some $\lambda > 0$ and some smooth function f defined on M^n. In the following, we will adopt the normalization $\lambda = \frac{1}{2}$. Hence, throughout this paper the fundamental equation will be given by

$$\mathrm{Ric} + \nabla^2 f \;=\; \frac{1}{2}\, g\,. \tag{1.1}$$

Shrinking gradient Ricci solitons turn out to be Type I finite time singularities of the Ricci flow. Therefore their classification is important to understand finite time singularities of the Ricci flow in the large. Since the seminal work of Perelman on the classification of 3-dimensional shrinking gradient Ricci solitons, there has been a vast amount of litterature on the subject (see [6] for a survey).

In order to state our main result, we introduce the following definition of *asymptotically cylindrical* gradient shrinking Ricci soliton.

Definition 1.1. *A complete noncompact gradient shrinking Ricci soliton (M^n, g, f) is said to be* asymptotically cylindrical *if for every sequence of marked points $(x_k)_{k\in\mathbb{N}}$ which tends to infinity, the sequence of pointed Riemannian manifolds (M^n, g, x_k) converges in the smooth Cheeger-Gromov sense to the cylinder $(\mathbb{R}\times\mathbb{S}^{n-1}, dt^2 + h)$, where h is the metric of positive constant curvature normalized by $\mathrm{Ric}_h = h/2$.*

We are now in the position to state our main result.

Theorem 1.2. *Let $(M^n, g, \nabla f)$ be a complete noncompact gradient shrinking Ricci soliton, which is asymptotically cylindrical. Then, $(M^n, g, \nabla f)$ is isometric to the cylinder $(\mathbb{R}\times\mathbb{S}^{n-1}, dt^2 + h)$, where h is the metric of positive constant curvature normalized by $\mathrm{Ric}_h = h/2$.*

In case of positive curvature operator, we have the following :

Corollary 1.3. *Let $(M^n, g, \nabla f)$ be a complete shrinking gradient Ricci soliton with bounded positive curvature operator. Either it is compact either $\limsup_{+\infty} \frac{\mathrm{Vol}\, B(p,r)}{r} = +\infty$ for any $p \in M^n$.*

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A first remark on Theorem 1.2 is that we do not use any non negativity assumptions on the curvature tensor. Moreover, according to the classification of locally conformally flat shrinking gradient Ricci solitons due to [13], we are reduced to prove that such an asymptotically cylindrical soliton is rotationally symmetric.

The core of the proof of Theorem 1.2 is essentially based on the proof of the Perelman conjecture on the Bryant soliton by Brendle [4]. Following the notations of [4], we define $X := \nabla f$. Even if we mimic Brendle's proof, our proof substantially differs in the interpolation of almost-Killing vector fields (section 4) and the analysis of Lichnerowicz equation (section 5) since we do not assume the metric to be non negatively curved.

Now, we give the main steps of the proof of Theorem 1.2 :

- It suffices to build $n(n-1)/2$ independent Killing vector fields orthogonal to X. Generally speaking, a Killing field U satisfies $\Delta U + \mathrm{Ric}(U) = 0$. In the case of a shrinking gradient Ricci soliton, this gives $\Delta U - \nabla_U X + U/2 = 0$. For technical reasons, it is easier to estimate the operator $\Phi(U) = \Delta U - \nabla_X U + U/2$. Using the assumption on the asymptotic behavior, sections 2 and 3 are devoted to build $n(n-1)/2$ vector fields $(U_i)_i$ which are almost Killing (proposition 3.3). In particular, it is shown that the vector fields U_i are bounded and the vector fields $\Phi(U_i)$ decay sufficiently fast.

- In Section 4, we prove Theorem 4.2 that establishes the surjectivity of Φ in the following sense : we prove the existence of vector fields V_i decaying sufficiently fast such that $\Phi(V_i) = \Phi(U_i)$ for any i. In particular, this ensures that vector fields $U_i - V_i$ are not trivial. For that purpose, we use the potential function as a barrier to establish a maximum principle at infinity : see proposition 4.1.

- Section 5 studies the rigidity of Lichnerowicz equation. In fact, the vector fields $W_i := V_i - U_i$ built previously lie in $\ker \Phi$. Now, $\ker \Phi$ is not reduced to Killing fields since $X \in \ker \Phi$. Theorem 5.3 shows that there exist real numbers λ_i such that the vector fields $W_i - \lambda_i X$ are Killing. Its proof consists in noting that the symmetric 2-tensors $\mathscr{L}_{W_i}(g) =: h_i$ satisfy Lichnerowicz equation $(\mathscr{L}_X(h_i)) - h_i = \Delta_L(h_i)$ where Δ_L is the Lichnerowicz Laplacian. Theorem 5.3 shows that the only solution decaying polynomially is, up to a homothety, $\mathscr{L}_X(g)$. As for vector fields, we need to establish a priori estimates (proposition 5.2) with the help of our favorite barrier function $v := f - n/2$ which is a positive eigenvalue of Δ_f : $\Delta_f v = -v$.

- Section 6 concludes the proof of theorem 1.2 and proves corollary 1.3.

We end this introduction by stating some remarks on related works and some open questions. Recently, Kotschwar and Wang [10] have proved that two shrinking gradient Ricci solitons whose asymptotic cones are isometric are actually isometric. Their method is completely different from ours. By using rescaling arguments, can one reprove the result of Kotschwar and Wang in the particular case where the asymptotic cone is the most symmetric? Actually, it seems that it is not a straightforward adaptation of the cylindrical case: indeed, the weighted laplacian we are dealing with, i.e. $\Delta_f := \Delta + \nabla f$, is not invariant under scalings.

The same question can be asked for other kinds of singularity of the Ricci flow called expanding gradient Ricci solitons (EGS for short). We recall that an EGS is a complete

Riemannian manifold (M^n, g) satisfying

$$\mathrm{Ric} - \nabla^2 f = -\frac{1}{2}\, g\,,$$

for some smooth function f defined on M^n. Such singularities naturally arise as blow-up of non compact non collapsed Type III solutions with non negative curvature operator according to the work of Schulze and Simon [14]. As a consequence of their work, studying the asymptotic geometry of non compact non collapsed Riemannian manifolds with nonnegative curvature operator reduces to the classification of (the asymptotic cones of) non negatively curved EGS. Now, Bryant, in unpublished notes, has also built rotational symmetric EGS on $\mathbb{R}^n$ for $n \geq 3$ with positive curvature [8, Section 5, Chap. 1]]. This construction gives the whole classification of non negatively curved rotationally symmetric EGS. Therefore, we ask for the analogue of the Kotschwar-Wang theorem in this setting. Namely, with the above notations, is an expanding gradient Ricci soliton asymptotic to the cone $(C(\mathbb{S}^{n-1}), dr^2 + (cr)^2 g_{\mathbb{S}^{n-1}})$ rotationally symmetric ? Recently, Chodosh [7] answered positively in the case the metric has nonnegative curvature operator. In that case, the main ingredient is the maximum principle for symmetric 2-tensors due to Hamilton which is not available in case of mixed sign curvatures. Now, one has always a barrier function on an EGS since $\Delta_f(f + n/2) = f + n/2$. Therefore, can one get rid of the sign assumption as well ?

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2. Asymptotic geometry

We start by recalling some basic and well known curvature identities that hold on a shrinking gradient Ricci soliton. A proof of these identities can be found for example in [11].

Lemma 2.1. *Let $(M^n, g, \nabla f)$ be a gradient shrinking Ricci soliton. Then, setting $X := \nabla f$, the following identities hold true.*

$$\Delta f + \mathrm{R} = \frac{n}{2}\,, \tag{2.1}$$

$$\nabla \mathrm{R} = 2\,\mathrm{Ric}(X, \cdot\,)\,, \tag{2.2}$$

$$|X|^2 + \mathrm{R} = f, \tag{2.3}$$

$$\mathrm{div}\,\mathrm{Rm}\,(Y, Z, W) = \mathrm{Rm}\,(Y, Z, W, X), \tag{2.4}$$

for every vector fields Y, Z, W. Moreover, setting $\Delta_f := \Delta - \nabla_X$, we have that

$$\begin{aligned} \mathrm{Rm} &= \Delta_f \mathrm{Rm} + \mathrm{Rm} * \mathrm{Rm}, && (2.5)\\ \mathrm{Ric} &= \Delta_f \mathrm{Ric} + 2\,\mathrm{Rm} * \mathrm{Ric}, && (2.6)\\ \mathrm{R} &= \Delta_f \mathrm{R} + 2|\mathrm{Ric}|^2, && (2.7) \end{aligned}$$

where, if A and B are two tensors, $A * B$ denotes some linear combination of contractions of the tensorial product of A and B.

Remark 2.2. We observe that the general form of identity (2.3) is $|X|^2 + \mathrm{R} = f + c$, where $c \in \mathbb{R}$ is a real constant. In the rest of this paper we will systematically make the normalization assumption $c = 0$.

We recall the following growth estimate due to Cao-Zhou [5] on the potential function of a noncompact gradient shrinking soliton.

Lemma 2.3. *Let $(M^n, g, \nabla f)$ be a complete noncompact gradient shrinking Ricci soliton. Then, the potential function f satisfies the estimates*

$$\frac{1}{4}(r(x) - c_1)^2 \le f(x) \le \frac{1}{4}(r(x) + c_2)^2\,,$$

where $r(x) = d(x_0, x)$ is the distance function from some fixed point $x_0 \in M$, c_1 and c_2 are positive constants depending only on n and the geometry of g on the unit ball $B(x_0, 1)$.

From now on, we assume that our complete noncompact gradient shrinking Ricci soliton (M^n, g, f) is *asymptotically cylindrical* in the sense of Definition 1.1 and with bounded curvature. In order to give a more careful estimate of how the soliton metric converges to the cylindrical one, it is convenient to introduce the following tensor

$$T \;:=\; \mathrm{Ric} \;-\; \frac{\mathrm{R}}{n-1}\left(g \;-\; \frac{df \otimes df}{|df|^2}\right). \qquad (2.8)$$

We observe that the tensor T is well defined whenever $\nabla f \neq 0$. In particular, from the results in [5], we have that T is well defined outside a compact set.

Lemma 2.4. *Let $(M^n, g, \nabla f)$ be a complete noncompact asymptotically cylindrical gradient shrinking Ricci soliton and let T be the tensor defined in (2.8) outside a compact set. Then, we have that $|T| = o(1)$, as $f \to +\infty$. This means that for every sequence of points $(x_k)_{k\in\mathbb{N}}$ such that $f(x_k) \to +\infty$, as $k \to +\infty$, one has that $|T|(x_k) \to 0$.*

Proof. We start by computing the quantity $|T|^2$.

$$\begin{aligned} |T|^2 &= |\mathrm{Ric}|^2 - \frac{\mathrm{R}^2}{n-1} + \frac{\mathrm{R}}{n-1}\,\frac{2\,\mathrm{Ric}(\nabla f, \nabla f)}{|\nabla f|^2}\\ &= |\mathrm{Ric}|^2 - \frac{\mathrm{R}^2}{n-1} + \frac{\mathrm{R}}{n-1}\,\frac{\langle \nabla \mathrm{R}, \nabla f\rangle}{|\nabla f|^2}\,, \end{aligned}$$

where we used the identity (2.3) in the last equality. By the fact that the soliton is asymptotically cylindrical, it is immediate to deduce that $|\mathrm{Ric}|^2 - \mathrm{R}^2/(n-1) = o(1)$, for $f \to +\infty$. For the same reason, we have that $|\nabla \mathrm{R}| = o(1)$, as $f \to +\infty$, whereas, by the results in [5], one has that $|\nabla f|^2 = O(f)$, as $f \to +\infty$. From these facts, we infer that also the third term in the right hand side tends to zero, as $f \to +\infty$. ∎

So far, we have shown that $|T| = o(1)$ for $f \to +\infty$. In order to improve this estimate we state the following lemma, in which we prove some basic but useful properties of the tensor T.

Lemma 2.5. *Let $(M^n, g, \nabla f)$ be a complete noncompact asymptotically cylindrical gradient shrinking Ricci soliton and let T be the tensor defined in (2.8) outside a compact set. Then, setting $\mathbf{n} := X/|X| = \nabla f/|\nabla f|$, we have*

$$\operatorname{tr} T = 0\,, \qquad |\,T(\mathbf{n},\cdot\,)\,| = o\,(f^{-1/2}) \quad \textit{and} \quad T(\mathbf{n},\mathbf{n}) = o\,(f^{-1})\,, \tag{2.9}$$

as $f \to +\infty$.

Proof. The fact that the tensor T is traceless follows immediately from its definition. Using the identities (2.2) and (2.3) in Lemma 2.1, we get

$$|\,T(\mathbf{n},\cdot\,)\,| \;=\; |\,\mathrm{Ric}(\mathbf{n},\cdot\,)\,| \;=\; \frac{|\nabla \mathrm{R}|}{2|X|} \;=\; o\,(f^{-1/2})\,,$$

where in the last equality we also used the fact that the soliton is *asymptotically cylindrical* and thus $|\nabla \mathrm{R}| \to 0$, as $f \to +\infty$. From the asymptotic behavior of the soliton it is also possible to deduce that $\Delta \mathrm{R} \to 0$, as $f \to +\infty$. Moreover, by the Definition 1.1, one has $|\,\mathrm{Ric} - \frac{1}{2}g\,| \to 0$ and thus $2|\,\mathrm{Ric}\,|^2 - \mathrm{R} \to 0$, as $f \to +\infty$. Combining this with the identities above and with equation (2.7), we deduce

$$T(\mathbf{n},\mathbf{n}) \;=\; \mathrm{Ric}(\mathbf{n},\mathbf{n}) \;=\; \frac{\langle \nabla \mathrm{R}, X\rangle}{|X|^2} \;=\; \frac{(\Delta \mathrm{R} + 2|\,\mathrm{Ric}\,|^2 - \mathrm{R})}{|X|^2} \;=\; o\,(f^{-1})\,.$$

This completes the proof of the lemma. □

In the next proposition, we derive a partial differential inequality for the quantity $|T|^2$. This will then be used to improve the estimates of the decay of $|T|$ at infinity.

Proposition 2.6. *Let $(M^n, g, \nabla f)$ be a complete noncompact asymptotically cylindrical gradient shrinking Ricci soliton and let T be the tensor defined in (2.8) outside a compact set. Then, there exists a positive constant $c(n)$, only depending on the dimension n, such that*

$$\Delta_f |T|^2 \;\geq\; c(n)\,|T|^2 \,+\, O(f^{-1})\,,$$

as $f \to \infty$. Moreover, we can choose $c(n) = 2/(n-2) - \eta$ for any positive η sufficiently small.

Proof. Using the same notations as in Lemma 2.5 and taking advantage of the equations (2.6) and (2.7), we compute the difference $\Delta_f T - T$, namely

$$\begin{aligned}
\Delta_f T - T \;&=\; \Delta_f \,\mathrm{Ric} - \mathrm{Ric} - \frac{1}{n-1}\big[\,\Delta_f(\mathrm{R}(g - \mathbf{n}\otimes\mathbf{n})) \,-\, \mathrm{R}(g - \mathbf{n}\otimes\mathbf{n})\,\big] \\
&=\; -\frac{1}{n-1}\big[\,(\Delta_f \mathrm{R} - \mathrm{R})(g - \mathbf{n}\otimes\mathbf{n}) \,+\, 2\,\nabla_{\nabla \mathrm{R}}(\mathbf{n}\otimes\mathbf{n}) \,-\, \mathrm{R}\,\Delta_f(\mathbf{n}\otimes\mathbf{n})\,\big] \\
&\quad\; -2\,\mathrm{Rm} * \mathrm{Ric} \\
&=\; \frac{1}{n-1}\big[\,2|\,\mathrm{Ric}\,|^2(g - \mathbf{n}\otimes\mathbf{n}) \,-\, 2\,\nabla_{\nabla \mathrm{R}}(\mathbf{n}\otimes\mathbf{n}) \,+\, \mathrm{R}\,\Delta_f(\mathbf{n}\otimes\mathbf{n})\,\big] \\
&\quad\; -2\,\mathrm{Rm} * \mathrm{Ric}\;.
\end{aligned}$$

Now, we recall that $(\mathrm{Rm} * \mathrm{Ric})_{ij} := \mathrm{R}_{iklj}\mathrm{R}_{kl}$. Therefore, we get

$$\begin{aligned}\langle \mathrm{Rm} * \mathrm{Ric}\,, T\rangle &= \langle \mathrm{Rm} * T\,, T\rangle + \frac{R}{n-1}\,\langle \mathrm{Rm} * (g - \mathbf{n}\otimes\mathbf{n})\,, T\rangle \\ &= \langle \mathrm{Rm} * T\,, T\rangle + \frac{\mathrm{R}}{n-1}\,\langle \mathrm{Ric} - \mathrm{Rm}(\mathbf{n},\cdot,\cdot,\mathbf{n})\,, T\rangle \\ &= \langle \mathrm{Rm} * T\,, T\rangle + \frac{\mathrm{R}}{n-1}\,|T|^2 - \frac{\mathrm{R}^2}{(n-1)^2}\,T(\mathbf{n},\mathbf{n}) - \frac{\mathrm{R}}{n-1}\,\langle \mathrm{Rm}(\mathbf{n},\cdot,\cdot,\mathbf{n})\,, T\rangle\,,\end{aligned}$$

where in the last equality we used the fact that the tensor T is traceless. Since $(M^n, g, \nabla f)$ is *asymptotically cylindrical*, we use the Lemma 2.4 and the fact that the Weyl part of the Riemmann tensor tends to zero in order to deduce that

$$\mathrm{Rm} = \frac{\mathrm{R}}{(n-1)(n-2)}(g - \mathbf{n}\otimes\mathbf{n}) \odot (g - \mathbf{n}\otimes\mathbf{n}) + o(1)\,,$$

where $\odot$ represents the Kulkarni Nomizu product. As a consequence, we get

$$\begin{aligned}\langle \mathrm{Rm} * T, T\rangle &= \frac{\mathrm{R}}{(n-1)(n-2)}\,\big[\,2\,|\,T(\mathbf{n},\cdot)|^2 - |T|^2\,\big] + o\,(|T|^2) \\ &= -\,\frac{\mathrm{R}}{(n-1)(n-2)}\,|T|^2 + o\,(|T|^2) + o\,(f^{-1})\,,\end{aligned}$$

where in the last equality we have used the estimates in Lemma 2.5. Moreover, using equation (2.4) shows that

$$\langle \mathrm{Rm}(\mathbf{n},\cdot,\cdot,\mathbf{n})\,, T\rangle = \frac{1}{|X|}\,\mathrm{div}\,\mathrm{Rm} * T = o\,(f^{-1/2})|T|\,.$$

Taking the sum, we obtain

$$\begin{aligned}\Delta_f|T|^2 - 2|T|^2 &= 2|\nabla T|^2 - \frac{4(n-3)\,\mathrm{R}}{(n-1)(n-2)}\,|T|^2 + o\,(|T|^2) + o\,(f^{-1/2})|T| \\ &\quad + \frac{2}{(n-1)}\Big\langle -2\,\nabla_{\nabla \mathrm{R}}(\mathbf{n}\otimes\mathbf{n}) + \mathrm{R}\,(\Delta_f(\mathbf{n}\otimes\mathbf{n})\,,\,T\Big\rangle + o\,(f^{-1})\,. \qquad (2.10)\end{aligned}$$

In order to proceed, we are going to analyze the asymptotic behavior of the second raw. We claim that

$$\langle -2\,\nabla_{\nabla \mathrm{R}}(\mathbf{n}\otimes\mathbf{n}) + \mathrm{R}\,\Delta_f(\mathbf{n}\otimes\mathbf{n})\,, T\rangle = O\,(f^{-1})\,,$$

as $f \to \infty$. We start with the estimate of the term $\langle 2\,\nabla_{\nabla \mathrm{R}}(\mathbf{n}\otimes\mathbf{n})\,, T\rangle$. First, we notice that

$$\langle \nabla_{\nabla \mathrm{R}}(\mathbf{n}\otimes\mathbf{n})\,, T\rangle = \langle \nabla_{\nabla \mathrm{R}}\,\mathbf{n}\,, T(\mathbf{n},\cdot)\rangle \leq |\nabla_{\nabla \mathrm{R}}\,\mathbf{n}|\,|T(\mathbf{n},\cdot)| \leq |\nabla\mathbf{n}|\,|\nabla \mathrm{R}|\,|T(\mathbf{n},\cdot)|\,.$$

On the other hand, we have that

$$|\nabla\mathbf{n}| = \left|\frac{\nabla\nabla f}{|\nabla f|} - \frac{\nabla\nabla f(\mathbf{n}\,,\cdot\,)\otimes\mathbf{n}}{|\nabla f|}\right| \leq 2\,\frac{|\nabla\nabla f|}{|\nabla f|} = O\,(f^{-1/2})\,.$$

Combining this with Lemma 2.5, we obtain that $\langle \nabla_{\nabla \mathrm{R}}(\mathbf{n}\otimes\mathbf{n})\,, T\rangle = o\,(f^{-1})$. We pass now to estimate the term $\langle \mathrm{R}\,\Delta_f(\mathbf{n}\otimes\mathbf{n})\,, T\rangle = \langle \mathrm{R}\,\Delta(\mathbf{n}\otimes\mathbf{n})\,, T\rangle - \langle \mathrm{R}\,\nabla_X(\mathbf{n}\otimes\mathbf{n})\,, T\rangle$. We start with the second term of the right hand side.

$$\begin{aligned}\langle \nabla_X(\mathbf{n}\otimes\mathbf{n}),T\rangle &= 2\,\langle \nabla_X\mathbf{n}\,,T(\mathbf{n},\cdot)\rangle\\ &\leq 2\left|\frac{\langle\nabla\nabla f(X,\cdot\,),T(\mathbf{n},\cdot)\rangle}{|\nabla f|}-\frac{\nabla\nabla f(\mathbf{n}\,,X)\,T(\mathbf{n},\mathbf{n})}{|\nabla f|}\right|\\ &\leq 2\left|\frac{\langle \mathrm{Ric}(X,\cdot\,),T(\mathbf{n},\cdot)\rangle}{|\nabla f|}\right| + |T(\mathbf{n},\mathbf{n})| + \left|\frac{\nabla\nabla f(\mathbf{n}\,,X)\,T(\mathbf{n},\mathbf{n})}{|\nabla f|}\right|\\ &\leq \frac{|\nabla \mathrm{R}|}{|\nabla f|}\,|T(\mathbf{n},\cdot)| + 2\,|T(\mathbf{n},\mathbf{n})| \;=\; o\,(f^{-1})\,,\end{aligned}$$

where we used the identity (2.2) and the estimates in Lemma 2.5. To estimate the term $\langle \mathrm{R}\,\Delta(\mathbf{n}\otimes\mathbf{n})\,,T\rangle$, we recall that

$$\Delta(\mathbf{n}\otimes\mathbf{n}) \;=\; (\Delta\mathbf{n})\otimes\mathbf{n} + \mathbf{n}\otimes(\Delta\mathbf{n}) + 2\,\nabla\mathbf{n} * \nabla\mathbf{n}$$

and we immediately notice that $\langle \nabla\mathbf{n} * \nabla\mathbf{n}\,,T\rangle = \nabla_k\mathbf{n}_i\,\nabla_k\mathbf{n}_j\,T_{ij} = O(f^{-1})$. Moreover, a direct computation shows that

$$\Delta\mathbf{n} \;=\; |X|\,\Delta\left(\frac{1}{|X|}\right)\mathbf{n} + 2\,\nabla_k\left(\frac{1}{|X|}\right)\cdot\nabla_k X + \frac{1}{|X|}\Delta X\,.$$

Using the identities in Lemma 2.5, it is easy to obtain $\Delta X = \Delta\nabla f = \nabla\Delta f + \mathrm{Ric}(\nabla f,\cdot\,) = -\nabla \mathrm{R} + \mathrm{Ric}(X\,,\cdot\,) = -\frac{1}{2}\nabla\mathrm{R}$, and thus

$$\frac{|\Delta X|}{|X|} \;=\; o\,(f^{-1/2})\,.$$

Similarly, $|\nabla(1/|X|)| = O(f^{-1})$ and $|X|\,\Delta(1/|X|) = O(f^{-1})$. Recasting all these estimates, it is straightforward to check that the claim is proven. Combining the claim with the estimate (2.10), we obtain

$$\Delta_f|T|^2 \;=\; 2\,|\nabla T|^2 + \frac{2}{(n-2)}\,|T|^2 + o\,(|T|^2) + O\,(f^{-1})\,.$$

The statement of the proposition follows at once. □

In order to analyze the partial differential inequality obtained in Proposition 2.6, we proof the following algebraic lemma about the evolution of the curvature tensor of a gradient Ricci soliton. A parabolic proof of this result can be found for example in [1], where the time derivative plays the role of the covariant derive along ∇f.

Lemma 2.7. *Let $(M^n,g,\nabla f)$ be a gradient Ricci soliton. Then, for every vector fields U, V, W and Y,*

$$\begin{aligned}\nabla_{\nabla f}\,\mathrm{Rm}(U,V,W,Y) &= -\nabla_U(\mathrm{div}\,\mathrm{Rm})(W,Y,V) + \nabla_V(\mathrm{div}\,\mathrm{Rm})(W,Y,U)\\ &\quad + \mathrm{Rm}(V,\nabla^2 f(U,\cdot\,),W,Y) - \mathrm{Rm}(U,\nabla^2 f(V,\cdot\,),W,Y).\end{aligned}$$

Proof. By the second Bianchi identity, we have

$$\nabla_{\nabla f}\,\mathrm{Rm}(U,V,W,Y) \;=\; -\nabla_U\,\mathrm{Rm}(V,\nabla f,W,Y) - \nabla_V\,\mathrm{Rm}(\nabla f,U,W,Y)\,.$$

On the other hand, one has

$$\begin{aligned}\nabla_U \operatorname{Rm}(V,\nabla f,W,Y) \;&=\; U\big(\operatorname{Rm}(V,\nabla f,W,Y)\big) - \operatorname{Rm}(\nabla_U V,\nabla f,W,Y)\\ &\quad - \operatorname{Rm}(V,\nabla^2 f(U,\cdot),W,Y) - \operatorname{Rm}(V,\nabla f,\nabla_U W,Y)\\ &\quad - \operatorname{Rm}(V,\nabla f,W,\nabla_U Y)\\ &=\; \nabla_U(\operatorname{div}\operatorname{Rm})(W,Y,V) - \operatorname{Rm}(V,\nabla^2 f(U,\cdot),W,Y)\,,\end{aligned}$$

where, in the last equality we used the identity (2.4) in Lemma 2.1. This concludes the proof of the lemma. □

As a corollary of this general lemma, we obtain estimates for the covariant derivatives of the Ricci tensor along ∇f.

Corollary 2.8. *Let $(M^n, g, \nabla f)$ be a complete noncompact asymptotically cylindrical gradient shrinking Ricci soliton. Then, we have that $|\nabla_{\nabla f}\operatorname{Ric}|$ and $|\nabla_{\nabla f}\nabla_{\nabla f}\operatorname{Ric}|$ are bounded.*

Proof. First of all we notice that, in local coordinates, the statement of the previous lemma reads

$$\begin{aligned}\nabla_p f\,\nabla_p \mathrm{R}_{ijkl} \;&=\; -\nabla_i(\operatorname{div}\operatorname{Rm})_{klj} + \nabla_j(\operatorname{div}\operatorname{Rm})_{kli} + \nabla_i\nabla_p f\,\mathrm{R}_{jpkl} - \nabla_j\nabla_p f\,\mathrm{R}_{ipkl}\\ &=\; -\nabla_i(\nabla_k \mathrm{R}_{lj} - \nabla_l \mathrm{R}_{kj}) + \nabla_j(\nabla_k \mathrm{R}_{li} - \nabla_l \mathrm{R}_{ki})\\ &\quad -\mathrm{R}_{ip}\,\mathrm{R}_{jpkl} + \mathrm{R}_{jp}\,\mathrm{R}_{ipkl} - \mathrm{R}_{ijkl}\,,\end{aligned}$$

where, in the last equality, we used the contracted second Bianchi identity and the soliton equation (1.1). By the fact that the soliton is asymptotically cylindrical, we obtain that $|\nabla_{\nabla f}\operatorname{Rm}|$ is bounded. In particular, we have that also $|\nabla_{\nabla f}\operatorname{Ric}|$ is bounded. Taking the trace of the identity above, we get

$$\nabla_p f\,\nabla_p \mathrm{R}_{jk} \;=\; -\nabla_i\nabla_k \mathrm{R}_{ij} + \Delta \mathrm{R}_{kj} + \frac{1}{2}\nabla_j\nabla_k \mathrm{R} - \mathrm{R}_{ip}\,\mathrm{R}_{jpki} + \mathrm{R}_{jp}\,\mathrm{R}_{pk} - \mathrm{R}_{jk}\,. \tag{2.11}$$

We are now in the position to estimate $|\nabla_{\nabla f}\nabla_{\nabla f}\operatorname{Ric}|$. In fact, taking the derivative of the previous expression, we obtain

$$\begin{aligned}\nabla_q f\,\nabla_p f\,\big(\nabla_q\nabla_p \mathrm{R}_{jk}\big) \;&=\; \nabla_q f\,\nabla_q\big(\nabla_p f\,\nabla_p \mathrm{R}_{jk}\big) - \nabla_q f\,(\nabla_q\nabla_p f)\,(\nabla_p \mathrm{R}_{jk})\\ &=\; \nabla_q f\;\nabla_q\big(-\nabla_i\nabla_k \mathrm{R}_{ij} + \Delta \mathrm{R}_{kj} + \frac{1}{2}\nabla_j\nabla_k \mathrm{R}\big)\\ &\quad +\nabla_q f\;\nabla_q\big(-\mathrm{R}_{ip}\,\mathrm{R}_{jpki} + \mathrm{R}_{jp}\,\mathrm{R}_{pk} - \mathrm{R}_{jk}\big)\\ &\quad +\frac{1}{2}\nabla_p \mathrm{R}\,(\nabla_p \mathrm{R}_{jk}) - \frac{1}{2}\nabla_p f\,(\nabla_p \mathrm{R}_{jk})\,.\end{aligned}$$

From the estimates obtained in the first part of the proof, we infer that the second and the third raw of the right hand side are bounded. To estimate the first raw, we first notice that, by the contracted second Bianchi identity, we have that

$$\frac{1}{2}\nabla_j\nabla_k \mathrm{R} - \nabla_i\nabla_k \mathrm{R}_{ij} \;=\; \mathrm{R}_{ikjl}\mathrm{R}_{il} - \mathrm{R}_{ik}\mathrm{R}_{ij}\,.$$

Hence, reasoning as before, it is easy to deduce that the term $\nabla_q f\,\nabla_q\big(\frac{1}{2}\nabla_j\nabla_k \mathrm{R} - \nabla_i\nabla_k \mathrm{R}_{ij}\big)$ is bounded. To complete the proof, we thus need to estimate the term $\nabla_q f\,(\nabla_q \Delta \mathrm{R}_{kj})$. By the usual formulae for the exchange of the derivatives, we get

$$\nabla_q \Delta \mathrm{R}_{kj} \;=\; \Delta\nabla_q \mathrm{R}_{kj} - (\operatorname{div}\operatorname{Rm})_{qkl}\mathrm{R}_{lj} - (\operatorname{div}\operatorname{Rm})_{qjl}\mathrm{R}_{lk} \tag{2.12}$$

$$\qquad + \mathrm{R}_{ql}(\nabla_l \mathrm{R}_{kj}) + \mathrm{R}_{qpkl}(\nabla_p \mathrm{R}_{lj}) + \mathrm{R}_{qpjl}(\nabla_p \mathrm{R}_{kl})\,. \tag{2.13}$$

Using the identites (2.2) and (2.4), it is immediate to check that the last raw contracted with $\nabla_q f$ gives rise to bounded terms. On the other hand we have that

$$\begin{aligned}\nabla_q f\,(\mathrm{div\,Rm})_{qkl} &= \nabla_q f\,\big(\nabla_q \mathrm{R}_{kl} - \nabla_k \mathrm{R}_{ql}\big)\\ &= \nabla_q f\,(\nabla_q \mathrm{R}_{kl}) \,-\, \nabla_k(\nabla_q f\,\mathrm{R}_{ql}) \,+\, \nabla_k\nabla_q f\,\mathrm{R}_{ql}\\ &= \nabla_q f\,(\nabla_q \mathrm{R}_{kl}) \,-\, \frac{1}{2}\nabla_k\nabla_l \mathrm{R} - \mathrm{R}_{kq}\mathrm{R}_{ql} + \frac{1}{2}\mathrm{R}_{kl}\,,\end{aligned}$$

and thus, by the previous discussion, it is evident that the second and the third terms in the first raw of (2.12) contracted with $\nabla_q f$ are bounded. Finally, we have

$$\begin{aligned}(\Delta\nabla_q \mathrm{R}_{kj})\,\nabla_q f &= \Delta\big(\nabla_q \mathrm{R}_{kj}\,\nabla_q f\big) - \nabla_q \mathrm{R}_{kj}(\Delta\nabla_q f) - 2\nabla_p\nabla_q R_{kj}\,\nabla_p\nabla_q f\\ &= \Delta\big(\nabla_q \mathrm{R}_{kj}\,\nabla_q f\big) - \nabla_q \mathrm{R}_{kj}(\nabla_q\Delta f) - \frac{1}{2}\nabla_q \mathrm{R}_{kj}\,\nabla_q \mathrm{R} + 2\nabla_p\nabla_q R_{kj}\,\mathrm{R}_{pq} - \Delta \mathrm{R}_{kj}\\ &= \Delta\big(\nabla_q \mathrm{R}_{kj}\,\nabla_q f\big) + \frac{1}{2}\nabla_q \mathrm{R}_{kj}\nabla_q \mathrm{R} + 2\nabla_p\nabla_q R_{kj}\,\mathrm{R}_{pq} - \Delta \mathrm{R}_{kj}\,.\end{aligned}$$

Again, from equation (2.11) and the previous observations, we have that all the terms of the right hand side are bounded and this completes the proof of the corollary. □

We are now in the position to prove the following quantitative decay estimates of $|T|$ and $|\nabla^k T|$ as $f \to \infty$.

Proposition 2.9. *Let $(M^n, g, \nabla f)$ be a complete noncompact asymptotically cylindrical gradient shrinking Ricci soliton and let T be the tensor defined in (2.8) outside a compact set. Then, there exists a positive constant $a(n)$ such that*

$$|T|^2 \,=\, O\,(f^{-a(n)}) \qquad \text{and} \qquad |\nabla^k T|^2 \,=\, O\,(f^{-a(n)+\varepsilon})\,,$$

for every $k > 0$ and every $\varepsilon > 0$. Moreover, we can choose $a(n) := \min\{1, 2/(n-2) - \eta\}$, for any positive η sufficiently small.

Proof. We have already observed that, for $f \geq f_0$ large enough, $|\nabla f| > 0$ and thus $\{f \geq f_0\}$ is diffeomorphic to $[f_0, +\infty) \times \{f = f_0\}$. Hence, the metric g can be written as

$$g \,=\, \frac{df \otimes df}{|\nabla f|^2} + h_{\alpha\beta}(f,\theta)\, d\theta^\alpha \otimes d\theta^\beta\,,$$

where $\theta = (\theta^1, \ldots, \theta^{n-1})$ are local coordinates on the regular level set $\{f = f_0\}$ and $h(f,\cdot)_{|p}$ is the metric induced on the regular level set $\{f = f(p)\}$. Let u be a smooth function on M. Then, it is well known that the Laplacian of u can be written as

$$\Delta u \,=\, \nabla^2 u\,(\mathbf{n},\mathbf{n}) + \mathrm{H}\,\langle \nabla u, \mathbf{n}\rangle + \Delta^h u\,, \tag{2.14}$$

where $\mathrm{H}(p)$ is the mean curvature of the level set $\{f = f(p)\}$. We notice that $\nabla^2 |T|^2\,(\mathbf{n},\mathbf{n}) = O\,(f^{-1})$ and that

$$H \,=\, \frac{\Delta f - \nabla^2 f(\mathbf{n},\mathbf{n})}{|\nabla f|} \,=\, \frac{n-1-\mathrm{R}+\mathrm{Ric}(\mathbf{n},\mathbf{n})}{2|\nabla f|} \,=\, O\,(f^{-1/2})\,.$$

To proceed, we notice that from Corollary 2.8 one has $\langle \nabla |T|^2, \nabla f\rangle \leq C_1$, for some $C_1 > 0$. In particular, from identity (2.2), $\partial_f |T|^2 \leq C_2 f^{-1}$ for some $C_2 > 0$. Morever, from

Proposition 2.6, we have

$$\begin{aligned} f\,\partial_f |T|^2 &= \langle \nabla |T|^2, \nabla f\rangle + R\partial_f |T|^2 \\ &\leq \Delta |T|^2 - c(n)|T|^2 + C_3 f^{-1} \\ &\leq \Delta^h |T|^2 - c(n)|T|^2 + C_4 f^{-1}\,, \end{aligned}$$

for some $C_3, C_4 > 0$. Setting $s := \log f$, $s_0 = \log f_0$ and $v(s,\theta) := |T|^2(e^s,\theta)$, we have

$$\partial_s v \,\leq\, \Delta^h v - c(n)\, v + C_4\, e^{-s}\,. \tag{2.15}$$

Since $\{f = f_0\}$ is compact, it is well defined $v_0 := \max_\theta v(s_0,\theta)$. By the parabolic maximum principle one has that $0 \leq v(s,\theta) \leq V(s)$, where $V(s)$ is the unique solution of the ODE associated to (2.15) with initial condition $V(0) = v_0$, namely

$$V(s) := \frac{C_4}{c(n)-1} e^{-s} + \Big(v_0\, e^{c(n)\, s_0} - \frac{C_4}{c(n)-1} e^{(c(n)-1)s_0} \Big)\, e^{-c(n)s}\,.$$

Recalling that $c(n) = 2/(n-2) - \eta$, we set $a(n) := \min\{1, 2/(n-2) - \eta\} > 0$ and we obtain $0 \leq v(s,\theta) = O\,(e^{-a(n)s})$ as $s \to \infty$. Rephrasing this in terms of $|T|$ and f, we have proved that $|T|^2 = O\,(f^{-a(n)})$. Combining the standard interpolation inequalities as in [9, Corollary 12.6] with Sobolev estimates and using the fact that the metric is asymptotically cylindrical it is possible to prove the C^k-estimate. For instance, if $k = 1$, we consider compact domains of the form $\Omega_{r,s} = \{p \in M^n |\, \mathrm{dist}(p, \{f = r\}) \leq s\}$ and a cutoff functions $\chi \in C^\infty_c(\Omega_{r,2})$ with $\chi \equiv 1$ on $\Omega_{r,1}$. By interpolation inequalities, we have

$$||\nabla^2(\chi T)||_{L^p(\Omega_{r,2})} \leq C(n,p)\, ||\chi T||^{1-2/p}_{L^\infty(\Omega_{r,2})}\, ||\nabla^p(\chi T)||^{2/p}_{L^2(\Omega_{r,2})} \leq C'(n,p) ||T||^{1-2/p}_{L^\infty(\Omega_{r,2})}\,,$$

where in the last inequality we have used the fact that the soliton is asymptotically cylindrical and hence the derivatives of the curvature are bounded. For p large enough, Sobolev inequality implies that there exist a constant $C_S(\Omega_{r,2})$ such that

$$||\nabla(\chi T)||_{L^\infty(\Omega_{r,2})} \leq C_S(\Omega_{r,2})\, ||\nabla^2(\chi T)||_{L^p(\Omega_{r,2})}\,.$$

Again, it is not hard to see that $C_S := \sup_{r>0} C_S(\Omega_{r,2}) < +\infty$, since the soliton is asymptotically cylindrical. In particular, this shows that

$$||\nabla(\chi T)||_{L^\infty(\Omega_{r,2})} \leq C_S\, C'(n,p) ||T||^{1-2/p}_{L^\infty(\Omega_{r,2})} \leq C'' r^{-a(n)(1-2/p)}\,,$$

for some positive constant C''. This concludes the proof of the proposition.

□

In dimension greater than three, one can estimate the scalar curvature from T.

Corollary 2.10. *Let $(M^n, g, \nabla f)$, $n \geq 4$, be a complete noncompact asymptotically cylindrical gradient shrinking Ricci soliton. Then, we have that*

$$|\nabla^k \mathrm{R}| = O(f^{-a(n)/2+\varepsilon}) \qquad \text{and} \qquad |\nabla^k \mathrm{Ric}\,| = O(f^{-a(n)/2+\varepsilon})$$

for every $k > 0$ and every $\varepsilon > 0$.

Proof. Reasoning as the previous proposition, it is not hard to see that

$$\nabla_i T_{ij} \,=\, \frac{n-3}{2(n-1)} \nabla_j \mathrm{R} + \frac{1}{n-1} \nabla_i \big(\mathrm{R}\, \mathbf{n}_i\, \mathbf{n}_j\big) \,=\, \frac{n-3}{2(n-1)} \nabla_j \mathrm{R} + O(f^{-1/2})\,.$$

By Proposition 2.9, there exists a positive constant $a(n)$ such that

$$|\,\mathrm{div}\, T| \leq |\nabla T| = O(f^{-a(n)/2+\varepsilon}),$$

for all $\varepsilon > 0$. And this proves the first estimate. To obtain the estimate for $|\nabla \mathrm{Ric}|$ it is sufficient to compute $|\nabla T|$ and use both the estimate for $|\nabla \mathrm{R}|$ and Proposition 2.9. This proves the case $k = 1$. For $k > 1$ it is sufficient to repeat the same argument, based on interpolation inequalities and uniform Sobolev estimates, as in the proof of Proposition 2.9. □

We are now in the position to improve the decay estimate of the scalar curvature at infinity.

Proposition 2.11. *Let $(M^n, g, \nabla f)$, $n \geq 3$, be a complete noncompact asymptotically cylindrical gradient shrinking Ricci soliton. Then, we have that*

$$\mathrm{R} = \frac{n-1}{2} + O(f^{-a(n)/2+\epsilon}).$$

Proof. Recall that $\Delta \mathrm{R} + 2|\mathrm{Ric}|^2 = \langle \nabla f, \nabla \mathrm{R} \rangle + \mathrm{R}$. Therefore, if $U := (n-1)/2 - \mathrm{R}$, then U satisfies

$$\begin{aligned} \langle \nabla f, \nabla U \rangle &= -\Delta \mathrm{R} + \mathrm{R} - 2\left(|T|^2 + \frac{\mathrm{R}^2}{n-1} - \frac{2\mathrm{R}}{n-1} T(\mathbf{n}, \mathbf{n}) \right) \\ &= \frac{2\mathrm{R}}{n-1} U + O(f^{-a(n)/2+\epsilon}), \end{aligned}$$

where we have used Proposition 2.5, 2.9 and Corollary 2.10. Integrating this equality along the flow generated by $\nabla f / |\nabla f|^2$ and using the fact that U tends to zero at infinity gives the result. □

Proposition 2.12. *Let $(M^n, g, \nabla f)$, $n \geq 3$, be a complete noncompact asymptotically cylindrical gradient shrinking Ricci soliton. Then, we have that*

$$\mathrm{Rm} = \frac{1}{2(n-2)} (g - \mathbf{n} \otimes \mathbf{n}) \odot (g - \mathbf{n} \otimes \mathbf{n}) + O(f^{-a(n)/2+\epsilon}).$$

Proof. In the case of a shrinking soliton, Lemma 2.7 tells us

$$\nabla_{\nabla f} \mathrm{Rm} = \nabla^2 \mathrm{Ric} - \mathrm{Rm} + \mathrm{Rm} * \mathrm{Ric}\,.$$

Now, by the very definition of T, we get

$$\nabla_{\nabla f} \mathrm{Rm} = \nabla^2 \mathrm{Ric} + \Big(\frac{2\mathrm{R}}{n-1} - 1 \Big) \mathrm{Rm} + \mathrm{Rm} * T.$$

Therefore, by Proposition 2.5, 2.9 and Corollary 2.10, one has

$$\nabla_{\nabla f} \mathrm{Rm} = O(f^{-a(n)/2+\epsilon}).$$

As

$$\nabla_{\nabla f} [(g - \mathbf{n} \otimes \mathbf{n}) \odot (g - \mathbf{n} \otimes \mathbf{n})] = O(\mathrm{Ric}(\mathbf{n}, \cdot)) = O(f^{-1/2})\,,$$

one has

$$\nabla_{\nabla f} (\mathrm{Rm} - \frac{1}{2(n-2)} (g - \mathbf{n} \otimes \mathbf{n}) \odot (g - \mathbf{n} \otimes \mathbf{n})) = O(f^{-a(n)/2+\epsilon}).$$

Integrating this estimate along the flow generated by $\nabla f / |\nabla f|^2$ we conclude the proof of the proposition. □

Let M_t be a connected component of the level set $\{f = t\}$ and let $g^{(t)}$ be the metric induced by g on M_t. We notice that the second fundamental form of M_t satisfies

$$h_{ij}^{(t)} = \frac{\nabla_i \nabla_j f}{|\nabla f|} = O(t^{-1/2})\,.$$

Thus, combining the last proposition with Gauss equations, we obtain

$$\begin{aligned} \mathrm{R}^{(t)}_{ijkl} &= \mathrm{R}_{ijkl} - h^{(t)}_{jk}h^{(t)}_{il} + h^{(t)}_{ik}h^{(t)}_{jl} = \frac{1}{2(n-2)}(g^{(t)} \odot g^{(t)})_{ijkl} + O(t^{-a(n)/2+\varepsilon}) \\ &= \frac{\mathrm{R}^{(t)}}{(n-1)(n-2)}(g^{(t)} \odot g^{(t)})_{ijkl} + O(t^{-a(n)/2+\varepsilon})\,, \end{aligned}$$

where we have used Proposition 2.11 in the last equality. As a consequence of the Riemann-Cartan uniformization theorem, for t large enough, there exists a family of diffeomorphisms $\phi_t : M_t \to \mathbb{S}^{n-1}$ such that

$$||2(n-2)g^{(t)} - \phi_t^* g^{\mathbb{S}^{n-1}}||_{C^k(M_t, g^{(t)})} = O(t^{-a(n)/2+\varepsilon})\,,$$

for every $k \geq 0$ and every $\varepsilon > 0$.

3. Almost Killing fields at infinity

It well known that on the $(n-1)$-dimensional sphere there are $n(n-1)/2$ linearly independent Killing vector fields, hence, by construction the same is true on $(M_t, \phi_t^* g^{\mathbb{S}^{n-1}})$, for t large enough. The aim of the following sections is to show that for some t_0, the $(n-1)$-dimensional manifold $(M_{t_0}, g^{(t_0)})$ admits $n(n-1)/2$ Killing vector fields as well. By classical results, this will imply that $(M_t, g^{(t)})$ must be homothetic to the round sphere $\mathbb{S}^{n-1}$.

We consider now the sequence $t_m = 2^m$ and the corresponding sequence of Riemannian manifolds $(M_{t_m^2/4}, \tilde{g}^m)$, where we set

$$\tilde{g}^m := \tfrac{1}{2(n-2)} \phi^*_{t_m^2/4} g^{\mathbb{S}^{n-1}}\,.$$

We then let $\{U_i^m\}_{i=1,\ldots,n(n-1)/2}$ be a collection of linearly independent Killing vector fields for $\tilde{g}^m$. By the results of the previous section, these vector fields can be regarded as approximate Killing vector fields on $M_{t_m^2/4}$ for the matrix $g^{t_m^2/4}$. In particular, it is possible to prove that

- $||U_i^m||_{C^k(\,M_{t_m^2/4}\,,\, g^{(t_m^2/4)}\,)} \,=\, O(1)$, for every $k \geq 0$.
- $\mathscr{L}_{U_i^m}\, g^{(t_m^2/4)} \,=\, O(t_m^{-a(n)+\epsilon})\,.$
- $\int_{M_{t_m^2/4}} \langle U_i^m, U_j^m\rangle \, d\tilde{\mu}_m \,=\, \mathrm{Vol}(M_{t_m^2/4}, \tilde{g}^m)\, \delta_{ij}.$

In the next proposition, we are going to extend these estimates to an annulus bounded by the level sets $M_{t_m^2/4}$. To do that, we define $\Omega_m := \{t_m^2/4 \leq f \leq t_m^2\}$, so that $\partial\Omega_m = M_{t_m^2/4} \cup M_{t_m^2}$ and we extend the vector fields $\{U_i^m\}$ on Ω_m by imposing the condition $[U_i^m, X] = 0$. With a small abuse of notation, we still denote by $\{U_i^m\}$ the extended vector fields.

Proposition 3.1. *Let $(M^n, g, \nabla f)$, $n \geq 3$, be a complete noncompact asymptotically cylindrical gradient shrinking Ricci soliton. Then, with the notation introduced above, we have that for every $m \in \mathbb{N}$ and every $i \in \{1, \ldots, n(n-1)/2\}$ the following estimates hold*

(i) $||U_i^m||_{C^k(\Omega_m\,,\,g\,)} \,=\, O(1)\,,$ *for every* $k \geq 0$,

(ii) $\sup_{\Omega_m} |\,\mathscr{L}_{U_i^m}\, g\,| \,=\, O(t_m^{-a(n)+\epsilon}),$

(iii) $\sup_{\Omega_m} |\langle U_i^m, \mathbf{n}\rangle| \,=\, O(t_m^{-1-a(n)+\epsilon}),$

(iv) $\Delta_f U_i^m + U_i^m/2 \,=\, O(t_m^{-a(n)+\epsilon}),$

(v) $\displaystyle\int_{M_t} \langle U_i^m, U_j^m\rangle d\mu_t \,=\, \mathrm{Vol}(M_t, g^{(t)})\, \delta_{ij} + O(t^{-a(n)/2+\epsilon}),$ *for every* $t \in [t_m^2/4, t_m^2]\,.$

Remark 3.2. We notice that in general, a Killing field U on a Riemannian manifold satisfies $\Delta U + \mathrm{Ric}(U, \cdot\,) = 0$. Using the shrinking solitons equation and recalling that, by construction we have $[U_i^m, X] = 0$, one can deduce that $\Delta U_i^m + \mathrm{Ric}(U_i^m, \cdot\,) = \Delta_f U_i^m + U_i^m/2$. Therefore, part (iv) of the statement can be thought as an estimate of how far the vector fields U_i^m are from being Killing.

Proof. (i) This statement is a consequence of the the equations $[U_i^m, X] = 0$, which we have used to extend our vector fields.

(ii) Let us check this estimate on $M_{t_m^2/4}$ first. Indeed, if V is orthogonal to X,

$$\begin{aligned} \big(\mathscr{L}_{U_i^m} g\big)\,(X,V) &= \langle \nabla_X U_i^m, V\rangle + \langle \nabla_V U_i^m, X\rangle \\ &= \langle \nabla_{U_i^m} X, V\rangle - \langle U_i^m, \nabla_V X\rangle = 0. \end{aligned}$$

To proceed, we compute

$$\big(\mathscr{L}_{U_i^m} g\big)\,(X,X) = 2\langle \nabla_X U_i^m, X\rangle = 2\nabla^2 f(U_i^m, X) = -2\,\mathrm{Ric}(U_i^m, X).$$

Therefore, $\big(\mathscr{L}_{U_i^m} g\big)\,(\mathbf{n},\mathbf{n}) = O(t_m^{-2-a(n)+\epsilon})$ on $M_{t_m^2/4}$. Now, as $[U_i^m, X] = 0$,

$$\mathscr{L}_X(\mathscr{L}_{U_i^m} g) = \mathscr{L}_{U_i^m}(\mathscr{L}_X g) = \mathscr{L}_{U_i^m}(g - 2\,\mathrm{Ric}).$$

Moreover, $(\mathscr{L}_X S) = \nabla_X S + S - \mathrm{Ric}\circ S - S\circ \mathrm{Ric}$, for any symmetric 2-tensor S. Therefore, for $S := \mathscr{L}_{U_i^m} g$, we get

$$\nabla_X(\mathscr{L}_{U_i^m} g) = -2(\mathscr{L}_{U_i^m}\mathrm{Ric}) + \mathrm{Ric}\circ(\mathscr{L}_{U_i^m} g) + (\mathscr{L}_{U_i^m} g)\circ \mathrm{Ric}\,.$$

Consequently, as $(\mathscr{L}_{U_i^m}\mathrm{Ric})$ is bounded, one has by Kato inquality,

$$\nabla_{X/|X|^2}\left|\big(\mathscr{L}_{U_i^m} g\big)\right| \le \left|\nabla_{X/|X|^2}\big(\mathscr{L}_{U_i^m} g\big)\right| \le \frac{2|\,\mathrm{Ric}\,|}{|X|^2}\,\left|(\mathscr{L}_{U_i^m} g)\right| + O(f^{-1}).$$

Hence, we obtain the result by integrating over Ω_m.

(iii) As before, we compute,

$$\begin{aligned} \nabla_X\langle U_i^m, X\rangle &= \langle \nabla_X U_i^m, X\rangle + \langle U_i^m, \nabla_X X\rangle \\ &= 2\langle \nabla_{U_i^m} X, X\rangle = \langle U_i^m, X\rangle - 2\,\mathrm{Ric}(X, U_i^m) \\ &= \langle U_i^m, X\rangle + O(f^{-a(n)/2+\epsilon}). \end{aligned}$$

Now, by construction, we have that $\langle U_i^m, X\rangle = 0$ on $M_{t_m^2/4}$. Hence, integrating the previous estimate on Ω_m, we obtain $\langle U_i^m, X\rangle = O(t_m^{-a(n)+\epsilon})$ on Ω_m and the result follows at once.

(iv) As we noticed in Remark 3.2, one has that $\Delta U_i^m + \mathrm{Ric}(U_i^m,\cdot) = \Delta_f U_i^m + U_i^m/2$. On the other hand, it holds the identity

$$\mathrm{div}(\mathscr{L}_{U_i^m} g) \;-\; \frac{1}{2}\nabla(\mathrm{tr}(\mathscr{L}_{U_i^m} g)) \;=\; \Delta U_i^m \;+\; \mathrm{Ric}(U_i^m,\cdot)\,.$$

To estimate the left hand side, we notice that (ii) gives $\sup_{\Omega_m}|\,\mathscr{L}_{U_i^m} g\,| = O\,(t_m^{-a(n)+\epsilon})$. Moreover, it is possible to deduce form (i) that $\sup_{\Omega_m}|\nabla^k\,\mathscr{L}_{U_i^m}(g)| = O\,(1)$, for every $k \ge 0$. By the interpolation inequalities, we know that $\sup_{\Omega_m}|\nabla\,\mathscr{L}_{U_i^m}(g)| = O(t_m^{-a(n)+\epsilon})$. This implies the desired estimates.

(v) To see the last estimate, we denote by $\mathrm{H}^{(t)}$ the mean curvature of M_t and we compute

$$\begin{aligned}
\frac{d}{dt}\int_{M_t}\langle U_i^m, U_j^m\rangle\, d\mu_t &= \int_{M_t}\langle U_i^m, U_j^m\rangle\,\frac{\mathrm{H}^{(t)}}{|X|}\, d\mu_t + \int_{M_t}\frac{\langle\nabla_X U_i^m, U_j^m\rangle + \langle U_i^m, \nabla_X U_j^m\rangle}{|X|^2}\, d\mu_t \\
&= \int_{M_t}\langle U_i^m, U_j^m\rangle\frac{(n-1)/2 - \mathrm{R} + \mathrm{Ric}(\mathbf{n},\mathbf{n})}{|X|^2} d\mu_t + \int_{M_t}\frac{2\,\langle\nabla_{U_i^m}X, U_j^m\rangle}{|X|^2}\, d\mu_t \\
&= O\,(t^{-1-a(n)/2+\epsilon}) + \int_{M_t}\frac{\langle U_i^m, U_j^m\rangle - 2\,\mathrm{Ric}(U_i^m, U_j^m)}{|X|^2}\, d\mu_t \\
&= O\,(t^{-1-a(n)/2+\epsilon}) + \int_{M_t}\frac{\langle U_i^m, \mathbf{n}\rangle\langle U_j^m, \mathbf{n}\rangle - 2\,T(U_i^m, U_j^m)}{|X|^2}\, d\mu_t \\
&= O\,(t^{-1-a(n)/2+\epsilon}),
\end{aligned}$$

where we used the estimates obtained in the previous section. The result follows now by a simple integration. □

For future convenience, we simplify the notations and summarize the results of this section in the following proposition.

Proposition 3.3. *Let $(M^n, g, \nabla f)$, $n \geq 3$, be a complete noncompact asymptotically cylindrical gradient shrinking Ricci soliton. Then, there exists a collection of $n(n-1)/2$ vector fields $\{U_i\}_{i=1,\ldots,n(n-1)/2}$ defined on (M^n, g) such that, for every $i \in \{1, \ldots, n(n-1)/2\}$, the following estimates hold*

$$\begin{aligned}
&(i) && |\nabla^k U_i\,| \,=\, O\,(1), \quad \textit{for every } k \geq 0, \\
&(ii) && |\,\mathscr{L}_{U_i} g\,| \,=\, O\,(f^{-a(n)/2+\epsilon}), \\
&(iii) && \Delta_f U_i + U_i/2 \,=\, O\,(f^{-a(n)/2+\epsilon}), \\
&(iv) && \int_{M_t}\langle U_i, U_j\rangle\, d\mu_t \,=\, \mathrm{Vol}(M_t, g^{(t)})\,\delta_{ij} + O\,(t^{-a(n)/2+\epsilon}).
\end{aligned}$$

4. Interpolating almost Killing vector fields

In this section, we will provide an a priori estimate as well as an existence result for solutions to the following equation

$$\Delta_f V + V/2 \,=\, Q, \tag{4.1}$$

provided the vector field Q has a suitable asymptotic behavior at infinity. As we have already seen, the almost Killing vector fields constructed in the previous section satisfy an equation of this type. We start with the a priori estimate.

Proposition 4.1. *Let $(M^n, g, \nabla f)$, $n \geq 3$, be a complete noncompact asymptotically cylindrical gradient shrinking Ricci soliton and let V be a vector field satisfying*

$$\Delta_f V + V/2 \,=\, Q,$$

where Q is vector field such that $Q = O(f^{-a(n)/2+\epsilon})$. Then, there exists positive constants A and t_0 such that, for $t_0 < t_1 < t_2$, one has the following estimate

$$\max_{t_1\leq f\leq t_2}\frac{|V| + Af^{-a(n)/2+\epsilon}}{f - n/2} = \max\left\{\max_{f=t_1}\frac{|V| + Af^{-a(n)/2+\epsilon}}{f-n/2};\max_{f=t_2}\frac{|V| + Af^{-a(n)/2+\epsilon}}{f-n/2}\right\}.$$

Proof. First of all,

$$\Delta_f |V|^2 \;=\; -|V|^2 + 2|\nabla V|^2 + 2\langle Q, V\rangle.$$

Therefore,

$$2|V|\Delta_f|V| \geq -|V|^2 + 2|\nabla V|^2 - 2|\nabla|V||^2 - 2|Q||V|.$$

Hence, by the Kato inequality,

$$\Delta_f|V| \geq -\frac{|V|}{2} - |Q|,$$

as soon as V does not vanish.

$$\begin{aligned}
\Delta_f(|V| + Af^{-\alpha}) + \frac{|V| + Af^{-\alpha}}{2} \;&\geq\; A\Delta_f f^{-\alpha} + \frac{A}{2}f^{-\alpha} - |Q| \\
&\geq\; A\alpha f^{-\alpha-1}(f - n/2) \\
&\quad\; + A\alpha(\alpha+1)f^{-\alpha-2}|X|^2 + \frac{A}{2}f^{-\alpha} - |Q| \\
&\geq\; f^{-\alpha}\left(A\left(\alpha + 1/2 - \frac{\alpha n}{2f}\right) - f^{\alpha}|Q|\right) \;>\; 0,
\end{aligned}$$

outside a compact set where $\alpha := a(n)/2 - \epsilon$. Finally, consider the function $v := f - n/2$, which satisfies $\Delta_f v = -v$, and define $u := |V| + Af^{-a(n)/2+\epsilon}$. For $v > 0$, a direct computation gives

$$\begin{aligned}
\Delta_f\left(\frac{u}{v}\right) \;&=\; \left(\frac{\Delta_f u}{u} - \frac{\Delta_f v}{v}\right)\frac{u}{v} + 2\langle\nabla v^{-1}, \nabla u\rangle \\
&>\; (-1/2 + 1)\frac{u}{v} + 2\langle\nabla v^{-1}, \nabla u\rangle \\
&>\; \left(1/2 - 2\frac{|X|^2}{f^2}\right)\frac{u}{v} - 2\langle\nabla \ln v, \nabla\left(\frac{u}{v}\right)\rangle \\
&>\; -2\langle\nabla \ln v, \nabla\left(\frac{u}{v}\right)\rangle,
\end{aligned}$$

outside a compact set. The result is now a consequence of the maximum principle. □

We are in the position to provide an existence result for the equation (4.1).

Theorem 4.2. *Let $(M^n, g, \nabla f)$, $n \geq 3$, be a complete noncompact asymptotically cylindrical gradient shrinking Ricci soliton and let Q be a vector field such that $Q = O(f^{-a(n)/2+\epsilon})$. Then there exists a vector field V such that,*

$$\Delta_f V + V/2 \;=\; Q \quad \text{on } M^n, \qquad \text{and} \qquad V \;=\; O\,(f^{-a(n)/2+\epsilon}).$$

Proof. Let $(t_m)_{m\in\mathbb{N}}$ be a sequence tending to $+\infty$ and, for every $m \in \mathbb{N}$, let V^m be a solution of the Dirichlet problem

$$\Delta_f V^m + V^m/2 \;=\; Q \quad \text{in } \{\, f \;\leq\; t_m^2/4\,\}, \qquad \text{with} \qquad V^m = 0 \quad \text{on } \{\, f = t_m^2/4\,\}.$$

To study the growth of V^m, we are going to prove the following claim.

Claim. *Given τ and α in $(0,1)$, there exists a positive constant ρ_0 such that, for every $t \in [\rho_0, t_m]$, it holds the estimate*

$$\alpha^{-1}\tau^{a(n)-2\epsilon}\,\bar{\mathrm{A}}^m(\tau t) \;\leq\; \bar{\mathrm{A}}^m(t) \,+\, t^{-a(n)+2\epsilon},$$

where we set $\bar{\mathrm{A}}^m(s) := \sup_{\{f = s^2/4\}} |V^m|$.

Assume by contradiction that there is a sequence $(r_m)_{m\in\mathbb{N}}$ going to $+\infty$ such that

$$\alpha^{-1}\tau^{a(n)-2\epsilon}\bar{\mathrm{A}}^m(\tau r_m) \geq \bar{\mathrm{A}}^m(r_m) + r_m^{-a(n)+2\epsilon} \tag{4.2}$$

an define

$$\bar{V}^m := \frac{V^m}{\max\{\bar{\mathrm{A}}^m(\tau r_m) + (\tau r_m)^{-a(n)+2\epsilon}, \tau^2(\bar{\mathrm{A}}^m(r_m) + r_m^{-a(n)+2\epsilon})\}},$$

$$f_m := \frac{f^{-a(n)/2+\epsilon}}{\max\{\bar{\mathrm{A}}^m(\tau r_m) + (\tau r_m)^{-a(n)+2\epsilon}, \tau^2(\bar{\mathrm{A}}^m(r_m) + r_m^{-a(n)+2\epsilon})\}}.$$

By Proposition 4.1, we get,

$$1 \leq \sup_{(\tau t_m)^2/4\leq f\leq t_m^2/4} |\bar{V}^m| + Af_m \leq \tau^{-2}. \tag{4.3}$$

Moreover, $\bar{V}^m$ satisfies

$$\Delta_f \bar{V}^m + \frac{\bar{V}^m}{2} = \bar{Q}^m, \tag{4.4}$$

$$\bar{Q}^m := \frac{Q}{\max\{\bar{\mathrm{A}}^m(\tau r_m) + (\tau r_m)^{-a(n)+2\epsilon}, \tau^2(\bar{\mathrm{A}}^m(r_m) + r_m^{-a(n)+2\epsilon})\}}. \tag{4.5}$$

By assumption on the growth of Q, the sequence $\bar{Q}^m$ is uniformly bounded. To blow-up this equation in order to reach a contradiction, we need first to control the covariant derivatives of $\bar{V}^m$.
Let $(\phi_t)_{t\in(-\infty,1)}$ be the flow generated by $\nabla f/(1-t)$. Then, define $\bar{V}^m(t) := \phi_t^*\bar{V}^m$, $\bar{Q}^m(t) := \phi_t^*\bar{Q}^m/(1-t)$ and the associated Ricci flow $g(t) := (1-t)\phi_t^* g$. Then, $\bar{V}^m(\cdot)$ satisfies the following heat equation:

$$\partial_t \bar{V}^m(t) \;=\; \Delta_{g(t)}\bar{V}^m(t) + \mathrm{Ric}_{g(t)}(\bar{V}^m(t)) - \bar{Q}^m(t). \tag{4.6}$$

Applying the classical interior parabolic estimates to the heat equation (4.6), we deduce that there exists a constant C_1 such that,

$$\sup_{\left\{\frac{(\tau t_m)^2}{4}\leq f\leq \frac{t_m^2}{4}\right\}} |\nabla^{g(0)}\bar{V}^m(0)|_{g(0)} \;\leq\; C_1 \sup_{\left\{\frac{((3\tau/4)t_m)^2}{4}\leq f\leq \frac{(3t_m/2)^2}{4}\right\}\times[s,0]} \left(|\bar{V}^m(t)|_{g(t)} + |\bar{Q}^m(t)|_{g(t)}\right).$$

It is worth pointing out that, by the fact that the soliton is asymptotically cylindrical, the constant C_1, which a priori depends on the ellipticity constants of $\Delta_{g(t)}$, the bounds on the coefficients of the zero order term in (4.6) as well as on the diameter of the domain, can be chosen uniformly. We claim that the right hand side can be further estimated to obtain

$$\sup_{(\tau t_m)^2/4\leq f\leq t_m^2/4} |\nabla\bar{V}^m| \leq C_2 \sup_{((\tau/2)t_m)^2/4\leq f\leq(2t_m)^2/4} \left(|\bar{V}^m| + |\bar{Q}^m|\right) \leq \frac{C_3}{\tau^2}. \tag{4.7}$$

In fact, the last inequality follows by (4.3). To prove the first inequality we first need some remarks about how the flow $(\phi_t)_{t\in(-\infty,1)}$ acts on the sublevels of f. As the scalar curvature is nonnegative and bounded by some constant C_4, one has, by the soliton identities,

$$\begin{aligned}\partial_t(f\circ\phi_t) &= \frac{|\nabla f|^2\circ\phi_t}{(1-t)}\leq\frac{(f\circ\phi_t)}{1-t},\\ \partial_t(f\circ\phi_t) &\geq \frac{(f\circ\phi_t)-C_4}{1-t}.\end{aligned}$$

Hence, by integrating the previous differential inequalities between a negative time s and 0,

$$(1-s)f(x)\leq f(\phi_s(x))\leq C_5+(1-s)(f(x)-C_6), \tag{4.8}$$

for $x\in M$. Thus, observing that $|\bar{V}^m(t)|_{g(t)}=(1-t)^{1/2}|\bar{V}^m|_{g(0)}\circ\phi_t$, one has

$$\begin{aligned}\sup_{\left\{\frac{((3\tau/4)t_m)^2}{4}\leq f\leq\frac{(3t_m/2)^2}{4}\right\}\times[s,0]}|\bar{V}^m(t)|_{g(t)} &\leq (1-s)^{1/2}\sup_{t\in[s,0]}\sup_{\left\{\frac{((3\tau/4)t_m)^2}{4}\leq f(\phi_t(x))\leq\frac{(3t_m/2)^2}{4}\right\}}|\bar{V}^m|\\ &\leq (1-s)^{1/2}\sup_{t\in[s,0]}\sup_{\left\{\frac{((3\tau/4)t_m)^2}{4(1-t)}-\frac{C_5}{1-t}+C_6\leq f(x)\leq\frac{(3t_m/2)^2}{4(1-t)}\right\}}|\bar{V}^m|\\ &\leq (1-s)^{1/2}\sup_{\left\{\frac{((3\tau/4)t_m)^2}{4}-\frac{C_5}{1-s}+C_6\leq f(x)\leq\frac{(3t_m/2)^2}{4}\right\}}|\bar{V}^m|\end{aligned}$$

Therefore, up to choose s such that for every m large enough,

$$\frac{((3\tau/4)t_m)^2}{4}-\frac{C_5}{1-s}+C_6\geq\frac{((\tau/2)t_m)^2}{4},$$

the claim is proved. To sum it up, we have obtained the uniform estimate

$$\sup_{(\tau t_m)^2/4\leq f\leq t_m^2/4}|\nabla\bar{V}^m|\leq\frac{C_7}{\tau^2}.$$

In particular, it means that the family of vector fields $(V^m)_m$ restricted to $(\tau t_m)^2/4\leq f\leq t_m^2/4$ is equi-Lipschitz.

Therefore, going back to the static equation (4.4) as (M^n,g) is asymptotically cylindrical, by blowing-up this equation we obtain that $(\bar{V}^m)_{m\in\mathbb{N}}$ converges to a vector field $\bar{V}^\infty$ which is radially constant, i.e. $\nabla_{\partial_r}\bar{V}^\infty=0$. Observe also that, if $f_\infty:=\lim_{m\to+\infty}f_m$,

$$\nabla_{\partial r}f_\infty=\lim_{m\to+\infty}\nabla_{\nabla 2\sqrt{f}}f_m=0.$$

Consequently, the supremum of $|\bar{V}^\infty|$ on each slice of the cylinder is a nonnegative constant c_∞ independent of the slice, the same holds for f_∞. Now, inequality (4.2) reads, as m tends to $+\infty$,

$$\alpha^{-1}\tau^{a(n)-2\epsilon}c_\infty\geq c_\infty+Af_\infty>0.$$

In particular, $c_\infty>0$, i.e. $\bar{V}^\infty$ does not vanish identically and

$$\alpha^{-1}\tau^{a(n)-2\epsilon}\geq 1,$$

which is a contradiction if α and τ are chosen properly. This proves the claim.

The claim ensures that $\sup_m\sup_{t\in[\rho_0,t_m]}\sup_{\{f=t^2/4\}}f^{a(n)/2-\epsilon}|V^m|<+\infty$. Therefore, if the sequence $(V^m)_{m\in\mathbb{N}}$ is not uniformly bounded on M, this can only happen on a fixed compact set. Assume on the contrary that $\sup_m\sup_{M^n}|V^m|=+\infty$.

Define $W^m := V^m/\sup_{M^n}|V^m|$. Since Q is bounded on M^n, this implies that $(W^m)_{m\in\mathbb{N}}$ uniformly converges on compact sets to a non vanishing vector field W^∞ with compact support satisfying $\Delta_f W^\infty + W^\infty/2 = 0$. Now, by the work of Bando [2], Δ_f is an elliptic operator with analytic coefficients, therefore, W^∞ must be analytic too. Since it has compact support, it must vanishes everywhere which is a contradiction. □

5. Rigidity of the Lichnerowicz equation

We start this section with the following proposition, which provides an elliptic equation for the Lie derivative of the metric along a vector field which satisfy equation (4.1) with $Q = 0$.

Proposition 5.1. *Let $(M^n, g, \nabla f)$, $n \geq 3$, be a gradient shrinking Ricci soliton. Assume a vector field V satisfies*

$$\Delta_f V + V/2 \,=\, 0. \tag{5.1}$$

Then the Lie derivative $h := (\mathscr{L}_V\, g)$ satisfies the Lichnerowicz equation

$$(\mathscr{L}_X\, h) - h = \Delta_L h, \tag{5.2}$$

where $\Delta_L h$ denotes the Lichnerowicz laplacian.

Proof. Consider the flow $\{\phi_t\}_t$ generated by the vector field V and the family of metrics $g(t) := \phi_t^* g$. By equation (2.31) in [1, Chapter 2] we obtain the variation of the Ricci curvature at the initial time.

$$\begin{aligned} \frac{\partial}{\partial t}\Big|_{t=0}(-2\,\mathrm{Ric}_{g(t)}) \;&=\; \Delta_L h + \mathrm{Hess}\,\mathrm{tr}(h) - \mathscr{L}_{\mathrm{div}(h)}(g) \\ &=\; \Delta_L h - \mathscr{L}_{\mathrm{div}(h) - \frac{1}{2}\nabla\,\mathrm{tr}(h)}(g), \end{aligned}$$

where we set $h := \frac{\partial}{\partial t} g(t)|_{t=0}$. Since $h = (\mathscr{L}_V\, g)$, we have that

$$\mathrm{div}(h) - \frac{1}{2}\nabla\,\mathrm{tr}(h) = \Delta V + \mathrm{Ric}(V). \tag{5.3}$$

Using both the equation (5.1) satisfied by the vector field V and the soliton equation, we deduce

$$\mathrm{div}(h) - \frac{1}{2}\nabla\,\mathrm{tr}(h) \;=\; \Delta V - \nabla_V X + V/2 \;=\; [X, V].$$

We can conclude that

$$-2(\mathscr{L}_V\,\mathrm{Ric}) \;=\; \Delta_L(\mathscr{L}_V\, g) - (\mathscr{L}_{[X,V]}\, g).$$

Recalling that $2\,\mathrm{Ric} = -(\mathscr{L}_X\, g) + g$, one has $-\mathscr{L}_V(-(\mathscr{L}_X\, g) + g) = \Delta_L(\mathscr{L}_V\, g) - (\mathscr{L}_{[X,V]}\, g)$ an thus

$$-(\mathscr{L}_V\, g) + \mathscr{L}_X(\mathscr{L}_V(g)) = \Delta_L(\mathscr{L}_V(g)),$$

which is the desired equation. □

We conclude this section with the analysis of the Lichnerowicz equation (5.2), providing an a priori estimate and a Liouville-type theorem.

Proposition 5.2. *Let $(M^n, g, \nabla f)$, $n \geq 3$, be a complete noncompact asymptotically cylindrical gradient shrinking Ricci soliton and let h be a symmetric 2-tensor satisfying the static Lichnerowicz equation*

$$\mathscr{L}_X(h) - h = \Delta_L h.$$

Then there exists $\alpha > 0$ large enough such that for $t_0 \leq t_1 < t_2$,

$$\max_{t_1 \leq f \leq t_2} \frac{|h|^2}{v^\alpha} = \max \left\{ \max_{f=t_1} \frac{|h|^2}{v^\alpha} ; \max_{f=t_2} \frac{|h|^2}{v^\alpha} \right\}.$$

Proof. We start by observing that on a shrinking soliton, the Lichnerowicz equation can be rewritten as

$$\Delta_f h + 2\,\mathrm{Rm} * h = 0.$$

Define as before $v := f - n/2$. Then, we have the following estimate

$$\begin{aligned}
\Delta_f \left(\frac{|h|^2}{v^\alpha} \right) &= v^{-\alpha} \langle 2\Delta_f h - \alpha \frac{\Delta_f v}{v} h, h \rangle + 2v^{-\alpha} |\nabla h|^2 + 2\langle \nabla v^{-\alpha}, \nabla |h|^2 \rangle \\
&\geq v^{-\alpha} \langle \alpha h - 2\,\mathrm{Rm} * h, h \rangle - 2\alpha \langle \nabla \ln v, \nabla \left(\frac{|h|^2}{v^\alpha} \right) \rangle - 2|\nabla \ln v|^2 \frac{|h|^2}{v^\alpha} \\
&\geq (\alpha - c(|\,\mathrm{Rm}\,|_\infty) - 2|\nabla \ln v|^2) \frac{|h|^2}{v^\alpha} - 2\alpha \langle \nabla \ln v, \nabla \left(\frac{|h|^2}{v^\alpha} \right) \rangle \\
&\geq -2\alpha \langle \nabla \ln v, \nabla \left(\frac{|h|^2}{v^\alpha} \right) \rangle,
\end{aligned}$$

outside a compact set for α large enough. The result, follows then by the maximum principle. □

Building on the previous a priori estimates, we are now in the position to present a Liouville-type theorem for solutions to the Lichnerowicz equation with a suitable decay at infinity.

Theorem 5.3. *Let $(M^n, g, \nabla f)$, $n \geq 3$, be a complete noncompact asymptotically cylindrical gradient shrinking Ricci soliton and let h be a symmetric 2-tensor satisfying the static Lichnerowicz equation i.e.*

$$\mathscr{L}_X(h) - h = \Delta_L h, \tag{5.4}$$

such that $h = O(f^{-\alpha_0})$, with $\alpha_0 > 0$. Then $h = 0$.

Proof. If there is a sequence $(t_m)_m$ tending to $+\infty$ such that $\sup_{M_{t_m^2/4}} |h| = 0$ for any m, then $h = 0$ on $f^{-1}([t_1, +\infty))$ by the a priori C^0 estimate given by proposition 5.2.

Assume by contradiction that $\mathrm{A}(t) := \sup_{M_{t^2/4}} |h|^2$ is positive for any large t. By the growth assumption on h, given $\beta \in (0,1)$ and $\tau \in (0,1)$, there is a sequence $(t_m)_m$ diverging to $+\infty$ such that

$$\beta^{-1} \tau^{2\alpha_0} \mathrm{A}(\tau t_m) \geq \mathrm{A}(t_m). \tag{5.5}$$

Now, define

$$h_m := \max \left\{ \max_{f=((\tau/2)t_m)^2/4} |h| ; \tau^\alpha \max_{f=(2t_m)^2/4} |h| \right\}.$$

By proposition 5.2, if $h^m := h/h_m$, there exist universal positive constants C_1 and C_2 such that

$$C_1 \leq \max_{((\tau/2)t_m)^2/4 \leq f \leq (2t_m)^2/4} |h^m|^2 \leq \frac{C_2}{\tau^{2\alpha}}. \tag{5.6}$$

Moreover, h^m still satisfies the Lichnerowicz equation.

Let $(\phi_t)_{t\in(-\infty,1)}$ be the flow generated by $\nabla f/(1-t)$. Then, define $h^m(t) := (1-t)\phi_t^* h^m$ and the associated Ricci flow $g(t) := (1-t)\phi_t^* g$. Then, $h^m(\cdot)$ satisfies the Lichnerowicz heat equation:

$$\partial_t h^m(t) \;=\; \Delta_{L,g(t)} h^m(t) \;=\; \Delta_{g(t)} h^m(t) + 2\,\mathrm{Rm}_{g(t)} * h^m(t) - 2\,\mathrm{Ric}_{g(t)} * h^m(t)\,. \tag{5.7}$$

Applying the classical interior parabolic estimates to the heat equation (5.7), we deduce that there exists a constant C_3 such that,

$$\sup_{\left\{\frac{(\tau t_m)^2}{4} \leq f \leq \frac{t_m^2}{4}\right\}} |\nabla^{g(0)} h^m(0)|_{g(0)} \;\leq\; C_3 \sup_{\left\{\frac{((3\tau/4)t_m)^2}{4} \leq f \leq \frac{(3t_m/2)^2}{4}\right\}\times[s,0]} |h^m(t)|_{g(t)}\,.$$

It is worth pointing out that, by the fact that the soliton is asymptotically cylindrical, the constant C_3, which a priori depends on the ellipticity constants of $\Delta_{g(t)}$, the bounds on the coefficients of the zero order term in (5.7) as well as on the diameter of the domain, can be chosen uniformly. We claim that the right hand side can be further estimated to obtain

$$\sup_{(\tau t_m)^2/4 \leq f \leq t_m^2/4} |\nabla h^m| \leq C_4 \sup_{((\tau/2)t_m)^2/4 \leq f \leq (2t_m)^2/4} |h^m| \leq \frac{C_5}{\tau^{2\alpha}}. \tag{5.8}$$

In fact, the last inequality follows by (5.6). To prove the first inequality we first need some remarks about how the flow $(\phi_t)_{t\in(-\infty,1)}$ acts on the sublevels of f. As the scalar curvature is nonnegative and bounded by some constant C_6, one has, by the soliton identities,

$$\begin{aligned}
\partial_t (f\circ\phi_t) &= \frac{|\nabla f|^2\circ\phi_t}{(1-t)} \leq \frac{(f\circ\phi_t)}{1-t},\\
\partial_t (f\circ\phi_t) &\geq \frac{(f\circ\phi_t) - C_6}{1-t}.
\end{aligned}$$

Hence, by integrating the previous differential inequalities between a negative time s and 0,

$$(1-s)f(x) \leq f(\phi_s(x)) \leq C_7 + (1-s)(f(x) - C_8), \tag{5.9}$$

for $x \in M$. Thus, observing that $|h^m(t)|_{g(t)} = |h^m|\circ\phi_t$, one has

$$\begin{aligned}
\sup_{\left\{\frac{((3\tau/4)t_m)^2}{4} \leq f \leq \frac{(3t_m/2)^2}{4}\right\}\times[s,0]} |h^m(t)|_{g(t)} &\leq \sup_{t\in[s,0]} \sup_{\left\{\frac{((3\tau/4)t_m)^2}{4} \leq f(\phi_t(x)) \leq \frac{(3t_m/2)^2}{4}\right\}} |h^m|\\
&\leq \sup_{t\in[s,0]} \sup_{\left\{\frac{((3\tau/4)t_m)^2}{4(1-t)} - \frac{C_7}{1-t} + C_8 \leq f(x) \leq \frac{(3t_m/2)^2}{4(1-t)}\right\}} |h^m|\\
&\leq \sup_{\left\{\frac{((3\tau/4)t_m)^2}{4} - \frac{C_7}{1-s} + C_8 \leq f(x) \leq \frac{(3t_m/2)^2}{4}\right\}} |h^m|
\end{aligned}$$

Therefore, up to choose s such that for every m large enough,

$$\frac{((3\tau/4)t_m)^2}{4} - \frac{C_7}{1-s} + C_8 \geq \frac{((\tau/2)t_m)^2}{4}\,,$$

the claim is proved. In synthesis, we have obtained the uniform estimate

$$\sup_{(\tau t_m)^2/4 \leq f \leq t_m^2/4} |\nabla h^m| \leq \frac{C}{\tau^{2\alpha}}.$$

In particular, it means that the family of symmetric 2-tensors $(h^m)_m$ restricted to $(\tau t_m)^2/4 \leq f \leq t_m^2/4$ is equi-Lipschitz. Going back to the static equation (5.4) and by rescaling this equation with t_m, as (M^n, g) is asymptotically cylindrical, $(h^m)_m$ converges to a symmetric 2-tensor h^∞ which is radially constant, i.e. $\nabla_{\partial_r} h^\infty = 0$. In particular, the maximum of the norm of h^∞ restricted to each slice of the cylinder is a positive constant denoted by c_∞. Now, as t_m goes to $+\infty$, inequality (5.5) reads :

$$\beta^{-1}\tau^{2\alpha_0} c_\infty \geq c_\infty,$$

which is a contradiction if we choose β and τ such that $\beta^{-1}\tau^{2\alpha_0} < 1$. Therefore, h vanishes outside a compact set and satisfies an elliptic equation with analytic coefficients, therefore, h vanishes everywhere. □

6. Conclusion

6.1. **Proof of the Theorem 1.2.** With the notations and the results of Proposition 3.3, one can apply Theorem 4.2 to each vector field U_i to ensure the existence of vector fields V_i satisfying

$$\Delta_f V_i + \frac{V_i}{2} = \Delta_f U_i + \frac{U_i}{2} \quad \text{on } M,$$
$$V_i = O(f^{-a(n)/2+\epsilon}).$$

Therefore, $W_i := U_i - V_i$ satisfies $\Delta_f W_i + W_i/2 = 0$. Moreover, the maximum principle applied to V_i gives $\nabla V_i = O(f^{-a(n)/2+\epsilon})$. Since, by construction, $\mathscr{L}_{U_i}(g) = O(f^{-a(n)/2+\epsilon})$, we get that $h_i := \mathscr{L}_{W_i}(g) = O(f^{-a(n)/2+\epsilon})$. Consequently, Theorem 5.3 ensures that $h_i = 0$ on M. Consequently, we have built $n(n-1)/2$ independent non trivial Killing vector fields on M. Now, it remains to show that they are orthogonal to X. Indeed,

$$\begin{aligned} [W_i, X] &= \nabla_{W_i} X - \nabla_X W_i = \frac{W_i}{2} - \mathrm{Ric}(W_i) - \nabla_X W_i \\ &= \frac{W_i}{2} + \Delta W_i - \nabla_X W_i = 0. \end{aligned}$$

Now,

$$\begin{aligned} 2\nabla^2 \langle W_i, X\rangle &= 2\nabla^2 \mathscr{L}_{W_i}(f) = 2\,\mathscr{L}_{W_i}(\nabla^2 f) \\ &= \mathscr{L}_{W_i}(\mathscr{L}_X(g)) = \mathscr{L}_X(\mathscr{L}_{W_i}(g)) = 0. \end{aligned}$$

We conclude by using the Bochner formula :

$$\begin{aligned} 0 &= \Delta\left(\frac{|\langle W_i, X\rangle|^2}{2}\right) = |\nabla^2\langle W_i, X\rangle|^2 + \mathrm{Ric}(\nabla\langle W_i, X\rangle, \nabla\langle W_i, X\rangle) \\ &\quad + \langle \nabla\Delta\langle W_i, X\rangle, \nabla\langle W_i, X\rangle\rangle \\ &= \mathrm{Ric}(\nabla\langle W_i, X\rangle, \nabla\langle W_i, X\rangle) > 0, \end{aligned}$$

unless $\nabla\langle W_i, X\rangle = 0$. Therefore, $\langle W_i, X\rangle$ are constants. Now, X vanishes somewhere, therefore, $\langle W_i, X\rangle = 0$ for any i. This completes the proof of Theorem 1.2.

6.2. **Proof of corollary 1.3.** Let $(M^n, g, \nabla f)$ be a shrinking gradient Ricci soliton with bounded positive curvature operator. Assume M^n is not compact. Then, by Naber [Corollary 4.1, [12]], we know that, for any sequence of points $(x_k)_k$ tending to infinity, $(M^n, g, x_k)_k$ subconverges to $(\mathbb{R} \times N, dt^2 + h, x_\infty)$ where (N, h) is a non flat shrinking gradient Ricci soliton with nonnegative curvature operator. If (M^n, g) has linear volume growth, N is compact and it is diffeomorphic to the levels of the potential function $f^{-1}(t)$ for large t. Now, since M^n has positive curvature operator, M^n is diffeomorphic to $\mathbb{R}^n$ by the Gromoll-Meyer theorem. In particular, $f^{-1}(t)$ is homeomorphic to a $(n-1)$-sphere. By the classification of compact manifolds with nonnegative curvature operator [3], we claim that N is diffeomorphic to the standard $(n-1)$-sphere and h is the metric of positive constant curvature normalized by $\mathrm{Ric}_h = h/2$ and therefore, by theorem 1.2, (M^n, g) is cylindrical, in particular, it is not positively curved. Contradiction.

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