Scuola Normale Superiore

Classe di Scienze Matematiche Fisiche e Naturali

Ph.D. Thesis

Optimization problems for solutions of elliptic equations and stability issues

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Contents

Introduction

Preamble

In this thesis we address some problems related to two topics in the Calculus of Variations which have attracted a growing interest in recent decades. In a general simplification we can refer to those two fields as *optimization problems for shapes and for solutions of elliptic equations* and *quantitative stability problems for geometric and functional inequalities*. It is worth bearing in mind that the historical and mathematical development of these two classes of problems have merged and looking at them as separate fields may not be the best approach to adopt. On the other hand a division of the works presented may simplify the reading, for this reason the thesis is divided into two main parts. In the first, Part I, containing three chapters, we deal with optimization problems related to the *shape optimization* field. In Part II, we address the quantitative stability of three problems, the first one regarding a class of isoperimetric inequalities, the second one about a spectral optimization problem and the third one concerning a class of functional inequalities.

The introduction, divided as well into two parts, is aimed to offer the basic background to get the reader into the two main fields taken into consideration, and to describe the organization and scopes of each chapter. More precisely in the first part we present a short introduction of the concept of *optimization problems for shapes and for solutions of elliptic equations* and recall some important examples, then we briefly describe the original contributions related to these subjects contained in the first three chapter of the thesis.

In the second part of the introduction, we describe the general concept of *quantitative stability* of a functional inequality, and then we delve into the state of the art of this subject by means of three important examples. After that, we describe the original contributions of the thesis in this topic, contained in Chapters 4, 5 and 6.

Eventually, we briefly describe the results of the last chapter, which is unrelated to the rest of the thesis. For the sake of brevity this introduction does not contain all the technical preliminaries needed in the thesis, which are given, when necessary, within each chapter. All the results contained in this thesis are part of papers published, submitted or in preparation, and are the outcome of various collaborations which are cited both in this introduction and at the beginning of each chapter.

Part I: Optimization problems for shapes and solutions of elliptic equations

An optimization problem in its general formulation takes the form

$$\min\{F(x) : x \in X\} \tag{0.1}$$

for a given set X and a cost functional $F : X \to \mathbb{R}$. The general definition of a shape optimization problem is simply the case where X is a subset of the powerset of $\mathbb{R}^d$, $\mathcal{P}(\mathbb{R}^d)$. So it is worth formulating problem (0.1) in the shape optimization's sense:

$$\min\{F(E) : E \subset \mathcal{A}\}, \qquad \mathcal{A} \subset \mathcal{P}(\mathbb{R}^d). \tag{0.2}$$

The main issues one usually addresses regarding problem (0.2) are

- existence and uniqueness of a minimizer;
- qualitative properties of a minimizer as regularity, symmetry or more specific properties (e.g. convexity).

This class contains another sub-class of problems, that is when $F(E) = G(u_E)$ where G is a suitable real function and u_E is the solution of a given PDE on the domain E. For the sake of clearness, and since the two problems are strictly connected in many cases, we consider in this introduction mainly the case of shape optimization problems, keeping in mind that most of the forthcoming assertions are valid for both the cases.

In any case it is surely a good starting point to wonder *how to prove the existence of a minimizer* of the problem (0.2). This is usually done by means of the *Direct Method in the Calculus of Variation* (or more briefly: Direct Method). Namely, we consider a minimizing sequence of sets $E_h \subset \mathcal{A}$, that is a sequence such that

$$F(E_h) \to \inf_{\mathcal{A}} F,$$

and try to construct a set E as limit point of the sets E_h's (this request is often abbreviated as a *compactness* request) such that F is lower semicontinuous at least along this sequence, that is $\underline{\lim}_h F(E_h) \geq F(E)$[1]. So the real problem in applying this classical and simple method is the choice of the right topology. Clearly the two requests needed to apply the Direct Method are in competition: to gain compactness we are led to ask the topology to be coarse enough, while to get semicontinuity, we would like it to be as fine as possible. Not surprisingly, it turns out that to attack different categories of problems we need different kinds of topologies. Immediately following we describe some of the main examples of problems in shape optimization which entail different kinds of topologies, with a twofold scope: to penetrate more deeply inside the world of the optimization problems for shapes and for solutions of elliptic equations, and to introduce some of the main mathematical tools and objects we deal with in Part I (but also in the rest of the thesis).

[1] Here as well as in the rest of the thesis, $\underline{\lim}$ indicates the limit inferior

The isoperimetric problem

One of the most natural topologies on the class of sets is the L^1 topology: we say that $E_h \to E$ in L^1 if the Lebesgue measure of their symmetric difference $|E\Delta E_h|$ converges to 0 as $h \to \infty$. A very classical problem making use of this topology is the *isoperimetric problem*. This highly fascinating and ancient problem is the godmother of a huge class of problems in Geometric Measure Theory, in Shape Optimization and about geometric and functional inequalities. Indeed it is present, explicitly or not, in several topics of this thesis (actually most of them are present in Part II of the thesis).

A rough formulation of the problem is the following: *find the container with minimal surface area which encompasses a fixed given volume.* To translate it into precise mathematical language, we introduce the (proper!) object needed to measure the *surface* area of a subset of $\mathbb{R}^d$: the perimeter. For a given set E we recall the definition of the perimeter of E relative to an open set A

$$P(E;A) = \sup\left\{\int_E \operatorname{div}(\phi)\,dx : \phi \in C_c^1(\mathbb{R}^d, A), \quad |\phi| \le 1\right\}, \tag{0.3}$$

first formulated by E. De Giorgi in [47]. If $A = \mathbb{R}^d$, we will denote the perimeter of E as $P(E)$. We do not enter into the details of this topic and we refer the interested reader to the (beautiful) theory of sets of finite perimeter to [2] and to the recent book [92], but we limit ourselves to state the properties sufficient to solve the isoperimetric problem. It is not difficult to prove that the functional P is lower semicontinuous with respect to the L^1-convergence, that is,

$$P(E) \le \underline{\lim}_{h\to\infty} P(E_h) \qquad \text{if} \quad |E\Delta E_h| \to 0,$$

where $|E\Delta E_h|$ is the $d-$dimensional Lebesgue measure of the symmetric difference between E and E_h: $E\Delta E_h := (E \setminus E_h) \cup (E_h \setminus E)$. Moreover one can prove that the class of sets of equi-bounded perimeter, contained in a big fixed ball, is compact[2]. Then, by means of the Direct Method we get existence for the following version of the isoperimetric problem

$$\min\{P(E) : E \subset B(0,R),\ |E| = c\} \tag{0.4}$$

for two given positive constants R and c. Although we will not enter into further detail, we mention the fact that the bound constraint on the sets is not necessary. Notice that in the definition of this problem we implicitly defined the topology we impose on the set: the L^1 topology (this notation is due to the fact that, if $\mathbf{1}_A$ and $\mathbf{1}_B$ are the characteristic functions of the Borel sets A and B, then $|A\Delta B| = \|\mathbf{1}_A - \mathbf{1}_B\|_{L^1(\mathbb{R}^d)}$). Moreover we also know what the explicit solutions are: the balls (not just *a* ball, since the problem is translation invariant). A complete proof of this fact has been given for the first time

[2]A proof of this fact can be easily performed by means of the theory about Functions of Bounded Variation, see for instance [2]. Anyway it is worth having a look to the original De Giorgi proof in [47].

in the celebrated paper of E. De Giorgi [47] by means of symmetrization techniques and followed later on by many other proofs.
An obvious but important property of the L^1 topology is that it is not sensible if we modify a set E by adding (or subtracting) a set with zero L^1 mass. This turns out to be a problem when one considers the other main class of problems related to elliptic operators described in the following example.

Spectral and Energy Function problems

A good guide to optimization problems related to spectral and Energy functionals can be the books [75], [77] and [25]. Here we introduce a couple of basic examples which are the main objects, together with the perimeter, we deal with in the first two chapters of Part I, and which engrafted a rich and interesting theory. Consider an open bounded Lipschitz set Ω and a function $f \in L^2(\Omega)$. Then we say that u is a solution of the Dirichlet problem:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{0.5}$$

if u is the (unique) solution of the variational problem

$$\begin{cases} \displaystyle\int_\Omega \langle \nabla u, \nabla v\rangle - fv\,dx = 0 \quad \forall v \in H^1_0(\Omega) \\ u \in H^1_0(\Omega) \end{cases} \tag{0.6}$$

By means of the Poincaré inequality, and the Lax-Milgram Theorem (see for instance [57]) we know that there exists a solution u_Ω for problem (0.5) characterized as the unique minimum of the following variational problem:

$$\min\left\{\int_\Omega |\nabla u|^2\,dx - 2\int_\Omega uf\,dx : u \in H^1_0(\Omega)\right\}. \tag{0.7}$$

Notice that by multiplying in equation (0.5) by the solution u_Ω and integrating by parts, we get that the quantity in (0.7) is equivalent to

$$\mathcal{E}_f(\Omega) = -\frac{1}{2}\int_\Omega u_\Omega f\,dx. \tag{0.8}$$

Before introducing the shape optimization problem we have in mind about the functional $\mathcal{E}_f$, we introduce the other class of functionals we are principally interested in: spectral functionals. To introduce them we recall that the *resolvent* of the Dirichlet-Laplacian $-\Delta$ on Ω, that is the functional that associates to every function $f \in L^2(\Omega)$ the solution of (0.5), is a compact, symmetric and (thanks to the minus sign) positive operator on $L^2(\Omega)$. Thus by classical results in functional analysis (see for instance [19]) it admits a positive spectrum, consisting in a sequence Λ_k of eigenvalues, converging to 0. As a consequence, there is a sequence of positive numbers $\lambda_k := \lambda_k(\Omega) = 1/\Lambda_k$

(eigenvalues) accumulating at infinity, and of functions $u_k \in H_0^1(\Omega)$ (eigenfunctions) such that

$$-\Delta u_k = \lambda_k u_k.$$

As in the case of the Energy Function, we would like to have a variational version to compute the eigenvalues. This is offered us by the Courant-Fischer Formulae, see [42]:

$$\lambda_k(\Omega) = \min \left\{ \mathcal{R}_\Omega[u] := \frac{\displaystyle\int_\Omega |\nabla u|^2\,dx}{\displaystyle\int_\Omega u^2\,dx} \; : \; \int_\Omega u u_i\,dx = 0,\; i = 0, \dots, k-1 \right\}. \tag{0.9}$$

The functional $\mathcal{R}_\Omega[u]$ is often referred to as Rayleigh quotient of u. With these examples in mind we can formulate optimization problems in the sense of (0.2) where $F(\Omega)$ is, for instance, λ_k or the Energy function $\mathcal{E}_f$. Other more general cost functionals F that can be considered are *spectral functionals* that is $F(\Omega) = \Phi(\lambda(\Omega))$ with $\Phi : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is a given function and where $\lambda(\Omega) = (\lambda_1(\Omega), \lambda_2(\Omega), \dots)$ is the spectrum of $-\Delta$ on Ω, and *integral functionals*, where F takes the form

$$F(\Omega) = \int_\Omega j(x, u_\Omega(x), \nabla u_\Omega(x))\,dx,$$

where u_Ω is the solution of (0.7) and j is a suitable integrand which is usually assumed to be convex in the gradient variable and bounded from below.

The L^1 topology is not the right topology to afford the existence problem for the functionals above since they are sensible under variation of zero L^1 volume (think for example to the case of a ball in $\mathbb{R}^2$ and the same ball without a diameter). Indeed the crucial quantity to look at while affording shape optimization problems related to elliptic equations, is the *capacity* of a set instead of its Lebesgue measure. Again, for the sake of brevity we will not enter into the details of this interesting theory, which will be however introduced together with its main properties, in Section 1.2 of Chapter 1, and we refer the reader to the book [25]. Concerning the aforementioned shape optimization problems, we cite [30] where a general theorem is proved which entails existence for a large class of functionals, under the constraint of equiboundedness of the sets. As in the case of the perimeter such bound is not necessary, but the proof of this fact is quite involved and very recent (see [96] and [24]).

Results of Part I of the Thesis

We describe hereafter the main results of the first three chapters of the thesis.

Optimization problems related to Schrödinger operators

In the first chapter, based on the joint work [31] together with Giuseppe Buttazzo, Augusto Gerolin and Bozhidar Velichkov, we consider a problem of the form

$$\min \{ F(V) \; : \; V \in \mathcal{V} \}, \tag{0.10}$$

where $V:\mathbb{R}^d\to\mathbb{R}$ is a non-negative real function and $\mathcal{V}$ a suitable constraint which can take different forms depending on the functional we deal with. For every admissible potential V we consider the Schrödinger equation formally written as

$$-\Delta u+Vu=f\qquad u\in H_0^1(\Omega),$$

and whose precise meaning has to be given in the weak form

$$\begin{cases}\displaystyle\int_\Omega\langle\nabla u,\nabla\varphi\rangle\,dx+\int_\Omega Vu\varphi\,dx=\int_\Omega f\varphi\,dx\quad\forall\varphi\in H_0^1(\Omega),\\ u\in H_0^1(\Omega)\cap L^2(V\,dx).\end{cases}$$

One of the main tools we exploit, to deal with this problem, is the $\gamma-$convergence for capacitary measures, introduced, together with its main properties, in Subsection 1.2. Thanks to this, we are able to prove two existence results in the case where Ω is bounded for the problem (0.10), valid for a large class of cost functions F, where the constraint is taken to be a weakly compact set in $L^p(\Omega)$. Then we consider a *volume constraint* for V (not necessarily bounded in $L^p(\Omega)$) of the form

$$\mathcal{V}=\left\{V:\int_\Omega\Psi(V)\,dx\le c\right\}$$

where $\Psi:[0,\infty]\to[0,\infty]$ is a given function and, under some mild hypotheses on F, we prove existence for problem (0.10) for a class of constraints where the main example is $\Psi(t)=t^{-p}$, $p>0$. Then, we afford the existence issue for the particular case of the Dirichlet Energy function under a convex L^p constraint for V and with Ω bounded. In this case we are able to prove existence and to characterize the solutions in terms of the solutions of a particular PDE in $\mathbb{R}^d$. Later on we turn our attention to the case of Ω unbounded. In this case it is more difficult to get existence (in particular, it is more delicate to get compactness for a minimizing sequence). So we focus to the particular cases of the minimization and the maximization (under suitable volume constraints) of the first eigenvalue and the Dirichlet Energy function related to the Schrödinger operator $-\Delta+V$. In the particular case where $\Omega=\mathbb{R}^d$ and $F=\lambda_1$, after solving the existence problem, we study some qualitative properties of the solutions, such as their symmetry. Moreover we state a relation between the eigenfunction u_V related the optimal potentials V and the solutions of a class of celebrated functional inequalities: the Gagliardo-Nirenberg-Sobolev inequalities (GNS). This lead us to study, as in the case of the GNS inequalities (see for instance [104]), the support of the optimal potentials. Eventually, thanks to the characterization of the supports of such functions, we are able to describe qualitatively the *shape* of the potentials V which minimize (under suitable constraints) the first and the second eigenvalue of the Schrödinger operator in $\mathbb{R}^d$.

Optimization problems for metric graphs

In the second chapter, based on the joint work [32] with Giuseppe Buttazzo and Bozhidar Velichkov we study the problem of minimizing the first eigenvalue of the Laplacian and its Energy function in the case where Ω is a graph. By a graph we will initially mean a connected one-dimensional set; thanks to classical theory about such sets, see for instance [4], these sets are rectifiable so that quantities as the Dirichlet Energy or as the first eigenvalue of the Dirichlet Laplacian are well defined. After having developed the main tools needed to state and afford the problem in such setting, we prove by means of a counterexample that in general an existence result without additional assumptions cannot be expected. Thus we consider the larger class of *metric graphs* (see Section 2.3) where we are able to prove a general existence theorem. Then we develop a symmetrization technique, which may be seen as analogous to the Pólya-Szëgo inequality in this particular setting, and by means of such technique, we describe some specific examples of optimal graph for the Energy Function.

A non-local isoperimetric problem

In the third chapter, based on a work in preparation with Michael Goldman and Matteo Novaga [70], we deal with a non-local isoperimetric problem. More precisely, we consider problem (0.2) where $\mathcal{A}$ is the class of sets with fixed volume, say c, and F is given by

$$F(E) = P(E) + \mathcal{Q}_\alpha(E) \tag{0.11}$$

where

$$\mathcal{Q}_\alpha(E) = \min\left\{\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{d\mu(x)\,d\mu(y)}{|x-y|^\alpha} : \mu(E) = Q\right\}$$

for $d \geq 2$ and $\alpha \in (0, d-1)$. Functionals of this kind are well known in the literature since they are related to the physical problem of finding the optimal shape of a liquid droplet in void, once it is provided with an electric charge Q. Similar problems have recently been addressed in several works, see for instance [80], [81] and the references therein. But the main case afforded is that where the charge is prescribed to be *uniform* on the set. In this chapter we prove that problem (0.11) is actually ill-posed: minimizing F is equivalent to minimizing separately the two functionals P and $\mathcal{Q}_\alpha$. This is not surprising if we notice that these two functionals are actually *governed* by different kind of measures: the first is invariant under L^1-negligible perturbations, while the second may be (depending on the optimal measure μ) a capacitary functional (that is sensible up to positive capacitary perturbations). However, we are able to prove that under further regularity conditions, the existence problem is solvable. Moreover (at least) in the *harmonic case* $\alpha = d-2 \geq 1$, if we restrict the set of competitors just to sets which are *small* $C^{1,1}$ perturbation of the ball, the ball is the unique minimum.

Part II: Stability problems

In recent years there has been renewed interest in the research of a *quantitative stability* version for several classes of inequalities. We can find several examples of quantitative versions of geometric inequalities see e.g [62], [58], [73], [72], [60], [61], functional inequalities, see e.g. [12], [8], [39], [34], [35], and of inequalities arising from the optimization of shapes and solutions of elliptic equations, see e.g. [18], [15]. The motivation of this interest probably resides in the recent work [62] where it has been proved a *sharp* quantitative version of the isoperimetric inequality.

Let us begin by explaining what we mean by quantitative stability of an inequality. The starting point is again an optimization problem of the form

$$\min\{F(x) : x \in X\}.$$

Suppose that we have solved this problem, and found a class of minimizers of F in X: $M \subset X$. Then we can translate our solution in terms of an inequality of the form

$$F(x) - F(y) \geq 0 \qquad \forall x \in X, \quad y \in M. \tag{0.12}$$

The quantitative stability question is then the following:
suppose that a point $z \in X$ is such that we almost get equality in (0.12)*. Can we claim then that z is near the set of minimizers M?*
The previous question, in the form it is stated, lacks at least formally into two points: first, we should formalize what we mean by *almost getting equality*; then we should specify in which sense we mean *nearness*. These are the crucial and starting points of any analysis about the quantitative version of an inequality, and can vary case by case.

To get into the concept of quantitative stability, we quote three examples of stability inequalities: the first one describing the state of the art for the isoperimetric inequality in quantitative version, the second one about the stability of some spectral problems, with particular regard to that related to the first eigenvalue of the Dirichlet Laplacian, and the third one regarding a class of functional inequalities, with particular attention to the Sobolev and Gagliardo-Nirenberg-Sobolev inequalities. After this overview, we pass to the description of the original results contained in the second part of the thesis: one stability problem about a class of isoperimetric inequalities, one related to the stability of the spectral problem of the Stekloff-Laplacian, and one which concerns the stability of the Gagliardo-Nirenberg-Sobolev inequalities.

The isoperimetric inequality in quantitative form

We recall once again that the isoperimetric inequality states that if E is a measurable subset of $\mathbb{R}^d$ of finite measure and if B_E is a ball of the same measure of E, then $P(E) \geq P(B_E)$. Thanks to the scaling laws satisfied by the perimeter and by the $d-$dimensional Lebesgue measure, we can avoid to prescribe the measure of E

$$|E|^{(1-d)/d} P(E) \geq d\omega_d^{1/d}$$

where ω_d is the measure of the unitary ball of $\mathbb{R}^d$. If we introduce the *isoperimetric deficit* of a set E as

$$\delta(E) = \frac{P(E)}{d\omega_d^{1/d}|E|^{(d-1)/d}} - 1$$

then the isoperimetric inequality becomes

$$\delta(E) \geq 0. \tag{0.13}$$

Then for a quantitative version of the isoperimetric inequality we mean an improvement of (0.13) of the form

$$\delta(E) \geq \Phi(\mathrm{dist}(E, M_E)) \tag{0.14}$$

where M_E is the manifold composed of all the balls of $\mathbb{R}^d$ of measure $|E|$, $\mathrm{dist}(\cdot, M)$ is a suitable distance of E from M_E and Φ is a modulus of continuity. Often the distance adopted for the isoperimetric problem is the so-called Fraenkel asymmetry defined as follows:

$$\mathcal{A}(E) = \inf\left\{\frac{|E\Delta B|}{|E|} : B \text{ ball}, |B| = |E|\right\}$$

where $E\Delta B = (E \setminus B) \cup (B \setminus E)$ denotes the symmetric difference between E and B. Notice that the choice of the isoperimetric deficit is not free: it is just the (rescaled) excess *offered* by the inequality itself. This is not the case of the asymmetry, whose choice is a crucial starting point for the analysis of the stability. Several partial results in the direction of proving an inequality of the form (0.14) have been obtained. For instance in [60], where the distance taken in consideration is actually *stronger* than the Fraenkel asymmetry, but the result is valid only for sets whose boundary is a small $W^{1,\infty}$ perturbation of the boundary of the ball (the so-called *nearly spherical sets*). The first remarkable results valid for any set of finite perimeter has been obtained in [72] and [73], where a quantitative version of the stability inequality of the form

$$\delta(E) \geq C(d)\mathcal{A}(E)^4$$

is proved. The inconvenience in this inequality is the exponent 4. Indeed by computing the deficit and the asymmetry on a family of ellipsoids converging to the ball, the guess is that the expected optimal exponent should be 2. This is actually the case, as proved recently in three works. The first one, in chronological order, takes its moves from the works [72] and [73] and exploits a clever symmetrization method. The second one [58] makes use of some results coming from the Mass Transport Theory and in particular of the proof of the isoperimetric inequality done by means of the Brenier map (see [110] for a comprehensive account on the subject). In this case the authors are able to prove the isoperimetric inequality also for anisotropic perimeters. The third paper [40], develops a strategy called *Selection Principle* which, combined with the results in [60] (the analysis of the nearly spherical sets) and classical regularity theory for sets of finite perimeter, brings again to the sharp exponent 2. It is worth mentioning that, although all three proofs have their own interests, the Selection Principle developed in [40] has been successfully applied to prove the stability of several other inequalities, as in [15], [1], [53], [13], [14].

Spectral inequalities in quantitative form

As mentioned, since the development of the techniques used for proofs of the sharp quantitative version of the isoperimetric inequality, there have been several authors who have tried to attack other inequalities. A class of inequalities which attracted some interest is that of the inequalities coming from the shape optimization theory, with particular regard to spectral problems. As a basic example we consider the first eigenvalue of the Dirichlet Laplacian, introduced in the previous subsection. We recall that for a given bounded set Ω with Lipschitz boundary, it is characterized as

$$\lambda_1(\Omega) = \min\left\{ \frac{\displaystyle\int_\Omega |\nabla u|^2\,dx}{\displaystyle\int_\Omega u^2\,dx} : 0 \neq u \in H^1_0(\Omega) \right\}.$$

The problem we introduce is then

$$\min\{\lambda_1(\Omega) : |\Omega| = c\}.$$

The solution of this problem can be stated by means of the celebrated Faber-Krahn inequality

$$\lambda_1(\Omega) \geq \lambda_1(B)$$

which holds for any ball B such that $|\Omega| = |B|$. As in the case of the perimeter, the first eigenvalue of the Dirichlet-Laplacian satisfies a scaling law: $\lambda_1(t\cdot) = t^{-2}\lambda_1(\cdot)$ so that the previous inequality becomes, without prescribing the volume of the sets,:

$$|\Omega|^{d/2}\lambda_1(\Omega) \geq \omega_d^{d/2}\lambda_1(B_1), \tag{0.15}$$

where B_1 is any ball of radius 1. It is worth mentioning the connection for this inequality with the isoperimetric one. Indeed (0.15) is usually proved via the *Schwarz-symmetrization* and in particular by means of the Pólya-Szëgo inequality: given a measurable function u, let u^* be the non-negative function which level sets $\{u^* > t\}$ are balls centred at the origin of measure $|\{|u| > t\}|$, then the Pólya-Szëgo inequality states that

$$\int |\nabla u|^2\,dx \geq \int |\nabla u^*|^2\,dx.$$

Moreover, in [23] is proved that the equality can holds only if u has not *flat regions*. These results immediately imply that the ball is the unique minimizer (up to translations and dilations) for λ_1. For a quantitative version of the Faber-Krahn inequality (0.15) we mean then an inequality of the form

$$|\Omega|^{d/2}\lambda_1(\Omega) - \omega_d^{d/2}\lambda_1(B_1) \geq \sigma\mathcal{A}(\Omega)^\alpha$$

where σ and α are fixed constants depending on d. Partial results in this direction has been obtained (in any dimension) in [64] where, by exploiting the sharp quantitative form of the isoperimetric inequality, the authors get the non-sharp exponent 4 (3 in dimension $d = 2$). For a sharp version of this inequality, that is with $\alpha = 2$, we refer to the recent work [15] which suitably exploits a *selection principle* argument, in the spirit of [40].

We will not enter into further details in this introduction about quantitative versions for spectral inequalities, but we mention that many other problems have been afforded about higher eigenvalues of the Dirichlet-Laplacian or about spectral problems related to the Laplacian with different kind of boundary conditions. See for instance [18] and the references therein.

Stability of functional inequalities

The last example we shall deal with and we briefly examine is that of the quantitative stability of functional inequalities. The principal examples in this area are given by the Sobolev and the Gagliardo-Nirenberg-Sobolev (briefly: GNS) inequalities. These inequalities, exhaustively described in Chapter 6, take the general form

$$\|u\|_{L^{dp/(d-p)}(\mathbb{R}^d)} \leq C(d,p)\|\nabla u\|_{L^p(\mathbb{R}^d)} \tag{0.16}$$

for the Sobolev case, and

$$\|u\|_{L^q(\mathbb{R}^d)} \leq C(d,p,q,s)\|\nabla u\|^{\theta}_{L^p(\mathbb{R}^d)}\|u\|^{1-\theta}_{L^s(\mathbb{R}^d)} \tag{0.17}$$

in the GNS case, for suitable parameters d, p, q, s and θ (for precise definitions, we refer again to Chapter 6). The solutions and optimal constants for the Sobolev inequality are known after the works of Aubin and Talenti [6], [109]. In the case of the GNS inequalities, instead, we know the solutions and the optimal constants only for a given one-parameter family of exponents [48]. See also [43] for an approach to the proofs of such inequalities using the Mass Transport theory. An important result about the quantitative form for the Sobolev inequality has been given in [12], in which only the case of $p = 2$ is considered and it is crucially exploited the fact the in this case the Euler-Lagrange equation associated to the Sobolev inequality is linear in the gradient part, and also that the set of solutions have an Hilbertian structure. The ideas developed in [12] have then been used to prove a stability version for a class of GNS inequality (again prescribing $p = 2$) in the recent work [35].

After the proof of the sharp isoperimetric inequality in quantitative form, there have been several attempts to prove a *sharp* quantitative version for Sobolev and GNS inequalities. Regarding the Sobolev case successful attempts have been done in [63] for $W^{1,1}$ functions (actually also for functions of bounded variation). See also the recent work [54] where the case $p = 2$ is considered. On the contrary, for the GNS case there have been proved quantitative versions only in the one-parameter class whose solutions are known or for the simplified structure obtained by prescribing $p = 2$, see [34], [35].

It is important then to remark about the work [39], where a non-sharp version of the Sobolev inequality is proved by means of the sharp version of the isoperimetric inequality. Indeed it is well known that the Sobolev inequality is implied by the isoperimetric one, and the proof of this fact basically relies on the co-area formula and a rearrangement argument (see [2]). The proof in [39] makes use of this implication and then exploits the stability version of the isoperimetric inequality. One benefit of this idea is the geometric flavour of the proof, which turns out to be adaptable also to inequalities whose optimal cases are not explicit, but whose qualitative geometric aspects (as being radial functions) are known. This is the case, as we will see in Chapter 6, of the GNS inequalities.

Results of Part II of the thesis

We describe hereafter the main results of the chapters contained in the second part of the thesis. Each one is somehow related to one of the three examples introduced above.

Weighted isoperimetric inequalities

The fourth chapter is based on part of a joint work with Lorenzo Brasco and Guido De Philippis [16] and addresses the problem of the quantitative stability of *weighted isoperimetric inequalities*. Namely, given a Lipschitz set E we consider its *weighted perimeter*

$$P_V(E) = \int_{\partial E} V(x)\, dx$$

where V is a non-negative weight function, and we address the problem

$$\min\{P_V(E) : |E| = c\}, \tag{0.18}$$

where $|\cdot|$ stands for the $d-$dimensional Lebesgue measure of a set. Such a problem has been solved for a class of radial functions V in [9] where it is proved, thanks to a symmetrization technique, that under some suitable hypotheses on V the ball centred at origin is the unique minimum. We first afford the same issue, offering a totally different proof of it (for the same class of functions V) by means of a sort of *calibration technique*, which allows us to get also *optimal* stability. Then we consider the analogous problem related to the exponential measures

$$\min\left\{P_{we^V}(E) : \int_E e^V\, dx = c\right\}$$

proving that under suitable hypotheses on V and w (having as main application the Gauss measure, i.e. $V = -|x|^2$) the *right* half spaces $\{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{d-1} : x_1 > t\}$ are solutions of the problem.

Stability of the first Stekloff-Laplacian eigenvalue

The fifth chapter (as well as the fourth) is based on the work [16]. In it we consider the first non-trivial eigenvalue of the Stekloff-Laplacian which can be characterized, for a given Lipschitz set Ω as the number $\sigma_2(\Omega)$ such that there exists a (unique) solution of the PDE

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ \langle \nabla u, \nu_\Omega \rangle = \sigma_2(\Omega) u & \text{on } \partial\Omega \end{cases}$$

where ν_Ω is the outer normal of Ω. Weinstock in dimension 2 [111][3] and later Brock [20] in any dimension proved that among all the sets of prescribed measure $c > 0$, the ball maximizes $\sigma_2(\cdot)$. This translates in the so-called Brock-Weinstock inequality

$$\sigma_2(\Omega) \leq \sigma_2(B_\Omega) \qquad B_\Omega \text{ ball such that } |B_\Omega| = |\Omega|.$$

A crucial tool in the Brock's proof, is the following *weighted* isoperimetric inequality:

$$\int_{\partial\Omega} |x|^2 \, d\mathcal{H}^{d-1} \geq \int_{\partial B_\Omega} |x|^2 \, d\mathcal{H}^{d-1}$$

where Ω is any set and B_Ω is the ball centred at the origin such that $|B_\Omega| = |\Omega|$. In this chapter, as a consequence of the the stability of the weighted isoperimetric inequalities proved in Chapter 4, we prove the stability of σ_2 as well. Then we prove that the quantitative version of the inequality is sharp. This is done thanks to a long and delicate proof modelled on the corresponding one related to Neumann eigenvalues in [18].

A reduction theorem for the stability of Gagliardo-Nirenberg-Sobolev inequalities

In the sixth chapter we deal with the Gagliardo-Nirenberg-Sobolev inequalities (0.17) and we prove a reduction theorem which entails that *to prove a quantitative version for such inequalities, it is sufficient to prove it only for radial decreasing functions.* Namely we address the problem of proving the following inequality

$$\delta_{GNS}(u) := \frac{\|\nabla u\|^\theta_{L^p(\mathbb{R}^d)} \|u\|^{1-\theta}_{L^s(\mathbb{R}^d)}}{G\|u\|_{L^q(\mathbb{R}^d)}} - 1 \geq C\lambda_{GNS}(u)^{\alpha_0} \tag{0.19}$$

where $G = G(d, p, q, s)$ is the optimal constant in (0.17), C and α_0 are positive constants (which do not depend on u), and

$$\lambda(u) = \inf\left\{ \frac{\|u - v\|_{L^q(\mathbb{R}^d)}}{\|u\|_{L^q(\mathbb{R}^d)}} : \|v\|_{L^q(\mathbb{R}^d)} = \|u\|_{L^q(\mathbb{R}^d)}, \quad v \text{ is optimal for (0.17)} \right\}.$$

[3] Actually Weinstock prescribes the perimeter of the set, implying so, in view of the isoperimetric inequality, a stronger version for the *Brock-Weinstock inequality* in dimension 2.

We prove that if (0.19) holds for radial decreasing functions, then, up to substitute α_0 with another suitable positive constant α_1, it holds true for any function. This allows us to simplify the problem in its general formulation, and in particular reduces a $d-$dimensional problem to a $1-$dimensional one. The results proved are valid also for the classes of parameters for which the optimal function of the GNS inequality (0.17) is not known. This is possible thanks to an adaptation of the symmetrization techniques developed in [39], where only the knowledge of the *qualitative shape* of the optimizer for the Sobolev functions (i.e. radial decreasing functions) is exploited.

Other works

In the final chapter we report a recent note written in collaboration with Camillo De Lellis and Matteo Focardi [51], and not related to the rest of the thesis. In this, we consider the localized Mumford-Shah functional for a bounded open domain Ω

$$\mathrm{MS}(v) = \int_\Omega |\nabla v|^2 dx + \mathcal{H}^{d-1}(S_v \cap \Omega), \qquad \text{for } v \in SBV(\Omega) \text{ and } A \subseteq \Omega \text{ open.}$$

where $SBV(\Omega)$ is the space of Special Bounded Variation functions and S_v is the jump set of the function v (see [2]). More precisely, we consider a local minimizer u for MS, that is a function in SBV such that $\mathrm{MS}(u) \le \mathrm{MS}(v)$ for any v such that $\{u \neq v\} \subset\subset \Omega$. Let us denote with Σ_u the set of points of S_u out of which u is locally regular and let

$$\Sigma'_u = \left\{ x \in \Sigma_u : \lim_{\rho \to 0} \frac{1}{\rho^{d-1}} \int_{B_\rho(x)} |\nabla u|^2 \, dx = 0 \right\}.$$

A classical result due to L. Ambrosio, N. Fusco and J.E. Hutchinson [3, Theorem 5.6], states that the dimension of Σ'_u is less or equal than $d-2$. The proof is quite involved and makes use of the concept of Almgren minimizers (see [2]). In this chapter we offer a simplified proof of the same result exploiting a recent work of C. De Lellis and M. Focardi [50] where it is proved that if u is a minimizer of MS, then the blow-up of its jump set in small gradient regime, that is Σ'_u, is a Caccioppoli partition.

Part I

Optimization problems for shapes and for solutions of elliptic equations

Chapter 1

Optimization problems related to Schrödinger operators

1.1 Introduction

This chapter is based on the joint work with Giuseppe Buttazzo, Augusto Gerolin and Bozhidar Velichkov [31]. The problem we consider takes the general form

$$\min\big\{F(V)\ :\ V\in\mathcal{V}\big\},\tag{1.1}$$

where F denotes a cost functional and $\mathcal{V}$ is an admissible class of functions $V:\mathbb{R}^d\to\mathbb{R}$. In this chapter, we will always suppose that V is non-negative and we will refer to it as *potential*. The principal examples we shall take into account, and which are also the leading motivation for the study of problem (1.1) in our case, are problems related to the Schrödinger operator $\mathcal{S}_V=-\Delta+V(x)$. More precisely the two principal cases we study are the following.

Energy functionals Consider the PDE concerning the energy of $\mathcal{S}_V$, related to a suitable function f:

$$-\Delta u+V(x)u=f,\qquad u\in H^1_0(\Omega).\tag{1.2}$$

If such equation admits a solution u_V, which is the case under suitable assumption on Ω and f, for instance if Ω is bounded and $f\in L^2(\Omega)$, then one can consider $F(V)$ as the energy of V:

$$\mathcal{E}_f(V)=\inf\left\{\int_\Omega\Big(\frac12|\nabla u|^2+\frac12V(x)u^2-f(x)u\Big)\,dx\ :\ u\in H^1_0(\Omega)\right\}=-\frac12\int_\Omega u_Vf.\tag{1.3}$$

The last equality in the previous formula follows by multiplying by u_V and integrating by part the Euler-Lagrange equation (1.2) of problem (1.3).

Spectral functionals The operator $\mathcal{S}_V=-\Delta+V(x)$ is positive definite (since $V\geq0$) and symmetric. Moreover, under suitable assumption on V (which we will always have in force), its resolvent operator is a compact operator from $L^2(\Omega)$ to $L^2(\Omega)$, whence $\mathcal{S}_V$ admits a discrete, positive (and unbounded) spectrum

$$\Lambda(V) = (\lambda_1(\Omega, V), \lambda_2(\Omega, V), \dots).$$

Since in all our analysis the set Ω will be fixed a priori, we will drop the dependence of Ω in the definition of λ_k simply writing $\lambda_k(V)$. Then we can consider functionals of the form

$$F(V) = \Phi(\Lambda(V)),$$

where $\Phi : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is an assigned function. A typical choice of such a function is given by $\Phi_k((x_1, x_2, \dots)) = x_k$, which, in terms of the functional F, corresponds to $F_k(V) = \lambda_k(V)$. In this case, we can express the functional also in a variational form:

$$\lambda_k(V) = \min\left\{\frac{\int_\Omega |\nabla u|^2 + V(x)u\,dx}{\int_\Omega u^2} \;:\; u \in H_0^1(\Omega),\ \int_\Omega uu_i\,dx = 0,\ i = 1, \dots, k-1\right\} \tag{1.4}$$

where u_i is an eigenfunction related to λ_i.

Although we do not intend to list the (many) known results (for which, a good guide is the book of Henrot [75, Chapter 8]), we mention that most of them are related to *spectral* problems, and in particular they are established for *bounded* domains Ω. As an example we recall the following general result proved in [5].

Theorem 1.1. *Let Ω be a $C^{1,1}$ bounded open set, $p > 1$ and $\mathcal{V}$ be a closed bounded convex subset of $L^p(\Omega)$. Then, there exists a unique $\widetilde{V}$ which maximizes $\lambda_1(V)$ in the class $\mathcal{V}$.*

Remark 1.2. The previous theorem addresses the *maximization* of the first eigenvalue $\lambda_1(V)$. The main reason is that the minimization of λ_1, as well as that of $\mathcal{E}_f$, often turns out to be a trivial or an ill-posed problem. For example, consider the constraint

$$\mathcal{V}_p = \{V : \|V\|_{L^p(\Omega)} \leq 1\}.$$

Then the problem

$$\min\{\lambda_1(V) : V \in \mathcal{V}_p\}$$

has the trivial solution $V = 0$ since, by (1.4) we have $\lambda_1(V) \geq \lambda_1(0) = \lambda_1(\Omega)$. In the maximization case, however, the convexity of the constraint is not necessary. Indeed, thanks to the inequality $\lambda_1(V_1) \leq \lambda_1(V_2)$ if $V_1 \leq V_2$ (due to (1.4)), maximizing $\lambda_1(V)$ on the constraint $\mathcal{V}_p$ is equivalent to maximize it under the constraint $\{V : \|V\|_{L^p(\Omega)} = 1\}$. Also the minimization problem may turn out to be not interesting, even with a non-convex constraint. For instance, consider a sequence of potentials V_n which converges weakly in L^2 to 0 and let u_Ω be the Dirichlet eigenfunction of Ω (or equivalently, the Schrödinger eigenfunction related to $V = 0$). Using u_Ω as a test function for the Rayleigh quotient in the definition of $\lambda_1(V_n)$ for each V_n, we get that

$$\inf\left\{\lambda_1(V) : \|V\|_{L^p(\Omega)} = 1\right\} = \lambda(\Omega).$$

But this implies that a minimizer V should be null almost everywhere, which is impossible since such a function cannot have L^p-norm equal to 1. Nevertheless, we will see, in Section 1.4, a class of natural constraints whose spectral and energy minimization problems are solvable.

Organization and main results of the chapter

The chapter is organized as follows. In Section 1.2 we introduce the notion of capacity, $\Gamma-$convergence, and $\gamma-$convergence. These tools, together with some related results, will be then exploited principally in the following Section 1.3.

In Section 1.3 we address the problem (1.1) in bounded domains where the constraint $\mathcal{V}$ is a weakly compact subset of $L^p(\mathbb{R}^d)$. As we have pointed out, many explicit minimization problems with such constraint are trivial. This is not the case if we consider *maximization problems*, see Remark 1.11. More precisely, in this section we prove two general results: in Theorem 1.10 we prove existence of solutions under some natural assumptions on the functional F and on the constraint $\mathcal{V}$. These assumptions are satisfied by a large class of integral functionals and of spectral functionals. Then we shall see a version of Theorem 1.10 (Theorem 1.13) valid in the class of capacitary measures. These theorems may be seen as a generalization of Theorem 1.1. The main tools we exploit to prove these results are contained in the theory of the $\gamma-$convergence discussed in Section 1.2.

In Section 1.4 we consider problem (1.1), this time seen as an actual minimization problem, that is with F non-negative, and, exploiting again some techniques borrowed by the $\gamma-$convergence theory, we prove an existence theorem (Theorem 1.16) for a wide class of non-convex constraints. Such theorem apply, for instance, to the case where the cost functional F is a spectral functional or the Energy functional, see Subsection 1.4.1.

In Section 1.5, we consider the problem of maximizing $\mathcal{E}(V)$. Namely we consider, for $p \geq 1$, the following problem

$$\max\left\{\mathcal{E}_f(V) : \int_\Omega V^p\,dx \leq 1, \quad V \geq 0\right\}.$$

Following some ideas developed in [75, Theorem 8.2.3], where the same problem related to $\lambda_1(V)$ is considered, we prove existence (and uniqueness, if $p > 1$) of optimal potentials (see Proposition 1.21 and Proposition 1.24). The class of constraints considered ($p \geq 1$) is sharp, in the sense that for $p < 1$ the problem has no solution, see Remark 1.26.

In Section 1.6 we consider the case $\Omega = \mathbb{R}^d$ and we prove existence of optimal potentials for the first eigenvalue $\lambda_1(V)$ and for the Energy function $\mathcal{E}_f(V)$ of the Schrödinger operator. For these two cases we are able to study qualitatively the *shape* of the optimal potentials, proving that they are radial functions and, in the minimization case, have compact support (see Proposition 1.31) while in the maximization case, they have support $\mathbb{R}^d$. Eventually, thanks to the previous results concerning the supports of $\lambda_1(V)$, we are able to characterize, at least qualitatively, the *shape* of the minimizers of the second eigenvalue $\lambda_2(V)$ (Proposition 1.34).

1.2 Capacitary measures and $\gamma-$convergence

For a subset $E \subset \mathbb{R}^d$ its *capacity* is defined by

$$\operatorname{cap}(E) = \inf\left\{\int_{\mathbb{R}^d} |\nabla u|^2\,dx + \int_{\mathbb{R}^d} u^2\,dx \ :\ u \in H^1(\mathbb{R}^d),\ u \geq 1 \text{ in a neighborhood of } E\right\}.$$

If a property $P(x)$ holds for all $x \in \Omega$, except for the elements of a set $E \subset \Omega$ of capacity zero, we say that $P(x)$ holds *quasi-everywhere* (shortly *q.e.*) in Ω, whereas the expression *almost everywhere* (shortly *a.e.*) refers, as usual, to the Lebesgue measure, which we often denote by $|\cdot|$.

A subset A of $\mathbb{R}^d$ is said to be *quasi-open* if for every $\varepsilon > 0$ there exists an open subset A_ε of $\mathbb{R}^d$, with $A \subset A_\varepsilon$, such that $\operatorname{cap}(A_\varepsilon \setminus A) < \varepsilon$. Similarly, a function $u : \mathbb{R}^d \to \mathbb{R}$ is said to be *quasi-continuous* (respectively *quasi-lower semicontinuous*) if there exists a decreasing sequence of open sets $(A_n)_n$ such that $\operatorname{cap}(A_n) \to 0$ and the restriction u_n of u to the set A_n^c is continuous (respectively lower semicontinuous). It is well known (see for instance [57]) that every function $u \in H^1(\mathbb{R}^d)$ has a quasi-continuous representative $\widetilde{u}$, which is uniquely defined up to a set of capacity zero, and given by

$$\widetilde{u}(x) = \lim_{\varepsilon\to 0} \frac{1}{|B_\varepsilon(x)|}\int_{B_\varepsilon(x)} u(y)\,dy\,,$$

where $B_\varepsilon(x)$ denotes the ball of radius ε centred at x. We identify the (a.e.) equivalence class $u \in H^1(\mathbb{R}^d)$ with the (q.e.) equivalence class of quasi-continuous representatives $\widetilde{u}$.

We denote by $\mathcal{M}^+(\mathbb{R}^d)$ the set of positive Borel measures on $\mathbb{R}^d$ (not necessarily finite or Radon) and by $\mathcal{M}^+_{\text{cap}}(\mathbb{R}^d) \subset \mathcal{M}^+(\mathbb{R}^d)$ the set of *capacitary measures*, i.e. the measures $\mu \in \mathcal{M}^+(\mathbb{R}^d)$ such that $\mu(E) = 0$ for any set $E \subset \mathbb{R}^d$ of capacity zero. We note that when μ is a capacitary measure, the integral $\int_{\mathbb{R}^d} |u|^2\,d\mu$ is well-defined for each $u \in H^1(\mathbb{R}^d)$, i.e. if $\widetilde{u}_1$ and $\widetilde{u}_2$ are two quasi-continuous representatives of u, then $\int_{\mathbb{R}^d} |\widetilde{u}_1|^2\,d\mu = \int_{\mathbb{R}^d} |\widetilde{u}_2|^2\,d\mu$.

For a subset $\Omega \subset \mathbb{R}^d$, we define the Sobolev space $H^1_0(\Omega)$ as

$$H^1_0(\Omega) = \left\{u \in H^1(\mathbb{R}^d) :\ u = 0 \text{ q.e. on } \Omega^c\right\}.$$

Alternatively, by using the capacitary measure I_Ω defined as

$$I_\Omega(E) = \begin{cases} 0 & \text{if } \operatorname{cap}(E \setminus \Omega) = 0 \\ +\infty & \text{if } \operatorname{cap}(E \setminus \Omega) > 0 \end{cases} \qquad \text{for every Borel set } E \subset \mathbb{R}^d, \tag{1.5}$$

the Sobolev space $H_0^1(\Omega)$ can be defined as

$$H_0^1(\Omega) = \left\{ u \in H^1(\mathbb{R}^d) \ : \ \int_{\mathbb{R}^d} |u|^2 \, dI_\Omega < +\infty \right\}.$$

More generally, for any capacitary measure $\mu \in \mathcal{M}^+_{\text{cap}}(\mathbb{R}^d)$, we define the space

$$H_\mu^1 = \left\{ u \in H^1(\mathbb{R}^d) \ : \ \int_{\mathbb{R}^d} |u|^2 \, d\mu < +\infty \right\},$$

which is a Hilbert space when endowed with the norm $\|u\|_{1,\mu}$, where

$$\|u\|_{1,\mu}^2 = \int_{\mathbb{R}^d} |\nabla u|^2 \, dx + \int_{\mathbb{R}^d} u^2 \, dx + \int_{\mathbb{R}^d} u^2 \, d\mu.$$

If $u \notin H_\mu^1$, then we set $\|u\|_{1,\mu} = +\infty$.

For $\Omega \subset \mathbb{R}^d$, we define $\mathcal{M}^+_{\text{cap}}(\Omega)$ as the space of capacitary measures $\mu \in \mathcal{M}^+_{\text{cap}}(\mathbb{R}^d)$ such that $\mu(E) = +\infty$ for any set $E \subset \mathbb{R}^d$ such that $\text{cap}(E \setminus \Omega) > 0$. For $\mu \in \mathcal{M}^+_{\text{cap}}(\mathbb{R}^d)$, we denote with $H_\mu^1(\Omega)$ the space $H^1_{\mu \vee I_\Omega} = H^1_\mu \cap H^1_0(\Omega)$, where $a \vee b = \max\{a, b\}$.

Definition 1.3. Given a metric space (X, d) and sequence of functionals $J_n : X \to \mathbb{R} \cup \{+\infty\}$, we say that J_n $\Gamma-$converges, in the topology provided by d, to the functional $J : X \to \mathbb{R} \cup \{+\infty\}$, if the following two conditions are satisfied:

- for every sequence x_n converging in to $x \in X$, we have

$$J(x) \leq \underline{\lim}_{n\to\infty} J_n(x_n);$$

- for every $x \in X$, there exists a sequence x_n converging to x, such that

$$J(x) = \lim_{n\to\infty} J_n(x_n).$$

For all details and properties of $\Gamma-$convergence we refer to [44]; here we simply recall that, whenever J_n $\Gamma-$converges to J,

$$\min_{x \in X} J(x) \leq \underline{\lim}_{n\to\infty} \min_{x \in X} J_n(x).$$

Definition 1.4. We say that the sequence of capacitary measures $\mu_n \in \mathcal{M}^+_{\text{cap}}(\Omega)$, $\gamma-$converges to the capacitary measure $\mu \in \mathcal{M}^+_{\text{cap}}(\Omega)$ if the sequence of functionals $\|\cdot\|_{1,\mu_n}$ $\Gamma-$converges to the functional $\|\cdot\|_{1,\mu}$ in $L^2(\Omega)$, i.e. if the following two conditions are satisfied:

- for every sequence $u_n \to u$ in $L^2(\Omega)$ we have

$$\int_{\mathbb{R}^d} |\nabla u|^2 \, dx + \int_{\mathbb{R}^d} u^2 \, d\mu \leq \underline{\lim}_{n\to\infty} \left\{ \int_{\mathbb{R}^d} |\nabla u_n|^2 \, dx + \int_{\mathbb{R}^d} u_n^2 \, d\mu_n \right\};$$

- for every $u \in L^2(\Omega)$, there exists $u_n \to u$ in $L^2(\Omega)$ such that

$$\int_{\mathbb{R}^d} |\nabla u|^2\,dx + \int_{\mathbb{R}^d} u^2\,d\mu = \lim_{n\to\infty} \left\{ \int_{\mathbb{R}^d} |\nabla u_n|^2\,dx + \int_{\mathbb{R}^d} u_n^2\,d\mu_n \right\}.$$

If $\mu \in \mathcal{M}^+_{\text{cap}}(\Omega)$ and $f \in L^2(\Omega)$ we define the functional $J_\mu(f,\cdot) : L^2(\Omega) \to \mathbb{R}\cup\{+\infty\}$ by

$$J_\mu(f,u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_\Omega u^2\,d\mu - \int_\Omega fu\,dx. \tag{1.6}$$

If $\Omega \subset \mathbb{R}^d$ is a bounded open set, $\mu \in \mathcal{M}^+_{\text{cap}}(\Omega)$ and $f \in L^2(\Omega)$, then the functional $J_\mu(f,\cdot)$ has a unique minimizer $u \in H^1_\mu$ that verifies the PDE formally written as

$$-\Delta u + \mu u = f, \qquad u \in H^1_\mu(\Omega), \tag{1.7}$$

and whose precise meaning is given in the weak form

$$\begin{cases} \displaystyle\int_\Omega \langle \nabla u, \nabla\varphi\rangle\,dx + \int_\Omega u\varphi\,d\mu = \int_\Omega f\varphi\,dx, \qquad \forall \varphi \in H^1_\mu(\Omega), \\ u \in H^1_\mu(\Omega). \end{cases}$$

The resolvent operator of $-\Delta+\mu$, that is the map $\mathcal{R}_\mu$ that associates to every $f \in L^2(\Omega)$ the solution $u \in H^1_\mu(\Omega) \subset L^2(\Omega)$, is a compact linear operator in $L^2(\Omega)$ and so, it has a discrete spectrum

$$0 < \cdots \le \Lambda_k \le \cdots \le \Lambda_2 \le \Lambda_1.$$

Their inverses $1/\Lambda_k$ are denoted by $\lambda_k(\mu)$ and are the eigenvalues of the operator $-\Delta + \mu$.

In the case $f = 1$ the solution will be denoted by w_μ and when $\mu = I_\Omega$ we will use the notation w_Ω instead of w_{I_Ω}. We also recall (see [25]) that if Ω is bounded, then the strong L^2-convergence of the minimizers w_{μ_n} to w_μ is equivalent to the $\gamma-$convergence of Definition 1.4.

Remark 1.5. An important well known characterization of the $\gamma-$convergence is the following: a sequence μ_n $\gamma-$converges to μ, if and only if, the sequence of resolvent operators $\mathcal{R}_{\mu_n}$ associated to $-\Delta + \mu_n$, converges (in the strong convergence of linear operators on L^2) to the resolvent $\mathcal{R}_\mu$ of the operator $-\Delta + \mu$. A consequence of this fact is that the spectrum of the operator $-\Delta + \mu_n$ converges (pointwise) to the one of $-\Delta + \mu$.

Remark 1.6. The space $\mathcal{M}^+_{\text{cap}}(\Omega)$ endowed with the $\gamma-$convergence is metrizable. If Ω is bounded, one may take $d_\gamma(\mu,\nu) = \|w_\mu - w_\nu\|_{L^2}$. Moreover, in this case, in [45] it is proved that the space $\mathcal{M}^+_{\text{cap}}(\Omega)$ endowed with the metric d_γ is compact.

1.3 Existence of optimal potentials in $L^p(\Omega)$

In this section we consider the optimization problem

$$\min\left\{F(V)\ :\ V:\Omega\to[0,+\infty],\ \int_\Omega V^p\,dx\le 1\right\},\tag{1.8}$$

where $p>0$ and $F(V)$ is a cost functional depending on the solution of some partial differential equation on Ω. Typically, $F(V)$ is the minimum of some functional $J_V : H_0^1(\Omega)\to\mathbb{R}$ depending on V. A natural assumption in this case is the lower semicontinuity of the functional F with respect to the $\gamma-$convergence, that is

$$F(\mu)\le \underline{\lim}_{n\to\infty}F(\mu_n),\qquad \text{whenever } \mu_n\to_\gamma \mu.\tag{1.9}$$

Proposition 1.7. *Let $\Omega\subset\mathbb{R}^d$ and let $V_n\in L^1(\Omega)$ be a sequence weakly converging in $L^1(\Omega)$ to a function V. Then the capacitary measures $V_n\,dx$ $\gamma-$converge to $V\,dx$.*

Proof. We have to prove that the solutions $u_n=R_{V_n}(1)$ of

$$\begin{cases}-\Delta u_n+V_n(x)u_n=1\\ u\in H_0^1(\Omega)\end{cases}$$

weakly converge in $H_0^1(\Omega)$ to the solution $u=R_V(1)$ of

$$\begin{cases}-\Delta u+V(x)u=1\\ u\in H_0^1(\Omega),\end{cases}$$

or equivalently that the functionals

$$J_n(u)=\int_\Omega|\nabla u|^2\,dx+\int_\Omega V_n(x)u^2\,dx$$

$\Gamma\big(L^2(\Omega)\big)$-converge to the functional

$$J(u)=\int_\Omega|\nabla u|^2\,dx+\int_\Omega V(x)u^2\,dx.$$

The $\Gamma-$liminf inequality (Definition 1.3) is immediate since, if $u_n\to u$ in $L^2(\Omega)$, we have

$$\int_\Omega|\nabla u|^2\,dx\le\underline{\lim}_{n\to\infty}\int_\Omega|\nabla u_n|^2\,dx$$

by the lower semicontinuity of the $H^1(\Omega)$ norm with respect to the $L^2(\Omega)-$convergence, and

$$\int_\Omega V(x)u^2\,dx\le\underline{\lim}_{n\to\infty}\int_\Omega V_n(x)u_n^2\,dx$$

by the strong-weak lower semicontinuity theorem for integral functionals (see for instance [27]).

Let us now prove the $\Gamma-$limsup inequality (see Definition 1.3) which consists, given $u \in H_0^1(\Omega)$, in constructing a sequence $u_n \to u$ in $L^2(\Omega)$ such that

$$\overline{\lim}_{n\to\infty} \int_\Omega |\nabla u_n|^2\,dx + \int_\Omega V_n(x)u_n^2\,dx \le \int_\Omega |\nabla u|^2\,dx + \int_\Omega V(x)u^2\,dx. \tag{1.10}$$

For every $t > 0$ let $u^t = (u \wedge t) \vee (-t)$; then, by the weak convergence of V_n, for t fixed we have

$$\lim_{n\to\infty} \int_\Omega V_n(x)|u^t|^2\,dx = \int_\Omega V(x)|u^t|^2\,dx,$$

and

$$\lim_{t\to+\infty} \int_\Omega V(x)|u^t|^2\,dx = \int_\Omega V(x)|u|^2\,dx.$$

Then, by a diagonal argument, we can find a sequence $t_n \to +\infty$ such that

$$\lim_{n\to\infty} \int_\Omega V_n(x)|u^{t_n}|^2\,dx = \int_\Omega V(x)|u|^2\,dx.$$

Taking now $u_n = u^{t_n}$, and noticing that for every $t > 0$

$$\int_\Omega |\nabla u^t|^2\,dx \le \int_\Omega |\nabla u|^2\,dx,$$

we obtain (1.10) and so the proof is complete. □

In the case of weak* convergence of measures the statement of Proposition 1.7 is no longer true, as the following proposition shows.

Proposition 1.8. *Let $\Omega \subset \mathbb{R}^d$ $(d \ge 2)$ be a bounded open set and let $V, W \in L^1_+(\Omega)$ be two functions such that $V \ge W$. Then, there is a sequence $V_n \in L^1_+(\Omega)$, uniformly bounded in $L^1(\Omega)$, such that the sequence of measures $V_n(x)\,dx$ converges weakly* to $V(x)\,dx$ and $\gamma-$converges to $W(x)\,dx$.*

Proof. Without loss of generality we can suppose $\int_\Omega (V - W)\,dx = 1$. Let μ_n be a sequence of probability measures on Ω weakly* converging to $(V - W)\,dx$ and such that each μ_n is a finite sum of Dirac masses. For each $n \in \mathbb{N}$ consider a sequence of positive functions $V_{n,m} \in L^1(\Omega)$ such that $\int_\Omega V_{n,m}\,dx = 1$ and $V_{n,m}dx$ converges weakly* to μ_n as $m \to \infty$. Moreover, we choose $V_{n,m}$ as a convex combination of functions of the form $|B_{1/m}|^{-1}\mathbf{1}_{B_{1/m}(x_j)}$. Here $B_r(x)$ is the ball of centre x and radius r, while $\mathbf{1}_A$ indicates the characteristic function of the set A.

We now prove that for fixed $n \in \mathbb{N}$, $(V_{n,m} + W)\,dx$ $\gamma-$converges, as $m \to \infty$, to $W\,dx$ or, equivalently, that the sequence $w_{W+V_{n,m}}$ converges in L^2 to w_W, as $m \to \infty$. Indeed, by the weak maximum principle, we have

$$w_{W+I_{\Omega_{m,n}}} \le w_{W+V_{n,m}} \le w_W,$$

where $\Omega_{m,n} = \Omega \setminus \cup_j B_{1/m}(x_j)$ and $I_{\Omega_{m,n}}$ is as in (1.5).

Since a point has zero capacity in $\mathbb{R}^d$ ($d \geq 2$) there exists a sequence $\phi_m \to 0$ strongly in $H^1(\mathbb{R}^d)$ with $\phi_m = 1$ on $B_{1/m}(0)$ and $\phi_m = 0$ outside $B_{1/\sqrt{m}}(0)$. We have

$$\begin{aligned}\int_\Omega |w_W - w_{W+I_{\Omega_{m,n}}}|^2\,dx &\leq 2\|w_W\|_{L^\infty}\int_\Omega (w_W - w_{W+I_{\Omega_{m,n}}})\,dx\\ &= 4\|w_W\|_{L^\infty}\big(E(W+I_{\Omega_{m,n}}) - E(W)\big) \qquad (1.11)\\ &\leq 4\|w_W\|_{L^\infty}\bigg(\int_\Omega \frac{1}{2}|\nabla w_m|^2 + \frac{1}{2}Ww_m^2 - w_m\,dx\\ &\qquad - \int_\Omega \frac{1}{2}|\nabla w_W|^2 + \frac{1}{2}Ww_W^2 - w_W\,dx\bigg),\end{aligned}$$

where w_m is any function in $\in H^1_0(\Omega_{m,n})$. Taking

$$w_m(x) = w_W(x)\prod_j (1 - \phi_m(x - x_j)),$$

since $\phi_m \to 0$ strongly in $H^1(\mathbb{R}^d)$, it is easy to see that $w_m \to w_W$ strongly in $H^1(\Omega)$ and so, by (1.11), $w_{W+I_{\Omega_{m,n}}} \to w_W$ in $L^2(\Omega)$ as $m \to \infty$. Since the weak convergence of probability measures and the $\gamma-$convergence are both induced by metrics, a diagonal sequence argument brings to the conclusion. □

We note that the hypotheses $V \geq W$ in Proposition 1.8 is necessary. Indeed, we have the following proposition, whose proof is contained in [33, Theorem 3.1] and we report it here for the sake of completeness.

Proposition 1.9. *Let $\mu_n \in \mathcal{M}^+_{cap}(\Omega)$ be a sequence of capacitary Radon measures weakly* converging to the measure ν and $\gamma-$converging to the capacitary measure $\mu \in \mathcal{M}^+_{cap}(\Omega)$. Then $\mu \leq \nu$ in Ω.*

Proof. We note that it is enough to show that $\mu(K) \leq \nu(K)$ whenever $K \subset\subset \Omega$ is a compact set. Let u be a non-negative smooth function with compact support in Ω such that $u \leq 1$ in Ω and $u = 1$ on K; we have

$$\mu(K) \leq \int_\Omega u^2\,d\mu \leq \underline{\lim}_{n\to\infty}\int_\Omega u^2\,d\mu_n = \int_\Omega u^2\,d\nu \leq \nu\left(\{u > 0\}\right).$$

Since u is arbitrary, we have the conclusion by the Borel regularity of ν. □

Theorem 1.10. *Let $F : L^1_+(\Omega) \to \mathbb{R}$ be a functional, lower semicontinuous with respect to the $\gamma-$convergence, and let $\mathcal{V}$ be a weakly $L^1(\Omega)$ compact set. Then the problem*

$$\min\left\{F(V) \ :\ V \in \mathcal{V}\right\}, \qquad (1.12)$$

admits a solution.

Proof. Let (V_n) be a minimizing sequence in $\mathcal{V}$. By the compactness assumption on $\mathcal{V}$, we may assume that V_n converges weakly $L^1(\Omega)$ to some $V \in \mathcal{V}$. By Proposition 1.7, we have that V_n $\gamma-$converges to V and so, by the semicontinuity of F,

$$F(V) \leq \underline{\lim}_{n\to\infty} F(V_n),$$

which gives the conclusion. $\square$

Remark 1.11. Theorem 1.10 applies for instance to the maximization of integral functionals and to the spectral functionals considered in the introduction, as for instance the Dirichlet eigenvalues and the Energy Function; it is not difficult to show that they are continuous with respect to the $\gamma-$convergence.

Remark 1.12. In some special cases the solution of (1.8) can be written explicitly in terms of the solution of some partial differential equation on Ω. This is the case of the Dirichlet Energy, that we shall discuss in Section 1.5, and of the first eigenvalue of the Dirichlet Laplacian λ_1 (see [75, Chapter 8]).

The compactness assumption on the admissible class $\mathcal{V}$ for the weak $L^1(\Omega)$ convergence in Theorem 1.10 is for instance satisfied if Ω has finite measure and $\mathcal{V}$ is a convex closed and bounded subset of $L^p(\Omega)$, with $p \geq 1$. In the case of measures an analogous result holds.

Theorem 1.13. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set and let $F : \mathcal{M}^+_{\text{cap}}(\Omega) \to \mathbb{R}$ be a functional lower semicontinuous with respect to the $\gamma-$convergence. Then the problem*

$$\min\left\{F(\mu)\ :\ \mu \in \mathcal{M}^+_{\text{cap}}(\Omega),\ \mu(\Omega) \leq 1\right\}, \tag{1.13}$$

admits a solution.

Proof. Let (μ_n) be a minimizing sequence. Then, up to a subsequence μ_n converges weakly* to some measure ν and $\gamma-$converges to some measure $\mu \in \mathcal{M}^+_{\text{cap}}(\Omega)$. By Proposition 1.9, we have that $\mu(\Omega) \leq \nu(\Omega) \leq 1$ and so μ is a solution of (1.13). $\square$

The following example shows that the optimal solution of problem (1.13) is not, in general, a function $V(x)$, even when the optimization criterion is the energy $\mathcal{E}_f$ introduced in (1.3). On the other hand, an explicit form for the optimal potential $V(x)$ will be provided in Proposition 1.21 and Proposition 1.24 assuming that f is in $L^2(\Omega)$.

Example 1.14. Let $\Omega = (-1, 1)$ and consider the functional

$$F(\mu) = -\min\left\{\frac{1}{2}\int_\Omega |u'|^2\,dx + \frac{1}{2}\int_\Omega u^2\,d\mu - u(0)\ :\ u \in H^1_0(\Omega)\right\}.$$

Then, for any μ such that $\mu(\Omega) \leq 1$, we have

$$F(\mu) \geq -\min\left\{\frac{1}{2}\int_\Omega |u'|^2\,dx + \frac{1}{2}\big(\sup_\Omega u\big)^2 - u(0)\ :\ u \in H^1_0(\Omega),\ u \geq 0\right\}. \tag{1.14}$$

By a symmetrization argument, the minimizer u of the right-hand side of (1.14) is radially decreasing; moreover, u is linear on the set $u < M$, where $M = \sup u$, and so it is of the form

$$u(x) = \begin{cases} \frac{M}{1-\alpha}x + \frac{M}{1-\alpha}, & x \in [-1, -\alpha], \\ M, & x \in [-\alpha, \alpha], \\ -\frac{M}{1-\alpha}x + \frac{M}{1-\alpha}, & x \in [\alpha, 1], \end{cases} \tag{1.15}$$

for some $\alpha \in [0, 1]$. A straightforward computation gives $\alpha = 0$ and $M = 1/3$. Thus, u is also the minimizer of

$$F(\delta_0) = -\min\left\{\frac{1}{2}\int_\Omega |u'|^2\,dx + \frac{1}{2}u(0)^2 - u(0) \ : \ u \in H^1_0(\Omega)\right\},$$

and so δ_0 is the solution of

$$\min\left\{F(\mu) \ : \ \mu(\Omega) \le 1\right\}.$$

1.4 Existence of optimal potentials for unbounded constraints

In this section we consider the optimization problem

$$\min\left\{F(V) \ : \ V \in \mathcal{V}\right\}, \tag{1.16}$$

where $\mathcal{V}$ is an admissible class of non-negative Borel functions on the bounded open set $\Omega \subset \mathbb{R}^d$ and F is a cost functional on the family of capacitary measures $\mathcal{M}^+_{\text{cap}}(\Omega)$. The admissible classes we study depend on a function $\Psi : [0, +\infty] \to [0, +\infty]$

$$\mathcal{V} = \left\{V : \Omega \to [0, +\infty] \ : \ V \text{ Lebesgue measurable}, \ \int_\Omega \Psi(V)\,dx \le 1\right\}.$$

Before we state the main existence result of this section, we need the following lemma which is a particular case of [27, Theorem 2.3.1].

Lemma 1.15. *Let $1 < p, q < \infty$ and let $u_n \in L^p(\Omega)$ and $v_n \in L^q(\Omega)$ be two sequences of positive functions on the open set $\Omega \subset \mathbb{R}^d$ such that u_n converges strongly in L^p to $u \in L^p(\Omega)$ and v_n converges weakly in L^q to $v \in L^q(\Omega)$. Suppose that $H : [0, +\infty] \to [0, +\infty]$ is a convex function. Then we have*

$$\int_\Omega uH(v)\,dx \le \varliminf_{n\to\infty} \int_\Omega u_n H(v_n)\,dx.$$

Theorem 1.16. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set and $\Psi : [0, +\infty] \to [0, +\infty]$ a strictly decreasing function such that there exists $\varepsilon > 0$ for which the function $x \to \Psi^{-1}(x^{1+\varepsilon})$, defined on $[0, +\infty]$, is convex. Then, for any increasing functional $F : \mathcal{M}^+_{\text{cap}}(\Omega) \to \mathbb{R}$ which is lower semicontinuous with respect to the $\gamma-$convergence, the problem (1.16) has a solution.*

Proof. Let $V_n \in \mathcal{V}$ be a minimizing sequence for problem (1.16). Then, $v_n := \Psi(V_n)^{\frac{1}{1+\varepsilon}}$ is a bounded sequence in $L^{1+\varepsilon}(\Omega)$ and so, up to a subsequence, we have that v_n converges weakly in $L^{1+\varepsilon}$ to some $v \in L^{1+\varepsilon}(\Omega)$. We will prove that $V := \Psi^{-1}(v^{1+\varepsilon})$ is a solution of (1.16). Clearly $V \in \mathcal{V}$ and so it remains to prove that $F(V) \leq \underline{\lim} F(V_n)$. In view of Remark 1.6, we can suppose that, up to a subsequence, V_n $\gamma-$converges to a capacitary measure $\mu \in \mathcal{M}^+_{\text{cap}}(\Omega)$. We claim that the following inequalities hold true:

$$F(V) \leq F(\nu) \leq \underline{\lim}_{n\to\infty} F(V_n). \tag{1.17}$$

In fact, the second inequality in (1.17) is the lower semicontinuity of F with respect to the $\gamma-$convergence, while the first needs a more careful examination. By the definition of $\gamma-$convergence, we have that for any $u \in H^1_0(\Omega)$, there is a sequence $u_n \in H^1_0(\Omega)$ which converges to u strongly in $L^2(\Omega)$ and is such that

$$\begin{aligned}
\int_\Omega |\nabla u|^2 dx + \int_\Omega u^2 d\mu &= \lim_{n\to\infty} \int_\Omega |\nabla u_n|^2 dx + \int_\Omega u_n^2 V_n\, dx \\
&= \lim_{n\to\infty} \int_\Omega |\nabla u_n|^2 dx + \int_\Omega u_n^2 \Psi^{-1}(v_n^{1+\varepsilon})\, dx \\
&\geq \int_\Omega |\nabla u|^2 dx + \int_\Omega u^2 \Psi^{-1}(v^{1+\varepsilon})\, dx \\
&= \int_\Omega |\nabla u|^2 dx + \int_\Omega u^2 V\, dx,
\end{aligned} \tag{1.18}$$

where the inequality in (1.18) is due to Lemma 1.15. Thus, for any $u \in H^1_0(\Omega)$, we have that

$$\int_\Omega u^2\, d\mu \geq \int_\Omega u^2 V\, dx,$$

and so, $V \leq \mu$. Since F is increasing, we obtain the first inequality in (1.17) which concludes the proof. □

Remark 1.17. The condition on the function Ψ in Theorem 1.16 is satisfied for instance by the following functions:

1. $\Psi(x) = x^{-p}$, for any $p > 0$;

2. $\Psi(x) = e^{-\alpha x}$, for any $\alpha > 0$.

1.4.1 Optimal potentials for the Dirichlet Energy and the first eigenvalue of the Dirichlet Laplacian

In the particular cases $F = \lambda_1$ and when $F = \mathcal{E}_f$, with $f \in L^2(\Omega)$ the solution of problem (1.16) can be expressed as a *double minimization* of a functional, where the first minimization is on $H^1_0(\Omega)$ and the second on $\mathcal{V}$. We note that, by the variational formulation

$$\lambda_1(V) = \min\left\{\int_\Omega |\nabla u|^2 dx + \int_\Omega u^2 V\,dx \ :\ u \in H^1_0(\Omega),\ \int_\Omega u^2\,dx = 1\right\}, \tag{1.19}$$

we can rewrite problem (1.16) as

$$\begin{aligned}\min\left\{\min_{\|u\|_2=1}\left\{\int_\Omega |\nabla u|^2\,dx + \int_\Omega u^2 V\,dx\right\}\ :\ V \geq 0,\ \int_\Omega \Psi(V)\,dx \leq 1\right\}\\ = \min_{\|u\|_2=1}\left\{\min\left\{\int_\Omega |\nabla u|^2\,dx + \int_\Omega u^2 V\,dx\ :\ V\geq 0,\ \int_\Omega \Psi(V)\,dx \leq 1\right\}\right\}.\end{aligned} \tag{1.20}$$

By computing the Euler-Lagrange equation for V with the constraint $\mathcal{V}$), and if Ψ is differentiable with Ψ' invertible, we get that the second minimum in (1.20), for a generic $u \in H^1_0(\Omega)$ is achieved for

$$V = (\Psi')^{-1}(\Lambda_u u^2), \tag{1.21}$$

where Λ_u is a constant such that $\int_\Omega \Psi\left((\Psi')^{-1}(\Lambda_u u^2)\right)\,dx = 1$. Thus, the solution of the problem on the right hand side of (1.20) is given through the solution of

$$\min\left\{\int_\Omega |\nabla u|^2\,dx + \int_\Omega u^2(\Psi')^{-1}(\Lambda_u u^2)\,dx :\ u \in H^1_0(\Omega),\ \int_\Omega u^2\,dx = 1\right\}. \tag{1.22}$$

Analogously, we obtain that the optimal potential for the Dirichlet Energy $\mathcal{E}_f$ is given by (1.21), where this time u is a solution of

$$\min\left\{\int_\Omega \frac{1}{2}|\nabla u|^2\,dx + \int_\Omega \frac{1}{2}u^2(\Psi')^{-1}(\Lambda_u u^2)\,dx - \int_\Omega fu\,dx\ :\ u \in H^1_0(\Omega)\right\}. \tag{1.23}$$

Thus we obtain the following result.

Corollary 1.18. *Under the assumptions of Theorem 1.16, for the functionals $F = \lambda_1$ and $F = \mathcal{E}_f$ there exists a solution of (1.16) given by $V = (\Psi')^{-1}(\Lambda_u u^2)$, where $u \in H^1_0(\Omega)$ is a minimizer of (1.22), in the case $F = \lambda_1$, and of (1.23), in the case $F = \mathcal{E}_f$.*

Example 1.19. If $\Psi(x) = x^{-p}$ with $p > 0$, the optimal potentials for λ_1 and $\mathcal{E}_f$ are given by

$$V = \left(\int_\Omega |u|^{2p/(p+1)}\,dx\right)^{1/p} u^{-2/(p+1)}, \tag{1.24}$$

where u is the minimizer of (1.22) and (1.23), respectively. We also note that, in this case

$$\int_\Omega u^2(\Psi')^{-1}(\Lambda_u u^2)\,dx = \left(\int_\Omega |u|^{2p/(p+1)}\,dx\right)^{(1+p)/p}.$$

Example 1.20. If $\Psi(x) = e^{-\alpha x}$ with $\alpha > 0$, the optimal potentials for λ_1 and $\mathcal{E}_f$ are given by

$$V = \frac{1}{\alpha}\left(\log\left(\int_\Omega u^2\,dx\right) - \log\left(u^2\right)\right), \tag{1.25}$$

where u is the minimizer of (1.22) and (1.23), respectively. We also note that, in this case

$$\int_\Omega u^2(\Psi')^{-1}(\Lambda_u u^2)\,dx = \frac{1}{\alpha}\left(\int_\Omega u^2\,dx \int_\Omega \log\left(u^2\right)\,dx - \int_\Omega u^2\log\left(u^2\right)\,dx\right).$$

1.5 Maximization problems in L^p concerning the Dirichlet Energy functional

In this section we study the a particular case of Theorem 1.10 where the functional F is the energy function (with a minus sign) $-\mathcal{E}_f$ and $\Omega \subset \mathbb{R}^d$ is an open bounded set and $f \in L^2(\Omega)$. The natural constraint used for similar problems is, in analogy with the problem related to the classical Dirichlet energy of a set, is *volume* constraint. More precisely we are going to study the class of problem, depending on the parameter p,

$$\max\left\{\mathcal{E}_f(V) : V \geq 0,\ \int_\Omega V^p \leq 1\right\}. \tag{1.26}$$

To link this problem to Theorem 1.10 we only have to observe that the $\mathcal{E}_f$ is continuous with respect to the $\gamma-$convergence, and that we can always formulate a maximization problem on $\mathbb{R}$ as a minimization problem, simply changing the sign of the functional we wish to optimize. We recall that replacing max by min, makes problem (1.26) trivial, with the only solution $V \equiv 0$. Analogous results for $F(V) = -\lambda_1(V)$ has been treated in a series of papers. We cite for instance [74] and [5] in addition to the survey [75, Chapter 8]. We start by considering the case $p > 1$.

Proposition 1.21. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, $1 < p < \infty$ and $f \in L^2(\Omega)$. Then the problem (1.26) has the unique solution*

$$V_p = \left(\int_\Omega |u_p|^{2p/(p-1)}\,dx\right)^{-1/p} |u_p|^{-1+(p+1)/(p-1)},$$

where $u_p \in H^1_0(\Omega) \cap L^{2p/(p-1)}(\Omega)$ is the minimizer of the functional

$$J_p(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\left(\int_\Omega |u|^{2p/(p-1)}\,dx\right)^{(p-1)/p} - \int_\Omega uf\,dx. \tag{1.27}$$

Moreover, we have $\mathcal{E}_f(V_p) = J_p(u_p)$.

Proof. To fix the notation let us define

$$\mathcal{V}_p = \left\{ V \geq 0 : \int_\Omega V^p \leq 1 \right\}.$$

We start noticing the trivial inequality

$$\max_{V\in\mathcal{V}_p} \min_{u\in H_0^1(\Omega)} \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + u^2 V - uf \right) dx \leq \min_{u\in H_0^1(\Omega)} \max_{V\in\mathcal{V}_p} \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + u^2 V - uf \right) dx. \tag{1.28}$$

Notice that by the Hölder inequality of parameter (p, p') applied on the integral of $u^2 V$, the maximum on the right hand side of (1.28) is finite. Moreover it is achieved by a function V such that $\Lambda p V^{p-1} = u^2$, where Λ is a suitable Lagrange multiplier. By the condition $\int_\Omega V^p \, dx = 1$ we precisely gets

$$V = \left(\int_\Omega |u|^{\frac{2p}{p-1}} \, dx \right)^{1/p} |u|^{\frac{2}{p-1}}.$$

Substituting in (1.28), we obtain

$$\max \{ \mathcal{E}_f(V) \ : \ V \in \mathcal{V} \} \leq \min \left\{ J_p(u) : \ u \in H_0^1(\Omega) \right\}. \tag{1.29}$$

Let u_n be a minimizing sequence for J_p. Since $\inf J_p \leq 0$, we can assume $J_p(u_n) \leq 0$ for each $n \in \mathbb{N}$. Thus, we have

$$\frac{1}{2} \int_\Omega |\nabla u_n|^2 \, dx + \frac{1}{2} \left(\int_\Omega |u_n|^{2p/(p-1)} \, dx \right)^{(p-1)/p} \leq \int_\Omega u_n f \, dx \leq C \|f\|_{L^2(\Omega)} \|\nabla u_n\|_{L^2}, \tag{1.30}$$

where C is a constant depending on Ω. Thus we obtain

$$\int_\Omega |\nabla u_n|^2 \, dx + \left(\int_\Omega |u_n|^{2p/(p-1)} \, dx \right)^{(p-1)/p} \leq 4C^2 \|f\|^2_{L^2(\Omega)}, \tag{1.31}$$

and so, up to subsequence u_n converges weakly in $H_0^1(\Omega)$ and $L^{2p/(p-1)}(\Omega)$ to some $u_p \in H_0^1(\Omega) \cap L^{2p/(p-1)}(\Omega)$. By the semicontinuity of the L^2-norm of the gradient and the $L^{\frac{2p}{p-1}}-$norm and the fact that $\int_\Omega f u_n \, dx \to \int_\Omega f u_p \, dx$, as $n \to \infty$, we have that u_p is a minimizer of J_p. By the strict convexity of J_p, we have that u_p is unique. Moreover, by (1.30) and (1.31), $J_p(u_p) > -\infty$. Writing down the Euler-Lagrange equation for u_p, we obtain

$$-\Delta u_p + \left(\int_\Omega |u_p|^{2p/(p-1)} \, dx \right)^{-1/p} |u_p|^{2/(p-1)} u_p = f.$$

Setting

$$V_p = \left(\int_\Omega |u_p|^{2p/(p-1)} \, dx \right)^{-1/p} |u_p|^{2/(p-1)},$$

we have that $\int_\Omega V_p^p\,dx = 1$ and u_p is the solution of

$$-\Delta u_p + V_p u_p = f. \tag{1.32}$$

In particular, we have $J_p(u_p) = \mathcal{E}_p(V_p)$ and so V_p solves (1.26). The uniqueness of V_p follows by the uniqueness of u_p and the characterization of the equality cases of the Hölder inequality

$$\int_\Omega u^2 V\,dx \le \left(\int_\Omega V^p\,dx\right)^{1/p}\left(\int_\Omega |u|^{2p/(p-1)}\,dx\right)^{(p-1)/p} \le \left(\int_\Omega |u|^{2p/(p-1)}\,dx\right)^{(p-1)/p}.$$

□

When the functional F is the energy $\mathcal{E}_f$, the existence result holds also in the case $p=1$, but its proof is a bit more complicated and is given in Proposition 1.24. It worth noticing the peculiar *shape* of the minimizer in this case given by (1.40). Before proving Proposition 1.24, we need some preliminary results. We also recall that analogous results has been obtained in the case $F=\lambda_1$ (see [75, Theorem 8.2.4]) and in the case $F=\mathcal{E}_f$, with the further request that $f\ge 0$. See [33].

Remark 1.22. Let u_p be the minimizer of J_p, defined in (1.27). By (1.31), we have the estimate

$$\|\nabla u_p\|_{L^2(\Omega)} + \|u_p\|_{L^{2p/(p-1)}(\Omega)} \le 4C^2\|f\|_{L^2(\Omega)}, \tag{1.33}$$

where C is the constant from (1.30). Moreover, we have $u_p\in H^2_{loc}(\Omega)$ and for each open set $\Omega'\subset\subset\Omega$, there is a constant C not depending on p such that

$$\|u_p\|_{H^2(\Omega')} \le C(f,\Omega').$$

Indeed, u_p satisfies the PDE

$$-\Delta u + c|u|^\alpha u = f, \tag{1.34}$$

with $c>0$ and $\alpha = 2/(p-1)$, and standard elliptic regularity arguments (see [56, Section 6.3]) give that $u\in H^2_{loc}(\Omega)$. To show that $\|u_p\|_{H^2(\Omega')}$ is bounded independently of p we apply the Nirenberg operator $\partial_k^h u = \frac{u(x+he_k)-u(x)}{h}$ on both sides of (1.34), and multiplying by $\phi^2\partial_k^h u$, where ϕ is an appropriate cut-off function which equals 1 on Ω', we have

$$\begin{aligned}\int_\Omega \phi^2|\nabla\partial_k^h u|^2\,dx + \int_\Omega \langle\nabla(\partial_k^h u),\nabla(\phi^2)\partial_k^h u\rangle\,dx + c(\alpha+1)\int_\Omega \phi^2|u|^\alpha|\partial_k^h u|^2\,dx \\ = -\int f\partial_k^h(\phi^2\partial_k^h u)\,dx,\end{aligned} \tag{1.35}$$

for all $k=1,\dots,d$. Some straightforward manipulations now give

$$\|\nabla^2 u\|^2_{L^2(\Omega')} \le \sum_{k=1}^d \int_\Omega \phi^2|\nabla\partial_k u|^2\,dx \le C(\Omega')\left(\|f\|_{L^2(\{\phi^2>0\})} + \|\nabla u\|_{L^2(\Omega)}\right). \tag{1.36}$$

Lemma 1.23. *Let $\Omega \subset \mathbb{R}^d$ be an open set and $f \in L^2(\Omega)$. Consider the functional $J_1 : L^2(\Omega) \to \mathbb{R}$ defined by*

$$J_1(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\|u\|_\infty^2 - \int_\Omega uf\,dx, \tag{1.37}$$

Then, J_p $\Gamma-$converges in $L^2(\Omega)$ to J_1, as $p \to 1$, where J_p is defined in (1.27).

Proof. Let $v_n \in L^2(\Omega)$ be a sequence of positive functions converging in L^2 to $v \in L^2(\Omega)$ and let $\alpha_n \to +\infty$. Then, we have that

$$\|v\|_{L^\infty(\Omega)} \le \underline{\lim}_{n\to\infty}\|v_n\|_{L^{\alpha_n}(\Omega)}. \tag{1.38}$$

In fact, suppose first that $\|v\|_{L^\infty} = M < +\infty$ and let $\omega_\varepsilon = \{v > M - \varepsilon\}$, for any $\varepsilon > 0$. Then, we have

$$\underline{\lim}_{n\to\infty}\|v_n\|_{L^{\alpha_n}(\Omega)} \ge \lim_{n\to\infty}|\omega_\varepsilon|^{(1-\alpha_n)/\alpha_n}\int_{\omega_\varepsilon} v_n\,dx = |\omega_\varepsilon|^{-1}\int_{\omega_\varepsilon} v\,dx \ge M - \varepsilon,$$

and so, letting $\varepsilon \to 0$, we have $\underline{\lim}_{n\to\infty}\|v_n\|_{L^{\alpha_n}(\Omega)} \le M$. If $\|v\|_{L^\infty(\Omega)} = +\infty$, then setting $\omega_k = \{v > k\}$, for any $k \ge 1$, and arguing as above, we obtain (1.38).
Let $u_n \to u$ in $L^2(\Omega)$. Then, by the semicontinuity of the L^2 norm of the gradient and (1.38) and the continuity of the term $\int_\Omega uf\,dx$, we have

$$J_1(u) \le \underline{\lim}_{n\to\infty}J_{p_n}(u_n), \tag{1.39}$$

for any decreasing sequence $p_n \to 1$. On the other hand, for any $u \in L^2$, we have $J_{p_n}(u) \to J_1(u)$ as $n \to \infty$ and so we have the conclusion. $\square$

Proposition 1.24. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set and $f \in L^2(\Omega)$. Then there is a unique solution of problem (1.26) with $p = 1$, given by*

$$V_1 = \frac{1}{M}\left(\mathbf{1}_{\omega_+}f - \mathbf{1}_{\omega_-}f\right), \tag{1.40}$$

where $M = \|u_1\|_{L^\infty(\Omega)}$, $\omega_+ = \{u_1 = M\}$, $\omega_- = \{u_1 = -M\}$, being $u_1 \in H_0^1(\Omega)\cap L^\infty(\Omega)$ the unique minimizer of the functional J_1, defined in (1.37). In particular, $\int_{\omega_+} f\,dx - \int_{\omega_-} f\,dx = M$, $f \ge 0$ on ω_+ and $f \le 0$ on ω_-.

Proof. For any $u \in H_0^1(\Omega)$ and any $V \ge 0$ with $\int_\Omega V\,dx \le 1$ we have

$$\int_\Omega u^2 V\,dx \le \|u\|_\infty^2\int_\Omega V\,dx \le \|u\|_\infty^2,$$

where for the sake of simplicity, we write $\|\cdot\|_\infty$ instead of $\|\cdot\|_{L^\infty(\Omega)}$. Arguing as in the proof of Proposition 1.21, we obtain the inequalities

$$\frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_\Omega u^2 V\,dx - \int_\Omega uf\,dx \le J_1(u),$$
$$\max\left\{\mathcal{E}_f(V)\ :\ \int_\Omega V \le 1\right\} \le \min\left\{J_1(u)\ :\ u \in H_0^1(\Omega)\right\}.$$

As in (1.30), we have that a minimizing sequence of J_1 is bounded in $H_0^1(\Omega)\cap L^\infty(\Omega)$ and thus by semicontinuity there is a minimizer $u_1 \in H_0^1(\Omega)\cap L^\infty(\Omega)$ of J_1, which is also unique, by the strict convexity of J_1. Let u_p denotes the minimizer of J_p as in Proposition 1.21. Then, by Remark 1.22, we have that the family u_p is bounded in $H_0^1(\Omega)$ and in $H^2(\Omega')$ for each $\Omega' \subset\subset \Omega$. Then, we have that each sequence u_{p_n} has a subsequence converging weakly in $L^2(\Omega)$ to some $u \in H^2_{loc}(\Omega)\cap H_0^1(\Omega)$. By Lemma 1.23, we have $u = u_1$ and so, $u_1 \in H^2_{loc}(\Omega)\cap H_0^1(\Omega)$. Thus $u_{p_n} \to u_1$ in $L^2(\Omega)$.

Let us define $M = \|u_1\|_\infty$ and $\omega = \omega_+ \cup \omega_-$. We claim that u_1 satisfies, on Ω the PDE

$$-\Delta u + \mathbf{1}_\omega f = f. \tag{1.41}$$

Indeed, setting $\Omega_t = \Omega \cap \{|u| < t\}$ for $t > 0$, we compute the variation of J_1 with respect to any function $\varphi \in H_0^1(\Omega_{M-\varepsilon})$. Namely we consider functions of the form $\varphi = \psi w_\varepsilon$ where w_ε is the solution of $-\Delta w_\varepsilon = 1$ on $\Omega_{M-\varepsilon}$, and $w_\varepsilon = 0$ on $\partial\Omega_{M-\varepsilon}$. Thus we obtain that $-\Delta u_1 = f$ on $\Omega_{M-\varepsilon}$ and letting $\varepsilon \to 0$ we conclude, thanks to the Monotone Convergence Theorem, that

$$-\Delta u_1 = f \qquad \text{on } \Omega_M = \Omega \setminus \omega.$$

Moreover, since $u_1 \in H^2_{loc}(\Omega)$, we have that $\Delta u_1 = 0$ on ω and so, we obtain (1.41). Since u_1 is the minimizer of J_1, we have that for each $\varepsilon \in \mathbb{R}$, $J_1((1+\varepsilon)u_1) - J_1(u_1) \geq 0$. Taking the derivative of this difference at $\varepsilon = 0$, we obtain

$$\int_\Omega |\nabla u_1|^2\,dx + M^2 = \int_\Omega f u_1\,dx. \tag{1.42}$$

By (1.41), we have $\int_\Omega |\nabla u_1|^2\,dx = \int_{\Omega\setminus\omega} f u_1\,dx$ and so

$$M = \int_{\omega_+} f\,dx - \int_{\omega_-} f\,dx. \tag{1.43}$$

Setting $V_1 := \frac{1}{M}\left(\mathbf{1}_{\omega_+} f - \mathbf{1}_{\omega_-} f\right)$, we have that $\int_\Omega V_1\,dx = 1$, $-\Delta u_1 + V_1 u_1 = f$ in $H^{-1}(\Omega)$ and

$$J_1(u_1) = \frac{1}{2}\int_\Omega |\nabla u_1|^2\,dx + \frac{1}{2}\int_\Omega u_1^2 V_1\,dx - \int_\Omega u_1 f\,dx.$$

We are left to prove that V_1 is admissible, i.e. $V_1 \geq 0$. To do this, consider w_ε the Energy function of the quasi-open set $\{u < M-\varepsilon\}$ and let $\varphi = w_\varepsilon \psi$ where $\psi \in C_c^\infty(\mathbb{R}^d)$, $\psi \geq 0$. Since $\varphi \geq 0$, we get that

$$0 \leq \lim_{t\to 0^+} \frac{J_1(u_1 + t\varphi) - J_1(u_1)}{t} = \int_\Omega \langle \nabla u_1, \nabla\varphi\rangle\,dx - \int_\Omega f\varphi\,dx.$$

This inequality holds for any ψ so that, integrating by parts, we obtain

$$-\Delta u_1 - f \geq 0$$

almost everywhere on $\{u_1 < M - \varepsilon\}$. In particular, since $\Delta u_1 = 0$ almost everywhere on $\omega_- = \{u = -M\}$, we obtain that $f \leq 0$ on ω_-. Arguing in the same way, and considering test functions supported on $\{u_1 \geq -M + \varepsilon\}$, we can prove that $f \geq 0$ on ω_+. This implies $V_1 \geq 0$ as required. $\square$

Remark 1.25. Under some additional assumptions on Ω and f one can obtain some more precise regularity results for u_1. In fact, in [55, Theorem A1] it was proved that if $\partial\Omega \in C^2$ and if $f \in L^\infty(\Omega)$ is positive, then $u_1 \in C^{1,1}(\overline{\Omega})$.

Remark 1.26. In the case $p < 1$ problem (1.26) does not admit, in general, a solution, even for regular f and Ω. We give a counterexample in dimension one, which can be easily adapted to higher dimensions.

Let $\Omega = (0,1)$, $f = 1$, and let $x_{n,k} = k/n$ for any $n \in \mathbb{N}$ and $k = 1, \ldots, n-1$. We define the (capacitary) measures

$$\mu_n = \sum_{k=1}^{n-1} +\infty\, \delta_{x_{n,k}},$$

where δ_x is the Dirac measure at the point x. Let w_n be the minimizer of the functional $J_{\mu_n}(1, \cdot)$, defined in (1.6). Then w_n vanishes at $x_{n,k}$, for $k = 1, \ldots, n-1$, and so we have

$$\mathcal{E}(\mu_n) = n \min\left\{ \frac{1}{2}\int_0^{1/n} |u'|^2\, dx - \int_0^{1/n} u\, dx \ :\ u \in H_0^1(0, 1/n) \right\} = -\frac{C}{n^2},$$

where $C > 0$ is a constant.

For any fixed n and j, let V_j^n be the sequence of positive functions such that $\int_0^1 |V_j^n|^p\, dx = 1$, defined by

$$V_j^n = C_n \sum_{k=1}^{n-1} j^{1/p} \mathbf{1}_{\left[\frac{k}{n}-\frac{1}{j}, \frac{k}{n}+\frac{1}{j}\right]} < \sum_{k=1}^{n-1} I_{\left[\frac{k}{n}-\frac{1}{j}, \frac{k}{n}+\frac{1}{j}\right]}, \tag{1.44}$$

where C_n is a constant depending on n and I is as in (1.5). By the compactness of the $\gamma-$convergence, we have that, up to a subsequence, $V_j^n\, dx$ $\gamma-$converges to some capacitary measure μ as $j \to \infty$. On the other hand it is easy to check that $\sum_{k=1}^{n-1} I_{\left[\frac{k}{n}-\frac{1}{j}, \frac{k}{n}+\frac{1}{j}\right]}(x)$ $\gamma-$converges to μ_n as $j \to \infty$. By (1.44), we have that $\mu \leq \mu_n$. In order to show that $\mu = \mu_n$ it is enough to check that each non-negative function $u \in H_0^1((0,1))$, for which $\int u^2\, d\mu < +\infty$, vanishes at $x_{n,k}$ for $k = 1, \ldots, n-1$. Suppose that $u(k/n) > 0$. By the definition of the $\gamma-$convergence, there is a sequence $u_j \in H_0^1(\Omega) = H^1_{V_j^n}(\Omega)$ such that $u_j \to u$ weakly in $H_0^1(\Omega)$ and $\int u_j^2 V_j^n\, dx \leq C$, for some constant C not depending on $j \in \mathbb{N}$. Since u_j are uniformly $1/2-$Hölder continuous, we can suppose that $u_j \geq \varepsilon > 0$ on some interval I containing k/n. But then for

j large enough I contains $[k/n - 1/j, k/n + 1/j]$ so that

$$C \geq \int_0^1 u_j^2 V_j^n \, dx \geq \int_{k/n-1/j}^{k/n+1/j} u_j^2 V_j^n \, dx \geq 2C_n \varepsilon^2 j^{1/p-1},$$

which is a contradiction for $p < 1$. Thus, we have that $\mu = \mu_n$ and so V_j^n $\gamma-$converges to μ_n as $j \to \infty$. In particular, $\mathcal{E}(\mu_n) = \lim_{j\to\infty} \mathcal{E}_1(V_j^n)$ and since the left-hand side converges to zero as $n \to \infty$, we can choose a diagonal sequence $V_{j_n}^n$ such that $\mathcal{E}(V_{j_n}^n) \to 0$ as $n \to \infty$. Since there is no admissible functional V such that $\mathcal{E}_1(V) = 0$, we have the conclusion.

1.6 Optimization problems in unbounded domains

In this section we consider optimization problems for which the domain region is the entire Euclidean space $\mathbb{R}^d$. General existence results, in the case when the design region Ω is unbounded, are hard to achieve since most of the cost functionals are not semicontinuous with respect to the $\gamma-$convergence in these domains. For example, it is not hard to check that if μ is a capacitary measure, infinite outside the unit ball B_1, then, for every sequence $(x_n)_n$ such that $|x_n| \to \infty$, the sequence of translated measures $\mu_n = \mu(\cdot + x_n)$ $\gamma-$converges to the capacitary measure

$$I_\emptyset(E) = \begin{cases} 0, & \text{if } \mathrm{cap}(E) = 0, \\ +\infty, & \text{if } \mathrm{cap}(E) > 0. \end{cases}$$

Thus increasing and translation invariant functionals are never lower semicontinuous with respect to the $\gamma-$convergence. In some special cases, as the Dirichlet Energy or the first eigenvalue of the Dirichlet Laplacian, one can obtain existence results by more direct methods, as those in Proposition 1.21.

For a potential $V \geq 0$ and a function $f \in L^q(\mathbb{R}^d)$, we define the Dirichlet Energy as

$$\mathcal{E}_f(V) = \inf \left\{ \int_{\mathbb{R}^d} \Big(\frac{1}{2}|\nabla u|^2 + \frac{1}{2} V(x) u^2 - f(x) u \Big) \, dx \ : \ u \in C_c^\infty(\mathbb{R}^d) \right\}. \tag{1.45}$$

In some cases it is convenient to work with the space $\dot{H}^1(\mathbb{R}^d)$, obtained as the closure of $C_c^\infty(\mathbb{R}^d)$ with respect to the L^2 norm of the gradient, instead of the classical Sobolev space $H^1(\mathbb{R}^d)$. From now on, with the scope of lightening the notation, and since there is no risk of confusion, we will write $\|\cdot\|_p$ in place of $\|\cdot\|_{L^p(\mathbb{R}^d)}$ to indicate the L^p-norm of a function on $\mathbb{R}^d$. We recall that if $d \geq 3$, the Sobolev inequality

$$\|u\|_{2d/(d-2)} \leq C_d \|\nabla u\|_2, \qquad \forall u \in \dot{H}^1(\mathbb{R}^d), \tag{1.46}$$

holds, while in the cases $d \leq 2$, we have respectively (see for instance [86])

$$\|u\|_\infty \leq \left(\frac{r+2}{2}\right)^{2/(r+2)} \|u\|_r^{r/(r+2)} \|u'\|_2^{2/(r+2)}, \qquad \forall r \geq 1,\ \forall u \in \dot{H}^1(\mathbb{R}); \tag{1.47}$$

$$\|u\|_{r+2} \leq \left(\frac{r+2}{2}\right)^{2/(r+2)} \|u\|_r^{r/(r+2)} \|\nabla u\|_2^{2/(r+2)}, \qquad \forall r \geq 1,\ \forall u \in \dot{H}^1(\mathbb{R}^2). \tag{1.48}$$

1.6.1 Optimal potentials in $L^p(\mathbb{R}^d)$

In this section we consider optimization problems for the Dirichlet Energy $\mathcal{E}_f$ among potentials $V \geq 0$ satisfying a constraint of the form $\|V\|_p \leq 1$. We note that the results contained in this section hold in a generic regular unbounded domain Ω. Nevertheless, for the sake of simplicity, we restrict our attention to the case $\Omega = \mathbb{R}^d$.

Proposition 1.27. *Let $p > 1$ and let q be in the interval with end-points $a = 2p/(p+1)$ and $b = \max\{1, 2d/(d+2)\}$ (with a included for every $d \geq 1$, and b included for every $d \neq 2$). Then, for every $f \in L^q(\mathbb{R}^d)$, there is a unique solution of the problem*

$$\max\left\{\mathcal{E}_f(V)\ :\ V \geq 0,\ \int_{\mathbb{R}^d} V^p\,dx \leq 1\right\}. \tag{1.49}$$

Proof. Arguing as in Proposition 1.21, we have that for $p > 1$ the optimal potential V_p is given by

$$V_p = \left(\int_{\mathbb{R}^d} |u_p|^{2p/(p-1)}\,dx\right)^{-1/p} |u_p|^{2/(p-1)}, \tag{1.50}$$

where u_p is the solution of the problem

$$\min\left\{\frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2\,dx + \frac{1}{2}\left(\int_{\mathbb{R}^d} |u|^{2p/(p-1)}\,dx\right)^{(p-1)/p} - \int_{\mathbb{R}^d} uf\,dx\ :\ u \in \dot{H}^1(\mathbb{R}^d) \cap L^{2p/(p-1)}(\mathbb{R}^d)\right\}. \tag{1.51}$$

Thus, it is enough to prove that there exists a solution of (1.51). For a minimizing sequence u_n we have

$$\frac{1}{2}\int_{\mathbb{R}^d} |\nabla u_n|^2\,dx + \frac{1}{2}\left(\int_{\mathbb{R}^d} |u_n|^{2p/(p-1)}\,dx\right)^{(p-1)/p} \leq \int_{\mathbb{R}^d} u_n f\,dx \leq C\|f\|_q\|u_n\|_{q'}. \tag{1.52}$$

Suppose that $d \geq 3$. Interpolating q' between $2p/(p-1)$ and $2d/(d-2)$ and using the Sobolev inequality (1.46), we obtain that there is a constant C, depending only on p, d and f, such that

$$\frac{1}{2}\int_{\mathbb{R}^d} |\nabla u_n|^2\,dx + \frac{1}{2}\left(\int_{\mathbb{R}^d} |u_n|^{2p/(p-1)}\,dx\right)^{(p-1)/p} \leq C.$$

Thus we can suppose that u_n converges weakly in $\dot{H}^1(\mathbb{R}^d)$ and in $L^{2p/(p-1)}(\mathbb{R}^d)$ and so, the problem (1.51) has a solution. In the case $d \le 2$, the claim follows since, by using (1.47), (1.48) and interpolating, we can still estimate $\|u_n\|_{q'}$ by means of $\|\nabla u_n\|_2$ and $\|u_n\|_{2p/(p-1)}$. $\square$

Repeating the argument of Section 1.5, one obtains an existence result for (1.49) in the case $p = 1$, too.

Proposition 1.28. *Let $f \in L^q(\mathbb{R}^d)$, where $q \in [1, \frac{2d}{d+2}]$, if $d \ge 3$, and $q = 1$, if $d = 1, 2$. Then there is a unique solution V_1 of problem (1.49) with $p = 1$, which is given by*

$$V_1 = \frac{f}{M}\left(\mathbf{1}_{\omega_+} - \mathbf{1}_{\omega_-}\right),$$

where $M = \|u_1\|_{L^\infty(\mathbb{R}^d)}$, $\omega_+ = \{u_1 = M\}$, $\omega_- = \{u_1 = -M\}$, and u_1 is the unique minimizer of

$$\min\left\{\frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2\,dx + \frac{1}{2}\|u\|_{L^\infty}^2 - \int_{\mathbb{R}^d} uf\,dx :\ u \in \dot{H}^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\right\}. \tag{1.53}$$

In particular, $\int_{\omega_+} f\,dx - \int_{\omega_-} f\,dx = M$, $f \ge 0$ on ω_+ and $f \le 0$ on ω_-.

We note that, when $p = 1$, the support of the optimal potential V_1 is contained in the support of the function f. This is not the case if $p > 1$, as the following example shows.

Example 1.29. Let $f = \mathbf{1}_{B_1(0)}$ and $p > 1$. By our previous analysis we know that there exist a solution u_p for problem (1.51) and a solution V_p for problem (1.49) given by (1.50). We note that u_p is positive, radially decreasing and satisfies the equation

$$-u''(r) - \frac{d-1}{r}u'(r) + Cu^\alpha = 0, \qquad r \in (1, +\infty),$$

where $\alpha = 2p/(p-1) > 2$ and C is a positive constant. Thus, we have that

$$u_p(r) = kr^{2/(1-\alpha)},$$

where k is an explicit constant depending on C, d and α. In particular u_p is not compactly supported on $\mathbb{R}^d$.

1.6.2 Optimal potentials in $\mathbb{R}^d$ with unbounded constraint

In this subsection we consider the problems

$$\min\left\{\mathcal{E}_f(V)\ :\ V \ge 0,\ \int_{\mathbb{R}^d} V^{-p}\,dx \le 1\right\}, \tag{1.54}$$

$$\min\left\{\lambda_1(V)\ :\ V \ge 0,\ \int_{\mathbb{R}^d} V^{-p}\,dx \le 1\right\}, \tag{1.55}$$

for $p > 0$ and $f \in L^q(\mathbb{R}^d)$. We will see in Proposition 1.30 that in order to have existence for (1.54) the parameter q must satisfy some constraints, depending on the value of p and on the dimension d. Namely, we need q to satisfy one of the following conditions

$$\begin{aligned} q \in [\frac{2d}{d+2}, \frac{2p}{p-1}], &\text{ if } d \geq 3 \text{ and } p > 1,\\ q \in [\frac{2d}{d+2}, +\infty], &\text{ if } d \geq 3 \text{ and } p \leq 1,\\ q \in (1, \frac{2p}{p-1}], &\text{ if } d = 2 \text{ and } p > 1,\\ q \in (1, +\infty], &\text{ if } d = 2 \text{ and } p \leq 1,\\ q \in [1, \frac{2p}{p-1}], &\text{ if } d = 1 \text{ and } p > 1,\\ q \in [1, +\infty], &\text{ if } d = 1 \text{ and } p \leq 1. \end{aligned} \tag{1.56}$$

We say that $q = q(p, d) \in [1, +\infty]$ is *admissible* if it satisfies (1.56). Note that $q = 2$ is admissible for any $d \geq 1$ and any $p > 0$.

Proposition 1.30. *Let $p > 0$ and $f \in L^q(\mathbb{R}^d)$, where q is admissible in the sense of (1.56). Then the minimization problem (1.54) has a solution V_p given by*

$$V_p = \left(\int_{\mathbb{R}^d} |u_p|^{2p/(p+1)}\,dx\right)^{1/p} |u_p|^{-2/(1+p)}, \tag{1.57}$$

where u_p is a minimizer of

$$\min\left\{\frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2\,dx + \frac{1}{2}\left(\int_{\mathbb{R}^d} |u|^{2p/(p+1)}\,dx\right)^{(p+1)/p} - \int_{\mathbb{R}^d} uf\,dx : \right. \tag{1.58}$$
$$\left. u \in \dot{H}^1(\mathbb{R}^d),\ |u|^{2p/(p+1)} \in L^1(\mathbb{R}^d)\right\}.$$

Moreover, if $p \geq 1$, then the functional in (1.58) is convex, its minimizer is unique and so is the solution of (1.54).

Proof. By means of (1.46), (1.47) and (1.48), and thanks to the admissibility of q, we get the existence of a solution of (1.58) through an interpolation argument similar to the one used in the proof of Proposition 1.27. The existence of an optimal potential follows reasoning as in Subsection 1.4.1. □

In Example 1.29, we showed that the optimal potentials for (1.49), may be supported on the whole $\mathbb{R}^d$. The analogous question for the problem (1.54) is whether the optimal potentials given by (1.57) have a bounded set of finiteness $\{V_p < +\infty\}$. In order to answer this question, it is sufficient to study the support of the solutions u_p of (1.58), which solve the equation

$$-\Delta u + C_p|u|^{-2/(p+1)}u = f, \tag{1.59}$$

where $C_p > 0$ is a constant depending on p.

Proposition 1.31. *Let $p > 0$ and let $f \in L^q(\mathbb{R}^d)$, for $q > d/2$, be a non-negative function with a compact support. Then every solution u_p of problem (1.58) has compact support.*

Proof. Without loss of generality we may assume that f is supported in the unit ball of $\mathbb{R}^d$. We first prove the result when f is radially decreasing. In this case u_p is also radially decreasing and non-negative. Let v be the function defined by $v(|x|) = u_p(x)$. Thus v satisfies the equation

$$\begin{cases} -v'' - \dfrac{d-1}{r}v' + C_p v^s = 0 \qquad r \in (1, +\infty), \\ v(1) = u_p(1), \end{cases} \tag{1.60}$$

where $s = (p-1)/(p+1)$ and $C_p > 0$ is a constant depending on p. Since $v \geq 0$ and $v' \leq 0$, we have that v is convex. Moreover, since

$$\int_1^{+\infty} v^2 r^{d-1}\,dr < +\infty, \qquad \int_1^{+\infty} |v'|^2 r^{d-1}\,dr < +\infty,$$

we have that v, v' and v'' vanish at infinity. Multiplying (1.60) by v' we obtain

$$\left(\frac{v'(r)^2}{2} - C_p\frac{v(r)^{s+1}}{s+1}\right)' = -\frac{d-1}{r}v'(r)^2 \leq 0.$$

Thus the function $v'(r)^2/2 - C_p v(r)^{s+1}/(s+1)$ is decreasing and vanishing at infinity and thus non-negative. Thus we have

$$-v'(r) \geq Cv(r)^{(s+1)/2}, \; r \in (1, +\infty), \tag{1.61}$$

where $C = \big(2C_p/(s+1)\big)^{1/2}$. Arguing by contradiction, suppose that v is strictly positive on $(1, +\infty)$. Dividing both sides of (1.61) and integrating, we have

$$-v(r)^{(1-s)/2} \geq Ar + B,$$

where $A = 2C/(1-s)$ and B is determined by the initial datum $v(1)$. This cannot occur, since the left hand side is negative, while the right hand side goes to $+\infty$, as $r \to +\infty$.

We now prove the result for a generic compactly supported and non-negative $f \in L^q(\mathbb{R}^d)$. Since the solution u_p of (1.58) is non-negative and is a weak solution of (1.59), we have that on each ball $B_R \subset \mathbb{R}^d$, $u_p \le u$, where $u \in H^1(B_R)$ is the solution of

$$-\Delta u = f \text{ in } B_R, \qquad u = u_p \text{ on } \partial B_R.$$

Since $f \in L^{d/2}(\mathbb{R}^d)$, by [67, Theorem 9.11] and a standard bootstrap argument on the integrability of u, we have that u is continuous on $B_{R/2}$. As a consequence, u_p is locally bounded in $\mathbb{R}^d$. In particular, it is bounded since $u_p \wedge M$, where $M = \|u_p\|_{L^\infty(B_1)}$, is a better competitor than u_p in (1.58). Let w be a radially decreasing minimizer of (1.58) with $f = \mathbf{1}_{B_1}$. Thus w is a solution of the PDE

$$-\Delta w + C_p w^s = \mathbf{1}_{B_1},$$

in $\mathbb{R}^d$, where C_p is as in (1.60). Then, the function $w_t(x) = t^{2/(1-s)} w(x/t)$ is a solution of the equation

$$-\Delta w_t + C_p w_t^s = t^{2s/(1-s)} \mathbf{1}_{B_t}.$$

Since u_p is bounded, there exists some $t \ge 1$ large enough such that $w_t \ge u_p$ on the ball B_t. Moreover, w_t minimizes (1.58) with $f = t^{2s/(1-s)} \mathbf{1}_{B_t}$ and so $w_t \ge u_p$ on $\mathbb{R}^d$ (otherwise $w_t \wedge u_p$ would be a better competitor in (1.58) than w_p). The conclusion follows since, by the first step of the proof, w_t has compact support. $\square$

The problems (1.55) and (1.54) are similar both in the questions of existence and the qualitative properties of the solutions.

Proposition 1.32. *For every $p > 0$ there is a solution of the problem (1.55) given by*

$$V_p = \left(\int_{\mathbb{R}^d} |u_p|^{2p/(p+1)} \, dx \right)^{1/p} |u_p|^{-2/(1+p)}, \tag{1.62}$$

where u_p is a radially decreasing minimizer of

$$\min \left\{ \int_{\mathbb{R}^d} |\nabla u|^2 \, dx + \left(\int_{\mathbb{R}^d} |u|^{2p/(p+1)} \, dx \right)^{(p+1)/p} \; : \; u \in H^1(\mathbb{R}^d), \; \int_{\mathbb{R}^d} u^2 \, dx = 1 \right\}. \tag{1.63}$$

Moreover, u_p has a compact support, hence the set $\{V_p < +\infty\}$ is a ball of finite radius in $\mathbb{R}^d$.

Proof. Let us first show that the minimum in (1.63) is achieved. Let $u_n \in H^1(\mathbb{R}^d)$ be a minimizing sequence of positive functions normalized in L^2. Note that by the Pólya-Szegö inequality we may assume that each of these functions is radially decreasing in $\mathbb{R}^d$ and so we will use the identification $u_n = u_n(r)$. In order to prove that the minimum is achieved it is enough to show that the sequence u_n converges in $L^2(\mathbb{R}^d)$. Indeed, since

u_n is a radially decreasing minimizing sequence, there exists $C > 0$ such that for each $r > 0$ we have

$$u_n(r)^{2p/(p+1)} \leq \frac{1}{|B_r|} \int_{B_r} u_n^{2p/(p+1)}\, dx \leq \frac{C}{r^d}.$$

Thus, for each $R > 0$, we obtain

$$\int_{B_R^c} u_n^2\, dx \leq C_1 \int_R^{+\infty} r^{-d(p+1)/p}\, r^{d-1}\, dr = C_2 R^{-1/p}, \tag{1.64}$$

where C_1 and C_2 do not depend on n and R. Since the sequence u_n is bounded in $H^1(\mathbb{R}^d)$, it converges locally in $L^2(\mathbb{R}^d)$ and, by (1.64), this convergence is also strong in $L^2(\mathbb{R}^d)$. Thus, we obtain the existence of a radially symmetric and decreasing solution u_p of (1.63) and so, of an optimal potential V_p given by (1.62).

We now prove that the support of u_p is a ball of finite radius. By the radial symmetry of u_p we can write it in the form $u_p(x) = u_p(|x|) = u_p(r)$, where $r = |x|$. With this notation, u_p satisfies the equation:

$$-u_p'' - \frac{d-1}{r} u_p' + C_p u_p^s = \lambda u_p,$$

where $s = (p-1)/(p+1) < 1$ and $C_p > 0$ is a constant depending on p. Arguing as in Proposition 1.31, we obtain that, for r large enough,

$$-u_p'(r) \geq \left(\frac{C_p}{s+1} u_p(r)^{s+1} - \frac{\lambda}{2} u_p(r)^2 \right)^{1/2} \geq \left(\frac{C_p}{2(s+1)} u_p(r)^{s+1} \right)^{1/2},$$

where, in the last inequality, we used the fact that $u_p(r) \to 0$, as $r \to \infty$, and $s+1 < 2$. Integrating both sides of the above inequality, we conclude that u_p has a compact support. □

Remark 1.33. We note that the solution $u_p \in H^1(\mathbb{R}^d)$ of (1.63) is the function for which the best constant C in the interpolated Gagliardo-Nirenberg-Sobolev inequality

$$\|u\|_2 \leq C \|\nabla u\|_2^{d/(d+2p)} \|u\|_{2p/(p+1)}^{2p/(d+2p)} \tag{1.65}$$

is achieved. Indeed, for any $u \in H^1(\mathbb{R}^d)$ and any $t > 0$, we define $u_t(x) := t^{d/2} u(tx)$. Thus, we have that $\|u\|_2 = \|u_t\|_2$, for any $t > 0$. Moreover, up to a rescaling, we may assume that the function $g : (0, +\infty) \to \mathbb{R}$, defined by

$$\begin{aligned} g(t) &= \int_{\mathbb{R}^d} |\nabla u_t|^2\, dx + \left(\int_{\mathbb{R}^d} |u_t|^{2p/(p+1)}\, dx \right)^{(p+1)/p} \\ &= t^2 \int_{\mathbb{R}^d} |\nabla u|^2\, dx + t^{-d/p} \left(\int_{\mathbb{R}^d} |u|^{2p/(p+1)}\, dx \right)^{(p+1)/p}, \end{aligned}$$

achieves its minimum in the interval $(0,+\infty)$ and, moreover, we have

$$\min_{t\in(0,+\infty)} g(t) = C\left(\int_{\mathbb{R}^d}|\nabla u|^2\,dx\right)^{d/(d+2p)}\left(\int_{\mathbb{R}^d}|u|^{\frac{2p}{p+1}}\,dx\right)^{2(p+1)/(d+2p)},$$

where C is a constant depending on p and d. In the case $u=u_p$, the minimum of g is achieved for $t=1$ and so, we have that u_p is a solution also of

$$\min\left\{\left(\int_{\mathbb{R}^d}|\nabla u|^2\,dx\right)^{d/(d+2p)}\left(\int_{\mathbb{R}^d}|u|^{2p/(p+1)}\,dx\right)^{2(p+1)/(d+2p)}\ :\ u\in H^1(\mathbb{R}^d),\ \int_{\mathbb{R}^d}u^2\,dx=1\right\},$$

which is just another form of (1.65).

1.6.3 Further remarks

We recall (see [26]) that the injection $H^1_V(\mathbb{R}^d)\hookrightarrow L^2(\mathbb{R}^d)$ is compact whenever the potential V satisfies $\int_{\mathbb{R}^d}V^{-p}\,dx<+\infty$ for some $0<p\le 1$. In this case the spectrum of the Schrödinger operator $-\Delta+V$ is discrete and we denote by $\lambda_k(V)$ its eigenvalues. The existence of an optimal potential for spectral optimization problems of the form

$$\min\left\{\lambda_k(V)\ :\ V\ge 0,\ \int_{\mathbb{R}^d}V^{-p}\,dx\le 1\right\}, \tag{1.66}$$

for general $k\in\mathbb{N}$, cannot be deduced by the direct methods used in Subsection 1.6.2. In this last section we make the following conjectures:

- For every $k\ge 1$, there is a solution V_k of the problem (1.66).
- The set of finiteness $\{V_k<+\infty\}$, of the optimal potential V_k, is bounded.

In what follows, we prove an existence result in the case $k=2$. We first recall that, by Proposition 1.32, there exists an optimal potential V_p, for λ_1, such that the set of finiteness $\{V_p<+\infty\}$ is a ball. Thus, we have a situation analogous to the Faber-Krahn inequality, which states that the minimum

$$\min\left\{\lambda_1(\Omega)\ :\ \Omega\subset\mathbb{R}^d,\ |\Omega|=c\right\}, \tag{1.67}$$

is achieved for the ball of measure c. We recall that, starting from (1.67), one may deduce, by a simple argument (see for instance [75]), the Krahn-Szegö inequality, which states that the minimum

$$\min\left\{\lambda_2(\Omega)\ :\ \Omega\subset\mathbb{R}^d,\ |\Omega|=c\right\}, \tag{1.68}$$

is achieved for a disjoint union of equal balls. In the case of potentials one can find two optimal potentials for λ_1 with disjoint sets of finiteness and then apply the argument from the proof of the Krahn-Szegö inequality. In fact, we have the following result.

Proposition 1.34. *There exists an optimal potential, solution of* (1.66) *with* $k = 2$. *Moreover, any optimal potential is of the form* $\min\{V_1, V_2\}$, *where* V_1 *and* V_2 *are optimal potentials for* λ_1 *which have disjoint sets of finiteness* $\{V_1 < +\infty\} \cap \{V_2 < +\infty\} = \emptyset$ *and are such that* $\int_{\mathbb{R}^d} V_1^{-p}\,dx = \int_{\mathbb{R}^d} V_2^{-p}\,dx = 1/2$.

Proof. Given V_1 and V_2 as above, we prove that for every $V : \mathbb{R}^d \to [0, +\infty]$ with $\int_{\mathbb{R}^d} V^{-p}\,dx = 1$, we have

$$\lambda_2(\min\{V_1, V_2\}) \leq \lambda_2(V).$$

Indeed, let u_2 be the second eigenfunction of $-\Delta + V$. We first suppose that u_2 changes sign on $\mathbb{R}^d$ and consider the functions $V_+ = \sup\{V, \infty_{\{u_2 \leq 0\}}\}$ and $V_- = \sup\{V, \infty_{\{u_2 \geq 0\}}\}$ where, for any measurable $A \subset \mathbb{R}^d$, we set

$$\infty_A(x) = \begin{cases} +\infty, & x \in A, \\ 0, & x \notin A. \end{cases}$$

We note that

$$1 = \int_{\mathbb{R}^d} V^{-p}\,dx = \int_{\mathbb{R}^d} V_+^{-p}\,dx + \int_{\mathbb{R}^d} V_-^{-p}\,dx.$$

Moreover, on the sets $\{u_2 > 0\}$ and $\{u_2 < 0\}$, the following equations are satisfied:

$$-\Delta u_2^+ + V_+ u_2^+ = \lambda_2(V) u_2^+, \qquad -\Delta u_2^- + V_- u_2^- = \lambda_2(V) u_2^-,$$

and so, multiplying respectively by u_2^+ and u_2^-, we obtain that

$$\lambda_2(V) \geq \lambda_1(V_+), \qquad \lambda_2(V) \geq \lambda_1(V_-), \tag{1.69}$$

where we have equalities if, and only if, u_2^+ and u_2^- are the first eigenfunctions corresponding to $\lambda_1(V_+)$ and $\lambda_1(V_-)$. Let now $\widetilde{V}_+$ and $\widetilde{V}_-$ be optimal potentials for λ_1 corresponding to the constraints

$$\int_{\mathbb{R}^d} \widetilde{V}_+^{-p}\,dx = \int_{\mathbb{R}^d} V_+^{-p}\,dx, \qquad \int_{\mathbb{R}^d} \widetilde{V}_-^{-p}\,dx = \int_{\mathbb{R}^d} V_-^{-p}\,dx.$$

By Proposition 1.32, the sets of finiteness of $\widetilde{V}_+$ and $\widetilde{V}_-$ are compact, hence we may assume (up to translations) that they are also disjoint. By the monotonicity of λ_1, we have

$$\max\{\lambda_1(V_1), \lambda_1(V_2)\} \leq \max\{\lambda_1(\widetilde{V}_+), \lambda_1(\widetilde{V}_-)\},$$

and so we obtain

$$\lambda_2(\min\{V_1, V_2\}) \leq \max\{\lambda_1(\widetilde{V}_+), \lambda_1(\widetilde{V}_-)\} \leq \max\{\lambda_1(V_+), \lambda_1(V_-)\} \leq \lambda_2(V),$$

as required. If u_2 does not change sign, then we consider $V_+ = \sup\{V, \infty_{\{u_2=0\}}\}$ and $V_- = \sup\{V, \infty_{\{u_1=0\}}\}$, where u_1 is the first eigenfunction of $-\Delta + V$. Then the claim follows by the same argument as above. $\square$

Chapter 2

Optimization problems for metric graphs

2.1 Introduction

This chapter is based on the paper [32], written in collaboration with Giuseppe Buttazzo and Bozhidar Velichkov. The issue we take into account, is an adaptation of the problem of minimizing the first eigenvalue and the Energy function of the Dirichlet Laplacian to the class of graphs. To give a precise meaning to the previous sentence we must define what is a graph, and introduce a suitable concept of differential operator on it. Initially, we define a graph C in $\mathbb{R}^d$ to be simply a closed connected subset of $\mathbb{R}^d$ with finite $1-$dimensional Hausdorff measure $\mathcal{H}^1(C)$. Since such sets are rectifiable (see for instance [4]) the standard theory on rectifiable sets allows us to define all the variational tools that are usually defined in the Euclidean setting:

- Dirichlet integral $\int_C \frac{1}{2}|u'|^2\,d\mathcal{H}^1$;
- Sobolev spaces

$$H^1(C)=\Big\{u\in L^2(C)\ :\ \int_C|u'|^2\,d\mathcal{H}^1<+\infty\Big\},$$
$$H^1_0(C;\mathcal{D})=\Big\{u\in H^1(C)\ :\ u=0\text{ on }\mathcal{D}\Big\};$$

- (Dirichlet) Energy

$$\mathcal{E}(C;\mathcal{D})=\inf\Big\{\int_C\Big(\frac{1}{2}|u'|^2-u\Big)\,d\mathcal{H}^1\ :\ u\in H^1_0(C,\mathcal{D})\Big\}.$$

In particular, for a fixed set $\mathcal{D}$ consisting of d points, $\mathcal{D}=\{D_1,\dots,D_d\}$, we consider the shape optimization problem

$$\min\big\{\mathcal{E}(C;\mathcal{D})\ :\ \mathcal{H}^1(C)=l,\ \mathcal{D}\subset C\big\},\tag{2.1}$$

where the total length l is fixed. Notice that in the problem above the unknown is the graph C and no a priori constraints on its topology are imposed.

In spite of the fact that the optimization problem (2.1) looks very natural, we show that in general an optimal graph may not exist (see Example 2.23); this leads us to consider another, larger, admissible class consisting of the so-called *metric graphs*, for which the embedding into $\mathbb{R}^d$ is not required. The precise definition of a metric graph is given in Section 2.3; roughly speaking they are metric spaces induced by combinatorial graphs with weighted edges.

Our main result is an existence theorem for optimal metric graphs, where the cost functional is the extension of the Energy functional defined above. In Section 2.4 we show some explicit examples of optimal metric graphs. The last section contains some remarks on possible extensions of our main result to other similar problems and on some open questions.

2.2 Sobolev space and Dirichlet Energy of a rectifiable set

Let $C \subset \mathbb{R}^d$ be a closed connected set of finite length, i.e. $\mathcal{H}^1(C) < \infty$, where $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure. On the set C we consider the metric

$$d(x,y) = \inf\left\{\int_0^1 |\dot\gamma(t)|\,dt \ : \ \gamma : [0,1] \to \mathbb{R}^d \text{ Lipschitz}, \ \gamma([0,1]) \subset C, \ \gamma(0) = x, \ \gamma(1) = y\right\},$$

which is finite since, by the First Rectifiability Theorem (see [4, Theorem 4.4.1]), there is at least one rectifiable curve in C connecting x to y. For any function $u : C \to \mathbb{R}$, Lipschitz with respect to the distance d (we also use the term $d-$Lipschitz), we define the norm

$$\|u\|^2_{H^1(C)} = \int_C |u(x)|^2 d\mathcal{H}^1(x) + \int_C |u'|(x)^2 d\mathcal{H}^1(x),$$

where

$$|u'|(x) = \overline{\lim}_{y\to x} \frac{|u(y) - u(x)|}{d(x,y)}.$$

The Sobolev space $H^1(C)$ is the closure of the $d-$Lipschitz functions on C with respect to the norm $\|\cdot\|_{H^1(C)}$.

Remark 2.1. The inclusion $H^1(C) \subset C_d(C)$ is compact, where $C_d(C)$ indicates the space of real-valued continuous functions on C, with respect to the metric d. In fact, for each $x, y \in C$, there is a rectifiable curve $\gamma : [0, d(x,y)] \to C$ connecting x to y, which we may assume arc-length parametrized. Thus, for any $u \in H^1(C)$, we have that

$$\begin{aligned}|u(x) - u(y)| &\leq \int_0^{d(x,y)} \left|\frac{d}{dt}u(\gamma(t))\right| dt \\ &\leq d(x,y)^{1/2} \left(\int_0^{d(x,y)} \left|\frac{d}{dt}u(\gamma(t))\right|^2 dt\right)^{1/2} \\ &\leq d(x,y)^{1/2}\|u'\|_{L^2(C)},\end{aligned}$$

and so, u is $1/2-$Hölder continuous. On the other hand, for any $x \in C$, we have that

$$\int_C u(y)d\mathcal{H}^1(y) \geq \int_C \left(u(x) - d(x,y)^{1/2}\|u'\|_{L^2(C)}\right) d\mathcal{H}^1(y) \geq lu(x) - l^{3/2}\|u'\|_{L^2(C)},$$

where $l = \mathcal{H}^1(C)$. Thus, we obtain the L^∞ bound

$$\|u\|_{L^\infty} \leq l^{-1/2}\|u\|_{L^2(C)} + l^{1/2}\|u'\|_{L^2(C)} \leq (l^{-1/2} + l^{1/2})\|u\|_{H^1(C)}.$$

and so, by the Ascoli-Arzelá Theorem, we have that the inclusion is compact.

Remark 2.2. By the same argument as in Remark 2.1 above, we have that for any $u \in H^1(C)$, the $(1,2)-$Poincaré inequality holds, i.e.

$$\int_C \left|u(x) - \frac{1}{l}\int_C u\, d\mathcal{H}^1\right| d\mathcal{H}^1(x) \leq l^{3/2}\left(\int_C |u'|^2 d\mathcal{H}^1\right)^{1/2}. \tag{2.2}$$

Moreover, if $u \in H^1(C)$ is such that $u(x) = 0$ for some point $x \in C$, then we have the Poincaré inequality:

$$\|u\|_{L^2(C)} \leq l^{1/2}\|u\|_{L^\infty(C)} \leq l\|u'\|_{L^2(C)}. \tag{2.3}$$

Since C is supposed connected, by the Second Rectifiability Theorem (see [4, Theorem 4.4.8]) there exists a countable family of injective arc-length parametrized Lipschitz curves $\gamma_i : [0, l_i] \to C$, $i \in \mathbb{N}$ and an $\mathcal{H}^1-$negligible set $N \subset C$ such that

$$C = N \cup \left(\bigcup_i Im(\gamma_i)\right),$$

where $Im(\gamma_i) = \gamma_i([0, l_i])$. By the chain rule (see Lemma 2.3 below) we have

$$\left|\frac{d}{dt}u(\gamma_i(t))\right| = |u'|(\gamma_i(t)), \qquad \forall i \in \mathbb{N}$$

and so, we obtain for the norm of $u \in H^1(C)$:

$$\|u\|^2_{H^1(C)} = \int_C |u(x)|^2 d\mathcal{H}^1(x) + \sum_i \int_0^{l_i} \left|\frac{d}{dt}u(\gamma_i(t))\right|^2 dt. \tag{2.4}$$

Moreover, we have the inclusion

$$H^1(C) \subset \oplus_{i\in} H^1([0,l_i]), \tag{2.5}$$

which gives the reflexivity of $H^1(C)$ and the lower semicontinuity of the $H^1(C)$ norm, with respect to the strong convergence in $L^2(C)$.

Lemma 2.3. *Let $\gamma : [0,l] \to \mathbb{R}^d$ be an injective arc-length parametrized Lipschitz curve with $\gamma([0,l]) \subset C$. Then we have*

$$\left|\frac{d}{dt}u(\gamma(t))\right| = |u'|(\gamma(t)), \qquad \textit{for } \mathcal{H}^1\textit{-a.e. } t \in [0,l]. \tag{2.6}$$

Proof. Let $u : C \to \mathbb{R}$ be a Lipschitz map with Lipschitz constant $Lip(u)$ with respect to the distance d. We prove that the chain rule (2.6) holds in all the points $t \in [0,l]$ which are Lebesgue points for $\left|\frac{d}{dt}u(\gamma(t))\right|$ and such that the point $\gamma(t)$ has density one, i.e.

$$\lim_{r\to 0} \frac{\mathcal{H}^1\big(C \cap B_r(\gamma(t))\big)}{2r} = 1, \tag{2.7}$$

(thus almost every points, see for istance [92]) where $B_r(x)$ indicates the ball of radius r in $\mathbb{R}^d$. Since, $\mathcal{H}^1-$almost all points $x \in C$ have this property, we obtain the conclusion. Without loss of generality, we consider $t = 0$. Let us first prove that $|u'|(\gamma(0)) \geq \left|\frac{d}{dt}u(\gamma(0))\right|$. We have that

$$|u'|(\gamma(0)) \geq \overline{\lim}_{t\to 0} \frac{|u(\gamma(t)) - u(\gamma(0))|}{d(\gamma(t),\gamma(0))} = \left|\frac{d}{dt}u(\gamma(0))\right|,$$

since γ is arc-length parametrized. On the other hand, we have

$$\begin{aligned}
|u'|(x) &= \overline{\lim}_{y\to x} \frac{|u(y) - u(x)|}{d(y,x)} \\
&= \lim_{n\to\infty} \frac{|u(y_n) - u(x)|}{d(y_n,x)} \\
&= \lim_{n\to\infty} \frac{|u(\gamma_n(r_n)) - u(\gamma_n(0))|}{r_n} \\
&\leq \lim_{n\to\infty} \frac{1}{r_n} \int_0^{r_n} \left|\frac{d}{dt}u(\gamma_n(t))\right| dt
\end{aligned} \tag{2.8}$$

where $y_n \in C$ is a sequence of points which realizes the $\limsup$ and $\gamma_n : [0,r_n] \to \mathbb{R}^d$ is a geodesic in C connecting x to y_n. Let $S_n = \{t : \gamma_n(t) = \gamma(t)\} \subset [0,r_n]$, then, we have

$$\begin{aligned}
\int_0^{r_n} \left|\frac{d}{dt}u(\gamma_n(t))\right|^2 dt &\leq \int_{S_n} \left|\frac{d}{dt}u(\gamma(t))\right|^2 dt + Lip(u)\,(r_n - |S_n|) \\
&\leq \int_0^{r_n} \left|\frac{d}{dt}u(\gamma(t))\right|^2 dt + Lip(u)\,(\mathcal{H}^1(B_{r_n}(\gamma(0)) \cap C) - 2r_n),
\end{aligned} \tag{2.9}$$

and so, since $\gamma(0)$ is of density 1, we conclude applying this estimate to (2.8). $\square$

Given a set of points $\mathcal{D} = \{D_1, \ldots, D_k\} \subset \mathbb{R}^d$ we define the admissible class $\mathcal{A}(\mathcal{D}; l)$ as the family of all closed connected sets C containing $\mathcal{D}$ and of length $\mathcal{H}^1(C) = l$. For any $C \in \mathcal{A}(\mathcal{D}; l)$ we consider the space of Sobolev functions which satisfy a Dirichlet condition at the points D_i:

$$H_0^1(C; \mathcal{D}) = \{u \in H^1(C) : u(D_j) = 0,\, j = 1 \ldots, k\},$$

which is well-defined by Remark 2.1. For the points D_i we use the term *Dirichlet points*. The *Dirichlet Energy* of the set C with respect to $D_1, \ldots, D_k$ is defined as

$$\mathcal{E}(C; \mathcal{D}) = \min \left\{ J(u) \ : \ u \in H_0^1(C; \mathcal{D}) \right\}, \tag{2.10}$$

where

$$J(u) = \frac{1}{2} \int_C |u'|(x)^2 \, d\mathcal{H}^1(x) - \int_C u(x) \, d\mathcal{H}^1(x). \tag{2.11}$$

Remark 2.4. For any $C \in \mathcal{A}(\mathcal{D}; l)$ there exists a unique minimizer of the functional $J : H_0^1(C; \mathcal{D}) \to \mathbb{R}$. In fact, by Remark 2.1 we have that a minimizing sequence is bounded in H^1 and compact in L^2. The conclusion follows by the semicontinuity of the L^2 norm of the gradient, with respect to the strong L^2 convergence, which is an easy consequence of equation (2.4). The uniqueness follows by the strict convexity of the L^2 norm and the sub-additivity of the gradient $|u'|$. We call the minimizer of J the *Energy function* of C with Dirichlet conditions in $D_1, \ldots, D_k$.

Remark 2.5. Let $u \in H^1(C)$ and $v : C \to \mathbb{R}$ be a positive Borel function. Applying the chain rule, as in (2.4), and the one dimensional co-area formula (see for instance [2]), we obtain a co-area formula for the functions $u \in H^1(C)$:

$$\begin{aligned} \int_C v(x)|u'|(x) \, d\mathcal{H}^1(x) &= \sum_i \int_0^{l_i} \left| \frac{d}{dt} u(\gamma_i(t)) \right| v(\gamma_i(t)) \, dt \\ &= \sum_i \int_0^{+\infty} \Big(\sum_{u \circ \gamma_i(t) = \tau} v \circ \gamma_i(t) \Big) \, d\tau \\ &= \int_0^{+\infty} \Big(\sum_{u(x) = \tau} v(x) \Big) \, d\tau. \end{aligned} \tag{2.12}$$

2.2.1 Optimization problem for the Dirichlet Energy on the class of connected sets

We study the following shape optimization problem:

$$\min \left\{ \mathcal{E}(C; \mathcal{D}) \ : \ C \in \mathcal{A}(\mathcal{D}; l) \right\}, \tag{2.13}$$

where $\mathcal{D} = \{D_1, ..., D_k\}$ is a given set of points in $\mathbb{R}^d$ and l is a prescribed length.

Remark 2.6. When $k = 1$ problem (2.13) reads as

$$\mathcal{E} = \min \left\{ \mathcal{E}(C; D) \ : \ \mathcal{H}^1(C) = l, \ D \in C \right\}, \tag{2.14}$$

where $D \in \mathbb{R}^d$ and $l > 0$. In this case the solution is a line of length l starting from D (see Figure 2.1). A proof of this fact, in a slightly different context, can be found in [59] and we report it here for the sake of completeness.

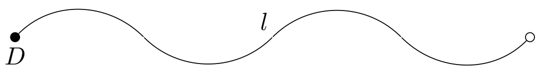

Figure 2.1: The optimal graph with only one Dirichlet point.

Let $C \in \mathcal{A}(D; l)$ be a generic connected set and let $w \in H^1_0(C; D)$ be its Energy function, i.e. the minimizer of J on C. Let $v : [0, l] \to \mathbb{R}$ be such that $\mu_w(\tau) = \mu_v(\tau)$, where μ_w and μ_v are the distribution function of w and v respectively, defined by

$$\mu_w(\tau) = \mathcal{H}^1(w \leq \tau) = \sum_i \mathcal{H}^1(w_i \leq \tau), \qquad \mu_v(\tau) = \mathcal{H}^1(v \leq \tau).$$

It is easy to see that, by the Cavalieri Formula, $\|v\|_{L^p([0,l])} = \|w\|_{L^p(C)}$, for each $p \geq 1$. By the co-area formula (2.12)

$$\int_C |w'|^2 \, d\mathcal{H}^1 = \int_0^{+\infty} \Big(\sum_{w=\tau} |w'| \Big) \, d\tau \geq \int_0^{+\infty} \Big(\sum_{w=\tau} \frac{1}{|w'|} \Big)^{-1} d\tau = \int_0^{+\infty} \frac{d\tau}{\mu'_w(\tau)}, \tag{2.15}$$

where we used the Cauchy-Schwartz inequality and the identity

$$\mu_w(t) = \mathcal{H}^1(\{w \leq t\}) = \int_{w \leq t} \frac{|w'|}{|w'|} \, ds = \int_0^t \Big(\sum_{w=s} \frac{1}{|w'|} \Big) \, ds$$

which implies that $\mu'_w(t) = \sum_{w=t} \frac{1}{|w'|}$. The same argument applied to v gives:

$$\int_0^l |v'|^2 \, dx = \int_0^{+\infty} \Big(\sum_{v=\tau} |v'| \Big) \, d\tau = \int_0^{+\infty} \frac{d\tau}{\mu'_v(\tau)}. \tag{2.16}$$

Since $\mu_w = \mu_v$, the conclusion follows.

The following theorem shows that it is enough to study the problem (2.13) on the class of finite graphs embedded in $\mathbb{R}^d$. Consider the subset $\mathcal{A}_N(\mathcal{D}; l) \subset \mathcal{A}(\mathcal{D}; l)$ of those sets C for which there exists a finite family $\gamma_i : [0, l_i] \to \mathbb{R}$, $i = 1, \ldots, n$ with $n \leq N$, of injective rectifiable curves such that $\cup_i \gamma_i([0, l_i]) = C$ and $\gamma_i((0, l_i)) \cap \gamma_j((0, l_j)) = \emptyset$, for each $i \neq j$.

Theorem 2.7. *Consider the set of distinct points* $\mathcal{D} = \{D_1, \dots, D_k\} \subset \mathbb{R}^d$ *and* $l > 0$. *We have that*

$$\inf\big\{\mathcal{E}(C;\mathcal{D}) \ :\ C \in \mathcal{A}(\mathcal{D};l)\big\} = \inf\big\{\mathcal{E}(C;\mathcal{D}) \ :\ C \in \mathcal{A}_N(\mathcal{D};l)\big\}, \tag{2.17}$$

where $N = 2k - 1$. *Moreover, if* C *is a solution of the problem* (2.13)*, then there is also a solution* $\widetilde{C}$ *of the same problem such that* $\widetilde{C} \in \mathcal{A}_N(\mathcal{D};l)$.

Proof. Consider a connected set $C \in \mathcal{A}(\mathcal{D};l)$. We show that there is a set $\widetilde{C} \in \mathcal{A}_N(\mathcal{D};l)$ such that $\mathcal{E}(\widetilde{C};\mathcal{D}) \leq \mathcal{E}(C;\mathcal{D})$. Let $\eta_1 : [0,a_1] \to C$ be a geodesic in C connecting D_1 to D_2 and let $\eta_2 : [0,a] \to C$ be a geodesic connecting D_3 to D_1. Let a_2 be the smallest real number such that $\eta_2(a_2) \in \eta_1([0,a_1])$. Then, consider the geodesic η_3 connecting D_4 to D_1 and the smallest real number a_3 such that $\eta_3(a_3) \in \eta_1([0,a_1]) \cup \eta_2([0,a_2])$. Repeating this operation, we obtain a family of geodesics η_i, $i = 1, \dots, k-1$ which intersect each other in a finite number of points. Each of these geodesics can be decomposed in several parts according to the intersection points with the other geodesics (see Figure 2.2).

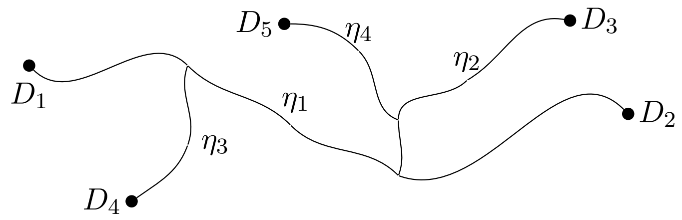

Figure 2.2: Construction of the set C'.

So, we can consider a new family of geodesics (still denoted by η_i), $\eta_i : [0,l_i] \to C$, $i = 1, \dots, n$, which does not intersect each other in internal points. Note that, by an induction argument on $k \geq 2$, we have $n \leq 2k - 3$. Let $C' = \cup_i \eta_i([0,l_i]) \subset C$. By the Second Rectifiability Theorem (see [4, Theorem 4.4.8]), we have that

$$C = C' \cup E \cup \Gamma,$$

where $\mathcal{H}^1(E) = 0$ and $\Gamma = \left(\bigcup_{j=1}^{+\infty} \gamma_j\right)$, where $\gamma_j : [0,l_j] \to C$ for $j \geq 1$ is a family of Lipschitz curves in C. Moreover, we can suppose that $\mathcal{H}^1(\Gamma \cap C') = 0$. In fact, if $\mathcal{H}^1(Im(\gamma_j) \cap C') \neq 0$ for some $j \in \mathbb{N}$, we consider the restriction of γ_j to (the closure of) each connected component of $\gamma_j^{-1}(\mathbb{R}^d \setminus C')$.

Let $w \in H_0^1(C;\mathcal{D})$ be the Energy function on C and let $v : [0, \mathcal{H}^1(\Gamma)] \to \mathbb{R}$ be a monotone increasing function such that $|\{v \leq \tau\}| = \mathcal{H}^1(\{w \leq \tau\} \cap \Gamma)$. Reasoning as in Remark 2.6, we have that

$$\frac{1}{2}\int_0^{\mathcal{H}^1(\Gamma)} |v'|^2\,dx - \int_0^{\mathcal{H}^1(\Gamma)} v\,dx \leq \frac{1}{2}\int_\Gamma |w'|^2\,d\mathcal{H}^1 - \int_\Gamma w\,d\mathcal{H}^1. \tag{2.18}$$

Let $\sigma : [0, \mathcal{H}^1(\Gamma)] \to \mathbb{R}^d$ be an injective arc-length parametrized curve such that $Im(\sigma) \cap C' = \sigma(0) = x'$, where $x' \in C'$ is the point where $w_{|C'}$ achieves its maximum. Let $\widetilde{C} = C' \cup Im(\sigma)$. Notice that $\widetilde{C}$ connects the points $D_1, \ldots, D_k$ and has length $\mathcal{H}^1(\widetilde{C}) = \mathcal{H}^1(C') + \mathcal{H}^1(Im(\sigma)) = \mathcal{H}^1(C') + \mathcal{H}^1(\Gamma) = l$. Moreover, we have

$$\mathcal{E}(\widetilde{C}; \mathcal{D}) \leq J(\widetilde{w}) \leq J(w) = \mathcal{E}(C; \mathcal{D}), \tag{2.19}$$

where $\widetilde{w}$ is defined by

$$\widetilde{w}(x) = \begin{cases} w(x), & \text{if } x \in C', \\ v(t) + w(x') - v(0), & \text{if } x = \sigma(t). \end{cases} \tag{2.20}$$

We have then (2.19), i.e. the energy decreases. We conclude by noticing that the point x' where we attach σ to C' may be an internal point for η_i, i.e. a point such that $\eta_i^{-1}(x') \in (0, l_i)$. Thus, the set $\widetilde{C}$ is composed of at most $2k - 1$ injective arc-length parametrized curves which does not intersect in internal points, i.e. $\widetilde{C} \in \mathcal{A}_{2k-1}(\mathcal{D}; l)$. $\square$

Remark 2.8. Theorem 2.7 above provides a nice class of admissible sets, where to search a minimizer of the energy functional $\mathcal{E}$. Indeed, according to its proof, we may limit ourselves to consider only graphs C such that:

1. C is a tree, i.e. it does not contain any closed loop;

2. the Dirichlet points D_i are vertices of degree one (endpoints) for C;

3. there are at most $k - 1$ other vertices; if a vertex has degree three or more, we call it Kirchhoff point;

4. there is at most one vertex of degree one for C which is not a Dirichlet point. In this vertex the energy function w satisfies Neumann boundary condition $w' = 0$ and so we call it Neumann point.

The previous properties are also necessary conditions for the optimality of the graph C (see Proposition 2.19 for more details).

As we show in Example 2.23, the problem (2.13) may not have a solution in the class of connected sets. It is worth noticing that the lack of existence only occurs for particular configurations of the Dirichlet points D_i and not because of some degeneracy of the cost functional $\mathcal{E}$. In fact, we are able to produce other examples in which an optimal graph exists (see Section 2.4).

2.3 Sobolev space and Dirichlet Energy of a metric graph

Let $V = \{V_1, \dots, V_N\}$ be a finite set and let $E \subset \{e_{ij} = \{V_i, V_j\}\}$ be a set of pairs of elements of V. We define combinatorial graph (or just graph) a pair $\Gamma = (V, E)$. We say the set $V = V(\Gamma)$ is the set of vertices of Γ and the set $E = E(\Gamma)$ is the set of edges. We denote with $|E|$ and $|V|$ the cardinalities of E and V and with $\deg(V_i)$ the degree of the vertex V_i, i.e. the number of edges incident to V_i.

A *path* in the graph Γ is a sequence $V_{\alpha_0}, \dots, V_{\alpha_n} \in V$ such that for each $k = 0, \dots, n-1$, we have that $\{V_{\alpha_k}, V_{\alpha_{k+1}}\} \in E$. With this notation, we say that the path connects V_{i_0} to V_{i_α}. The path is said to be *simple* if there are no repeated vertices in $V_{\alpha_0}, \dots, V_{\alpha_n}$. We say that the graph $\Gamma = (V, E)$ is connected, if for each pair of vertices $V_i, V_j \in V$ there is a path connecting them. We say that the connected graph Γ is a tree, if after removing any edge, the graph becomes not connected.

If we associate a non-negative length (or weight) to each edge, i.e. a map $l : E(\Gamma) \to [0, +\infty)$, then we say that the couple (Γ, l) determines a metric graph of length

$$l(\Gamma) := \sum_{i<j} l(e_{ij}).$$

A function $u : \Gamma \to \mathbb{R}^d$ on the metric graph Γ is a collection of functions $u_{ij} : [0, l_{ij}] \to \mathbb{R}$, for $1 \le i \ne j \le d$, such that:

1. $u_{ji}(x) = u_{ij}(l_{ij} - x)$, for each $1 \le i \ne j \le d$,
2. $u_{ij}(0) = u_{ik}(0)$, for all $\{i, j, k\} \subset \{1, \dots, d\}$,

where we used the notation $l_{ij} = l(e_{ij})$. A function $u : \Gamma \to \mathbb{R}$ is said continuous ($u \in C(\Gamma)$), if $u_{ij} \in C([0, l_{ij}])$, for all $i, j \in \{1, \dots, n\}$. We call $L^p(\Gamma)$ the space of $p-$summable functions ($p \in [1, +\infty)$), i.e. the functions $u = (u_{ij})_{ij}$ such that

$$\|u\|^p_{L^p(\Gamma)} := \frac{1}{2} \sum_{i,j} \|u_{ij}\|^p_{L^p(0, l_{ij})} < +\infty,$$

where $\|\cdot\|_{L^p(a,b)}$ denotes the usual L^p norm on the interval $[a, b]$. As usual, the space $L^2(\Gamma)$ has a Hilbert structure endowed by the scalar product:

$$\langle u, v \rangle_{L^2(\Gamma)} := \frac{1}{2} \sum_{i,j} \langle u_{ij}, v_{ij} \rangle_{L^2(0, l_{ij})}.$$

function defined on $\mathbb{R}^d$ to the set $\cup_{i,j} \gamma_{ij}([0, l_{ij}])$: this is mainly due to the fact that we allow intersections.

We define the Sobolev space $H^1(\Gamma)$ as:

$$H^1(\Gamma) = \{u \in C(\Gamma) : u_{ij} \in H^1([0, l_{ij}]),\ \forall i, j \in \{1, \dots, n\}\}, \tag{2.21}$$

which is a Hilbert space with the norm

$$\|u\|^2_{H^1(\Gamma)}=\frac{1}{2}\sum_{i,j}\|u_{ij}\|^2_{H^1([0,l_{ij}])}=\frac{1}{2}\sum_{i,j}\left(\int_0^{l_{ij}}|u_{ij}|^2dx+\int_0^{l_{ij}}|u'_{ij}|^2dx\right). \tag{2.22}$$

Remark 2.9. Note that for $u\in H^1(\Gamma)$ the family of derivatives $\big(u'_{ij}\big)_{1\le i\ne j\le d}$ is not a function on Γ, since $u'_{ij}(x)=\frac{\partial}{\partial x}u_{ji}(l_{ij}-x)=-u'_{ji}(l_{ij}-x)$. Thus, we work with the function $|u'|=\big(|u'_{ij}|\big)_{1\le i\ne j\le d}\in L^2(\Gamma)$.

Remark 2.10. The inclusions $H^1(\Gamma)\subset C(\Gamma)$ and $H^1(\Gamma)\subset L^2(\Gamma)$ are compact, since the corresponding inclusions, for each of the intervals $[0,l_{ij}]$, are compact. By the same argument, the H^1 norm is lower semicontinuous with respect to the strong L^2 convergence of the functions in $H^1(\Gamma)$.

For any subset $W=\{W_1,\dots,W_k\}$ of the set of vertices $V(\Gamma)=\{V_1,\dots,V_N\}$, we introduce the Sobolev space with *Dirichlet boundary conditions* on W:

$$H^1_0(\Gamma;W)=\big\{u\in H^1(\Gamma)\ :\ u(W_1)=\dots=u(W_k)=0\big\}. \tag{2.23}$$

Remark 2.11. Arguing as in Remark 2.1 we have that for each $u\in H^1_0(\Gamma;W)$ and, more generally, for each $u\in H^1(\Gamma)$ such that $u(V_\alpha)=0$ for some $\alpha=1,\dots,d$, the Poincaré inequality

$$\|u\|_{L^2(\Gamma)}\le l^{1/2}\|u\|_{L^\infty}\le l\|u'\|_{L^2(\Gamma)}, \tag{2.24}$$

holds, where

$$\|u'\|^2_{L^2(\Gamma)}:=\int_\Gamma|u'|^2\,dx:=\sum_{i,j}\int_0^{l_{ij}}|u'_{ij}|^2\,dx.$$

On the metric graph Γ, we consider the Dirichlet Energy with respect to W:

$$\mathcal{E}(\Gamma;W)=\inf\big\{J(u)\ :\ u\in H^1_0(\Gamma;W)\big\}, \tag{2.25}$$

where the functional $J:H^1_0(\Gamma;W)\to\mathbb{R}$ is defined by

$$J(u)=\frac{1}{2}\int_\Gamma|u'|^2dx-\int_\Gamma u\,dx. \tag{2.26}$$

Lemma 2.12. *Given a metric graph Γ of length l and Dirichlet points $\{W_1,\dots,W_k\}\subset V(\Gamma)=\{V_1,\dots,V_N\}$, there is a unique function $w=(w_{ij})_{1\le i\ne j\le d}\in H^1_0(\Gamma;W)$ which minimizes the functional J. Moreover, we have*

1. *for each $1\le i\ne j\le d$ and each $t\in(0,l_{ij})$, $-w''_{ij}=1$;*
2. *at every vertex $V_i\in V(\Gamma)$, which is not a Dirichlet point, w satisfies the Kirchhoff's law:*

$$\sum_j w'_{ij}(0)=0,$$

where the sum is over all j for which the edge e_{ij} exists;

Furthermore, the conditions (i) *and* (ii) *uniquely determine* w.

Proof. The existence is a consequence of Remark 2.10 and the uniqueness is due to the strict convexity of the L^2 norm. for each $n \in \mathbb{N}$ γ_{ij} to a function w_{ij}. Defining $w = (w_{ij})_{ij} \in H_0^1(\Gamma; D_1, \dots, D_k)$, by the lower semicontinuity of the L^2 norm of the gradient on each edge, we obtain the existence. For any $\varphi \in H_0^1(\Gamma; W)$, we have that 0 is a critical point for the function

$$\varepsilon \mapsto \frac{1}{2}\int_\Gamma |(w + \varepsilon\varphi)'|^2 dx - \int_\Gamma (w + \varepsilon\varphi)\, dx.$$

Since φ is arbitrary, we obtain the first claim. The Kirchhoff's law at the vertex V_i follows by choosing φ supported in a "small neighborhood" of V_i. The last claim is due to the fact that if $u \in H_0^1(\Gamma; W)$ satisfies (i) and (ii), then it is an extremal for the convex functional J and so, $u = w$. □

Remark 2.13. As in Remark 2.5 we have that the co-area formula holds for the functions $u \in H^1(\Gamma)$ and any positive Borel (on each edge) function $v : \Gamma \to \mathbb{R}$:

$$\begin{aligned}\int_\Gamma v(x)|u'|(x)\, dx &= \sum_{1 \le i < j \le d} \int_0^{l_{ij}} |u'_{ij}(x)|\, v(x)\, dx \\ &= \sum_{1 \le i < j \le d} \int_0^{+\infty} \Big(\sum_{u_{ij}(x) = \tau} v(x) \Big)\, d\tau \\ &= \int_0^{+\infty} \Big(\sum_{u(x)=\tau} v(x) \Big)\, d\tau.\end{aligned} \tag{2.27}$$

2.3.1 Optimization problem for the Dirichlet Energy on the class of metric graphs

We say that the continuous function $\gamma = (\gamma_{ij})_{1 \le i \ne j \le d} : \Gamma \to \mathbb{R}^d$ is an *immersion* of the metric graph Γ into $\mathbb{R}^d$, if for each $1 \le i \ne j \le d$ the function $\gamma_{ij} : [0, l_{ij}] \to \mathbb{R}^d$ is an injective arc-length parametrized curve. We say that $\gamma : \Gamma \to \mathbb{R}^d$ is an *embedding*, if it is an immersion which is also injective, i.e. for any $i \ne j$ and $i' \ne j'$, we have

1. $\gamma_{ij}((0, l_{ij})) \cap \gamma_{i'j'}([0, l_{i'j'}]) = \emptyset$,
2. $\gamma_{ij}(0) = \gamma_{i'j'}(0)$, if and only if, $i = i'$.

Remark 2.14. Suppose that Γ is a metric graph of length l and that $\gamma : \Gamma \to \mathbb{R}^d$ is an embedding. Then the set $C := \gamma(\Gamma)$ is rectifiable of length $\mathcal{H}^1(\gamma(\Gamma)) = l$ and the spaces $H^1(\Gamma)$ and $H^1(C)$ are isometric as Hilbert spaces, where the isomorphism is given by the composition with the function γ.

Indeed, the topology of the embedded is uniquely determined by the topology of its representation in $\mathbb{R}^d$. This is not the case of the larger class of immersed graphs, where the intersections are allowed and new vertices may appear.

Consider a finite set of distinct points $\mathcal{D} = \{D_1, \dots, D_k\} \subset \mathbb{R}^d$ and let $l \geq St(\mathcal{D})$, where $St(\mathcal{D})$ is the length of the Steiner set, the minimal among the ones connecting all the points D_i (see [4] for more details on the Steiner problem). Consider the shape optimization problem:

$$\min \Big\{ \mathcal{E}(\Gamma; \mathcal{V}) \ : \ \Gamma \in CMG,\ l(\Gamma) = l,\ \mathcal{V} \subset V(\Gamma),\ \exists \gamma : \Gamma \to \mathbb{R}^d \text{ immersion},\ \gamma(\mathcal{V}) = \mathcal{D} \Big\}, \tag{2.28}$$

where CMG indicates the class of connected metric graphs. Note that since $l \geq St(\mathcal{D})$, there is a metric graph and an *embedding* $\gamma : \Gamma \to \mathbb{R}^d$ such that $\mathcal{D} \subset \gamma(V(\Gamma))$ and so the admissible set in the problem (2.28) is non-empty, as well as the admissible set in the problem

$$\min \Big\{ \mathcal{E}(\Gamma; \mathcal{V}) \ : \ \Gamma \in CMG,\ l(\Gamma) = l,\ \mathcal{V} \subset V(\Gamma),\ \exists \gamma : \Gamma \to \mathbb{R}^d \text{ embedding},\ \gamma(\mathcal{V}) = \mathcal{D} \Big\}. \tag{2.29}$$

We will see in Theorem 2.18 that problem (2.28) admits a solution, while Example 2.23 shows that in general an optimal embedded graph for problem (2.29) may not exist.

Remark 2.15. By Remark 2.14 and by the fact that the functionals we consider are invariant with respect to the isometries of the Sobolev space, we have that the problems (2.13) and (2.29) are equivalent, i.e. if $\Gamma \in CMG$ and $\gamma : \Gamma \to \mathbb{R}^d$ is an embedding such that the pair (Γ, γ) is a solution of (2.29), then the set $\gamma(\Gamma)$ is a solution of the problem (2.13). On the other hand, if C is a solution of the problem (2.13), by Theorem 2.7, we can suppose that $C = \bigcup_{i=1}^d \gamma_i([0, l_i])$, where γ_i are injective arc-length parametrized curves, which does not intersect internally. Thus, we can construct a metric graph Γ with vertices the set of points $\{\gamma_i(0), \gamma_i(l_i)\}_{i=1}^d \subset \mathbb{R}^d$, and d edges of lengths l_i such that two vertices are connected by an edge, if and only if they are the endpoints of the same curve γ_i. The function $\gamma = (\gamma_i)_{i=1,\dots,d} : \Gamma \to \mathbb{R}^d$ is an embedding by construction and by Remark 2.14, we have $\mathcal{E}(C; \mathcal{D}) = \mathcal{E}(\Gamma; \mathcal{D})$.

Theorem 2.16. *Let $\mathcal{D} = \{D_1, \dots, D_k\} \subset \mathbb{R}^d$ be a finite set of points and let $l \geq St(\mathcal{D})$ be a positive real number. Suppose that Γ is a connected metric graph of length l, $\mathcal{V} \subset V(\Gamma)$ is a set of vertices of Γ and $\gamma : \Gamma \to \mathbb{R}^d$ is an immersion (embedding) such that $\mathcal{D} = \gamma(\mathcal{V})$. Then there exist a connected metric graph $\widetilde{\Gamma}$ of at most $2k$ vertices and $2k-1$ edges, a set $\widetilde{\mathcal{V}} \subset V(\widetilde{\Gamma})$ of vertices of $\widetilde{\Gamma}$ and an immersion (embedding) $\widetilde{\gamma} : \widetilde{\Gamma} \to \mathbb{R}^d$ such that $\mathcal{D} = \widetilde{\gamma}(\widetilde{\mathcal{V}})$ and*

$$\mathcal{E}(\widetilde{\Gamma}; \widetilde{\mathcal{V}}) \leq \mathcal{E}(\Gamma; \mathcal{V}). \tag{2.30}$$

Proof. We repeat the argument from Theorem 2.7. We first construct a connected metric graph Γ' such that $V(\Gamma') \subset V(\Gamma)$ and the edges of Γ' are appropriately chosen paths in Γ. The edges of Γ, which are not part of any of these paths, are symmetrized in a single edge, which we attach to Γ' in a point, where the restriction of w to Γ' achieves its maximum, where w is the Energy function for Γ.

Suppose that $V_1,\dots,V_k \in \mathcal{V} \subset V(\Gamma)$ are such that $\gamma(V_i) = D_i$, $i = 1,\dots,k$. We start constructing Γ' by taking $\widetilde{\mathcal{V}} := \{V_1,\dots,V_k\} \subset V(\Gamma')$. Let $\sigma_1 = \{V_{i_0}, V_{i_1},\dots,V_{i_s}\}$ be a path of different vertices (i.e. simple path) connecting $V_1 = V_{i_s}$ to $V_2 = V_{i_0}$ and let $\tilde{\sigma}_2 = \{V_{j_0}, V_{j_1},\dots,V_{j_t}\}$ be a simple path connecting $V_1 = V_{j_t}$ to $V_3 = V_{j_0}$. Let $t' \in \{1,\dots,t\}$ be the smallest integer such that $V_{j_{t'}} \in \sigma_1$. Then we set $V_{j_{t'}} \in V(\Gamma')$ and $\sigma_2 = \{V_{j_0}, V_{j_1},\dots,V_{j_{t'}}\}$. Consider a simple path $\tilde{\sigma}_3 = \{V_{m_0}, V_{m_1},\dots,V_{m_r}\}$ connecting $V_1 = V_{m_r}$ to $V_3 = V_{m_0}$ and the smallest integer r' such that $V_{m_{r'}} \in \sigma_1 \cup \sigma_2$. We set $V_{m_{r'}} \in V(\Gamma')$ and $\sigma_3 = \{V_{m_0}, V_{m_1},\dots,V_{m_{r'}}\}$. We continue the operation until each of the points $V_1,\dots,V_k$ is in some path σ_j. Thus we obtain the set of vertices $V(\Gamma')$. We define the edges of Γ' by saying that $\{V_i, V_{i'}\} \in E(\Gamma')$ if there is a simple path σ connecting V_i to $V_{i'}$ and which is contained in some path σ_j from the construction above; the length of the edge $\{V_i, V_{i'}\}$ is the sum of the lengths of the edges of Γ which are part of σ. We notice that $\Gamma' \in CMG$ is a tree with at most $2k-2$ vertices and $2k-2$ edges. Moreover, even if Γ' is not a subgraph of Γ ($E(\Gamma')$ may not be a subset of $E(\Gamma)$), we have the inclusion $H^1(\Gamma') \subset H^1(\Gamma)$.

Consider the set $E'' \subset E(\Gamma)$ composed of the edges of Γ which are not part of any of the paths σ_j from the construction above. We denote with l'' the sum of the lengths of the edges in E''. For any $e_{ij} \in E''$ we consider the restriction $w_{ij} : [0, l_{ij}] \to \mathbb{R}$ of the Energy function w on e_{ij}. Let $v : [0, l''] \to \mathbb{R}$ be the monotone function defined by the equality $|\{v \geq \tau\}| = \sum_{e_{ij} \in E''} |\{w_{ij} \geq \tau\}|$. Using the co-area formula (2.27) and repeating the argument from Remark 2.14, we have that

$$\frac{1}{2}\int_0^{l''} |v'|^2 dx - \int_0^{l''} v(x)\,dx \leq \sum_{e_{ij}\in E''} \left(\frac{1}{2}\int_0^{l_{ij}} |w_{ij}'|^2 dx - \int_0^{l_{ij}} w_{ij}\,dx\right). \tag{2.31}$$

Let $\widetilde{\Gamma}$ be the graph obtained from Γ by creating a new vertex W_1 in the point, where the restriction $w_{|\Gamma'}$ achieves its maximum, and another vertex W_2, connected to W_1 by an edge of length l''. It is straightforward to check that $\widetilde{\Gamma}$ is a connected metric tree of length l and that there exists an immersion $\widetilde{\gamma} : \widetilde{\Gamma} \to \mathbb{R}^d$ such that $\mathcal{D} = \widetilde{\gamma}(\widetilde{\mathcal{V}})$. The inequality (2.30) follows since, by (2.31), $J(\widetilde{w}) \leq J(w)$, where $\widetilde{w}$ is defined as w on the edges $E(\Gamma') \subset E(\widetilde{\Gamma})$ and as v on the edge $\{W_1, W_2\}$. □

Before we prove our main existence result, we need a preliminary Lemma.

Lemma 2.17. *Let Γ be a connected metric tree and let $\mathcal{V} \subset V(\Gamma)$ be a set of Dirichlet vertices. Let $w \in H_0^1(\Gamma;\mathcal{V})$ be the Energy function on Γ with Dirichlet conditions in $\mathcal{V}$, i.e. the function that realizes the minimum in the definition of $\mathcal{E}(\Gamma;\mathcal{V})$. Then, we have the bound $\|w'\|_{L^\infty} \leq l(\Gamma)$.*

Proof. Up to adding vertices in the points where $|w'| = 0$, we can suppose that on each edge $e_{ij} := \{V_i, V_j\} \in E(\Gamma)$ the function $w_{ij} : [0, l_{ij}] \to \mathbb{R}^+$ is monotone. Moreover, up to relabel the vertices of Γ we can suppose that if $e_{ij} \in V(\Gamma)$ and $i < j$, then $w(V_i) \leq w(V_j)$. Fix $V_i, V_{i'} \in V(\Gamma)$ such that $e_{ii'} \in E(\Gamma)$. We will prove that $|w'_{ii'}(0)| \leq l(\Gamma)$. It is enough to consider the case $i < i'$, i.e. $w'_{ii'}(0) > 0$. We construct the graph $\widetilde{\Gamma}$ inductively, as follows (see Figure 2.3):

1. $V_i \in V(\widetilde{\Gamma})$;
2. if $V_j \in V(\widetilde{\Gamma})$ and $V_k \in V(\Gamma)$ are such that $e_{jk} \in E(\Gamma)$ and $j < k$, then $V_k \in V(\widetilde{\Gamma})$ and $e_{jk} \in E(\widetilde{\Gamma})$.

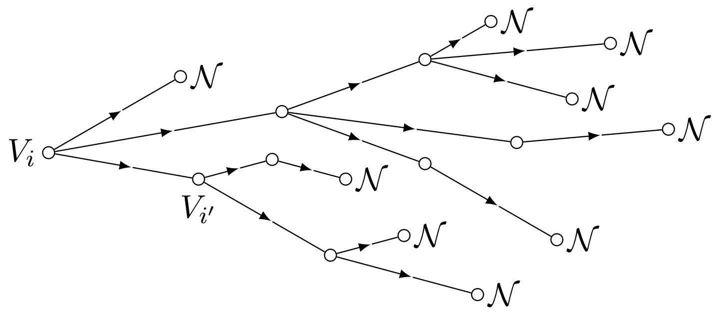

Figure 2.3: The graph $\widetilde{\Gamma}$; the letter $\mathcal{N}$ labels the vertices in which $w' = 0$.

The graph $\widetilde{\Gamma}$ constructed by the above procedure and the restriction $\widetilde{w} \in H^1(\widetilde{\Gamma})$ of w to $\widetilde{\Gamma}$ have the following properties:

1. On each edge $e_{jk} \in E(\widetilde{\Gamma})$, the function $\widetilde{w}_{jk} \geq 0$ is positive, monotone and $\widetilde{w}''_{jk} = -1$;
2. $\widetilde{w}(V_j) > \widetilde{w}(V_k)$ whenever $e_{jk} \in E(\widetilde{\Gamma})$ and $j > k$;
3. if $V_j \in V(\widetilde{\Gamma})$ and $j > i$, then there is exactly one $k < j$ such that $e_{kj} \in E(\widetilde{\Gamma})$;
4. for j and k as in the previous point, we have that
$$0 \leq \widetilde{w}'_{kj}(l_{kj}) \leq \sum_s \widetilde{w}'_{js}(0),$$
where the sum on the right-hand side is over all $s > j$ such that $e_{sj} \in E(\widetilde{\Gamma})$. If there are not such s, we have that $\widetilde{w}'_{kj}(l_{kj}) = 0$.

We prove that for any graph $\widetilde{\Gamma}$ and any function $\widetilde{w} \in H^1(\widetilde{\Gamma})$, for which the conditions (a), (b), (c) and (d) are satisfied, we have that
$$\sum_j \widetilde{w}'_{ij}(0) \leq l(\widetilde{\Gamma}),$$

where the sum is over all $j \geq i$ and $e_{ij} \in E(\widetilde{\Gamma})$. It is enough to observe that each of the operations (i) and (ii) described below, produces a graph which still satisfies (a), (b), (c) and (d). Let $V_j \in V(\widetilde{\Gamma})$ be such that for each $s > j$ for which $e_{js} \in E(\widetilde{\Gamma})$, we have that $\widetilde{w}'_{js}(l_{js}) = 0$ and let $k < j$ be such that $e_{jk} \in E(\widetilde{\Gamma})$.

1. If there is only one $s > j$ with $e_{js} \in E(\widetilde{\Gamma})$, then we erase the vertex V_j and the edges e_{kj} and e_{js} and add the edge e_{ks} of length $l_{ks} := l_{kj} + l_{js}$. On the new edge we define $\widetilde{w}_{ks} : [0, l_{sk}] \to \mathbb{R}^+$ as
$$\widetilde{w}_{ks}(x) = -\frac{x^2}{2} + l_{ks}\, x + \widetilde{w}_{kj}(0),$$
which still satisfies the conditions above since $\widetilde{w}'_{kj} - l_{kj} \leq l_{js}$, by (d), and $\widetilde{w}'_{ks} = l_{ks} \geq \widetilde{w}'_{kj}(0)$.

2. If there are at least two $s > j$ such that $e_{js} \in E(\widetilde{\Gamma})$, we erase all the vertices V_s and edges e_{js}, substituting them with a vertex V_S connected to V_j by an edge e_{jS} of length
$$l_{jS} := \sum_s l_{js},$$
where the sum is over all $s > j$ with $e_{js} \in E(\widetilde{\Gamma})$. On the new edge, we consider the function $\widetilde{w}_{jS}$ defined by
$$\widetilde{w}_{jS}(x) = -\frac{x^2}{2} + l_{jS}\, x + \widetilde{w}(V_j),$$
which still satisfies the conditions above since
$$\sum_{s:s>j} \widetilde{w}'_{js}(0) = \sum_{s:s>j} l_{js} = l_{jS} = \widetilde{w}'_{jS}(0).$$

We apply (i) and (ii) until we obtain a graph with vertices V_i, V_j and only one edge e_{ij} of length $l(\widetilde{\Gamma})$. The function we obtain on this graph is $-\frac{x^2}{2} + l(\widetilde{\Gamma})x$ with derivative in 0 equal to $l(\widetilde{\Gamma})$. Since, after applying (i) and (ii), the sum $\sum_{j>i} \widetilde{w}'_{ij}(0)$ does not decrease, we get that the claim is true. $\square$

Theorem 2.18. *Consider a set of distinct points $\mathcal{D} = \{D_1, \dots, D_k\} \subset \mathbb{R}^d$ and a positive real number $l \geq St(\mathcal{D})$. Then there exists a connected metric graph Γ, a set of vertices $\mathcal{V} \subset V(\Gamma)$ and an immersion $\gamma : \Gamma \to \mathbb{R}^d$ which are solution of the problem* (2.28)*. Moreover, Γ can be chosen to be a tree of at most $2k$ vertices and $2k-1$ edges.*

Proof. Consider a minimizing sequence (Γ_n, γ_n) of connected metric graphs Γ_n and immersions $\gamma_n : \Gamma_n \to \mathbb{R}^d$. By Theorem 2.16, we can suppose that each Γ_n is a tree with at most $2k$ vertices and $2k-1$ edges. Up to extracting a subsequence, we may assume that the metric graphs Γ_n are the same graph Γ but with different lengths l_{ij}^n of the edges e_{ij}. We can suppose that for each $e_{ij} \in E(\Gamma)$ $l_{ij}^n \to l_{ij}$ for some $l_{ij} \geq 0$ as $n \to \infty$. We construct the graph $\widetilde{\Gamma}$ from Γ identifying the vertices $V_i, V_j \in V(\Gamma)$ such that $l_{ij} = 0$. The graph $\widetilde{\Gamma}$ is a connected metric tree of length l and there is an immersion $\widetilde{\gamma} : \widetilde{\Gamma} \to \mathbb{R}^d$ such that $\mathcal{D} \subset \widetilde{\gamma}(\widetilde{\Gamma})$. In fact if $\{V_1, \dots V_N\}$ are the vertices of Γ, up to extracting a subsequence, we can suppose that for each $i = 1, \dots, d$ $\gamma_n(V_i) \to X_i \in \mathbb{R}^d$. We define $\widetilde{\gamma}(V_i) := X_i$ and $\gamma_{ij} : [0, l_{ij}] \to \mathbb{R}^d$ as any injective arc-length parametrized curve connecting X_i and X_j, which exists, since

$$l_{ij} = \lim l_{ij}^n \geq \lim |\gamma_n(V_i) - \gamma_n(V_j)| = |X_i - X_j|.$$

To prove the theorem, it is enough to check that

$$\mathcal{E}(\widetilde{\Gamma}; \mathcal{V}) = \lim_{n\to\infty} \mathcal{E}(\Gamma_n; \mathcal{V}).$$

Let $w^n = (w_{ij}^n)_{ij}$ be the Energy function on Γ_n. Up to a subsequence, we may suppose that for each $i = 1, \dots, N$, $w^n(V_i) \to a_i \in \mathbb{R}$ as $n \to \infty$. Moreover, by Lemma 2.17, we have that if $l_{ij} = 0$, then $a_i = a_j$. On each of the edges $e_{ij} \in E(\widetilde{\Gamma})$, where $l_{ij} > 0$, we define the function $w_{ij} : [0, l_{ij}] \to \mathbb{R}$ as the parabola such that $w_{ij}(0) = a_i$, $w_{ij}(l_{ij}) = a_j$ and $w_{ij}'' = -1$ on $(0, l_{ij})$. Then, we have

$$\frac{1}{2}\int_0^{l_{ij}^n} |(w_{ij}^n)'|^2\,dx - \int_0^{l_{ij}^n} w_{ij}^n\,dx \xrightarrow[n\to\infty]{} \frac{1}{2}\int_0^{l_{ij}} |(w_{ij})'|^2\,dx - \int_0^{l_{ij}} w_{ij}\,dx,$$

and so, it is enough to prove that $\widetilde{w} = (w_{ij})_{ij}$ is the Energy function on $\widetilde{\Gamma}$, i.e. (by Lemma 2.12) that the Kirchoff's law holds in each vertex of $\widetilde{\Gamma}$. This follows since for each $1 \leq i \neq j \leq N$ we have

1. $(w_{ij}^n)'(0) \to w_{ij}'(0)$, as $n \to \infty$, if $l_{ij} \neq 0$;
2. $|(w_{ij}^n)'(0) - (w_{ij}^n)'(l_{ij}^n)| \leq l_{ij}^n \to 0$, as $n \to \infty$, if $l_{ij} = 0$.

The proof is then concluded. $\square$

The proofs of Theorem 2.16 and Theorem 2.18 suggest that a solution $(\Gamma, \mathcal{V}, \gamma)$ of the problem (2.28) must satisfy some optimality conditions. We summarize these additional informations in the following Proposition.

Proposition 2.19. *Consider a connected metric graph Γ, a set of vertices $\mathcal{V} \subset V(\Gamma)$ and an immersion $\gamma : \Gamma \to \mathbb{R}^d$ such that $(\Gamma, \mathcal{V}, \gamma)$ is a solution of the problem (2.28). Moreover, suppose that all the vertices of degree two are in the set $\mathcal{V}$. Then we have that:*

1. *the graph Γ is a tree;*

2. *the set* $\mathcal{V}$ *has exactly* k *elements, where* k *is the number of Dirichlet points* $\{D_1, \dots, D_k\}$*;*

3. *there is at most one vertex* $V_j \in V(\Gamma) \setminus \mathcal{V}$ *of degree one;*

4. *if there is no vertex of degree one in* $V(\Gamma) \setminus \mathcal{V}$*, then the graph* Γ *has at most* $2k-2$ *vertices and* $2k-3$ *edges;*

5. *if there is exactly one vertex of degree one in* $V(\Gamma) \setminus \mathcal{V}$*, then the graph* Γ *has at most* $2k$ *vertices and* $2k-1$ *edges.*

Proof. We use the notation $V(\Gamma) = \{V_1, \dots, V_N\}$ for the vertices of Γ and e_{ij} for the edges $\{V_i, V_j\} \in E(\Gamma)$, whose lengths are denoted by l_{ij}. Moreover, we can suppose that for $j = 1, \dots, k$, we have $\gamma(V_j) = D_j$, where $D_1, \dots, D_k$ are the Dirichlet points from problem (2.28) and so, $\{V_1, \dots, V_k\} \subset \mathcal{V}$. Let $w = (w_{ij})_{ij}$ be the Energy function on Γ with Dirichlet conditions in the points of $\mathcal{V}$.

1. Suppose that we can remove an edge $e_{ij} \in E(\Gamma)$, such that the graph $\Gamma' = (V(\Gamma), E(\Gamma) \setminus e_{ij})$ is still connected. Since $w''_{ij} = -1$ on $[0, l_{ij}]$ we have that at least one of the derivatives $w'_{ij}(0)$ and $w'_{ij}(l_{ij})$ is not zero. We can suppose that $w'_{ij}(l_{ij}) \neq 0$. Consider the new graph $\widetilde{\Gamma}$ to which we add a new vertex: $V(\widetilde{\Gamma}) = V(\Gamma) \cup V_0$, then erase the edge e_{ij} and create a new one $e_{i0} = \{V_i, V_0\}$, of the same length, connecting V_i to V_0: $E(\widetilde{\Gamma}) = (E(\Gamma) \setminus e_{ij}) \cup e_{i0}$. Let $\widetilde{w}$ be the Energy function on $\tilde{\Gamma}$ with Dirichlet conditions in $\mathcal{V}$. When seen as a subspaces of $\oplus_{ij} H^1([0, l_{ij}])$, we have that $H^1_0(\Gamma; \mathcal{V}) \subset H^1_0(\widetilde{\Gamma}; \mathcal{V})$ and so $\mathcal{E}(\widetilde{\Gamma}; \mathcal{V}) \leq \mathcal{E}(\Gamma; \mathcal{V})$, where the equality occurs, if and only if the Energy functions w and $\widetilde{w}$ have the same components in $\oplus_{ij} H^1([0, l_{ij}])$. In particular, we must have that $w_{ij} = \widetilde{w}_{i0}$ on the interval $[0, l_{ij}]$, which is impossible since $w'_{ij}(l_{ij}) \neq 0$ and $\widetilde{w}'_{i0}(l_{ij}) = 0$.

2. Suppose that there is a vertex $V_j \in \mathcal{V}$ with $j > k$ and let $\widetilde{w}$ be the Energy function on Γ with Dirichlet conditions in $\{V_1, \dots, V_k\}$. We have the inclusion $H^1_0(\Gamma; \mathcal{V}) \subset H^1_0(\Gamma; \{V_1, \dots, V_k\})$ and so, the inequality $J(\widetilde{w}) = \mathcal{E}(\Gamma; \{V_1, \dots, V_k\}) \leq \mathcal{E}(\Gamma; \mathcal{V}) = J(w)$, which becomes an equality if and only if $\widetilde{w} = w$, which is impossible. Indeed, if the equality holds, then in V_j, w satisfies both the Dirichlet condition and the Kirchoff's law. Since w is positive, for any edge e_{ji} we must have $w_{ji}(0) = 0$, $w'_{ji}(0) = 0$, $w''_{ji} = -1$ ad $w_{ji} \geq 0$ on $[0, l_{ji}]$, which is impossible.

3. Suppose that there are two vertices V_i and V_j of degree one, which are not in $\mathcal{V}$, i.e. $i, j > k$. Since Γ is connected, there are two edges, $e_{ii'}$ and $e_{jj'}$ starting from V_i and V_j respectively. Suppose that the Energy function $w \in H^1_0(\Gamma; \{V_1, \dots, V_k\})$ is such that $w(V_i) \geq w(V_j)$. We define a new graph $\tilde{\Gamma}$ by erasing the edge $e_{jj'}$ and creating the edge e_{ij} of length $l_{jj'}$. On the new edge e_{ij} we consider the function $w_{ij}(x) = w_{jj'}(x) + w(V_i) - w(V_j)$. The function $\widetilde{w}$ on $\widetilde{\Gamma}$ obtained by this construction is such that $J(\widetilde{w}) \leq J(w)$, this concludes the proof.

The points (iv) and (v) follow by the construction in Theorem 2.16 and the previous claims (i), (ii) and (iii). $\square$

Remark 2.20. Suppose that $V_j \in V(\Gamma) \setminus \mathcal{V}$ is a vertex of degree one and let V_i be the vertex such that $e_{ij} \in E(\Gamma)$. Then the Energy function w with Dirichlet conditions in $\mathcal{V}$ satisfies $w'_{ji}(0) = 0$. In this case, we call V_j a Neumann vertex. By Proposition 2.19, an optimal graph has at most one Neumann vertex.

2.4 Some examples of optimal metric graphs

In this section we show three examples. In the first one we deal with two Dirichlet points, the second concerns three aligned Dirichlet points and the third one deals with the case in which the Dirichlet points are vertices of an equilateral triangle. In the first and the third one we find the minimizer explicitly as an embedded graph, while in the second one we limit ourselves to prove that there is no embedded minimizer of the energy, i.e. that problem (2.29) does not admit a solution.

In the following example we use a symmetrization technique similar to the one from Remark 2.6.

Example 2.21. Let D_1 and D_2 be two distinct points in $\mathbb{R}^d$ and let $l \geq |D_1 - D_2|$ be a real number. Then the problem

$$\begin{aligned} \min\{\mathcal{E}(\Gamma; \{V_1, V_2\}) : \Gamma \in CMG,\ l(\Gamma) = l,\ V_1, V_2 \in V(\Gamma), \\ \text{exists } \gamma : \Gamma \to \mathbb{R} \text{ immersion},\ \gamma(V_1) = D_1, \gamma(V_2) = D_2\}. \end{aligned} \tag{2.32}$$

has a solution (Γ, γ), where Γ is a metric graph with vertices $V(\Gamma) = \{V_1, V_2, V_3, V_4\}$ and edges $E(\Gamma) = \{e_{13} = \{V_1, V_3\}, e_{23} = \{V_2, V_3\}, e_{43} = \{V_4, V_3\}\}$ of lengths $l_{13} = l_{23} = \frac{1}{2}|D_1 - D_2|$ and $l_{34} = l - |D_1 - D_2|$, respectively. The map $\gamma : \Gamma \to \mathbb{R}^d$ is an embedding such that $\gamma(V_1) = D_1$, $\gamma(V_2) = D_2$ and $\gamma(V_3) = \frac{D_1+D_2}{2}$ (see Figure 2.4).

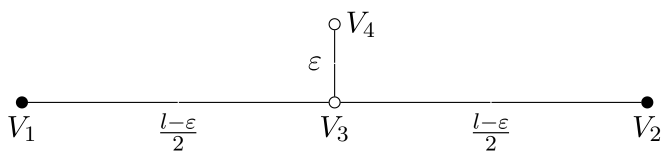

Figure 2.4: The optimal graph with two Dirichlet points.

To fix the notations, we suppose that $|D_1 - D_2| = l - \varepsilon$. Let $u = (u_{ij})_{ij}$ be the Energy function of a generic metric graph Σ and immersion $\sigma : \Sigma \to \mathbb{R}^d$ with $D_1, D_2 \in \sigma(V(\Sigma))$. Let $M = \max\{u(x) : x \in \Sigma\} > 0$. We construct a candidate $v \in H^1_0(\Gamma; \{V_1, V_2\})$ such that $J(v) \leq J(u)$, which immediately gives the conclusion.

We define v through the following three *increasing* functions

$$v_{13} = v_{23} \in H^1([0,(l-\varepsilon)/2]), \qquad v_{34} \in H^1([0,\varepsilon]),$$

with boundary values

$$v_{13}(0) = v_{23}(0) = 0, \qquad v_{13}((l-\varepsilon)/2) = v_{23}((l-\varepsilon)/2) = v_{34}(0) = m < M,$$

and level sets uniquely determined by the equality $\mu_u = \mu_v$, where μ_u and μ_v are the distribution functions of u and v respectively, defined by

$$\mu_u(t) = \mathcal{H}^1(\{u \le t\}) = \sum_{e_{ij}\in E(\Sigma)} \mathcal{H}^1(\{u_{ij} \le t\}),$$
$$\mu_v(t) = \mathcal{H}^1(\{v \le t\}) = \sum_{j=1,2,4} \mathcal{H}^1(\{v_{j3} \le t\}).$$

As in Remark 2.6 we have $\|v\|_{L^1(\Gamma)} = \|u\|_{L^1(C)}$ and

$$\int_\Sigma |u'|^2\,dx = \int_0^M \Big(\sum_{u=\tau}|u'|\Big)\,d\tau \ge \int_0^M n_u^2(\tau)\left(\sum_{u=\tau}\frac{1}{|u'|(\tau)}\right)^{-1} d\tau = \int_0^M \frac{n_u^2(\tau)}{\mu_u'(\tau)}\,d\tau \quad (2.33)$$

where $n_u(\tau) = \mathcal{H}^0(\{u=\tau\})$. The same argument holds for v on the graph Γ but, this time, with the equality sign:

$$\int_\Gamma |v'|^2 dx = \int_0^M \Big(\sum_{v=\tau}|v'|\Big)\,d\tau = \int_0^M \frac{n_v^2(\tau)}{\mu_v'(\tau)}\,d\tau, \qquad (2.34)$$

since $|v'|$ is constant on $\{v=\tau\}$, for every τ. Then, in view of (2.33) and (2.34), to conclude it is enough to prove that $n_u(\tau) \ge n_v(\tau)$ for almost every τ. To this aim we first notice that, by construction $n_v(\tau) = 1$ if $\tau \in [m,M]$ and $n_v(\tau) = 2$ if $\tau \in [0,m)$. Since n_u is decreasing and greater than 1 on $[0,M]$, we only need to prove that $n_u \ge 2$ on $[0,m]$. To see this, consider two vertices $W_1, W_2 \in V(\Sigma)$ such that $\sigma(W_1) = D_1$ and $\sigma(W_2) = D_2$. Let η be a simple path connecting W_1 to W_2 in Σ. Since σ is an immersion we know that the length $l(\eta)$ of η is at least $l-\varepsilon$. By the continuity of u, we know that $n_u \ge 2$ on the interval $[0, \max_\eta u)$. Since $n_v = 1$ on $[m,M]$, we need to show that $\max_\eta u \ge m$. Otherwise, we would have

$$l(\eta) \le |\{u \le \max_\eta u\}| < |\{u \le m\}| = |\{v \le m\}| = |D_1 - D_2| \le l(\eta),$$

which is impossible.

Remark 2.22. In the previous example the optimal metric graph Γ is such that for any (admissible) immersion $\gamma : \Gamma \to \mathbb{R}^d$, we have $|\gamma(V_1) - \gamma(V_3)| = l_{13}$ and $|\gamma(V_2) - \gamma(V_3)| = l_{23}$, i.e. the point $\gamma(V_3)$ is necessary the midpoint $\frac{D_1+D_2}{2}$, so we have a sort of *rigidity* of the graph Γ. More generally, we say that an edge e_{ij} is *rigid*, if for any admissible immersion $\gamma : \Gamma \to \mathbb{R}^d$, i.e. an immersion such that $\mathcal{D} = \gamma(\mathcal{V})$, we have $|\gamma(V_i) - \gamma(V_j)| = l_{ij}$, in other words the realization of the edge e_{ij} in $\mathbb{R}^d$ via any immersion γ is a segment. One may expect that in the optimal graph all the edges, except the one containing the Neumann vertex, are rigid. Unfortunately, we are able to prove only the weaker result that:

1. if the Energy function w, of an optimal metric graph Γ, has a local maximum in the interior of an edge e_{ij}, then the edge is rigid; if the maximum is global, then Γ has no Neumann vertices;

2. if Γ contains a Neumann vertex V_j, then w achieves its maximum at it.

To prove the second claim, we just observe that if it is not the case, then we can use an argument similar to the one from point (iii) of Proposition 2.19, erasing the edge e_{ij} containing the Neumann vertex V_j and creating an edge of the same length that connects V_j to the point, where w achieves its maximum, which we may assume a vertex of Γ (possibly of degree two).

For the first claim, we apply a different construction which involves a symmetrization technique. In fact, if the edge e_{ij} is not rigid, then we can create a new metric graph of smaller energy, for which there is still an immersion which satisfies the conditions in problem (2.28). In this there are points $0 < a < b < l_{ij}$ such that $l_{ij} - (b-a) \geq |\gamma(V_i) - \gamma(V_j)|$ and $\min_{[a,b]} w_{ij} = w_{ij}(a) = w_{ij}(b) < \max_{[a,b]} w_{ij}$. Since the edge is not rigid, there is an immersion γ such that $|\gamma_{ij}(a) - \gamma_{ij}(b)| > |b-a|$. The problem (2.32) with $D_1 = \gamma_{ij}(a)$ and $D_2 = \gamma_{ij}(b)$ has as a solution the $T-$like graph described in Example 2.21. This shows that the original graph could not be optimal, which is a contradiction.

Example 2.23. Consider the set of points $\mathcal{D} = \{D_1, D_2, D_3\} \subset \mathbb{R}^2$ with coordinates respectively $(-1, 0)$, $(1, 0)$ and $(n, 0)$, where n is a positive integer. Given $l = (n+2)$, we aim to show that for n large enough there is no solution of the optimization problem

$$\min\{\mathcal{E}(\Gamma;\mathcal{V}) : \Gamma \in CMG,\ l(\Gamma) = l,\ \mathcal{V} \subset V(\Gamma),\ \exists\gamma : \Gamma \to \mathbb{R} \text{ embedding},\ \mathcal{D} = \gamma(\mathcal{V})\}\,. \tag{2.35}$$

In fact, we show that all the possible solutions of the problem

$$\min\{\mathcal{E}(\Gamma;\mathcal{V}) : \Gamma \in CMG,\ l(\Gamma) = l,\ \mathcal{V} \subset V(\Gamma),\ \exists\gamma : \Gamma \to \mathbb{R} \text{ immersion},\ \mathcal{D} = \gamma(\mathcal{V})\} \tag{2.36}$$

are metric graphs Γ for which there is no embedding $\gamma : \Gamma \to \mathbb{R}^2$ such that $\mathcal{D} \subset \gamma(V(\Gamma))$. Moreover, there is a sequence of embedded metric graphs which is a minimizing sequence for the problem (2.36).

More precisely, we show that the only possible solution of (2.36) is one of the following metric trees:

1. Γ_1 with vertices $V(\Gamma_1) = \{V_1, V_2, V_3, V_4\}$ and edges $E(\Gamma_1) = \{e_{14} = \{V_1, V_4\}, e_{24} = \{V_2, V_4\}, e_{34} = \{V_3, V_4\}\}$ of lengths $l_{14} = l_{24} = 1$ and $l_{34} = n$, respectively. The set of vertices in which the Dirichlet condition holds is $\mathcal{V}_1 = \{V_1, V_2, V_3\}$.

2. Γ_2 with vertices $V(\Gamma_2) = \{W_i\}_{i=1}^6$, and edges $E(\Gamma_2) = \{e_{14}, e_{24}, e_{35}, e_{45}, e_{56}\}$,where $e_{ij} = \{W_i, W_j\}$ for $1 \leq i \neq j \leq 6$ of lengths $l_{14} = 1+\alpha$, $l_{24} = 1-\alpha$, $l_{35} = n-\beta$, $l_{45} = \beta - \alpha$, $l_{56} = \alpha$, where $0 < \alpha < 1$ and $\alpha < \beta < n$. The set of vertices in which the Dirichlet condition holds is $\mathcal{V}_1 = \{V_1, V_2, V_3\}$. A possible immersion γ is described in Figure 2.5.

Figure 2.5: The two candidates for a solution of (2.36).

We start showing that if there is an optimal metric graph with no Neumann vertex, then it must be Γ_1. In fact, by Proposition 2.19, we know that the optimal metric graph is of the form Γ_1, but we have no information on the lengths of the edges, which we set as $l_i = l(e_{i4})$, for $i = 1, 2, 3$ (see Figure 2.6). We can calculate explicitly the minimizer of the Energy functional and the energy itself in function of l_1, l_2 and l_3.

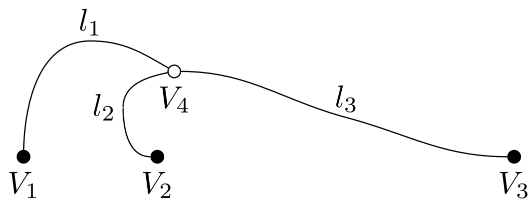

Figure 2.6: A metric tree with the same topology as Γ_1.

The minimizer of the energy $w : \Gamma \to \mathbb{R}$ is given by the functions $w_i : [0, l_i] \to \mathbb{R}$, where $i = 1, 2, 3$ and

$$w_i(x) = -\frac{x^2}{2} + a_i x. \tag{2.37}$$

where

$$a_1 = \frac{l_1}{2} + \frac{l_2 l_3 (l_1 + l_2 + l_3)}{2(l_1 l_2 + l_2 l_3 + l_3 l_1)}, \tag{2.38}$$

and a_2 and a_3 are defined by a cyclic permutation of the indices. As a consequence, we obtain that the derivative along the edge e_{14} in the vertex V_4 is given by

$$w_1'(l_1) = -l_1 + a_1 = -\frac{l_1}{2} + \frac{l_2 l_3 (l_1 + l_2 + l_3)}{2(l_1 l_2 + l_2 l_3 + l_3 l_1)}, \tag{2.39}$$

and integrating the Energy function w on Γ, we obtain

$$\mathcal{E}(\Gamma;\{V_1,V_2,V_3\}) = -\frac{1}{12}(l_1^3+l_2^3+l_3^3) - \frac{(l_1+l_2+l_3)^2 l_1 l_2 l_3}{4(l_1l_2+l_2l_3+l_3l_1)}. \tag{2.40}$$

Studying this function using Lagrange multipliers is somehow complicated due to the complexity of its domain. Thus we use a more geometric approach applying the symmetrization technique described in Remark 2.6 in order to select the possible candidates. We prove that if the graph is optimal, then all the edges must be rigid (this would force the graph to coincide with Γ_1). Suppose that the optimal graph Γ is not rigid, i.e. there is a non-rigid edge. Then, for $n>4$, we have that $l_2<l_1<l_3$ and so, by (2.39), we obtain $w_3'(l_3)<w_1'(l_1)<w_2'(l_2)$. As a consequence of the Kirchoff's law we have $w_3'(l_3)<0$ and $w_2'(l_2)>0$ and so, w has a local maximum on the edge e_{34} and is increasing on e_{14}. By Remark 2.22, we obtain that the edge e_{34} is rigid.

We first prove that $w_1'(l_1)>0$. In fact, if this is not the case, i.e. $w_1'(l_1)<0$, by Remark 2.22, we have that the edges e_{14} is also rigid and so, $l_1+l_3=|D_1-D_3|=n+1$, i.e. $l_2=1$. Moreover, by (2.39), we have that $w_1'(l_1)<0$, if and only if $l_1^2>l_2l_3=l_3$. The last inequality does not hold for $n>11$, since, by the triangle inequality, $l_2+l_3\geq |D_2-D_3|=n-1$, we have $l_1\leq 3$. Thus, for n large enough, we have that w is increasing on the edge e_{14}.

We now prove that the edges e_{14} and e_{24} are rigid. In fact, suppose that e_{24} is not rigid. Let $a\in(0,l_1)$ and $b\in(0,l_2)$ be two points close to l_1 and l_2 respectively and such that $w_{14}(a)=w_{24}(b)<w(V_4)$ since w_{14} and w_{24} are strictly increasing. Consider the metric graph $\widetilde{\Gamma}$ whose vertices and edges are

$$V(\widetilde{\Gamma}) = \{V_1=\widetilde{V}_1,\ V_2=\widetilde{V}_2,\ V_3=\widetilde{V}_3,\ V_4=\widetilde{V}_4,\ \widetilde{V}_5,\ \widetilde{V}_6\},$$

$$E(\widetilde{\Gamma}) = \{e_{15},\ e_{25},\ e_{45},\ e_{34},\ e_{46}\},$$

where $e_{ij}=\{\widetilde{V}_i,\widetilde{V}_j\}$ and the lengths of the edges are respectively (see Figure 2.7)

$$\widetilde{l}_{15}=a,\ \widetilde{l}_{25}=b,\ \widetilde{l}_{45}=l_2-b,\ \widetilde{l}_{34}=l_3,\ \widetilde{l}_{46}=l_1-a.$$

Figure 2.7: The graph Γ (on the left) and the modified one $\widetilde{\Gamma}$ (on the right).

The new metric graph is still a competitor in the problem (2.36) and there is a function $w\in H_0^1(\widetilde{\Gamma};\{V_1,V_2,V_3\})$ such that $\mathcal{E}(\widetilde{\Gamma};\{V_1,V_2,V_3\})<J(\widetilde{w})=J(w)$, which is a contradiction with the optimality of Γ. In fact, it is enough to define $\widetilde{w}$ as

$$\widetilde{w}_{15}=w_{14}|_{[0,a]},\ \widetilde{w}_{25}=w_{24}|_{[0,b]},\ \widetilde{w}_{54}=w_{24}|_{[b,l_2]},\ \widetilde{w}_{34}=w_{34},\ \widetilde{w}_{64}=w_{14}|_{[a,l_1]},$$

and observe that $\widetilde{w}$ is not the Energy function on the graph $\widetilde{\Gamma}$ since it does not satisfy the Neumann condition in $\widetilde{V}_6$. In the same way, if we suppose that w_{14} is not rigid, we obtain a contradiction, and so all the three edges must be rigid, i.e. $\Gamma = \Gamma_1$.

In a similar way we prove that a metric graph Γ with a Neumann vertex can be a solution of (2.36) only if it is of the same form as Γ_2. We proceed in two steps: first, we show that, for n large enough, the edge containing the Neumann vertex has a common vertex with the longest edge of the graph; then we can conclude reasoning analogously to the previous case. Let Γ be a metric graph with vertices $V(\Gamma) = \{V_i\}_{i=1}^6$, and edges $E(\Gamma) = \{e_{15}, e_{24}, e_{34}, e_{45}, e_{56}\}$, where $e_{ij} = \{V_i, V_j\}$ for $1 \leq i \neq j \leq 6$.

We prove that $w(V_6) \leq \max_{e_{34}} w$, i.e. the graph Γ is not optimal, since, by Remark 2.22, the maximum of w must be achieved in the Neumann vertex V_6 (the case $E(\Gamma) = \{e_{14}, e_{25}, e_{34}, e_{45}, e_{56}\}$ is analogous). Let $w_{15} : [0, l_{15}] \to \mathbb{R}$, $w_{65} : [0, l_{65}] \to \mathbb{R}$ and $w_{34} : [0, l_{34}] \to \mathbb{R}$ be the restrictions of the Energy function w of Γ to the edges e_{15}, e_{65} and e_{34} of lengths l_{15}, l_{65} and l_{34}, respectively. Let $u : [0, l_{15} + l_{56}] \to \mathbb{R}$ be defined as

$$u(x) = \begin{cases} w_{15}(x), \ x \in [0, l_{15}], \\ w_{56}(x - l_{15}), \ x \in [l_{15}.l_{15} + l_{56}]. \end{cases} \tag{2.41}$$

If the metric graph Γ is optimal, then the Energy function on w_{54} on the edge e_{45} must be decreasing and so, by the Kirchhoff's law in the vertex V_5, we have that $w_{15}'(l_{15}) + w_{65}'(l_{65}) \leq 0$, i.e. the left derivative of u at l_{15} is less than the right one:

$$\partial_- u(l_{15}) = w_{15}'(l_{15}) \leq w_{56}'(0) = \partial_+ u(l_{15}).$$

By the maximum principle, we have that

$$u(x) \leq \widetilde{u}(x) = -\frac{x^2}{2} + (l_{15} + l_{56})x \leq \frac{1}{2}(l_{15} + l_{56})^2.$$

On the other hand, $w_{34}(x) \geq v(x) = -\frac{x^2}{2} + \frac{l_{34}}{2}x$, again by the maximum principle on the interval $[0, l_{34}]$. Thus we have that

$$\max_{x \in [0, l_{34}]} w_{34}(x) \geq \max_{x \in [0, l_{34}]} v(x) = \frac{1}{8} l_{34}^2 > \frac{1}{2}(l_{15} + l_{56})^2 \geq w(V_6),$$

for n large enough.

Repeating the same argument, one can show that the optimal metric graph Γ is not of the form $V(\Gamma) = \{V_1, V_2, V_3, V_4, V_5\}$, $E(\Gamma) = \{\{V_1, V_4\}, \{V_2, V_4\}, \{V_3, V_4\}, \{V_4, V_5\}\}$.

Thus, we obtained that the if the optimal graph has a Neumann vertex, then the corresponding edge must be attached to the longest edge. To prove that it is of the same form as Γ_2, there is one more case to exclude, namely: Γ defined by $V(\Gamma) = \{V_1, V_2, V_3, V_4, V_5\}$, $E(\Gamma) = \{\{V_1, V_2\}, \{V_2, V_4\}, \{V_3, V_4\}, \{V_4, V_5\}\}$ (see Figure 2.8). By Example 2.21, the only possible candidate of this form is the graph with lengths

$l(\{V_1, V_2\}) = |D_1 - D_2| = 2$, $l(\{V_2, V_4\}) = \frac{n-1}{2}$, $l(\{V_3, V_4\}) = \frac{n-1}{2}$, $l(\{V_4, V_5\}) = 2$. In this case, we compare the energy of Γ and Γ_1 by an explicit calculation:

$$\mathcal{E}(\Gamma; \{V_1, V_2, V_3\}) = -\frac{n^3 - 3n^2 + 6n}{24} > -\frac{n^2(n+1)^2}{12(2n+1)} = \mathcal{E}(\Gamma_1; \{V_1, V_2, V_3\}), \qquad (2.42)$$

for n large enough.

Figure 2.8: The graph Γ_1 (on the left) has lower energy than the graph Γ (on the right).

Before we pass to our last example, we need the following Lemma.

Lemma 2.24. *Let $w_a : [0, 1] \to \mathbb{R}$ be given by $w_a(x) = -\frac{x^2}{2} + ax$, for some positive real number a. If $w_a(1) \le w_A(1) \le \max_{x\in[0,1]} w_a(x)$, then $J(w_A) \le J(w_a)$, where $J(w) = \frac{1}{2}\int_0^1 |w'|^2\,dx - \int_0^1 w\,dx$.*

Proof. It follows by performing the explicit calculations. □

Example 2.25. Let D_1, D_2 and D_3 be the vertices of an equilateral triangle of side 1 in $\mathbb{R}^2$, i.e.

$$D_1 = (-\frac{\sqrt{3}}{3}, 0),\ D_2 = (\frac{\sqrt{3}}{6}, -\frac{1}{2}),\ D_3 = (\frac{\sqrt{3}}{6}, \frac{1}{2}).$$

We study the problem (2.28) with $\mathcal{D} = \{D_1, D_2, D_3\}$ and $l > \sqrt{3}$. We show that the solutions may have different qualitative properties for different l and that there is always a symmetry breaking phenomenon, i.e. the solutions does not have the same symmetries as the initial configuration $\mathcal{D}$. We first reduce our study to the following three candidates (see Figure 2.9):

1. The metric tree Γ_1, defined by with vertices $V(\Gamma) = \{V_1, V_2, V_3, V_4\}$ and edges $E(\Gamma) = \{e_{14}, e_{24}, e_{34}\}$, where $e_{ij} = \{V_i, V_j\}$ and the lengths of the edges are respectively $l_{24} = l_{34} = x$, $l_{14} = \frac{\sqrt{3}}{2} - \sqrt{x^2 - \frac{1}{4}}$, for some $x \in [1/2, 1/\sqrt{3}]$. Note that the length of Γ_1 is less than $1 + \sqrt{3}/2$, i.e. it is a possible solution only for $l \le 1 + \sqrt{3}/2$. The new vertex V_4 is of Kirchhoff type and there are no Neumann vertices. A symmetrization argument around the point P shows that there is a better candidate among the graphs of the type described in point (2) below.

2. The metric tree Γ_2 with vertices $V = (V_1, V_2, V_3, V_4, V_5)$ and $E(\Gamma) = \{e_{14}, e_{24}, e_{34}, e_{45}\}$, where $e_{ij} = \{V_i, V_j\}$ and the lengths of the edges $l_{14} = l_{24} = l_{34} = 1/\sqrt{3}$, $l_{45} = l - \sqrt{3}$, respectively. The new vertex V_4 is of Kirchhoff type and V_5 is a Neumann vertex.

3. The metric tree Γ_3 with vertices $V(\Gamma) = \{V_1, V_2, V_3, V_4, V_5, V_6\}$ and edges $E(\Gamma) = \{e_{15}, e_{24}, e_{34}, e_{45}, e_{56}\}$, where $e_{ij} = \{V_i, V_j\}$ and the lengths of the edges are $l_{24} = l_{34} = x$, $l_{15} = \frac{lx}{2(2l-3x)} + \frac{\sqrt{3}}{4} - \frac{1}{4}\sqrt{4x^2-1}$, $l_{45} = \frac{\sqrt{3}}{4} - \frac{lx}{2(2l-3x)} - \frac{1}{4}\sqrt{4x^2-1}$ and $l_{56} = l - 2x - \sqrt{3}/2 + \frac{1}{2}\sqrt{4x^2-1}$. The new vertices V_4 and V_5 are of Kirchhoff type and V_6 is a Neumann vertex.

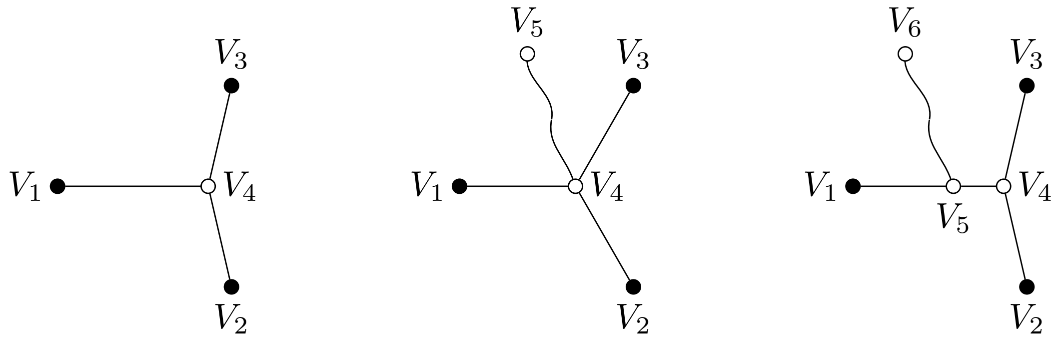

Figure 2.9: The three competing graphs.

Suppose that the metric graph Γ is optimal and has the same vertices and edges as Γ_1. Without loss of generality, we can suppose that the maximum of the Energy function w on Γ is achieved on the edge e_{14}. If $l_{24} \neq l_{34}$, we consider the metric graph $\widetilde{\Gamma}$ with the same vertices and edges as Γ and lengths $\widetilde{l}_{14} = l_{14}$, $\widetilde{l}_{24} = \widetilde{l}_{34} = (l_{24}+l_{34})/2$. An immersion $\widetilde{\gamma} : \widetilde{\Gamma} \to \mathbb{R}^2$, such that $\widetilde{\gamma}(V_j) = D_j$, for $j = 1, 2, 3$ still exists and the energy decreases, i.e. $\mathcal{E}(\widetilde{\Gamma}; \{V_1, V_2, V_3\}) < \mathcal{E}(\Gamma; \{V_1, V_2, V_3\})$. In fact, let $v = \widetilde{w}_{24} = \widetilde{w}_{34} : [0, \frac{l_{24}+l_{34}}{2}] \to \mathbb{R}$ be an increasing function such that $2|\{v \geq \tau\}| = |\{w_{24} \geq \tau\}| + |\{w_{34} \geq \tau\}|$. By the classical Pólya-Szegö inequality and by the fact that w_{24} and w_{34} are never constant in an open region, we obtain that

$$J(\widetilde{w}_{24}) + J(\widetilde{w}_{34}) < J(w_{24}) + J(w_{34}),$$

and so it is enough to construct a function $\widetilde{w}_{14} : [0, l_{14}] \to \mathbb{R}$ such that $\widetilde{w}_{14}(l_{14}) = \widetilde{w}_{24} = \widetilde{w}_{34}$ and $J(\widetilde{w}_{14}) \leq J(w_{14})$. Consider a function such that $\widetilde{w}''_{14} = -1$, $\widetilde{w}_{14}(0) = 0$ and $\widetilde{w}_{14}(l_{14}) = \widetilde{w}_{24}(l_24) = \widetilde{w}_{34}(l_{34})$. Since we have the inequality $w_{14}(l_{14}) \leq \widetilde{w}_{14}(l_{14}) \leq \max_{[0,l_{14}]} w_{14} = \max_\Gamma w$, we can apply Lemma 2.24 and so, $J(\widetilde{w}_{14}) \leq J(w_{14})$. Thus, we obtain that $l_{24} = l_{34}$ and that both the functions w_{24} and w_{34} are increasing (in particular, $l_{14} \geq l_{24} = l_{34}$). If the maximum of w is achieved in the interior of the edge e_{14} then, by Remark 2.22, the edge e_{14} must be rigid and so, all the edges must be rigid. Thus, Γ coincides with Γ_1 for some $x \in (\frac{1}{2}, \frac{1}{\sqrt{3}}]$. If the maximum of w is achieved in the vertex V_4, then applying one more time the above argument, we obtain $l_{14} = l_{24} = l_{34} = \frac{1}{\sqrt{3}}$, i.e. Γ is Γ_1 corresponding to $x = \frac{1}{\sqrt{3}}$.

Suppose that the metric graph Γ is optimal and that has the same vertices as Γ_2. If $w = (w_{ij})_{ij}$ is the Energy function on Γ with Dirichlet conditions in $\{V_1, V_2, V_3\}$, we have that w_{14}, w_{24} an w_{34} are increasing on the edges e_{14}, e_{24} and e_{34}. As in the previous situation $\Gamma = \Gamma_1$, by a symmetrization argument, we have that $l_{14} = l_{24} = l_{34}$. Since any level set $\{w = \tau\}$ contains exactly 3 points, if $\tau < w(V_4)$, and 1 point, if $\tau \geq w(V_4)$, we can apply the same technique as in Example 2.21 to obtain that $l_{14} = l_{24} = l_{34} = \frac{1}{\sqrt{3}}$.

Suppose that the metric graph Γ is optimal and that has the same vertices and edges as Γ_3. Let w be the Energy function on Γ with Dirichlet conditions in $\{V_1, V_2, V_3\}$. Since we assume Γ optimal, we have that w_{45} is increasing on the edge e_{45} and $w(V_5) \geq w_{ij}$, for any $\{i, j\} \neq \{5, 6\}$. Applying the symmetrization argument from the case $\Gamma = \Gamma_1$ and Lemma 2.24, we obtain that $l_{24} = l_{34} = x$ and that the functions $w_{24} = w_{34}$ are increasing on $[0, l_{24}]$. Let $a \in [0, l_{15}]$ be such that $w_{15}(a) = w(V_4)$. By a symmetrization argument, we have that necessarily $l_{15} - a = l_{45}$ an that $w_{45}(x) = w_{15}(x - a)$. Moreover, the edges e_{15} and e_{45} are rigid. Indeed, for any admissible immersion $\gamma = (\gamma_{ij})_{ij} : \Gamma \to \mathbb{R}^2$, we have that the graph $\widetilde{\Gamma}$ with vertices $V(\widetilde{\Gamma}) = \{\widetilde{V}_1, V_4, V_5, V_6\}$ and edges $E(\widetilde{\Gamma}) = \Big\{\{\widetilde{V}_1, V_5\}, \{V_4, V_5\}, \{V_5, V_6\}\Big\}$, is a solution of the problem (2.32) with $D_1 := \gamma_{15}(a)$ and $D_2 := \gamma(V_4)$. By Example 2.21 and Remark 2.22, we have $|\gamma_{15}(a) - \gamma(V_4)| = 2l_{45}$ and, since this holds for every admissible γ, we deduce the rigidity of e_{15} and e_{45}. Using this information one can calculate explicitly all the lengths of the edges of Γ using only the parameter x, obtaining the third class of possible minimizers.

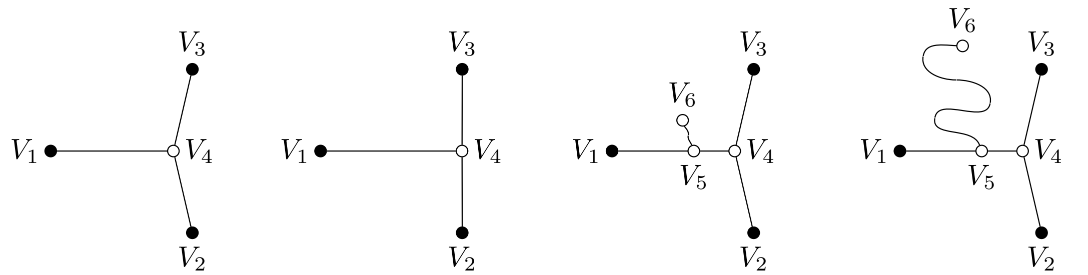

Figure 2.10: The optimal graphs for $l < 1 + \sqrt{3}/2$, $l = 1 + \sqrt{3}/2$, $l > 1 + \sqrt{3}/2$ and $l >> 1 + \sqrt{3}/2$.

An explicit estimate of the energy shows that:

1. If $\sqrt{3} \leq l \leq 1 + \sqrt{3}/2$, we have that the solution of the problem (2.28) with $\mathcal{D} = \{D_1, D_2, D_3\}$ is of the form Γ_1 (see Figure 2.10).

2. If $l > 1 + \sqrt{3}/2$, then the solution of the problem (2.28) with $\mathcal{D} = \{D_1, D_2, D_3\}$

is of the form Γ_3.

In both cases,the parameter x is uniquely determined by the total length l and so, we have uniqueness up to rotation on $\frac{2\pi}{3}$. Moreover, in both cases the solutions are metric graphs, for which there is an embedding γ with $\gamma(V_i) = D_i$, i.e. they are also solutions of the problem (2.29) with $\mathcal{D} = \{D_1, D_2, D_3\}$ and $l \geq \sqrt{3}$.

2.5 Complements and further results

In this Section we present two generalizations of Theorem 2.18. The first one deals with a more general class of constraints $D_1, \dots, D_k$ while in the second one we consider a larger class of admissible sets.

Corollary 2.26. *Let $D_1, \dots, D_k$ be k disjoint compact sets in $\mathbb{R}^d$ and let $l \geq St(d_1, \dots, d_k)$, i.e. such that there exists a closed connected set C of length $\mathcal{H}^1(C) = l$, which intersects all the sets $D_1, \dots, D_k$. Then the optimization problem*

$$\min\{\mathcal{E}(\Gamma;\mathcal{V}) : \ \Gamma \in CMG, \ l(\Gamma) = l, \mathcal{V} \subset V(\Gamma), \ \Gamma \in Adm(\mathcal{V}; D_1, \dots, D_k)\} \tag{2.43}$$

admits a solution, where we say that $\Gamma \in Adm(\mathcal{V}; D_1, \dots, D_k)$, if there exists an immersion $\gamma : \Gamma \to \mathbb{R}^d$ such that for each $j = 1, \dots, k$ there is $V_j \in \mathcal{V}$ such that $\gamma(V_j) \in D_j$.

Proof. As in Theorem 2.16, we can restrict our attention to the connected metric trees Γ with the same vertices $V(\Gamma) = \{V_1, \dots, V_N\}$ and edges $E(\Gamma) = \{e_{ij}\}_{ij}$. Moreover, we can suppose that $\mathcal{V} = \{V_1, \dots, V_k\}$ is fixed. By the compactness of the sets D_j, we can take a minimizing sequence Γ_n and immersions γ_n such that for each $j = 1, \dots, k$, we have $\gamma_n(V_j) \to X_j \in D_j$, as $n \to \infty$. The claim follows by the same argument as in Theorem 2.18. □

Theorem 2.18 can be restated in the more general framework of the metric spaces of finite Hausdorff measure, which is the natural extension of the class of the one dimensional subspaces of $\mathbb{R}^d$ of finite length. In fact, for any compact connected metric space (shortly CCMS) (C, d), we consider the one dimensional Hausdorff measure $\mathcal{H}^1_d$ with respect to the metric d and the Sobolev space $H^1(C)$ obtained by the closure of the Lipschitz functions on C, with respect to the norm $\|u\|^2_{H^1(C)} = \|u\|^2_{L^2(\mathcal{H}^1_d)} + \|u'\|^2_{L^2(\mathcal{H}^1_d)}$, where u' is defined as in the case $C \subset \mathbb{R}^d$. The energy $\mathcal{E}(C;\mathcal{V})$ with respect to the set $\mathcal{V} \subset C$ is defined as in (2.10). As in the case of metric graphs, we define an immersion $\gamma : C \to \mathbb{R}^d$ as a continuous map such that for any arc-length parametrized curve $\eta : (-\varepsilon, \varepsilon) \to C$, we have that $|(\gamma \circ \eta)'(t)| = 1$ for almost every $t \in (-\varepsilon, \varepsilon)$. As a consequence of Theorem 2.18, we have the following:

Corollary 2.27. *Consider the set of points $\mathcal{D} = \{D_1, \dots, D_k\} \subset \mathbb{R}^d$ and a positive real number $l \geq St(D_1, \dots, D_k)$. Then the following optimization problem has solution:*

$$\min\left\{\mathcal{E}(C;\mathcal{V}) : \ (C, d) \in CCMS, \ \mathcal{H}^1_d(C) \leq l, \ C \in Adm(\mathcal{V}; D_1, \dots, D_k)\right\}, \tag{2.44}$$

where the admissible set $Adm(\mathcal{V};\{D_1,\dots,D_k\})$ is the set of connected metric spaces, for which there exists an immersion $\gamma:\Gamma\to\mathbb{R}^d$ such that $\gamma(\mathcal{V})=\{D_1,\dots,D_k\}$. Moreover, the solution of the problem (2.44) is a connected metric graph, which is a tree of at most $2k$ vertices and $2k-1$ edges.

Proof. Repeating the construction from Theorem 2.7, we can restrict our attention to the class of metric graphs. The conclusion follows from Theorem 2.18. □

The results from Theorem 2.7 and Theorem 2.18, hold also for other cost functionals as, for example, the first eigenvalue of the Dirichlet Laplacian:

$$\lambda_1(\Gamma;\mathcal{V})=\min\left\{\int_\Gamma |u'|^2\,dx:\ u\in H_0^1(\Gamma),\ \int_\Gamma u^2\,dx=1\right\},\tag{2.45}$$

where Γ is a metric graph and $\mathcal{V}\subset V(\Gamma)$ is a set of vertices, where a Dirichlet boundary conditions are imposed. Reasoning as in Remark 2.6, we have that among all connected metric graphs (shortly, CMG) of fixed length l and with at least one Dirichlet vertex, the one with the lowest first eigenvalue is given by the segment $[0,l]$, with Dirichlet condition in 0. Moreover, for any pair $D_1,D_2\in\mathbb{R}^d$ and any $l\geq|D_1-D_2|=:l-\epsilon$ the solution of

$$\min\left\{\lambda_1(\Gamma;\mathcal{V}):\ \Gamma\in CMG, l(\Gamma)=l,\mathcal{V}\subset V(\Gamma),\exists\gamma:\Gamma\to\mathbb{R}^d\text{ immersion},\ \gamma(\mathcal{V})=\mathcal{D}\right\},\tag{2.46}$$

is the graph described in Figure 2.4, i.e. the solution of (2.32) from Example 2.21. In the case when the set $\mathcal{D}$ is given by three points disposed in the vertices of an equilateral triangle, the solutions of (2.46) are quantitatively the same (see Figure 2.10) as the solutions of (2.36) from Example 2.25. In general, we have the following existence result

Theorem 2.28. *Consider a set of distinct points $\mathcal{D}=\{D_1,\dots,D_k\}\subset\mathbb{R}^d$ and a positive real number $l\geq St(\mathcal{D})$. Then there exists a connected metric graph Γ, a set of vertices $\mathcal{V}\subset V(\Gamma)$ and an immersion $\gamma:\Gamma\to\mathbb{R}^d$ which are solution of the problem (2.46). Moreover, Γ can be chosen to be a tree of at most $2k$ vertices and $2k-1$ edges.*

Proof. The proof is identical to the one of Theorem 2.18. □

Remark 2.29. The question of existence of an optimal graph is open for general cost functionals J spectral type, i.e. $J=F(\lambda_1,\lambda_2,\dots,\lambda_k,\dots)$, where $F:\mathbb{R}\to\mathbb{R}$ is a real function and λ_k is the $k-$th eigenvalue of the Dirichlet Laplacian:

$$\lambda_k(\Gamma;\mathcal{V})=\min_{K\subset H_0^1(\Gamma)}\max\left\{\int_\Gamma |u'|^2\,dx:\ u\in K,\ \int_\Gamma u^2\,dx=1\right\},\tag{2.47}$$

where the minimum is over all k dimensional subspaces K of $H_0^1(\Gamma)$. In fact, the crucial point in the proof of Theorem 2.18 is the reduction to the class of connected metric trees with number of vertices bounded by some universal constant. This reduction becomes a rather involved question even for the simplest spectral functionals λ_k for $k\geq 2$.

Chapter 3

A non-local isoperimetric problem

3.1 Introduction

This chapter is based on a work (in preparation), in collaboration with Michael Goldman and Matteo Novaga [70].

In a series of recent works, among which we cite [80], [81], [41] and [66], and references therein, are investigated functionals of the form

$$\mathcal{E}(E) = P(E) + \mathcal{NL}(E) \tag{3.1}$$

where $E \subset \mathbb{R}^d$, $P(\cdot)$ is the perimeter functional and $\mathcal{NL}(\cdot)$ is a non-local term which usually is translation and rotation invariant, and repulsive. In [80] and [81] it is considered the case

$$\mathcal{NL}_Q(E) = \int_{E\times E} \frac{Q^2}{|x-y|^\alpha}\, dxdy, \qquad 0 < \alpha < d, \tag{3.2}$$

where Q is a real, non-negative number. The study of functionals of this type under volume constraint is often motivated by physical reasons. For example, when the dimension d is 3 (or greater) and $\alpha = d-2$, the functional $P + \mathcal{NL}_Q$ is related to the question of which shapes can assume a conductive liquid drop once it is provided with a uniform charge Q.

Notice that if $Q = 0$ we fall in the classical isoperimetric problem. Moreover an easy computation shows that if $Q_n \to 0$ as $n \to \infty$, then $\mathcal{NL}_{Q_n}$ $\Gamma-$converges, in the L^1 topology of the sets, to 0. These observations suggest that if the charge Q is small enough, the perimeter plays the *leading role*. In [80] and [81] it is indeed proved that for small charges the non-local term is actually irrelevant and thus the minimizers are balls. It is worth mentioning that the isoperimetric inequality in *sharp* quantitative form (see [62], [58] or [40]),

$$\frac{P(E) - P(B_E)}{P(B_E)} \geq c(d) \left(\frac{|E \Delta B_E|}{|E|} \right)^2,$$

where B_E is a suitable ball of measure $|E|$, together with the regularity theory of the perimeter quasi-minimizers (see [92], [2]), plays a crucial role in the proof. Another remarkable (but not surprising) result is that if Q is big enough, the roles of P and $\mathcal{NL}_Q$ are overturned. In particular, there is a Q_0 such that if $Q > Q_0$ a minimum cannot be connected and, since the non-local term is repulsive, it cannot exist (indeed two connected components would tend to have infinite mutual distance). Moreover, empirical laws suggest that given a body E, the charge Q tends to distribute on the boundary ∂E. This brings to consider the case where the charge is a priori distributed on the boundary of the set:

$$\mathcal{NL}'_Q(E) = \int_{\partial E \times \partial E} \frac{Q^2}{|x-y|^\alpha}\, dxdy. \tag{3.3}$$

In [66] it is taken into account the aforementioned case and it is proved that if one prescribes the *potential*[1]

$$v_\alpha(x) = \int_{\partial E} \frac{dy}{|x-y|^\alpha}$$

to be constant on the boundary of E, then the ball is a local minimum, where the locality is settled among under small $C^{2,\beta}$ perturbations of the ball, for some $\beta > 0$. Motivated by these results, we consider the case where the total charge Q is fixed, but *it is free to move in E*. More precisely we consider the problem of minimizing, under volume constraint, the quantity

$$P(E) + \min_{\mu(E)=Q} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{d\mu(x)\, d\mu(y)}{|x-y|^\alpha} = P(E) + \mathcal{Q}_\alpha(E)$$

As we will see in Section 3.2, depending on the value of α, the charge measure can distributes on the whole E or just in its boundary ∂E.

A remarkable case among those where the charge distributes on the boundary of the set, is the *harmonic* one: $\alpha = d - 2$.

Organization and main results of the chapter

In Section 3.2 we set the main definitions and preliminary results needed in the rest of the chapter. In particular we analyse the *Riesz potential* related to a measure μ and a parameter α:

$$v^\mu_\alpha(x) = \int_{\mathbb{R}^d} \frac{d\mu(y)}{|x-y|^\alpha} \tag{3.4}$$

[1] Here we follow the classic notation about *Riesz potentials*. Although they share such a notation, the concept of potential discussed in this chapter is not related to the *Schrödinger potentials* of Chapter 1.

and its associated *potential energy*

$$\int_{\mathbb{R}^d} v_\alpha^\mu(x)\, d\mu(x).$$

Many results of the section are well known in literature and are reported for the reader convenience.

Section 3.3 is devoted to the proof of Theorem 3.21 and Theorem 3.22, where the following problem

$$\min_{E\subset\Omega,\quad |E|=1} \{P(E) + \mathcal{Q}_\alpha(E)\} \tag{3.5}$$

is considered. Precisely, in Theorem 3.21 we prove that problem (3.5) does not admit a minimizer when $\Omega = \mathbb{R}^d$. Then, in Theorem 3.22, we deal with a bounded, regular domain Ω. In this case we show that problem (3.5) can be *decoupled* into the isoperimetric problem and that of minimizing $\mathcal{Q}_\alpha$ on Ω (see equations (3.15) and (3.16)). This means that the problem is ill-posed and a minimum cannot exist in general, see Remark 3.23.

In Section 3.4 we prove that if we impose some regularity (in our case, the $\delta-$ball condition, see Definition 3.16, and the connectedness of the admissible sets) to the class of minimization, the minimum is achieved.

Eventually, in Section 3.5 we prove that the ball is a local minimum, i.e. it is the minimum among all sets whose boundary is a $C^{1,1}$ perturbation of the boundary of the ball itself.

3.2 The Riesz potential energy

In this section we study the main properties of the Riesz potential (3.4) and of the interaction energy (see Definition 3.1 below). Many of the results of this section are probably well known in literature. An interesting and useful guide to this subject may be the book [84].

In the following, given an open set $\Omega \subset \mathbb{R}^d$, we denote by $\mathcal{M}(\Omega)$ the set of all Borel measures with support in Ω.

Definition 3.1. Let $d \geq 2$ and $\alpha > 0$. For any Borel measures μ and ν with $\mu(\mathbb{R}^d) = \nu(\mathbb{R}^d) = 1$, we define the interaction energy between μ and ν by

$$\mathcal{Q}_\alpha(\mu,\nu) := \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{d\mu(x)\, d\mu(y)}{|x-y|^\alpha}.$$

When $\mu = \nu$, we simply write $\mathcal{Q}_\alpha(\mu) := \mathcal{Q}_\alpha(\mu,\mu)$. When $\mu = f\mathcal{H}^d \llcorner E$ and $\nu = g\mathcal{H}^d \llcorner E$, we denote $\mathcal{Q}_\alpha^E(\mu,\nu) = \mathcal{Q}_\alpha^E(f,g)$ (and when $f = g$ we denote it by $\mathcal{Q}_\alpha^E(f)$). When $\mu = f\mathcal{H}^{d-1} \llcorner \partial E$ and $\nu = g\mathcal{H}^{d-1} \llcorner \partial E$ we simply write $\mathcal{Q}_\alpha(\mu,\nu) = \mathcal{Q}_\alpha^{\partial E}(f,g)$ (and when $f = g$ we denote it by $\mathcal{Q}_\alpha^{\partial E}(f)$).

Definition 3.2. Let $d \geq 2$ and $\alpha > 0$. Then for every Borel set A we define the Riesz potential energy of A by

$$\mathcal{Q}_\alpha(A) := \inf\{\mathcal{Q}_\alpha(\mu) : \mu \in \mathcal{M}(\mathbb{R}^d),\, \mu(A) = 1\}. \tag{3.6}$$

Remark 3.3. Notice that by changing μ in $Q\mu$, for every charge $Q > 0$ and every Borel set A, there holds

$$Q^2 \mathcal{Q}_\alpha(A) = \inf\{\mathcal{Q}_\alpha(\mu) : \mu \in \mathcal{M}(\mathbb{R}^d),\, \mu(A) = Q\}.$$

Lemma 3.4. *If E is compact then the infimum in (3.6) is achieved.*

Proof. The kernel $1/|x|^\alpha$ is positive and lower semicontinuous. Thus the energy

$$\int_{\bar{E}\times\bar{E}} \frac{d\mu(x)\, d\mu(y)}{|x-y|^\alpha}$$

is semicontinuous for the weak* topology. Moreover the constraint in the definition of $\mathcal{Q}_\alpha$ is weakly* compact (since E is bounded), whence it follows that a minimizer μ exists.

□

Remark 3.5. When the set is unbounded, there does not always exist an optimal measure μ. It is for example possible to construct a set E of finite volume with $\mathcal{Q}_\alpha(E) = 0$. Indeed, for $\alpha \in (0, d-1)$ and $\gamma \in (\frac{1}{d-1}, +\infty)$, consider the set $E = \{(x, x') \in \mathbb{R} \times \mathbb{R}^{d-1} : |x'| \leq 1 \text{ and } |x'| \leq \frac{1}{|x|^\gamma}\}$ then E has finite volume and taking N balls of radius $r = N^{-\beta}$, at mutual distance $\ell = N^{\frac{\beta}{\gamma}-1}$ inside E with charge $1/N$ distributed uniformly on each ball, we get

$$\mathcal{Q}_\alpha(E) \leq c\left(N \frac{1}{N^2} r^{-\alpha} + N^2 \frac{1}{N^2} \ell^{-\alpha}\right) \leq C\left(N^{\alpha\beta-1} + N^{(1-\frac{\beta}{\gamma})\alpha}\right)$$

for suitable constants $c, C > 0$, so that if $\frac{1}{d-1} < \gamma < \beta < \frac{1}{\alpha}$, we obtain $\mathcal{Q}_\alpha(E) = 0$.

Definition 3.6. Given a non-negative Borel measure μ on $\mathbb{R}^d$ and $\alpha \in (0, d-1)$, we define the function

$$v_\alpha^\mu(x) = \int_{\mathbb{R}^d} \frac{d\mu(y)}{|x-y|^\alpha} = \mu * k_\alpha(x)$$

where $k_\alpha(x) = |x|^{-\alpha}$. We will sometime drop the dependence of μ and/or α in the definition of v_α^μ and we will refer to it as *potential*.

Lemma 3.7. *Let E be a bounded Borel set and let μ be a minimizer of $\mathcal{Q}_\alpha(E)$. Then $v^\mu = \mathcal{Q}_\alpha(E)$ q.e. on* $\mathrm{spt}(\mu)$.

Proof. It follows by the minimality of μ, by computing the first variation of $\mathcal{Q}_\alpha(E)$. □

The following lemma is proven in [84, Chapter I].

Lemma 3.8. *For every $\alpha > 0$ and every measure μ, the function v_α^μ is lower semicontinuous*

We recall another important result which will be exploited in Section 3.4 (for its proof we refer to [84, Theorem 1.15] or [86, Corollary 5.10]).

Theorem 3.9. *For any signed measure μ and for any $\alpha \in (0, d)$, there holds,*

$$\mathcal{Q}_\alpha(\mu) = \int_{\mathbb{R}^d} \left(v_{\alpha/2}^\mu(x)\right)^2 \, dx$$

and therefore,

$$\mathcal{Q}_\alpha(\mu) \geq 0.$$

Moreover equality holds if and only if $\mu = 0$.

Remark 3.10. A useful consequence of Theorem 3.9, is that the functional $\mathcal{Q}_\alpha(\cdot, \cdot)$ is a *positive*, bilinear operator on the product space of Borel measures on $\mathbb{R}^d$, $\mathcal{M}(\mathbb{R}^d) \times \mathcal{M}(\mathbb{R}^d)$. In particular it satisfies the Cauchy-Schwarz inequality

$$\mathcal{Q}_\alpha(\mu, \nu) \leq \mathcal{Q}_\alpha(\mu, \mu)^{1/2} \mathcal{Q}(\nu, \nu)^{1/2}. \tag{3.7}$$

Some properties of the potential energy $\mathcal{Q}_\alpha$ are in order.

Lemma 3.11. *For every bounded Borel set E the measure minimizing $\mathcal{Q}_\alpha(E)$ is unique.*

Proof. Let μ and ν be two minimizing measures. Then since

$$\mathcal{Q}(\mu - \nu, \nu) = \int_{\bar{E}} d(\mu - \nu)(x) \int_{\bar{E}} \frac{d\nu(y)}{|x - y|^\alpha} = \int_{\bar{E}} v_\alpha^\nu(x) d(\mu - \nu)(x) = \mathcal{Q}_\alpha(E)(\mu(E) - \nu(E)) = 0,$$

we find that

$$\begin{aligned}\mathcal{Q}_\alpha(\mu, \mu) = \mathcal{Q}_\alpha((\mu - \nu) + \nu, (\mu - \nu) + \nu) &= \mathcal{Q}_\alpha(\mu - \nu, \mu - \nu) + \mathcal{Q}_\alpha(\nu, \nu) - 2\mathcal{Q}_\alpha(\mu - \nu, \nu) \\ &= \mathcal{Q}_\alpha(\mu - \nu, \mu - \nu) + \mathcal{Q}_\alpha(\mu, \mu)\end{aligned}$$

from which $\mathcal{Q}_\alpha(\mu - \nu) = 0$. Hence, by Theorem 3.9, $\mu = \nu$. □

We will also make use of the following density result which is an adaptation of [84, Theorem 1.11, Lemma 1.2].

Proposition 3.12. *Let E be a smooth connected closed set of $\mathbb{R}^d$. Then for every $\alpha \in (0, d-2)$,*

$$\mathcal{Q}_\alpha(E) = \inf \left\{ \mathcal{Q}_\alpha^E(f) \ : \ \mu = f dx, \ f \in L^\infty(E), \ \int_E f \ dx = 1 \right\}.$$

Proof. Let μ be such that $\mu(\bar{E}) = 1$, $\mathrm{spt}(\mu) \subset \bar{E}$ and $\mathcal{Q}_\alpha(\mu) < +\infty$. Then for $\varepsilon > 0$ consider the measure $\mu_\varepsilon \, dx$ where

$$\mu_\varepsilon(x) := \frac{1}{|B_\varepsilon(x) \cap E|} \int_{B_\varepsilon(x)} d\mu(y).$$

Since $\|\mu_\varepsilon\|_{L^\infty(E)} \leq \frac{1}{\min_{x\in E} |B_\varepsilon(x)\cap E|} \leq C \frac{1}{\varepsilon^d}$, we are left to prove that $\mathcal{Q}^E_\alpha(\mu_\varepsilon) \to \mathcal{Q}_\alpha(\mu)$. By Theorem 3.9, $\mathcal{Q}^E_\alpha(\mu_\varepsilon) = \int_{\mathbb{R}^d} \left(v^{\mu_\varepsilon}_{\alpha/2}\right)^2 (x) \, dx$. Let us show that for all $x \in \mathbb{R}^d$,

$$v^{\mu_\varepsilon}_{\alpha/2}(x) \leq C v^{\mu}_{\alpha/2}(x) \quad \text{and} \quad \lim_{\varepsilon\to 0} v^{\mu_\varepsilon}_{\alpha/2}(x) = v^{\mu}_{\alpha/2}(x)$$

from which we can conclude using the Dominated Convergence Theorem. Let us start by noticing that for $x \in \mathbb{R}^d$,

$$\begin{aligned}
v^{\mu_\varepsilon}_{\alpha/2}(x) &= \int_E \int_E \frac{1}{|B_\varepsilon(y)\cap E|} 1_{B_\varepsilon}(y-z) \frac{d\mu(z)}{|x-y|^{\alpha/2}} \, dy \\
&= \int_E \left(\int_{B_\varepsilon(z)\cap E} \frac{1}{|B_\varepsilon(y)\cap E|} \frac{|x-z|^{\alpha/2}}{|x-y|^{\alpha/2}} \, dy \right) \frac{d\mu(z)}{|x-z|^{\alpha/2}} \\
&\leq \int_E \left(\frac{C}{\varepsilon^d} \int_{B_\varepsilon(z)} \frac{|x-z|^{\alpha/2}}{|x-y|^{\alpha/2}} \, dy \right) \frac{d\mu(z)}{|x-z|^{\alpha/2}}.
\end{aligned}$$

As in the proof of [84, Theorem 1.11], we observe that the function $(x,z) \to \frac{1}{\varepsilon^d} \int_{B_\varepsilon(z)} \frac{|x-z|^{\alpha/2}}{|x-y|^{\alpha/2}} \, dy$ is uniformly bounded in (x, z, ε) so that $v^{\mu_\varepsilon}_{\alpha/2}(x) \leq C v^{\mu}_{\alpha/2}(x)$ for some universal constant $C > 0$. Consider now a point $x \in \mathbb{R}^d$ such that $v^{\mu}_{\alpha/2} < +\infty$ then for every $\delta > 0$ we can find a ball $B_\eta(x)$ such that $v^{\mu'}_{\alpha/2} < \delta$ where $\mu' = \mu \llcorner B_\eta(x)$. By the previous computations, we then find $v^{(\mu')_\varepsilon}_{\alpha/2}(x) \leq C\delta$. Since $\lim_{\varepsilon\to 0} v^{(\mu-\mu')_\varepsilon}_{\alpha/2}(x) = v^{\mu-\mu'}_{\alpha/2}(x)$, there holds

$$\begin{aligned}
v^{\mu}_{\alpha/2}(x) &= v^{\mu'}_{\alpha/2}(x) + v^{\mu-\mu'}_{\alpha/2}(x) \leq \delta + \lim_{\varepsilon\to 0} v^{(\mu-\mu')_\varepsilon}_{\alpha/2}(x) \\
&\leq (1+C)\delta + \underline{\lim}_{\varepsilon\to 0} v^{\mu_\varepsilon}_{\alpha/2}(x) \leq (1+C)\delta + \overline{\lim_{\varepsilon\to 0}}\, v^{\mu_\varepsilon}_{\alpha/2}(x) \\
&\leq (1+C)\delta + \overline{\lim_{\varepsilon\to 0}}\, v^{(\mu')_\varepsilon}_{\alpha/2}(x) + \overline{\lim_{\varepsilon\to 0}}\, v^{(\mu-\mu')_\varepsilon}_{\alpha/2}(x) \\
&\leq 2(1+C)\delta + v^{\mu}_{\alpha/2}(x)
\end{aligned}$$

so that letting $\delta \to 0$ we find that $\lim_{\varepsilon\to 0} v^{\mu_\varepsilon}_{\alpha/2}(x) = v^{\mu}_{\alpha/2}(x)$ as claimed.

□

With the same ideas, the following proposition can be proven.

Proposition 3.13. *Let E be a smooth connected closed set of $\mathbb{R}^d$ then for every $\alpha \in [d-2, d-1)$,*

$$\mathcal{Q}_\alpha(\partial E) = \inf \left\{ \mathcal{Q}_\alpha^{\partial E}(f) \;:\; \mu = f d\mathcal{H}^{d-1},\; f \in L^\infty(\partial E),\; \int_{\partial E} f \; d\mathcal{H}^{d-1} = 1 \right\}.$$

Lemma 3.14. *Let $\alpha \in (0, d-1)$. For every bounded open set E, the minimizer μ of $\mathcal{Q}_\alpha(E)$ satisfies:*

- *if $\alpha \leq d-2$ then $\operatorname{spt}(\mu) \subset \partial E$, and thus $\mathcal{Q}_\alpha(E) = \mathcal{Q}_\alpha(\partial E)$,*
- *if $\alpha > d-2$ then $\operatorname{spt}(\mu) = \bar{E}$.*

Moreover, when $\alpha \geq d-2$, $v_\alpha^\mu = \mathcal{Q}_\alpha(E)$ on $\bar{E}$.

Proof. The case $\alpha \leq d-2$ can be found in [84, pages 132 and 162] but we give a proof for the reader's convenience. Let $k_\alpha = 1/|x|^\alpha$ then:

· $\Delta k_\alpha > 0$ for $|x| > 0$ and $\alpha > d-2$,

· $\Delta k_{d-2} = 0$ for $|x| > 0$,

· $\Delta k_\alpha < 0$ for $\alpha < d-2$ in $\mathbb{R}^d \setminus \operatorname{spt}(\mu)$.

Thus for every connected set E:

· $\Delta v_\alpha^\mu > 0$ in $\mathbb{R}^d \backslash \operatorname{spt}(\mu)$ for $\alpha > d-2$,

· $-\Delta v_{d-2}^\mu = \mu$,

· $\Delta v_\alpha^\mu < 0$ in $\mathbb{R}^d \backslash \operatorname{spt}(\mu)$ for $\alpha < d-2$.

Moreover, $\Delta v_\mu^\alpha < 0$ in $\mathbb{R}^d$ for $\alpha < d-2$, that is v_α^μ is strictly superharmonic. Since $v_\alpha^\mu = \mathcal{Q}_\alpha(E)$ in $\operatorname{spt}(\mu) \subset \bar{E}$ and $v_\alpha^\mu \geq \mathcal{Q}_\alpha(E)$ in $\bar{E}$,

· for $\alpha > d-2$, $v_\alpha^\mu = \mathcal{Q}_\alpha(E)$ in $\bar{E}$ and $\operatorname{spt}(\mu) = \bar{E}$ (since otherwise v_a^μ would have a maximum out of $\operatorname{spt}(\mu)$ which would contradict $\Delta v_\alpha^\mu > 0$ in $\mathbb{R}^d \backslash \operatorname{spt}(\mu)$).

· for $\alpha = d-2$, $v_{d-2}^\mu = \mathcal{Q}_\alpha(E)$ in $\bar{E}$ since otherwise it would have a maximum in the open set $\bar{E} \backslash \operatorname{spt}(\mu)$ which would contradict the fact that $\Delta v_{d-2}^\mu = 0$ in $\bar{E} \backslash \operatorname{spt}(\mu)$. Now since $0 = -\Delta v_{d-2}^\mu = \mu$ in $\bar{E}$, it implies that $\operatorname{spt}(\mu) \subset \partial E$. Since $v_{d-2}^\mu = \mathcal{Q}_\alpha(E)$ in $\bar{E}$, $v_{d-2}^\mu \geq \mathcal{Q}_\alpha(E)$ in ∂E and by lower semicontinuity of v_{d-2}^μ, we find that $v_{d-2}^\mu = \mathcal{Q}_\alpha(E)$ on ∂E.

· for $\alpha < d-2$, v_α^μ has necessarily a minimum in $\bar{E}$ on the boundary of E and by superharmonicity, $v_\alpha^\mu > \mathcal{Q}_\alpha(E)$ in E so that $\operatorname{spt}(\mu) \subset \{v_\alpha^\mu = \mathcal{Q}_\alpha(E)\} \subset \partial E$.

□

Lemma 3.15. *Let $\alpha \in (0, d-2]$. Then the uniform measure on the sphere ∂B*

$$d\mathcal{U}_B = \frac{1}{P(B)} d\mathcal{H}^{d-1} \llcorner \partial B$$

is the unique optimizer for $\mathcal{Q}_\alpha(\partial B)$. Similarly, for $d-1 > \alpha > d-2$, the uniform measure on the ball B,

$$d\tilde{\mathcal{U}}_B = \frac{1}{|B|} d\mathcal{H}^d \llcorner B$$

is the unique optimizer for $\mathcal{Q}_\alpha(B)$.

Proof. Let μ_B be a minimum for the problem $\mathcal{Q}_\alpha(\partial B)$. By Lemma 3.14 we know that $v^{\mu_B} = \mathcal{Q}_\alpha(\partial B)$ on ∂B. Thus for every $\bar{x}$, $x \in \partial B$ we have

$$\begin{aligned} v_B(\bar{x}) &= \int_{\partial B} \frac{d\mu_B(y)}{|x-y|^\alpha} = \frac{1}{P(B)} \int_{\partial B} \int_{\partial B} \frac{d\mu_B(y)\, d\mathcal{H}^{d-1}(x)}{|x-y|^\alpha} \\ &= \int_{\partial B} \left(\frac{1}{P(B)} \int_{\partial B} \frac{d\mathcal{H}^{d-1}(x)}{|x-y|^\alpha} \right) d\mu_B(y) \\ &= \frac{1}{P(B)} \int_{\partial B} \frac{d\mathcal{H}^{d-1}(x)}{|x-y|^\alpha}. \end{aligned}$$

Hence

$$\mathcal{Q}_\alpha(\partial B) = \int_{\partial B} v_B(\bar{x})\, d\mu_B(\bar{x}) = \frac{1}{P(B)} \int_{\partial B} \frac{d\mathcal{H}^{d-1}(x)}{|x-y|^\alpha} = \int_{\partial B \times \partial B} \frac{d\mathcal{U}_B(x)\, d\mathcal{U}_B(y)}{|x-y|^\alpha},$$

which means that $\mathcal{U}_B$ is an optimizer for $\mathcal{Q}_\alpha(\partial B)$. The proof for the cases $\alpha \in [d-2, d-1)$ is analogous. □

Definition 3.16. Given $\delta > 0$, we say that E satisfies the $\delta-$ball condition if for any $x \in \partial E$, there are two balls of radius δ, one contained in E and one in E^c, both tangent to ∂E in x.

Remark 3.17. A set which satisfies the $\delta-$ball condition has $C^{1,1}$ boundary, see [49].

Lemma 3.18. *Let $\delta > 0$, then every set $E \in \mathcal{K}_\delta^{co}$ with $|E| = m$ satisfies*

$$\operatorname{diam}(E) \leq \sqrt{d}\, 2^{d+2} \frac{m}{|B|} \delta^{1-d}.$$

Proof. Consider the tiling of $\mathbb{R}^d$ given by $[0, 2\delta[^d + 2\delta\mathbb{Z}^d$ and for $k \in \mathbb{Z}^d$ let $C_k = 2\delta k + [0, 2\delta[^d$. For every $k \in \mathbb{Z}^d$ such that $C_k \cap E \neq \emptyset$, let $B_\delta(x_k)$ be a ball of radius δ such that $B_\delta(x_k) \subset E$ and $B_\delta(x_k) \cap C_k \neq \emptyset$. the existence of such a ball is guaranteed by the $\delta-$ball condition. Any such a ball can intersect at most 2^d cubes C_j so that

$$\sharp\{k \in \mathbb{Z}^d \,:\, E \cap C_k \neq \emptyset\} = \frac{1}{|B_\delta|} \sum_{k: C_k \cap E \neq \emptyset} |B_\delta(x_k)| \leq \frac{2^d}{|B_\delta|}|E|,$$

where $\sharp A$ stands for the cardinality of the set A. Since E is connected we can assume that $E \subset [0, 4\delta \frac{2^d}{|B_\delta|} m]^d$, and so

$$\operatorname{diam}(E) \leq \sqrt{d}\, 2^{d+2} \frac{m}{|B|} \delta^{1-d}.$$

$\square$

Proposition 3.19. *Let $d \geq 3$, $\alpha = d-2$, $\delta > 0$ and $E \subset \mathbb{R}^d$ be a bounded set which satisfies the $\delta-$ball condition. Then the optimal measure μ for $\mathcal{Q}_\alpha(E) = \mathcal{Q}_\alpha(\partial E)$ can be written as $\mu = f\mathcal{H}^{d-1} \llcorner \partial E$ with $\|f\|_{L^\infty(\partial E)} \leq \mathcal{Q}_\alpha(E)(d-2)\delta^{-1}$.*

Proof. By Lemma 3.14 we know that an optimizer μ is concentrated on ∂E. Denote by $v = v^\mu_{d-2}$ the potential related to μ on E. By Lemma 3.14, we get that $v = \mathcal{Q}_\alpha(E)$ on E, and that $-\Delta v = \mu$. By classical elliptic regularity (see for instance [67, Corollary 8.36]), v is regular in $\mathbb{R}^d \backslash E$ (and at least $C^{1,\beta}$ up to the boundary). Consider now a point $x \in \partial E$ and let $y \in E$ be such that the ball $B_\delta(y)$ is contained in E and is tangent to ∂E in x. The existence of such an y is guaranteed by the $\delta-$ball condition satisfied by E. Let u be a function satisfying

$$\Delta u = 0 \quad \text{in } B^c_\delta(y); \qquad u = v(x) = \mathcal{Q}_\alpha(E) \quad \text{on } \partial B_\delta(y).$$

Notice that $u(z) = \frac{\mathcal{Q}_\alpha(E)\delta^{d-2}}{|z-y|^{d-2}}$ out of $B_\delta(y)$. By the maximum principle for harmonic functions, $u \leq \mathcal{Q}_\alpha(E)$ on ∂E. Thus, again by the maximum principle, applied to $u - v$, we get that $v \geq u$ on $\mathbb{R}^d \setminus E$. Since $u(x) = v(x)$, we get

$$|\nabla v(x)| \leq |\nabla u(x)| = \mathcal{Q}_\alpha(E)(d-2)\delta^{-1}. \tag{3.8}$$

Let us prove that $\mu = |\nabla v| \mathcal{H}^{d-1} \llcorner \partial E$. For this, let $x \in \partial E$ and $r > 0$ and consider a test function $\varphi \in C^\infty_c(B_r(x))$. Then

$$\begin{aligned} \int_{\partial E \cap B_r(x)} \varphi d\mu &= -\int_{B_r(x)} \varphi \Delta v = \int_{B_r(x)} \langle \nabla\varphi, \nabla v\rangle \; dy \\ &= \int_{B_r(x) \cap E^c} \langle \nabla\varphi, \nabla v\rangle \; dy = \int_{\partial E} \varphi \langle \nabla v, \nu^E\rangle d\mathcal{H}^{d-1} \end{aligned}$$

where ν^E is the external normal to E. Since v is constant on ∂E, its tangential derivative is zero; moreover, since $v < \mathcal{Q}_\alpha(E)$ on $\mathbb{R}^d \backslash \bar{E}$ we have that $\langle \nabla v, \nu^E \rangle \geq 0$.

Therefore, $\langle \nabla v, \nu^E \rangle = |\nabla v|$ on ∂E. Putting all this together, we find that for every test function φ,

$$\int_{\partial E \cap B_r(x)} \varphi d\mu = \int_{\partial E} \varphi |\nabla v| d\mathcal{H}^{d-1},$$

which is equivalent to the claim $\mu = |\nabla v| \mathcal{H}^{d-1} \llcorner \partial E$.

□

3.3 Relaxation of the problem

Definition 3.20. Let $d \geq 2$ and $\alpha > 0$. For every $Q > 0$ and every Borel set $E \subset \mathbb{R}^d$ we define the fuctionals,

$$\mathcal{F}(E) = P(E) + Q^2 \mathcal{Q}_\alpha(E), \tag{3.9}$$

and

$$\mathcal{G}(E) = P(E) + Q^2 \mathcal{Q}_\alpha(\partial E). \tag{3.10}$$

Notice that for $\alpha \in (0, d-2]$, by Lemma 3.14, the functionals $\mathcal{F}$ and $\mathcal{G}$ coincide.

In this section we consider a closed, connected, regular set $\Omega \subset \mathbb{R}^d$ (not necessarily bounded) of measure $|\Omega| > m$ and address the following problems:

$$\inf_{|E|=m,\, E \subset \Omega} \mathcal{F}(E), \tag{3.11}$$

and

$$\inf_{|E|=m,\, E \subset \Omega} \mathcal{G}(E), \tag{3.12}$$

where the (implicit) parameter α belongs to $(0, d-1)$. Let us start by investigating the case $\Omega = \mathbb{R}^d$ (despite for simplicity we will consider only this case, the proof of the following theorem can be easily adapted to any unbounded regular set).

Theorem 3.21. *For every $\alpha \in (0, d-1)$,*

$$\inf_{|E|=m} \mathcal{F}(E) = \inf_{|E|=m} \mathcal{G}(E) = \min_{|E|=m} P(E) = m^{d-1} P(B).$$

In other words, (3.11) and (3.12) do not admit a minimizer.

Proof. Let $N \in \mathbb{N}$ and $\beta > 0$ (to be fixed later). Consider N balls of radius $N^{-\beta}$ infinitely far away from each other and put on each of these balls a charge $\frac{1}{N}$. Let $V_N := N r^d |B|$ be their total volume and consider the set E given by the union of these balls with a (non-charged) ball of volume $m - V_N$. We want to choose β in such a way that

$$\lim_{N \to +\infty} N r^{d-1} = 0 \qquad \text{and} \qquad \lim_{N \to +\infty} \frac{1}{N^2} \frac{1}{r^\alpha} = 0. \tag{3.13}$$

Indeed, with (3.13) in force, we would get that $V_N \to 0$ and

$$m^{d-1}P(B) \leq P(E) + Q^2\mathcal{Q}_\alpha(E) \leq (m - V_N)^{d-1}P(B) + Nr^{d-1} + \frac{Q^2}{N^2}\frac{C}{r^\alpha} \to m^{d-1}P(B).$$

Conditions (3.13) are satisfied as soon as $\frac{1}{d-1} < \beta < \frac{2}{\alpha}$. Such choice of β is allowed since $0 < \alpha < d-1$. $\square$

We now investigate what happens when Ω is bounded.

Theorem 3.22. *Let E_0 be a solution of the constrained isoperimetric problem*

$$\min\{P(E) : E \subset \Omega,\ |E| = m\} \tag{3.14}$$

where Ω is a regular set. Then, for $\alpha \in (d-2, d-1)$, we have

$$\inf_{|E|=m,\, E\subset\Omega} \mathcal{F}(E) = P(E_0) + \mathcal{Q}_\alpha(\Omega). \tag{3.15}$$

Similarly, for $\alpha \in (0, d-2]$,

$$\inf_{|E|=m,\, E\subset\Omega} \mathcal{G}(E) = P(E_0) + \mathcal{Q}_\alpha(\partial\Omega). \tag{3.16}$$

Proof. We only prove (3.15) since (3.16) can be proved exactly in the same way. We divide the proof into three steps.

Step 1. For $\varepsilon > 0$ and $f \in L^\infty(A)$, with $f \geq 0$ and $\int_A f = 1$, we shall construct a measure $\tilde{\mu}_\varepsilon$ with $\mathrm{spt}(\tilde{\mu}_\varepsilon) \subset A$, $\tilde{\mu}_\varepsilon(A) = 1$, satisfying

$$P(\mathrm{spt}(\tilde{\mu}_\varepsilon)) \leq \varepsilon \tag{3.17}$$

and

$$\mathcal{Q}_\alpha(\tilde{\mu}_\varepsilon) \leq \mathcal{Q}_\alpha^A(f) + \varepsilon. \tag{3.18}$$

Let $\delta > \lambda > 0$ be small parameters to be fixed later and consider the tiling of the space given by $[0,\lambda)^d + \lambda\mathbb{Z}^d$. For every $k \in \mathbb{Z}^d$ such that $(\lambda k + [0,\lambda)^d) \cap A \neq \emptyset$, we let $C_k := \lambda k + [0,\lambda)^d$ and let x_k be the centre of C_k. Notice that the number N of such squares C_k is bounded by $C(A)\lambda^{-d}$. Letting $f_k := \int_{C_k} f\, dx$, it holds

$$\begin{aligned}
\sum_{|x_k-x_j|\geq 2\delta} \frac{f_k f_j}{|x_k-x_j|^\alpha} &= \sum_{|x_k-x_j|\geq 2\delta} \int_{C_k\times C_j} \frac{f(x)f(y)}{|x-y|^\alpha}\,\frac{|x-y|^\alpha}{|x_k-x_j|^\alpha}dx\ dy \\
&\leq \sum_{|x_k-x_j|\geq 2\delta} \int_{C_k\times C_j} \frac{f(x)f(y)}{|x-y|^\alpha}\,\frac{(|x_k-x_j|+2\lambda)^\alpha}{|x_k-x_j|^\alpha}dx\ dy \\
&\leq \sum_{|x_k-x_j|\geq 2\delta} \int_{C_k\times C_j} \frac{f(x)f(y)}{|x-y|^\alpha}\left(1 + C(\alpha)\frac{\lambda}{\delta}\right)dx\ dy \\
&\leq \mathcal{Q}_\alpha^A(f)\left(1 + C(\alpha)\frac{\lambda}{\delta}\right).
\end{aligned} \tag{3.19}$$

where we used the fact

$$\sum_{|x_k-x_j|\geq 2\delta}\int_{C_k\times C_j}\frac{f(x)f(y)}{|x-y|^\alpha}\leq\int_{A\times A}\frac{f(x)f(y)}{|x-y|^\alpha}=\mathcal{Q}_\alpha^A(f)<\infty.$$

Let now $r=(\lambda/2)^\beta$, with $\beta>1$. If $\operatorname{dist}(x_k,\mathbb{R}^d\setminus A)\leq r$, we replace the point x_k with a point $\tilde{x}_k\in C_{j(k)}$, with $|\tilde{x}_k-x_{j(k)}|\geq\lambda/4$, where $C_{j(k)}\subset A$ is a cube adjacent to C_k. For simplicity of notation, we still denote $\tilde{x}_k$ by x_k. We consider N balls of radius r centred at the points x_k, and we set

$$\tilde{\mu}_\varepsilon:=\sum_k\frac{f_k}{|B_r|}\chi_{B_r(x_k)}.$$

Notice that, by construction, $\operatorname{spt}(\tilde{\mu}_\varepsilon)\subset A$ and $\tilde{\mu}_\varepsilon(A)=\int_A f=1$. Then we get

$$\begin{aligned}\mathcal{Q}_\alpha(\tilde{\mu}_\varepsilon)&=\sum_{j,k}\frac{f_kf_j}{|B_r|^2}\int_{B_r(x_j)\times B_r(x_k)}\frac{dxdy}{|x-y|^\alpha}\\&=\sum_k\frac{f_k^2}{|B_r|^2}\int_{B_r(x_k)\times B_r(x_k)}\frac{dxdy}{|x-y|^\alpha}\\&+\sum_{|x_j-x_k|<2\delta,\,k\neq j}\frac{f_kf_j}{|B_r|^2}\int_{B_r(x_j)\times B_r(x_k)}\frac{dxdy}{|x-y|^\alpha}\\&+\sum_{|x_j-x_k|\geq 2\delta}\frac{f_kf_j}{|B_r|^2}\int_{B_r(x_j)\times B_r(x_k)}\frac{dxdy}{|x-y|^\alpha}\\&=I_1+I_2+I_3.\end{aligned}$$

Moreover we have

$$I_1\leq CN\|f\|^2_{L^\infty(A)}|C_k|^2\frac{1}{r^\alpha}\leq C\|f\|^2_{L^\infty(A)}\lambda^{d-\alpha\beta},\tag{3.20}$$

and

$$I_2\leq C\delta^dN^2\|f\|^2_{L^\infty(A)}|C_k|^2\frac{1}{\lambda^\alpha}\leq C\|f\|^2_{L^\infty(A)}\frac{\delta^d}{\lambda^\alpha}.\tag{3.21}$$

Eventually, from (3.19) it follows

$$\begin{aligned}I_3&=\sum_{|x_j-x_k|\geq 2\delta}\frac{f_kf_j}{|x_k-x_j|^\alpha}\frac{1}{|B_r|^2}\int_{B_r(x_j)\times B_r(x_k)}\frac{|x_k-x_j|^\alpha}{|x-y|^\alpha}\,dxdy\\&\leq\sum_{|x_k-x_j|\geq 2\delta}\frac{f_kf_j}{|x_k-x_j|^\alpha}\left(1+C(\alpha)\frac{r}{\delta}\right)\\&\leq\mathcal{Q}_\alpha^A(f)\left(1+C(\alpha)\frac{\lambda}{\delta}\right)\left(1+C(\alpha)\frac{r}{\delta}\right)\\&\leq\mathcal{Q}_\alpha^A(f)+C(\alpha)\mathcal{Q}_\alpha^A(f)\frac{\lambda}{\delta}.\end{aligned}\tag{3.22}$$

Letting $\lambda = \delta^\gamma$, from (3.20), (3.21), (3.22), we then get

$$\mathcal{Q}_\alpha(\tilde{\mu}_\varepsilon) = I_1 + I_2 + I_3 \leq \mathcal{Q}_\alpha^A(f) + C(\alpha)\mathcal{Q}_\alpha^A(f)\delta^{\gamma-1} + C\|f\|^2_{L^\infty(A)}\left(\delta^{\gamma(d-\alpha\beta)} + \delta^{d-\alpha\gamma}\right).$$

Choosing $1 < \beta < d/\alpha$ and $1 < \gamma < d/\alpha$, for δ small enough we obtain (3.18).

Let us show that (3.17) holds as well. To this aim, we notice that

$$P(\mathrm{spt}(\tilde{\mu}_\varepsilon)) \leq CNr^{d-1} = CN\lambda^{\beta(d-1)} = C\lambda^{\beta(d-1)-d}. \tag{3.23}$$

Hence, for λ small enough, (3.17) follows from (3.23) by choosing $d/\alpha > \beta > d/(d-1)$, which is possible since $\alpha < d-1$.

Step 2. Let now E_0 be a solution of the constrained isoperimetric problem (3.14), and let

$$E_\varepsilon := \left(E_0 \cup \bigcup_k B_r(x_k)\right) \setminus B_\eta \qquad \mu_\varepsilon := \frac{\tilde{\mu}_\varepsilon \llcorner E_\varepsilon}{1 - \tilde{\mu}_\varepsilon(B_\eta)},$$

where $B_\eta \subset E_0$ is a ball such that $|E_\varepsilon| = m$. Notice that $\mathrm{spt}(\mu_\varepsilon) \subset E_\varepsilon$ and $\mu_\varepsilon(E_\varepsilon) = 1$. By (3.23) we have

$$|B_\eta|^{\frac{d-1}{d}} \leq \left|\bigcup_k B_r(x_k)\right|^{\frac{d-1}{d}} \leq CP\left(\bigcup_k B_r(x_k)\right) \leq C\lambda^{\beta(d-1)-d},$$

so that $\eta \leq C\lambda^{\beta-1}$. In particular, recalling (3.18), for λ sufficiently small the measure μ_ε satisfies

$$\mathcal{Q}_\alpha(\mu_\varepsilon) \leq \mathcal{Q}_\alpha(\tilde{\mu}_\varepsilon) + \varepsilon \leq \mathcal{Q}_\alpha^A(f) + 2\varepsilon. \tag{3.24}$$

From (3.24) we then get

$$\varlimsup_{\varepsilon\to 0} P(E_\varepsilon) + Q^2\mathcal{Q}_\alpha(\mu_\varepsilon) = P(E_0) + Q^2\mathcal{Q}_\alpha^A(f). \tag{3.25}$$

Step 3. By Proposition 3.12, for any $\varepsilon > 0$ we can find a function $f \in L^\infty(A)$ such that $\int_A f = 1$ and $\mathcal{Q}_\alpha^A(f) \leq \mathcal{Q}_\alpha(A) + \varepsilon$. Thus (3.15) follows by (3.25) and a diagonal argument.

□

Remark 3.23. An interpretation of Theorem 3.22 is that the two problems in the definition of $\mathcal{F}$, the isoperimetric one and the *charge-minimizing* one, are actually decoupled. A consequence of this fact is that the minimization problem

$$\min\left\{P(E) + Q^2\int_{E\times E}\frac{d\mu(x)\,d\mu(y)}{|x-y|^\alpha} : |E| = m,\, E \subset \Omega \;\; \mu(E) = 1\right\}$$

is ill-posed. This is due by the fact that the perimeter is invariant under perturbation by sets of null L^1 volume, while the Energy functional is sensible to perturbations which change *only* the capacity of the set, see [84, Chapter 2].

3.4 Existence and characterization of minimizers under a regularity condition

In the previous section we have seen that we cannot hope to get existence for problem (3.11) without some further assumption on the class of minimization. In this section we investigate the existence question in the class K_δ.
We consider the four problems

$$\min\{\mathcal{F}(E):\ |E|=m,\ E\in\mathcal{K}_\delta^{co}\}, \tag{3.26}$$

$$\min\{\mathcal{G}(E):\ |E|=m,\ E\in\mathcal{K}_\delta^{co}\}, \tag{3.27}$$

$$\min\{\mathcal{F}(E):\ |E|=m,\ E\in\mathcal{K}_\delta\}, \tag{3.28}$$

and

$$\min\{\mathcal{G}(E):\ |E|=m,\ E\in\mathcal{K}_\delta\}. \tag{3.29}$$

Up to rescaling, we can assume that $|E|=|B|$ where B is the unit ball. Indeed, for (3.26) for example, for every E of volume m and every f minimizing $\mathcal{Q}_\alpha(\partial E)$, considering $\tilde{E}=\left(\frac{|B|}{m}\right)^{1/d}E$ and $\tilde{f}(x)=\left(\frac{|B|}{m}\right)^{\frac{1}{d(d-1)}}f\left(\left(\frac{m}{|B|}\right)^{1/d}x\right)$ we have $|\tilde{E}|=|B|$, $\int_{\partial\tilde{E}}\tilde{f}=1$ and

$$P(E)+Q^2\mathcal{Q}_\alpha^{\partial E}(f)=\left(\frac{m}{|B|}\right)^{d-1/d}\left(P(\tilde{E})+Q^2\left(\frac{|B|}{m}\right)^{\frac{d-1}{d}-\alpha}\mathcal{Q}_\alpha^{\partial\tilde{E}}(\tilde{f})\right).$$

Given a set E with $|E|=|B|$, we let $\delta P(E):=P(E)-P(B)$ be the (non-rescaled) isoperimetric deficit.

Theorem 3.24. *Problem* (3.26) *has solution.*

Proof. Let $E_n\in\mathcal{K}_\delta^{co}$ be a minimizing sequence. And let μ_n be the corresponding optimal measures for $\mathcal{Q}_\alpha(E_n)$. We can then assume that

$$\delta P(E_n)\le Q^2\mathcal{Q}_\alpha(B),$$

therefore $P(E_n)$ is uniformly bounded. By Lemma 3.18, the sets E_n are also uniformly bounded so that by BV compactness, there exists a subsequence converging in L^1 to some E with $|E|=m$. Similarly, up to subsequence, μ_n is weakly* converging to some probability measure μ. Let us prove that E_n converges to E also in the Kuratowski convergence, or equivalently, in the Hausdorff metric (see for instance [4]). Namely we have to check the following two conditions:

$$(i)\, x_n\to x,\quad x_n\in E_n\ \Rightarrow\ x\in E;$$
$$(ii)\, x\in E\ \Rightarrow\ \exists x_n\in E_n\,\text{such that}\ x_n\to x.$$

The second condition is an easy consequence of the L^1-convergence. To prove the first one, we notice that by the $\delta-$ball condition, up to choose a radius r small enough, there exists a constant $c = c(d,\delta) > 0$ such that $|B_r(x_n) \cap E_n| \geq cr^d$, which implies, together with the L^1-convergence, statement (i). Similarly one can also prove the Hausdorff convergence of ∂E_n to ∂E. Since the set $\mathcal{K}_\delta^{co}$ is stable under Hausdorff convergence, $E \in \mathcal{K}_\delta^{co}$.
Since the perimeter P is lower semicontinuous under L^1 convergence and since $\mathcal{Q}_\alpha(\mu)$ is lower semicontinuous under weak* convergence,

$$\underline{\lim}_{n\to+\infty} P(E_n) + Q^2\mathcal{Q}_\alpha(\mu_n) \geq P(E) + Q^2\mathcal{Q}_\alpha(\mu),$$

which concludes the proof. □

Similarly, one can show the following theorem.

Theorem 3.25. *Problem (3.27) has solution.*

Corollary 3.26. *There exists Q_0 such that for every $Q \leq Q_0$, problems (3.28) and (3.29) have a solution.*

Proof. We will only consider (3.28), since (3.29) can be treated in a similar way. As noticed before, for every minimizing sequence $E_n \in \mathcal{K}_\delta$, we can assume that there holds

$$\delta P(E_n) \leq Q^2\mathcal{Q}_\alpha(B),$$

so that by the quantitative isoperimetric inequality [62], up to a translation we can assume that

$$|B\Delta E_n|^2 \leq C(d)Q^2\mathcal{Q}_\alpha(B)$$

so that $|E_n \cap B^c| \leq CQ$ but since every connected component of E_n has volume at least $|B_\delta|$ by the $\delta-$ball condition, we see that for Q small enough, E_n must be connected. Thus the existence of minimizers follows as in Theorem 3.24. □

It is natural to expect that for large enough charge Q, it is more favorable to have two connected components rather than one, which leads to non-existence of minimizers in $\mathcal{K}_\delta$. Let us prove that it is indeed the case (at least for certain α). We start with the following lemma.

Lemma 3.27. *Let $\alpha > 0$ and let E be a bounded Borel set. Then*

$$\mathcal{Q}_\alpha(E) \geq \frac{1}{\mathrm{diam}(E)^\alpha}.$$

This implies

$$\min\left\{P(E) + Q^2\mathcal{Q}_\alpha(E) : \; |E| = |B|, \; E \in \mathcal{K}_\delta^{co}\right\} \geq P(B) + \left(\sqrt{d}\,2^{d+2}\right)^{-\alpha} Q^2\delta^{(d-1)\alpha} \tag{3.30}$$

and similarly

$$\min\left\{P(E)+Q^2\mathcal{Q}_\alpha(\partial E):\ |E|=|B|,\ E\in\mathcal{K}_\delta^{co}\right\}\geq P(B)+\left(\sqrt{d}\,2^{d+2}\right)^{-\alpha}Q^2\delta^{(d-1)\alpha}. \tag{3.31}$$

Proof. Let μ be any positive measure with support in E such that $\mu(E)=1$. Then

$$\mathcal{Q}_\alpha(E)\geq\int_{E\times E}\frac{d\mu(x)d\mu(y)}{|x-y|^\alpha}\geq\int_{E\times E}\frac{d\mu(x)d\mu(y)}{\mathrm{diam}(E)^\alpha}=\frac{1}{\mathrm{diam}(E)^\alpha}.$$

An application of Lemma 3.18 and the isoperimetric inequality lead to (3.30) and to (3.31). □

Theorem 3.28. *Let $\alpha<2$. Then there exists $\delta_0>0$ such that for every $\delta\leq\delta_0$, there exists $Q_0(\delta)$ such that for every $Q\geq Q_0(\delta)$ problems (3.28) and (3.29) do not have a solution.*

Proof. If there exists a minimizer, then it must be connected. Thus, in order to prove non-existence, it is enough to construct a competitor $E\in\mathcal{K}_\delta$ with energy less than

$$P(B)+\left(\sqrt{d}\,2^{d+2}\right)^{-\alpha}Q^2\delta^{(d-1)\alpha}$$

which bounds from below the energy of any set in $\mathcal{K}_\delta^{co}$. To this aim, consider the set E given by $N=\frac{|B|}{\delta^d}$ balls of radius δ. Up to increase their mutual distances, we can suppose that the Riesz potential energy of E is only made of the self interactions of each ball with itself. Considering (3.28), we have

$$P(E)+Q^2\mathcal{Q}_\alpha(E)=N\delta^{d-1}P(B)+N^2Q^2\mathcal{Q}_\alpha(B_\delta)=|B|P(B)\delta^{-1}+\frac{\mathcal{Q}_\alpha(B)}{|B|^2}\,Q^2\delta^{2d-\alpha}.$$

We see that if $2d-\alpha>(d-1)\alpha$, i.e. $2>\alpha$, then for δ small enough there holds $\frac{\mathcal{Q}_\alpha(B)}{|B|^2}\,\delta^{2d-\alpha}<\left(\sqrt{d}\,2^{d+2}\right)^{-\alpha}\delta^{(d-1)\alpha}$. Thus, for Q large enough,

$$P(E)+Q^2\mathcal{Q}_\alpha(E)<P(B)+\left(\sqrt{d}\,2^{d+2}\right)^{-\alpha}Q^2\delta^{(d-1)\alpha}.$$

The non-existence for problem (3.29) follows similarly. □

Remark 3.29. Notice that in dimension 3, the previous theorem contains the Coulomb interaction potential case, $\alpha=1, d=3$.

3.5 Stability of the ball

In this section we prove that in the harmonic case $\alpha = d-2$, the ball is an optimizer for problem (3.11) among sets in the family of the *nearly spherical sets* belonging to $\mathcal{K}_\delta^{co}$ introduced in Definition 3.16, that is, the sets which are a small $W^{1,\infty}$ perturbation of the ball and that satisfy the $\delta-$ball condition.

Let E be such that $|E| = |B|$ and ∂E can be written as a graph over ∂B. In polar coordinates we have

$$E = \big\{R(x)x \,:\, R(x) = 1 + \varphi(x),\, x \in \partial B\big\}.$$

The condition $|E| = |B|$ then is equivalent to

$$\int_{\partial B} \Big((1+\varphi(x))^d - 1\Big)\, d\mathcal{H}^{d-1}(x) = 0$$

which implies that if $\|\varphi\|_{L^\infty(\partial B)}$ is small enough, then

$$\int_{\partial B} \varphi d\mathcal{H}^{d-1} = O(\|\varphi\|^2_{L^2(\partial B)}). \tag{3.32}$$

Letting

$$\bar\varphi = \frac{1}{|\partial B|}\int_{\partial B} \varphi d\mathcal{H}^{d-1},$$

the Poincaré Inequality gives

$$\begin{aligned}\int_{\partial B} |\nabla\varphi|^2 d\mathcal{H}^{d-1} &\geq C\int_{\partial B} |\varphi - \bar\varphi|^2 d\mathcal{H}^{d-1} = C\int_{\partial B}\varphi^2\mathcal{H}^{d-1} - \frac{C}{|\partial B|}\left(\int_{\partial B}\varphi\right)^2 d\mathcal{H}^{d-1}\\ &= C\int_{\partial B}\varphi^2 d\mathcal{H}^{d-1} - \frac{C}{4|\partial B|}\left(\int_{\partial B}\varphi^2 d\mathcal{H}^{d-1}\right)^2\\ &\geq \frac{3}{4}C\int_{\partial B}\varphi^2 d\mathcal{H}^{d-1}\end{aligned} \tag{3.33}$$

as soon as

$$\int_{\partial B}\varphi^2 d\mathcal{H}^{d-1} \leq 1, \tag{3.34}$$

for some constant C depending only on the dimension d.
Up to translation, we can also assume that the barycentre of E coincides with that of B, say 0. This implies that

$$\left|\int_{\partial B} x\varphi(x) d\mathcal{H}^{d-1}(x)\right| = O(\|\varphi\|^2_{L^2(\partial B)}). \tag{3.35}$$

Lemma 3.30. *Suppose that $\varphi : \partial B \to \mathbb{R}^d$ parametrizes ∂E and satisfies (3.32), (3.33), (3.35) and (3.34). Then we have*

$$\delta P(E) \geq c_0 \int_{\partial B} |\nabla \varphi|^2 d\mathcal{H}^{d-1} \geq c_1 \int_{\partial B} |\varphi|^2 d\mathcal{H}^{d-1} = c_2 \left| \int_{\partial B} \varphi d\mathcal{H}^{d-1} \right|. \tag{3.36}$$

Proof. The first inequality is quite a well known fact, and we refer to [60] or [40] for its proof. The second one is exactly (3.33), while the third one follows from (3.32). □

A consequence of Lemma 3.30 is the following corollary.

Corollary 3.31. *Suppose that ∂E is parametrized on ∂B by a function φ which satisfies the hypotheses of Lemma 3.30. Then there exists a positive constant $C = C(d)$ such that*

$$|\mathcal{Q}_\alpha^{\partial B}(\varphi)| \leq C \delta P(E), \tag{3.37}$$

and, for any positive constant λ,

$$|\mathcal{Q}_\alpha^{\partial B}(\lambda, \varphi)| \leq C \lambda \delta P(E). \tag{3.38}$$

Proof. Inequality (3.38) is an immediate consequence of (3.36). Concerning the first one we have, by the Hölder inequality and the Fubini Theorem,

$$\begin{aligned}
&\mathcal{Q}_\alpha^{\partial B}(\varphi) = \int_{\partial B \times \partial B} \frac{\varphi(x)\varphi(y)}{|x-y|^\alpha} \, d\mathcal{H}^{d-1}(x) d\mathcal{H}^{d-1}(y) \\
&\leq \left(\int_{\partial B \times \partial B} \frac{\varphi(x)^2}{|x-y|^\alpha} \, d\mathcal{H}^{d-1}(x) d\mathcal{H}^{d-1}(y) \right)^{1/2} \left(\int_{\partial B \times \partial B} \frac{\varphi(y)^2}{|x-y|^\alpha} \, d\mathcal{H}^{d-1}(x) d\mathcal{H}^{d-1}(y) \right)^{1/2} \\
&\leq C(\alpha) \int_{\partial B} \varphi(x)^2 \, d\mathcal{H}^{d-1}(x).
\end{aligned}$$

So (3.37) follows again from (3.36). □

The following (technical) lemma will be exploited in Proposition 3.35.

Lemma 3.32. *Let $E = \{R(x)x \,:\, R(x) = 1 + \varphi(x),\, x \in \partial B\}$ and let $g \in L^\infty(\partial B)$, then there exists $\varepsilon_0(\alpha, d)$ and a constant $C = C(\alpha, d) > 0$ such that if $\|\varphi\|_{W^{1,\infty}(\partial B)} \leq \varepsilon_0$, then*

$$\begin{aligned}
&\left| \int_{\partial B \times \partial B} \left(\frac{1}{|R(x) - R(y)|^\alpha} - \frac{(1 - \frac{\alpha}{2}\varphi(x))(1 - \frac{\alpha}{2}\varphi(y))}{|x-y|^\alpha} \right) g(x) g(y) d\mathcal{H}^{d-1}(x) \, d\mathcal{H}^{d-1}(y) \right| \\
&\qquad \leq C(\alpha, d) \|g\|^2_{L^\infty(\partial B)} \delta P(E).
\end{aligned} \tag{3.39}$$

Proof. First, notice that since $|x| = |y| = 1$, we get

$$|R(x)x - R(y)y|^2 = |x-y|^2\left(1 + \varphi(x) + \varphi(y) + \varphi(x)\varphi(y) + \psi(x,y)\right) \tag{3.40}$$

where $\psi(x,y) = \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^2}$. Thus, for any $x, y \in \partial B$,

$$|R(x)x - R(y)y|^{-\alpha} = \frac{(1-\frac{\alpha}{2}\varphi(x))(1-\frac{\alpha}{2}\varphi(y)) + \frac{\alpha(4-\alpha)}{4}\varphi(x)\varphi(y) - \frac{\alpha}{2}(\psi(x,y)+\eta(x,y))}{|x-y|^\alpha} \tag{3.41}$$

where

$$\eta(x,y) \le C\left(\varphi^2(x) + \varphi^2(y) + \psi^2(x,y)\right).$$

By (3.39) and (3.41) we get

$$\begin{aligned}
&\int_{\partial B\times\partial B}\left(\frac{1}{|R(x)-R(y)|^\alpha} - \frac{(1-\frac{\alpha}{2}\varphi(x))(1-\frac{\alpha}{2}\varphi(y))}{|x-y|^\alpha}\right) g(x)g(y)d\mathcal{H}^{d-1}(x)\, d\mathcal{H}^{d-1}(y)\\
&= \frac{\alpha(4-\alpha)}{4}\int_{\partial B\times\partial B}\frac{\varphi(x)\varphi(y)}{|x-y|^\alpha} g(x)g(y)\, d\mathcal{H}^{d-1}(x)d\mathcal{H}^{d-1}(y)\\
&\quad -\frac{\alpha}{2}\int_{\partial B\times\partial B}\frac{\psi(x,y)+\eta(x,y)}{|x-y|^\alpha} g(x)g(y)\, d\mathcal{H}^{d-1}(x)d\mathcal{H}^{d-1}(y).
\end{aligned} \tag{3.42}$$

By (3.37),

$$\int_{\partial B\times\partial B}\frac{\varphi(x)\varphi(y)}{|x-y|^\alpha}\, d\mathcal{H}^{d-1}(x)d\mathcal{H}^{d-1}(y) = \mathcal{Q}_\alpha^{\partial B}(\varphi) \le C\delta P(E).$$

Furthermore, since

$$\psi(x,y) \le C\|\nabla\varphi\|^2_{L^\infty(\partial B)},$$

and $\eta(x,y) \le C\varphi^2(x) + \varphi^2(y) + \psi(x,y)$, for a suitable C, we only have to check that

$$\int_{\partial B\times\partial B}\frac{\psi(x,y)}{|x-y|^\alpha}\, d\mathcal{H}^{d-1}(x)d\mathcal{H}^{d-1}(y) \le C\delta P(E).$$

For x, y in ∂B let us denote by $\Gamma_{x,y}$ the geodesic going from x to y and by $\ell(x,y)$ the geodesic distance between x and y (that is the length of $\Gamma_{x,y}$). Notice that on ∂B, the euclidean distance and ℓ are equivalent so that proving the previous inequality is equivalent to prove that it holds

$$\int_{\partial B\times\partial B}\ell(x,y)^{-(\alpha+2)}(\varphi(x)-\varphi(y))^2 \le C\delta P(E).$$

We have

$$
\begin{aligned}
&\int_{\partial B\times\partial B}\ell(x,y)^{-(\alpha+2)}(\varphi(x)-\varphi(y))^2\\
&\qquad\le c(d)\int_{\partial B\times\partial B}\ell(x,y)^{-(\alpha+1)}\int_{\Gamma_{x,y}}|\nabla\varphi|^2(z)dz\, d\mathcal{H}^{d-1}(x)d\mathcal{H}^{d-1}(y)\\
&\qquad= c(d)\int_{\partial B}\int_0^{2\pi}t^{-(\alpha+1)}t^{d-1}\left(\int_{\{\ell(x,z)\le t\}}|\nabla\varphi|^2(z)d\mathcal{H}^{d-1}(z)\right)dtd\mathcal{H}^{d-1}(x)\\
&\qquad= c(d)\int_0^{2\pi}t^{(d-1)-(\alpha+1)}\left(\int_{\partial B}\int_{\{\ell(x,z)\le t\}}|\nabla\varphi|^2(z)d\mathcal{H}^{d-1}(x)d\mathcal{H}^{d-1}(z)\right)dt\\
&\qquad= c(d)\mathcal{H}^{d-2}(\mathbb{S}^{d-2})\int_0^{2\pi}t^{(d-1)-\alpha}\left(\int_{\partial B}|\nabla\varphi|^2(z)d\mathcal{H}^{d-1}(z)\right)dt\\
&\qquad= c(d)\mathcal{H}^{d-2}(\mathbb{S}^{d-2})\int_0^{2\pi}t^{(d-1)-\alpha}dt\left(\int_{\partial B}|\nabla\varphi|^2(z)d\mathcal{H}^{d-1}(z)\right)\\
&\qquad\le C(d)\delta P(E)
\end{aligned}
$$

where $\mathbb{S}^{d-2}$ is the $(d-2)-$dimensional sphere and where we used that $\alpha<d-1$. $\square$

Before we pass to our main stability estimates, let us recall the following simple interpolation inequality

Lemma 3.33. *For every $0\le p<q<r<+\infty$, there exists a constant $C(r,p,q)$ such that for every $\varphi\in H^r(\mathbb{R}^d)$, it holds*

$$
\|\varphi\|_{H^q}\le C\left(\|\varphi\|_{H^r}\right)^{\frac{r-q}{r-p}}\left(\|\varphi\|_{H^p}\right)^{\frac{q-p}{r-p}}. \tag{3.43}
$$

Remark 3.34. In the previous lemma we adopted the notation $\|u\|_{H^p}:=\||\xi|^p\hat{u}\|_{L^2(\mathbb{R}^d)}$ and $H^p(\mathbb{R}^d):=\{u\in L^2(\mathbb{R}^d):\|u\|_{H^p}<\infty\}$. Such notation will be used *only* in the remaining part of this chapter.

Proof. Using Fourier transform, for every $\varphi\in H^r$ and every $\lambda>0$, there holds

$$
\begin{aligned}
\|\varphi\|_{H^q}^2=\int_{\mathbb{R}^d}|\hat{\varphi}|^2|\xi|^{2q}d\xi&=\int_{|\xi|\le\lambda}|\hat{\varphi}|^2|\xi|^{2p}|\xi|^{2(q-p)}d\xi+\int_{|\xi|\ge\lambda}|\hat{\varphi}|^2|\xi|^{2r}|\xi|^{2(q-r)}d\xi\\
&\le\lambda^{2(q-p)}\|\varphi\|_{H^p}^2+\lambda^{-2(r-q)}\|\varphi\|_{H^r}^2.
\end{aligned}
$$

Optimizing in λ yields (3.43). $\square$

Proposition 3.35. *Let $\alpha\in(0,d-1)$ and $r>\alpha/2$. For $\partial E=\big\{R(x)x\,:\,R(x)=1+\varphi(x),\ x\in\partial B\big\}$ and $f\in L^\infty(\partial E)$, there exists $\varepsilon_0(\alpha,d)$ and a constant $C=C(\alpha,d)>0$ such that if $\|\varphi\|_{W^{1,\infty}(\partial B)}\le\varepsilon_0$ and $\|\varphi\|_{H^r}\le\varepsilon_0$ then*

$$
\mathcal{Q}_\alpha^{\partial E}(f)-\mathcal{Q}_\alpha^{\partial B}(\bar{f})\ge-C\|f\|_{L^\infty(\partial E)}^2\delta P(E),
$$

where $\bar{f}=\frac{1}{P(E)}\int_{\partial E}fd\mathcal{H}^{d-1}$.

Proof. We have

$$\int_{\partial E\times\partial E}\frac{f(x)f(y)}{|x-y|^{\alpha}}\,d\mathcal{H}^{d-1}(x)\,d\mathcal{H}^{d-1}(y)$$
$$=\int_{\partial B\times\partial B}\frac{f(R(x)x)f(R(y)y)R(x)^{d-2}R(y)^{d-2}\sqrt{R(x)^2+|\nabla R(x)|^2}\sqrt{R(y)^2+|\nabla R(y)|^2}}{|R(x)x-R(y)y|^{\alpha}}. \tag{3.44}$$

Let $g(x):=f(R(x)x)R(x)^{d-2}\sqrt{R(x)^2+|\nabla R(x)|^2}$, then (3.44) can be rewritten as

$$\mathcal{Q}_\alpha^{\partial E}(f)=\int_{\partial B\times\partial B}\frac{g(x)g(y)}{|R(x)-R(y)|^{\alpha}}\,d\mathcal{H}^{d-1}(x)d\mathcal{H}^{d-1}(y). \tag{3.45}$$

We shall assume that E is close enough to B so that

$$\|g\|_{L^\infty(\partial B)}\le 2\|f\|_{L^\infty(\partial E)}. \tag{3.46}$$

A simple computation shows that there is a positive constant $C=C(\alpha,d)$ such that letting $\bar g:=\frac{1}{P(B)}\int_{\partial B}g d\mathcal{H}^{d-1}$, there holds

$$\mathcal{Q}_\alpha^{\partial B}(\bar g)-\mathcal{Q}_\alpha^{\partial E}(\bar f)\le C\|f\|^2_{L^\infty(\partial E)}\delta P(E), \tag{3.47}$$

so, up to sum and subtract $\mathcal{Q}_\alpha^{\partial E}(\bar f)$, thanks to (3.46) and (3.47), the proposition is proven if we can show that

$$\mathcal{Q}_\alpha^{\partial E}(f)\ge\mathcal{Q}_\alpha^{\partial B}(\bar g)-\|g\|^2_{L^\infty(\partial E)}\delta P(E).$$

By Lemma 3.32, we obtain

$$\mathcal{Q}_\alpha^{\partial E}(f)=\mathcal{Q}_\alpha^{\partial B}\left(g(1-\frac{\alpha}{2}\varphi)\right)+\mathcal{R}(f,\varphi)$$

with

$$|\mathcal{R}(f,\varphi)|\le c\|f\|^2_{L^\infty(\partial E)}\delta P(E).$$

Hence we get

$$\mathcal{Q}_\alpha^{\partial E}(f)\ge\mathcal{Q}_\alpha^{\partial B}\left(g(1-\frac{\alpha}{2}\varphi)\right)-c\|f\|^2_{L^\infty(\partial E)}\delta P(E). \tag{3.48}$$

We now estimate

$$
\begin{aligned}
\mathcal{Q}_\alpha^{\partial B}(g(1-\frac{\alpha}{2}\varphi)) =&\mathcal{Q}_\alpha^{\partial B}(g(1-\frac{\alpha}{2}\varphi), g(1-\frac{\alpha}{2}\varphi)) \\
=&\mathcal{Q}_\alpha^{\partial B}(g,g) - \alpha\mathcal{Q}_\alpha^{\partial B}(g,g\varphi) + \frac{\alpha^2}{4}\mathcal{Q}_\alpha^{\partial B}(g\varphi,g\varphi) \\
=&\mathcal{Q}_\alpha^{\partial B}(\bar g,\bar g) + \mathcal{Q}_\alpha^{\partial B}(g-\bar g,g-\bar g) - \alpha\mathcal{Q}_\alpha^{\partial B}(g-\bar g,g\varphi) - \alpha\mathcal{Q}_\alpha^{\partial B}(\bar g,g\varphi) \\
&+ \frac{\alpha^2}{4}\mathcal{Q}_\alpha^{\partial B}(\bar g\varphi,\bar g\varphi) + \frac{\alpha^2}{2}\mathcal{Q}_\alpha^{\partial B}(\bar g\varphi,(g-\bar g)\varphi) + \frac{\alpha^2}{4}\mathcal{Q}_\alpha^{\partial B}((g-\bar g)\varphi,(g-\bar g)\varphi) \\
=&\mathcal{Q}_\alpha^{\partial B}(\bar g) + \mathcal{Q}_\alpha^{\partial B}(g-\bar g) + \frac{\alpha^2}{4}\mathcal{Q}_\alpha^{\partial B}((g-\bar g)\varphi) - \alpha\mathcal{Q}_\alpha^{\partial B}(g-\bar g,(g-\bar g)\varphi) \\
&- \alpha\mathcal{Q}_\alpha^{\partial B}(\bar g,(g-\bar g)\varphi) - \alpha\mathcal{Q}_\alpha^{\partial B}(g-\bar g,\bar g\varphi) + \frac{\alpha^2}{2}\mathcal{Q}_\alpha^{\partial B}(\bar g\varphi,(g-\bar g)\varphi) \\
&- \alpha\mathcal{Q}_\alpha^{\partial B}(\bar g,\bar g\varphi) + \frac{\alpha^2}{4}\mathcal{Q}_\alpha^{\partial B}(\bar g\varphi).
\end{aligned}
\tag{3.49}
$$

The last two terms in the right hand side of (3.49), in view of Lemma 3.30 and the bilinearity of $\mathcal{Q}_\alpha^{\partial B}$, satisfy:

$$
-\mathcal{Q}_\alpha^{\partial B}(\bar g,\bar g\varphi) + \frac{\alpha}{2}\mathcal{Q}_\alpha^{\partial B}(\bar g\varphi) \geq -c\bar g^2\delta P(E). \tag{3.50}
$$

By inequality (3.7) and Young's inequality, we get that for any functions h_1 and h_2 we have

$$
\mathcal{Q}_\alpha^{\partial B}(h_1,h_2) \leq \mathcal{Q}_\alpha^{\partial B}(h_1,h_1)^{\frac{1}{2}}\mathcal{Q}_\alpha^{\partial B}(h_2,h_2)^{\frac{1}{2}} \leq \varepsilon\mathcal{Q}_\alpha^{\partial B}(h_1,h_1) + \frac{1}{4\varepsilon}\mathcal{Q}_\alpha^{\partial B}(h_2,h_2) \tag{3.51}
$$

for any $\varepsilon > 0$. In particular, applying such inequality to the functions $h_1 = g - \bar g$ and $h_2 = (g-\bar g)\varphi$ on the fourth term in the right hand side of (3.49), then on the sixth one, with $h_1 = g - \bar g$ and $h_2 = \bar g\varphi$ and exploiting (3.50), we get that there exists a positive constant C such that

$$
\begin{aligned}
&\mathcal{Q}_\alpha^{\partial B}(g(1-\frac{\alpha}{2}\varphi)) - \mathcal{Q}_\alpha^{\partial B}(\bar g) \\
&\geq C\left(\frac{1}{2}\mathcal{Q}_\alpha^{\partial B}(g-\bar g) - \mathcal{Q}_\alpha^{\partial B}(\bar g,(g-\bar g)\varphi) - \mathcal{Q}_\alpha^{\partial B}((g-\bar g)\varphi) - \bar g^2\delta P(E)\right).
\end{aligned}
\tag{3.52}
$$

Again by Lemma 3.30, we have that

$$
-\mathcal{Q}_\alpha^{\partial B}((g-\bar g)\varphi) \geq -\|g\|^2_{L^\infty(\partial B)}\mathcal{Q}_\alpha^{\partial B}(\varphi) \geq -C\|g\|^2_{L^\infty(\partial B)}\delta P(E).
$$

Let us show that the term $\mathcal{Q}_\alpha^{\partial B}(\bar g,(g-\bar g)\varphi)$ can be estimated by the term $\mathcal{Q}_\alpha^{\partial B}(g-\bar g)$. Let $\widetilde\varphi : \mathbb{R}^d \to \mathbb{R}$ be a regular extension of φ, and let $g = (g-\bar g)d\mathcal{H}^{d-1}\llcorner\partial B$. By a

Fourier transform we get

$$\begin{aligned}\mathcal{Q}_\alpha^{\partial B}(\bar g,(g-\bar g)\varphi) =&\bar g\int_{\partial B}(g-\bar g)\varphi = \bar g\int_{\mathbb{R}^d}\widehat{\widetilde{\varphi}}\widehat{g}\\ \le&\bar g\left(\int_{\mathbb{R}^d}\widehat{\widetilde{\varphi}}^2|\xi|^{d-\alpha}\right)^{\frac12}\left(\int_{\mathbb{R}^d}\frac{\widehat{g}^2}{|\xi|^{d-\alpha}}\right)^{\frac12}\\ =&\bar g\|\widetilde{\varphi}\|_{H^{\frac{d-\alpha}{2}}}\mathcal{Q}_\alpha^{\partial B}(g-\bar g,g-\bar g)^{\frac12}\\ \le&C(\alpha,d)\bar g\|\varphi\|_{H^{\frac{d-\alpha}{2}}}\mathcal{Q}_\alpha^{\partial B}(g-\bar g)^{\frac12}.\end{aligned}$$

We remark now that, if

$$\mathcal{Q}_\alpha^{\partial B}(\bar g,(g-\bar g)\varphi)\le\frac12\mathcal{Q}_\alpha^{\partial B}(g-\bar g), \tag{3.53}$$

then we would get

$$\mathcal{Q}_\alpha^{\partial B}(g(1-\frac{\alpha}{2}\varphi))-\mathcal{Q}_\alpha^{\partial B}(\bar g)\ge -C\|\bar g\|^2_{L^\infty(\partial B)}\delta P(E),$$

which would conclude the proof. On the other hand if (3.53) does not hold, then

$$\mathcal{Q}_\alpha^{\partial B}(g-\bar g)<C(\alpha,d)\bar g\|\varphi\|_{H^{\frac{d-\alpha}{2}}}\mathcal{Q}_\alpha^{\partial B}(g-\bar g)^{\frac12},$$

which implies

$$\mathcal{Q}_\alpha^{\partial B}(g-\bar g)^{\frac12}<C\bar g\|\varphi\|_{H^{\frac{d-\alpha}{2}}},$$

so that

$$\mathcal{Q}_\alpha^{\partial B}(\bar g,(g-\bar g)\varphi)\le C\bar g\|\widetilde{\varphi}\|_{H^{\frac{d-\alpha}{2}}}\mathcal{Q}_\alpha^{\partial B}(g-\bar g)^{\frac12}\le C\bar g^2\|\varphi\|^2_{H^{\frac{d-\alpha}{2}}}.$$

If $\frac{d-\alpha}{2}\le 1$ then using (3.43) with $p=0$, $q=\frac{d-\alpha}{2}$ and $r=1$, up to extend again φ to a regular function on $\mathbb{R}^d$, we obtain

$$\|\varphi\|^2_{H^{\frac{d-\alpha}{2}}}\le c\left(\|\varphi\|^2_{H^1}\right)^{1-\alpha/2}\left(\|\varphi\|^2_{L^2(\mathbb{R}^d)}\right)^{\frac{d-\alpha}{2}}\le c\|\varphi\|^2_{H^1}+\|\varphi\|^2_{L^2(\mathbb{R}^d)}\le C\delta P(E).$$

Otherwise, we use (3.43) with $p=1$, $q=\frac{d-\alpha}{2}$ and r to obtain

$$\|\varphi\|^2_{H^{\frac{d-\alpha}{2}}}\le c\left(\|\varphi\|^2_{H^1}\right)^{\frac{\alpha/2-1}{r-1}}\left(\|\varphi\|^2_{H^r}\right)^{\frac{r-\alpha/2}{r-1}}\le C\delta P(E),$$

which concludes the proof. □

Corollary 3.36. *Let $d\ge 3$. Then for any $\delta>0$ there exists a $Q_\delta>0$ such that if $Q<Q_\delta$ then the ball is the unique minimizer of problem (3.10) with $\alpha=d-2$ among the sets in $\mathcal{K}_\delta$ with charge Q.*

Proof. Let $Q > 0$ and let E_Q be a minimizer in $\mathcal{K}_\delta$. Since $|E_Q \Delta B|^2 \leq C\delta P(E_Q) \leq CQ^2 \mathcal{Q}_\alpha(B)$, E_Q converges in L^1 to B as $Q \to 0$. As before, thanks to the $\delta-$ball condition, there is also convergence in the sense of Hausdorff of E_Q and ∂E_Q. By the $\delta-$ball condition again and the Hausdorff convergence of the boundaries, for Q small enough, ∂E_Q is a graph over ∂B of some $C^{1,1}$ function. Moreover, by the $\delta-$ball condition we have that $\|\kappa_{E_Q}\|_{L^\infty(\partial E_Q)} \leq \frac{1}{\delta}$, where $\kappa_A(x)$ is the mean curvature of the boundary of the set A in x, so that by classical Elliptic Regularity Theory (see for instance [67]), ∂E_Q is $C^{1,\beta}$ with uniform $C^{1,\beta}$ bound depending only on δ. From this we see that if $\partial E_Q = \{(1 + \varphi_Q(x))x \ : \ x \in \partial B\}$ then $\|\varphi_Q\|_{C^{1,\beta}(\partial B)}$ is converging to 0. We can thus assume that φ_Q satisfies the conditions of Proposition 3.35.
For simplicity of notations we drop the index Q in the rest of the proof. Let $\mu = f d\mathcal{H}^{d-1} \llcorner \partial E$ be the minimizer of $\mathcal{Q}_\alpha(E)$. Since $\mathcal{Q}_\alpha(E) \leq P(B) + Q^2 \mathcal{Q}_\alpha(B)$, by Proposition 3.19, $\|f\|_{L^\infty(\partial E)} \leq (d-2)\delta^{-1}(P(B) + Q^2 \mathcal{Q}_\alpha(B))$. On the other hand, since $\int_{\partial E} f = 1$, we have

$$\bar{f} = \frac{1}{P(E)}.$$

Recall that by Lemma 3.15, the optimal measure for $\mathcal{Q}_\alpha(B)$ is given by $\frac{\mathcal{H}^{d-1} \llcorner \partial B}{P(B)}$. By the minimality of E we then have

$$\begin{aligned} \delta P(E) = P(E) - P(B) &\leq Q^2(\mathcal{Q}_\alpha(B) - \mathcal{Q}_\alpha(E)) \\ &= Q^2 \left(\mathcal{Q}_\alpha^{\partial B}(\bar{f}) - \mathcal{Q}_\alpha^{\partial E}(f) + \mathcal{Q}_\alpha^{\partial B}(1/P(B)) - \mathcal{Q}_\alpha^{\partial B}(1/P(E)) \right). \end{aligned}$$

A simple computation shows that

$$\mathcal{Q}_\alpha^{\partial B}(1/P(B)) - \mathcal{Q}_\alpha^{\partial B}(1/P(E)) \leq C^2 \delta P(E)$$

for a suitable positive constant $C = C(\alpha, d)$. So, by Proposition 3.35 we have that

$$\delta P(E) \leq CQ^2 \delta P(E)(1 + \|f\|^2_{L^\infty(\partial E)}) \leq CQ^2 \delta P(E),$$

which implies that $E = B$ for Q small enough. □

Part II

Quantitative stability problems

Chapter 4

Weighted isoperimetric inequalities in quantitative form

4.1 Introduction

This chapter is devoted to the study of two classes of *weighted perimeters*, and their related isoperimetric problems. Section 4.2 is based on the joint work with Lorenzo Brasco and Guido De Philippis [16].

We recall that the perimeter, at least for Lipschitz sets is simply the $d-1$ dimensional measure of the topological boundary:

$$P(E) = \int_{\partial E} d\mathcal{H}^{d-1}(x).$$

To introduce the issue of this chapter, we recall (once again) the isoperimetric inequality

$$P(E) \geq P(B) \qquad \text{for any ball } B \text{ of measure } |E|.$$

Exploiting the rescaling law of the perimeter $P(tE) = t^{d-1}P(E)$ and of the $d-$dimensional Lebesgue measure $|tE| = t^d|E|$, the isoperimetric inequality can be stated avoiding to prescribe the measure of the sets E and B:

$$|E|^{\frac{1-d}{d}} P(E) \geq d\omega_d^{1/d},$$

where ω_d is the Lebesgue measure of the unit ball in $\mathbb{R}^d$. We remark that, also if the Lebesgue measure is a priori fixed, the family of minimizer is not unique. This is clearly due to the invariance under translation of the perimeter. We introduce now a notion of perimeter which in general does not share with the classical one such property: *the weighted perimeter.*

Definition 4.1 (Weighted perimeter)**.** Let $V : \mathbb{R}^d \to [0, \infty)$ be a non-negative Borel function. Then for every open bounded Lipschitz set $E \subset \mathbb{R}^d$, its *weighted perimeter* is given by

$$P_V(E) = \int_{\partial E} V(x)\, d\mathcal{H}^{d-1}(x).$$

With this definition in force, we can write down the main problem we shall investigate in this chapter: given a positive constant c, find (if there) a solution to the problem

$$\min\{P_V(E) : |E| = c\}. \tag{4.1}$$

We stress that the study of (4.1) is slightly related to a family of problems about weighted isoperimetric inequalities, as for instance those considered in [65]. However the issue we study presents a fundamental difference: the measure constraint and the weight assigned to the perimeter are not homogeneous! In order to study problem (4.1), we impose the weight V to satisfy some conditions. A natural hypothesis for V is to be a radial function, that is $V(x) = v(|x|)$ for some $v : [0, \infty) \to [0, \infty]$. Clearly with this sole requirement we cannot even expect to get existence for problem (4.1). Under further assumptions on V, in [9] it has been proved that the ball centred at the origin is the unique minimizer, or equivalently, that

$$P_V(E) \geq P_V(B_{r_E}(0)), \tag{4.2}$$

where r_E is such that $|E| = |B_{r_E}|$. In Section 4.2, Theorem 4.2, we obtain an improvement of the same result: a quantitative stability version of the inequality. More precisely we prove that, under suitable assumptions on the weight V, the inequality

$$P_V(E) \geq P_V(B_{r_E}(0)) + c_{d,V,|E|} \left(\frac{|E \Delta B_{r_E}(0)|}{|E|}\right)^2 \tag{4.3}$$

holds true, where $c_{d,V,|E|}$ is a constant. Clearly such result entails the weighted isoperimetric inequality (4.2). Moreover, in Subsection 4.2.1 we shall prove that such inequality is sharp, in the sense that the exponent 2 cannot be substituted by any lower exponent.

Our proof of (4.3) (and thus of (4.2)), based on a sort of *calibration* method, is completely different from the one in [9], which is based on symmetrization techniques. Even if the hypotheses imposed on V in both proofs (our and that in [9]) seem to be needed just for technical reasons, such conditions turn out to be (basically) the same; this may suggest that they could be optimal (see Remark 4.3). Despite this, we are not able to prove this fact by means of a counterexample.

In Section 4.3 we study a family of weighted measures linked to exponential measures. Namely we deal with the problem

$$\min\left\{P_{we^V}(E) : \int_E e^V\, dx = c\right\}. \tag{4.4}$$

In particular we prove in Theorem 4.5 (respectively Theorem 4.7) that, under suitable assumptions on w and V, the half-spaces of the form $\{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{d-1} : x_1 > t\}$

(respectively $\{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{d-1} : x_1 < t\}$) are the only minimizer of problem (4.4). The proof is based on the calibration technique developed in the case of the Lebesgue measure, and exploits the particular structure of exponential measures. For this latter problem we will not address the stability issue.

4.2 Weighted isoperimetric inequality for the Lebesgue measure

The main result of this section is the following theorem. Later, in Subsection 4.2.1, we will prove that the exponent 2 appearing at the right-hand side of (4.6) is sharp.

Theorem 4.2. *Let $d \geq 2$ and $V : [0, \infty) \to [0, \infty)$ be a weight function such that $V \in C^2((0, \infty))$ and satisfying the following properties:*

$$V(0) = 0 \quad \textit{and} \quad W(t) := V'(t) + (d-1)\frac{V(t)}{t} \quad \textit{is such that} \quad W'(t) > 0, \quad t > 0. \tag{4.5}$$

Then for every open bounded set $E \subset \mathbb{R}^d$ with Lipschitz boundary, we have

$$P_V(E) \geq d\,\omega_d^{1/d}\,|E|^{1-\frac{1}{d}} \left[V\left(\left(\frac{|E|}{\omega_d}\right)^{\frac{1}{d}} \right) + c_{d,V,|E|} \left(\frac{|E \Delta B|}{|E|} \right)^2 \right], \tag{4.6}$$

where B is the ball centred at the origin and such that $|B| = |E|$. Here $c_{d,V,|E|}$ is a constant depending on d, the weight V and the measure of E, defined by

$$c_{d,V,|E|} = \frac{1}{4} \left(\min_{t \in \left[r_E,\, r_E \sqrt[d]{2}\right]} W'(t) \right) \frac{\sqrt[d]{2} - 1}{d} \left(\frac{|E|}{\omega_d} \right)^{\frac{2}{d}},$$

where for simplicity we set

$$r_E := \left(\frac{|E|}{\omega_d} \right)^{\frac{1}{d}}. \tag{4.7}$$

Proof. Let B be the ball centred at the origin and having radius r_E, so that $|B| = |E|$. The key idea of the proof is to use a sort of *calibration technique*, adapted to the case of weighted perimeters; related ideas can be found in the recent paper [82]. Namely, we consider the following vector field

$$x \mapsto V(|x|)\,\frac{x}{|x|}, \qquad x \in \mathbb{R}^d \setminus \{0\},$$

whose divergence is given by

$$\operatorname{div}\left(V(|x|)\,\frac{x}{|x|} \right) = V'(|x|) + (d-1)\frac{V(|x|)}{|x|} = W(|x|), \quad x \in \mathbb{R}^d \setminus \{0\}.$$

Recall that, by hypotheses, W is an increasing function. Integrating W on E and then applying the Divergence Theorem, we obtain

$$\int_E W(|x|)\,dx = \int_{\partial E} V(|x|)\,\left\langle \frac{x}{|x|}, \nu_E(x)\right\rangle\,d\mathcal{H}^{d-1} \leq P_V(E),$$

and in the same way, integrating on B we get

$$\int_B W(|x|)\,dx = \int_{\partial B} V(|x|)\,d\mathcal{H}^{d-1} = P_V(B).$$

Subtracting $P_V(B)$ from the previous inequality, we then obtain

$$\int_E W(|x|)\,dx - \int_B W(|x|)\,dx \leq P_V(E) - P_V(B).$$

We now observe that thanks to the fact that $|B| = |E|$, we have $|E \setminus B| = |B \setminus E|$ and then

$$\begin{aligned}\int_E W(|x|)\,dx - \int_B W(|x|)\,dx &= \int_{E\setminus B} W(|x|)\,dx - \int_{B\setminus E} W(|x|)\,dx \\ &= \int_{E\setminus B} \left[W(|x|) - W(r_E)\right]\,dx - \int_{B\setminus E} \left[W(|x|) - W(r_E)\right]\,dx \\ &= \int_{E\Delta B} |W(|x|) - W(r_E)|\,dx =: \mathcal{R}(E),\end{aligned}$$

where in the last equality we used the monotone behaviour of W. Resuming, we have obtained the following

$$P_V(E) - P_V(B) \geq \mathcal{R}(E), \tag{4.8}$$

and the right-hand side is just the integral of a given function over the region $E\Delta B$, so very likely this gives the desired estimate (4.6). Note that since the functional $\mathcal{R}$ is non-negative, we already proved the isoperimetric inequality. In order to make this precise, let us introduce the radius

$$r_2 = \left(r_E^d + \frac{|E \setminus B|}{\omega_d}\right)^{\frac{1}{d}},$$

and the annular region

$$T = \{x \in \mathbb{R}^d \,:\, r_E < |x| < r_2\},$$

which by construction satisfies $|T| = |E \setminus B| = |B \setminus E|$. Notice that

$$r_2 \leq r_E \sqrt[d]{2}.$$

Using the monotonicity of the function $t \mapsto W(t)$, we get

$$\begin{aligned}\mathcal{R}(E) &= \int_{\{x\in E\,:\,|x|>r_E\}} [W(|x|) - W(r_E)]\,dx + \int_{\{x\notin E\,:\,|x|<r_E\}} [W(r_E) - W(|x|)]\,dx \\ &\geq \int_T [W(|x|) - W(r_E)]\,dx,\end{aligned}$$

so that

$$\mathcal{R}(E) \geq d\,\omega_d \int_{r_E}^{r_2} [W(\varrho) - W(r_E)]\,\varrho^{d-1}\,d\varrho. \tag{4.9}$$

Thanks to the hypotheses $W'(t) > 0$ if $t > 0$, if we set

$$c_1 = \min_{t\in\left[r_E,\, r_E\sqrt[d]{2}\right]} W'(t),$$

this is a strictly positive constant, depending on d, V and $|E|$, then from (4.9) we can infer

$$\mathcal{R}(E) \geq d\,\omega_d\,c_1 \int_{r_E}^{r_2} (\varrho - r_E)\,\varrho^{d-1}\,d\varrho.$$

We now develope the computations for this integral: keeping into account that $|E| = \omega_d\,r_E^d$, we have

$$\begin{aligned}\int_{r_E}^{r_2} (\varrho - r_E)\,\varrho^{d-1}\,d\varrho &= \frac{r_2^{d+1} - r_E^{d+1}}{d+1} - r_E\,\frac{r_2^d - r_E^d}{d} \\ &= r_E^{d+1}\left[\frac{1}{d+1}\left(\left(1 + \frac{|E\setminus B|}{|E|}\right)^{\frac{d+1}{d}} - 1\right) - \frac{1}{d}\,\frac{|E\setminus B|}{|E|}\right].\end{aligned}$$

Let us now focus on the function $\varphi(t) = (1+t)^\alpha - 1$, for $t \in [0,1]$ and with $1 < \alpha < 2$: we have the following elementary estimate

$$(1+t)^\alpha - 1 \geq \alpha\,t + c_2\,t^2, \qquad t \in [0,1],$$

with constant c_2 given by

$$c_2 = \frac{\alpha}{4}\,(2^{\alpha-1} - 1) > 0.$$

Applying this inequality with the choices $t = |E\setminus B|/|E|$ and $\alpha = 1 + 1/d$, we then obtain

$$\int_{r_E}^{r_2} (\varrho - r_E)\,\varrho^{d-1}\,d\varrho \geq r_E^{d+1}\,\frac{\sqrt[d]{2} - 1}{d}\,\left(\frac{|E\setminus B|}{|E|}\right)^2.$$

Thus, we arrive at the following estimate

$$P_V(E) - P_V(B) \geq \mathcal{R}(E) \geq d\,\omega_d\,r_E^{d+1}\,\frac{C}{4}\,\left(\frac{|E\Delta B|}{|E|}\right)^2,$$

where we have set

$$C = \left(\min_{t \in \left[r_E, r_E \sqrt[d]{2} \right]} W'(t) \right) \frac{\sqrt[d]{2} - 1}{d}.$$

This finally gives (4.6), keeping into account that

$$P_V(B) = d\, \omega_d^{1/d}\, |E|^{(d-1)/d}\, V(r_E).$$

and recalling the definition of r_E. $\square$

Remark 4.3 (Assumptions on the weight V)**.** In [9] it is proven under the following assumptions

$$V \text{ strictly increasing} \qquad \text{and} \qquad v(t) := V(t^{1/d})\, t^{1-1/d}, \quad t \geq 0 \qquad \text{convex}, \tag{4.10}$$

the following sharp lower bound for the weighted perimeter

$$P_V(E) \geq d\, \omega_d^{1/d}\, |E|^{1-\frac{1}{d}}\, V\left(\left(\frac{|E|}{\omega_d} \right)^{\frac{1}{d}} \right), \tag{4.11}$$

with equality if and only if E is a ball centred at the origin. This precisely implies that the ball centred at the origin is the only minimizer of P_V, under volume constraint. It is not difficult to see that this is slightly more general than our (4.5), since (4.10) is equivalent to require that W is non-decreasing. Anyway, our hypotheses could be somehow relaxed: first of all, from the estimate (4.9), we easily see that our proof still characterizes the ball as the unique isoperimetric set, simply requiring that W is strictly increasing, in particular avoiding the requirement $W' > 0$ and the C^2 regularity of V. Secondly, a closer inspection of our proof reveals that it provides the stronger lower bound

$$P_V(E) - P_V(B) \geq \frac{1}{2} \int_{\partial E} \left| \nu_E(x) - \frac{x}{|x|} \right|^2 V(|x|)\, \mathcal{H}^{d-1} + \mathcal{R}(E). \tag{4.12}$$

Then a characterization of equality cases and a stability estimate seems still feasible, by simply requiring W non-decreasing (as in [9]) and exploiting the first term in the right-hand side of (4.12). A stability estimate of this type – i.e. containing the L^2 distance of the normal versors – has been derived in [61] for the standard isoperimetric inequality. However, in our case some additional difficulties arise, due to the presence of the weight V.

In connection with our later application in Chapter 5, a significant instance of function V satisfying (4.5) is given by any strictly convex power function, i.e. $V(|x|) = |x|^p$ with $p > 1$. In this case, we use the distinguished notation

$$P_p(E) = \int_{\partial E} |x|^p\, \mathcal{H}^{d-1},$$

and occasionally we will call $P_p(E)$ the *$p-$perimeter of E*. We have $P_p(\lambda\, E) = \lambda^{p+d-1}\, P_p(E)$, for every $\lambda > 0$, which implies in particular that the shape functional

$$E \mapsto |E|^{(1-d-p)/d}\, P_p(E),$$

is dilation invariant. Then inequality (4.11) can be equivalently written in scaling invariant form as

$$|E|^{\frac{1-p-d}{d}}\, P_p(E) \geq d\, \omega_d^{\frac{1-p}{d}}. \tag{4.13}$$

As a corollary of the previous theorem, we have the following quantitative improvement of (4.13).

Corollary 4.4. *Let $p > 1$, then for every open bounded set $E \subset \mathbb{R}^d$ with Lipschitz boundary, we have*

$$|E|^{\frac{1-p-d}{d}}\, P_p(E) \geq d\, \omega_d^{\frac{1-p}{d}} \left[1 + c_{d,p} \left(\frac{|E\Delta B|}{|E|}\right)^2\right], \tag{4.14}$$

where B is the ball centred at the origin such that $|E| = |B|$ and $c_{d,p}$ is a constant depending only on d and p, given by

$$c_{d,p} = \frac{(d+p-1)\,(p-1)}{4}\, \frac{\sqrt[d]{2}-1}{d} \left(\min_{t\in[1,\sqrt[d]{2}]} t^{p-2}\right).$$

Proof. We start observing that if $V(t) = t^p$, then

$$W(t) = (d+p-1)\, t^{p-1} \qquad \text{and} \qquad W'(t) = (d+p-1)\,(p-1)\, t^{p-2}.$$

In particular, using the homogeneity of W' we get that

$$\min_{t\in[r_E, r_E\sqrt[d]{2}]} W'(t) = r_E^{p-2} \min_{t\in[1,\sqrt[d]{2}]} W'(t) = \left(\frac{|E|}{\omega_d}\right)^{\frac{p-2}{d}} (d+p-1)\,(p-1) \left(\min_{t\in[1,\sqrt[d]{2}]} t^{p-2}\right).$$

Then in order to obtain (4.14), it is sufficient to insert the previous into (4.6), to use that

$$V\left(\left(\frac{|E|}{\omega_d}\right)^{\frac{1}{d}}\right) = \left(\frac{|E|}{\omega_d}\right)^{\frac{p}{d}},$$

and to divide both members of (4.6) by $|E|^{(p+d-1)/d}$. $\square$

4.2.1 Nearly spherical ellipsoids

We now show that the exponent 2 for the term $|E\Delta B|$ in inequality (4.6) is optimal. To this aim, we simply exhibit for every radius R a sequence of sets E_ε^R, such that $|E_\varepsilon^R| = \omega_d\, R^d$ and that

$$\overline{\lim}_{\varepsilon\to 0}\frac{P_V(E_\varepsilon^R) - P_V(B_R)}{|B_R \Delta E_\varepsilon^R|^2} \le C, \tag{4.15}$$

where B_R is the ball of radius R centred in the origin. For the sake of simplicity, we confine ourselves to consider the case $d = 2$: the very same arguments still work for every $d \ge 3$.

First of all, we aim to prove (4.15) for $R = 1$, then we will show how to obtain it for a general $R > 0$. Let us consider the following family of ellipses

$$E_\varepsilon = \left\{ \left(x\,\sqrt{1+\varepsilon}, \frac{y}{\sqrt{1+\varepsilon}} \right) \,:\, x^2 + y^2 < 1 \right\},$$

whose boundary can be parametrized by

$$\gamma_\varepsilon(\vartheta) = \left(\sqrt{1+\varepsilon}\, \cos\vartheta, \frac{1}{\sqrt{1+\varepsilon}}\, \sin\vartheta \right), \qquad \vartheta \in [0, 2\pi].$$

Observe that by construction we have $|E_\varepsilon| = |B_1| = \pi$, since

$$E_\varepsilon = \mathcal{M}_\varepsilon(B_1)$$

with $\mathcal{M}_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$ linear application, having (with a slight abuse of notation) $\det \mathcal{M}_\varepsilon = 1$. Now, we need to expand the term

$$P_V(E_\varepsilon) = \int_0^{2\pi} V(|\gamma_\varepsilon(\vartheta)|)\, |\gamma_\varepsilon'(\vartheta)|\, d\vartheta.$$

At this aim we use the following second-order Taylor expansions for $|\gamma_\varepsilon|$, $|\gamma_\varepsilon'|$ and $V(|\gamma_\varepsilon|)$:

$$\begin{aligned} |\gamma_\varepsilon(\vartheta)| &= (1+\varepsilon)^{-1/2}\,\sqrt{1 + 2\varepsilon\,\cos^2\vartheta + \varepsilon^2\,\cos^2\vartheta} \\ &\simeq 1 + \varepsilon\,\left(\cos^2\vartheta - \frac{1}{2}\right) + \frac{\varepsilon^2}{2}\,\left(\frac{3}{4} - \cos^4\vartheta\right) \end{aligned}$$

and similarly

$$|\gamma_\varepsilon'(\vartheta)| \simeq 1 + \varepsilon\,\left(\sin^2\vartheta - \frac{1}{2}\right) + \frac{\varepsilon^2}{2}\,\left(\frac{3}{4} - \sin^4\vartheta\right),$$

while

$$V(|\gamma_\varepsilon(\vartheta)|) \simeq V(1) + \varepsilon\, V'(1)\,\left[\cos^2\vartheta - \frac{1}{2}\right] + \frac{\varepsilon^2}{2}\,\left[V'(1)\,\left(\frac{3}{4} - \cos^4\vartheta\right) + V''(1)\,\left(\frac{1}{2} - \cos^2\vartheta\right)^2\right].$$

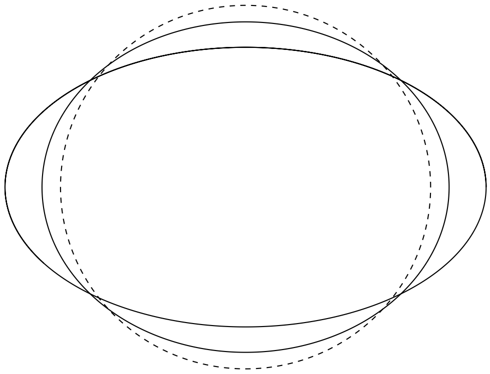

Figure 4.1: The family of ellipses E_ε.

Thus we have the following second-order expansion for the integrand defining $P_V(E_\varepsilon)$:

$$\begin{aligned} V(|\gamma_\varepsilon(\vartheta)|)\,|\gamma'_\varepsilon(\vartheta)| \simeq V(1) + \varepsilon \left[V'(1)\left(\cos^2\vartheta - \frac{1}{2}\right) + V(1)\left(\sin^2\vartheta - \frac{1}{2}\right)\right] \\ + \varepsilon^2 \left[V'(1)\left(\cos^2\vartheta - \frac{1}{2}\right)\left(\sin^2\vartheta - \frac{1}{2}\right) + \frac{V(1)}{2}\left(\frac{3}{4} - \sin^4\vartheta\right)\right. \\ \left. + \frac{V''(1)}{2}\left(\frac{1}{2} - \cos^2\vartheta\right)^2 + \frac{V'(1)}{2}\left(\frac{3}{4} - \cos^4\vartheta\right)\right]. \end{aligned}$$

Finally, we observe that

$$\int_0^{2\pi}\left(\cos^2\vartheta - \frac{1}{2}\right)d\vartheta = \int_0^{2\pi}\left(\sin^2\vartheta - \frac{1}{2}\right)d\vartheta = 0,$$

and

$$\int_0^{2\pi}\left(\cos^2\vartheta - \frac{1}{2}\right)^2 d\vartheta = -\int_0^{2\pi}\left(\cos^2\vartheta - \frac{1}{2}\right)\left(\sin^2\vartheta - \frac{1}{2}\right)d\vartheta = \frac{\pi}{4}$$

while

$$\int_0^{2\pi}\left(\frac{3}{4} - \cos^4\vartheta\right)d\vartheta = \int_0^{2\pi}\left(\frac{3}{4} - \sin^4\vartheta\right)d\vartheta = \frac{3}{4}\,\pi.$$

Summarizing, we have obtained

$$P_V(E_\varepsilon) - P_V(B_1) \simeq \frac{\pi}{8}\,\varepsilon^2\left[3\,V(1) + V'(1) + V''(1)\right]. \tag{4.16}$$

On the other hand it is easily seen that $|E_\varepsilon \Delta B_1| = O(\varepsilon)$, thus we get (4.15) for $R = 1$.

To obtain this result for a generic $R > 0$, we notice that for every set E,

$$P_V(R\,E) = R\,P_{V_R}(E),$$

where $V_R(t) = V(R\,t)$, $t \geq 0$. Hence, if we set $\widetilde{E}_\varepsilon := R\,E_\varepsilon$ we have

$$\begin{aligned} P_V(\widetilde{E}_\varepsilon) - P_V(B_R) &= R\,[P_{V_R}(E_\varepsilon) - P_{V_R}(B_1)] \\ &\simeq \varepsilon^2\,\frac{\pi\,R}{8}\,\left[3\,V(R) + R\,V'(R) + R^2\,V''(R)\right], \end{aligned}$$

thanks to (4.16), thus giving (4.15) also in the general case. Observe that (4.5) implies that

$$R^2\,V''(R) + RV'(R) > V(R),$$

and thus in particular

$$3\,V(R) + R\,V'(R) + R^2\,V''(R) > 4\,V(R) > 0.$$

4.3 Weighted isoperimetric inequalities for exponential measures

In this section we adapt the *calibration method* developed in last section to another class of isoperimetric problems, related this time to a Gauss-type family of measures, instead of the Lebesgue one. In order to introduce our main results, stated in Theorem 4.5 and Theorem 4.7, we introduce some notations. Let μ be a finite positive Radon measure on $\mathbb{R}^d$ and let A be a Borel subset of $\mathbb{R}^d$, the *right rearrengement* of the set A, denoted by R^μ_A, is the open right half-space $\{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{d-1} \,:\, x_1 > t_A\}$, having the same measure of A: $\mu(R_A) = \mu(A)$. Notice that, if $d\mu = f dx$ for some integrable function f, then the value t_A is uniquely determined.
Given a measurable function $V : \mathbb{R}^d \to \mathbb{R}$ we denote by γ the measure whose density equals e^V, and for any measurable set $E \subset \mathbb{R}^d$ we define the $V-$volume of E as

$$\gamma(E) = \int_E e^{V(x)} dx. \tag{4.17}$$

In what follows we will often adopt the notation $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{d-1}$. Consider now a Borel *weight* function $w : \mathbb{R} \to [0, +\infty]$ and define, for any open set A with Lipschitz boundary, the weighted $V-$perimeter as follows:

$$P_{we^V}(A) = \int_{\partial A} w(x_1) e^{V(x)} d\mathcal{H}^{d-1}(x).$$

In the following theorem we show that, under further conditions on both w and V, right half-spaces are minimizers of such a perimeter among the sets of fixed measure with respect to the measure γ.

Theorem 4.5. *Let $A \subset \mathbb{R}^d$ be a Lipschitz set, and $w : \mathbb{R} \to \mathbb{R}^+$, $V : \mathbb{R}^d \to \mathbb{R}$ be C^1 functions satisfying the following assumption:*

(i) $\int_A e^V dx$ *and* $\int_A w(x_1)e^V\, dx$ *are finite,*

(ii) $g(x) := -w'(x_1) - w(x_1)\partial_1 V(x)$ *depends only on x_1, and it is a decreasing function on the real line.*

(iii) $\int_{x_1 \geq t} e^V\, dx < +\infty$, $\int_{x_1 \geq t} we^V\, dx < +\infty$ *for any $t \in \mathbb{R}$.*

Then

$$P_{we^V}(A) \geq P_{we^V}(R_A^\mu),$$

where R_A is the right half-space $R_A = \{(x_1, x') =: x_1 > t_A\}$ which satisfies

$$\gamma(R_A) = \gamma(A).$$

Proof. Let $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^d$ and consider the vector field $-e_1 w(x_1)e^V(x)$. Its divergence is given by

$$\operatorname{div}(-e_1 w(x_1)e^V(x)) = (-w'(x_1) - w(x_1)\partial_1 V(x))e^{V(x)} = g(x)e^{V(x)}.$$

By an application of the Divergence Theorem we have

$$\begin{aligned} \int_A g(x) d\gamma(x) &= \int_A \operatorname{div}(-e_1 w(x_1)e^{V(x)})dx \\ &= \int_{\partial A} w(x_1)e^{V(x)}\langle \nu_A(x), -e_1\rangle d\mathcal{H}^{d-1}(x) \\ &\leq \int_{\partial A} w(x_1)e^{V(x)} d\mathcal{H}^{d-1}(x) = P_{we^V}(A). \end{aligned} \tag{4.18}$$

Let t_A be a real number such that the right half-space $R_A = \{(x_1, x') : x_1 \geq t_A\}$ satisfies $\gamma(R_A) = \gamma(A)$. Then, since the outer normal of R_A is the constant vector field $-e_1$, we have that the inequality in (4.18) turns into an equality if tested on R_A instead of A. Hence we get

$$P_{we^V}(A) - P_{we^V}(R_A) \geq \int_A g(x)d\gamma(x) - \int_{R_A} g(x)d\gamma(x).$$

Since $\gamma(A) = \gamma(R_A)$, then it holds $\gamma(A \setminus R_A) = \gamma(R_A \setminus A)$ as well. Thus

$$\begin{aligned} \int_A g(x)d\gamma(x) - \int_{R_A} g(x)d\gamma(x) &= \int_{A\setminus R_A} g(x)d\gamma(x) - \int_{R_A\setminus A} g(x)d\gamma(x) \\ &= \int_{A\setminus R_A} (g(x) - g(t_A e_1))d\gamma(x) - \int_{R_A\setminus A} (g(x) - g(t_A e_1))d\gamma(x). \end{aligned}$$

Since every $x \in A \setminus R_A$ (respectively $x \in R_A \setminus A$) satisfies $\langle x, e_1 \rangle < t_A$ (respectively $> t_A$), by condition *(ii)* we deduce

$$\begin{aligned} P_{we^V}(A) - P_{we^V}(R_A) &\geq \int_{A \setminus R_A} |g(x) - g(t_A e_1)| d\gamma(x) + \int_{R_A \setminus A} |g(x) - g(t_A e_1)| d\gamma(x) \\ &= \int_{A \Delta R_A} |g(x) - g(t_A e_1)| d\gamma(x) \geq 0 \end{aligned}$$

where $A \Delta R_A = (A \setminus R_A) \cup (R_A \setminus A)$ stands for the symmetric difference between A and R_A. □

Condition *(ii)* of the previous theorem is satisfied by functions of the form

$$V(x_1, x') = V_1(x_1) + V_2(x'),$$

where the real function $V_1 \in C^2(\mathbb{R})$ and, together with the weight $w \geq 0$ satisfies the ordinary differential inequality

$$w'' + V_1'' w + V_1' w' \geq 0. \tag{4.19}$$

Although Theorem 4.5 is stated in the whole space $\mathbb{R}^d$, inequality (4.19) is seldom (if ever) satisfied by functions which fulfil also the integrability property *(i)*. This is not the case if we restrict our attention to the half space

$$\Omega = \{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{d-1} : x_1 > 0\}.$$

In this case we can find a big class of functions w and V satisfying (4.19).

Corollary 4.6. *Let $w \in C^2([0, \infty], [0, \infty])$ and $V = V_1(x_1) + V_2(x') : \mathbb{R}^d \to \mathbb{R}$ be such that $V_1 \in C^1(\mathbb{R})$ and*

$$w'' + w' V_1' + w V_1'' \geq 0.$$

Suppose moreover that the functions w and V satisfy the integrability condition (i) in Theorem 4.5. Then the solution of problem

$$\min \left\{ P_{we^V}(A) : 1_A \in L^1(\Omega, we^V), \int_A e^V = c \right\}$$

is given by the right space $R_c = \{x_1 \geq t_c\}$ such that $\int_{R_c} we^V = c$.

A class of particular interest is that where $V(x) = -|x|^2$, that is where $e^V\, dx$ corresponds to the (non-rescaled) Gauss measure. In this case there is a large class of real functions w which adapt to the hypotheses of Corollary 4.6. This is the case, for instance of $w(t) = e^{-t}$, $t^{-\alpha}$ with $\alpha \geq 1$ and $w(t) = -\log(t)\mathbf{1}_{(0,1]}$. Similar results have been considered in [21] where the authors deal with the case $w(t) = t^k$, $k > 0$ and the substantial difference with our problem is that the space Ω is weighted as the perimeter.

If we suitably overturn the hypotheses of Theorem 4.5, we get an analogous reversed result, related to left half spaces.

Theorem 4.7. *Let $A \subset \mathbb{R}^d$ be a Lipschitz set, and $w : \mathbb{R} \to \mathbb{R}^+$, $V : \mathbb{R}^d \to \mathbb{R}$ be C^1 functions satisfying the assumptions (i) and (ii) of Theorem 4.5 and*

(iii)′ $\int_{x_1 \le t} e^V\,dx < +\infty$, $\int_{x_1 \le t} we^V\,dx < +\infty$ for any $t \in \mathbb{R}$.

Then

$$P_{we^V}(A) \ge P_{we^V}(L_A),$$

where L_A is the left half space $L_A = \{(x_1, x') =: x_1 < t_A\}$ which satisfies

$$\gamma(R_A) = \gamma(A).$$

Proof. It is analogous to the proof of Theorem 4.5, considering this time the vector field $e_1 we^V$. $\square$

The same considerations made for Theorem 4.5 hold also in this case. Again some interesting examples can be made considering sets lying in the *right* half space Ω. Observe that this request is not completely analogous to the previous one, where Ω was itself an optimal subspace. Indeed if we restrict as before our attention to the right half space $\Omega = \{x_1 > 0\}$, the minimizer are *vertical* strips of the form $\{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{d-1} : 0 < x_1 < b\}$.

Corollary 4.8. *Let $w \in C^2([0,\infty],[0,\infty])$ and $V = V_1(x_1) + V_2(x') : \mathbb{R}^d \to \mathbb{R}$ be such that $V_1 \in C^1(\mathbb{R})$ and*

$$w'' + w'V_1' + wV_1'' \ge 0.$$

Suppose moreover that the functions w and V satisfy the integrability condition (iii)′ of Theorem 4.7. Then the solution of problem

$$\min\left\{P_{we^V}(A) : \mathbf{1}_A \in L^1(\Omega, w\gamma_d), \gamma_d(A) = c\right\}$$

is given by the strip $S_c = \{0 < x_1 < t_c\}$ such that $\int_{S_c} we^V = c$.

In this case instances of weights V are those of the form $V(x_1, x') = x_1^2 + V_2(x')$, for a suitable integrable function V_2. These examples are slightly related to the recent work [22], where the weight considered is $V(x) = |x|^2$, the space (and not only the perimeter) is endowed with the measure $\varphi = x_1^k e^{|x|^2}$, $k > 0$ and the minimizer of the isoperimetric problem

$$\min\left\{\int_A \varphi(x)\,dx : A \subset \Omega, \int_A \varphi(x)\,dx = c\right\}$$

is the centred half ball $B_c = \{x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{d-1} : x_1 > 0, |x| < r_c\}$ such that $\int_{B_c} \varphi(x)\,dx = c$. In our case some examples of weights w which fulfil the hypotheses of Corollary 4.8 are $w(t) = e^{-t^\alpha}$, with $\alpha > 1$ and $w(t) = t^k$, $k \in (0,1)$.

Chapter 5

Stability for the first Stekloff-Laplacian eigenvalue

5.1 Introduction

The scope of this chapter, based on the joint work with Lorenzo Brasco and Guido De Philippis [16], is to study the stability of the spectral problem related to the first non-trivial *Stekloff eigenvalue*. The first non-trivial Stekloff eigenvalue can be formulated as follows

$$\sigma_2(E) = \inf_{u \in W^{1,2}(E)\setminus\{0\}} \left\{ \frac{\displaystyle\int_E |\nabla u(x)|^2\, dx}{\displaystyle\int_{\partial E} u(x)^2\, d\mathcal{H}^{d-1}} \ :\ \int_{\partial E} u(x)\, d\mathcal{H}^{d-1} = 0 \right\},$$

i.e. $1/\sigma_2(E)$ is the best constant in the Poincaré-Wirtinger trace inequality

$$\int_{\partial E} \left| u(x) - \left(\fint_{\partial E} u(x) \right) \right|^2 d\mathcal{H}^{d-1} \le C_E \int_E |\nabla u(x)|^2\, dx, \qquad u \in W^{1,2}(E). \tag{5.1}$$

The *Brock-Weinstock inequality* asserts that in the class of sets with given volume, σ_2 is *maximized* by a ball, i.e.

$$|E|^{1/d}\, \sigma_2(E) \le |B|^{1/d}\, \sigma_2(B), \tag{5.2}$$

where B is any ball and equality holds if and only if E itself is a ball. Notice that the quantity $|E|^{1/d}\, \sigma_2(E)$ is scaling invariant. The main result of this chapter is a *sharp quantitative version* of (5.2):

Theorem 5.1 (Quantitative Brock-Weinstock inequality)**.** *For every open bounded Lipschitz set $E \subset \mathbb{R}^d$, there holds*

$$|E|^{1/d}\, \sigma_2(E) \le |B|^{1/d}\, \sigma_2(B)\, \left[1 - \alpha_d\, \mathcal{A}(E)^2\right], \tag{5.3}$$

where α_d is an explicit dimensional constant and $\mathcal{A}(E)$ is the Fraenkel asymmetry

$$\mathcal{A}(E) = \inf\left\{ \frac{\|1_E - 1_B\|_{L^1(\mathbb{R}^d)}}{|E|} : |E| = |B| \right\}.$$

To be precise, we will prove, in Theorem 5.6, a slightly stronger result, which entails Theorem 5.1. Some words on the proof of this result are in order: the maximality of the ball for σ_2 is a consequence of a particular case of the weighted isoperimetric inequality analysed in Section 4.2 of the previous chapter. Namely, the crucial point is that the ball centred at the origin (uniquely) minimizes the shape functional

$$E \mapsto \int_{\partial E} |x|^2 \, d\mathcal{H}^{d-1},$$

among sets with given measure, result that, as we have seen in Chapter 4, is proved in [9]. This further isoperimetric characterization of the ball is the key tool in Brock's proof of (5.2) in [20]: then in order to derive (5.3), we can take advantage of the stability result proved in the previous chapter for such a weighted perimeter. Namely, we will make use of the following result (stated in this chapter as a lemma, but which is a direct consequence of Theorem 4.2 and, in particular, of Corollary 4.4). .

Lemma 5.2. *For every open bounded Lipschitz set $E \subset \mathbb{R}^d$, we have*

$$\int_{\partial E} |x|^2 \, d\mathcal{H}^{d-1} \geq \int_{\partial B} |x|^2 \, d\mathcal{H}^{d-1} \left[1 + \beta_d \left(\frac{|E \Delta B|}{|E|} \right)^2 \right],$$

where β_d is an explicit dimensional constant, B is the ball centred at the origin such that $|B| = |E|$ and $E\Delta B$ denotes the symmetric difference.

Concerning the *sharpness* of the exponent 2 for the Fraenkel asymmetry in (5.3), we stress that its proof (Theorem 5.9) is much longer than the same result for weighted perimeters proved in Subsection 4.2.1 of the previous chapter. A possible explanation is the following: since, differently from the perimeter, the Stekloff eigenvalues do not have a straightforward geometrical meaning, it is much more complicated to understand how they are affected by deformations of an optimal shape. If the eigenvalue is differentiable in the sense of the *shape derivative* (see [77]) – like in the case of the first Dirichlet eigenvalue λ_1 – one can use the following argument. Any perturbation of the type $E_\varepsilon := (\mathrm{Id} + \varepsilon X)(B)$, for some smooth vector field X, should provide a Taylor expansion of the form

$$|E|^{2/d} \lambda_1(E_\varepsilon) \simeq |B|^{2/d} \lambda_1(B) + O(\varepsilon^2), \qquad \varepsilon \ll 1, \tag{5.4}$$

since the first derivative of $|\cdot|^{2/d} \lambda_1(\cdot)$ has to vanish at the minimum "point" B. Then one observes that for such a family of sets, the Fraenkel asymmetry satisfies $\mathcal{A}(E_\varepsilon) =$

$O(\varepsilon)$. This explains, for instance, why the following inequality was expected (and in fact proved in the recent work [15]) in the (sharp) form

$$|E|^{2/d}\,\lambda_1(E) \geq |B|^{2/d}\,\lambda_1(B)\,\left[1 + c_d\,\mathcal{A}(E)^2\right].$$

For the case of the first non-trivial Stekloff eigenvalue σ_2, proving the sharpness is more complicated: indeed, the most basic example – nearly spherical ellipsoids – leads to an expansion with a non-trivial first order term, i.e.

$$|E_\varepsilon|^{1/d}\,\sigma_2(E_\varepsilon) \simeq |B|^{1/d}\,\sigma_2(B) + O(\varepsilon).$$

The same phenomenon has already been observed in [18, Section 5] for the Neumann case. A possible explanation for this fact is the following: at the maximum point, i.e. for a ball B, the eigenvalue σ_2 is multiple and thus *not differentiable* (see [75, Section 2]). Roughly speaking, this implies that along some "directions" (i.e. for some deformations of the ball) the functional σ_2 could have a non-trivial "super-differential". In order to show that the exponent 2 in (5.3) is indeed sharp, one has to exclude that this happens for every direction: namely, one has to exhibit a particular family of deformations E_ε for which a correct expansion like (5.4) is guaranteed. We will achieve our aim by suitably refining a construction introduced in [18, Section 6] to solve the same problem in the Neumann case.

The chapter is organized as follows: in Section 5.2 we recall some basic facts about Stekloff eigenvalues, as well as the spectral optimization problems we are concerned with. Thanks to our quantitative estimates for weighted perimeters, we will prove that optimal shapes for these spectral problems are stable (Section 5.3). The corresponding stability estimates happen to be sharp as well, as shown in the final Section 5.4.

5.2 Spectral optimization for Stekloff eigenvalues

We start recalling some basic definitions. We will refer to [75, Chapter 7] as the reference to this subject.

Let $E \subset \mathbb{R}^d$ be an open bounded set with Lipschitz boundary. Thanks to the compactness of the embedding of $W^{1,2}(E)$ into $L^2(\partial E)$, we have that the resolvant operator $\mathcal{R} : L^2(\partial E) \to L^2(\partial E)$ defined by

$$\mathcal{R}g \in W^{1,2}(E) \qquad \text{solves in weak sense} \qquad \left\{\begin{array}{rcll} -\Delta u &=& 0, & \text{in } E,\\ \langle \nabla u, \nu_E\rangle &=& g & \text{on } \partial E,\end{array}\right.$$

is a compact, symmetric and positive linear operator. Hence $\mathcal{R}$ has a discrete spectrum, made only of real positive eigenvalues accumulating at 0. As a consequence, we have that the following boundary value problem for harmonic functions

$$\left\{\begin{array}{rcll} -\Delta u &=& 0, & \text{in } E,\\ \langle \nabla u, \nu_E\rangle &=& \sigma\, u, & \text{on } \partial E,\end{array}\right.$$

has non-trivial solutions only for a discrete set of values $\sigma_1(E) \le \sigma_2(E) \le \sigma_3(E)\dots$ accumulating at ∞: these are the so-called *Stekloff eigenvalues of E*. Here solutions are intended in the usual weak sense, i.e.

$$\int_E \langle \nabla u(x), \nabla \varphi(x)\rangle\, dx = \sigma_k(E) \int_{\partial E} u(x)\, \varphi(x)\, dx, \qquad \text{for every } \varphi \in W^{1,2}(E),\ k \in \mathbb{N}.$$

The corresponding solutions $\{\xi_k\}_{k\ge 1}$ are called *eigenfunctions of the Stekloff-Laplacian* and they give an orthonormal basis of $L^2(\partial E)$, once renormalized by $\|\xi_k\|_{L^2(\partial E)} = 1$, for every $k \ge 1$. Throughout the next sections we will use the classical convention of counting the eigenvalues with their multiplicities: this means that if for a certain $k \in \mathbb{N}\setminus\{0\}$, there exist m linearly independent non-trivial solutions for $\sigma_k(E)$, then we will write $\sigma_k(E) = \sigma_{k+1}(E) = \dots = \sigma_{k+m-1}(E)$.

Observe that if E has k connected components $E_1, \dots, E_k$, then $\sigma_1(E) = \dots = \sigma_k(E) = 0$ and the corresponding renormalized eigenfunctions are given by

$$\xi_i(x) = \frac{1_{E_i}(x)}{\sqrt{\mathcal{H}^{d-1}(\partial E_i)}}, \qquad i = 1, \dots, k.$$

In particular *the first Stekloff eigenvalue of a set is always trivial* and corresponds to constant functions. For this reason, given $k \in \mathbb{N}\setminus\{0\}$, we always have that

$$0 = \inf\{\sigma_k(E) \,:\, |E| = c\},$$

and this infimum is attained for every open set having k connected components.

Remark 5.3. For what follows, it is important to remark that the functions $\{\xi_k\}_{k\ge 2}$ also give an orthogonal basis for the following closed subspace of $W^{1,2}(E)$

$$\mathrm{Har}(E) = \left\{ u \in W^{1,2}(E) \,:\, \int_{\partial E} u = 0 \ \text{ and } \int_E \langle \nabla u, \nabla \varphi\rangle = 0 \text{ for every } \varphi \in W^{1,2}_0(E) \right\}, \tag{5.5}$$

on which $u \mapsto \|\nabla u\|_{L^2}$ and $u \mapsto \|u\|_{W^{1,2}}$ are equivalent norms, thanks to the Poincaré-Wirtinger inequality (5.1) and to inequality

$$\|u\|_{L^2(E)} \le C_E \left(\|\nabla u\|_{L^2(E)} + \|u\|_{L^2(\partial E)}\right), \qquad u \in W^{1,2}(E),$$

which can be proved by means of a standard compactness argument. Notice that for every $u \in \mathrm{Har}(E)$, its Dirichlet integral can be written as

$$\int_E |\nabla u(x)|^2\, dx = \sum_{k\ge 2} \alpha_k^2\, \sigma_k(E), \qquad \text{where} \quad \alpha_k = \int_{\partial E} \xi_k(x)\, u(x)\, d\mathcal{H}^{d-1}. \tag{5.6}$$

For any ball B of radius R, its first non-trivial Stekloff eigenvalue is given by

$$\sigma_2(B) = \frac{1}{R},$$

which corresponds to the eigenfunctions $\xi_i(x) = x_{i-1}$, with $i = 2, \dots, d+1$, i.e. the eigenvalue $\sigma_2(B)$ has multiplicity d. Also, we notice that the shape functional $E \mapsto |E|^{1/d}\,\sigma_2(E)$ is scaling invariant, thus in particular

$$|B|^{1/d}\,\sigma_2(B) = \omega_d^{1/d},$$

for any ball B. About the first non-trivial Stekloff eigenvalue of a set E, we have the following sharp estimate, first derived in [111] for dimension $d = 2$ and then generalized to any dimension in [20].

Brock-Weinstock inequality. For every $E \subset \mathbb{R}^d$ open bounded set with Lipschitz boundary, we have

$$|E|^{1/d}\,\sigma_2(E) \le \omega_d^{1/d}, \tag{5.7}$$

and equality holds if and only if E is a ball. In other words, for every $c > 0$ the unique solution of the following spectral optimization problem

$$\max\{\sigma_2(E)\,:\,|E| \ge c\},$$

is given by a ball of measure c.

Remark 5.4. As already remarked in the Introduction, $1/\sigma_2(E)$ can be characterized as the sharp constant in the Poincaré-Wirtinger trace inequality (5.1). We then notice that the Brock-Weinstock inequality can be extended to any set supporting such an inequality and for which the trace of a $W^{1,2}$ function is well-defined: in these cases, it is still meaningful speaking of $\sigma_2(E)$, though the embedding $W^{1,2}(E) \hookrightarrow L^2(\partial E)$ could not be compact and hence its Stekloff-Laplacian could have a continuous spectrum.

Actually, the Brock-Weinstock inequality is a straightforward consequence of a stronger estimate proved by Brock in [20], involving the first d non-trivial Stekloff eigenvalues: namely, for every $E \subset \mathbb{R}^d$ bounded open set with Lipschitz boundary, we have

$$\frac{1}{|E|^{1/d}} \sum_{i=2}^{d+1} \frac{1}{\sigma_i(E)} \ge \frac{d}{\omega_d^{1/d}}, \tag{5.8}$$

i.e. any ball minimizes the sum of the reciprocal of the first d non-trivial Stekloff eigenvalues, among sets of given measure.

Remark 5.5. In the case of convex sets, an even stronger estimate is possible [78]: the ball maximizes the product of the first d non-trivial Stekloff eigenvalues, under measure constraint

$$|E| \prod_{i=2}^{d+1} \sigma_i(E) \le \omega_d. \tag{5.9}$$

A simple application of the arithmetic-geometric mean inequality shows that the previous implies (5.8): it should be noticed that in dimension $d = 2$, the convexity assumption can be dropped (see [76]), while for higher dimensions it is still an open problem to know whether (5.9) holds for all sets or not.

5.3 The stability issue

The main goal of this section is to show how (5.8) and (5.7) can be improved by means of a quantitative stability estimate. At this aim, for every $E \subset \mathbb{R}^d$ open set with finite measure, we recall the definition of *Fraenkel asymmetry*

$$\mathcal{A}(E) := \inf \left\{ \frac{\|1_E - 1_B\|_{L^1(\mathbb{R}^d)}}{|E|} \, : \, B \text{ ball with } |B| = |E| \right\},$$

i.e. this is the distance in the L^1 sense of a generic set E from the "manifold" of balls, renormalized in order to make it scaling invariant: observe that $0 \le \mathcal{A}(E) < 2$. Then the main result of this section is the following quantitative improvement of (5.8).

Theorem 5.6. *For every $E \subset \mathbb{R}^d$ open bounded set with Lipschitz boundary, we have*

$$\frac{1}{|E|^{1/d}} \sum_{i=2}^{d+1} \frac{1}{\sigma_i(E)} \ge \frac{d}{\omega_d^{1/d}} \left[1 + c_{d,2}\, \mathcal{A}(E)^2\right], \tag{5.10}$$

where

$$c_{d,2} = \frac{d+1}{d} \, \frac{\sqrt[d]{2} - 1}{4}.$$

Proof. We start reviewing the proof of Brock in [20]: the first step is to have a variational characterization for the sum of inverses of eigenvalues. In the case of Stekloff eigenvalues, the following formula holds (see [79, Theorem 1], for example):

$$\sum_{i=2}^{d+1} \frac{1}{\sigma_i(E)} = \max_{(v_2,\dots,v_{d+1}) \in \mathcal{I}} \sum_{i=2}^{d+1} \int_{\partial E} v_i(x)^2 \, d\mathcal{H}^{d-1},$$

where the set of admissible functions is given by

$$\mathcal{I} = \left\{ (v_2, \dots, v_{d+1}) \in (W^{1,2}(E))^d \, : \, \int_{\partial E} v_i(x) \, d\mathcal{H}^{d-1} = 0, \, \int_E \langle \nabla v_i(x), \nabla v_j(x) \rangle \, dx = \delta_{ij} \right\}.$$

Observe that the quantities $\sigma_i(E)$ are invariant under translations, so without loss of generality we can suppose that the barycentre of ∂E is in the origin, i.e.

$$\int_{\partial E} x_i \, d\mathcal{H}^{d-1} = 0, \qquad i = 1, \dots, d.$$

This implies that the eigenfunctions ξ_i relative to $\sigma_2(B) = \cdots = \sigma_{d+1}(B)$ are admissible in the previous maximization problem, thus as admissible functions we take

$$v_i(x) = \frac{x_{i-1}}{\sqrt{|E|}}, \qquad i = 2, \dots, d+1.$$

In this way, we obtain

$$\frac{1}{|E|^{1/d}} \sum_{i=2}^{d+1} \frac{1}{\sigma_i(E)} \geq \frac{1}{|E|^{1+1/d}} \int_{\partial E} |x|^2 \, d\mathcal{H}^{d-1} = |E|^{-\frac{d+1}{d}} \, P_2(E),$$

which implies

$$\frac{1}{|E|^{1/d}} \sum_{i=2}^{d+1} \frac{1}{\sigma_i(E)} - \frac{d}{\omega_d^{1/d}} \geq |E|^{-\frac{d+1}{d}} \, P_2(E) - \frac{d}{\omega_d^{1/d}}.$$

This means that the deficit of this spectral inequality is controlling from above the deficit of the 2−perimeter. Thus it is sufficient to use the quantitative estimate (5.2) for the 2−perimeter, so to obtain

$$\frac{1}{|E|^{1/d}} \sum_{i=2}^{d+1} \frac{1}{\sigma_i(E)} - \frac{d}{\omega_d^{1/d}} \geq \frac{d}{\omega_d^{1/d}} \, c_{d,2} \left(\frac{|E \Delta B|}{|E|} \right)^2,$$

where B is the ball centred at the origin and such that $|E| = |B|$. Using the definition of $\mathcal{A}(E)$, we can conclude the proof. □

A straightforward consequence of the previous result is the following quantitative version of the Brock-Weinstock inequality.

Corollary 5.7. *For every $E \subset \mathbb{R}^d$ open bounded set with Lipschitz boundary, we have*

$$|E|^{1/d} \, \sigma_2(E) \leq \omega_d^{1/d} \, \left[1 - \delta_d \, \mathcal{A}(E)^2\right], \tag{5.11}$$

where δ_d is a constant depending only on the dimension, given by

$$\delta_d = \frac{1}{8} \, \min \left\{ 1, \frac{d+1}{d} \, \left(\sqrt[d]{2} - 1 \right) \right\}.$$

Proof. First of all, we can suppose that

$$|E|^{1/d} \, \sigma_2(E) \geq \frac{1}{2} \, \omega_d^{1/d}, \tag{5.12}$$

otherwise estimate (5.11) is trivially true with constant $\delta_d = 1/8$, just by using the fact that $\mathcal{A}(E) < 2$. So, let us suppose that (5.12) holds true: since $\sigma_2(E) \leq \sigma_i(E)$ for every $i \geq 3$, from (5.10) we can infer

$$\frac{d}{|E|^{1/d} \, \sigma_2(E)} \geq \frac{d}{\omega_d^{1/d}} \, \left[1 + c_{d,2} \, \mathcal{A}(E)^2\right],$$

which can be rewritten as

$$|E|^{1/d}\,\sigma_2(E)\,\left[1+c_{d,2}\,\mathcal{A}(E)^2\right]\leq \omega_d^{1/d}.$$

The previous formula easily implies (5.11), thanks to (5.12). □

Remark 5.8. In the next section we will prove that both the estimates derived in Theorem 5.6 and Corollary 5.7 are sharp. We point out that defining the two deficit functionals

$$\mathrm{Inv}(E):=\frac{|B|^{1/d}}{d\,|E|^{1/d}}\sum_{i=2}^{d+1}\frac{\sigma_2(B)}{\sigma_i(E)}-1\qquad\text{and}\qquad BW(E):=\frac{|B|^{1/d}\,\sigma_2(B)}{|E|^{1/d}\,\sigma_2(E)}-1,\quad(5.13)$$

we have that

$$c_{d,2}\,\mathcal{A}(E)^2\leq \mathrm{Inv}(E)\leq BW(E),$$

where in the first inequality we used Theorem 5.6. Then if one can prove that the exponent 2 for $\mathcal{A}(E)$ is sharp in the quantitative Brock-Weinstock inequality, this will automatically prove the optimality of the power 2 for inequality (5.10).

5.4 Sharpness of the quantitative Brock-Weinstock inequality

In this section, we will show the sharpness of the quantitative Brock-Weinstock inequality (5.11): as remarked, this in turn will give the sharpness of (5.10) as well. Namely, we are going to prove the following result.

Theorem 5.9. *There exists a family $\{E_\varepsilon\}_{\varepsilon>0}$ of smooth sets approaching the ball B of unit radius in such a way that*

$$\mathcal{A}(E_\varepsilon)\simeq\frac{\big|E_\varepsilon\Delta B\big|}{\big|E_\varepsilon\big|}\simeq\varepsilon\qquad\textit{and}\qquad BW(E_\varepsilon)\simeq\varepsilon^2,\qquad\varepsilon\ll 1,\qquad(5.14)$$

where $BW(E)$ is defined by (5.13).

The rest of this section is devoted to construct such a family of deformations E_ε. Since the whole construction is quite complicate, for the sake of readability we will divide it into 4 main steps.

5.4.1 Step 1: setting of the construction and basic properties

In what follows, $B\subset\mathbb{R}^d$ stands for the open unit ball, centred at the origin. We consider a general nearly circular domain, given by

$$E_\varepsilon=\{x\in\mathbb{R}^d\,:\,x=0\ \text{ or }\ |x|<1+\varepsilon\,\psi(x/|x|)\},$$

where $\psi\in C^\infty(\partial B)$ satisfies the following assumptions.

Key assumptions. For every $a \in \mathbb{R}^d$, there holds

$$\int_{\partial B} \psi(x)\, d\mathcal{H}^{d-1} = 0, \qquad \int_{\partial B} \langle a, x\rangle\, \psi(x)\, d\mathcal{H}^{d-1} = 0, \tag{5.15}$$

and

$$\int_{\partial B} \langle a, x\rangle^2\, \psi(x)\, d\mathcal{H}^{d-1} = 0. \tag{5.16}$$

We start with a basic result of geometric type.

Lemma 5.10. *Let $\psi \in C^\infty(\partial B)$ satisfying (5.15). Then*

$$|E_\varepsilon| - |B| \simeq \varepsilon^2 \qquad \text{and} \qquad \mathcal{A}(E_\varepsilon) \simeq \varepsilon \simeq \frac{|E_\varepsilon \Delta B|}{|E_\varepsilon|}. \tag{5.17}$$

Proof. Using polar coordinates, the measure $|E_\varepsilon|$ can be expressed as follows

$$\begin{aligned} |E_\varepsilon| &= \frac{1}{d}\int_{\partial B} (1+\varepsilon\, \psi(x))^d\, d\mathcal{H}^{d-1} \\ &\simeq |B| + \varepsilon \int_{\partial B} \psi(x)\, d\mathcal{H}^{d-1} + \varepsilon^2\, \frac{d-1}{2} \int_{\partial B} \psi(x)^2\, d\mathcal{H}^{d-1}, \end{aligned}$$

which gives the first relation in (5.17), thanks to the fact that ψ has zero-mean on ∂B.

For the second one, we start observing that $|E_\varepsilon \Delta B| \simeq \varepsilon$: still using polar coordinates, we get

$$\begin{aligned} |E_\varepsilon \Delta B| = &\frac{1}{d}\int_{\{x\in\partial B\,:\,\psi(x)>0\}} \Big[(1+\varepsilon\,\psi(x))^d - 1\Big]\, d\mathcal{H}^{d-1} \\ &+ \frac{1}{d}\int_{\{x\in\partial B\,:\,\psi(x)<0\}} \Big[1-(1+\varepsilon\,\psi(x))^d\Big]\, d\mathcal{H}^{d-1} \simeq \varepsilon \int_{\partial B} |\psi(x)|\, d\mathcal{H}^{d-1}. \end{aligned}$$

Now, let $B(x_0, r_\varepsilon)$ be a ball realizing the asymmetry, i.e. such that $\mathcal{A}(E_\varepsilon)\,|E_\varepsilon| = |E_\varepsilon \Delta B(x_0, r_\varepsilon)|$: in particular, we have $|B(x_0, r_\varepsilon)| = |E_\varepsilon|$. It is easily seen that

$$\mathcal{A}(E_\varepsilon) \le c\, \frac{|E_\varepsilon \Delta B|}{|E_\varepsilon|}, \tag{5.18}$$

for some constant c independent of ε: indeed, by definition of $\mathcal{A}(E_\varepsilon)$ and triangular inequality, we get

$$\mathcal{A}(E_\varepsilon) \le \frac{|E_\varepsilon \Delta B(0, r_\varepsilon)|}{|E_\varepsilon|} \le \frac{|E_\varepsilon \Delta B|}{|E_\varepsilon|} + \frac{|B \Delta B(0, r_\varepsilon)|}{|E_\varepsilon|} \le c\, \frac{|E_\varepsilon \Delta B|}{|E_\varepsilon|},$$

since $|B \Delta B(0, r_\varepsilon)| = \big|\,|B(x_0, r_\varepsilon)| - |B|\,\big| \simeq \varepsilon^2$, while $|E_\varepsilon \Delta B| \simeq \varepsilon$.

Using the symmetries of B and (5.15), for every $a \in \mathbb{R}^d$ we can infer

$$\int_{E_\varepsilon} \langle a, y\rangle \, dy = \frac{1}{d+1} \int_{\partial B} (1 + \varepsilon\, \psi(x))^{d+1} \langle a, x\rangle \, d\mathcal{H}^{d-1} \simeq \varepsilon^2 \, \frac{d}{2} \int_{\partial B} \psi(x)^2 \langle a, x\rangle \, d\mathcal{H}^{d-1}.$$

Choosing $a = \mathbf{e}_i$, i.e. the coordinate directions, the previous formula implies that the barycentre of E_ε coincides with the origin, up to an error of order ε^2. Since the barycentre of $B(x_0, r_\varepsilon)$ is given by its centre x_0, we then get

$$\begin{aligned} |B(x_0, r_\varepsilon)|\, |x_0| = \left| \int_{B(x_0,r_\varepsilon)} y \, dy \right| &\le \left| \int_{B(x_0,r_\varepsilon)} y \, dy - \int_{E_\varepsilon} y \, dy \right| + \left| \int_{E_\varepsilon} y \, dy \right| \\ &\le \int_{B(x_0,r_\varepsilon)\Delta E_\varepsilon} |y| \, dy + \left| \int_{E_\varepsilon} y \, dy \right| \le C \, |B(x_0, r_\varepsilon)\Delta E_\varepsilon| + C\, \varepsilon^2, \end{aligned}$$

for some constant C independent of ε. In other words, we get $|x_0| \le C\, \mathcal{A}(E_\varepsilon) + C\, \varepsilon^2$ – possibly with a different constant C, but still independent of ε – then we can estimate

$$\frac{|B\Delta E_\varepsilon|}{|E_\varepsilon|} \le \frac{|E_\varepsilon \Delta B(x_0, r_\varepsilon)|}{|E_\varepsilon|} + \frac{|B(x_0, r_\varepsilon)\Delta B|}{|E_\varepsilon|} \le \mathcal{A}(E_\varepsilon) + C' \, |x_0| + C'' \, \varepsilon^2, \tag{5.19}$$

for some C', C'' not depending on ε: here we used that $|B(x_0, r_\varepsilon)\Delta B|$ is comparable to the distance of their centres – that is, comparable to $|x_0|$ – up to an error of order ε^2, due to the difference of the measures. It is only left to use the estimate on $|x_0|$ in (5.19), in conjunction with (5.18) and the fact that $|E_\varepsilon \Delta B| \simeq \varepsilon$: we then get

$$\frac{1}{c'}\, \varepsilon \le \mathcal{A}(E_\varepsilon) + C\, \varepsilon^2 \le c' \, \varepsilon,$$

for some $c' > 1$, which finally gives $\mathcal{A}(E_\varepsilon) \simeq \varepsilon$, as desired. □

Remark 5.11 (Meaning of the key assumptions)**.** We point out that conditions (5.15) and (5.16) are equivalent to require that ψ is orthogonal in the $L^2(\partial B)$ sense to the first three eigenspace of the Laplace-Beltrami operator on ∂B, i.e. to spherical harmonics of order $0, 1$ and 2 respectively (see [99] for a comprehensive account on spherical harmonics). Each of these conditions will play a precise role in our construction: thanks to the previous result, the first one implies that E_ε has the same measure as B, up to an error of order ε^2. The second condition in (5.15) implies that E_ε has the same barycentre as B, still up to an error of order ε^2: then this order coincides with the magnitude of $\mathcal{A}(E_\varepsilon)^2$. Eventually, recalling that every Stekloff eigenfunction ξ relative to $\sigma_2(B)$ has the form $\xi(x) = \langle a, x\rangle$, condition (5.16) implies

$$\int_{\partial B} \psi(x)\, |\xi(x)|^2 \, d\mathcal{H}^{d-1} = 0 \qquad \text{and} \qquad \int_{\partial B} \psi(x)\, |\nabla_\tau \xi(x)|^2 \, d\mathcal{H}^{d-1} = 0, \tag{5.20}$$

where ∇_τ is the tangential gradient. Relations (5.20) will be crucially exploited in order to prove that $\sigma_2(B) - \sigma_2(E_\varepsilon) \simeq \varepsilon^2$.

Let us fix now an eigenfunction u_ε for $\sigma_2(E_\varepsilon)$, normalized in such a way that

$$\int_{\partial E_\varepsilon} u_\varepsilon(x)^2 \, d\mathcal{H}^{d-1} = 1 \qquad \text{and} \qquad \int_{E_\varepsilon} \left|\nabla u_\varepsilon(x)\right|^2 dx = \sigma_2(E_\varepsilon). \tag{5.21}$$

Remark 5.12. Thanks to the fact that ∂E_ε is of class C^∞, we obtain that $u_\varepsilon \in C^\infty(\overline{E_\varepsilon})$. Moreover, the domains E_ε are uniformly of class C^k, for every $k \geq 0$, hence we can assume the functions u_ε to satisfy uniform C^k estimates, i.e.

$$\|u_\varepsilon\|_{C^k(\overline{E_\varepsilon})} \leq H_k\,, \tag{5.22}$$

for some constants $H_k > 0$ depending only on $k \in \mathbb{N}$.

We now give the basic estimate of $\sigma_2(B)$ from above in terms of $\sigma_2(E_\varepsilon)$: this is the cornerstone of the whole construction.

Lemma 5.13. *Let $\varepsilon_0 \ll 1$, there exist two functions $d, Q : [0, \varepsilon_0] \to \mathbb{R}$ with*

$$\lim_{\varepsilon \to 0} (|d(\varepsilon)| + |Q(\varepsilon)|) = 0,$$

and a constant $K > 0$ such that for every ε, we have

$$\sigma_2(B) \leq \frac{\sigma_2(E_\varepsilon) + d(\varepsilon)}{1 + Q(\varepsilon) - K\varepsilon^2}. \tag{5.23}$$

Proof. Since we want to compare $\sigma_2(E_\varepsilon)$ with $\sigma_2(B)$, we have to suitably adapt the eigenfuction u_ε, in order to let it be admissible for the Rayleigh quotient defining $\sigma_2(B)$. To do so, we start considering a C^k extension $\widetilde{u}_\varepsilon$ of u_ε with $k = [d/2] + 3$ to the larger set [1]

$$D_\varepsilon = \left\{x \,:\, |x| \leq 1 + \varepsilon \|\psi\|_{L^\infty(\partial B)}\right\} \supset \overline{B \cup E_\varepsilon},$$

and we can make such an extension in such a way that

$$\|\widetilde{u}_\varepsilon\|_{C^k(D_\varepsilon)} \leq K \|u_\varepsilon\|_{C^k(E_\varepsilon)}\,. \tag{5.24}$$

Then, we estimate the mean value of this extension on the boundary ∂B: we set

$$\delta := \int_{\partial B} \widetilde{u}_\varepsilon(x)\, d\mathcal{H}^{d-1},$$

and we define the application $\phi_\varepsilon : \partial B \to \partial E_\varepsilon$, given by

$$\phi_\varepsilon(x) = x + \varepsilon\, \psi(x)\, x, \qquad x \in \partial B. \tag{5.25}$$

Observe that we have

$$\widetilde{u}_\varepsilon(\phi_\varepsilon(x)) = u_\varepsilon(\phi_\varepsilon(x)), \qquad x \in \partial B,$$

[1] The choice of k will be clear in the proof of Lemma 5.19.

so that our uniform estimates (5.22) and (5.24) yield

$$\widetilde{u}_\varepsilon(x) = u_\varepsilon(\phi_\varepsilon(x)) + O(\varepsilon), \qquad x \in \partial B. \tag{5.26}$$

Using this information in the definition of δ, we get

$$\delta = \int_{\partial B} u_\varepsilon(\phi_\varepsilon(x))\, d\mathcal{H}^{d-1} + O(\varepsilon) = \int_{\partial B} u_\varepsilon(\phi_\varepsilon(x))\, J_\varepsilon(x)\, d\mathcal{H}^{d-1} + O(\varepsilon)$$

where in the last equality we have set

$$J_\varepsilon(x) = (1+\varepsilon\,\psi(x))^{d-2}\sqrt{(1+\varepsilon\,\psi(x))^2 + \varepsilon^2\,|\nabla_\tau\,\psi(x)|^2}, \qquad x \in \partial B,$$

and we used the following straightforward estimate

$$\|J_\varepsilon(y) - 1\|_{L^\infty(\partial B)} = O(\varepsilon), \tag{5.27}$$

the quantity $\nabla_\tau\psi$ being the *tangential gradient* of ψ on ∂B. With the change of variable $y = \phi_\varepsilon(x)$, we then arrive at

$$\delta = \frac{1}{\mathcal{H}^{d-1}(\partial B)}\int_{\partial E_\varepsilon} u_\varepsilon(y)\, d\mathcal{H}^{d-1} + O(\varepsilon) = O(\varepsilon), \tag{5.28}$$

thanks to the fact that $\int_{\partial E_\varepsilon} u_\varepsilon = 0$. We are now ready to define an admissible function for $\sigma_2(B)$: we set

$$v_\varepsilon := \widetilde{u}_\varepsilon \cdot 1_{\bar{B}} - \delta, \tag{5.29}$$

and we immediately notice that

$$\|v_\varepsilon\|_{C^k(\overline{B})} \leq K(d)\,, \tag{5.30}$$

thanks to (5.22), (5.24) and (5.28) (recall that k depends only on d). In words, v_ε is the original eigenfunction u_ε extended to the whole D_ε, then restricted to the ball B and finally vertically translated in order to satisfy the zero-mean condition on ∂B. By its very definition and using (5.28), we immediately observe that

$$\left|\int_{\partial B} v_\varepsilon^2 - \int_{\partial B} \widetilde{u}_\varepsilon^2\right| = \left|-2\,\delta\int_{\partial B}\widetilde{u}_\varepsilon + \delta^2\,\mathcal{H}^{d-1}(\partial B)\right| = \delta^2\,\mathcal{H}^{d-1}(\partial B) \leq K\varepsilon^2. \tag{5.31}$$

Now we set

$$d(\varepsilon) := \int_{B\setminus E_\varepsilon} |\nabla v_\varepsilon|^2 - \int_{E_\varepsilon\setminus B} |\nabla u_\varepsilon|^2,$$

so that we can write

$$\int_B |\nabla v_\varepsilon(x)|^2 = \int_{E_\varepsilon} |\nabla u_\varepsilon|^2 + d(\varepsilon) = \sigma_2(E_\varepsilon) + d(\varepsilon), \tag{5.32}$$

where we used that $\nabla v_\varepsilon = \nabla u_\varepsilon$ on $B \cap E_\varepsilon$. Moreover, using (5.26) and (5.31), we have

$$\int_{\partial B} v_\varepsilon(x)^2 \geq \int_{\partial B} \widetilde{u}_\varepsilon(x)^2 - K\,\varepsilon^2 = \int_{\partial E_\varepsilon} u_\varepsilon(x)^2 + Q(\varepsilon) - K\varepsilon^2 = 1 + Q(\varepsilon) - K\varepsilon^2, \quad (5.33)$$

having defined

$$Q(\varepsilon) := \int_{\partial B} \widetilde{u}_\varepsilon(x)^2 - \int_{\partial E_\varepsilon} u_\varepsilon(x)^2.$$

We are now able to estimate $\sigma_2(B)$: since

$$\sigma_2(B) \leq \frac{\displaystyle\int_B |\nabla v_\varepsilon(x)|^2\,dx}{\displaystyle\int_{\partial B} v_\varepsilon(x)^2\,d\mathcal{H}^{d-1}},$$

using (5.32) and (5.33), we finally obtain (5.23). □

Remark 5.14. Thanks to the uniform estimates (5.22) with $k = 0$, 1 and to (5.27), it is immediate to infer

$$\big|d(\varepsilon)\big| \leq K\,\varepsilon\,, \qquad\qquad \big|Q(\varepsilon)\big| \leq K\,\varepsilon\,, \quad (5.34)$$

which inserted in (5.23) gives the easy estimate

$$\sigma_2(B) \leq \sigma_2(E_\varepsilon) + K\varepsilon,$$

possibly with a different constant $K > 0$.

The previous observation shows that in order to exhibit the sharp decay rate of the deficit along the sequence E_ε, we need a precise control of the decay rate of the error terms d and Q. Indeed, each estimate on them automatically translates into an estimate of the same order for $\sigma_2(B) - \sigma_2(E_\varepsilon)$. Let us state precisely this observation, whose proof is immediate from (5.23).

Lemma 5.15. *There exist two constants C_1 and C_2 such that*

$$|\sigma_2(B) - \sigma_2(E_\varepsilon)| \leq C_1\,(|d(\varepsilon)| + |Q(\varepsilon)|) + C_2\,\varepsilon^2, \qquad \textit{for every } \varepsilon \ll 1.$$

Keeping in mind Corollary 5.7 and (5.17), we know that

$$C_3\,\varepsilon^2 \leq BW(E_\varepsilon) \leq C_4\,|\sigma_2(B) - \sigma_2(E_\varepsilon)| + C_5\,\varepsilon^2, \quad (5.35)$$

hence to conclude the optimality of the exponent 2 in (5.11) one would like to enforce (5.34), proving that

$$|d(\varepsilon)| + |Q(\varepsilon)| \leq K\,\varepsilon^2.$$

5.4.2 Step 2: improving the decay rate

In order to gain this improvement, the following Lemma will be of crucial importance. This guarantees that if the distance in C^1 between v_ε and the eigenspace corresponding to $\sigma_2(B)$ has a certain rate of decaying at 0, then the decays of $d(\varepsilon)$ and $Q(\varepsilon)$ are improved of the same order. *It is precisely here*, in the proof of this result, *that the Key Assumption* (5.16) *on* ψ *will heavily come into play.*

Lemma 5.16. *Let* $\omega : [0,1] \to \mathbb{R}^+$ *be a continuous function such that* $t^2/K \le \omega(t) \le K\sqrt{t}$. *Suppose that for every* $\varepsilon \ll 1$, *there exists an eigenfunction* ξ_ε *for* $\sigma_2(B)$ *such that*

$$\|v_\varepsilon - \xi_\varepsilon\|_{C^1(\overline{B})} \le C\,\omega(\varepsilon), \tag{5.36}$$

for some constant C *independent of* ε. *Then there exists a constant* C_6, *still independent of* ε, *such that*

$$|d(\varepsilon)| + |Q(\varepsilon)| \le C_6\,\omega(\varepsilon)\,\varepsilon \qquad \textit{for every } \varepsilon \ll 1.$$

Proof. We start estimating the term $|d(\varepsilon)|$: the computations are similar to that in [18], but we have to pay attention to some extra terms, which come from the fact that we are facing a Stekloff problem.

Using the uniform estimates (5.22) and recalling the definition (5.25) of ϕ_ε, we have

$$|\nabla u_\varepsilon(x)|^2 = \left|\nabla u_\varepsilon\left(\phi_\varepsilon\left(\frac{x}{|x|}\right)\right)\right|^2 + O(\varepsilon), \qquad \text{for every } x \in E_\varepsilon \setminus B,$$

and observe that, alignedting the gradient in its radial and tangential components, the right-hand side can be written as

$$\begin{aligned} |\nabla u_\varepsilon\left(\phi_\varepsilon(x/|x|)\right)|^2 &= |\partial_\varrho u_\varepsilon\left(\phi_\varepsilon(x/|x|)\right)|^2 + \frac{1}{(1+\varepsilon\,\psi(x/|x|))^2}\,|\nabla_\tau u_\varepsilon(\phi_\varepsilon(x/|x|))|^2 \\ &= |\partial_\varrho u_\varepsilon(\phi_\varepsilon(x/|x|))|^2 + |\nabla_\tau u_\varepsilon(\phi_\varepsilon(x/|x|))|^2 + O(\varepsilon). \end{aligned}$$

Using once again (5.22), the latter in turn can be estimated as follows

$$|\partial_\varrho u_\varepsilon(\phi_\varepsilon(x/|x|))|^2 + |\nabla_\tau u_\varepsilon(\phi_\varepsilon(x/|x|))|^2 = \sigma_2(E_\varepsilon)^2\,|u_\varepsilon(x/|x|)|^2 + |\nabla_\tau u_\varepsilon(x/|x|)|^2 + O(\varepsilon).$$

Notice that we also used that u_ε satisfies the boundary condition

$$\langle \nabla u_\varepsilon(x), \nu_{E_\varepsilon}(x)\rangle = \sigma_2(E_\varepsilon)\,u_\varepsilon(x), \qquad x \in \partial E_\varepsilon,$$

and that the normal vector on ∂E_ε is radial up to an error of order ε, since we have

$$\nu_{E_\varepsilon}(x) = \frac{(1+\varepsilon\,\psi(x/|x|))\,x/|x| - \varepsilon\,\nabla_\tau\psi(x/|x|)}{\sqrt{(1+\varepsilon\,\psi(x/|x|))^2 + |\nabla_\tau\psi(x/|x|)|^2}} = \frac{x}{|x|} + O(\varepsilon), \qquad x \in \partial E_\varepsilon,$$

Therefore, recalling also that $|E_\varepsilon \setminus B| \simeq \varepsilon$, one obtains

$$\begin{aligned}\int_{E_\varepsilon\setminus B} |\nabla u_\varepsilon(x)|^2\,dx &= \varepsilon \int_{\partial B\cap\{\psi>0\}} \psi(x)\,\left[\sigma_2(E_\varepsilon)^2\,u_\varepsilon(x)^2 + |\nabla_\tau u_\varepsilon(x)|^2\right]\,d\mathcal{H}^{d-1} + O(\varepsilon^2)\\ &= \varepsilon \int_{\partial B\cap\{\psi>0\}} \psi(x)\,\left[\sigma_2(B)^2\,v_\varepsilon(x)^2 + |\nabla_\tau v_\varepsilon(x)|^2\right]\,d\mathcal{H}^{d-1} + O(\varepsilon^2)\,,\end{aligned} \tag{5.37}$$

where the last equality comes from the fact that $v_\varepsilon = u_\varepsilon$ on $E_\varepsilon \cap B$ up to the additive constant δ, which is of order ε thanks to (5.28), and from the fact that $|\sigma_2(B)-\sigma_2(E_\varepsilon)| \le C\,\varepsilon$. In the very same way, recalling that by definition of v_ε one has

$$\nabla v_\varepsilon(\phi_\varepsilon(x)) = \nabla u_\varepsilon(\phi_\varepsilon(x)), \qquad \text{for every } x \in \partial B \setminus E_\varepsilon,$$

and that the uniform estimates holds also for v_ε by (5.30), one gets

$$\int_{B\setminus E_\varepsilon} \left|\nabla v_\varepsilon(x)\right|^2 dx = -\varepsilon \int_{\partial B\cap\{\psi>0\}} \psi(x)\,\left[\sigma_2(B)^2\,v_\varepsilon(x)^2 + |\nabla_\tau v_\varepsilon(x)|^2\right]\,d\mathcal{H}^{d-1} + O(\varepsilon^2)\,. \tag{5.38}$$

Finally, recalling the definition of $d(\varepsilon)$, from (5.36), (5.37) and (5.38) one obtains

$$\begin{aligned}|d(\varepsilon)| &\le \varepsilon\,\sigma_2(B)^2 \left|\int_{\partial B} \psi(x)\,v_\varepsilon(x)^2\,d\mathcal{H}^{d-1}\right|\\ &\quad + \varepsilon \left|\int_{\partial B} \psi(x)\,|\nabla_\tau v_\varepsilon(x)|^2\,d\mathcal{H}^{d-1}\right| + O(\varepsilon^2)\\ &= \varepsilon\,\sigma_2(B)^2 \left|\int_{\partial B} \psi(x)\,\xi_\varepsilon(x)^2\,d\mathcal{H}^{d-1}\right|\\ &\quad + \varepsilon \left|\int_{\partial B} \psi(x)\,|\nabla_\tau \xi_\varepsilon(x)|^2\,d\mathcal{H}^{d-1}\right| + C'\,\varepsilon\,\omega(\varepsilon) + O(\varepsilon^2) \le \widetilde{C}\,\varepsilon\,\omega(\varepsilon),\end{aligned}$$

where in the last estimate we used property (5.16).

We now come to the estimate of $|Q(\varepsilon)|$: remember that this is given by

$$Q(\varepsilon) = \int_{\partial B} \left[\widetilde{u}_\varepsilon(x)^2 - \widetilde{u}_\varepsilon(x+\varepsilon\,\psi(x)\,x)^2\,J_\varepsilon(x)\right]\,d\mathcal{H}^{d-1},$$

i.e. this error term contains a boundary integral, then estimates are a bit different from the Neumann case treated in [18].

In order to handle this term Q, for ease of computations it could be more useful to rewrite it as follows

$$Q(\varepsilon) = Q_1(\varepsilon) + Q_2(\varepsilon),$$

where we set

$$Q_1(\varepsilon) := \int_{\partial B} \left[\widetilde{u}_\varepsilon(x)^2 - \widetilde{u}_\varepsilon(\phi_\varepsilon(x))^2\right]\,d\mathcal{H}^{d-1},$$

and

$$Q_2(\varepsilon) := \int_{\partial B} \widetilde{u}_\varepsilon(\phi_\varepsilon(x/|x|))^2 \, [1 - J_\varepsilon(x)] \, d\mathcal{H}^{d-1}.$$

Let us start with $Q_1(\varepsilon)$: by construction $\nabla \widetilde{u}_\varepsilon(x) = \nabla v_\varepsilon(x)$, then using the uniform estimates (5.22), (5.24) and the hypotheses (5.36), we have

$$\begin{aligned} |Q_1(\varepsilon)| &= \left| \int_{\partial B} \left[\widetilde{u}_\varepsilon(x)^2 - \widetilde{u}_\varepsilon\,(\phi_\varepsilon(x))^2 \right] \, d\mathcal{H}^{d-1} \right| \\ &\leq 2\,\varepsilon \left| \int_{\partial B} \widetilde{u}_\varepsilon(x)\, \partial_\varrho \widetilde{u}_\varepsilon(x)\, \psi(x)\, d\mathcal{H}^{d-1} \right| + O(\varepsilon^2) \\ &\leq 2\,\varepsilon \left| \int_{\partial B} \xi_\varepsilon(x)\, \partial_\varrho \xi_\varepsilon(x)\, \psi(x)\, d\mathcal{H}^{d-1} \right| + C\,\omega(\varepsilon)\,\varepsilon \\ &= 2\,\varepsilon\, \sigma_2(B) \left| \int_{\partial B} \xi_\varepsilon(x)^2\, \psi(x)\, d\mathcal{H}^{d-1} \right| + C\,\omega(\varepsilon)\,\varepsilon, \end{aligned}$$

which yields the estimate $|Q_1(\varepsilon)| \leq C\,\omega(\varepsilon)\,\varepsilon$, again thanks to property (5.16). Observe that in the last equality we have exploited the fact that ξ_ε satisfies the Stekloff boundary condition. Finally, it is left to estimate the term $Q_2(\varepsilon)$: first of all, we have

$$1 - J_\varepsilon(x) = -(d-1)\,\varepsilon\,\psi(\vartheta) + O(\varepsilon^2),$$

while using the definition of v_ε, the uniform estimates (5.22) and (5.24) and the fact that $\delta = O(\varepsilon)$, we get

$$\widetilde{u}_\varepsilon(\phi_\varepsilon(x)) = \widetilde{u}_\varepsilon(x) + O(\varepsilon) = v_\varepsilon(x) + \delta + O(\varepsilon) = v_\varepsilon(x) + O(\varepsilon), \qquad x \in \partial B.$$

Inserting these into the definition of $Q_2(\varepsilon)$ and using (5.36), we finally obtain

$$\begin{aligned} |Q_2(\varepsilon)| &\leq (d-1)\,\varepsilon \left| \int_{\partial B} v_\varepsilon(x)^2\, \psi(x)\, d\mathcal{H}^{d-1} \right| + O(\varepsilon^2) \\ &\leq (d-1)\,\varepsilon \left| \int_{\partial B} \xi_\varepsilon(x)^2\, \psi(x)\, d\mathcal{H}^{d-1} \right| + C\,\omega(\varepsilon)\,\varepsilon, \end{aligned}$$

which concludes the proof, again thanks to property (5.16). □

Remark 5.17. Observe that if on the contrary ψ violates condition (5.16), we cannot assure that all the first-order term in the previous estimates cancel out: then we would not get any improvement on d and Q. For example, for the case of the ellipsoids E_ε considered in Section 4.2.1, their boundaries can be described as follows

$$\partial E_\varepsilon = \left\{ y = \varrho_\varepsilon(x)\, x \in \mathbb{R}^2 \,:\, x \in \partial B \quad \text{and} \quad \varrho_\varepsilon(x) = \sqrt{(1+\varepsilon)\, x_1^2 + \frac{x_2^2}{1+\varepsilon}} \right\},$$

and observe that

$$\varrho_\varepsilon(x) \simeq 1 + \varepsilon\,(x_1^2 - x_2^2), \qquad x \in \partial B.$$

It is not difficult to see that $\psi(x) = x_1^2 - x_2^2$ does not satisfy (5.16): and in fact, in analogy with the Neumann case (see [18, Section 5]), one can show that

$$\sigma_2(B) - \sigma_2(E_\varepsilon) \simeq \varepsilon,$$

i.e. ellipsoids do not exhibit the sharp decay rate for the Brock-Weinstock inequality.

5.4.3 Step 3: nearness estimates

Thanks to the previous step, we know that to improve (5.34) it is sufficient to estimate the C^1 distance of v_ε from the eigenspace relative to $\sigma_2(B)$, in terms of ε: the main point is that *we can perform such an estimation, in terms of $|d(\varepsilon)|$ and $|Q(\varepsilon)|$ themselves.* This is the content of the third step.

We start with an easy $W^{1,2}(B)$ estimate, whose proof is based on a Fourier decomposition on the basis $\{\xi_k\}_{k\ge 2}$ of Stekloff eigenfunctions for B: the idea is quite the same as in [18], but an extra difficulty arises, since we cannot directly decompose v_ε in $W^{1,2}$ on the basis $\{\xi_k\}_{k\ge 2}$. Rather, we have to project it on the space of harmonic functions and to control, in terms of ε, both the Dirichlet integral of this projection and the distance between v_ε and the space of harmonic functions.

Lemma 5.18. *For every $\varepsilon \ll 1$, there exists an eigenfunction ξ_ε relative to $\sigma_2(B)$ such that*

$$\|v_\varepsilon - \xi_\varepsilon\|_{W^{1,2}(B)} \le C\,\sqrt{|d(\varepsilon)| + |Q(\varepsilon)|} + K\,\varepsilon, \qquad \text{for every } \varepsilon \ll 1, \tag{5.39}$$

for some constant C independent of ε.

Proof. First of all, let us set $f_\varepsilon := \Delta v_\varepsilon = \Delta \widetilde{u}_\varepsilon$. Thanks to the fact that $\widetilde{u}_\varepsilon$ is a C^k extension of u_ε and that the latter is harmonic on $E_\varepsilon \cap B$, we get that f_ε is a C^{k-2} function on B such that

$$f_\varepsilon(x) = 0, \qquad x \in E_\varepsilon \cap B.$$

Moreover, on $B \setminus E_\varepsilon$ we have

$$\begin{aligned} |f_\varepsilon(x)| &\le \left| f_\varepsilon\left(\phi_\varepsilon\left(\frac{x}{|x|}\right)\right)\right| + \|\nabla f_\varepsilon\|_{L^\infty(B)} \left|\phi_\varepsilon\left(\frac{x}{|x|}\right) - x\right| \\ &= \|\nabla f_\varepsilon\|_{L^\infty(B)} \left|\phi_\varepsilon\left(\frac{x}{|x|}\right) - x\right| \le C\left|\phi_\varepsilon\left(\frac{x}{|x|}\right) - \frac{x}{|x|}\right| \le C\,\varepsilon\,\|\psi\|_{L^\infty}, \qquad x \in B\setminus E_\varepsilon, \end{aligned} \tag{5.40}$$

so that in conclusion $\|f_\varepsilon\|_{L^\infty(B)} \le C\,\varepsilon$. We now introduce the *harmonic projection* φ_ε of v_ε, i.e. φ_ε solves

$$\left\{\begin{array}{rcll} \Delta\varphi_\varepsilon &=& 0, & \text{in } B, \\ \varphi_\varepsilon &=& v_\varepsilon, & \text{on } \partial B, \end{array}\right.$$

and observe that we have

$$\|v_\varepsilon - \varphi_\varepsilon\|_{W^{1,2}(B)} \le C\,\|f_\varepsilon\|_{L^2(B)} \le C\,\varepsilon, \tag{5.41}$$

where we used the previous estimate on f_ε. Since φ_ε is harmonic and $v_\varepsilon - \varphi_\varepsilon \in W^{1,2}_0(B)$, we obtain following estimate on the Dirichlet integrals:

$$\|\nabla v_\varepsilon - \nabla\varphi_\varepsilon\|^2_{L^2(B)} = \int_B |\nabla v_\varepsilon(x)|^2\,dx - \int_B \langle\nabla v_\varepsilon, \nabla\varphi_\varepsilon\rangle\,dx$$

Keeping into account (5.41), we finally obtain

$$\left|\int_B |\nabla v_\varepsilon(x)|^2\,dx - \int_B |\nabla\varphi_\varepsilon(x)|^2\,dx\right| \le C\,\varepsilon^2. \tag{5.42}$$

Since $\varphi_\varepsilon \in \mathrm{Har}(B)$ – remember the definition (5.5) – we can use a spectral decomposition for it and write

$$\varphi_\varepsilon = \sum_{k\ge 2} \alpha_k(\varepsilon)\,\xi_k, \qquad \text{where} \qquad \alpha_k(\varepsilon) = \int_{\partial B} \varphi_\varepsilon(x)\,\xi_k(x)\,d\mathcal{H}^{d-1},\ \ k \ge 2,$$

then

$$\|\varphi_\varepsilon\|^2_{L^2(\partial B)} = \sum_{k\ge 2} \alpha_k(\varepsilon)^2 \qquad \text{and} \qquad \|\nabla\varphi_\varepsilon\|^2_{L^2(B)} = \sum_{k\ge 2} \sigma_k(B)\,\alpha_k(\varepsilon)^2,$$

where for the second decomposition we used (5.6). By (5.31) and the definition of $Q(\varepsilon)$, we have

$$\begin{aligned}\left|\int_{\partial B} v_\varepsilon(x)^2 - 1\right| &\le \left|\int_{\partial B} \widetilde{u}_\varepsilon(x)^2 - \int_{\partial E_\varepsilon} u_\varepsilon(x)^2\right| + \left|\int_{\partial B} v_\varepsilon(x)^2 - \int_{\partial B} \widetilde{u}_\varepsilon(x)^2\right| \\ &\le |Q(\varepsilon)| + K\,\varepsilon^2,\end{aligned}$$

and since $\varphi_\varepsilon = v_\varepsilon$ on ∂B, the previous implies

$$\left|\|\varphi_\varepsilon\|^2_{L^2(\partial B)} - 1\right| \le |Q(\varepsilon)| + K\,\varepsilon^2.$$

In particular, we get

$$\left|\sum_{k=2}^{d+1} \alpha_k(\varepsilon)^2 - 1\right| \le \sum_{k\ge d+2} \alpha_k(\varepsilon)^2 + |Q(\varepsilon)| + K\varepsilon^2,$$

and multiplying both members by $\sigma_2(B)$ we have

$$\sigma_2(B)\left|\sum_{k=2}^{d+1} \alpha_k(\varepsilon)^2 - 1\right| \le \sigma_2(B) \sum_{k\ge d+2} \alpha_k(\varepsilon)^2 + c_1\,|Q(\varepsilon)| + K\varepsilon^2. \tag{5.43}$$

On the other hand, by (5.32) and (5.42) we have

$$\begin{aligned}\Big|\|\nabla\varphi_\varepsilon\|^2_{L^2(B)}-\sigma_2(B)\Big| &\le \Big|\|\nabla v_\varepsilon\|^2_{L^2(B)}-\sigma_2(B)\Big|+\Big|\|\nabla v_\varepsilon\|^2_{L^2(B)}-\|\nabla\varphi_\varepsilon\|^2_{L^2(B)}\Big|\\ &\le |\sigma_2(E_\varepsilon)-\sigma_2(B)|+|d(\varepsilon)|+C\,\varepsilon^2\\ &\le C\,\left(|d(\varepsilon)|+|Q(\varepsilon)|\right)+K\,\varepsilon^2,\end{aligned}$$

which can be rewritten as

$$\left|\sigma_2(B)\left(\sum_{k=2}^{d+1}\alpha_k(\varepsilon)^2-1\right)+\sum_{k\ge d+2}\sigma_k(B)\,\alpha_k(\varepsilon)^2\right|\le c_2\,\left(|d(\varepsilon)|+|Q(\varepsilon)|\right)+K\,\varepsilon^2,$$

and this implies

$$\sum_{k\ge d+2}\sigma_k(B)\,\alpha_k(\varepsilon)^2\le c_2\,\left(|d(\varepsilon)|+|Q(\varepsilon)|\right)+K\,\varepsilon^2+\sigma_2(B)\left|\sum_{k=2}^{d+1}\alpha_k(\varepsilon)^2-1\right|. \tag{5.44}$$

We can now combine (5.43) and (5.44), so to obtain

$$\sum_{k\ge d+2}\left(\sigma_k(B)-\sigma_2(B)\right)\alpha_k(\varepsilon)^2\le (c_1+c_2)\,\left(|d(\varepsilon)|+|Q(\varepsilon)|\right)+K\,\varepsilon^2.$$

Notice that

$$1-\frac{\sigma_2(B)}{\sigma_k(B)}>0,\qquad k\ge d+2,$$

since $\sigma_2(B)$ has multiplicty d and this forms a non-decreasing sequence, then from the previous we can infer

$$\sum_{k\ge d+2}\sigma_k(B)\,\alpha_k(\varepsilon)^2\le C\,\left(|d(\varepsilon)|+|Q(\varepsilon)|\right)+K\,\varepsilon^2,$$

possibly with different constants C and K, depending on the spectral gap $\sigma_{d+2}(B)-\sigma_2(B)$, but not on ε. If we set

$$\xi_\varepsilon=\sum_{k=2}^{d+1}\alpha_k(\varepsilon)\xi_k,$$

we have

$$\|\varphi_\varepsilon-\xi_\varepsilon\|^2_{L^2(\partial B)}\le\sigma_{d+2}(B)\,\|\nabla v_\varepsilon-\nabla\xi_\varepsilon\|^2_{L^2(B)}$$

and

$$\|\nabla\varphi_\varepsilon-\nabla\xi_\varepsilon\|^2_{L^2(B)}=\sum_{k\ge d+2}\sigma_k(B)\,\alpha_k(\varepsilon)^2\le C\,\left(|d(\varepsilon)|+|Q(\varepsilon)|\right)+K\,\varepsilon^2,$$

which yields

$$\|\varphi_\varepsilon-\xi_\varepsilon\|_{W^{1,2}(B)}\le C\,\sqrt{|d(\varepsilon)|+|Q(\varepsilon)|}+K\,\varepsilon,$$

thanks to the fact that $u \mapsto \|u\|_{L^2(\partial B)} + \|\nabla u\|_{L^2(B)}$ is equivalent to the standard norm of $W^{1,2}(B)$. Finally, it is only left to observe that

$$\|v_\varepsilon - \xi_\varepsilon\|_{W^{1,2}(B)} \leq \|\varphi_\varepsilon - \xi_\varepsilon\|_{W^{1,2}(B)} + \|v_\varepsilon - \varphi_\varepsilon\|_{W^{1,2}(B)},$$

thus we have obtained (5.39). $\square$

We show how the previous Sobolev estimate (5.39) can be enhanced, replacing the $W^{1,2}(B)$ norm with the C^1 one.

Lemma 5.19. *For every $\varepsilon \ll 1$, there exists an eigenfunction ξ_ε relative to $\sigma_2(B)$ such that*

$$\|v_\varepsilon - \xi_\varepsilon\|_{C^1(\overline{B})} \leq C_7 \sqrt{|d(\varepsilon)| + |Q(\varepsilon)|} + C_8\, \varepsilon, \qquad \text{for every } \varepsilon \ll 1, \tag{5.45}$$

for some positive constants C_7, C_8 independent of ε.

Proof. First of all, let us write down the the Neumann boundary value problems solved by v_ε and ξ_ε: these are given respectively by

$$\begin{cases} \Delta v_\varepsilon = f_\varepsilon, & \text{in } \partial B \\ \langle \nabla v_\varepsilon, \nu\rangle = \sigma_2(B)\, g_\varepsilon, & \text{on } \partial B \end{cases} \qquad \text{and} \qquad \begin{cases} \Delta \xi_\varepsilon = 0, & \text{in } \partial B \\ \langle \nabla \xi_\varepsilon, \nu\rangle = \sigma_2(B)\, \xi_\varepsilon, & \text{on } \partial B \end{cases}$$

where

$$f_\varepsilon(x) = \Delta \widetilde{u}_\varepsilon(x), \qquad x \in B,$$

and the boundary value g_ε is given by (recall that $\nabla v_\varepsilon = \nabla \widetilde{u}_\varepsilon$)

$$\begin{aligned} g_\varepsilon(x) = v_\varepsilon(x) &+ \big[u_\varepsilon(\phi_\varepsilon(x)) - v_\varepsilon(x)\big] \\ &+ \left(\frac{\sigma_2(E_\varepsilon)}{\sigma_2(B)} - 1\right) u_\varepsilon(\phi_\varepsilon(x)) \\ &+ \frac{1}{\sigma_2(B)} \langle \nabla \widetilde{u}_\varepsilon(x) - \nabla u_\varepsilon(\phi_\varepsilon(x)), \nu_{E_\varepsilon}(\phi_\varepsilon(x))\rangle \\ &+ \frac{1}{\sigma_2(B)} \langle \nabla \widetilde{u}_\varepsilon(x), \nu(x) - \nu_{E_\varepsilon}(\phi_\varepsilon(x))\rangle =: v_\varepsilon(x) + \sum_{i=1}^{4} g_{\varepsilon,i}(x) \qquad x \in \partial B. \end{aligned}$$

Thus in order to gain informations on the distance between v_ε and ξ_ε, it suffices to estimate f_ε and the boundary term $g_\varepsilon - \xi_\varepsilon$: indeed, by standard Elliptic Regularity (see [109, Proposition 7.5]) and by the triangular inequality, for every $k \geq 1$ we have

$$\begin{aligned} \|v_\varepsilon - \xi_\varepsilon\|_{W^{k,2}(B)} &\leq C\Big(\|v_\varepsilon - \xi_\varepsilon\|_{L^2(B)} + \|g_\varepsilon - \xi_\varepsilon\|_{W^{k-3/2,2}(\partial B)} + \|f_\varepsilon\|_{W^{k-2,2}(B)}\Big) \\ &\leq C\Big(\|v_\varepsilon - \xi_\varepsilon\|_{L^2(B)} + \|v_\varepsilon - \xi_\varepsilon\|_{W^{k-3/2,2}(\partial B)} \\ &+ \sum_{i=1}^{4} \|g_{\varepsilon,i}\|_{W^{k-3/2,2}(\partial B)} + \|f_\varepsilon\|_{W^{k-2,2}(B)}\Big). \end{aligned} \tag{5.46}$$

The first term on the right-hand side can be easily estimated as follows

$$\|v_\varepsilon - \xi_\varepsilon\|_{L^2(B)} \le \|v_\varepsilon - \xi_\varepsilon\|_{W^{1,2}(B)} \le C\,\sqrt{|d(\varepsilon)| + |Q(\varepsilon)|} + K\,\varepsilon,$$

where we used (5.39) in the second inequality: then to obtain (5.45) it suffices to prove that

$$\|v_\varepsilon - \xi_\varepsilon\|_{W^{k-3/2,2}(\partial B)} \le C\,\sqrt{|d(\varepsilon)| + |Q(\varepsilon)|} + K\,\varepsilon, \tag{5.47}$$

$$\sum_{i=1}^{4} \|g_{\varepsilon,i}\|_{W^{k-3/2,2}(\partial B)} \le C\,\varepsilon, \tag{5.48}$$

$$\|f_\varepsilon\|_{W^{k-2,2}(B)} \le C\,\varepsilon, \tag{5.49}$$

with $k = [d/2] + 2$. Indeed, using the Sobolev Imbedding Theorem, this would yield

$$\|v_\varepsilon - \xi_\varepsilon\|_{C^1(\overline{B})} \le C\,\|v_\varepsilon - \xi_\varepsilon\|_{W^{[d/2]+2,2}(B)},$$

and combining (5.46) and (5.47)–(5.49), we would conclude the proof.

We now begin to estimate the terms $g_{\varepsilon,i}$: recalling that $u_\varepsilon \circ \phi_\varepsilon = \widetilde{u}_\varepsilon \circ \phi_\varepsilon$ on ∂B and using (5.26) and the uniform estimates on $\widetilde{u}_\varepsilon$, we get that

$$\|g_{\varepsilon,1}\|_{W^{k-3/2,2}(\partial B)} \le \|\widetilde{u}_\varepsilon \circ \phi_\varepsilon - \widetilde{u}_\varepsilon\|_{W^{k-3/2,2}(\partial B)} + \delta\,\left(\mathcal{H}^{d-1}(\partial B)\right)^{1/2} = O(\varepsilon).$$

For the second, we use (5.22) and Lemma 5.15, to obtain

$$\|g_{\varepsilon,2}\|_{W^{k-3/2,2}(\partial B)} \le K\,\frac{|\sigma_2(E_\varepsilon) - \sigma_2(B)|}{\sigma_2(B)} \le K\,\left(|d(\varepsilon)| + |Q(\varepsilon)|\right),$$

possibly with a different constant K, still not depending on ε. For the the third term, we just use a triangular inequality and the uniform estimates (5.22), (5.24)

$$\begin{aligned}
\|g_{\varepsilon,3}\|_{W^{k-3/2,2}(\partial B)} &\le C\,\|\nabla\widetilde{u}_\varepsilon - \nabla u_\varepsilon \circ \phi_\varepsilon\|_{W^{k-3/2,2}(\partial B)} \\
&\le C\,\|\nabla\widetilde{u}_\varepsilon - \nabla(u_\varepsilon \circ \phi_\varepsilon)\|_{W^{k-3/2,2}(\partial B)} \\
&+ C\,\|\nabla(u_\varepsilon \circ \phi_\varepsilon) - \nabla u_\varepsilon \circ \phi_\varepsilon\|_{W^{k-3/2,2}(\partial B)} \le C\,\varepsilon,
\end{aligned}$$

again thanks to the fact that $\widetilde{u}_\varepsilon \circ \phi_\varepsilon = u_\varepsilon \circ \phi_\varepsilon$ on ∂B. Finally, still using the uniform estimates (5.24) and (5.22), we have

$$\|g_{\varepsilon,4}\|_{W^{k-3/2,2}(\partial B)} \le C\,\|\nu_B - \nu_{E_\varepsilon} \circ \phi_\varepsilon\|_{W^{k-3/2,2}(\partial B)}.$$

The term $\nu_{E_\varepsilon} \circ \phi_\varepsilon$ can be explicitly written as

$$\nu_{E_\varepsilon}(\phi_\varepsilon(x)) = \frac{(1 + \varepsilon\,\psi(x))\,\nu_B(x) - \varepsilon\,\nabla_\tau\psi(x)}{\sqrt{(1 + \varepsilon\,\psi(x))^2 + \varepsilon^2\,|\nabla_\tau\psi(x)|^2}}, \qquad x \in \partial B,$$

In this way

$$\nu_B(x) - \nu_{E_\varepsilon}(\phi_\varepsilon(x)) = \nu_B(x) \left(1 - \frac{1+\varepsilon\,\psi(x)}{\sqrt{(1+\varepsilon\,\psi(x))^2 + \varepsilon^2\,|\nabla_\tau \psi(x)|^2}} \right) - \varepsilon\,\frac{\nabla_\tau \psi(x)}{\sqrt{(1+\varepsilon\,\psi(x))^2 + \varepsilon^2\,|\nabla_\tau \psi(x)|^2}}.$$

Then observe that

$$\varphi_1(x) = 1 - \frac{1+\varepsilon\,\psi(x)}{\sqrt{(1+\varepsilon\,\psi(x))^2 + \varepsilon^2\,|\nabla_\tau \psi(x)|^2}}, \qquad x \in \partial B,$$

and

$$\varphi_2(x) = \varepsilon\,\frac{\nabla_\tau \psi(x)(x)}{\sqrt{(1+\varepsilon\,\psi(x))^2 + \varepsilon^2\,|\nabla_\tau \psi(x)|^2}}, \qquad x \in \partial B,$$

are two C^∞ applications on ∂B, such that for every $m \in \mathbb{N}$

$$\|\varphi_i\|_{C^m(\partial B)} \le C_m\,\varepsilon, \qquad i = 1, 2,$$

where C_m is a constant depending on the $C^{m+1}(\partial B)$ norm of ψ, but not on ε. This permits to conclude the estimate on $g_{\varepsilon,4}$: we finally have

$$\|g_{\varepsilon,4}\|_{W^{k-3/2,2}(\partial B)} \le C\,\|\nu_B - \nu_{E_\varepsilon} \circ \phi_\varepsilon\|_{W^{k-3/2,2}(\partial B)} \le C\,\varepsilon,$$

so collecting all these estimates we end up with (5.48), for any k.

Concerning the term f_ε, we have already seen that $\|f_\varepsilon\|_{L^\infty(B)} \le C\,\varepsilon$: repeating the argument (5.40) for every derivative and using that the $C^{[d/2]+1}$ norm of f_ε is uniformly bounded[2], we obtain

$$\|f_\varepsilon\|_{C^{k-2}(\overline{B})} \le C\,\varepsilon,$$

for $k = [d/2] + 2$, so that the $W^{k-2,2}$ norm is estimated as follows

$$\|f_\varepsilon\|_{W^{k-2,2}(B)} \le C\,\|f_\varepsilon\|_{C^{k-2}(\overline{B})} \le C\,\varepsilon.$$

Finally, we aim to prove (5.47): by the trace inequality and (5.39) we have

$$\|v_\varepsilon - \xi_\varepsilon\|_{W^{1/2,2}(\partial B)} \le C\,\|v_\varepsilon - \xi_\varepsilon\|_{W^{1,2}(B)} \le C\,\sqrt{|d(\varepsilon)| + |Q(\varepsilon)|} + K\,\varepsilon.$$

A first application of (5.46) with $k = 2$, gives

$$\|v_\varepsilon - \xi_\varepsilon\|_{W^{2,2}(E)} \le C\,\sqrt{|d(\varepsilon)| + |Q(\varepsilon)|} + K\,\varepsilon,$$

and applying the trace inequality we obtain

$$\|v_\varepsilon - \xi_\varepsilon\|_{W^{3/2,2}(\partial B)} \le C\,\|v_\varepsilon - \xi_\varepsilon\|_{W^{2,2}(E)} \le C\,\sqrt{|d(\varepsilon)| + |Q(\varepsilon)|} + K\,\varepsilon,$$

thus the validity of (5.47) with $k = 3$. Finitely many repetitions of the previous argument give (5.47) with $k = [d/2] + 2$ and thus the proof is concluded. □

[2]This is the reason why we choose $\widetilde{u}_\varepsilon$ to be a C^{k+1} extension of u_ε with $k = [d/2] + 2$.

5.4.4 Step 4: conclusion

Thanks to Lemma 5.15, we know that

$$|\sigma_2(B) - \sigma_2(E_\varepsilon)| \leq C_1 \left(|d(\varepsilon)| + |Q(\varepsilon)|\right) + C_2\,\varepsilon^2.$$

First applying Lemma 5.19 and then Lemma 5.16 with $\omega(\varepsilon) = C_7\sqrt{|d(\varepsilon)| + |Q(\varepsilon)|} + C_8\,\varepsilon$, we obtain

$$|d(\varepsilon)| + |Q(\varepsilon)| \leq \widetilde{C}\,\varepsilon\,\sqrt{|d(\varepsilon)| + |Q(\varepsilon)|} + \widetilde{C}\,\varepsilon^2. \tag{5.50}$$

Let us set

$$t(\varepsilon) = \frac{\varepsilon}{\sqrt{|d(\varepsilon)| + |Q(\varepsilon)|}},$$

then from (5.50) we can infer

$$\frac{1}{\widetilde{C}} \leq t(\varepsilon) + t(\varepsilon)^2,$$

which easily implies that $t(\varepsilon) \geq c$ for some costant $c > 0$, i.e.

$$\sqrt{|d(\varepsilon)| + |Q(\varepsilon)|} \leq \frac{\varepsilon}{c}.$$

A further application of Lemma 5.15 finally shows that

$$|\sigma_2(B) - \sigma_2(E_\varepsilon)| \leq C\,\varepsilon^2,$$

possibly with a different constant C, still independent of ε. Inserting this into (5.35), we can conclude the proof of Theorem 5.9.

Chapter 6

A reduction theorem for the stability of Gagliardo-Nirenberg-Sobolev inequalities

6.1 Introduction

In this chapter we address the problem of the stability of the Gagliardo-Nirenberg-Sobolev inequalities (briefly: GNS). To lighten the notation, among the whole chapter we will drop the dependence on the set of integration when it is $\mathbb{R}^d$, and, analogously, we will always write $\|\cdot\|_p$ instead of $\|\cdot\|_{L^p(\mathbb{R}^d)}$ to indicate the L^p norm of a function on $\mathbb{R}^d$.

The sharp Gagliardo-Nirenberg-Sobolev inequality in $\mathbb{R}^d$, with $d \geq 2$, takes the form, for a suitable $G = G(d, p, s, q) > 0$,

$$G\|u\|_q \leq \|\nabla u\|_p^\theta \|u\|_s^{1-\theta} \tag{6.1}$$

where the parameters s, q, p satisfies

$$\begin{aligned} &1 < p < d, \\ &1 \leq s < q < p^\star, \qquad p^\star = \frac{dp}{d-p} \qquad \text{and} \\ &\frac{\theta}{p^\star} + \frac{1-\theta}{s} = \frac{1}{q}, \end{aligned} \tag{6.2}$$

and where u is taken in $D^{p,s}(\mathbb{R}^d)$, that is the closure of $C_c^\infty(\mathbb{R}^d)$ under the norm $\|u\|_{D^{p,s}} = \|\nabla u\|_p + \|u\|_s$. Inequality (6.1) can be derived by combining the interpolation inequality between the L^s, L^q and $L^{p^\star}$ norms on $\mathbb{R}^d$ with the Sobolev inequality:

$$S(n,p)\|u\|_{p^\star} \leq \|\nabla u\|_p. \tag{6.3}$$

Here $S(n,p)$ is the optimal Sobolev constant, namely

$$S(n,p) = \inf\left\{ \frac{\|\nabla u\|_p}{\|u\|_{p^\star}} : \quad u \in W^{1,p}(\mathbb{R}^d) \setminus \{0\} \right\}. \tag{6.4}$$

Explicit formulas for $S(n,p)$ and minimizers in (6.4) are known since the work of Aubin [6] and Talenti [109]. We stress that the same result *is not available* for the optimal constant and functions in (6.1) with the exception of the one-parameter family

$$p = 2, \quad q = 2t, \quad s = t+1, \tag{6.5}$$

see [48], [43]. What can be said in full generality, is that optimal functions in (6.1) exist and, according to the Pólya-Szegö inequality (see for instance [86, Chapter 3]), are non-negative, radially symmetric functions with decreasing profile (see [23], [106]). Furthermore, they are unique up to translations, rescaling and multiplication by (non-zero) constants. Now we want to address the quantitative stability of inequality (6.1). To state rigorously this problem, we introduce the (GNS) *deficit* $\delta(u)$ of a function $u \in D^{p,s}(\mathbb{R}^d)$ as

$$\delta(u) = \frac{\|\nabla u\|_p^\theta \|u\|_s^{1-\theta}}{G\|u\|_q} - 1, \tag{6.6}$$

and notice that inequality (6.1) reads, in terms of $\delta(u)$, as

$$\delta(u) \geq 0. \tag{6.7}$$

Then, for a quantitative version of the GNS inequality, we mean an improvement of inequality (6.7) of the form

$$\delta(u) \geq \kappa_0 \mathrm{dist}(\mathrm{u}, \mathrm{M})^{\alpha_0}, \tag{6.8}$$

κ_0, α_0 are positive constants independent of u and $\mathrm{dist}(\cdot, \mathrm{M})$ indicates an appropriate distance from M, the set of the optimizers for (6.1). The concept of distance we will adopt is the following (note the analogy with the Fraenkel asymmetry defined in Chapter 4 and Chapter 5):

$$\lambda(u) = \inf\left\{ \frac{\|u-v\|_q^q}{\|u\|_q^q} : v \text{ is optimal for (6.1)}, \quad \|v\|_q = \|u\|_q \right\}. \tag{6.9}$$

Results in this direction have been recently obtained with some ad-hoc techniques valid for special classes of parameters among those in (6.2), see [34] and [35]. In particular, the parameters considered in [34] are those introduced in (6.5) (although the authors focus, because of their later applications, just on the particular case $p = 2, q = 6, s = 4$), and the knowledge of minimizers is exploited in a crucial way. In [35] the authors address a class of parameters, $p = s = 2, q > 2$, for which the minimizers are not explicitly known and they follow a strategy developed by Bianchi and Egnell in [12], which heavily relies on the Hilbertian structure corresponding to $p = 2$ and seems complicate to generalize. Thus the above techniques seem adaptable to prove the stability of the GNS inequalities only for a particular class of parameters. In this chapter, thanks to a general symmetrization technique introduced by Cianchi, Fusco, Maggi and Pratelli in [39], we are able to prove a reduction principle which is valid for the *whole class* of parameters (6.2). Namely we reduce the problem to that of showing the stability just for radial symmetric functions, reducing the complexity of the task in its generality from a $d-$dimensional to a $1-$dimensional problem. Although this does not solve completely the issue, it offers a more simple way to attack it.

The main result we shall prove is the following

Theorem 6.1. *Consider the functionals $\delta(\cdot)$ and $\lambda(\cdot)$ defined in (6.6) and (6.9) respectively. Suppose that there exist two positive constant k_0 and α_0 such that the stability inequality*

$$\delta(u) \geq \kappa_0 \lambda(u)^{\alpha_0} \tag{6.10}$$

holds for any radial non-increasing function $u \in D^{p,s}(\mathbb{R}^d)$. Then there are two positive constants k_1 and α_1 such that the inequality

$$\delta(u) \geq \kappa_1 \lambda(u)^{\alpha_1} \tag{6.11}$$

holds true for any function in $D^{p,s}(\mathbb{R}^d)$.

The first step needed to prove Theorem 6.1 is a sort of *continuity at* 0 of the asymmetry λ with respect to the deficit δ. Namely we will prove, in Corollary 6.5 of Section 6.2, that given a sequence of functions $(u_h)_h$ such that $\delta(u_h)$ converges to 0 as h goes to infinity, then also $\lambda(u_h)$ converges to 0. This will be done by means of the compactness Theorem 6.2, where it is proved that a sequence of functions whose deficits are infinitesimal, up to be (suitably) rescaled and translated, is compact in $L^q(\mathbb{R}^d)$. Then we will pass to the proof of Theorem 6.1. Its proof is done in Section 6.3 and Section 6.4, each of them devoted to obtain a simplification of the class of the functions we deal with. In particular in Section 6.3 we prove a further reduction step, stating that if the stability inequality (6.10) holds for radial decreasing functions, it holds as well for $d-$symmetric functions, that is functions which are symmetric with respect to d orthogonal hyperplanes. More precisely we prove that if there exist positive constants κ_0 and α_0 such that for any radial decreasing function $u \in D^{p,s}(\mathbb{R}^d)$ inequality (6.10) holds true, then there exist positive constants $\widetilde{\kappa}_0$ and $\widetilde{\alpha}_0$ depending on d, p, q and s such that for any $d-$symmetric function $u \in D^{p,s}(\mathbb{R}^d)$ we have

$$\delta(u) \geq \widetilde{\kappa}_0 \lambda(u)^{\widetilde{\alpha}_0}. \tag{6.12}$$

Eventually, in Section 6.4, we prove that to get the stability of GNS inequality, it is not restrictive to consider only $d-$symmetric functions. Namely we prove the existence of two positive constants κ_2 and α_2 such that for every function $u \in D^{p,s}(\mathbb{R}^d)$ there exists a $d-$symmetric function $\bar{u}$ such that the following reduction inequalities hold true

$$\lambda(u) \leq \kappa_2 \lambda(\bar{u})^{\alpha_2}, \qquad \delta(\bar{u}) \leq \kappa_2 \delta(u)^{\alpha_2}. \tag{6.13}$$

It is then easy to see that combining (6.12) with the reduction inequalities (6.13) we prove the claim of Theorem 6.1.

6.2 Continuity of λ with respect to δ via a compactness theorem

We devote this section to the proof of the following Theorem.

Theorem 6.2. *Let $(u_h)_h$ be a sequence in $D^{p,s}(\mathbb{R}^d)$ such that $\delta(u_h)$ converges to 0 as $h \to \infty$. Then there exist $(\lambda_h)_h \subset (0, +\infty)$ and $(x_h)_h \subset \mathbb{R}^d$ such that the rescaled sequence*

$$w_h(x) = \tau_{\lambda_h} u_h(x - x_h) = \lambda_h^{d/q} u_h(\lambda_h(x - x_h)) \tag{6.14}$$

satisfies:

(i) $\delta(w_h) = \delta(u_h)$; $\lambda(w_h) = \lambda(u_h)$;

(ii) $\|w_h\|_q = \|u_h\|_q \quad \forall h \in N$;

(iii) there exist constants $C_0, C_1 > 0$, depending only on d, p, q, s, such that

$$\frac{1}{C_0} \leq \|\nabla w_h\|_p \leq C_0, \qquad \frac{1}{C_1} \leq \|w_h\|_s \leq C_1;$$

(iv) $w_h \to w$ *strongly in $L^q(\mathbb{R}^d)$ as $h \to \infty$ with $w \in D^{p,s}(\mathbb{R}^d)$.*

In this chapter the parameters d, p, s, q and θ are always intended to satisfy conditions (6.11). We start our analysis defining the following functionals:

$$G(u) = \|\nabla u\|_p^{\theta} \|u\|_s^{1-\theta}, \qquad F(u) = \int |\nabla u|^p + \int |u|^s \tag{6.15}$$

defined for $u \in D^{p,s}(\mathbb{R}^d)$. Given $m > 0$, we consider the infimum problems:

$$\psi(m) = \inf\{G(u) : \|u\|_q^q = m\}, \qquad \varphi(m) = \inf\{F(u) : \|u\|_q^q = m\}. \tag{6.16}$$

Lemma 6.3. *There exists $\eta_0 = \eta_0(d, p, q, s) > 0$ with the following property. For any $u \in D^{p,s}(\mathbb{R}^d)$ there exists $\lambda > 0$ such that, if $\tau_\lambda(u) = \lambda^{d/q} u(\lambda x)$, then*

$$F(\tau_\lambda u) = \eta_0 G(u)^k \qquad \textit{where} \qquad k = q \cdot \frac{np + ps - ns}{np + pq - ns} < q. \tag{6.17}$$

Proof. For $u \in D^{p,s}(\mathbb{R}^d)$ we have

$$\|\tau_\lambda u\|_q^q = \|u\|_q^q, \quad \|\tau_\lambda u\|_s^s = \lambda^{-d+ds/q} \|u\|_s^s \quad \text{and} \quad \|\nabla \tau_\lambda u\|_p^p = \lambda^{-d+p+\frac{dp}{q}} \|\nabla u\|_p^p; \tag{6.18}$$

hence

$$F(\tau_\lambda u) = \lambda^a A + \lambda^b B = f(\lambda)$$

where $A = \|\nabla u\|_p^p$, $B = \|u\|_s^s$, $a = -n + p + np/q$ and $b = -n + ns/q$. The function f attains its minimum at

$$\lambda_m = \Big(-\frac{b}{a}\Big)^{\frac{1}{a-b}} \Big(\frac{B}{A}\Big)^{\frac{1}{a-b}}$$

with the value

$$f(\lambda_m) = \eta_0 \Big(A^{\theta/p}\Big)^{q\nu} \Big(B^{(1-\theta)/s}\Big)^{q\nu}$$

where $\eta_0 = \eta_0(d, p, s, q)$ and $\nu = \frac{dp+ps-ds}{dp+pq-ds}$, that is, the claim of the lemma. □

Lemma 6.4. *There exists $\alpha = \alpha(d, p, q, s) \in (0, 1)$ such that*

$$\varphi(m) = m^\alpha \varphi(1) \qquad \forall m > 0.$$

In particular φ is strictly super-additive in $(0, 1)$.

Proof. Let $u \in D^{p,s}(\mathbb{R}^d)$ be such that $\|u\|_q^q = m$. Let $v = u/m^{1/q}$ and set $\tau_\lambda v(x) = \lambda^{d/q} v(\lambda x)$. Setting $\lambda = m^{\frac{p-s}{dp+pq-ds}}$ we get, after some calculation analogous to those in Lemma 6.3,

$$F(\tau_\lambda v) = m^{-\alpha} F(u)$$

where $\alpha = \frac{dp+ps-ds}{dp+pq-ds}$. If we now consider a minimizing sequence $(u_h)_h$ for $\varphi(m)$ such that $\|u\|_q^q = m$ and as above $v_h = u_h/m^{1/q}$, we obtain

$$\varphi(m) = \lim_{h\to\infty} F(u_h) = m^\alpha \lim_{h\to\infty} F(\tau_\lambda v_h) \geq m^\alpha \varphi(1).$$

The opposite inequality can be proved with an analogous argument considering a sequence $(v_h)_h$ minimizing for $\varphi(1)$, and setting $u_h = m^{1/q} v_h$. □

We pass now to the proof of Theorem 6.2.

Proof of Theorem 6.2. Let φ be the function defined in (6.16). We recall the Lions's Concentration-Compactness Theorem (see [89] or [107, Theorem 4.3 and Theorem 4.8]): given a non-negative sequence $(\rho_h)_h$ in $L^1(\mathbb{R}^d)$ with fixed L^1 norm, say 1, there is a subsequence $(\rho_{h_k})_k$ which satisfies one of the following properties:

(1) (concentration) there exists a sequence $(y_k)_k \in \mathbb{R}^d$ such that for every $\varepsilon > 0$ there exists $R \in (0, \infty)$ such that $\int_{B_R(y_k)} \rho_{h_k} \geq 1 - \varepsilon \quad$ for every $k \in \mathbb{N}$

(2) (vanishing) $\lim_{k\to\infty} \sup_{y\in\mathbb{R}^d} \int_{B_R(y)} \rho_{h_k} = 0 \quad \forall \quad 0 < R < \infty$

(3) (dichotomy) there exist $\alpha \in (0, 1)$ and two sequences $R_h \to +\infty$ and $y_h \in \mathbb{R}^d$ such that $\int_{B_{R_{h_k}}(y_{h_k})} \rho_{h_k} \to \alpha$ and $\int_{\mathbb{R}^d \setminus B_{2R_{h_k}}(y_{h_k})} \rho_{h_k} \to 1 - \alpha$.

Let $u_h \in D^{p,s}(\mathbb{R}^d)$ be such that $\delta(u_h) \to 0$. We can suppose that $\|u_h\|_q = 1$. Consider $w_h(x - x_h) = \tau_{\lambda_h} u_h(x - x_h)$ where $\lambda_h > 0$ is defined, for every h, as in Lemma 6.3. Thanks to formulas (6.18), each function w_h satisfies statements (i) and (ii) of the theorem. Moreover, Lemma 6.3 provides us two positive constants η_0 and k such that $F(w_h) = \eta_0 G(u_h)^k$. Since $\delta(w_h) + 1 = G(w_h)/G$, and $\delta(w_h)$ tends to 0, it follows that the sequence $F(w_h) \to G^k = \varphi(1)$ as h tends to ∞, where φ is defined in (6.16). In particular the sequence $(w_h)_h$ must satisfy statement (iii) of the theorem. In order to prove point *(iv)*, we apply the Concentration-Compactness Theorem to the sequence $(|w_h|^q)_h$ aiming to exclude cases (2) and (3).

If the sequence vanishes, by Hölder inequality we would get vanishing also for the sequence $(|w_h|^s)_h$, since $s < q$. It is not difficult to see that these conditions, together to the equiboundedness of $(w_h)_h$ in $D^{p,s}(R^n)$, guarantees that $w_h \to 0$ strongly in L^q as $h \to \infty$ (see for istance [90, Lemma I.1]). Since $\|w_h\|_q = \|u_h\|_q = 1$, we would get a contradiction. So we can exclude case (2).

The dichotomy case is more complicated and requires a longer analysis. Suppose to have dichotomy for the sequence $(|w_h|^q)_h$. Then there exist $\alpha \in (0,1)$ and a sequence of positive numbers $R_h \to \infty$ as $h \to \infty$ such that

$$\int_{B_{R_h}} |w_h|^q \to \alpha; \quad \int_{B^c_{2R_h}} |w_h|^q \to 1-\alpha; \quad \int_{B_{2R_h}\setminus B_{R_h}} |w_h|^q \to 0.$$

Let $f \in C^1_c(B(0,2);[0,1])$ such that $f = 1$ on $B(0,1)$ and consider $f_h(x) = f(x/R_h) \in C^1_c(B(0,2R_h);[0,1])$. Choose also f such that $|\nabla f_h| \le C/R_h$ for some $C > 0$. Then we have

$$\begin{aligned}
F(w_h) &= \int |\nabla w_h|^p + \int |w_h|^s \\
&= \int_{B_{R_h}} |\nabla(f_h w_h)|^p + \int_{B^c_{2R_h}} |\nabla((1-f_h)w_h)|^p + \int_{B_{2R_h}\setminus B_{R_h}} |\nabla w_h|^p \\
&\quad + \int_{B_{R_h}} |f_h w_h|^s + \int_{B^c_{2R_h}} |(1-f_h)w_h|^s + \int_{B_{2R_h}\setminus B_{R_h}} |w_h|^s \\
&= \int |\nabla(f_h w_h)|^p + \int |\nabla[(1-f_h)w_h]|^p \\
&\quad + \int_{B_{2R_h}\setminus B_{R_h}} \Big[|\nabla w_h|^p - |\nabla(f_h w_h)|^p - |\nabla[(1-f_h)w_h]|^p\Big] \\
&\quad + \int \Big[|f_h w_h|^s + |(1-f_h)w_h|^s\Big] \\
&\quad + \int_{B_{2R_h}\setminus B_{R_h}} \Big[|w_h|^s - f_h^s|w_h|^s - (1-f_h)^s|w_h|^s\Big].
\end{aligned}$$

Since f_h assume values only in $[0,1]$, the last integral is non-negative. Neglecting this quantity and reordering the terms, we get

$$\begin{aligned} F(w_h) &\geq F(f_h w_h) + F((1-f_h)w_h) - \epsilon(h) \\ &\geq \varphi(\|f_h w_h\|_q^q) + \varphi(\|(1-f_h)w_h\|_q^q) - \epsilon(h), \end{aligned} \tag{6.19}$$

where

$$\epsilon(h) = \int_{B_{2R_h}\setminus B_{R_h}} \Big[|\nabla(f_h w_h)|^p + |\nabla[(1-f_h)w_h]|^p - |\nabla w_h|^p \Big].$$

We claim that the error $\epsilon(h) \to 0$ is controlled from above by a quantity which converges to 0 as $h \to \infty$. Indeed

$$\begin{aligned} \epsilon(h) &= \int_{B_{2R_h}\setminus B_{R_h}} \Big[|\nabla(f_h w_h)|^p + |\nabla[(1-f_h)w_h)]|^p - |\nabla w_h|^p \Big] \\ &= \int_{B_{2R_h}\setminus B_{R_h}} \Big[|f_h \nabla w_h + w_h \nabla f_h|^p + |(1-f_h)\nabla w_h - w_h \nabla f_h|^p - |\nabla w_h|^p \Big] \\ &\leq \int_{B_{2R_h}\setminus B_{R_h}} \Big[\Big(|f_h \nabla w_h + w_h \nabla f_h| + |(1-f_h)\nabla w_h - w_h \nabla f_h| \Big)^p - |\nabla w_h|^p \Big] \\ &\leq \int_{B_{2R_h}\setminus B_{R_h}} \Big[\Big(|\nabla w_h| + 2|w_h||\nabla f_h| \Big)^p - |\nabla w_h|^p \Big] \\ &\leq \int_{B_{2R_h}\setminus B_{R_h}} \Big[\varepsilon C_p |\nabla w_h|^p + C_\varepsilon |\nabla f_h|^p |w_h|^p \Big] \\ &\leq \varepsilon C_p \sup_{h\in\mathbb{N}} \int |\nabla w_h|^p + C_\varepsilon \int_{B_{2R_h}\setminus B_{R_h}} |\nabla f_h|^p |w_h|^p. \end{aligned} \tag{6.20}$$

where the first inequality is due to the super additivity of the map $t \mapsto t^p$ on $\mathbb{R}_+$, the second to the triangle inequality and the third one is the Young inequality of (suitable) parameters $\varepsilon > 0$ and C_ε. We need to estimate the quantity

$$g(h) = \int_{B_{2R_h}\setminus B_{R_h}} |\nabla f_h|^p |w_h|^p.$$

If $p > s$ then interpolating the L^p norm of the w_h's between the L^s norm and the $L^{p^\star}$ norm and recalling that $|\nabla f_h| \leq C/R_h$, we get

$$\int_{B_{2R_h}\setminus B_{R_h}} |\nabla f_h|^p |w_h|^p \leq \frac{C^p}{R_h^p} \|w_h\|_{p^\star}^{p\theta} \|w_h\|_s^{p(1-\theta)}.$$

Since we already know that w_h satisfies the statement (iii) of the theorem, we get that $g(h) \to 0$ as $h \to \infty$. If $s \geq p$ we divide $g(h)$ into two terms:

$$g(h) = \int_{(B_{2R_h}\setminus B_{R_h})\cap\{w_h\geq 1\}} |\nabla f_h|^p |w_h|^p + \int_{(B_{2R_h}\setminus B_{R_h})\cap\{w_h< 1\}} |\nabla f_h|^p |w_h|^p = g_1(h) + g_2(h).$$

Since $q > p$ we have

$$g_1(h) \leq \frac{C^p}{R_h^p} \int_{B_{2R_h} \setminus B_{R_h}} |w_h|^q \leq \frac{C^p}{R_h^p}$$

and so $g_1(h) \to 0$ as $h \to \infty$. Moreover, by Hölder inequality of parameter $p^\star/p$, we get

$$g_2(h) \leq \frac{C^p}{R_h^p} \left(\int_{(B_{2R_h} \setminus B_{R_h}) \cap \{w_h < 1\}} |w_h|^{p^\star} \right)^{p/p^\star} |(B_{2R_h} \setminus B_{R_h})|^{\frac{1}{(p/p^\star)'}}$$

where

$$(p/p^\star)' = \frac{p/p^\star}{-1 + p/p^\star} = n/p.$$

Since $p^\star > q$, we obtain

$$g_2(h) \leq (\omega_d(2^d - 1))^{p/d} C^p \left(\int_{(B_{2R_h} \setminus B_{R_h}) \cap \{w_h < 1\}} |w_h|^q \right)^{p/p^\star}$$

where ω_d is the measure of the unit ball of $\mathbb{R}^d$. So also g_2 converges to 0 as $h \to \infty$. Thus, passing to the limit in (6.20), first in $h \to \infty$ and then in $\varepsilon \to 0$ we obtain that $\epsilon(h) \to 0$. Since $\|w_h f_h\|_q^q$ and $\|(1 - f_h)w_h\|_q^q$ converges respectively to λ and $1 - \lambda$, we can conclude thanks to Lemma 6.4 that

$$\varphi(1) \geq \varphi(\lambda) + \varphi(1 - \lambda) > \varphi(1)$$

obtaining a contradiction.
So we can exclude also the dichotomy phenomenon. Since w_h is equibounded in $L^q(\mathbb{R}^d)$, we can consider its weak-L^q limit w. This is also a strong limit in L^q. Indeed by concentration, up to translations and since $q > 1$, we have

$$1 - \varepsilon \leq \lim_{h \to \infty} \int_{B_R} |w_h|^q = \int_{B_R} |w|^q \leq \int |w|^q \leq \underline{\lim}_{h \to \infty} \int |w_h|^q = 1.$$

This conclude the proof of the theorem. $\square$

Corollary 6.5. *Consider a sequence $(u_h)_h \subset D^{p,s}(\mathbb{R}^d)$ such that $\delta(u_h) \to \infty$ as $h \to 0$. Then also $\lambda(u_h) \to 0$ for $h \to \infty$.*

Proof. As a consequence of Theorem 6.2, up to subsequence and to a rescaling of the form $\tau_\lambda u_h(x) = \lambda^{d/q} u(\lambda x)$, we can suppose that $u_h \to u$ strongly in $L^q(\mathbb{R}^d)$. Since the map $u \mapsto \lambda(u)$ is strongly continuous in $L^q(\mathbb{R}^d)$, $\lambda(u_h)$ converges to $\lambda(u)$. Furthermore, by the semicontinuity of the deficit (under L^q-convergence), $\delta(u) = 0$. Hence u is optimal for (6.1) and $\lambda(u) = 0$. $\square$

6.3 Reduction to $d-$symmetric functions

In this section we will prove that under the hypothesis that inequality (6.10) holds true for radial decreasing functions, up to change the values of α_0 and k_0, it holds also for $d-$symmetric functions (recall that a function is $k-$symmetric in $\mathbb{R}^d$, $d \geq k$, if it is symmetric with respect to k mutually orthogonal hyperplanes). We begin with a brief overview of the strategy we shall adopt. Given a function $u \in D^{p,s}(\mathbb{R}^d)$, the natural radial symmetric function to look at is its spherical rearrangement $u^\star$ (see [86, Chapter 3] for its the definition and main properties). Suppose that inequality (6.11) holds true for radial symmetric decreasing functions (thus for $u^\star$). Then by the triangle inequality we get

$$\lambda(u)^{1/q} \leq \|u - u^\star\|_q + \lambda(u^\star)^{1/q} \leq \|u - u^\star\|_q + \kappa_0^{1/q}\delta(u^\star)^{\alpha_0/q}.$$

We notice that by the Pólya-Szegö inequality we have that $\delta(u^\star) \leq \delta(u)$. But it is not clear if we can estimate the L^q distance between u and $u^\star$ in terms of $\delta(u)$. Indeed this turns out to be true *only if a function is already $d-$symmetric.* We shall prove, in Lemma 6.6, that $\delta(u)$ controls the Pólya-Szegö deficit, defined as

$$\delta_{PS}(u) = \frac{\|\nabla u\|_p - \|\nabla u^\star\|_p}{\|\nabla u^\star\|_p}, \tag{6.21}$$

and then, in Lemma 6.7, we will obtain an estimate of the $L^{p^\star}$ distance between u and $u^\star$ in terms of the L^p distance between $|\nabla u|$ and $|\nabla u^\star|$.

Lemma 6.6. *There exist two positive constants δ_0 and C_0 such that for every $u \in D^{p,s}(\mathbb{R}^d)$ such that $\|u\|_q = 1$, with $\delta(u) \leq \delta_0$, up to the rescales (6.14), we have*

$$\delta_{PS}(u) \leq C\delta(u)^{1/\theta} \tag{6.22}$$

where $\theta \in (0,1)$ is the parameter introduced in (6.2).

Proof. By Theorem 6.2, up to rescaling it we can suppose that u satisfies properties $(i)-(iii)$ in (6.14). If we choose $\delta(u) \leq 1/G$, we obtain:

$$G \leq \|\nabla u^\star\|_p^\theta \|u^\star\|_s^{1-\theta} \leq \|\nabla u\|_p^\theta \|u\|_s^{1-\theta} \leq 1 + G. \tag{6.23}$$

Then,

$$\begin{aligned} G\delta(u) &= \Big(\|\nabla u\|_p^\theta - \|\nabla u^\star\|_p^\theta\Big)\|u\|_s^{1-\theta} + G\delta(u^\star) \\ &\geq C_1^{\theta-1}\Big(\|\nabla u\|_p^\theta - \|\nabla u^\star\|_p^\theta\Big) + G\delta(u^\star) \end{aligned} \tag{6.24}$$

where we used the fact that $\|u\|_q = 1$ (statement (ii)) and $\|u\|_s \geq C_1^{-1}$ (statement (iii)). By (6.23) there exists a positive constant c such that,

$$\|\nabla u\|_p^\theta - \|\nabla u^\star\|_p^\theta \geq c\Big(\|\nabla u\|_p - \|\nabla u^\star\|_p\Big)^\theta.$$

Now the conclusion follows from (6.24) and definition (6.21), with $C = c^{1/\theta}$. □

To obtain the desired estimate of the L^p distance between u and $u^\star$ we shall use the following result, whose proof can be found in [39, Theorem 3].

Lemma 6.7. *Let $n \geq 2$, $1 < p < n$ and $z = \max\{p, 2\}$. Then there exists a positive constant C such that*

$$\int |u - u^\star|^{p^\star} \leq C \left(\int |u|^{p^\star} \right)^{\frac{p}{n}} \left(\int |\nabla u^\star|^p \right)^{\frac{1}{z'}} \left(\int |\nabla u|^p - \int |\nabla u^\star|^p \right)^{\frac{1}{z}} \tag{6.25}$$

holds for every non-negative $u \in W^{1,p}(\mathbb{R}^d)$ which is symmetric with respect to the coordinate hyperplanes.

We are now able to proceed with the proof of (6.12) when u is taken over the class of $d-$symmetric functions.

Theorem 6.8. *Suppose that there exist positive constants κ_0 and α_0 such that for any radial decreasing function $u \in D^{p,s}(\mathbb{R}^d)$ inequality (6.11) holds true. Then, there exist positive constants κ_1 and α_1 depending on d, p, q and s such that for any $d-$symmetric function $u \in D^{p,s}(\mathbb{R}^d)$ we have*

$$\delta(u) \geq \widetilde{\kappa}_0 \lambda(u)^{\widetilde{\alpha}_0}. \tag{6.26}$$

Proof. Since $\lambda(u) \leq 2^{q-1}$, if $\delta > 0$ and $\delta(u) \geq \min\{\delta, 1/G\}$, then we have $\lambda(u) \leq (2^{q-1}/\delta)\delta(u)$ and so (6.12) holds true with $\kappa_1 = 2^{q-1}/\delta$ and $\alpha_1 = 1$. Hence we may assume that $\delta(u) \leq \min\{\delta, 1/G\}$ for a suitably small δ. Moreover, by Theorem 6.2 we can suppose $\|u\|_q = 1$ and $\|u\|_s \in [1/C_0, C_0]$ where $C_0 \geq 1$ is a constant independent of u. Keeping in mind these remarks, we divide the proof into two steps:
Step 1: We first assume that $u \geq 0$. In this case we have, by an interpolation inequality, Lemma 6.7, and Theorem 6.2,

$$\begin{aligned} \int |u - u^\star|^q &\leq \left(\int |u - u^\star|^s \right)^{(1-\theta)q/s} \left(\int |u - u^\star|^{p^\star} \right)^{\theta q/p^\star} \\ &\leq C^{\frac{\theta q}{p^\star}} 2^{\frac{(1-\theta)q(s-1)}{s}} \left(\int u^s \right) \left(\int |u|^{p^\star} \right)^{\frac{p\theta q}{np^\star}} \left(\int |\nabla u^\star|^p \right)^{\frac{\theta q}{p^\star z'}} \left(\int |\nabla u|^p - \int |\nabla u^\star|^p \right)^{\frac{\theta q}{p^\star z}} \\ &\leq C(d,p,q,s) \left(\int |\nabla u|^p - \int |\nabla u^\star|^p \right)^\gamma . \end{aligned} \tag{6.27}$$

where $C(d, p, q, s)$ and γ are suitable positive constants depending on d, p, s and q. Notice that we use the boundedness of the L^p norm of ∇u and of the L^s norm of u granted by Theorem 6.2 (up to choose δ small enough). Moreover we exploited Lemma 6.7 and thus the assumption that $u \geq 0$. If we suppose $\delta(u) \leq \delta \leq G$, again by Theorem 6.2 we get

$$G \leq C_2 \|\nabla u^\star\|_p^\theta \leq C_2 \|\nabla u\|_p^\theta = C_3 G(\delta(u) + 1) \leq C_3 G(1 + G),$$

for suitable positive constants C_2 and C_3. Hence there exists $C_4 > 0$ such that

$$\|\nabla u\|_p^p - \|\nabla u^\star\|_p^p \leq C_4(\|\nabla u\|_p - \|\nabla u^\star\|_p)$$

By the triangle inequality, estimates (6.22) and (6.27), the Pólya-Szegö inequality and the assumption on radial functions that we have as hypotheses, we can find constants C_5, C_6 and C_7 independent of u such that

$$\begin{aligned}\lambda(u)^q &\leq 2^{q-1}\left(\lambda(u^\star) + \|u-u^\star\|_q^q\right)\\ &\leq C_5\left[\delta(u)^{\alpha_0} + \left(\int |\nabla u|^p - \int |\nabla u^\star|^p\right)^\gamma\right]\\ &\leq C_6\left[\delta(u)^{\alpha_0} + \delta(u)^{\gamma/\theta}\right] \leq C_7\delta(u)^\xi.\end{aligned}$$

where $\xi = \min\{\alpha_0, \gamma/\theta\}$.

Step 2: We assume now that u changes sign. In this case consider the positive and the negative part of u: $u^+ = u1_{\{u>0\}}$ and $u^- = -u1_{\{u<0\}}$ (here 1_A denotes the characteristic function of the set A). By Lemma 6.3 we are provided a positive constant $\lambda = \lambda(u)$ such that

$$\eta_0 G(u)^\kappa = F(\tau_\lambda u).$$

Moreover we have that

$$F(\tau_\lambda u^\pm) \geq \inf_\mu F(\tau_\mu u^\pm) = \eta_0 G(u^\pm)^\kappa.$$

where F and G are defined in (6.15). So we get

$$\begin{aligned}G^\kappa(\delta(u)+1)^\kappa = \left(\|\nabla u\|_p^\theta \|u\|_s^{1-\theta}\right)^\kappa &= \frac{1}{\eta_0}\left(\int |\nabla\tau_\lambda u|^p + \int |\tau_\lambda u|^s\right)\\ &= \frac{1}{\eta_0}\left(\int |\nabla\tau_\lambda u^+ + \nabla\tau_\lambda u^-|^p + |\tau_\lambda u^+ + \tau_\lambda u^-|^s\right)\\ &= \frac{1}{\eta_0}\left(\int |\nabla\tau_\lambda u^+|^p + \int |\nabla\tau_\lambda u^-|^p + \int |\tau_\lambda u^+|^s + \int |\tau_\lambda u^-|^s\right)\\ &\geq \left(\|\nabla u^+\|_p^\theta \|u^+\|_s^{1-\theta}\right)^\kappa + \left(\|\nabla u^-\|_p^\theta \|u^-\|_s^{1-\theta}\right)^\kappa\\ &\geq G^\kappa\left[\|u^+\|_q^\kappa + \|u^-\|_q^\kappa\right] = G^\kappa\left[\|u^+\|_q^\kappa + \|u^-\|_q^\kappa\right].\end{aligned}$$

The last equality is due to the fact that $\tau_\lambda u^+$ and $\tau_\lambda u^-$ have disjoint supports while in the last inequality we exploited the GNS inequality. Let us set $\int (u^+)^q = t$ and $\int (u^-)^q = 1-t$. We can suppose $t \in (0,1)$, since u changes sign. Then the previous formula takes the form

$$f(t) = \left(t^{\kappa/q} + (1-t)^{\kappa/q}\right)^{1/\kappa} - 1 \leq \delta(u).$$

The function f symmetric in $[0,1]$, is null on 0 and 1 and since $\kappa < q$ (by Lemma 6.3) is concave. Thus there exists $a > 0$ such that

$$f(t) \geq \frac{1}{a} \min\{t, 1-t\},$$

so that

$$\min\left\{ \int (u^+)^q, \int (u^-)^q \right\} \leq a\delta(u). \tag{6.28}$$

Suppose that the minimum in (6.28) is achieved by $\int (u^-)^q$ (being analogous the other case). Since $\delta(|u|) = \delta(u)$, we can conclude, thanks to the triangle inequality and to (6.28), that

$$\lambda(u)^{1/q} \leq \lambda(|u|)^{1/q} + \left(\int |u - |u||^q \right)^{1/q} \leq C_8(\delta(u)^{\xi/q} + \delta(u)^{1/q}) \leq \widetilde{\kappa}_0 \delta(u)^{\alpha/q}$$

where $\alpha = \min\{\xi, 1\}$ and C_8 a positive constant independent of u. The last inequality holds for $\delta(u) < 1$. So (6.12) holds with $\widetilde{\alpha}_0 = \alpha/q$. Eventually if $\delta(u) \geq 1$, then inequality (6.26) follows easily (possibly increasing the value of $\tilde{\kappa}_0$) since $\lambda(u) < 2^{q-1}$. $\square$

6.4 Reduction inequalities

The goal of this section is to prove the reduction inequalities (6.13). Namely we will prove the following result.

Theorem 6.9. *Assume the hypotheses of Theorem 6.8. Then there exist two positive constants κ_2 and α_2 such that for every non-negative function $u \in D^{p,s}(\mathbb{R}^d)$ there exists a $d-$symmetric function $\bar{u}$ such that the following reduction inequalities hold true*

$$\lambda(u) \leq \kappa_2 \lambda(\bar{u})^{\alpha_2}, \qquad \delta(\bar{u}) \leq \kappa_2 \delta(u)^{\alpha_2}. \tag{6.29}$$

This will be done arguing similarly to the Sobolev case considered in [39], although some technical modifications shall be needed. We begin recalling that if v is a optimal function for (6.1), then any other optimal function is of the form

$$v_{a,b,x_0}(x) = av(b(x - x_0)),$$

where a and b are non-null constant and $x_0 \in \mathbb{R}^d$. We define the *relative asymmetry* of a function on an affine subspace S of $\mathbb{R}^d$ as

$$\lambda(u|S) = \inf_{(a,b,x_0)\in\mathbb{R}^2\times S} \left\{ \frac{\|u - v_{a,b,x_0}\|_q^q}{\|u\|_q^q} : \text{v optimal for (6.1)}, \|v_{a,b,x_0}\|_q = \|u\|_q \right\}.$$

Next Lemma will show that the infimum in the definition of the relative asymmetry (and so of the asymmetry) is achieved.

Lemma 6.10. *Let $u \in D^{p,s}(\mathbb{R}^d)$ and S an affine space contained in $\mathbb{R}^d$. Then the infima in the definition of $\lambda(u)$ and $\lambda(u|S)$ are achieved.*

Proof. Since the two cases are analogous, we show a proof just for the asymmetry. We can suppose without loss of generality that $\|u\|_q = 1$. Let us start observing that $\lambda(u) < 2$. Indeed, if v is a competitor in the definition of $\lambda(u)$, then, up to translate the centre of symmetry of v, that u and v do not have disjointed supports. Then

$$\lambda(u) \le \int |v-u|^q = \int_{\{u>v\}} (u-v)^q + \int_{\{v>u\}} (v-u)^q < \int u^q + \int v^q = 2. \tag{6.30}$$

Let now $v_h(x) = a_h v(b_h(x-x_h))$ be a sequence of functions such that $\|v_h\|_q = 1$ and $\|u-v_h\|_q^q \to \lambda(u)$ as $h \to \infty$. We want to show that, up to subsequences, $(a_h, b_h, x_h) \to (a, b, x_0) \subset \mathbb{R}^2 \times \mathbb{R}^d$ as $h \to \infty$. We have that

$$1 = \int |v_h|^q = a_h^q \int |v(b_h(x-x_h))|^q = \frac{a_h^q}{b_h^n} \int |v|^q = \frac{a_h^q}{b_h^n} \tag{6.31}$$

so $a_h^q = b_h^n$. Since $v \in L^q(\mathbb{R}^d)$ there exists a function $\rho(\varepsilon)$ converging to 0 as $\varepsilon \to 0$ such that for each $z \in \mathbb{R}^d$ we have

$$\int_{B(z,\varepsilon)} |v|^q \le \rho(\varepsilon), \qquad \int_{B(0,1/\varepsilon)} |v|^q \ge 1 - \rho(\varepsilon). \tag{6.32}$$

Set now $b_- = \underline{\lim}_{h\to\infty} b_h$ and $b_+ = \overline{\lim}_{h\to\infty} b_h$. We claim that $b_- > 0$ and that $b_+ < \infty$. Suppose $b_- = 0$; then, recalling that v_h is a radial function, we have

$$\begin{aligned} \int_{B(0,1/\varepsilon)} |v_h|^q \le \int_{B(x_h,1/\varepsilon)} |v_h|^q &= \frac{a_h^q}{b_h^n} \int_{B(0,b_h/\varepsilon)} |v(y)|^q dy \\ &= \int_{B(0,b_h/\varepsilon)} |v(y)|^q dy. \end{aligned} \tag{6.33}$$

and the last quantity, up to pass to a subsequence, converge to 0 as $h \to \infty$. So we can suppose that for a fixed ε with h big enough,

$$\int_{B(0,1/\varepsilon)} |v_h|^q \le \varepsilon. \tag{6.34}$$

So, thanks to (6.34) and (6.32) we have

$$\begin{aligned} \|u - v_h\|_q^q &= \int_{B(0,1/\varepsilon)} |u-v_h|^q + \int_{B(0,1/\varepsilon)^c} |u-v_h|^q \\ &\ge \left| \left(\int_{B(0,1/\varepsilon)} |u|^q \right)^{1/q} - \left(\int_{B(0,1/\varepsilon)} |v_h|^q \right)^{1/q} \right|^q \\ &+ \left| \left(\int_{B(0,1/\varepsilon)^c} |v_h|^q \right)^{1/q} - \left(\int_{B(0,1/\varepsilon)^c} |u|^q \right)^{1/q} \right|^q \\ &\ge [(1-\rho(\varepsilon))^{1/q} - \varepsilon^{1/q}]^q + [(1-\varepsilon)^{1/q} - \rho(\varepsilon)^{1/q}]^q. \end{aligned}$$

Passing to the limit in h and then in ε we obtain that $\lambda(u) \geq 2$, that is a contradiction. Suppose now that $b_+ = \infty$. Then

$$\begin{aligned} \int_{B(x_h,\varepsilon)^c} |v_h|^q &= \int_{B(x_h,\varepsilon)^c} a_h^q |v(b_h(x-x_h))|^q dx = a_h^q \int_{B(0,\varepsilon)^c} |v(b_h x)|^q dx \\ &= \frac{a_h^q}{b_h^n} \int_{B(0,\varepsilon)^c} |v(z)|^q dz = \int_{B(0,b_h\varepsilon)^c} |v|^q \end{aligned} \tag{6.35}$$

and arguing as before we can suppose that $\int_{B(x_h,\varepsilon)^c} |v_h|^q \leq \varepsilon$ for h big enough. By (6.32) we get

$$\begin{aligned} \|u-v_h\|_q^q &= \int_{B(x_h,\varepsilon)} |u-v_h|^q + \int_{B(x_h,\varepsilon)^c} |u-v_h|^q \\ &\geq \left| \left(\int_{B(x_h,\varepsilon)} |v_h|^q \right)^{1/q} - \left(\int_{B(x_h,\varepsilon)} |u|^q \right)^{1/q} \right|^q \\ &+ \left| \left(\int_{B(x_h,\varepsilon)^c} |u|^q \right)^{1/q} - \left(\int_{B(x_h,\varepsilon)^c} |v_h|^q \right)^{1/q} \right|^q \\ &\geq (1-\varepsilon) - \rho(\varepsilon) + (1-\rho(\varepsilon)) - \varepsilon \end{aligned}$$

so that we obtain again a contradiction. Suppose now that $(x_h)_h$ is not bounded and extract a subsequence (not relabelled) such that $|x_h| \to \infty$. Then given $N > 0$, if h is big enough we would get $\int_{B(x_h,N)} |u|^q \leq 1/N$. If we choose N such that $\int_{B(x_h,N)} |v_h| \geq \varepsilon$ for all $h \in \mathbb{N}$, we obtain

$$\begin{aligned} \|u-v_h\|_q^q &\geq \int_{B(x_h,N)} |u-v_h|^q + \int_{B(x_h,N)^c} |u-v_h|^q \\ &\geq \left| \left(\int_{B(x_h,N)} |v_h|^q \right)^{1/q} - \left(\int_{B(x_h,N)} |u|^q \right)^{1/q} \right|^q \\ &+ \left| \left(\int_{B(x_h,N)^c} |u|^q \right)^{1/q} - \left(\int_{B(x_h,N)} |v_h|^q \right)^{1/q} \right|^q \\ &\geq \left[(1-\varepsilon)^{1/q} - \frac{1}{N^{1/q}} \right]^q + \left[\left(1 - \frac{1}{N}\right)^{1/q} - \varepsilon^{1/q} \right]^q . \end{aligned}$$

and again we get a contradiction. □

Clearly the asymmetry of a function controls its relative asymmetry. But if we consider a $d-$symmetric function is true also the opposite, as shown in next lemma.

Lemma 6.11. *Let $u \in D^{p,s}(\mathbb{R}^d)$ $k-$symmetric with respect to k mutually orthogonal hyperplanes and let S be the intersection of such hyperplanes. Then*

$$\lambda(u|S) \leq 3^q \lambda(u).$$

Proof. Suppose as usual that $\|u\|_q = 1$. Let $v_{a,b,x}$ a minimum for $\lambda(u)$. We consider now the orthogonal projection x_S of x on S and y the symmetric point of x_S with respect to S. Notice that since u is symmetric with respect to S, also $v_{a,b,y}$ is a minimum for $\lambda(u)$. Moreover, since the minima of the asymmetry are radial symmetric functions with decreasing profile we have

$$\|v_{a,b,x} - v_{a,b,x_S}\|_q \leq \|v_{a,b,x} - v_{a,b,y}\|_q.$$

This observation and the triangle inequality imply

$$\begin{aligned}
\lambda(u)^{1/q} &\leq \|u - v_{a,b,x_S}\|_q \leq \|u - v_{a,b,x}\|_q + \|v_{a,b,x} - v_{a,b,x_S}\|_q \\
&= \lambda(u)^{1/q} + \|v_{a,b,x} - v_{a,b,x_S}\|_q \leq \lambda(u)^{(1/q)} + \|v_{a,b,x} - v_{a,b,y}\|_q \\
&\leq \lambda(u)^{1/q} + \|v_{a,b,x} - u\|_q^q + \|u - v_{a,b,y}\|_q = 3\lambda(u)^{1/q}
\end{aligned}$$

and the conclusion follows. □

Next result shows that the $d-$symmetry condition in the last Lemma can in some sense be relaxed:

Lemma 6.12. *There exists a constant C_0 with the following property. Consider a function $u \in L^q(\mathbb{R}^d)$, $u \geq 0$, H an hyperplane of $\mathbb{R}^d$ and H^+ and H^- the half spaces in which $\mathbb{R}^d$ is divided by H. Suppose that*

$$\int_{H^+} |u|^q = \int_{H^-} |u|^q = \frac{1}{2}\int_H |u|^q,$$

then

$$\lambda(u|H) \leq C_0 \lambda(u)^{1/q}. \tag{6.36}$$

with a constant C_0 depending only on q and d. Moreover, if T_H denote the reflection with respect to H of $\mathbb{R}^d$, it holds

$$\int |u \circ T_H - u|^q \leq C_0 \|u\|_q^q \lambda(u)^{1/q}. \tag{6.37}$$

Proof. Suppose without loss of generality that $\|u\|_q = 1$ and let $v_0 = v_{a,b,x_0}$ a minimum for $\lambda(u)$ centred at x_0. Suppose that $x_0 \in H^+$, being the other case analogous, and let $\bar{x}$ the projection of x_0 on H and $\bar{v} = v_{a,b,\bar{x}}$. Then

$$\lambda(u|H) \leq \int |u - \bar{u}|^q \leq 2^{q-1}\left(\lambda(u) + \int |v_0 - \bar{v}|^q\right). \tag{6.38}$$

Consider the translated half spaces $K^{\pm} = H^{\pm} + (x_0 - \bar{x})$. Since $x_0 \in H^+$ it follows that $K^+ \subseteq H^+$ and $H^- \subseteq K^-$. We have that

$$\frac{1}{2} = \int_{K^{\pm}} v_0^q = \int_{H^{\pm}} u^q = \int_{H^{\pm}} \bar{v}^q$$

and

$$\int_{H^-} |v_0 - \bar{v}|^q = \int_{K^+} |v_0 - \bar{v}|^q \leq \int_{H^+} |v_0 - \bar{v}|^q;$$

so that

$$\int |v_0 - \bar{v}|^q \leq 2 \int_{H^+} |v_0 - \bar{v}|^q. \tag{6.39}$$

Since $v_0 \geq \bar{v}$ on K^+ we get that $|v_0 - \bar{v}|^q \leq v_0^q - \bar{v}^q$ on K^+. Then

$$\begin{aligned} \int_{K^+} |v_0 - \bar{v}|^q &\leq \int_{K^+} v_0^q - \int_{K^+} \bar{v}^q = \frac{1}{2} - \int_{H^-} v_0^q \\ &\leq C\Big(\|u\|_{L^q(H^-)} - \|v_0\|_{L^q(H^-)}\Big) \leq C\|u - v_0\|_q = C\lambda(u)^{1/q}, \end{aligned} \tag{6.40}$$

for a suitable positive constant C. Moreover

$$\begin{aligned} \int_{H^+\setminus K^+} |v_0 - \bar{v}|^q &\leq 2^{q-1} \int_{H^+\setminus K^+} (v_0^q + \bar{v}^q) = 2^q \int_{H^+\setminus K^+} v_0^q \\ &= 2^q \left[\int_{H^+} v_0^q - \frac{1}{2}\right] = 2^q \left[\int_{H^+} v_0^q - \int_{H^+} u^q\right] \end{aligned}$$

and reasoning as in (6.40) we obtain

$$\int_{H^+\setminus K^+} |v - \bar{v}|^q \leq C\lambda(u)^{1/q}. \tag{6.41}$$

Inequality (6.36) is then a consequence of (6.38), (6.39), (6.40), and (6.41). We are left to show inequality (6.37). Let $\hat{u}$ be the minimal function for $\lambda(u|H)$. Then

$$\begin{aligned} \int_{H^{\pm}} |u \circ T_H - u|^q &\leq 2^{q-1} \left(\int_{H^{\pm}} |u \circ T_H - \hat{u}|^q + \int_{H^{\pm}} |u - \hat{u}|^q \right) \\ &= 2^{q-1} \int |u - \hat{u}|^q = 2^{q-1} \lambda(u|H) \leq C_0 \lambda(u)^{1/q}. \end{aligned}$$

□

Before passing to the proof of Theorem 6.9 we need another technical lemma which, roughly speaking, states that if two optimal functions for the GNS inequality are near in L^q norm, then their L^q distance on the whole $\mathbb{R}^d$ can be controlled just by that on a *quarter* of $\mathbb{R}^d$. Its proof is quite technical but it is essentially based on a Taylor expansion.

Lemma 6.13. *Let u be an optimal function for the GNS inequality of parameters s, q, p centred in 0 with $\|u\|_q = 1$, and set $u_{\alpha,z}(x) = \alpha^{d/q} u(\alpha(x-z))$ (for simplicity, $u_{1,z} = u_z$). Consider two orthogonal half spaces H and K containing the origin in their boundaries. There exist two constants $k = k(d, s, q, p) > 0$ and $\tilde{\rho} << 1$ such that if*

$$\int |u_{\lambda,x_0} - u_{\mu,y_0}|^q \leq \tilde{\rho}, \quad \int_{H\cap K} |u_{\mu,y_0}|^q \geq \frac{1}{8}$$

then

$$\int_{H\cap K} |u_{\lambda,x_0} - u_{\mu,y_0}|^q \geq k \int |u_{\lambda,x_0} - u_{\mu,y_0}|^q. \tag{6.42}$$

Proof. Up to a rotation we can consider $H = \{e_1 \geq 0\}$ and $K = \{e_2 \geq 0\}$. Let us set $Q = H \cap K = \{e_1 \geq 0, e_2 \geq 0\}$. Consider the sets

$$T_k = \{x : 1/k < |\nabla u(x)| < M\} \qquad M = \max_{\mathbb{R}^d} |\nabla u(x)|.$$

By the radial shape of u we know that each T_k is a radial set composed of a countable union of centred annuli, i.e. there exists a non-decreasing sequence of positive numbers $(r_{k,j})_j$ such that

$$T_k = \bigcup_{j\in\mathbb{N}} \left(B_{r_{k,j+1}} \setminus B_{r_{k,j}}\right)$$

where B_r is the ball centred at the origin of radius r. Let now $I_k = \{j \in \mathbb{N} : r_{k,j+1} - r_{k,j} \geq 1/k\}$ and set

$$S_k = \bigcup_{j\in I_k} \left(B_{r_{k,j+1}} \setminus B_{r_{k,j}}\right).$$

We consider, for $z \in \mathbb{R}^d$, the greater centred annulus contained in $S_k \cap (S_k + z)$:

$$\Sigma(k, z) = \left\{x \in S_k \cap (S_k + z) : \partial B_{|x|} \subseteq S_k \cap (S_k + z)\right\}.$$

Notice, that $\Sigma(k, z_1) = \Sigma(k, z_2)$ whenever $|z_1| = |z_2|$ so we may define $\Sigma(k, r) = \Sigma(k, z)$ if $|z| = r$. Clearly $\bigcup_{k\in\mathbb{N}} S_k = \mathbb{R}^d$. Moreover, since $|\nabla u|$ is continuous, we have that

$$\bigcup_{|z|>0} \Sigma(k, z) = S_k. \tag{6.43}$$

Indeed, if $x \in S_k$, there exists $r = r(x) > 0$ such that $|\nabla u|(y) > 1/k$ for every $y \in B_{r(x)}(x)$, that is $\overline{B_{r(x)}(x)} \subseteq S_k$. Thus it is sufficient to choose $|z| < \operatorname{dist}(x, \partial S_k)$ and we get that $\partial B_{|x|} \subseteq S_k \cap (S_k + z)$, i.e $x \in \Sigma(k, z)$. With these definitions in mind, we pass to prove inequality (6.42). Up to a change of variables we can consider just the case $\lambda = 1 + l > 1$, $\mu = 1$. For any Borel set A we have:

$$
\begin{aligned}
&\int_A |\lambda^{d/q} u(\lambda(x-x_0)) - u(x-y_0)|^q = \int_{A+y_0} |(1+l)^{d/q} u((1+l)(x+y_0-x_0)) - u(x)|^q \\
&= \int_{A+y_0} \Big|(1+l)^{d/q} \left[u(x) + \big\langle \nabla u(x), lx + (1+l)(y_0-x_0)\big\rangle\right] - u(x) + R(x)\Big|^q \\
&= \int_{A+y_0} \Big|\langle \nabla u(x), lx + (1+l)(y_0-x_0)\rangle + \frac{n}{q} l \langle \nabla u(x), lx + (1+l)(y_0-x_0)\rangle \\
&+ \frac{n}{q} l u(x) + R(x)\Big|^q \\
&= \int_{A+y_0} \left| |\nabla u(x)| \left\langle \frac{x}{|x|}, y_0 - x_0 \right\rangle + l \left[|x||\nabla u(x)| + \frac{n}{q} u(x)\right] + R(x)\right|^q \\
&:= \int_{A+y_0} |E_{x_0,y_0,l}(x) + R(x)|^q
\end{aligned}
\tag{6.44}
$$

where the last inequality is due to the radial symmetry of ∇u and the error term $R(x)$ is given by

$$
R(x) = (1+l)^{d/q} \left[\frac{|\nabla^2 u(x)|}{|x|^2} (x \otimes x)(lx + (1+l)(y_0-x_0))^2 \right] = O(l^2) + O(|lx+(1+l)(y_0-x_0)|^2).
$$

We notice that there exists $\tilde{\rho}_1$ such that for every $x \in S_k$

$$
|R(x)| \leq \frac{1}{2} |E_{x_0,y_0,l}(x)| \tag{6.45}
$$

if $|x_0| + |y_0| + l \leq \tilde{\rho}_1$, since R contains any terms of $E_{x_0,y_0,l}$ with an higher power (so it is of higher order). We aim now to find $\tilde{\rho}$ such that, for $|x_0| + |y_0| + l \leq \tilde{\rho}$, the following chain of inequalities hold true:

$$
\int |E_{x_0,y_0,l} + R|^q \leq c_1 \int_{\Sigma_{k,y_0}} |E_{x_0,y_0,l} + R|^q \leq c_2 \int_{\Sigma_{k,y_0} \cap Q} |E_{x_0,y_0,l} + R|^q \leq c_2 \int_Q |E_{x_0,y_0,l} + R|^q \tag{6.46}
$$

for suitable constants c_1, c_2 and k. This would immediately imply inequality (6.42). We start remarking that

$$
\lim_{k \to \infty} \int_{S_k} |E_{x_0,y_0,l} + R|^q = \int |E_{x_0,y_0,l} + R|^q
$$

so there exists $\overline{k}$ such that for $k \geq \overline{k}$ we have

$$
\int_{S_k} |E_{x_0,y_0,l} + R|^q \geq \frac{1}{2} \int |E_{x_0,y_0,l} + R|^q.
$$

Moreover, in view of (6.43) we get,

$$
\int_{\Sigma(k,y_0)} |E_{x_0,y_0,l} + R|^q \geq \frac{1}{2} \int_{S_k} |E_{x_0,y_0,l} + R|^q.
$$

So the first inequality in (6.46) holds with $c_1 = 1/4$. Since the last inequality in (6.46) is trivial, we are left to prove the central one. Consider now, by contradiction, a sequence $(x_h, y_h, l_h) \to 0 \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$ as $h \to \infty$ such that (6.46) does not hold. Integrating on the whole strip we get, for h big enough, the following estimate:

$$\begin{aligned} &\int_{\Sigma(k,y_h)} |E_{x_h,y_h,l_h} + R|^q \leq (3/2)^q \int_{\Sigma(k,y_h)} |E_{x_h,y_h,l_h}|^q \\ &\leq (3/2)^q \int_{\Sigma(k,y_h)} |M \left\langle \frac{x}{|x|}, y_h - x_h \right\rangle + l(|x||\nabla u(x)| + (d/q)u(x))|^q \\ &\leq (3/2)^q \int_{\Sigma(k,y_h)} |M|y_h - x_h| + l_h(|x||\nabla u(x)| + (d/q)u(x))|^q \\ &\leq (3/2)^q \int_{\Sigma(k,y_h)} |M|v_h| + c_0 l|^q \end{aligned} \tag{6.47}$$

where we set $v_h = y_h - x_h$ and

$$c_0 = \overline{\lim}_{h\to\infty} \max_{\Sigma(k,y_h)} |x||\nabla u(x)| + \frac{n}{q}u(x) > 0.$$

So there are constant k_0, k_1 such that

$$\int_{\Sigma(k,y_h)} |E_{x_h,y_h,l_h} + R|^q \leq k_0 \int_{\Sigma(k,y_h)} |k_1 l + |v_h||^q \tag{6.48}$$

By the other hand we get

$$\begin{aligned} &\int_{\Sigma(k,y_h)\cap Q} |E_{x_h,y_h,l_h} + R|^q \geq \frac{1}{2^q} \int_{\Sigma(k,y_h)\cap Q} |E_{x_h,y_h,l_h}|^q \\ &= \frac{1}{2^q} \int_{\Sigma(k,y_h)} \left| |\nabla u(x)| \left\langle \frac{x}{|x|}, \frac{v_h}{|v_h|} \right\rangle |v_h| + l(|x||\nabla u(x)| + (d/q)u(x)) \right|^q . \end{aligned}$$

We consider now three possible situation: $|v_h| << l_h$, $l_h << |v_h|$ or $l_h \simeq |v_h|$. In the first case we have, thanks to (6.48), that

$$\int_{\Sigma(k,y_h)} |E_{x_h,y_h,l_h} + R|^q \leq k_1 \int_{\Sigma(k,y_h)} l^q$$

for a suitable k_1. Moreover it is easy to find positive constants k_2 and k_3 independent of l_h and v_h such that

$$\int_{\Sigma(k,y_h)\cap Q} |E_{x_h,y_h,l_h} + R|^q \geq k_2 \int_{\Sigma(k,y_h)\cap Q} l^q \geq k_3 \int_{\Sigma(k,y_h)} l^q.$$

So in this case (6.46) holds with $c_2 = \frac{k_2}{k_3} c_1$ (or, in other terms, we get a contradiction). The second case, $l_h << |v_h|$ can be treated with the same argument, with the only observation that, for the estimate from above, we must restrict furtherly the set of integration to the set $U = \{x \in Q : |\langle x/|x|, v_0/|v_0|\rangle| \geq 1/10\}$. We are left to study the case where $|v_h| \simeq l_h$. If $\underline{\lim}_h |v_h|/l_h \geq \widetilde{c} > 0$, we have that

$$\int_{\Sigma(k,y_h)} |E_{x_h,y_h,l_h} + R|^q = l_h^q \int_{\Sigma(k,y_h)} \left| c_0 \frac{l_h}{|v_h|} + M \right|^q .$$

Let us define $V = \{x \in Q : |\langle x/|x|, v_h/|v_h|\rangle| \leq \alpha\}$ where α is a constant (depending on $\widetilde{c}$) that will be chosen later. We have:

$$\begin{aligned} &\int_{\Sigma(k,y_h)\cap Q} |E_{x_h,y_h,l_h} + R|^q \geq \int_{\Sigma(k,y_h)\cap V} |E_{x_h,y_h,l_h} + R|^q \\ &\geq \kappa_0 l^q \int_{\Sigma(k,y_h)\cap V} \left| c_1 \frac{l}{|v_h|} - M\alpha \right|^q . \end{aligned}$$

Choosing α small enough, since $l_h/|v_h| >> 0$, we can find constant independent of l_h and v_h such that

$$\int_{\Sigma(k,y_h)\cap Q} |E_{x_h,y_h,l_h} + R|^q \geq \kappa_1 l^q \int_{\Sigma(k,y_h)\cap V} \left| c_0 \frac{l_h}{|v_h|} + M \right|^q \geq \kappa_2 l^q \int_{\Sigma(k,y_h)} \left| c_0 \frac{l_h}{|v_h|} + M \right|^q ,$$

so again (6.46) holds with $c_2 = k_2/c_1$ (thus again a contradiction). Eventually, if $\underline{\lim}_h l_h/|v_h| \geq \widetilde{c} > 0$, we repeat a similar argument integrating, in the estimate from above, over $V_2 = \{x \in Q : |\langle x/|x|, v_h/|v_h|\rangle| \geq 1/10\}$. □

We pass now to prove Theorem 6.9. For the sake of clearness we divide its proof into two parts. We first prove a proposition which provides us a method to transform a generic function in $D^{p,s}(\mathbb{R}^d)$ in an $(d-1)-$symmetric function which satisfies the reduction inequalities (6.29). Then we will see how to obtain the last symmetry.

Proposition 6.14. *There exists a positive constant C such that for every function $u \in D^{p,s}(\mathbb{R}^d)$ there is an $(d-1)-$symmetric function $\tilde{u}$ such that*

$$\lambda(u) \leq C\lambda(\tilde{u}), \qquad \delta(\tilde{u}) \leq 2^{d-1}\delta(u). \tag{6.49}$$

Proof. As usual, by homogeneity of the deficit and the asymmetry, we can consider $\|u\|_q = 1$. Moreover we can suppose that $\delta(u) < \bar{\delta}$ for an arbitrary small $\bar{\delta}$. Indeed, if $\delta(u) \geq \bar{\delta}$, let v be a radial (and so $d-$symmetric!) function such that $0 < \delta(v) < 2^{d-1}\bar{\delta}$. Then

$$\lambda(u) \leq 2^q = \frac{2^q}{\lambda(v)}\lambda(v) \leq \bar{C}\lambda(v), \qquad \delta(v) \leq 2^{d-1}\bar{\delta} \leq 2^{d-1}\delta(u).$$

Consider, for $k = 1, \ldots d$, the d hyperplanes orthogonal to the coordinate axis such that

$$\int_{H_k^+} u^q = \int_{H_k^-} u^q = \frac{1}{2}$$

where $H_k^{\pm}$ are the two half spaces in which $\mathbb{R}^d$ is divided by H_k. Denoting T_k the reflection with respect to H_k, we define

$$u_k^+(x) = \begin{cases} u(x) & \text{if } x \in H_k^+ \\ u(T_k(x)) & \text{if } x \in H_k^- \end{cases}$$
$$u_k^-(x) = \begin{cases} u(x) & \text{if } x \in H_k^- \\ u(T_k(x)) & \text{if } x \in H_k^+ \end{cases} \tag{6.50}$$

By construction $u_k^{\pm}$ are symmetric with respect to H_k. We observe now that

$$\int u^s = \frac{\int (u_k^+)^s + \int (u_k^-)^s}{2}, \qquad \int |\nabla u|^p = \frac{\int |\nabla u_k^+|^p + \int |\nabla u_k^-|^p}{2},$$

and since $t \to t^{1/p}$ and $t \to t^{1/s}$ are concave functions, we have that

$$\|u\|_s \geq \frac{\|u_k^+\|_s + \|u_k^-\|_s}{2}, \qquad \|\nabla u\|_p \geq \frac{\|\nabla u_k^+\|_p + \|\nabla u_k^-\|_p}{2}. \tag{6.51}$$

By the definition of $\delta(u)$ and since $(x,y) \mapsto x^\theta y^{1-\theta}$ is concave on $\mathbb{R}^2_+$ and strictly increasing in x and y, we get that

$$\begin{aligned} G\delta(u) &\geq \left(\frac{\|\nabla u_k^+\|_p + \|\nabla u_k^-\|_p}{2}\right)^\theta \left(\frac{\|u_k^+\|_s + \|u_k^-\|_s}{2}\right)^{1-\theta} - G \\ &\geq \frac{G}{2}\delta(u_k^+) + \frac{G}{2}\delta(u_k^-), \end{aligned}$$

and so

$$\delta(u) \geq \frac{\delta(u_k^+) + \delta(u_k^-)}{2}.$$

In particular for every $k = 1, \ldots, d$

$$\max\{\delta(u_k^+), \delta(u_k^-)\} \leq 2\delta(u).$$

Let v_k^+ and v_k^- be the functions which minimize $\lambda(u_k^{\pm}|H_k)$. Then, by triangle inequality and Lemma 6.11 we have

$$\begin{aligned} \lambda(u) &\leq \int |u - v_k^+|^q = \int_{H_k^+} |u_k^+ - v_k^+|^q + \int_{H_k^-} |u_k^- - v_k^+|^q \\ &\leq 2^{q-1}\left(\frac{\lambda(u_k^+|H_k) + \lambda(u_k^-|H_k)}{2} + \int_{H_k^-} |v_k^+ - v_k^-|^q\right) \\ &\leq 2^{q-2}3^q\left(\lambda(u_k^+) + \lambda(u_k^-) + \int_{H_k^-} |v_k^+ - v_k^-|^q\right). \end{aligned}$$

We claim now that for $\delta(u)$ is small enough, chosen any couple of indexes $1 \le i < j \le d$, for $k_1 = i$ or $k_1 = j$ the following inequality holds:

$$\int_{H_{k_1}^-} |v_{k_1}^+ - v_{k_1}^-|^q \le C\left(\int_{H_{k_1}^+} |u_{k_1}^+ - v_{k_1}^+|^q + \int_{H_{k_1}^-} |u_{k_1}^- - v_{k_1}^-|^q\right). \tag{6.52}$$

Let us show how this concludes: by (6.51) and (6.52) we would have

$$\lambda(u) \le C \max\{\lambda(u_{k_1}^+),\ \lambda(u_{k_1}^-)\}, \quad \max\{\delta(u_{k_1}^+), \delta(u_{k_1}^-)\} \le 2\delta(u).$$

So we would obtain, among $u_{k_1}^\pm$, a $1-$symmetric function with L^q norm equal to 1, say $u_{k_1}^+$. We can now iterate this procedure exploiting two hyperplanes between the $(d-1)$ we did not use yet. We would then obtain a $2-$symmetric function which would satisfy the reductions inequalities (6.49) (with respect to $u_{k_1}^+$). We can iterate such construction until we have just one hyperplane left. But then we would have an $(d-1)-$symmetric function which satisfies inequalities (6.49). Thus we are only left to prove (6.52). We divide such proof into two further steps:

<u>Step 1</u> There exists a positive constant C_0 such that for all $1 \le i < j \le d$, σ, $\tau \in \{+,-\}$ it holds

$$\int |v_i^\sigma - v_j^\tau|^q \le C_0 \int_{H_i^\sigma \cap H_j^\tau} |v_i^\sigma - v_j^\tau|^q. \tag{6.53}$$

<u>Step 2</u> (6.53) implies (6.52).

To prove (6.53) (thus Step 1) we notice that, thanks to Lemma 6.13, it is verified if there are two positive constants ρ and C_1 such that:

(i) $\int (v_i^\sigma)^q = \int (v_j^\tau)^q = 1$;

(ii) H_i^σ ed H_j^τ are two orthogonal half spaces which contains on their boundary the centre of symmetry of v_i^σ e v_j^τ;

(iii) $\int |v_i^\sigma - v_j^\tau|^q \le \rho$.

So we are left to check (iii). We have

$$\|v_i^\sigma - v_j^\tau\|_q \le \|v_i^\sigma - u_i^\sigma\|_q + \|u_i^\sigma - u\|_q + \|u - u_j^\tau\|_q + \|u_j^\tau - v_j^\tau\|_q. \tag{6.54}$$

Thanks to Lemma 6.12

$$\int |u_i^\sigma - u|^q = \frac{1}{2}\int |u \circ T_i - u|^q \le C\lambda(u)^{1/q}. \tag{6.55}$$

Moreover

$$\int |v_i^\sigma - u_i^\sigma|^q \le 2^{q-1}\Big(\lambda(u|H_i) + \|u - u_i^\sigma\|_p^p\Big) \le C\lambda(u)^{1/q}. \tag{6.56}$$

Clearly the same estimate holds for the two addends not yet considered in (6.54). Putting together (6.55) and (6.56) we obtain (*iii*) and this conclude the proof of Step 1. Let us prove now Step 2. Suppose to fix the ideas that $i = 1$ and $j = 2$. For $k = 1, 2$ we set

$$h_k = v_k^+ 1_{H_k^+} + v_k^- 1_{H_k^-}.$$

Thanks to (6.53),

$$\int |h_1 - h_2|^q \geq \int_{H_1^+ \cap H_2^+} |h_1 - h_2|^q = \int_{H_1^+ \cap H_2^+} |v_1^+ - v_2^+|^q \geq \frac{1}{C} \int |v_1^+ - v_2^+|^q$$

With an similar argument, using $H_1^- \cap H_2^+$ instead of $H_1^+ \cap H_2^+$ we get

$$\int |h_1 - h_2|^q \geq \frac{1}{C} \int |v_1^- - v_2^+|^q.$$

Hence

$$\int |v_1^+ - v_1^-|^q \leq 2^q C \int |h_1 - h_2|^q. \tag{6.57}$$

Similarly we can see that

$$\int |v_2^+ - v_2^-|^q \leq 2^q C \int |h_1 - h_2|^q. \tag{6.58}$$

Furthermore we have

$$\begin{aligned}
\int |h_1 - h_2|^q &\leq 2^{q-1} \left(\int |h_1 - u|^q + \int |h_2 - u|^q \right) \\
&= 2^{q-1} \left(\int_{H_1^+} |v_1^+ - u_1^+|^q + \int_{H_1^-} |v_1^- - u_1^-|^q \right. \\
&\quad \left. + \int_{H_2^+} |v_2^+ - u_2^+|^q + \int_{H_2^-} |v_2^- - u_2^-|^q \right) \\
&\leq 2^q \max \left\{ \int_{H_1^+} |v_1^+ - u_1^+|^q + \int_{H_1^-} |v_1^- - u_1^-|^q, \right. \\
&\quad \left. \int_{H_2^+} |v_2^+ - u_2^+|^q + \int_{H_2^-} |v_2^- - u_2^-|^q \right\} \\
&= 2^q \left(\int_{H_k^+} |v_k^+ - u_k^+|^q + \int_{H_k^-} |v_k^- - u_k^-|^q \right).
\end{aligned} \tag{6.59}$$

putting together (6.57), (6.58) and (6.59) we obtain the claim of Step 2. □

We stress here that we cannot symmetrize directly our function once again. A formal argument which shows a problem that may occur is the following: consider a function v such that $\delta(v) = \lambda(v) = 0$ and construct a new function u as follows:

$$u(x) = v(x)1_{\{x_1 \geq 0\}}(x) + 2^{d/q}v(2x)\mathbf{1}_{\{x_1<0\}}(x).$$

Such a function is $(d-1)-$symmetric with respect to $H_k = \{x_k = 0\}$ for $k \neq 1$. if we try to symmetrize such function with respect to H_1 we would obtain $u_+(x) = 2^{d/q}(v(2x))$ and $u_-(x) = v(x)$. Clearly none of those functions satisfy the first inequality in (6.29). However we are going to see that a more refined symmetrization can bypass this problem.

Proof of Theorem 6.9. Assume as usual that $\|u\|_q = 1$. We can assume, thanks to Proposition 6.14, that u is an $(d-1)-$symmetric function and that $\delta(u) < \bar{\delta}$ with $\bar{\delta}$ arbitrarily small. Up to a rotation and a translation, we can consider u to be symmetric with respect to the coordinate axes $\{x_k = 0\}$ for $k = 2, \ldots, d$ and such that

$$\int_{\{x_k>0\}} u^q = \frac{1}{2} = \int_{\{x_k<0\}} u^q.$$

Let $u^\pm$ be the two symmetrizations of u with respect to the hyperplane $\{x_1 = 0\}$, constructed as in Proposition 6.14 . We have that $\max\{\delta(u^+), \delta(u^-)\} \leq 2\delta(u)$. So, if $\min\{\lambda(u^+), \lambda(u^-)\} \geq C_0\lambda(u)$, we would be done. Thus we suppose that

$$\max\{\lambda(u^+), \lambda(u^-)\} < \varepsilon\lambda(u) \tag{6.60}$$

for some constant ε to be chosen. Consider $Q = \{|x_1| \leq x_2\}$, $Q^+ = Q \cap \{x_1 > 0\}$ and $Q^- = Q \cap \{x_1 < 0\}$ and define a function $\hat{u}$ as follows:

$$\hat{u}(x) = \begin{cases} u(x) & \text{if} \quad x \in Q \\ u(R_1 x) & \text{if} \quad x \in R_1(Q) \\ \hat{u}(R_2 x) & \text{if} \quad x \in R_2(Q \cup R_1(Q)) \end{cases}$$

where R_1 and R_2 are the reflection with respect to $\{x_1 = x_2\}$ and $\{x_1 = -x_2\}$ respectively. The function $\hat{u}$ satisfies all the symmetries of u with the exception of the one related to the hyperplane $\{x_2 = 0\}$, but by construction it is symmetric also with respect to $\{x_1 = \pm x_2\}$. So it is $d-$symmetric. It remains to show that $\hat{u}$ satisfies the reduction inequalities (6.29). Let us start with the one relative to the asymmetry. To this aim we will denote $\hat{v}$, v^+ and v^- as the functions who achieve the minima of $\lambda(\hat{u}|\{0\})$, $\lambda(u^+|\{0\})$ and $\lambda(u^-|\{0\})$ respectively. Since $\int \hat{u}^q \leq 4$, we get that

$$\begin{aligned} 3^q\lambda(\hat{u}) \geq \lambda(\hat{u}|\{0\}) = \frac{\int |\hat{u} - \hat{v}|^q}{\int \hat{u}^q} &= \frac{4}{\int \hat{u}^q} \int_Q |u - \hat{v}|^q \\ &\geq \int_{Q^+} |u^+ - \hat{v}|^q + \int_{Q^-} |u^- - \hat{v}|^q \\ &= \int_{Q^+} |u^+ - \hat{v}|^q + \int_{Q^+} |u^- - \hat{v}|^q \geq \frac{1}{2^{q-1}} \int_{Q^+} |u^+ - u^-|^q. \end{aligned}$$

The first inequality in (6.29) is then true if we can estimate $\lambda(u)$ in terms of $\int_{Q^+} |u^+ - u^-|^q$. To this aim we observe that

$$\begin{aligned}\|u^+ - u^-\|_{L^q(Q^+)} &= \frac{1}{2}\|u^+ - u^-\|_{L^q(Q)} \\ &= \frac{1}{2}\|(v^+ - v^-) - (v^+ - u^+) - (u^- - v^-)\|_{L^q(Q)} \\ &\geq \frac{1}{2}\Big[\|v^+ - v^-\|_{L^q(Q)} - \|u^+ - v^+\|_{L^q(Q)} - \|u^- - v^-\|_{L^q(Q)}\Big].\end{aligned} \tag{6.61}$$

Moreover

$$\int_Q |u^\pm - v^\pm|^q \leq \int |u^\pm - v^\pm|^q = \lambda(u^\pm|\{0\}) \leq 3^q\lambda(u^\pm) \leq \varepsilon 3^q \lambda(u) \tag{6.62}$$

where we used Lemma 6.11 and Lemma 6.12 and the fact that $u^\pm$ are $d-$symmetric functions. Thanks to (6.61) and (6.62) we obtain

$$\|u^+ - u^-\|_{L^q(Q^+)} \geq \frac{1}{2}\Big[\|v^+ - v^-\|_{L^q(Q)} - 2\Big(\frac{3^q}{C(\varepsilon)}\lambda(u)\Big)^{1/q}\Big], \tag{6.63}$$

where $C(\varepsilon)$ is a suitable positive constant. Furthermore, always thanks to (6.62) we have

$$\begin{aligned}\lambda(u) &\leq \int |u - v^+|^q = \frac{1}{2}\int |v^+ - u^+|^q + \frac{1}{2}\int |v^+ - u^-|^q \\ &\leq \varepsilon\frac{3^q}{2}\lambda(u) + 2^{q-2}\left(\varepsilon 3^q\lambda(u) + \int |v^+ - v^-|^q\right),\end{aligned}$$

and so

$$\int |v^+ - v^-|^q \geq \frac{1}{2^{q-2}}\Big[\lambda(u) - \varepsilon\frac{3^q}{2}\lambda(u) - \varepsilon 3^q\lambda(u)\Big] \geq \frac{1}{2^q}\lambda(u) \tag{6.64}$$

where last inequality is justified choosing ε small enough. Summarying we obtained

$$\|u^+ - u^-\|_{L^q(Q^+)} \geq C_0\lambda(u)^{1/q}.$$

where C_0 is a suitable positive constant independent of u. Let us consider now the second inequality in (6.29), that is the one concerning the deficit. We stress here that we cannot say that $\|\hat{u}\|_q = 1$. We have

$$\Big|\|\hat{u}\|_{L^q(Q^+)} - (1/8)^{1/q}\Big| = \Big|\|u^+\|_{L^q(Q^+)} - \|v^+\|_{L^q(Q^+)}\Big| \leq \|u^+ - v^+\|_{L^q(Q^+)}.$$

and, since $|s^q - t^q| \leq C_1|s-t|$ for a suitable C_1 if s and t are in $[0,1]$,

$$\begin{aligned}\left|\int_{Q^+}\hat{u}^q - \frac{1}{8}\right| &\leq C_2\|u^+ - v^+\|_q = C_2\lambda(u^+|\{0\})^{1/q} \\ &\leq 3^{1/q}C_3\lambda(u^+)^{1/q} \leq 3^{1/q}C_4\delta(u^+)^{\alpha/q} \leq 2C_4\delta(u)^{\alpha/q},\end{aligned} \tag{6.65}$$

where we used Lemma 6.11, the fact that u is $d-$symmetric and that $\delta(u^+) \leq 2\delta(u)$ and where C_2, C_3 and C_4 are positive constants. An analogous estimate holds on Q^-, on $U^+ = \{x_1 > 0, x_2 > 0\} \setminus Q$ and on $U^- = \{x_1 < 0, x_2 > 0\} \setminus Q$. Then, by triangle inequality we obtain that

$$\left| \int \hat{u}^q - 1 \right| \leq C\delta(u)^{\alpha/q} \tag{6.66}$$

for a suitable $\alpha > 0$. Let us recall the definition of the functionals F and G given in (6.15):

$$G(u) = \|\nabla u\|_p^\theta \|u\|_s^{1-\theta}, \qquad F(u) = \int |\nabla u|^p + \int |u|^s. \tag{6.67}$$

By Lemma 6.3 we know that there exist two positive constants κ and η_0 such that for every u there exists $\lambda > 0$ such that $F(\tau_\lambda u) = \eta_0 G(u)^\kappa$ where $\tau_\lambda u(x) = \lambda^{d/q} u(\lambda x)$. Furthermore such λ minimize the function $\mu \mapsto F(\tau_\mu u)$ in $\mathbb{R}^+$. So we get

$$G^\kappa \|\hat{u}\|_q^\kappa (1 + \delta(\hat{u}))^\kappa = G(\hat{u})^\kappa \leq \left(\frac{1}{\eta_0} \int \left|\nabla \tau_\lambda \hat{u}\right|^p + \frac{1}{\eta_0} \int \left|\tau_\lambda \hat{u}\right|^s \right). \tag{6.68}$$

for all $\lambda > 0$. Then

$$\begin{aligned} \frac{1}{\eta_0}\left(\int \left|\nabla \tau_\lambda \hat{u}\right|^p + \int \left|\tau_\lambda \hat{u}\right|^s \right) &= \frac{4}{\eta_0}\left(\int_{Q^+ \cup Q^-} \left|\nabla \tau_\lambda \hat{u}\right|^p + \int_{Q^+ \cup Q^-} \left|\tau_\lambda \hat{u}\right|^s \right) \\ = \frac{4}{\eta_0}\left(\int_{\{x_2 > 0\}} \left(\left|\nabla \tau_\lambda u\right|^p + \left|\tau_\lambda u\right|^s \right) - \int_{U^+ \cup U^-} \left(\left|\nabla \tau_\lambda u\right|^p + \left|\tau_\lambda u\right|^s \right) \right). \end{aligned} \tag{6.69}$$

Choosing $\lambda > 0$ such that $F(\tau_\lambda u) = \eta_0 G(u)^\kappa$, since u is symmetric with respect to $\{x_2 = 0\}$, we get

$$\begin{aligned} \int_{\{x_2 > 0\}} \left(\left|\nabla \tau_\lambda u\right|^p + \left|\tau_\lambda u\right|^s \right) &= \frac{1}{2} F(\tau_\lambda u) \\ &= \frac{\eta_0}{2} G(u)^\kappa = \frac{\eta_0 G^\kappa}{2} (\delta(u) + 1)^\kappa \\ &\leq \frac{\eta_0 G^\kappa}{2} (1 + C_5 \delta(u)), \end{aligned} \tag{6.70}$$

where the last passage is true for $\delta(u)$ small enough. Let us consider now the function

$$v(x) = \begin{cases} \tau_\lambda u(x) & x \in U^+ \\ \tau_\lambda u(R_1 x) & x \in R_1 U^+ \\ v(S_1 x) & x \in \{x_1 < 0, x_2 > 0\} \\ v(S_2 x) & x \in \{x_2 < 0\} \end{cases}$$

where S_i is the symmetrization with respect to $\{x_i = 0\}$. We have

$$\begin{aligned}\int_{U^+} \left(\left|\nabla\tau_\lambda(u)\right|^p + \left|\tau_\lambda(u)\right|^s\right) &= \frac{1}{8}\left(\int \left(\left|\nabla v\right|^p + \left|v\right|^s\right)\right) \geq \frac{1}{8}\min_{\mu\in\mathbb{R}^+} F(\tau_\mu v) \\ &= \frac{\eta_0}{8}G(v)^\kappa \geq \frac{\eta_0 G^\kappa}{8}\left(\int v^q\right)^{\kappa/q} = \frac{\eta_0 G^\kappa}{8}\cdot 8^{\kappa/q}\left(\int_{U^+} u^q\right)^{\kappa/q} \\ &\geq \eta_0 G^\kappa 8^{-1+\kappa/q}\left(\frac{1}{8} - C_6\delta(u)^\alpha\right)^{\kappa/q} \geq \eta_0 G^\kappa\left(\frac{1}{8} - C_7\delta(u)^\alpha\right).\end{aligned} \tag{6.71}$$

with suitable constants C_6 and C_7 and for $\delta(u)$ small enough. Notice that in the last passage we used (6.65). An analogous estimate can be obtained on U^-. By (6.68), (6.69), (6.70) and (6.71) we get

$$\begin{aligned}G^\kappa\|\hat{u}\|_q^\kappa(1+\delta(\hat{u}))^\kappa &\leq \frac{4}{\eta_0}\left[\frac{\eta_0 G^\kappa}{2}(1+C_5\delta(u)) - 2\eta_0 G^\kappa\left(\frac{1}{8} - C_7\delta(u)\right)^\beta\right] \\ &= G^\kappa\left[1 + C_8\delta(u)^\alpha\right]\end{aligned} \tag{6.72}$$

for a suitable $C_8 > 0$ and where α is the minumum between 1 and β. By (6.72) and thanks to (6.66) we conclude that

$$\delta(\hat{u}) \leq \frac{1 + C_9\delta(u)^\alpha}{1 - C_9\delta(u)^\alpha} - 1 \leq C\delta(u)^\alpha$$

that is true again for $\delta(u)$ small enough. We conclude setting $\alpha_2 = \alpha$ and $\kappa_2 = C$. □

Chapter 7

Estimate of the dimension of the singular set of the MS functional: a short proof

7.1 Introduction

In this chapter we report a short note written in collaboration with Camillo De Lellis and Matteo Focardi. In this we observe that a classical result proved in [3] about the Hausdorff dimension, $\dim_{\mathcal{H}}$, of the singular set of a minimizer of the Mumford-Shah functional, can be easily derived from the fact that the blow-up of such a set is a Caccioppoli partition (see Section 7.2), as recently proved in [50]. This work is unrelated to the rest of the Thesis and we enclose it without properly introducing all the technical preliminaries exploited, which are, anyway, mentioned and recalled whenever needed.

The (localized) Mumford-Shah energy on a bounded open subset $\Omega \subset \mathbb{R}^d$ is given by

$$\mathrm{MS}(v, A) = \int_A |\nabla v|^2 dx + \mathcal{H}^{d-1}(S_v \cap A), \qquad \text{for } v \in SBV(\Omega) \text{ and } A \subseteq \Omega \text{ open} \tag{7.1}$$

In the previous definition, the space $SBV(\Omega)$ (Special functions of Bounded Variation) is that of the classical BV functions (functions of Bounded Variation) which Cantor part vanishes, while the set S_v denotes the jump set of the function $v \in SBV(\Omega)$, that is the set where the derivative of v is not absolutely continuous with respect to the Lebesgue measure. A comprehensive account on the subject is [2, Chapter 4]. In what follows if $A = \Omega$ we shall drop the dependence on the set of integration.

We recall the following result due to L. Ambrosio, N. Fusco and J. E. Hutchinson [3, Theorem 5.6].

Theorem 7.1. *Let u be a local minimizer of the Mumford-Shah energy, i.e. any function $u \in SBV(\Omega)$ with $\mathrm{MS}(u) < \infty$ and such that*

$$\mathrm{MS}(u) \leq \mathrm{MS}(w) \qquad \textit{whenever } \{w \neq u\} \subset\subset \Omega.$$

Let $\Sigma_u \subseteq \overline{S_u}$ be the set of points out of which $\overline{S_u}$ is locally regular, and let

$$\Sigma'_u := \left\{ x \in \Sigma_u : \lim_{\rho \to 0} \rho^{1-d} \int_{B_\rho(x)} |\nabla u|^2 = 0 \right\}.$$

Then, $\dim_{\mathcal{H}} \Sigma'_u \leq d - 2$.

The main interest in establishing such an estimate on the set Σ'_u, the so-called subset of triple-junctions, is related to the understanding of the Mumford-Shah conjecture (see [2, Chapter 6] for a related discussion, see also [50, Section 7]).

Indeed, Theorem 7.1, together with the higher integrability property of the approximate gradients enjoyed by minimizers as established in $2-$dimensions by [50] and more recently in any dimension by [52], imply straightforwardly an analogous estimate on the full singular set Σ_u. More precisely, in view of [50, Theorem 1] and [52, Theorem 1.1] any local minimizer u of the Mumford-Shah energy is such that $|\nabla u| \in L^p_{\mathrm{loc}}(\Omega)$ for some $p > 2$, therefore [3, Corollary 5.7] yields that

$$\dim_{\mathcal{H}} \Sigma_u \leq \max\{d - 2, d - p/2\},$$

where $\dim_{\mathcal{H}}(E)$ is the Hausdorff dimension of the set E. A characterization (of a suitable version) of the Mumford-Shah conjecture in $2-$dimensions in terms of a refined higher integrability property of the gradient in the finer scale of weak Lebesgue spaces has been recently established in [50, Proposition 5].

Our proof of Theorem 7.1 rests on a compactness result proved by C. De Lellis and M. Focardi (see [50, Theorem 13]) showing that the blow-up limits of the jump set S_u in points in the regime of small gradients, i.e. in points of Σ'_u, are minimal Caccioppoli partitions. The original approach in [3], instead, relies on the notion of Almgren's area mimizing sets, for which an involved analysis of the composition of SBV functions with Lipschitz deformations (not necessarily one-to-one) and a revision of the regularity theory for those sets are needed (cp. with [3, Sections 2, 3 and 4]).

Given [50, Theorem 13], the regularity theory of minimal Caccioppoli partitions developed in [94], [87] and [88], and standard arguments in geometric measure theory yield the conclusion, thus bypassing the above mentioned technical complications.

We describe briefly the plan of this chapter: in Section 7.2 we introduce the necessary definitions and recall some well-known facts about Caccioppoli partitions. In Section 7.3 we prove our main result, Theorem 7.1.

7.2 Caccioppoli partitions

In what follows $\Omega \subset \mathbb{R}^d$ will denote a bounded open set.

Definition 7.2. A Caccioppoli partition of Ω is a countable partition $\mathscr{E} = \{E_i\}_{i=1}^{\infty}$ of Ω in sets of (positive Lebesgue measure and) finite perimeter with $\sum_{i=1}^{\infty} P(E_i;\Omega) < \infty$.

For each Caccioppoli partition $\mathscr{E}$ we define its set of interfaces as

$$J_{\mathscr{E}} := \bigcup_{i\in} \partial^* E_i \,.$$

Here $\partial^* E$ denotes the essential boundary of the set E:

$$\partial^* E = \left\{ x \in \mathbb{R}^d : \lim_{\rho\to 0^+} \frac{D\mathbf{1}_E(B_\rho(x))}{|D\mathbf{1}_E|(B_\rho(x))} := \nu_E(x) \text{ exists and satisfies } |\nu_E(x)| = 1 \right\},$$

where $D\mathbf{1}_E$ is the distributional derivative of the characteristic function of the set E while $|D\mathbf{1}_E|$ is its total variation. Also about the main properties of $\partial^* E$ we refer to [2] and [92]. The partition $\mathscr{E}$ is said to be minimal if

$$\mathcal{H}^{d-1}(J_{\mathscr{E}}) \le \mathcal{H}^{d-1}(J_{\mathscr{F}})$$

for all Caccioppoli partitions $\mathscr{F}$ for which there exists an open subset $\Omega' \subset\subset \Omega$ with

$$\sum_{i=1}^{\infty} \big|(F_i \triangle E_i) \cap (\Omega \setminus \Omega')\big| = 0.$$

Definition 7.3. Given a Caccioppoli partition $\mathscr{E}$ we define its singular set $\Sigma_{\mathscr{E}}$ as the set of points for which the approximate tangent plane to $J_{\mathscr{E}}$ does not exist. That is, the complementary of $\Sigma_{\mathscr{E}}$ is the set of points x such that $(\mathscr{E} - x)/\rho$ has a weak limit point as $\rho \to 0^+$, as a Radon measure (called approximate tangent space). See [2, Section 2.11].

A characterization of the singular set $\Sigma_{\mathscr{E}}$ for minimal Caccioppoli partitions in the spirit of $\varepsilon-$regularity results is provided in the ensuing statement (cp. with [88, Corollary 4.2.4] and [92, Theorem III.6.5]).

Theorem 7.4. *Let Ω be an open set and $\mathscr{E} = \{E_i\}_{i\in}$ a minimal Caccioppoli partition of Ω.*

Then, there exists a dimensional constant $\varepsilon = \varepsilon(d) > 0$ such that

$$\Sigma_{\mathscr{E}} = \left\{ x \in \Omega \cap \overline{J_{\mathscr{E}}} : \inf_{B_\rho(x) \subset\subset \Omega} e(x,\rho) \ge \varepsilon \right\}, \tag{7.2}$$

where $e(x,\rho)$ denotes the spherical excess of $\mathscr{E}$ at the point $x \in J_{\mathscr{E}}$ at the scale $\rho > 0$, that is

$$e(x,\rho) := \min_{\nu\in\mathbb{S}^{d-1}} \frac{1}{\rho^{d-1}} \int_{B_\rho(x)\cap J_{\mathscr{E}}} \frac{|\nu_{\mathscr{E}}(y) - \nu|^2}{2} d\mathcal{H}^{d-1}(y).$$

We recall next a result that is probably well-known in literature.

Theorem 7.5. *Let $\mathscr{E}$ be a minimal Caccioppoli partition in Ω, then* $\dim_{\mathcal{H}}\Sigma_{\mathscr{E}} \leq d-2$. *If, in addition, $d = 2$, then $\Sigma_{\mathscr{E}}$ is locally finite.*

Proof. We apply the abstract version of Federer's reduction argument in [105, Theorem A.4] with the set of functions

$$\mathcal{F} = \{\mathbf{1}_{J_{\mathscr{E}}} : \mathscr{E} \text{ is a minimal Caccioppoli partition}\}$$

endowed with the convergence

$$\mathbf{1}_{J_{\mathscr{E}_h}} \to \mathbf{1}_{J_{\mathscr{E}}} \iff \lim_{h\to\infty}\int_{J_{\mathscr{E}_h}} g\, d\mathcal{H}^{d-1} = \int_{J_{\mathscr{E}}} g\, d\mathcal{H}^{d-1}, \quad \text{for all } g \in C^1_c(\Omega).$$

and singularity map $\mathrm{sing}(\mathbf{1}_{\mathscr{E}}) = \Sigma_{\mathscr{E}}$.

It is easy to see that condition $A.1$ (closure under scaling) and $A.3(2)$ hold true. Moreover, the blow-ups of a minimal Caccioppoli partition converge to a minimizing cone (see [87, Theorem 3.5], or [88, Theorem 4.4.5 (a)]), so that $A.2$ holds as well. About $A.3(1)$, we notice that the singular set of an hyperplane is empty. Eventually, if a sequence $(\mathbf{1}_{J_{\mathscr{E}_h}})_{h\in\mathbb{N}} \subseteq \mathcal{F}$ converges to $\mathbf{1}_{J_{\mathscr{E}}}$ and $(x_h)_{h\in\mathbb{N}}$ converges to x, with $x_h \in \Sigma_{\mathscr{E}_h}$ for all h, then by the continuity of the excess and the characterization in (7.2), $x \in \Sigma_{\mathscr{E}}$, so that condition $A.3(3)$ is satisfied as well.

To conclude, we recall that [105, Theorem A.4] itself ensures that the set $\Sigma_{\mathscr{E}}$ is locally finite being in this setting $\dim_{\mathcal{H}}\Sigma_{\mathscr{E}} = 0$. □

7.3 Proof of the main result

We are now ready to prove the main result of the chapter following the approach exploited in [3, Theorem 5.6]. To this aim we recall that in [2, Theorems 8.1-8.3] it is characterized alternatively the singular set Σ_u as follows

$$\Sigma_u = \{x \in \overline{S_u} : \underline{\lim}_{\rho\to 0}\, (\mathscr{D}(x,\rho) + \mathscr{A}(x,\rho)) \geq \varepsilon_0\}, \tag{7.3}$$

where ε_0 is a dimensional constant, and the scaled Dirichlet Energy and the scaled mean-flatness are respectively defined as

$$\mathscr{D}(x,\rho) := \rho^{1-d}\int_{B_\rho(x)} |\nabla u|^2 dy, \quad \mathscr{A}(x,\rho) := \rho^{-1-d}\min_{T\in\Pi}\int_{S_u\cap B_\rho(x)} \mathrm{dist}^2(y,T)d\mathcal{H}^{d-1}(y),$$

with Π the set of all affine $(d-1)-$hyperplanes in $\mathbb{R}^d$.

Proof of Theorem 7.1. We argue by contradiction: suppose that there exists $s > d-2$ such that $\mathcal{H}^s(\Sigma'_u) > 0$. From this, we infer that $\mathcal{H}^s_\infty(\Sigma'_u) > 0$, (here $\mathcal{H}^s_\infty$ is the pre-Hausdorff measure, see [2, Definition 2.46]) and moreover that for $\mathcal{H}^s-$a.e. $x \in \Sigma'_u$ it holds

$$\overline{\lim}_{\rho\to 0^+}\frac{\mathcal{H}^s_\infty(\Sigma'_u \cap B_\rho(x))}{\rho^s} \geq \frac{\omega_s}{2^s} \tag{7.4}$$

where ω_s is the $s-$dimensional Hausdorff measure of the unitary ball (see for instance [2, Theorem 2.56 and formula (2.43)] or [92, Lemma III.8.15]). Without loss of generality, suppose that (7.4) holds at $x = 0$, and consider a sequence $\rho_h \to 0$ for which

$$\mathcal{H}^s_\infty(\Sigma'_u \cap B_{\rho_h}) \geq \frac{\omega_s}{2^{s+1}} \rho_h^s \qquad \text{for all } h \in \mathbb{N}. \tag{7.5}$$

[50, Theorem 13] provides a subsequence, not relabeled for convenience, and a minimal Caccioppoli partition $\mathscr{E}$ such that

$$\mathcal{H}^{d-1} \llcorner \rho_h^{-1} S_u \overset{*}{\rightharpoonup} \mathcal{H}^{d-1} \llcorner J_{\mathscr{E}}, \ \text{ and } \quad \rho_h^{-1}\overline{S_u} \to \overline{J_{\mathscr{E}}} \ \text{ locally Hausdorff.} \tag{7.6}$$

In turn, from the latter we claim that if $\mathcal{F}$ is any open cover of $\Sigma_{\mathscr{E}} \cap \overline{B}_1$, then for some $h_0 \in \mathbb{N}$

$$\rho_h^{-1}\Sigma'_u \cap \overline{B}_1 \subseteq \bigcup_{F \in \mathcal{F}} F \qquad \text{for all } h \geq h_0. \tag{7.7}$$

Indeed, if this is not the case we can find a sequence $x_{h_j} \in \rho_{h_j}^{-1}\Sigma'_u \cap \overline{B}_1$ converging to some point $x_0 \notin \Sigma_{\mathscr{E}}$. If $\pi^{\mathscr{E}}_{x_0}$ is the approximate tangent plane to $J_{\mathscr{E}}$ at x_0, that exists by the very definition of $\Sigma_{\mathscr{E}}$, then for some ρ_0 we have

$$\rho^{1-d} \int_{B_\rho(x_0) \cap J_{\mathscr{E}}} \mathrm{dist}^2(y, \pi^{\mathscr{E}}_{x_0}) d\mathcal{H}^{d-1} < \varepsilon_0, \qquad \text{for all } \rho \in (0, \rho_0).$$

In turn, from the latter inequality it follows for $\rho \in (0, \rho_0 \wedge 1)$

$$\overline{\lim}_{j \to \infty} \int_{B_\rho(x_{h_j}) \cap \rho_{h_j}^{-1} S_u} \mathrm{dist}^2(y, \pi^{\mathscr{E}}_{x_0})\, d\mathcal{H}^{d-1} < \varepsilon_0.$$

Therefore, as $x_{h_j} \in \rho_{h_j}^{-1}\Sigma'_u$, we get for j large enough

$$\overline{\lim}_{\rho \to 0} \big(\mathscr{D}(x_{h_j}, \rho) + \mathscr{A}(x_{h_j}, \rho)\big) < \varepsilon_0,$$

a contradiction in view of the characterization of the singular set in (7.3).

To conclude, we note that by (7.7) we get

$$\mathcal{H}^s_\infty(\Sigma_{\mathscr{E}} \cap \overline{B_1}) \geq \overline{\lim}_{h \to \infty} \mathcal{H}^s_\infty(\rho_h^{-1}\Sigma'_u \cap \overline{B_1});$$

given this, (7.5) and (7.6) yield that

$$\mathcal{H}^s(\Sigma_{\mathscr{E}} \cap \overline{B_1}) \geq \mathcal{H}^s_\infty(\Sigma_{\mathscr{E}} \cap \overline{B_1}) \geq \overline{\lim}_{h \to \infty} \mathcal{H}^s_\infty(\rho_h^{-1}\Sigma'_u \cap \overline{B_1}) \geq \frac{\omega_s}{2^{s+1}},$$

thus contradicting Theorem 7.5. □

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