Noname manuscript No.
(will be inserted by the editor)

A structure property of "vertical" integral currents, with an application to the distributional determinant

Domenico Mucci

Received: date / Accepted: date

Abstract We deal with integral currents in Cartesian products of Euclidean spaces that satisfy a "verticality" assumption. The main example is the boundary of the graph of some classes vector-valued and non-smooth Sobolev maps, provided that the boundary current has finite mass. In fact, the action of such currents is non-zero only on forms with a high number (depending on the Sobolev regularity) of differentials in the direction of the vertical space. We prove that such vertical currents live on a set that projects on the horizontal space into a nice set with integer dimension. The dimension of the concentration set is related to the level of verticality that is assumed. Therefore, for boundary of graphs of Sobolev maps, this dimension decreases as the Sobolev exponent increases. As an application, we then prove a concentration property concerning the singular part of the distributional determinant and minors.

Keywords Rectifiable currents · Boundary of the graph · Distributional determinant

Mathematics Subject Classification (2000) 49Q15 · 28A75

1 Introduction

In this paper we discuss some structure properties concerning "vertical" integral currents. Roughly speaking, since we consider currents T in the product $\mathbb{R}^n \times \mathbb{R}^N$ of a "horizontal" and "vertical" Euclidean space, the adjective "vertical" refers to the property that the action of T is null on forms that contain a bounded number of differentials in the vertical directions.

More precisely, denoting by x and y the variables in $\mathbb{R}^n$ and $\mathbb{R}^N$, respectively, for any integers $0 \leq h \leq k \leq n+N$, we define by $T_{(h)}$ the restriction of a current T in $\mathcal{D}_k(\mathbb{R}^n \times \mathbb{R}^N)$ to the k-forms in $\mathbb{R}^n \times \mathbb{R}^N$ that contain exactly h differentials dy^j in the vertical directions y, compare formula (2.3) below.

D. Mucci
Dipartimento di Matematica e Informatica dell'Università di Parma, Parco Area delle Scienze 53/A, I-43124 Parma, Italy.
E-mail: domenico.mucci@unipr.it

Referring to Sec. 2 for the standard notation of Geometric Measure Theory, we now simply observe that the component $T_{(h)}$ makes sense only if $\max\{0,k-n\}\leq h\leq\min\{k,N\}$. Therefore, for simplicity we shall possibly denote $T_{(h)}:=0$ for h strictly lower than $\max\{0,k-n\}$ or strictly larger than $\min\{k,N\}$.

A STRUCTURE PROPERTY. We first consider "completely vertical" currents T, i.e., satisfying

$$T_{(h)}=0 \qquad \text{for } h=0,\dots,k-1\,. \tag{1.1}$$

By the previous remark, if $T\in\mathcal{D}_k(\mathbb{R}^n\times\mathbb{R}^N)$ satisfies (1.1), one automatically has $T=0$ if $N\leq k-1$. Therefore, in the following result we assume $N\geq k$.

Theorem 1.1 (Structure property I) *Let $n\geq 1$ and $N\geq k\geq 1$ integers. Let $T\in\mathcal{R}_k(\mathbb{R}^n\times\mathbb{R}^N)$ be an i.m. rectifiable current satisfying $\mathbf{M}(T)+\mathbf{M}(\partial T)<\infty$ and the "verticality" property* (1.1)*. Then there exists an at most countable set of points $\{a_i\}_i\subset\mathbb{R}^n$ and of i.m. rectifiable currents $\Sigma_i\in\mathcal{R}_k(\mathbb{R}^N)$ such that*

$$T=\sum_{i=0}^{\infty}\delta_{a_i}\times\Sigma_i\,,\qquad \mathbf{M}(T)=\sum_{i=0}^{\infty}\mathbf{M}(\Sigma_i)<\infty\,. \tag{1.2}$$

If in particular T is an integral cycle, i.e., $\partial T=0$, then $\partial\Sigma_i=0$ for each i, and $T=0$ in the case $N=k$.

A GENERAL STRUCTURE PROPERTY. We now replace the "verticality" assumption (1.1) with the following more general one:

$$T_{(h)}=0 \qquad \text{for } h=0,\dots,\mathbf{q}-1\,, \tag{1.3}$$

where $\mathbf{q}$ is any positive integer such that $1\leq\mathbf{q}\leq k$. As before, if a current $T\in\mathcal{D}_k(\mathbb{R}^n\times\mathbb{R}^N)$ satisfies (1.3), in low dimension $N<\mathbf{q}$ one automatically has $T=0$, so that we assume $N\geq\mathbf{q}$. We shall then prove:

Theorem 1.2 (Structure property II) *Let $n\geq 1$ and $N\geq\mathbf{q}$ integers, with $1\leq\mathbf{q}\leq k$. Let $T\in\mathcal{R}_k(\mathbb{R}^n\times\mathbb{R}^N)$ an i.m. rectifiable current satisfying $\mathbf{M}(T)+\mathbf{M}(\partial T)<\infty$ and the "verticality" property* (1.3)*. Then there exists a countably $\mathcal{H}^{k-\mathbf{q}}$-rectifiable subset $S_{k-\mathbf{q}}$ of $\mathbb{R}^n$ such that*

$$\operatorname{set}(T)\subset S_{k-\mathbf{q}}\times\mathbb{R}^N\,.$$

If in particular T is an integral cycle, i.e., $\partial T=0$, then $T=0$ in the case $N=\mathbf{q}$.

Remark 1.3 In the structure theorems we do not assume that the current T is compactly supported. We now see that this generality allows us to apply such results to the boundary of the current carried by the "graph" of Sobolev maps.

BOUNDARY OF GRAPHS. In fact, if $u:\mathbb{R}^n\to\mathbb{R}^N$ is smooth, the current G_u carried by the graph of u is well-defined by the integration of compactly supported smooth n-forms ω in $\mathbb{R}^n\times\mathbb{R}^N$ over the naturally oriented n-manifold given by the graph $\mathcal{G}_u$ of u:

$$G_u(\omega):=\int_{\mathcal{G}_u}\omega\,,\qquad \omega\in\mathcal{D}^n(\mathbb{R}^n\times\mathbb{R}^N)\,. \tag{1.4}$$

Moreover, G_u is locally i.m. rectifiable in $\mathcal{R}_{n,\text{loc}}(\mathbb{R}^n \times \mathbb{R}^N)$, and denoting by $(\text{Id}_{\mathbb{R}^n} \bowtie u)(x) := (x, u(x))$ the graph map, the area formula yields

$$G_u(\omega) = \int_{\mathbb{R}^n} (\text{Id}_{\mathbb{R}^n} \bowtie u)^{\#} \omega \qquad \forall \omega \in \mathcal{D}^n(\mathbb{R}^n \times \mathbb{R}^N). \tag{1.5}$$

Assume now that u is a Sobolev map in $W^{1,1}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^N)$ such that *each minor of the Jacobian matrix* ∇u *is in* $L^1_{\text{loc}}(\mathbb{R}^n)$. Following the theory by Giaquinta-Modica-Souček [9], see also [10], the above definition (1.4) holds true in a measure-theoretic sense, and (1.5) is obtained by means of the approximate gradient ∇u.[1]

Now, if u is smooth, by Stokes' theorem the current G_u has null boundary.[2] Moreover, by a density argument, the null-boundary condition $\partial G_u = 0$ extends to maps in $W^{1,p}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^N)$, where $p = \min\{n, N\}$. However, in general $\partial G_u \neq 0$, as the example 1.5 below taken from [9, Sec. 3.2.2] shows. Arguing as in [9, Sec. 3.2.3] or [10, Prop. 4.22], it turns out that a summability assumption yields to a verticality property of the boundary current ∂G_u, namely:[3]

Proposition 1.4 *Let* $1 \leq p < n$ *and* $\mathbf{q}$ *the integer part of* p*. If in addition* $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^N)$*, the boundary current* $T := \partial G_u$ *satisfies the verticality property* (1.3).

Assume now that u is smooth outside some compact set K of $\mathbb{R}^n$, and that $\mathbf{M}(\partial G_u) < \infty$. Then by the boundary rectifiability theorem 2.4, the boundary current $T = \partial G_u$ is i.m. rectifiable in $\mathcal{R}_{n-1}(\mathbb{R}^n \times \mathbb{R}^N)$, with support $\text{spt}\, \partial G_u \subset K \times \mathbb{R}^N$, whereas $\partial T = \partial(\partial G_u) = 0$. Therefore, if in addition u satisfies the summability hypothesis of Proposition 1.4 we can apply Theorem 1.2, with $k = n-1$, and obtain the existence of a countably $\mathcal{H}^{n-1-\mathbf{q}}$-rectifiable subset $S_{n-1-\mathbf{q}}$ of K such that

$$\text{set}(\partial G_u) \subset S_{n-1-\mathbf{q}} \times \mathbb{R}^N. \tag{1.6}$$

Example 1.5 Let $\mathbf{q} \geq 2$ integer and $u : \mathbb{R}^{\mathbf{q}} \to \mathbb{R}^{\mathbf{q}}$ given by $u(x) := x/|x|$, so that $u \in W^{1,q}_{\text{loc}}$ for each $q < \mathbf{q}$. We have $\partial G_u = -\delta_0 \times [\![\mathbb{S}^{\mathbf{q}-1}]\!]$, where δ_0 is the unit Dirac mass at the origin and $[\![\mathbb{S}^{\mathbf{q}-1}]\!]$ in the $(\mathbf{q}-1)$-current integration on the (positively oriented) unit $(\mathbf{q}-1)$-sphere in the target space. Notice that $\det \nabla u = 0$ a.e., but $u \notin W^{1,\mathbf{q}}_{\text{loc}}$. By adding $n - \mathbf{q}$ dumb x-variables to the map in previous example, one easily infers that (1.6) fails to hold for maps u outside the Sobolev class $W^{1,\mathbf{q}}_{\text{loc}}$.

Therefore, the Sobolev regularity in Proposition 1.4 is optimal for $\mathbf{q} \geq 2$, whereas for $\mathbf{q} = 1$ the optimality follows from [9, Sec. 3.2.3, Prop. 1].

Distributional determinant. As an application, in Sec. 8 below we shall discuss some new properties concerning the singular part of the *distributional determinant* $\text{Det}\, \nabla u$, first introduced by J.M. Ball [5].

Let $n = N$ and $u \in L^q_{\text{loc}} \cap W^{1,p}_{\text{loc}}(\mathbb{R}^n, \widehat{\mathbb{R}}^n)$ for some exponents q and p satisfying $n - 1 \leq p < n$ and $1/q + (n-1)/p \leq 1$. Then the distributional determinant

[1] The countably $\mathcal{H}^n$-rectifiable set $\mathcal{G}_u$ is the subset of $\mathbb{R}^n \times \mathbb{R}^N$ given by the points $(x, u(x))$, where x is a Lebesgue point of both u and ∇u and $u(x)$ is the Lebesgue value of u. Recall that for $W^{1,1}_{\text{loc}}$-maps the approximate gradient ∇u agrees with the distributional derivative Du.

[2] We have $\partial G_u(\eta) := G_u(d\eta) = \int_{\mathcal{G}_u} d\eta = \int_{\partial \mathcal{G}_u} \eta = 0$ for every $\eta \in \mathcal{D}^{n-1}(\mathbb{R}^n \times \mathbb{R}^N)$.

[3] In fact, (1.5) yields that for any $\eta \in \mathcal{D}^{n-1}(\mathbb{R}^n \times \mathbb{R}^N)$ the integral representing $\partial G_u(\eta^{(h)}) := G_u(d(\eta^{(h)}))$ involves minors of ∇u of order at most $h+1$. Choosing a sequence $\{u_j\} \subset C^\infty(\mathbb{R}^n, \mathbb{R}^N)$ converging to u strongly in $W^{1,\mathbf{q}}_{\text{loc}}$, by dominated convergence $G_{u_j}(d(\eta^{(h)})) \to G_u(d(\eta^{(h)}))$ if $h \leq \mathbf{q} - 1$. Since $\partial G_{u_j} = 0$ for each j, one obtains (1.3).

is well-defined by the formula (8.1) below. De Lellis-Ghiraldin [6] extended a decomposition property first obtained by S. Müller [16], and conjectured by Ball [5], showing that if in addition the pointwise determinant $\det \nabla u$ is locally summable, then $\operatorname{Det} \nabla u$ is a signed Radon measure, the density w.r.t. the Lebesgue measure $\mathcal{L}^n$ being $\det \nabla u$. With the above assumptions, if u is smooth outside some compact set $K \subset \mathbb{R}^n$, and $\mathbf{M}(\partial G_u) < \infty$, we have seen that (1.6) holds, where $N = n$ and $\mathbf{q} = n - 1$, so that S_0 in (1.6) is a countable set. By means of Theorem 1.1 we shall then prove, Theorem 8.1, that *the singular part of* $\operatorname{Det} \nabla u$ *is concentrated on an at most countable set of points*, namely on S_0.

We also deal with the *distributional minors of order* m. For $m = N < n$, they agree with the components of the distributional Jacobian, first studied by Jerrard-Soner [12]. An interesting review concerning the distributional Jacobian and singularities of Sobolev maps into spheres can be found in [1]. In Theorem 8.2, under suitable (and optimal) summability hypotheses, see (8.9), as a consequence of Theorem 1.2 we shall prove that *the singular part of each distributional minors of order* m *is concentrated on a countably rectifiable set of codimension* m.

Remark 1.6 Condition $\mathbf{M}(\partial G_u) < \infty$ *is necessary to the validity of Theorems* 8.1 *and* 8.2. In fact, S. Müller [17] showed that for $n = N = 2$ the singular part of the distributional determinant may in general concentrate on a set of Hausdorff dimension α, for any prescribed $0 < \alpha < 1$. More precisely, there exist bounded Hölder continuous Sobolev functions u in $W^{1,p}(\Omega, \mathbb{R}^2)$ for every $p < 2$, where $\Omega = (0,1)^2 \subset \mathbb{R}^2$, such that $\det \nabla u = 0$ and $|\nabla u^1| |\nabla u^2| = 0$ a.e. in Ω, but $\operatorname{Det} \nabla u = V' \otimes V'$, where V is the Cantor-Vitali function. Therefore, the distributional determinant has a "Cantor-type" part and the role played by V' in the Cantor set C is here played by $\operatorname{Det} \nabla u$ in $C \times C$.

The "graph" of u is very similar to the graph of the Cantor-Vitali function V and, actually, has infinitely many holes. In fact, in [9, Sec. 4.2.5] it is shown that in such an example one has $\mathbf{M}(\partial G_u) = \infty$.

PLAN OF THE PAPER. Sec. 2 contains some notation and preliminary results. In Sec. 3, we extend the *isoperimetric inequality* from [15, Prop. 2.1]. In Sec. 4, we consider a *projection argument* that allows to recover the action of a current in terms of the projected currents onto suitable coordinate subspaces. These results are used in Sec. 5 to prove Theorem 1.1 in the case of integral currents, i.e., when $\partial T = 0$. Sec. 6 contains the proof of Theorem 1.2, whereas in Sec. 7 we deal with the more general case of normal currents, i.e., when $\mathbf{M}(\partial T) < \infty$. Finally, in Sec. 8 we shall prove the already mentioned concentration property concerning the singular part of the distributional determinant and minors.

2 Notation and preliminary results

In this section we collect some notation and preliminary results. We refer to [2, 7, 9, 13, 18] for general facts about Geometric Measure Theory, whereas further details concerning currents carried by graphs can be found in [9] or [10].

RECTIFIABLE SETS. Let U an open set in $\mathbb{R}^D$ and $\mathcal{H}^k$ the k-dimensional Hausdorff measure on $\mathbb{R}^D$. For $1 \leq k \leq D$ integer, a set $\mathcal{M} \subset U$ is said to be *countably* $\mathcal{H}^k$*-rectifiable* if it is $\mathcal{H}^k$-measurable and $\mathcal{H}^k$-almost all of $\mathcal{M}$ is contained in the union of the images of countably many Lipschitz functions from $\mathbb{R}^k$ to U, compare [7, 3.2.14]. The set $\mathcal{M}$ is said to be *k-rectifiable* if in addition $\mathcal{H}^k(\mathcal{M}) < \infty$.

Remark 2.1 The rectifiability criterium by Besicovitch-Marstrand-Mattila, see [7] or [4, Thm. 2.63], states that if $A \subset \mathbb{R}^D$ is a Borel set satisfying $\mathcal{H}^k(A) < \infty$, then A is k-rectifiable if and only if the *k-dimensional density* $\Theta^k(\mathcal{H}^k, A, x)$ is equal to one for $\mathcal{H}^k$-a.e. $x \in A$. This yields that k-rectifiable sets can be “fractured”.

General area-coarea formula. The following theorem by Federer [7, 3.2.2] subsumes both the area and coarea formulas, compare [13, 3.13].

Theorem 2.2 *Let $\mathcal{M} \subset \mathbb{R}^{D_1}$ a k-rectifiable set and $\mathcal{N}$ a μ-rectifiable subset of $\mathbb{R}^{D_2}$, where $D_1 \geq D_2 \geq 1$ and $k \geq \mu$. Let $f : \mathbb{R}^{D_1} \to \mathbb{R}^{D_2}$ a Lipschitz function such that $f(\mathcal{M}) = \mathcal{N}$. Then, for any $\mathcal{H}^k \llcorner \mathcal{M}$-integrable function $\psi : \mathcal{M} \to \mathbb{R}$ we have*

$$\int_{\mathcal{M}} J_f^{\mathcal{M}}(w)\, \psi(w)\, d\mathcal{H}^k(w) = \int_{\mathcal{N}} \Big(\int_{\mathcal{M}\cap f^{-1}(\{z\})} \psi \, d\mathcal{H}^{k-\mu} \Big)\, d\mathcal{H}^{\mu}(z)\,.$$

In this formula, $J_f^{\mathcal{M}}$ denotes the *k-dimensional tangential Jacobian* of $f_{|\mathcal{M}} : \mathcal{M} \to \mathbb{R}^{D_2}$, compare e.g. [9, Sec. 2.1.5].[4]

Rectifiable currents. We shall denote by $\mathcal{E}^k(U)$, $\mathcal{E}_b^k(U)$, and $\mathcal{D}^k(U)$ the spaces of smooth, bounded smooth, and compactly supported smooth k-forms in U, respectively. The (strong) dual space to $\mathcal{D}^k(U)$ is the class of k-currents $\mathcal{D}_k(U)$.[5] For each open set $V \subset U$ the *mass of a current* $T \in \mathcal{D}_k(U)$ *in* V is[6]

$$\mathbf{M}_V(T) := \sup\{T(\omega) \mid \omega \in \mathcal{D}^k(U)\,,\ \|\omega\| \leq 1\,,\ \operatorname{spt}\omega \subset V\}\,,$$

and $\mathbf{M}(T) := \mathbf{M}_U(T)$ denotes the *mass* of T. If a current $T \in \mathcal{D}_k(U)$ has locally finite mass, i.e., $\mathbf{M}_V(T) < \infty$ for each open set $V \subset\subset U$, then

$$T(\omega) = \int_{\mathcal{M}} \langle \omega(z), \overrightarrow{\xi}(z)\rangle\, \theta(z)\, d\mathcal{H}^k(z) \qquad \forall\, \omega \in \mathcal{D}^k(U)$$

where $\mathcal{M} \subset U$ is a countably $\mathcal{H}^k$-rectifiable set, the multiplicity $\theta : \mathcal{M} \to]0, +\infty]$ is $\mathcal{H}^k$-measurable and locally $(\mathcal{H}^k \llcorner \mathcal{M})$-summable, and $\overrightarrow{\xi} : \mathcal{M} \to \Lambda_k\mathbb{R}^m$ is $\mathcal{H}^k$-measurable with $|\overrightarrow{\xi}| = 1$ $(\mathcal{H}^k \llcorner \mathcal{M})$-a.e.. In this case, one writes $T = \tau(\mathcal{M}, \theta, \overrightarrow{\xi})$.

A current T is said to be an *integer multiplicity* (i.m) *rectifiable current*, $T \in \mathcal{R}_k(U)$, if in addition T has finite mass, the density θ takes integer values, and for $\mathcal{H}^k$-a.e. $z \in \mathcal{M}$ the unit k-vector $\overrightarrow{\xi}(z) \in \Lambda_k\mathbb{R}^m$ provides an orientation to the approximate tangent space to $\mathcal{M}$ at z. Moreover, $\operatorname{set}(T)$ denotes the *set of positive multiplicity* θ *in* $\mathcal{M}$, so that $\mathcal{H}^k(\operatorname{set}(T)) \leq \mathbf{M}(T) < \infty$ for every $T \in \mathcal{R}_k(U)$, and $\mathbf{M}(T) = \int_{\mathcal{M}} \theta\, d\mathcal{H}^k$. Notice that for such currents the support of T agrees with the closure of $\operatorname{set}(T)$, and in general $\mathcal{H}^k(\operatorname{spt} T) \leq \infty$.

If $T \in \mathcal{D}_k(U)$ has finite mass, by dominated convergence the action of T extends to forms $\omega \in \mathcal{E}_b^k(U)$, or even to k-forms with bounded Borel coefficients in U. In particular, the restriction $T \llcorner B$ is well-defined for each Borel set $B \in \mathcal{B}(U)$. Since we shall work with *currents with no compact support*,[7] we shall use the

[4] For $k = \mu$ one has $J_f^{\mathcal{M}}(w) := (\det[(d^{\mathcal{M}} f_w)^*(d^{\mathcal{M}} f_w)])^{1/2}$ for $\mathcal{H}^k$-a.e. $w \in \mathcal{M}$.

[5] Therefore, $\mathcal{D}_0(U)$ is the usual space of distributions in U.

[6] Here we have denoted by $\|\omega\|$ the *comass norm* of ω. Using the standard Euclidean norm of ω, one obtains an equivalent notion of mass that agrees with the previous one for i.m. rectifiable currents.

[7] The support of T is defined exactly as for distributions.

symbol "$_{,c}$" when referring to subclasses of currents with compact support. Also, $\mathcal{R}_{k,\mathrm{loc}}(U)$ denotes the class of currents T with locally finite mass and such that $T \llcorner K \in \mathcal{R}_{k,\mathrm{loc}}(U)$ for each compact set $K \subset U$. Moreover, a current $T \in \mathcal{R}_k(U)$ is a *normal* current if in addition $\mathbf{M}((\partial T) \llcorner U) < \infty$,[8] and T is an *integral cycle* if $(\partial T) \llcorner U = 0$. Finally, the subclass $\mathcal{P}_k(U)$ of *integral polyhedral chains* is the abelian group (with integer coefficients) generated by oriented k-simplices in U.

Main properties. The fundamental theorem by Federer-Fleming [8] makes i.m. rectifiable currents very natural and important, especially in connection with the calculus of variations:[9]

Theorem 2.3 (Closure-compactness) *Let $\{T_j\} \subset \mathcal{R}_k(U)$ a sequence of i.m. rectifiable currents satisfying $\sup_j [\,\mathbf{M}(T_j \llcorner V) + \mathbf{M}((\partial T_j) \llcorner V)\,] < \infty$ for each open set $V \subset\subset U$. If T_j weakly converges to some current $T \in \mathcal{D}_k(U)$, then $T \in \mathcal{R}_k(U)$. Otherwise, there exists a subsequence $\{T_{j'}\}$ of $\{T_j\}$ and an i.m. rectifiable current $T \in \mathcal{R}_k(U)$ such that $T_{j'} \rightharpoonup T$.*

Since the Deformation theorem holds true for normal currents $T \in \mathcal{D}_k(U)$, not necessarily with compact support, compare [2, 1.16], one obtains:

Theorem 2.4 (Boundary rectifiability) *Let $T \in \mathcal{R}_k(U)$ satisfy $\mathbf{M}((\partial T) \llcorner U) < \infty$. Then the boundary of T is i.m. rectifiable too, i.e., $(\partial T) \llcorner U \in \mathcal{R}_{k-1}(U)$.*

As a consequence, compare [2, 2.11], arguing as in [7, 4.2.20] one also proves:

Theorem 2.5 (Strong polyhedral approximation) *Let $T \in \mathcal{R}_k(U)$ such that $\mathbf{M}((\partial T) \llcorner U) < \infty$. Then for each $j \in \mathbb{N}^+$ we can find an integral polyhedral chain $P_j \in \mathcal{P}_k(U)$ and a C^1-diffeomorphism g_j of U onto itself such that $\mathrm{Lip}(g_j) \leq 1+1/j$, $\mathrm{Lip}(g_j^{-1}) \leq 1+1/j$, and $\mathbf{M}(g_{j\#}T - P_j) + \mathbf{M}(\partial(g_{j\#}T - P_j) \llcorner U) \leq 1/j$.*

Integral cycles. We shall need the following

Lemma 2.6 *Let $T \in \mathcal{R}_k(U)$ satisfy $(\partial T) \llcorner U = 0$. Then we have $T(d\eta) = 0$ for every $(k-1)$-form η with Lipschitz coefficients and support contained in U.*

Proof For $R > 0$, we choose a cut-off function $\chi_R \in C_c^\infty([0,+\infty))$ such that $\chi_R(t) = 1$ for $0 \leq t \leq R$, $\chi_R(t) = 0$ for $t \geq R+1$, $0 \leq \chi_R \leq 1$ and $|\chi_R'| \leq 2$. Since $\chi_R(|z|)\,\eta$ is compactly supported in U, condition $(\partial T) \llcorner U = 0$ yields that $T(d[\chi_R(|y|)\,\eta]) = 0$, whence

$$T(d\eta) = T\big(d[(1-\chi_R(|y|))\,\eta]\big)\,. \tag{2.1}$$

Set $U_R := \{z \in U \,:\, |z| \geq R\}$ and $W_j := U_j \setminus U_{j+1}$, for $j \in \mathbb{N}$. Since T has finite mass, one has

$$\lim_{R\to\infty} \mathbf{M}(T \llcorner U_R) = 0\,, \qquad \liminf_{j\to\infty} j \cdot \mathbf{M}(T \llcorner W_j) = 0\,. \tag{2.2}$$

8 If $k \geq 1$ the boundary current $\partial T \in \mathcal{D}_{k-1}(U)$ is defined by duality for any $T \in \mathcal{D}_k(U)$ through the formula $\partial T(\eta) := T(d\eta)$ for every $\eta \in \mathcal{D}^{k-1}(U)$.

9 The weak convergence $T_j \rightharpoonup T$ in $\mathcal{D}_k(U)$ is defined in the dual sense by requiring that $T_j(\omega) \to T(\omega)$ for every test form $\omega \in \mathcal{D}^k(U)$, so that the mass is sequentially weakly lower semicontinuous. Therefore if a sequence $\{T_j\} \subset \mathcal{D}_k(U)$ satisfies $\sup_j \mathbf{M}(T_j) < \infty$, there exists a subsequence $\{T_{j'}\}$ of $\{T_j\}$ and a current $T \in \mathcal{D}_k(U)$ with finite mass such that $T_{j'} \rightharpoonup T$.

Moreover,

$$d[(1-\chi_R(|z|))\,\eta] = -\chi_R'(|z|)\,d|z|\wedge\eta + (1-\chi_R(|z|)\,d\eta\,.$$

Therefore, taking $R=j$, by (2.1) we estimate for each j

$$|T(d\eta)| \le c\,\|\eta\|_{\infty,W_j}\,\mathbf{M}(T \llcorner W_j) + \|d\eta\|_\infty\,\mathbf{M}(T \llcorner U_j)\,.$$

Since η has Lipschitz coefficients and support contained in U, we get

$$\|\eta\|_{\infty,W_j} \le c_1(1+\|z\|_{\infty,W_j}) \le c_2(1+j)\,, \qquad \|d\eta\| \le c_3$$

for some absolute constants $c_i>0$. Hence for each j

$$|T(d\eta)| \le c_2(1+j)\,\mathbf{M}(T \llcorner W_j) + c_3\,\mathbf{M}(T \llcorner U_j)$$

and the claim follows by taking a subsequence according to (2.2). □

Notation for multi-indices. Recall that x and y denote the variables in the horizontal and vertical spaces $\mathbb{R}^n$ and $\mathbb{R}^N$, respectively. If $\alpha=(\alpha_1,\dots,\alpha_p)$, where $1\le\alpha_1<\cdots<\alpha_p\le n$, is a multi-index of length $|\alpha|=p\le n$, we set $x_\alpha:=(x_{\alpha_1},\dots,x_{\alpha_p})$ and $dx^\alpha:=dx^{\alpha_1}\wedge\cdots\wedge dx^{\alpha_p}$, where $x=(x_1,\dots,x_n)$. We also say that the positive integer i belongs to α if it is one of the indices $\alpha_1,\dots,\alpha_p$. If $i\in\alpha$ we denote by $\alpha-i$ the multi-index of length $p-1$ obtained by removing i from α. Also, $\overline{\alpha}$ is the complement of α in $(1,\dots,n)$, we set $\overline{0}:=(1,\dots,n)$, and $\sigma(\alpha,\overline{\alpha})$ is the sign of the permutation which reorders α and $\overline{\alpha}$, e.g., $\sigma(\alpha,\overline{\alpha})=(-1)^{i-1}$ if $\alpha=i$. For $\alpha=i$ we finally set $\widehat{x}_i:=x_{\overline{\alpha}}$ and $\widehat{dx^i}:=dx^{\overline{\alpha}}$. A similar notation holds for β and dy^β, with n replaced by N. Moreover, we shall denote by $(e_1,\dots,e_n)$ and $(\varepsilon_1,\dots,\varepsilon_N)$ the canonical bases in $\mathbb{R}^n$ and $\mathbb{R}^N$, respectively, so that e.g. $e_\alpha:=e_{\alpha_1}\wedge\cdots\wedge e_{\alpha_p}$.

Splitting of currents. Assume now $U=\mathbb{R}^n\times\mathbb{R}^N$. Every k-form $\eta\in\mathcal{D}^k(\mathbb{R}^n\times\mathbb{R}^N)$ splits as a sum $\omega=\sum_h\omega^{(h)}$, where the $\omega^{(h)}$'s are the components that contain exactly h differentials in the vertical y-variables.[10] Therefore, the above summation is restricted to $\max\{0,n-k\}\le h\le\min\{k,N\}$.

Every current $T\in\mathcal{D}_k(\mathbb{R}^n\times\mathbb{R}^N)$ then splits as

$$T=\sum_{h=\max\{0,n-k\}}^{\min\{k,N\}} T_{(h)}\,, \qquad \text{where} \quad T_{(h)}(\omega):=T(\omega^{(h)})\,. \tag{2.3}$$

Homotopy formula. Let $f,g:U\to V$ be two smooth maps defined between open sets $U\subset\mathbb{R}^D$ and $V\subset\mathbb{R}^\mu$, and let $h:U\times[0,1]\to V$ denote the affine homotopy map

$$h(z,t):=t\,f(z)+(1-t)\,g(z)\,,\quad z\in U\,,\quad t\in[0,1]\;.$$

[10] We thus have for some $\omega_{\alpha,\beta}\in C_c^\infty(\mathbb{R}^n\times\mathbb{R}^N)$

$$\omega^{(h)}:=\sum_{\substack{|\alpha|+|\beta|=k\\|\beta|=h}}\omega_{\alpha,\beta}\,dx^\alpha\wedge dy^\beta \qquad \text{if} \qquad \omega=\sum_{|\alpha|+|\beta|=k}\omega_{\alpha,\beta}\,dx^\alpha\wedge dy^\beta\,.$$

If a current $T \in \mathcal{D}_k(U)$ has finite mass, by dominated convergence the action of T is well-defined on smooth forms $\omega \in \mathcal{E}^k_b(U)$ with bounded coefficients, e.g. for $\omega = f^\# \eta$ for any $\eta \in \mathcal{D}^k(V)$ and for f as above. Hence the image current $f_\# T \in \mathcal{D}_k(V)$ is well-defined by $f_\# T(\eta) := T(f^\# \eta)$, for $\eta \in \mathcal{D}^k(V)$. Moreover, if T is a normal current, i.e., $\mathbf{M}(T) + \mathbf{M}((\partial T) \llcorner U) < \infty$, the image currents $h_\#(T \times [\![0,1]\!])$ and $h_\#(\partial T \times [\![0,1]\!])$ are both well defined provided that f and g are bounded or the restriction of h to the support of $T \times [\![0,1]\!]$ is proper. In particular, if T has compact support, the *homotopy formula* [18, 26.22] yields

$$\partial h_\#(T \times [\![0,1]\!]) = h_\#(\partial T \times [\![0,1]\!]) + (-1)^k (f_\# T - g_\# T) . \tag{2.4}$$

To our purposes, assume now $U = V = \mathbb{R}^n \times \mathbb{R}^N$. Dealing with currents that are not compactly supported, in general (2.4) fails to hold. However, for suitable choices of f and g (the identity and a projection map, respectively) we overcome this problem by restricting the range of t to intervals of the type $[\varepsilon, 1]$.

Proposition 2.7 *Let $\varepsilon \in]0,1[$ and $h_\varepsilon : (\mathbb{R}^n \times \mathbb{R}^N) \times [\varepsilon,1] \to \mathbb{R}^n \times \mathbb{R}^N$ denote the affine homotopy map*

$$h_\varepsilon(x,y,t) := t\,(x,y) + (1-t)(x,0) , \quad (x,y) \in \mathbb{R}^n \times \mathbb{R}^N , \quad t \in [\varepsilon,1] . \tag{2.5}$$

If $T \in \mathcal{D}_k(\mathbb{R}^n \times \mathbb{R}^N)$ has finite mass, the image current $h_{\varepsilon\#}(T \times [\![\varepsilon,1]\!])$ is well-defined in $\mathcal{D}_{k+1}(\mathbb{R}^n \times \mathbb{R}^N)$ and it has locally finite mass*, i.e., for every compact set $K \subset \mathbb{R}^n \times \mathbb{R}^N$*

$$\mathbf{M}((h_{\varepsilon\#}(T \times [\![\varepsilon,1]\!])) \llcorner K) < \infty .$$

Similarly, if $\mathbf{M}(\partial T) < \infty$ the image current $h_{\varepsilon\#}(\partial T \times [\![\varepsilon,1]\!])$ is well defined in $\mathcal{D}_k(\mathbb{R}^n \times \mathbb{R}^N)$ and it has locally finite mass. Finally, setting $f_\varepsilon(x,y) := (x, \varepsilon y)$, if $\mathbf{M}(T) + \mathbf{M}(\partial T) < \infty$, the following homotopy formula *holds:*

$$\partial h_{\varepsilon\#}(T \times [\![\varepsilon,1]\!]) = h_{\varepsilon\#}(\partial T \times [\![\varepsilon,1]\!]) + (-1)^k (T - f_{\varepsilon\#} T) . \tag{2.6}$$

Proof Since T has finite mass, we deduce that $h_{\varepsilon\#}(T \times [\![\varepsilon,1]\!])$ is well-defined provided that $\|h_\varepsilon^\# \omega\| < \infty$ for every $\omega \in \mathcal{D}^{k+1}(\mathbb{R}^n \times \mathbb{R}^N)$. To prove this property, by a density argument we may and do assume that ω is a linear combinations of forms of the type $\varphi(x)\,\psi(y)\,dx^\alpha \wedge dy^\beta$, where $\varphi \in C^\infty_c(\mathbb{R}^n)$, $\psi \in C^\infty_c(\mathbb{R}^N)$, and $|\alpha| + |\beta| = k+1$. If $|\beta| > 0$, we have

$$h_\varepsilon^\#(\varphi(x)\,\psi(y) dx^\alpha \wedge dy^\beta) = \varphi(x)\,dx^\alpha \wedge \psi(\widetilde{h}_\varepsilon(y,t))\,\widetilde{h}_\varepsilon^\#\,dy^\beta ,$$

where $\widetilde{h}_\varepsilon : \mathbb{R}^N \times [\varepsilon,1] \to \mathbb{R}^N$ is given by $\widetilde{h}_\varepsilon(y,t) = ty$, and we compute

$$\widetilde{h}_\varepsilon^\#\,dy^\beta = dy^\beta - (-1)^{|\beta|}\,\widetilde{\omega}_\beta \wedge dt , \quad \text{where } \widetilde{\omega}_\beta := \sum_{j \in \beta} \sigma(j, \beta - j)\, y_j\, dy^{\beta - j} \in \mathcal{E}^{|\beta|-1}(\mathbb{R}^N) .$$

Since moreover $\psi \in C^\infty_c(\mathbb{R}^N)$, there exists $R > 0$ such that $\psi(y) = 0$ if $|y| > R$, hence $\psi(\widetilde{h}_\varepsilon(y,t)) = 0$ for every $(y,t) \in \mathbb{R}^N \times [\varepsilon,1]$ provided that $|y| > R/\varepsilon$. Using that $|\widetilde{\omega}_\beta(y)| \leq |y|$, this yields

$$\|h_\varepsilon^\#(\varphi(x)\,\psi(y) dx^\alpha \wedge dy^\beta)\| \leq c \cdot \|\varphi\|_\infty \|\psi\|_\infty \frac{R}{\varepsilon} < \infty \qquad \text{on} \quad \mathbb{R}^n \times \mathbb{R}^N \times [\varepsilon,1] .$$

If $|\beta| = 0$, we have $\|h_\varepsilon^\#(\varphi(x)\,\psi(y)\,dx^\alpha)\| = \|\varphi(x)\,\psi(\widetilde{h}_\varepsilon(y,t))\,dx^\alpha\| \leq \|\varphi\|_\infty \|\psi\|_\infty$.

In particular, denoting by $\widetilde{B}_R$ the closed ball in $\mathbb{R}^N$ centered at the origin and with radius R, we deduce that for each $R > 1$

$$\mathbf{M}((h_{\varepsilon\#}(T \times [\![\varepsilon, 1]\!])) \llcorner \mathbb{R}^n \times \widetilde{B}_R) \leq c \cdot \frac{R}{\varepsilon}\,\mathbf{M}(T) < \infty\,,$$

and hence that $h_{\varepsilon\#}(T \times [\![\varepsilon, 1]\!])$ has locally finite mass. The second assertion is proved in a similar way. As a consequence, if $\mathbf{M}(T) + \mathbf{M}(\partial T) < \infty$, property (2.6) follows from the standard homotopy formula (2.4), with 0 replaced by ε, using the dominated convergence theorem. □

Remark 2.8 In general the image currents $h_{\varepsilon\#}(T \times [\![\varepsilon, 1]\!])$ and $h_{\varepsilon\#}(\partial T \times [\![\varepsilon, 1]\!])$ from Proposition 2.7 do not have finite mass, if T does not have compact support.

ORTHOGONAL PROJECTIONS. We shall denote by $\pi : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}^n$ and $\widehat{\pi} : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}^N$ the *orthogonal projections* onto the x and y coordinates, respectively. Let $T \in \mathcal{D}_k(\mathbb{R}^n \times \mathbb{R}^N)$ a current with finite mass, $\mathbf{M}(T) < \infty$. Let h denote an integer with $\max\{0, k-n\} \leq h \leq \min\{k, N\}$. For any $\omega \in \mathcal{D}^h(\mathbb{R}^N)$, we shall denote by $\pi_\#(T \llcorner \widehat{\pi}^\# \omega)$ the current in $\mathcal{D}_{k-h}(\mathbb{R}^n)$ such that

$$\langle \pi_\#(T \llcorner \widehat{\pi}^\# \omega), \varphi \rangle := T(\widehat{\pi}^\# \omega \wedge \pi^\# \varphi) = T(\omega \wedge \varphi)\,,^{11} \qquad \varphi \in \mathcal{D}^{k-h}(\mathbb{R}^n)\,.$$

Similarly, for $\varphi \in \mathcal{D}^{k-h}(\mathbb{R}^n)$, we shall denote by $\widehat{\pi}_\#(T \llcorner \pi^\# \varphi)$ the current in $\mathcal{D}_h(\mathbb{R}^N)$ such that

$$\langle \widehat{\pi}_\#(T \llcorner \pi^\# \varphi), \omega \rangle := T(\pi^\# \varphi \wedge \widehat{\pi}^\# \omega) = T(\varphi \wedge \omega)\,, \qquad \omega \in \mathcal{D}^h(\mathbb{R}^N)\,.$$

3 An isoperimetric inequality

In this section we extend the isoperimetric inequality from [15, Prop. 2.1]. It will be used in the case $k = N - 1$ of the proof of Theorem 1.1. The main difficulty is due to the fact that we do not require the current T in Proposition 3.1 below to have compact support.

Any form $\omega \in \mathcal{D}^{N-1}(\mathbb{R}^N)$ is identified by a compactly supported smooth vector field $g \in C_c^\infty(\mathbb{R}^N, \mathbb{R}^N)$ via the formula

$$\omega_g(y) := \sum_{j=1}^{N} (-1)^{j-1} g^j(y)\, \widehat{dy^j}\,, \qquad g = (g^1, \ldots, g^N)\,, \tag{3.1}$$

where $\widehat{dy^j} := dy^1 \wedge \cdots \wedge dy^{j-1} \wedge dy^{j+1} \wedge \cdots \wedge dy^N$, so that $d\omega_g = \operatorname{div} g\, dy$, where $dy := dy^1 \wedge \cdots \wedge dy^N$. If $T \in \mathcal{R}_{N-1}(\mathbb{R}^n \times \mathbb{R}^N)$, we let μ_g correspondingly denote the signed measure given on Borel sets $B \in \mathcal{B}(\mathbb{R}^n)$ by

$$\langle \mu_g, B \rangle := (-1)^{N-1} \langle \pi_\#(T \llcorner \widehat{\pi}^\# \omega_g), B \rangle\,,$$

so that for functions $\varphi \in C_c^\infty(\mathbb{R}^n)$

$$(-1)^{N-1} \langle \mu_g, \varphi \rangle = (T \llcorner \widehat{\pi}^\# \omega_g)(\pi^\# \varphi) = T(\varphi \wedge \omega_g)\,.$$

We shall denote by $B_r(x_0)$ the open ball in $\mathbb{R}^n$ of radius r and centered at $x_0 \in \mathbb{R}^n$.

[11] We shall often omit to write the action of the pull-back by π and $\widehat{\pi}$.

Proposition 3.1 *Let* $N \geq 2$ *and* $T \in \mathcal{R}_{N-1}(\mathbb{R}^n \times \mathbb{R}^N)$ *an i.m. rectifiable current satisfying the property* $T_{(N-2)} = 0$ *and the* null-boundary condition $\partial T = 0$*. Then for every* $x_0 \in \mathbb{R}^n$ *and a.e.* $r > 0$ *we have*

$$|\langle \mu_g, \overline{B}_r(x_0)\rangle| \leq c_N \, \|\operatorname{div} g\|_\infty \, \mathbf{M}(T \llcorner \overline{B}_r(x_0) \times \mathbb{R}^N)^{N/(N-1)} \tag{3.2}$$

for all $g \in C_c^\infty(\mathbb{R}^N, \mathbb{R}^N)$*, where* $c_N > 0$ *is an absolute constant, not depending on* g*.*

Proof Fix $\varepsilon \in]0,1[$, define $h_\varepsilon : \mathbb{R}^n \times \mathbb{R}^N \times [\varepsilon, 1] \to \mathbb{R}^n \times \mathbb{R}^N$ as in (2.5) and denote

$$H_T^\varepsilon := h_{\varepsilon\#}(T \times [\![\, \varepsilon, 1 \,]\!]) \in \mathcal{D}_N(\mathbb{R}^n \times \mathbb{R}^N)\,. \tag{3.3}$$

By Proposition 2.7, we may and do introduce for $k = 0, \ldots, \min\{n, N\}$ and $\eta \in \mathcal{D}^k(\mathbb{R}^n)$ the $(N-k)$-current

$$H_T^\varepsilon \llcorner \eta := \widehat{\pi}_\#(H_T^\varepsilon \llcorner \pi^\# \eta) \in \mathcal{D}_{N-k}(\mathbb{R}^N)\,.$$

Setting $\widetilde{h}_\varepsilon(y,t) := ty$ for $(y,t) \in \mathbb{R}^N \times [\varepsilon, 1]$, we thus equivalently have:

$$H_T^\varepsilon \llcorner \eta(\omega) := (T \times [\![\, \varepsilon, 1 \,]\!])(\eta \wedge \widetilde{h}_\varepsilon^{\#} \omega)\,, \qquad \omega \in \mathcal{D}^{N-k}(\mathbb{R}^N)\,, \tag{3.4}$$

where we have omitted to write the pull-back of the orthogonal projection maps. Even if in general the current H_T^ε from (3.3) does not have finite mass, see Remark 2.8, by Proposition 2.7 we deduce that for every $\eta \in \mathcal{D}^k(\mathbb{R}^n)$, the current $H_T^\varepsilon \llcorner \eta$ in $\mathcal{D}_{N-k}(\mathbb{R}^N)$ has locally finite mass. Choosing $k = 1$, we shall make use of the following extension of [15, Lemma 2.3], the proof of which is postponed.

Lemma 3.2 *Let* $T \in \mathcal{R}_{N-1}(\mathbb{R}^n \times \mathbb{R}^N)$ *be such that* $T_{(N-2)} = 0$*. Then* $H_T^\varepsilon \llcorner \eta = 0$ *for every* $\eta \in \mathcal{D}^1(\mathbb{R}^n)$*.*

Setting now $f_\varepsilon(x,y) := (x, \varepsilon y)$, we let μ_g^ε denote for every $g \in C_c^\infty(\mathbb{R}^N, \mathbb{R}^N)$ the signed measure given on Borel sets $B \in \mathcal{B}(\mathbb{R}^n)$ by

$$\langle \mu_g^\varepsilon, B\rangle := (-1)^{N-1} \langle \pi_\#(f_{\varepsilon\#} T \llcorner \widehat{\pi}^\# \omega_g), B\rangle\,,$$

so that for functions $\varphi \in C_c^\infty(\mathbb{R}^n)$

$$(-1)^{N-1} \langle \mu_g^\varepsilon, \varphi\rangle = f_{\varepsilon\#} T(\varphi \wedge \omega_g)\,.$$

Property $\partial T = 0$ implies that $h_{\varepsilon\#}(\partial T \times [\![\, \varepsilon, 1 \,]\!]) = 0$. Therefore, using the above definitions, the general homotopy formula (2.6) gives

$$\langle \mu_g - \mu_g^\varepsilon, \varphi\rangle = H_T^\varepsilon \llcorner d\varphi(\omega_g) + H_T^\varepsilon \llcorner \varphi(d\omega_g)$$

for every $\varphi \in C_c^\infty(\mathbb{R}^n)$ and $g \in C_c^\infty(\mathbb{R}^N, \mathbb{R}^N)$, whereas Lemma 3.2 yields that $H_T^\varepsilon \llcorner d\varphi(\omega_g) = 0$, so that

$$\langle \mu_g - \mu_g^\varepsilon, \varphi\rangle = H_T^\varepsilon \llcorner \varphi(d\omega_g) = H_T^\varepsilon \llcorner \varphi(\operatorname{div} g(y)\, dy)\,.$$

Therefore, taking a sequence $\{\varphi_j\} \subset C_c^\infty(\mathbb{R}^n)$ converging in L^1 to the characteristic function χ of the closed ball $\overline{B}_r(x_0)$, and setting $B_r := B_r(x_0)$ for simplicity, we deduce that

$$\langle \mu_g - \mu_g^\varepsilon, \overline{B}_r\rangle = H_T^\varepsilon \llcorner \chi_{\overline{B}_r}(\operatorname{div} g(y)\, dy)\,. \tag{3.5}$$

Also, setting $\widetilde{f}_\varepsilon(y) = \varepsilon y$ and $K_\varphi = \operatorname{spt}\varphi$, since $\|\omega_g\| = \|g\|_\infty$ we estimate

$$|f_{\varepsilon\#}T(\varphi\wedge\omega_g)| = |T(\varphi\wedge\widetilde{f}_\varepsilon^{\#}\omega_g)| \le \|\varphi\|_\infty\,\|g\|_\infty\,\varepsilon^{N-1}\,\mathbf{M}(T \llcorner K_\varphi\times\mathbb{R}^N)\,,$$

so that the measures μ_g and μ_g^ε have finite total variation, as

$$\begin{aligned} |\mu_g|(\overline{B}_r) &\le \|g\|_\infty\,\mathbf{M}(T\llcorner\overline{B}_r\times\mathbb{R}^N) < \infty\,,\\ |\mu_g^\varepsilon|(\overline{B}_r) &\le \varepsilon\,\|g\|_\infty\,\mathbf{M}(T\llcorner\overline{B}_r\times\mathbb{R}^N) < \infty\,. \end{aligned} \tag{3.6}$$

On the other hand, for each $\omega\in\mathcal{D}^N(\mathbb{R}^N)$, by (3.3) and (3.4) we have

$$H_T^\varepsilon\llcorner\chi_{\overline{B}_r}(\omega) = (T\times[\![\,\varepsilon,1\,]\!])(\chi_{\overline{B}_r}\wedge\widetilde{h}_\varepsilon^{\#}\omega) = ((T\llcorner\overline{B}_r\times\mathbb{R}^N)\times[\![\,\varepsilon,1\,]\!])(\widetilde{h}_\varepsilon^{\#}\omega)\,.$$

Therefore, since by Proposition 2.7 the current $H_T^\varepsilon\llcorner\varphi$ in $\mathcal{D}_N(\mathbb{R}^N)$ has locally finite mass, and T is i.m. rectifiable in $\mathcal{R}_{N-1}(\mathbb{R}^n\times\mathbb{R}^N)$, we deduce that the current $H_T^\varepsilon\llcorner\chi_{\overline{B}_r}$ is locally i.m. rectifiable in $\mathcal{R}_{N,\mathrm{loc}}(\mathbb{R}^N)$. We then proceed in a way similar to the second part of the proof of [14, Prop. 3.1].

More precisely, by using the *degree theory* from [9, Sec. 4.3.2], for a.e. $r>0$ small there exists an *integer valued* and locally summable function $\Delta_r^\varepsilon\in L^1_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{Z})$ such that

$$H_T^\varepsilon\llcorner\chi_{\overline{B}_r}(\psi(y)\,dy) = \int_{\mathbb{R}^N}\Delta_r^\varepsilon(y)\,\psi(y)\,dy \qquad \forall\,\psi\in C_c^\infty(\mathbb{R}^N)\,.$$

By (3.5), this yields that for every $g\in C_c^\infty(\mathbb{R}^N,\mathbb{R}^N)$

$$\langle\mu_g-\mu_g^\varepsilon,\overline{B}_r\rangle = \int_{\mathbb{R}^N}\Delta_r^\varepsilon(y)\,\operatorname{div}g(y)\,dy\,. \tag{3.7}$$

Moreover, by (3.6) the measure $\mu_g-\mu_g^\varepsilon$ has finite total variation, and

$$|\langle\mu_g-\mu_g^\varepsilon,\overline{B}_r\rangle| \le \|g\|_\infty\,(1+\varepsilon)\,\mathbf{M}(T\llcorner\overline{B}_r\times\mathbb{R}^N) < \infty\,.$$

Therefore, Δ_r^ε is a function of bounded variation in $\mathbb{R}^N$, with

$$\begin{aligned} |D\Delta_r^\varepsilon|(\mathbb{R}^N) &:= \sup_{\|g\|_\infty\le 1}\int_{\mathbb{R}^N}\Delta_r^\varepsilon(y)\,\operatorname{div}g(y)\,dy\\ &\le \sup_{\|g\|_\infty\le 1}|\langle\mu_g-\mu_g^\varepsilon,\overline{B}_r\rangle| \le (1+\varepsilon)\,\mathbf{M}(T\llcorner\overline{B}_r\times\mathbb{R}^N) < \infty\,. \end{aligned} \tag{3.8}$$

By Sobolev embedding theorem, and by density of smooth maps in $BV_{\mathrm{loc}}(\mathbb{R}^N)$, compare [4, Thm. 3.47], we can find a real constant $m_r^\varepsilon\in\mathbb{R}$ such that

$$\|\Delta_r^\varepsilon - m_r^\varepsilon\|_{L^{N/(N-1)}(\mathbb{R}^N)} \le c_N\,|D\Delta_r^\varepsilon|(\mathbb{R}^N)\,.$$

Since Δ_r^ε is integer-valued, the constant $m_r^\varepsilon\in\mathbb{Z}$ and hence we can estimate the L^1-norm of the integer-valued function $y\mapsto(\Delta_r^\varepsilon(y)-m_r^\varepsilon)$ by

$$\int_{\mathbb{R}^N}|\Delta_r^\varepsilon(y)-m_r^\varepsilon|\,dy \le \int_{\mathbb{R}^N}|\Delta_r^\varepsilon(y)-m_r^\varepsilon|^{N/(N-1)}\,dy = \|\Delta_r^\varepsilon-m_r^\varepsilon\|^{N/(N-1)}_{L^{N/(N-1)}(\mathbb{R}^N)}\,.$$

Using that $\int_{\mathbb{R}^N}\operatorname{div}g(y)\,dy = 0$, by (3.7) we thus obtain

$$\begin{aligned} |\langle\mu_g-\mu_g^\varepsilon,\overline{B}_r\rangle| &\le \int_{\mathbb{R}^N}|(\Delta_r^\varepsilon(y)-m_r^\varepsilon)\,\operatorname{div}g(y)|\,dy \le \|\operatorname{div}g\|_\infty\int_{\mathbb{R}^N}|\Delta_r^\varepsilon(y)-m_r^\varepsilon|\,dy\\ &\le \|\operatorname{div}g\|_\infty\,c_N\,(|D\Delta_r^\varepsilon|(\mathbb{R}^N))^{N/(N-1)} \end{aligned}$$

and definitively, by (3.8),

$$|\langle \mu_g - \mu_g^\varepsilon, \overline{B}_r\rangle| \le \|\operatorname{div} g\|_\infty \, c_N \, (1+\varepsilon)^{N/(N-1)} \mathbf{M}(T \llcorner \overline{B}_r \times \mathbb{R}^N)^{N/(N-1)} . \qquad (3.9)$$

Finally, since $|\langle \mu_g \overline{B}_r\rangle| \le |\langle \mu_g - \mu_g^\varepsilon, \overline{B}_r\rangle| + |\langle \mu_g^\varepsilon, \overline{B}_r\rangle|$, using the second line in (3.6), the isoperimetric inequality (3.2) follows by letting $\varepsilon \to 0$ in the above formula (3.9). □

Proof of Lemma 3.2 We have (3.4) with $k = 1$. Using (3.1), write $\omega \in \mathcal{D}^{N-1}(\mathbb{R}^N)$ as $\omega = \omega_g$ for some $g \in C_c^\infty(\mathbb{R}^N, \mathbb{R}^N)$. By linearity, without loss of generality we may and do assume that $g^j = 0$ for $j > 1$, and let $g^1(y) = f(y)$, so that $\omega_g = \omega := f(y)\,\widehat{dy^1}$, where $f \in C_c^\infty(\mathbb{R}^N)$. We compute

$$\widetilde{h}_\varepsilon^{\#}\omega = f(ty)[t^{N-1}\widehat{dy^1} + (-1)^N \, \omega^1 \wedge t^{N-2}dt] ,$$

where $\omega^1 := \displaystyle\sum_{l=2}^N (-1)^l \, y_l \, dy^{\overline{(1,l)}} \in \mathcal{E}^{N-2}(\mathbb{R}^N)$. Since the form $\eta \wedge f(ty)\, t^{N-1}\widehat{dy^1}$ does not contain the differential dt, by definition of Cartesian product of currents and the dominated convergence theorem we get $(T \times [\![\, \varepsilon, 1]\!])(\eta \wedge f(ty)\, t^{N-1}\widehat{dy^1}) = 0$ and hence

$$\begin{aligned} H_T^\varepsilon \llcorner \eta(\omega) &= (-1)^N (T \times [\![\, \varepsilon, 1]\!])(\eta(x) \wedge f(ty)\, \omega^1 \wedge t^{N-2}dt]) \\ &= (-1)^N T(\eta(x) \wedge \omega^1(y)\, F_\varepsilon(y)) , \end{aligned}$$

where $F_\varepsilon(y) := \displaystyle\int_\varepsilon^1 f(ty)\, t^{N-2}\, dt$. Arguing as in the proof of Proposition 2.7, using that $\omega^1 \in \mathcal{E}^{N-2}(\mathbb{R}^N)$ satisfies $|\omega^1(y)| \le |y|$, we deduce that

$$\|\eta(x) \wedge \omega^1(y)\, F_\varepsilon(y)\| \le c\, \|\eta\|\, \|f\|_\infty \, \frac{R}{\varepsilon} < \infty \qquad \text{on} \quad \mathbb{R}^n \times \mathbb{R}^N ,$$

where $R > 0$ is chosen so that $f(y) = 0$ if $|y| > R$. Since $\mathbf{M}(T) < \infty$, property $T_{(N-2)} = 0$ and the dominated convergence yield that $T(\eta(x) \wedge \omega^1(y)\, F_\varepsilon(y)) = 0$, as required. □

4 A projection argument

In this section we discuss a projection argument that will be used in the proof of Theorem 1.1 in the case $N > k+1$, see Step 2 in Sec. 5. We first introduce some notation.

Let β an ordered multi-index in $\{1, \dots, N\}$ of length $|\beta| = k+1$, and define the corresponding projection maps

$$\begin{aligned} &\Pi^\beta : \mathbb{R}^N \to \mathbb{R}^{k+1}_\beta \simeq \mathbb{R}^{k+1} , \quad \Pi^\beta(y) = y_\beta := (y_{\beta_1}, \dots, y_{\beta_{k+1}}) , \\ &\Psi_\beta : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}^n \times \mathbb{R}^{k+1}_\beta , \, \Psi_\beta(x,y) := (\mathrm{Id}_{\mathbb{R}^n} \bowtie \Pi^\beta)(x,y) = (x, \Pi^\beta(y)) . \end{aligned} \qquad (4.1)$$

For $T \in \mathcal{R}_k(\mathbb{R}^n \times \mathbb{R}^N)$, we let $T^\beta := \Psi_{\beta\#}T \in \mathcal{D}_k(\mathbb{R}^n \times \mathbb{R}^{k+1}_\beta)$ denote the corresponding image current, see Lemma 4.1 below.

If T satisfies the hypotheses of Theorem 1.1, as in the case $k = N-1$, see Step 1 in Sec. 5, we deduce that $\mathrm{set}(T^\beta) \subset S_0^\beta \times \mathbb{R}^{k+1}_\beta$ for an at most countable set of points $S_0^\beta \subset \mathbb{R}^n$. Making use of the general area-coarea formula, we thus aim at recovering the action of T in terms of the action of the currents T^β on suitably related forms, see Proposition 4.4 below. This would allow to conclude that $\mathrm{set}(T) \subset S_0 \times \mathbb{R}^N$ for an at most countable set of points $S_0 \subset \mathbb{R}^n$.

Unfortunately, this strategy may fail in general, due to the possible occurrence of cancellations when projecting T to T^β. However, denoting $\mathcal{M} := \mathrm{set}(T)$, one easily deduces that such a cancellation phenomenon is avoided provided that the *multiplicity function* $\mathbf{N}(\Psi_\beta|\mathcal{M}; z) := \mathcal{H}^0(\mathcal{M} \cap \Psi_\beta^{-1}(\{z\})$ is equal to one for $\mathcal{H}^k$-a.e. point z in the shadow $\Psi_\beta(\mathcal{M})$.

We now see that this property is obtained by suitably rotating the target space $\mathbb{R}^N$. To this purpose, we shall first consider the case of polyhedral chains, Proposition 4.2.

Projection of currents. We first point out the following fact:

Lemma 4.1 *Let $T \in \mathcal{R}_k(\mathbb{R}^n \times \mathbb{R}^N)$ satisfying $\partial T = 0$. Then the image current $T^\beta := \Psi_{\beta\#} T$ is i.m. rectifiable in $\mathcal{R}_k(\mathbb{R}^n \times \mathbb{R}^{k+1}_\beta)$ and satisfies the null-boundary condition $\partial T^\beta = 0$.*

Proof Since T is i.m. rectifiable, the first assertion follows if we show that $\mathbf{M}(T^\beta) < \infty$. To prove this, observe that for every $\omega \in \mathcal{D}^k(\mathbb{R}^n \times \mathbb{R}^{k+1}_\beta)$ the pull-back form $\Psi_\beta^\# \omega$ belongs to the class $\mathcal{E}^k_b(\mathbb{R}^n \times \mathbb{R}^N)$ and satisfies $\|\Psi_\beta^\# \omega\| \leq \|\omega\|$. Therefore, by dominated convergence we estimate

$$T^\beta(\omega) := T(\Psi_\beta^\# \omega) \leq \mathbf{M}(T)\, \|\Psi_\beta^\# \omega\| \leq \mathbf{M}(T)\, \|\omega\|$$

that gives $\mathbf{M}(T^\beta) \leq \mathbf{M}(T)$. As to the second assertion, for every $\eta \in \mathcal{D}^{k-1}(\mathbb{R}^n \times \mathbb{R}^{k+1}_\beta)$ we have

$$\partial T^\beta(\eta) = T^\beta(d\eta) = T(\Psi_\beta^\# d\eta) = T(d\Psi_\beta^\# \eta) = T(d\widetilde{\eta})\,,$$

where the smooth form $\widetilde{\eta} := \Psi_\beta^\# \eta$ belongs to $\mathcal{E}^{k-1}_b(\mathbb{R}^n \times \mathbb{R}^N)$. Since $\|\widetilde{\eta}\| + \|d\widetilde{\eta}\| < \infty$, Lemma 2.6 gives $T(d\widetilde{\eta}) = 0$, as required. □

Projection of polyhedral chains. For $N > k+1$, denote by $\mathbf{O}^*(N, k+1)$ the set of orthogonal projections $\mathbf{p}$ of $\mathbb{R}^N$ onto the $(k+1)$-dimensional subspaces of $\mathbb{R}^N$. There is a unique measure on $\mathbf{O}^*(N, k+1)$ that is invariant under Euclidean motions of $\mathbb{R}^N$ and normalized to have total measure 1.

Proposition 4.2 *Let $N > k+1$ and $P \in \mathcal{P}_k(\mathbb{R}^n \times \mathbb{R}^N)$ be an integral polyhedral chain, and let $\mathcal{M} := \mathrm{set}(P)$. Then for a.e. projection $\mathbf{p} \in \mathbf{O}^*(N, k+1)$ and for $\mathcal{H}^k$-a.e. $z \in (\mathrm{Id}_{\mathbb{R}^n} \bowtie \mathbf{p})(\mathcal{M})$ we have*

$$\mathbf{N}(\mathrm{Id}_{\mathbb{R}^n} \bowtie \mathbf{p}|\mathcal{M}; z) := \mathcal{H}^0(\mathcal{M} \cap (\mathrm{Id}_{\mathbb{R}^n} \bowtie \mathbf{p})^{-1}(\{z\})) = 1\,.$$

Proof Every projection of the type $\mathrm{Id}_{\mathbb{R}^n} \bowtie \mathbf{q}$, where $\mathbf{q} \in \mathbf{O}^*(N, N-1)$, is clearly determined by a couple $\pm\nu$ of opposite unit normals in $\mathbb{R}^n \times \mathbb{R}^N$, i.e., $\pm\nu \in$

$\mathbb{S}^{n+N-1}$, where ν is orthogonal to the "horizontal" space $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^n \times \mathbb{R}^N$. Hence, the couple $\pm\nu$ belongs to the "vertical" $(N-1)$-sphere

$$\mathbb{S}_v^{N-1} := \{(x,y) \in \mathbb{S}^{N+n-1} \subset \mathbb{R}^n \times \mathbb{R}^N \mid x = 0\}\,.$$

Using this identification, we write $\mathrm{Id}_{\mathbb{R}^n} \bowtie \mathbf{q} = \pi_{\pm\nu}$.

Since P is a k-dimensional integral polyhedral chain, and $k \leq N-2$, it is readily checked that the property

$$\mathbf{N}(\pi_{\pm\nu}|\mathcal{M}; z) := \mathcal{H}^0(\mathcal{M} \cap {\pi_{\pm\nu}}^{-1}(\{z\})) = 1 \qquad \forall\, z \in \pi_{\pm\nu}(\mathcal{M})$$

holds true for every choice of $\pm\nu \in \mathbb{S}_v^{N-1}$ except for a "bad" set $B \subset \mathbb{S}_v^{N-1}$ of null $\mathcal{H}^{k+1}$-measure, $\mathcal{H}^{k+1}(B)$=0. This proves the claim for $N = k+2$. If $N = k+m$ with $m \geq 3$, it suffices to iterate $m-2$ times the above argument. □

Remark 4.3 Proposition 4.2 is false for projections $\mathbf{p} \in \mathbf{O}^*(N,k)$. If e.g. $N = k+2$, it suffices to take $P = \delta_0 \times [\![Q]\!]$, where Q is a k-dimensional cube in $\mathbb{R}^{k+2}$.

THE AREA-COAREA FORMULA ON CURRENTS. Let now $T \in \mathcal{R}_k(\mathbb{R}^n \times \mathbb{R}^N)$, where $N > k+1$, and write $T := \tau(\mathcal{M}, \theta, \overrightarrow{\xi})$. Moreover, for any index β with $|\beta| = m$, where $1 \leq m \leq N-1$, denote by ξ_β the component of the tangent k-vector field $\overrightarrow{\xi}$ corresponding to the base k-vectors $e_\alpha \wedge \varepsilon_\gamma$, where β contains all the entries of γ, i.e.,

$$\xi_\beta := \sum_{\substack{|\alpha|+|\gamma|=k \\ \gamma\subset\beta}} \xi_\alpha^\gamma\, e_\alpha \wedge \varepsilon_\gamma \qquad \text{if} \qquad \overrightarrow{\xi} = \sum_{|\alpha|+|\gamma|=k} \xi_\alpha^\gamma\, e_\alpha \wedge \varepsilon_\gamma\,.$$

Define

$$\mathcal{M}_\beta := \{(x,y) \in \mathcal{M} \mid \xi_\beta(x,y) \neq 0\} \tag{4.2}$$

and observe that the set $\mathcal{M}_\beta$ is k-rectifiable, see Remark 2.1. According to (4.1), this yields that $\mathcal{N}_\beta := \Psi_\beta(\mathcal{M}_\beta)$ is a k-rectifiable subset of $\mathbb{R}^n \times \mathbb{R}^{k+1}_\beta$. Let $\overrightarrow{\zeta_\beta}$ denote an $\mathcal{H}^k \llcorner \mathcal{N}_\beta$-measurable function such that $\overrightarrow{\zeta_\beta}(x, y_\beta)$ is a unit k-vector orienting the approximate tangent space to $\mathcal{N}_\beta$ at $\mathcal{H}^k$-a.e. point $(x, y_\beta) \in \mathcal{N}_\beta$. By applying the general area-coarea formula, Theorem 2.2, we obtain:

Proposition 4.4 *Let $|\beta| = m \in \{1, \ldots, N-1\}$. Let $|\alpha| + |\gamma| = k$, with $\gamma \subset \beta$. Let $\eta_\alpha^\gamma \in \mathcal{D}^k(\mathbb{R}^n \times \mathbb{R}^N)$ given by*

$$\eta_\alpha^\gamma := \phi(x)\, f(y_{\overline{\beta}})\, g(y_\beta)\, dx^\alpha \wedge dy^\gamma\,,$$

where $\phi \in C_c^\infty(\mathbb{R}^n)$, $f \in C_c^\infty(\mathbb{R}^{N-m}_{\overline{\beta}})$, $g \in C_c^\infty(\mathbb{R}^m_\beta)$. With the previous notation, we have:

$$T(\eta_\alpha^\gamma) = \int_{\mathcal{N}_\beta} \langle \phi(x)\, \widehat{\Phi}(x, y_\beta)\, g(y_\beta)\, dx^\alpha \wedge dy^\gamma, \overrightarrow{\zeta_\beta}(x, y_\beta)\rangle\; d\mathcal{H}^k(x, y_\beta)\,, \tag{4.3}$$

where we have set

$$\widehat{\Phi}(x, y_\beta) := \int_{\mathcal{M}_\beta \cap (\psi_\beta^{-1}(\{(x,y_\beta)\})} \sigma(x,y)\, f(y_{\overline{\beta}})\, \theta(x,y)\, d\mathcal{H}^0(x,y) \tag{4.4}$$

for a suitable sign $\sigma(x,y) = \pm 1$, see formula (4.6) below.

PROOF Since $\gamma \subset \beta$, we clearly have

$$T(\eta_\alpha^\gamma) = \int_{\mathcal{M}_\beta} \langle \eta_\alpha^\gamma, \xi_\beta \rangle \, \theta \, d\mathcal{H}^k \,. \tag{4.5}$$

The function Ψ_β being an orthogonal projection, it is readily checked that the k-dimensional tangential Jacobian of Ψ_β agrees with the norm of the k-vector ξ_β:

$$J_{\Psi_\beta}^{\mathcal{M}_\beta}(x,y) = |\xi_\beta(x,y)| \qquad \text{for } \mathcal{H}^k\text{-a.e. } (x,y) \in \mathcal{M}_\beta \,.$$

Furthermore, for $\mathcal{H}^k$-a.e. $(x,y_\beta) \in \mathcal{N}_\beta$ and $(x,y) \in \mathcal{M}_\beta \cap \Psi_\beta^{-1}(\{(x,y_\beta)\})$ we have

$$\frac{\xi_\beta(x,y)}{|\xi_\beta(x,y)|} = \sigma(x,y)\, \overrightarrow{\zeta_\beta}(x,y_\beta)\,, \quad \text{where} \quad \sigma(x,y) := \pm 1 \,. \tag{4.6}$$

We then apply Theorem 2.2, where $\mathcal{M} = \mathcal{M}_\beta$, $\mathcal{N} = \mathcal{N}_\beta$, $\mu = k$, $D_1 = n+N$, $D_2 = n+m$, $f = \Psi_\beta$, $w = (x,y)$, $z = (x,y_\beta)$, to the $\mathcal{H}^k \llcorner \mathcal{M}_\beta$-integrable function

$$\Phi(x,y) := \theta(x,y) \langle \phi(x)\, f(y_{\overline{\beta}})\, g(y_\beta)\, dx^\alpha \wedge dy^\gamma, \xi_\beta(x,y) \rangle \, |\xi_\beta(x,y)|^{-1} \,.$$

Since $\langle \eta_\alpha^\gamma, \xi_\beta \rangle \, \theta = J_{\psi_\beta}^{\mathcal{M}_\beta} \cdot \Phi$, by (4.5) we then obtain

$$\begin{aligned} T(\eta_\alpha^\gamma) &= \int_{\mathcal{M}_\beta} J_{\psi_\beta}^{\mathcal{M}_\beta}(x,y)\, \Phi(x,y)\, d\mathcal{H}^k(x,y) \\ &= \int_{\mathcal{N}_\beta} \Big(\int_{\mathcal{M}_\beta \cap \psi_\beta^{-1}(\{(x,y_\beta)\})} \Phi \, d\mathcal{H}^0(x,y) \Big)\, d\mathcal{H}^k(x,y_\beta) \\ &= \int_{\mathcal{N}_\beta} \langle \phi(x)\, \widehat{\Phi}(x,y_\beta)\, g(y_\beta)\, dx^\alpha \wedge dy^\gamma, \overrightarrow{\zeta_\beta}(x,y_\beta) \rangle \; d\mathcal{H}^k(x,y_\beta)\,, \end{aligned}$$

where $\widehat{\Phi}$ is given by (4.4). □

GOOD PROJECTIONS. We now restrict to the case $m = k+1$ of our interest. Assume that $T = P$ is an integral polyhedral chain in $\mathcal{P}_k(\mathbb{R}^n \times \mathbb{R}^N)$. On account of Proposition 4.2, possibly slightly rotating the target space $\mathbb{R}^N$, and denoting without loss of generality by $(y_1, \ldots, y_n)$ the rotated coordinates, using (4.2) we may and do assume that

$$\mathbf{N}(\Psi_\beta | \mathcal{M}; (x,y_\beta)) := \mathcal{H}^0(\mathcal{M}_\beta \cap \Psi_\beta^{-1}\{(x,y_\beta)\}) = 1 \quad \text{for } \mathcal{H}^k\text{-a.e. } (x,y_\beta) \in \mathcal{N}_\beta \,.$$

This gives $\mathcal{N}_\beta := \Psi_\beta(\mathcal{M}_\beta) = \operatorname{set}(P^\beta)$, where $P^\beta := \Psi_{\beta\#} P \in \mathcal{P}_k(\mathbb{R}^n \times \mathbb{R}_\beta^{k+1})$, see Lemma 4.1. Writing as before $P := \tau(\mathcal{M}, \theta, \overrightarrow{\xi})$, we also may and do choose the orienting unit k-vector field $\overrightarrow{\zeta_\beta}$ is such a way that the sign $\sigma(x,y) \equiv 1$ in the formula (4.4). We thus have $P^\beta = \tau(\mathcal{N}_\beta, \theta_\beta, \overrightarrow{\zeta_\beta})$, where the multiplicity function $\theta_\beta(x,y_\beta) = \theta(x,y)$ for the unique point $(x,y) \in \mathcal{M}_\beta$ such that $\Psi_\beta(x,y) = (x,y_\beta) \in \mathcal{N}_\beta$. Since (4.4) becomes

$$\widehat{\Phi}(x,y_\beta) = \int_{\mathcal{M}_\beta \cap (\psi_\beta^{-1}(\{(x,y_\beta)\})} f(y_{\overline{\beta}})\, \theta_\beta(x,y_\beta)\, d\mathcal{H}^0(x,y)\,,$$

we conclude that (4.3) can be equivalently written as

$$P(\eta_\alpha^\gamma) = P^\beta(\phi(x)\, \widetilde{\Phi}(x,y_\beta)\, g(y_\beta)\, dx^\alpha \wedge dy^\gamma)\,,$$

where we have set

$$\widetilde{\Phi}(x,y_\beta) := \int_{\mathcal{M}_\beta \cap (\psi_\beta^{-1}(\{(x,y_\beta)\})} f(y_{\overline{\beta}})\, d\mathcal{H}^0(x,y)\,. \tag{4.7}$$

Projection of integral currents. We finally show the way to extend the previous features to i.m. rectifiable currents with finite boundary mass.

Proposition 4.5 *Assume $N > k+1$. Let $T \in \mathcal{R}_k(\mathbb{R}^n \times \mathbb{R}^N)$ such that $\mathbf{M}(\partial T) < \infty$. Following the notation from Proposition 4.4, write for $|\beta| = k+1$*

$$T = \tau(\mathcal{M},\theta,\overrightarrow{\xi})\,, \qquad \Psi_{\beta\#}T = \tau(\mathcal{N}_\beta,\theta_\beta,\overrightarrow{\zeta_\beta})\,.$$

Then, possibly by slightly rotating the target space, for $|\alpha| + |\gamma| = k$, with $\gamma \subset \beta$, we have

$$T(\eta_\alpha^\gamma) = \Psi_{\beta\#}T(\phi(x)\,\widetilde{\Phi}(x,y_\beta)\,g(y_\beta)\,dx^\alpha \wedge dy^\gamma)\,, \tag{4.8}$$

where $\widetilde{\Phi}(x,y_\beta)$ is defined as in (4.7), with $\mathcal{M}_\beta$ given by (4.2).

Proof By the strong polyhedral approximation theorem 2.5, for each $j \in \mathbb{N}^+$ we find an integral polyhedral chain $P_j \in \mathcal{P}_k(\mathbb{R}^n \times \mathbb{R}^N)$ and a C^1-diffeomorphism g_j of $\mathbb{R}^n \times \mathbb{R}^N$ onto itself such that $\mathrm{Lip}(g_j) \le 1 + 1/j$, $\mathrm{Lip}(g_j^{-1}) \le 1 + 1/j$, and $\mathbf{M}(g_{j\#}T - P_j) + \mathbf{M}(\partial(g_{j\#}T - P_j)) \le 1/j$.

Denote $\mathcal{M}_j = \mathrm{set}(P_j)$. By applying Proposition 4.2 to the sequence $\{P_j\}_j$, we deduce that for a.e. projection $\mathbf{p} \in \mathbf{O}^*(N,k+1)$, for each $j \in \mathbb{N}^+$, and for $\mathcal{H}^k$-a.e. $z \in (\mathrm{Id}_{\mathbb{R}^n} \bowtie \mathbf{p})(\mathcal{M}_j)$

$$\mathbf{N}(\mathrm{Id}_{\mathbb{R}^n} \bowtie \mathbf{p} | \mathcal{M}_j; z) := \mathcal{H}^0(\mathcal{M}_j \cap (\mathrm{Id}_{\mathbb{R}^n} \bowtie \mathbf{p})^{-1}(\{z\})) = 1\,.$$

As a consequence, possibly by slightly rotating the target space, we deduce that for each multi-index β with $|\beta| = k+1$ the projections Ψ_β are "good" for each P_j in the above sense, i.e.,

$$\mathbf{N}(\Psi_\beta | \mathcal{M}_j; z) := \mathcal{H}^0(\mathcal{M}_j \cap {\Psi_\beta}^{-1}(\{z\})) = 1 \tag{4.9}$$

for each $j \in \mathbb{N}^+$ and for $\mathcal{H}^k$-a.e. $z \in \Psi_\beta(\mathcal{M}_j)$.

Define now $\widetilde{P}_j := f_{j\#}P_j$, where $f_j = g_j^{-1}$, and write $\widetilde{P}_j = \tau(\widetilde{\mathcal{M}}_j,\theta_j,\xi_j)$, where $\widetilde{\mathcal{M}}_j := \mathrm{set}(\widetilde{P}_j)$. Formula (4.9) yields that for each $j \in \mathbb{N}^+$ and for $\mathcal{H}^k$-a.e. $z \in \Psi_\beta \circ g_j(\widetilde{\mathcal{M}}_j)$

$$\mathbf{N}(\Psi_\beta \circ g_j | \widetilde{\mathcal{M}}_j; z) := \mathcal{H}^0(\widetilde{\mathcal{M}}_j \cap (\Psi_\beta \circ g_j)^{-1}(\{z\})) = 1\,.$$

By applying the general area-coarea formula, Theorem 2.2, we thus infer that

$$\int_{\widetilde{\mathcal{M}}_j} J^{\widetilde{\mathcal{M}}_j}_{\Psi_\beta \circ g_j}(z)\,\widetilde{\theta}_j(z)\,d\mathcal{H}^k(z) = \mathbf{M}((\Psi_\beta \circ g_j)_\# \widetilde{P}_j)\,.$$

By the strong convergence, and again by the area-coarea formula, we also have

$$\lim_{j\to\infty} \mathbf{M}((\Psi_\beta \circ g_j)_\# \widetilde{P}_j) = \mathbf{M}(\psi_{\beta\#}T) \le \int_{\mathcal{M}} J^{\mathcal{M}}_{\Psi_\beta}(z)\,\theta(z)\,d\mathcal{H}^k(z)$$

where, we recall, $T = \tau(\mathcal{M},\theta,\overrightarrow{\xi})$, and we can assume without loss of generality $\mathcal{M} = \mathrm{set}(T)$. Since moreover $\mathbf{M}(\widetilde{P}_j - T) \to 0$, denoting by $\triangle$ the symmetric difference, we also infer that $\mathcal{H}^k(\widetilde{\mathcal{M}}_j \triangle \mathcal{M}) \to 0$ as $j \to \infty$. Using that $\mathrm{Lip}(g_j) \leq 1 + 1/j$ and $\mathrm{Lip}(g_j^{-1}) \leq 1 + 1/j$, we thus deduce that

$$\int_{\mathcal{M}} J^{\mathcal{M}}_{\Psi_\beta}(z)\, \theta(z)\, d\mathcal{H}^k(z) \leq \liminf_{j\to\infty} \int_{\widetilde{\mathcal{M}}_j} J^{\widetilde{\mathcal{M}}_j}_{\Psi_\beta \circ g_j}(z)\, \widetilde{\theta}_j(z)\, d\mathcal{H}^k(z)$$

and definitively that

$$\mathbf{M}(\psi_{\beta\#}T) = \int_{\mathcal{M}} J^{\mathcal{M}}_{\Psi_\beta}(z)\, \theta(z)\, d\mathcal{H}^k(z)\,.$$

Using again the general area-coarea formula, this yields that for each β

$$\mathbf{N}(\Psi_\beta|\mathcal{M}; z) := \mathcal{H}^0(\mathcal{M} \cap {\Psi_\beta}^{-1}(\{z\})) = 1$$

for $\mathcal{H}^k$-a.e. $z \in \Psi_\beta(\mathcal{M})$. This means exactly that each Ψ_β is a “good” projection in the above sense. The claim follows from Proposition 4.4 and from the above argument concerning “good” projections. □

Remark 4.6 For future use, we notice that the function $\widetilde{\Phi}$ in (4.7) is bounded and $\mathcal{H}^k \llcorner \mathcal{N}_\beta$-summable, hence it can be extended to a bounded Borel function $\widetilde{\Phi}$ on $\mathbb{R}^n \times \mathbb{R}^{k+1}_\beta$. Since moreover T^β has finite mass, the action of T^β is uniquely extended to such class of forms $\omega = \phi(x)\, \widetilde{\Phi}(x, y_\beta)\, g(y_\beta)\, dx^\alpha \wedge dy^\gamma$.

5 The structure theorem I for integral cycles

In this section we prove the structure theorem 1.1 in the case of integral cycles, i.e. satisfying $\partial T = 0$. In Step 1, we deal with the easier case $k = N - 1$, where we directly apply Proposition 3.1. In Step 2 we consider the case of higher codimension $N \geq k + 2$, and make use of the projection argument from Sec. 4. In Step 3 we conclude that $T = 0$ if $N = k$.

Step 1: the case $k = N - 1$. Since $N = k + 1 \geq 2$, we follow the notation from Sec. 3, and apply this concentration property:

Lemma 5.1 *Let λ and μ be respectively a non-negative and a signed Radon measure on $\mathbb{R}^n$, with finite total variation, such that for every $x_0 \in \mathbb{R}^n$ and a.e. $r > 0$ we have*

$$|\mu(\overline{B}_r(x_0)| \leq c\, \lambda(\overline{B}_r(x_0))^\alpha$$

for some fixed constants $c > 0$ and $\alpha > 1$. Then μ is purely atomic, and it is concentrated on the at most countable set of atoms of λ.

For a proof of Lemma 5.1, we refer to [14, Lemma 4.4] and also [11, Lemma 6.3], where a gap (the absolute continuity of μ with respect to λ) is filled.

Now, by Proposition 3.1 we obtain the isoperimetric inequality (3.2). We can thus apply Lemma 5.1 with $\alpha = N/(N-1)$, $\mu = \mu_g$, and $\lambda = \lambda(T)$ given by

$$\langle \lambda(T), B \rangle := \mathbf{M}(T \llcorner B \times \mathbb{R}^N)\,, \qquad B \in \mathcal{B}(\mathbb{R}^n)\,.$$

Denoting by $\{a_i\}_i \subset \mathbb{R}^n$ the at most countable family of atoms of $\lambda(T)$, we deduce that for every i there exists a signed Radon measure λ_i on $\mathbb{R}^N$ such that for every $\phi \in C_c^\infty(\mathbb{R}^n)$ and $g \in C_c^\infty(\mathbb{R}^N, \mathbb{R}^N)$

$$T(\phi \wedge \omega_g) = (-1)^{N-1} \langle \mu_g, \phi \rangle = \sum_{i=1}^\infty \delta_{a_i}(\phi) \cdot \lambda_i(g)\,,$$

where $\omega_g \in \mathcal{D}^{N-1}(\mathbb{R}^N)$ is given by (3.1). Also, forms of the type $\phi \wedge \omega_g$ are dense in the space of forms $\eta = \eta^{(N-1)}$ in $\mathcal{D}^{N-1}(\mathbb{R}^n \times \mathbb{R}^N)$, whereas $T(\eta^{(h)}) = 0$ for $h \le N-2$, by the assumption (1.1) with $k = N$. Define $\Sigma_i \in \mathcal{D}_{N-1}(\mathbb{R}^N)$ by

$$\Sigma_i(\omega_g) := \lambda_i(g)\,, \qquad g \in C_c^\infty(\mathbb{R}^N, \mathbb{R}^N)\,.$$

Now, for every $x \in \mathbb{R}^n$ and for all but an at most countable set of "bad" radii $r > 0$, the boundary $\partial B_r(x)$ does not contain atoms of $\lambda(T)$. Hence, by Lemma 5.1, for any "good" radius we have $\langle \mu_g, \partial B_r(x) \rangle = 0$ for every $g \in C_c^\infty(\mathbb{R}^N, \mathbb{R}^N)$. Taking a smooth sequence $\{\phi_j\} \in C_c^\infty(\mathbb{R}^n)$ strongly converging in L^1 to the characteristic function of $\overline{B}_r(x)$, we find that

$$\lim_{j\to\infty} T(\phi_j \wedge \omega_g) = \sum \{\Sigma_i(\omega_g) \mid i \text{ is such that } a_i \in B_r(x)\}$$

for each $g \in C_c^\infty(\mathbb{R}^N, \mathbb{R}^N)$. Since $T \in \mathcal{R}_{N-1}(\mathbb{R}^n \times \mathbb{R}^N)$ with $\partial T = 0$, this yields that (1.2) holds true, where $\Sigma_i \in \mathcal{R}_{N-1}(\mathbb{R}^N)$ satisfies $\partial \Sigma_i = 0$ for each i.

Step 2: the case $N > k+1$. For a multi-index β of length $|\beta| = k+1$, let Ψ_β denote the projection map given by (4.1), and define $T^\beta := \Psi_{\beta\#} T$. By the assumption, Lemma 4.1 yields that T^β is i.m. rectifiable in $\mathcal{R}_k(\mathbb{R}^n \times \mathbb{R}^{k+1}_\beta)$ and satisfies $\partial T^\beta = 0$. Moreover, by (1.1) it is readily checked that $T^\beta(\eta) = T^\beta(\eta^{(k)})$ for every form $\eta \in \mathcal{D}^k(\mathbb{R}^n \times \mathbb{R}^{k+1}_\beta)$. Then, by using the case $N = k+1$, we deduce the existence of an at most countable subset S_0^β of $\mathbb{R}^n$ such that

$$\operatorname{set}(T^\beta) \subset S_0^\beta \times \mathbb{R}^{k+1}_\beta\,. \tag{5.1}$$

It then remains to show that

$$\operatorname{set}(T) \subset S_0 \times \mathbb{R}^N\,, \quad \text{where} \quad S_0 := \bigcup_{|\beta|=k+1} S_0^\beta\,. \tag{5.2}$$

To this purpose, possibly by slightly rotating the target space, we may and do apply Proposition 4.5 with $\gamma = \beta - j$ for some $j \in \beta$, so that $|\alpha| = 0$. The current $T^\beta = \tau(\mathcal{N}_\beta, \theta_\beta, \overrightarrow{\zeta_\beta})$ satisfies (5.1), whereas in (4.8) we have just obtained that

$$T(\phi(x)\, f(y_{\overline{\beta}})\, g(y_\beta)\, dy^{\beta-j}) = T^\beta(\phi(x)\, \widetilde{\Phi}(x, y_\beta)\, g(y_\beta) dy^{\beta-j})\,, \tag{5.3}$$

with $\widetilde{\Phi}$ given by (4.7). Moreover, linear combinations of forms of the type

$$\phi(x)\, f(y_{\overline{\beta}})\, g(y_\beta)\, dy^{\beta-j}\,, \text{ where } \phi \in C_c^\infty(\mathbb{R}^n)\,,\ f \in C_c^\infty(\mathbb{R}^{N-k-1})\,,\ g \in C_c^\infty(\mathbb{R}^{k+1})$$

yield a dense subclass of forms $\eta = \eta^{(k)} \in \mathcal{D}^k(\mathbb{R}^n \times \mathbb{R}^N)$, see Remark 4.6. Therefore, we deduce that (5.2) follows from (5.1). In conclusion, the structure property (1.2) is obtained by means of the same argument that is used at the end of Step 1.

STEP 3: THE CASE $N = k$. We show that $T = 0$ if $N = k$. Recall that the claim is trivial for $N < k$, by the verticality property (1.1).

Assume then $N = k$, and consider the injection map $\mathrm{i} : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n \times \mathbb{R}^{k+1}$ such that $\mathrm{i}(x, y) := (x, y, 0)$. On account of Lemma 2.6, it is readily checked that the current $\widetilde{T} := \mathrm{i}_\# T$ satisfies the hypotheses of Theorem 1.1. However, the corresponding currents $\Sigma_i \in \mathcal{R}_k(\mathbb{R}^{k+1})$ in (1.2) are supported in $\mathbb{R}^k \times \{0\}$ and satisfy $\partial\Sigma_i = 0$. By the Constancy theorem, see [18, 26.27], any integral k-cycle with finite mass in $\mathbb{R}^k \times \{0\}$ is equal to zero. Therefore, $\Sigma_i = 0$ for all i, hence $\mathrm{i}_\# T = 0$ and finally $T = 0$.

6 The structure theorem II for integral cycles

In this section we prove the more general structure theorem 1.2 for the subclass of integral currents, i.e., satisfying $\partial T = 0$. The proof relies on some arguments from *slicing theory*, for which we refer to [18, Sec. 28] and [9, Sec. 2.5].

We make use of an induction argument on $\mathbf{p} \in \mathbb{N}$ in order to deal with the case $\mathbf{q} = k - \mathbf{p}$ in (1.3), for any choice of the dimensions n, N of the domain and target spaces, respectively, and k of the current T. Notice that for $\mathbf{p} = 0$ the claim has been proved in Theorem 1.1.

We thus fix $\mathbf{p}$ a positive integer, and assume that we have proved the claim if T satisfies (1.3) with $\mathbf{q} = k - \nu$ for each natural $\nu = 0, 1, \ldots, \mathbf{p} - 1$.

In Step 1, using a slicing argument we easily solve the case $N < k$. In Step 2, we assume $N = k$ and exploit the assumption $\partial T = 0$. In Steps 3 and 4, the hardest part of the proof, we deal with the case $N = k + 1$. In Step 5, using the projection argument from Sec. 4, we readily recover the case $N > k + 1$.

Let $T \in \mathcal{R}_k(\mathbb{R}^n \times \mathbb{R}^N)$ satisfying $\partial T = 0$ and property $T_{(h)} = 0$ for $h = 0, \ldots, \mathbf{q} - 1$, where $\mathbf{q} = k - \mathbf{p}$. Since $k - \mathbf{q} = \mathbf{p}$, we have show the existence of a countably $\mathcal{H}^{\mathbf{p}}$-rectifiable subset $S_{\mathbf{p}}$ of $\mathbb{R}^n$ such that

$$\mathrm{set}(T) \subset S_{\mathbf{p}} \times \mathbb{R}^N \,. \tag{6.1}$$

Now, every form $\eta \in \mathcal{D}^k(\mathbb{R}^n \times \mathbb{R}^N)$ decomposes as $\eta = \sum_m \eta^{(m)}$, where $\max\{0, k - n\} \le m \le \min\{k, N\}$ and

$$\eta^{(m)} = \sum_{|\alpha| = k - m} \eta_\alpha \,, \qquad \eta_\alpha := \sum_{|\beta| = m} \eta^{\alpha,\beta}(x, y)\, dy^\beta \wedge dx^\alpha \tag{6.2}$$

for some $\eta^{\alpha,\beta} \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^N)$. Since (1.3) gives $T(\eta^{(m)}) = 0$ for $m < \mathbf{q}$, we shall analyze the action of T on the components $\eta^{(m)}$, where we assume $m \ge \mathbf{q}$.

STEP 1: THE CASE $N < k$. According to (6.2), denote by $\pi_\alpha : \mathbb{R}^n \to \mathbb{R}^{k-m}$ the orthogonal projection onto the α-components of x, i.e., $\pi_\alpha(x) = x_\alpha$, and by Π_{x_α} the $(n + m - k)$-plane $\pi_\alpha^{-1}(\{x_\alpha\})$. Notice that $m < k$ if $N < k$. For $\mathcal{H}^{k-m}$-a.e. $x_\alpha \in \mathbb{R}^{k-m}$, we thus define the sliced current

$$T_{x_\alpha} := \langle T, \pi_\alpha \bowtie \mathrm{Id}_{\mathbb{R}^N}, x_\alpha \rangle \,, \qquad (\pi_\alpha \bowtie \mathrm{Id}_{\mathbb{R}^N})(x, y) := (x_\alpha, y) \,.$$

Remark 6.1 Recall that a dense sub-class of smooth forms is given by linear combinations of forms with coefficients of the type $\eta^{\alpha,\beta}(x,y)=\varphi(x_{\overline{\alpha}})\,\widetilde{\varphi}(x_\alpha)\,\psi(y)$, where $\varphi\in C^\infty(\mathbb{R}^{n+m-k})$, $\widetilde{\varphi}\in C^\infty(\mathbb{R}^{k-m})$, and $\psi\in C_c^\infty(\mathbb{R}^N)$.

By assumption, and using that the slicing map is an orthogonal projection only involving the "horizontal" coordinates x, for $\mathcal{H}^{k-m}$-a.e. $x_\alpha\in\mathbb{R}^{k-m}$ the following properties hold:

1. T_{x_α} belongs to $\mathcal{R}_m(\Pi_{x_\alpha}\times\mathbb{R}^N)$;
2. the boundary of the slice agrees (up to the sign) with the slice of the boundary, hence condition $\partial T=0$ yields $\partial T_{x_\alpha}=0$;
3. $T_{x_\alpha}(\eta^{(h)})=0$ for every $h<\mathbf{q}$ and $\eta\in\mathcal{D}^m(\Pi_{x_\alpha}\times\mathbb{R}^N)$.

This yields that the sliced m-current T_{x_α} satisfies the hypothesis of Theorem 1.2. Moreover, we have $k-\mathbf{p}=\mathbf{q}=m-\nu$, where $\nu:=\mathbf{p}-(k-m)$, and by the assumptions $0<k-m\le\mathbf{p}$, hence $0\le\nu<\mathbf{p}$. Therefore, by the inductive hypothesis, and since $m-\mathbf{q}=\mathbf{p}-(k-m)$, we find the existence of a countably $\mathcal{H}^{\mathbf{p}-(k-m)}$-rectifiable subset $S_{\mathbf{p}-(k-m)}$ of Π_{x_α} such that

$$\mathrm{set}(T_{x_\alpha})\subset S_{\mathbf{p}-(k-m)}\times\mathbb{R}^N\,. \tag{6.3}$$

Now, the slicing formula gives

$$T(\varphi(x_{\overline{\alpha}})\,\widetilde{\varphi}(x_\alpha)\,\psi(y)\,dy^\beta\wedge dx^\alpha)=\int_{\mathbb{R}^{k-m}}\Big(T_{x_\alpha}(\varphi(x_{\overline{\alpha}})\,\psi(y)\,dy^\beta)\Big)\,\widetilde{\varphi}(x_\alpha)\,dx_\alpha\,;$$

therefore the property (6.1) follows from (6.3), on account of Remark 6.1.

Remark 6.2 If $N=\mathbf{q}$, i.e. $\mathbf{p}=k-N$, we thus obtain that $T=0$. This property follows from the argument in Step 3 from Sec. 5 in the case $\mathbf{p}=0$, whereas for $\mathbf{p}>0$, whence $N<k$, it is an immediate consequence of the previous slicing argument.

STEP 2: THE CASE $N=k$. Since the previous slicing argument holds for $m<k$, in this case it suffices to consider the action of T on forms of the type $\eta=\eta^{(k)}$.

We have $\eta^{(k)}=\phi(x,y)\,dy$ for some $\phi\in C_c^\infty(\mathbb{R}^n\times\mathbb{R}^k)$, where $dy:=dy^1\wedge\cdots\wedge dy^k$. By linearity and density, we may and do assume $\phi(x,y)=\varphi(x)\,f(y_1)\,g(\widehat{y_1})$, where $\varphi\in C_c^\infty(\mathbb{R}^n)$, $f\in C_c^\infty(\mathbb{R})$, and $g\in C_c^\infty(\mathbb{R}^{k-1})$. We thus denote by F a primitive of f, and set

$$\xi:=\varphi(x)\,F(y_1)\,g(\widehat{y_1})\,\widehat{dy^1}\in\mathcal{E}_b^{n-1}(\mathbb{R}^n\times\mathbb{R}^k)\,.$$

Using the usual convention of summation on the repeated indices, we compute

$$d\xi=\varphi_{,x_i}(x)\,F(y_1)\,g(\widehat{y_1})\,dx^i\wedge\widehat{dy^1}+\varphi(x)\,f(y_1)\,g(\widehat{y_1})\,dy\,.$$

Since ξ has bounded Lipschitz coefficients, property $\partial T=0$ and Lemma 2.6 yield that $T(d\xi)=0$, hence

$$T(\varphi(x)\,f(y_1)\,g(\widehat{y_1})\,dy)=-T(\varphi_{,x_i}(x)\,F(y_1)\,g(\widehat{y_1})\,dx^i\wedge\widehat{dy^1})\,.$$

Therefore, the argument that we used for the component $\eta^{(k-1)}$, applied this time to the k-form $\varphi_{,x_i}(x)\,F(y_1)\,g(\widehat{y_1})\,dx^i\wedge\widehat{dy^1}$, yields the assertion, thanks to the dominated convergence theorem.

Step 3: the case $N=k+1$. Fix $j\in\{1,\dots,k+1\}$. For $t_1<t_2$, denote

$$\{t_1<y_j<t_2\}:=\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^{k+1}\mid t_1<y_j<t_2\}\,.$$

For a.e. choice of $t_1<t_2$ it turns out that the sliced current $T\llcorner\{t_1<y_j<t_2\}$ is i.m. rectifiable and with boundary of finite mass. Write as usual $T=\tau(\mathcal{M},\theta,\overrightarrow{\xi})$, where we may and do assume $\mathcal{M}=\operatorname{set}(T)$. In Step 4, we shall prove the following

Proposition 6.3 *For a.e. real numbers $t_1<t_2$ there exists a k-rectifiable set $\widetilde{\mathcal{M}}\subset\mathbb{R}^n\times\mathbb{R}^{k+1}$, with $\widetilde{\mathcal{M}}\subset\operatorname{set}(T\llcorner\{t_1<y_j<t_2\})$, and a countably $\mathcal{H}^{\mathbf{p}}$-rectifiable subset $S_{\mathbf{p}}$ of $\mathbb{R}^n$ satisfying*

$$\widetilde{\mathcal{M}}\subset S_{\mathbf{p}}\times\mathbb{R}^{k+1}$$

such that for every k-form ω of the type $\omega:=\phi(x,\widehat{y}_j)\,\widehat{dy^j}$, where $\phi\in C^\infty_c(\mathbb{R}^n\times\mathbb{R}^k_{\widehat{y}_j})$,

$$T\llcorner\{t_1<y_j<t_2\}(\omega)=\int_{\widetilde{\mathcal{M}}}\langle\omega,\overrightarrow{\xi}\rangle\,\theta\,d\mathcal{H}^k\,. \tag{6.4}$$

Now, any completely vertical k-form in $\mathcal{D}^k(\mathbb{R}^n\times\mathbb{R}^{k+1})$ can be written as

$$\eta=\eta^{(k)}=\sum_{j=1}^{k+1}\psi_j(x,y)\,\widehat{dy^j}\,,\qquad\psi_j\in C^\infty_c(\mathbb{R}^n\times\mathbb{R}^{k+1})\,.$$

Fix $j\in\{1,\dots,n\}$. By a density argument, we may and do assume that $\psi_j(x,y)=\phi(x,\widehat{y}_j)\,f(y_j)$ for some $\phi\in C^\infty_c(\mathbb{R}^n\times\mathbb{R}^k_{\widehat{y}_j})$ and $f\in C^\infty_c(\mathbb{R})$.

For $\nu\in\mathbb{N}$ and $h\in\mathbb{Z}$, denote $t^\nu_h:=h\,2^{-\nu}$. Possibly by slightly moving the points t^ν_h, we may and do assume that for each ν and h we can apply Proposition 6.3 to the restricted current $T\llcorner\{t^\nu_h<y_j<t^\nu_{h+1}\}$. We then find a k-rectifiable set $\widetilde{\mathcal{M}}^\nu_h\subset\mathbb{R}^n\times\mathbb{R}^{k+1}$, with $\widetilde{\mathcal{M}}^\nu_h\subset\mathcal{M}$, where $\mathcal{M}=\operatorname{set}(T)$, and a countably $\mathcal{H}^{\mathbf{p}}$-rectifiable subset $S_{\mathbf{p}}(\nu,h)$ of $\mathbb{R}^n$ satisfying

$$\widetilde{\mathcal{M}}^\nu_h\subset S_{\mathbf{p}}(\nu,h)\times\mathbb{R}^{k+1}$$

and such that (the sliced currents having finite mass) for every $\phi\in C^\infty_c(\mathbb{R}^n\times\mathbb{R}^k_{\widehat{y}_j})$

$$T\llcorner\{t^\nu_h<y_j<t^\nu_{h+1}\}(\phi(x,\widehat{y}_j)\,\widehat{dy^j})=\int_{\widetilde{\mathcal{M}}^\nu_h}\langle\phi(x,\widehat{y}_j)\,\widehat{dy^j},\overrightarrow{\xi}\rangle\,\theta\,d\mathcal{H}^k\,.$$

Since moreover $f\in C^\infty_c(\mathbb{R})$, there exists a sequence $\{f_\nu\}_\nu$ of piecewise constant and bounded functions $f_\nu:\mathbb{R}\to\mathbb{R}$ satisfying:

i) f_ν is constant on $I^\nu_h:=]t^\nu_h,t^\nu_{h+1}[$ for each h;
ii) f_ν has compact support contained in the support of f;
iii) $f_\nu\to f$ uniformly as $\nu\to\infty$.

As a consequence, using that $T=\tau(\mathcal{M},\theta,\overrightarrow{\xi})$ is i.m. rectifiable, we have

$$T(f(y_j)\,\phi(x,\widehat{y}_j)\,\widehat{dy^j})=\lim_{\nu\to\infty}T(f_\nu(y_j)\,\phi(x,\widehat{y}_j)\,\widehat{dy^j})\,. \tag{6.5}$$

Since moreover $f_\nu(y_j)\equiv a^\nu_h\in\mathbb{R}$ for each $y_j\in I^\nu_h$ and each h, we have

$$T(f_\nu(y_j)\,\phi(x,\widehat{y}_j)\,\widehat{dy^j})=\sum_h a^\nu_h\cdot T\llcorner\{t^\nu_h<y_j<t^\nu_{h+1}\}(\phi(x,\widehat{y}_j)\,\widehat{dy^j})\,,$$

where the sum in h is finite for each f_ν. Setting $\widetilde{\mathcal{M}}^\nu := \bigcup_h \widetilde{\mathcal{M}}^\nu_h$ and $S_{\mathbf{p}}(\nu) := \bigcup_h S_{\mathbf{p}}(\nu,h)$, then $\widetilde{\mathcal{M}}^\nu$ is a k-rectifiable subset of $\mathcal{M}$, and $S_{\mathbf{p}}(\nu)$ a countably $\mathcal{H}^{\mathbf{p}}$-rectifiable subset of $\mathbb{R}^n$ satisfying $\widetilde{\mathcal{M}}^\nu \subset S_{\mathbf{p}}(\nu)\times\mathbb{R}^{k+1}$ and such that

$$T(f_\nu(y_j)\,\phi(x,\widehat{y}_j)\,\widehat{dy^j}) = \int_{\widetilde{\mathcal{M}}^\nu} \langle f_\nu(y_j)\,\phi(x,\widehat{y}_j)\,\widehat{dy^j},\,\overrightarrow{\xi}\rangle\,\theta\,d\mathcal{H}^k\,.$$

Therefore, setting $\widetilde{\mathcal{M}}^{(j)} := \bigcup_\nu \widetilde{\mathcal{M}}^\nu$ and $S^j_{\mathbf{p}} := \bigcup_\nu S_{\mathbf{p}}(\nu)$, again $\widetilde{\mathcal{M}}^{(j)}$ is a k-rectifiable subset of $\mathcal{M}$, and $S^j_{\mathbf{p}}$ a countably $\mathcal{H}^{\mathbf{p}}$-rectifiable subset of $\mathbb{R}^n$ satisfying $\widetilde{\mathcal{M}}^{(j)} \subset S^j_{\mathbf{p}}\times\mathbb{R}^{k+1}$ and such that

$$T(f_\nu(y_j)\,\phi(x,\widehat{y}_j)\,\widehat{dy^j}) = \int_{\widetilde{\mathcal{M}}^{(j)}} \langle f_\nu(y_j)\,\phi(x,\widehat{y}_j)\,\widehat{dy^j},\,\overrightarrow{\xi}\rangle\,\theta\,d\mathcal{H}^k \qquad \forall\,\nu\in\mathbb{N}\,.$$

By (6.5), we thus obtain

$$T(f(y_j)\,\phi(x,\widehat{y}_j)\,\widehat{dy^j}) = \int_{\widetilde{\mathcal{M}}^{(j)}} \langle f(y_j)\,\phi(x,\widehat{y}_j)\,\widehat{dy^j},\,\overrightarrow{\xi}\rangle\,\theta\,d\mathcal{H}^k\,.$$

By linearity and density, letting $\widetilde{\mathcal{M}} = \cup_j \widetilde{\mathcal{M}}^{(j)}$ and $S_{\mathbf{p}} := \bigcup_j S^j_{\mathbf{p}}$, we have just shown that

$$T(\eta^{(k)}) = \int_{\widetilde{\mathcal{M}}} \langle \eta^{(k)},\,\overrightarrow{\xi}\rangle\,\theta\,d\mathcal{H}^k$$

where $\widetilde{\mathcal{M}}$ is a k-rectifiable subset of $\mathcal{M}$, and $S_{\mathbf{p}}$ a countably $\mathcal{H}^{\mathbf{p}}$-rectifiable subset of $\mathbb{R}^n$ satisfying $\widetilde{\mathcal{M}} \subset S_{\mathbf{p}}\times\mathbb{R}^{k+1}$. Arguing as in Step 1 for the components $\eta^{(m)}$, where $m<k$, the claim (6.1) follows.

Step 4: proof of proposition 6.3. By slicing theory, for a.e. radius $R>0$ the i.m. rectifiable current

$$T^{j,R} := T \llcorner \{(x,y)\in\mathbb{R}^n\times\mathbb{R}^{k+1} \mid |y_h|<R \text{ for any } h\neq j\}$$

satisfies $\mathbf{M}(\partial T^{j,R})<\infty$. Moreover, for any such "good" radius R the current

$$T^{j,R}_{s_1,s_2} := T \llcorner \{(x,y)\in\mathbb{R}^n\times\mathbb{R}^{k+1} \mid s_1<y_j<s_2,\ |y_h|<R \text{ for any } h\neq j\}$$

satisfies $M(\partial T^{j,R}_{s_1,s_2})<\infty$ for a.e. $s_1<s_2$. Therefore, for a.e. $t_1<t_2$ we can find an increasing sequence of good radii $R_h\nearrow\infty$ such that the compactly supported i.m. rectifiable current $T^{j,R_h}_{t_1,t_2}\in\mathcal{R}_{k,c}(\mathbb{R}^n\times\mathbb{R}^{k+1})$ satisfies $M(\partial T^{j,R_h}_{t_1,t_2})<\infty$ for each h.

Consider the affine homotopy map $h^{j,R_h}:(\mathbb{R}^n\times\mathbb{R}^{k+1})\times[0,1]\to\mathbb{R}^n\times\mathbb{R}^{k+1}$

$$h^{j,R_h}(x,y,t) := t\,(x,y) + (1-t)\,f^{j,R_h}(x,y)\,,$$

where $f^{j,R_h}(x,y) := (x,R_h+1,\dots,R_h+1,y_j,R_h+1,\dots,R_h+1)$. The current $h^{j,R_h}_{\#}(T^{j,R_h}_{t_1,t_2}\times[\![0,1]\!])$ is compactly supported in $\mathcal{R}_{k+1,c}(\mathbb{R}^n\times\mathbb{R}^{k+1})$. Similarly, both the currents

$$\begin{aligned} S^{j,R_h}_{t_1,t_2} &:= (-1)^k\,h^{j,R_h}_{\#}(\partial T^{j,R_h}_{t_1,t_2}\times[\![0,1]\!]) - f^{j,R_h}_{\#}(T^{j,R_h}_{t_1,t_2})\,, \\ \widetilde{T}^{j,R_h}_{t_1,t_2} &:= T^{j,R_h}_{t_1,t_2} + S^{j,R_h}_{t_1,t_2} \end{aligned} \tag{6.6}$$

are compactly supported in $\mathcal{R}_{k,c}(\mathbb{R}^n\times\mathbb{R}^{k+1})$. Moreover, using that $\partial f^{j,R_h}_{\#}(T^{j,R_h}_{t_1,t_2})$ agrees with $f^{j,R_h}_{\#}(\partial T^{j,R_h}_{t_1,t_2})$, the homotopy formula (2.4) gives $\partial\widetilde{T}^{j,R_h}_{t_1,t_2}=0$.

We claim that

$$\mathcal{H}^k\big(\mathrm{set}(T^{j,R_h}_{t_1,t_2})\triangle\,\mathrm{set}(S^{j,R_h}_{t_1,t_2})\big)=0\,. \tag{6.7}$$

In fact, $\mathrm{set}(f^{j,R_h}_{\#}(T^{j,R_h}_{t_1,t_2}))$ is contained in $\{(x,y)\mid y_h=R_h+1 \text{ if } h\neq j\}$, hence it is $\mathcal{H}^k$-essentially disjoint with $\mathrm{set}(T^{j,R_h}_{t_1,t_2})$. Since moreover $\mathbf{M}(\partial T^{j,R_h}_{t_1,t_2})<\infty$, by our construction we also get

$$\mathcal{H}^k\big(\mathrm{set}(T^{j,R_h}_{t_1,t_2})\triangle\,\mathrm{set}\big(h^{j,R_h}_{\#}\big(\partial T^{j,R_h}_{t_1,t_2}\times[\![0,1]\!]\big)\big)\big)=0\,.$$

By (6.7) we thus infer that there is no cancellation in the sum in the second line of the definition (6.6), i.e.,

$$\mathbf{M}(\widetilde{T}^{j,R_h}_{t_1,t_2})=\mathbf{M}\big(T^{j,R_h}_{t_1,t_2}\big)+\mathbf{M}\big(S^{j,R_h}_{t_1,t_2}\big)\,.$$

Therefore, writing as usual

$$T^{j,R_h}_{t_1,t_2}=\tau(\mathcal{M}_h,\theta,\overrightarrow{\xi}\,)\,,\qquad \widetilde{T}^{j,R_h}_{t_1,t_2}=\tau(\mathcal{N}_h,\widetilde{\theta},\overrightarrow{\zeta}\,)\,, \tag{6.8}$$

and assuming without loss of generality that $\theta\neq 0$ on $\mathcal{M}_h$ and $\widetilde{\theta}\neq 0$ on $\mathcal{N}_h$, this yields that

$$\mathcal{H}^k(\mathcal{N}_h)=\mathcal{H}^k(\mathcal{M}_h)+\mathcal{H}^k(\mathcal{N}_h\setminus\mathcal{M}_h)\,. \tag{6.9}$$

If e.g. $j\neq 1$, setting $\widetilde{y}:=y_{\overline{(1,j)}}$, by a density argument we may and do choose $\phi(x,\widehat{y}_j)=\varphi(x)\,f(y_1)\,g(\widetilde{y})$ in (6.4), where $\varphi\in C^\infty_c(\mathbb{R}^n)$, $f\in C^\infty_c(\mathbb{R})$, and $g\in C^\infty_c(\mathbb{R}^{k-1})$. Denote by F a primitive of f, and let

$$\xi:=\varphi(x)\,F(y_1)\,g(\widetilde{y})\,d\widetilde{y}\,,\qquad d\widetilde{y}:=dy^{\overline{(1,j)}}\,,$$

so that $\xi\in\mathcal{E}^{k-1}_b(\mathbb{R}^n\times\mathbb{R}^k_{\widehat{y}_j})$ satisfies $d\xi=\omega+\widetilde{\omega}$, where

$$\omega:=\varphi(x)\,f(y_1)\,g(\widetilde{y})\,\widehat{dy^j}\,,\quad \widetilde{\omega}:=\varphi_{,x_i}(x)\,F(y_1)\,g(\widetilde{y})\,dx^i\wedge d\widetilde{y}\,.$$

Condition $\partial\widetilde{T}^{j,R_h}_{t_1,t_2}=0$ yields $\widetilde{T}^{j,R_h}_{t_1,t_2}(d\xi)=0$, whence $\widetilde{T}^{j,R_h}_{t_1,t_2}(\omega)=-\widetilde{T}^{j,R_h}_{t_1,t_2}(\widetilde{\omega})$.

Now, in the formulas (6.8) we denote

$$\begin{aligned}\overrightarrow{\xi}(z)&=\sum_{|\alpha|+|\beta|=k}\xi^{\alpha,\beta}(z)\,e_\alpha\wedge\varepsilon_\beta\,,&&z\in\mathcal{M}_h\\ \overrightarrow{\zeta}(z)&=\sum_{|\alpha|+|\beta|=k}\zeta^{\alpha,\beta}(z)\,e_\alpha\wedge\varepsilon_\beta\,,&&z\in\mathcal{N}_h\end{aligned}$$

and correspondingly define

$$\begin{aligned}\widetilde{\mathcal{M}}_h&:=\mathcal{M}_h\setminus\{z\in\mathcal{M}_h\mid \xi^{\alpha,\beta}(z)=0\quad\text{for each }\alpha\text{ and }\beta\text{ s.t. }\beta=\overline{j}\text{ or }\beta=\overline{(1,j)}\ \}\\ \widetilde{\mathcal{N}}_h&:=\mathcal{N}_h\setminus\{z\in\mathcal{N}_h\mid \zeta^{\alpha,\beta}(z)=0\quad\text{for each }\alpha\text{ and }\beta\text{ s.t. }\beta=\overline{j}\text{ or }\beta=\overline{(1,j)}\ \}\,.\end{aligned}$$

On account or Remark 2.1, the set $\widetilde{\mathcal{N}}_h$ is k-rectifiable and moreover

$$\widetilde{T}^{j,R_h}_{t_1,t_2}(\omega)=\int_{\widetilde{\mathcal{N}}_h}\langle\omega,\overrightarrow{\zeta}\,\rangle\,\widetilde{\theta}\,d\mathcal{H}^k\,,\quad \widetilde{T}^{j,R_h}_{t_1,t_2}(\widetilde{\omega})=\int_{\widetilde{\mathcal{N}}_h}\langle\widetilde{\omega},\overrightarrow{\zeta}\,\rangle\,\widetilde{\theta}\,d\mathcal{H}^k\,.$$

Since $\widetilde{\omega}$ "contains" the differentials dx^i, and $\partial \widetilde{T}^{j,R_h}_{t_1,t_2} = 0$, by applying to the term $\widetilde{T}^{j,R_h}_{t_1,t_2}(\widetilde{\omega})$ the slicing argument that we used in Step 1 for the component $\eta^{(k-1)}$, we thus deduce the existence of a countably $\mathcal{H}^{\mathbf{p}}$-rectifiable subset $S^h_{\mathbf{p}}$ of $\mathbb{R}^n$ such that $\widetilde{\mathcal{N}}_h \subset S^h_{\mathbf{p}} \times \mathbb{R}^{k+1}$. Since moreover the property (6.9) yields

$$\mathcal{H}^k(\widetilde{\mathcal{N}}_h) = \mathcal{H}^k(\widetilde{\mathcal{M}}_h) + \mathcal{H}^k(\widetilde{\mathcal{N}}_h \setminus \widetilde{\mathcal{M}}_h)\,,$$

we also obtain that $\widetilde{\mathcal{M}}_h \subset S^h_{\mathbf{p}} \times \mathbb{R}^{k+1}$.

Finally, since $T \llcorner \{t_1 < y_j < t_2\}$ has finite mass, we deduce that $T^{j,R_h}_{t_1,t_2} \rightharpoonup T \llcorner \{t_1 < y_j < t_2\}$ weakly in $\mathcal{D}^k(\mathbb{R}^n \times \mathbb{R}^{k+1})$ as $h \to \infty$. Therefore, Proposition 6.3 follows by taking $\widetilde{\mathcal{M}} = \cup_h \widetilde{\mathcal{M}}_h$ and $S_{\mathbf{p}} := \cup_h S^h_{\mathbf{p}}$.

Step 5: the case $N > k+1$. Exactly as in Step 2 of the proof of Theorem 1.1 from Sec. 5, we make use of the projection argument from Sec. 4. We thus fix a multi-index β of length $|\beta| = k+1$, consider the projection map Ψ_β given by (4.1), and on account of Lemma 4.1 define

$$T^\beta := \Psi_{\beta\#} T \in \mathcal{R}_k(\mathbb{R}^n \times \mathbb{R}^{k+1}_\beta)\,.$$

By the assumption, we deduce that T^β satisfies the hypotheses of Theorem 1.2, with $\mathbf{q} = k - \mathbf{p}$. Then, by using the case $N = k+1$, we find a countably $\mathcal{H}^{\mathbf{p}}$-rectifiable subset $S^\beta_{\mathbf{p}}$ of $\mathbb{R}^n$ such that

$$\operatorname{set}(T^\beta) \subset S^\beta_{\mathbf{p}} \times \mathbb{R}^{k+1}_\beta\,, \qquad \mathbb{R}^{k+1}_\beta \hookrightarrow \mathbb{R}^N\,. \tag{6.10}$$

It then remains to show that

$$\operatorname{set}(T) \subset S_{\mathbf{p}} \times \mathbb{R}^N\,, \quad \text{where} \quad S_{\mathbf{p}} := \bigcup_{|\beta|=k+1} S^\beta_{\mathbf{p}}\,. \tag{6.11}$$

To this purpose, possibly by slightly rotating the target space, we again apply Proposition 4.5. The current $T^\beta = \tau(\mathcal{N}_\beta, \theta_\beta, \overrightarrow{\zeta_\beta})$ satisfies (6.10), whereas (4.8) holds true, with $\widetilde{\Phi}$ given by (4.7). By Remark 4.6, we conclude that (6.11) follows from (6.10), as required.

7 The structure theorems for normal currents

In this section we prove Theorems 1.1 and 1.2 for the wider classes of currents $T \in \mathcal{R}_k(\mathbb{R}^n \times \mathbb{R}^N)$ satisfying $\mathbf{M}(\partial T) < \infty$. We thus assume that

$$T_{(h)} = 0 \qquad \text{for } h = 0, \ldots, \mathbf{q}-1\,, \tag{7.1}$$

where $1 \le \mathbf{q} \le k$, and show that $\operatorname{set}(T) \subset S_{k-\mathbf{q}} \times \mathbb{R}^N$ for some countably $\mathcal{H}^{k-\mathbf{q}}$-rectifiable subset $S_{k-\mathbf{q}}$ of $\mathbb{R}^n$.

To this purpose, consider the injection map $\mathfrak{i} : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}^n \times \mathbb{R}^{N+1}$ given by $\mathfrak{i}(x,y) := (x,y,0)$. Let $B_R \subset \mathbb{R}^n \times \mathbb{R}^N$ denote the open ball of radius $R > 0$ centered at the origin. By slicing theory, for a.e. $R > 0$, the restriction $T_R := T \llcorner B_R$ is a compactly supported i.m. rectifiable current in $\mathcal{R}_{k,c}(\mathbb{R}^n \times \mathbb{R}^N)$ such that $\mathbf{M}(\partial T_R) < \infty$. Then, the image current $\mathfrak{i}_\# T_R$ belongs to $\mathcal{R}_{k,c}(\mathbb{R}^n \times \mathbb{R}^{N+1})$, it has compact support contained in $\overline{B}_R \times \{0\}$, and it satisfies $\mathbf{M}(\partial\, \mathfrak{i}_\# T_R) < \infty$. Therefore, by the boundary rectifiability theorem 2.4, the current $\overline{T}_R := \partial\, \mathfrak{i}_\# T_R$ is i.m. rectifiable in $\mathcal{R}_{k-1,c}(\mathbb{R}^n \times \mathbb{R}^{N+1})$.

Remark 7.1 The assumption (7.1) yields that $\mathfrak{i}_\# T_{R\,(h)} = 0$ for $h = 0, \dots, \mathbf{q} - 1$.

Consider the affine homotopy map $\widehat{h} : [0,1] \times (\mathbb{R}^n \times \mathbb{R}^{N+1}) \to \mathbb{R}^n \times \mathbb{R}^{N+1}$

$$\widehat{h}(t,x,y,z) := \widehat{h}_t(x,y,z) := (x, ty, t(z-1)+1)\,, \quad t \in [0,1]\,,\ (x,y) \in \mathbb{R}^n \times \mathbb{R}^N\,,\ z \in \mathbb{R}$$

and let $\widehat{T}_R := \widehat{h}_\#([\![0,1]\!] \times \overline{T}_R)$, so that $\widehat{T}_R$ is i.m. rectifiable in $\mathcal{R}_{k,c}(\mathbb{R}^n \times \mathbb{R}^{N+1})$. At the end of this section, we shall prove the following

Lemma 7.2 *The current $\widehat{T}_R$ satisfies the verticality property* (7.1).

Now, by the definition we have $\widehat{h}_{0\#} \overline{T}_R = \partial\, \widehat{h}_{0\#}(\mathfrak{i}_\# T_R)$ on $\mathcal{D}^{k-1}(\mathbb{R}^n \times \mathbb{R}^{N+1})$. Since $\widehat{h}_0(x,y,z) = (x,0,1)$ and $k \geq 1$, the Remark 7.1 yields that $\widehat{h}_{0\#}(\mathfrak{i}_\# T_R) = 0$, whence $\widehat{h}_{0\#}\overline{T}_R = 0$. Therefore, since $\partial \overline{T}_R = 0$, the homotopy formula (2.4) yields

$$\partial \widehat{T}_R = \widehat{h}_{1\,\#}\overline{T}_R - \widehat{h}_{0\,\#}\overline{T}_R = \overline{T}_R =: \partial\, \mathfrak{i}_\# T_R\,.$$

We thus deduce that the current $\Sigma_R := \mathfrak{i}_\# T_R - \widehat{T}_R \in \mathcal{R}_{k,c}(\mathbb{R}^n \times \mathbb{R}^{N+1})$ satisfies the null-boundary condition $\partial \Sigma_R = 0$ and the verticality property (7.1). Since the structure theorems 1.1 and 1.2 have already been proved in the case of boundaryless currents, we infer that

$$\mathrm{set}(\Sigma_R) \subset S_R \times \mathbb{R}^{N+1} \tag{7.2}$$

for some countably $\mathcal{H}^{k-\mathbf{q}}$-rectifiable subset $S_R \subset \mathbb{R}^n$. We now claim that

$$\mathrm{set}(T_R) \subset S_R \times \mathbb{R}^N\,, \qquad T_R := T \llcorner B_R\,. \tag{7.3}$$

In fact, using that $\widehat{h}_0(x,y,z) = (x,0,1)$, by our construction

$$\mathrm{set}(\mathfrak{i}_\# T_R) \subset \mathbb{R}^n \times \mathbb{R}^N \times \{0\}\,, \qquad \mathcal{H}^k(\mathrm{set}(\widehat{T}_R) \cap (\mathbb{R}^n \times \mathbb{R}^N \times \{0\})) = 0\,.$$

Denoting by $\triangle$ the symmetric difference, this yields that

$$\mathcal{H}^k(\mathrm{set}(\mathfrak{i}_\# T_R) \triangle\, \mathrm{set}(\widehat{T}_R)) = 0\,.$$

Therefore, $\mathbf{M}(\Sigma_R) = \mathbf{M}(\mathfrak{i}_\# T_R) + \mathbf{M}(\widehat{T}_R)$, i.e., there is no cancellation in the sum $\Sigma_R := \mathfrak{i}_\# T_R - \widehat{T}_R$. Using (7.2), we can thus conclude that $\mathrm{set}(\mathfrak{i}_\# T_R) \subset S_R \times \mathbb{R}^{N+1}$ and definitely that (7.3) holds true.

Since $\mathrm{set}(T_R)$ is increasing with R, and S_R is countably $\mathcal{H}^{k-\mathbf{q}}$-rectifiable, by choosing an increasing sequence of “good” radii $R_j \nearrow \infty$ we obtain

$$\mathrm{set}(T) \subset S_{k-\mathbf{q}} \times \mathbb{R}^N\,, \qquad S_{k-\mathbf{q}} = \cup_j S_{R_j}\,,$$

where $S_{k-\mathbf{q}}$ is countably $\mathcal{H}^{k-\mathbf{q}}$-rectifiable, as required.

Proof of Lemma 7.2 By Remark 7.1, the boundary $\overline{T}_R := \partial\, \mathfrak{i}_\# T_R$ satisfies

$$\overline{T}_{R\,(h)} = 0 \qquad \text{for } h = 0, \dots, \mathbf{q} - 2\,. \tag{7.4}$$

Moreover, the current $[\![0,1]\!] \times \overline{T}_R$ has compact support, and

$$\widehat{T}_R(\widetilde{\omega}) = ([\![0,1]\!] \times \overline{T}_R)(\widehat{h}^{\#}\widetilde{\omega}) \qquad \forall\, \widetilde{\omega} \in \mathcal{D}^k(\mathbb{R}^n \times \mathbb{R}^{N+1})\,.$$

Assume that $\widetilde{\omega} = \widetilde{\omega}^{(j)}$, where $1 \le j \le \mathbf{q} - 1$, and in particular that $\widetilde{\omega} = \eta \wedge \omega$, where $\eta \in \mathcal{D}^{k-j}(\mathbb{R}^n)$ and $\omega \in \mathcal{D}^j(\mathbb{R}^{N+1})$. We decompose the pull-back of $\widetilde{\omega}$ as

$$\widehat{h}^{\#}\widetilde{\omega} = \eta(x) \wedge (\Phi(\widetilde{y},t) \wedge dt + \Psi(\widetilde{y},t))\,, \qquad \widetilde{y} \in \mathbb{R}^{N+1},$$

where the forms $\Phi(\cdot,t) \in \mathcal{E}^{j-1}(\mathbb{R}^{N+1})$ and $\Psi(\cdot,t) \in \mathcal{E}^j_b(\mathbb{R}^{N+1})$ for every $t \in (0,1)$. We have

$$([\![0,1]\!] \times \overline{T}_R)(\eta(x) \wedge \Psi(\widetilde{y},t)) = 0\,,$$

as $\eta \wedge \Psi(\widetilde{y},t)$ does not contain the differential dt, whereas

$$([\![0,1]\!] \times \overline{T}_R)(\eta(x) \wedge \Phi(\widetilde{y},t) \wedge dt) = \overline{T}_R(\eta(x) \wedge \widetilde{\Phi}(\widetilde{y}))$$

for some $(j-1)$-form $\widetilde{\Phi} \in \mathcal{E}^{j-1}(\mathbb{R}^N)$. Since $j \le \mathbf{q}-1$, the verticality property (7.4) gives $\overline{T}_R(\eta(x) \wedge \widetilde{\Phi}(\widetilde{y})) = 0$. The case $j = 0$ being trivial, Lemma 7.2 follows by linearity and density. □

8 Distributional determinant and minors

In this final section we discuss some new results concerning the *distributional determinant* and the *distributional minors*, extending some properties proved in [15].

THE DISTRIBUTIONAL DETERMINANT. Let $N = n \ge 2$ and $u : \mathbb{R}^n \to \widehat{\mathbb{R}}^n$ satisfy the following properties:

(i) $u \in L^\infty_{\rm loc} \cap W^{1,n-1}_{\rm loc}$ or $u \in L^q_{\rm loc} \cap W^{1,p}_{\rm loc}$ for some exponents q and p such that

$$n-1 < p < n \qquad \text{and} \qquad \frac{1}{q} + \frac{n-1}{p} \le 1\,;$$

(ii) $\det \nabla u \in L^1_{\rm loc}$;
(iii) u is smooth outside some compact set $K \subset \mathbb{R}^n$;
(iv) the boundary current ∂G_u has finite mass.

If (i) holds, the distributional determinant is well defined by

$$\operatorname{Det} \nabla u := \frac{1}{n} \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(u^j \, (\operatorname{adj} \nabla u)^j_i \right), \tag{8.1}$$

where $\operatorname{adj} \nabla u$ is the matrix of the adjoints of ∇u, and it is a signed Radon measure. One has $\operatorname{Det} \nabla u = \det \nabla u \, \mathcal{L}^n$ if u is locally Lipschitz and hence, by a standard density argument, if $u \in W^{1,n}_{\rm loc}$. More generally, as we have seen in the introduction, if (ii) holds the graph current G_u is well-defined by (1.4) and (1.5). Therefore, the distributional determinant can be described by means of the action of G_u.

In fact, following [14] we compute that

$$\langle \operatorname{Det} \nabla u, \varphi \rangle = (-1)^n \int_{\mathbb{R}^n} u^{\#}\omega_n \wedge d\varphi \qquad \forall\, \varphi \in C^\infty_c(\mathbb{R}^n) \tag{8.2}$$

where $\omega_n := \dfrac{1}{n}\sum_{j=1}^{n}(-1)^{j-1}y_j\,\widehat{dy^j} \in \mathcal{E}^{n-1}(\widehat{\mathbb{R}}^n)$. Therefore, by (1.5) it turns out that

$$\langle \mathrm{Det}\,\nabla u, \varphi\rangle = (-1)^n\, G_u(\omega_n \wedge d\varphi)\,.$$

Since moreover $d(\omega_n\wedge\varphi) = d\omega_n\wedge\varphi + (-1)^{n-1}\omega_n\wedge d\varphi$, if $u \in L^\infty_{\mathrm{loc}}\cap W^{1,n-1}_{\mathrm{loc}}$

$$\langle \mathrm{Det}\,\nabla u, \varphi\rangle = G_u(d\omega_n\wedge\varphi) - \partial G_u(\omega_n\wedge\varphi)\,. \tag{8.3}$$

Using that $d\omega_n = dy^1\wedge\cdots\wedge dy^n$, by (1.5) we also have

$$G_u(d\omega_n\wedge\varphi) = G_u(\varphi\,dy^1\wedge\cdots\wedge dy^n) = \int_{\mathbb{R}^n}\varphi(x)\,\det\nabla u(x)\,dx\,.$$

Assume now in addition that (iii) and (iv) hold. Then, by the boundary rectifiability theorem 2.4 it turns out that ∂G_u is i.m. rectifiable in $\mathcal{R}_{n-1}(\mathbb{R}^n\times\widehat{\mathbb{R}}^n)$, whence the second addendum in the right-hand side of (8.3) agrees with the *singular part* $(\mathrm{Det}\,\nabla u)^s$ with respect to the Lebesgue measure $\mathcal{L}^n$. We thus deduce for any bounded Borel function φ with compact support

$$\langle(\mathrm{Det}\,\nabla u)^a,\varphi\rangle = \langle \det\nabla u\,\mathcal{L}^n,\varphi\rangle\,,\qquad \langle(\mathrm{Det}\,\nabla u)^s,\varphi\rangle = -\partial G_u(\omega_n\wedge\varphi) \tag{8.4}$$

i.e. the decomposition into absolute continuous and singular parts.

Recall from the introduction that by Proposition 1.4, the boundary current $T = \partial G_u$ satisfies the assumptions of the structure theorem 1.1, with $k = n-1$ and $N = n$. Therefore, (1.6) holds true for some countable subset S_0 of K.

As a consequence, we now prove the following:

Theorem 8.1 *Let $u:\mathbb{R}^n\to\widehat{\mathbb{R}}^n$ satisfy the properties* (i)–(iv). *Then the singular part $(\mathrm{Det}\,\nabla u)^s$ w.r.t. the Lebesgue measure $\mathcal{L}^n$ has finite total variation and is concentrated on the at most countable subset S_0 of K.*

Proof For $R>0$, choose a cut-off function $\chi_R\in C^\infty_c([0,+\infty))$ as in the proof of Lemma 2.6. By (8.2) and (1.5), for every $\varphi\in C^\infty_c(\mathbb{R}^n)$ we have

$$\langle \mathrm{Det}\,\nabla u,\varphi\rangle = (-1)^n\,G_u(\chi_R(|y|)\,\omega_n\wedge d\varphi) + (-1)^n\,G_u((1-\chi_R(|y|))\,\omega_n\wedge d\varphi)\,. \tag{8.5}$$

The form $\chi_R(|y|)\,\omega_n\wedge\varphi$ has bounded Lipschitz coefficients, compact support, and

$$d(\chi_R(|y|)\,\omega_n\wedge\varphi) = d(\chi_R(|y|)\,\omega_n)\wedge\varphi + (-1)^{n-1}\chi_R(|y|)\,\omega_n\wedge d\varphi\,.$$

By (ii)–(iv), we can write

$$(-1)^n\,G_u(\chi_R(|y|)\,\omega_n\wedge d\varphi) = G_u(d(\chi_R(|y|)\,\omega_n)\wedge\varphi) - \partial G_u(\chi_R(|y|)\,\omega_n\wedge\varphi)\,, \tag{8.6}$$

compare (8.3) for the case $u\in L^\infty_{\mathrm{loc}}\cap W^{1,n-1}_{\mathrm{loc}}$. Now, we have

$$d(\chi_R(|y|)\,\omega_n) = \chi_R(|y|)\,d\omega_n + \chi_R'(|y|)\,d|y|\wedge\omega_n$$

and recalling that $d\omega_n = dy^1\wedge\cdots\wedge dy^n$

$$d|y|\wedge\omega_n = \Big(\sum_{j=1}^n\frac{y_j}{|y|}\,dy^j\Big)\wedge\omega_n = \frac{1}{n}\,|y|\,d\omega_n\,.$$

Let $K_\varphi \subset \mathbb{R}^n$ denote the support of $\varphi \in C_c^\infty(\mathbb{R}^n)$, so that $\det \nabla u \in L^1(K_\varphi)$. Since $u^\# d\omega_n \wedge \varphi = \varphi \cdot \det \nabla u \, dx$, by (1.5) we get

$$G_u(d(\chi_R(|y|)\,\omega_n) \wedge \varphi) = \int_{\mathbb{R}^n} \varphi(x) \Big(\chi_R(|u|) + \frac{1}{n}\chi'_R(|u|)\,|u|\Big) \det \nabla u(x)\, dx\,.$$

We claim that there exists an increasing sequence $\{R_j\}$ of integer radii $R_j \nearrow \infty$ such that

$$\lim_{j\to\infty} G_u(d(\chi_{R_j}(|y|)\,\omega_n) \wedge \varphi) = \int_{\mathbb{R}^n} \varphi(x) \det \nabla u(x)\, dx\,. \tag{8.7}$$

In fact, since $u \in L^1_{\mathrm{loc}}$, we have that $\chi_R(|u|) \to 1$ a.e. in K_φ. Whence, by the dominated convergence

$$\lim_{R\to\infty} \int_{\mathbb{R}^n} \varphi(x)\,\chi_R(|u|) \det \nabla u(x)\, dx = \int_{\mathbb{R}^n} \varphi(x) \det \nabla u(x)\, dx\,.$$

Moreover, the restriction of $\chi'_R(|u|)$ to K_φ is uniformly bounded and supported in $K_\varphi^R := \{x \in K_\varphi \mid R \le |u(x)| < R+1\}$. Setting for $R = j \in \mathbb{N}$

$$a_j := \int_{K_\varphi^j} |\det \nabla u(x)|\, dx\,,$$

condition $\det \nabla u \in L^1(K_\varphi)$ yields that $\sum_j a_j < \infty$, whence $\liminf\limits_{j\to\infty} (j+1)\,a_j = 0$. Therefore, the claim (8.7) follows by observing that

$$\left| \int_{\mathbb{R}^n} \varphi(x)\, \frac{1}{n}\chi'_R(|u|)\,|u| \det \nabla u(x)\, dx \right| \le c\,\|\varphi\|_\infty\,(j+1)\,a_j\,.$$

Similarly, we get

$$G_u((1-\chi_R(|y|))\,\omega_n \wedge d\varphi) = \int_{\mathbb{R}^n} ((1-\chi_R(|u|))\,u^\#\omega_n \wedge d\varphi\,.$$

Since u satisfies (i), by dominated convergence, and using that $(1-\chi_R(|u|)) \to 0$ a.e. in K_φ as $R \to \infty$, we deduce

$$\lim_{R\to\infty} G_u((1-\chi_R(|y|))\,\omega_n \wedge d\varphi) = 0\,. \tag{8.8}$$

By (8.5), (8.6), (8.7), and (8.8), we obtain that for every $\varphi \in C_c^\infty(\mathbb{R}^n)$

$$\langle \mathrm{Det}\,\nabla u, \varphi\rangle = \int_{\mathbb{R}^n} \det \nabla u(x)\,\varphi(x)\, dx + \lim_{j\to\infty} \langle \mu_{R_j}, \varphi\rangle\,,$$

where the increasing sequence $R_j \nearrow \infty$ is chosen as in (8.7) and

$$\langle \mu_{R_j}, \varphi\rangle := -\partial G_u(\chi_{R_j}(|y|)\,\omega_n \wedge \varphi)\,.$$

Since by Theorem 1.1 all the measures μ_{R_j} are concentrated on the countable set S_0, the claim follows. □

Distributional minors. More generally, let $n, N \ge 2$ integers and let us fix the order $2 \le m \le \min(n,N)$. We assume that $u : \mathbb{R}^n \to \mathbb{R}^N$ satisfies:

(i') $u \in L^\infty_{\rm loc} \cap W^{1,m-1}_{\rm loc}$ or $u \in L^q_{\rm loc} \cap W^{1,p}_{\rm loc}$ for some exponents q and p such that

$$m-1 < p < m \qquad \text{and} \qquad \frac{1}{q} + \frac{m-1}{p} \le 1\,; \tag{8.9}$$

(ii') all the minors of the Jacobian matrix ∇u are in $L^1_{\rm loc}$;

and again the above properties (iii) and (iv).

Following the notation about multi-indices, for an $(N \times n)$-matrix G, if $|\alpha| = n - m$ and $|\beta| = m$, we denote by $G^\beta_{\overline{\alpha}}$ the square $(m \times m)$-submatrix obtained by selecting the rows and columns by β and $\overline{\alpha}$, respectively, and by $M^\beta_{\overline{\alpha}}(G)$ its determinant. We also define the matrix of adjoints of $G^\beta_{\overline{\alpha}}$ by the formula

$$(\operatorname{adj} G^\beta_{\overline{\alpha}})^j_i := \sigma(j, \beta - j)\, \sigma(i, \overline{\alpha} - i)\, \det G^{\beta - j}_{\overline{\alpha} - i}\,, \qquad j \in \beta\,, \quad i \in \overline{\alpha}\,.$$

If (i') holds, the *distributional minor* of indices $\overline{\alpha}$ and β of ∇u is well-defined by

$$\operatorname{Div}^\beta_{\overline{\alpha}} u := \frac{1}{|\beta|} \sum_{j\in\beta} \sum_{i \in \overline{\alpha}} \frac{\partial}{\partial x_i} \big(u^j\, (\operatorname{adj}(\nabla u)^\beta_{\overline{\alpha}})^j_i \big)\,.$$

Moreover, $\operatorname{Div}^\beta_{\overline{\alpha}} u$ is a signed Radon measure, that agrees with $M^\beta_{\overline{\alpha}}(\nabla u)\, \mathcal{L}^n$ if u is locally Lipschitz or even in $W^{1,m}_{\rm loc}$.

Denote by $\omega^\alpha_\varphi \in \mathcal{D}^{n-m}(\mathbb{R}^n)$ the form $\omega^\alpha_\varphi(x) := (-1)^{|\alpha|} \sigma(\alpha, \overline{\alpha})\, \varphi(x)\, dx^\alpha$, so that $\omega^\alpha_\varphi := \varphi$ if $m = n$, and let $\omega_\beta := \dfrac{1}{|\beta|} \displaystyle\sum_{j \in \beta} \sigma(j, \beta - j)\, y_j\, dy^{\beta - j} \in \mathcal{E}^{m-1}(\mathbb{R}^N)$. Following [14], similarly to (8.2) we get

$$\langle \operatorname{Div}^\beta_{\overline{\alpha}} u, \varphi \rangle = (-1)^m \int_{\mathbb{R}^n} u^\# \omega_\beta \wedge d\omega^\alpha_\varphi \qquad \forall\, \varphi \in C^\infty_c(\mathbb{R}^n)\,.$$

Therefore, if (ii') holds, by (1.5) we similarly obtain

$$\langle \operatorname{Div}^\beta_{\overline{\alpha}} u, \varphi \rangle = (-1)^m\, G_u(\omega_\beta \wedge d\omega^\alpha_\varphi)\,.$$

If $u \in L^\infty_{\rm loc} \cap W^{1,m-1}_{\rm loc}$, we thus can write

$$\langle \operatorname{Div}^\beta_{\overline{\alpha}} u, \varphi \rangle = G_u(dy^\beta \wedge \omega^\alpha_\varphi) - \partial G_u(\omega_\beta \wedge \omega^\alpha_\varphi) \tag{8.10}$$

where by (1.5) we compute

$$G_u(dy^\beta \wedge \omega^\alpha_\varphi) = \int_{\mathbb{R}^n} \varphi(x)\, M^\beta_{\overline{\alpha}}(\nabla u(x))\, dx\,.$$

Again, if (iii) and (iv) hold, ∂G_u is i.m. rectifiable in $\mathcal{R}_{n-1}(\mathbb{R}^n \times \mathbb{R}^N)$, and we thus obtain the decomposition

$$\langle (\operatorname{Div}^\beta_{\overline{\alpha}} u)^a, \varphi \rangle = \langle M^\beta_{\overline{\alpha}}(\nabla u)\, \mathcal{L}^n, \varphi \rangle\,, \qquad \langle (\operatorname{Div}^\beta_{\overline{\alpha}} u)^s, \varphi \rangle = -\partial G_u(\omega_\beta \wedge \omega^\alpha_\varphi)$$

into absolute continuous and singular parts.

Furthermore, by Proposition 1.4, this time $T = \partial G_u$ satisfies the assumptions of the structure theorem 1.2, with $k = n-1$ and $\mathbf{q} = m - 1$, whence (1.6) holds, with S_{n-m} a countably $\mathcal{H}^{n-m}$-rectifiable subset of K. Therefore, arguing as in the proof of Theorem 8.1, and with the obvious modifications, we similarly obtain:

Theorem 8.2 *Let* $n, N \geq 2$ *and* $2 \leq m \leq \min(n, N)$. *Let* $u : \mathbb{R}^n \to \mathbb{R}^N$ *satisfy the properties* (i′), (ii′), (iii), *and* (iv), *and let* $|\alpha| = n - m$ *and* $|\beta| = m$. *Then the singular part* $(\mathrm{Div}^{\beta}_{\overline{\alpha}} u)^s$ *w.r.t. the Lebesgue measure* $\mathcal{L}^n$ *has finite total variation and is concentrated on a countably* $\mathcal{H}^{n-m}$*-rectifiable subset* S_{n-m} *of* K.

Remark 8.3 In the case $m = 1$, if $\beta = j$ and $\overline{\alpha} = i$, we have $(\mathrm{adj}\,\nabla u)^{\beta}_{\overline{\alpha}} = 1$ and $\mathrm{Div}^{\beta}_{\overline{\alpha}} u = D_i u^j$. Therefore, Theorem 8.2 describes in some sense the higher order counterpart of some features concerning the class SBV_0 studied in Thm. 3.1 and Thm. 3.4 from [3].

Acknowledgements I thank M. Giaquinta for some useful discussions, and L. Biliotti for giving me suggestions concerning the projection argument from Sec. 4. I also thank the referee of a first version of the paper who helped me to deeply improve the results.

References

1. Alberti G.: Distributional Jacobian and singularities of Sobolev maps. Ricerche Mat. **54** (2005), no. 2, 375-394 (2006).
2. Almgren F.J.: Deformations and multiple-valued functions. In: "Geometric measure theory and the calculus of variations", Proc. Symposia Pure Math. Amer. Math. Soc., R.I., **44** (1986), 29–130.
3. Ambrosio L., Braides A., Garroni A.: Special functions with bounded variation and with weakly differentiable traces on the jump set. NoDEA Nonlinear Differential Equations Appl. **5** (1998), 219–243.
4. Ambrosio L., Fusco N., Pallara D.: Functions of bounded variation and free discontinuity problems. Oxford Math. Monographs, Oxford 2000.
5. Ball J.M.: Convexity conditions and existence theorems in nonlinear elasticity. Arch. Rational Mech. Anal. **63** (1977), 337–403.
6. De Lellis C., Ghiraldin F.: An extension of the identity Det = det. C. R. Acad. Sci. Paris, Ser. 1 **348** (2010), 973–976.
7. Federer H.: Geometric measure theory. Grundlehren math. wissen. 153, Springer, New York (1969).
8. Federer H., Fleming W.: Normal and integral currents. Ann. of Math. **72** (1960), 458–520.
9. Giaquinta M., Modica G., Souček J.: Cartesian currents in the calculus of variations I, *Cartesian currents*. Ergebnisse Math. Grenzgebiete (III Ser), 37, Springer, Berlin (1998).
10. Giaquinta M., Mucci D.: Maps into manifolds and currents: area and $W^{1,2}$-, $W^{1/2}$-, BV-energies. Edizioni della Normale, C.R.M. Series, Sc. Norm. Sup. Pisa (2006).
11. Henao D., Mora-Corral C.: Lusin's condition and the distributional determinant for deformations with finite energy. Adv. Calc. Var. **5** 4 (2012), 355–409.
12. Jerrard R.L., Soner H.M.: Functions of higher bounded variation. Indiana Univ. Math. J. **51** (2002), 645–677.
13. Morgan F.: Geometric measure theory. A beginner's guide. Academic Press, Inc. Boston, MA (1988).
14. Mucci D.: Fractures and vector valued maps. Calc. Var. Partial Differential Equations **22** (2005), 391–420. Erratum to: Fractures and vector valued maps. Ibid. **37** (2010), 547–548.
15. Mucci D.: A variational problem involving the distributional determinant. Riv. Mat. Univ. Parma New Series **1** no. 2 (2010), 321–345.
16. Müller S.: Det = det. A remark on the distributional determinant. C. R. Acad. Paris Sér. I Math. **311** 1 (1990), 13-17.
17. Müller S.: On the singular support of the distributional determinant. Ann. Inst. H. Poincaré Anal. Non Linéaire **10** (1993), 657-696.
18. Simon L.: Lectures on geometric measure theory. Proc. of the Centre for Math. Analysis, Vol. 3, Australian National University, Canberra (1983).