ON THE REGULARITY OF CRITICAL AND MINIMAL SETS OF A FREE INTERFACE PROBLEM

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Abstract. We study a free interface problem of finding the optimal energy configuration for mixtures of two conducting materials with an additional perimeter penalization of the interface. We employ the regularity theory of linear elliptic equations to give a short proof of a partial regularity proved by Fan Hua Lin in [8], and to study the possible shapes of conical sets which are critical points.

1. Introduction

In this paper we consider the functional

$$\mathcal{F}(E,v) = \varepsilon P(E,\Omega) + \int_\Omega \sigma_E(x)|Dv|^2\,dx, \tag{1.1}$$

where $\varepsilon > 0$, $\Omega \subset \mathbb{R}^n$ is an open set, $v \in W^{1,2}(\Omega)$ and $P(E,\Omega)$ stands for the perimeter of E in Ω. Moreover, $\sigma_E(x) = \beta\chi_E(x) + \alpha\chi_{\Omega\setminus E}(x)$, where $0 < \alpha < \beta < \infty$ are given constants.

Given a function $u_0 \in W^{1,2}(\Omega)$ and a measurable $E \subset \Omega$, we denote by u_E, or simply by u if no confusion arises, the corresponding elastic equilibrium, i.e., the minimizer in $W^{1,2}(\Omega)$ of the functional

$$\int_\Omega \sigma_E(x)|Dv|^2\,dx$$

under the boundary condition $v = u_0$ on $\partial\Omega$. The function u solves the linear equation

$$\int_\Omega \langle \sigma_E Du, D\varphi\rangle\,dx = 0 \qquad \text{for every } \varphi \in W_0^{1,2}(\Omega). \tag{1.2}$$

If we denote by u_β and u_α the restriction of u on E and $\Omega \setminus E$, respectively, they are harmonic in their domains. Moreover, equation (1.2) implies the transition condition

$$\alpha\partial_\nu u_\alpha(x) = \beta\partial_\nu u_\beta(x) \qquad \text{for all } x \in \partial E \cap \Omega, \tag{1.3}$$

where ∂_ν denotes the differential of u in the direction of the exterior normal to ∂E.

Note that if (E,u) is a smooth critical point of the functional (1.1) the following Euler-Lagrange equation holds

$$\varepsilon H_{\partial E} + \beta|Du_\beta|^2 - \alpha|Du_\alpha|^2 = \lambda \quad \text{on } \partial E \cap \Omega, \tag{1.4}$$

where $H_{\partial E}$ stands for the mean curvature of ∂E and λ is either zero or a Langrange multiplier (in case of a volume constraint).

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In the physical literature critical points of the functional (1.1), i.e, solutions of equations (1.2) and (1.4) are studied to model the shape of liquid drops exposed to an electric or a magnetic field. In the model the set E represents a liquid drop with dielectric permittivity β, surrounded by a fluid with smaller permittivity α, and u stands for the electrostatic potential induced by an applied electric field. At the interface, assumed to be in static equilibrium, the normal component of the electric displacement field $\sigma_E Du$ has to be continuous. This implies that u has to satisfy (1.3) or equivalently that it has to be a solution of equation (1.2). On the other hand, the interface has to balance the electric stress and the surface tension: this leads to (1.4).

The occurrence of conical tips at an interface exposed to an electric field has been observed by several authors (see e.g. [15]). Moreover, theoretical investigations ([7], [11], [12], [14]) suggest that conical critical points, the so called *Taylor cones*, may only occur if the ratio β/α is sufficiently large and if the angle at the vertex belongs to a certain range, which is independent of the penalization factor ε.

In the mathematical literature people considered both the problem (P) of minimizing (1.1) under the boundary condition $u = u_0$ on $\partial\Omega$ and the constrained problem

$$(P_c) \qquad \min\{\mathcal{F}(E, v) \,:\, v = u_0 \text{ on } \partial\Omega,\, |E| = d\}$$

for some given $d < |\Omega|$. The partial regularity of minimizers of the unconstrained problem (P) was proved by Fan-Hua Lin in [8] (see also [1], [9]). In the special case $n = 2$ the result of Lin has been improved by Larsen in [5], [6]. However, the full regularity of the free interface ∂E in two dimensions still remains open.

In this paper give a short proof of the following regularity theorem.

Theorem 1.1. *There exists $\sigma > 0$, depending only on n and the ratio β/α, such that, if (E, u) is a minimizer of either problem (P) or problem (P_c), then*

- (a) *the reduced boundary $\partial^* E \cap \Omega$ is a $C^{1,\sigma}$ hypersurface,*
- (b) *$(\partial E \setminus \partial^* E) \cap \Omega$ is relatively closed in $\partial E \cap \Omega$ and $\mathcal{H}^{n-1}((\partial E \setminus \partial^* E) \cap \Omega) = 0$.*

The above statement slightly generalizes the regularity result proved in [8], where only the unconstrained problem (P) was considered. However, in our opinion the interesting feature of this result is that the degree of regularity of the free interface is independent of the penalization factor ε. The proof of Theorem 1.1, which can be found in Section 3, also gives a characterization of the singular set with no dependence on ε. Note also that, once the $C^{1,\sigma}$ regularity of $\partial^* E \cap \Omega$ is obtained, then using [8, Lemma 5.3] and a standard bootstrap argument one obtains that $\partial^* E \cap \Omega$ is in fact C^∞.

Our proof is based on a decay estimate for the gradient of a minimizer of the Dirichlet energy (see Proposition 2.4). Roughly speaking, we prove that if u is a solution of (1.2) and x_0 is a point in Ω, where either the density of E is close to 0 or 1, or the set E is asymptotically close to a hyperplane, then for sufficiently small ρ we have

$$\int_{B_\rho(x_0)} |Du|^2 \, dx \leq C\rho^{n-1+2\sigma} \tag{1.5}$$

for some $\sigma > 0$ only depending on n and β/α. With this estimate in hands Theorem 1.1 then follows as a consequence of well known properties of *perimeter almost minimizers* (see [13] and [10]).

The decay estimate (1.5) is also the main ingredient of the proof of our next result. In order to state it, for a given $\theta \in (0, \pi/2)$, we denote by E_θ the spherical cone with angle θ which has vertex at the origin, i.e.,

$$E_\theta = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,:\, x_n > \frac{1}{\tan\theta}|x'|\}.$$

Theorem 1.2. *Let $n \geq 3$. There exist two positive constants $\delta_0 = \delta_0(n, \beta/\alpha) > 0$ and $\gamma_0 = \gamma_0(n) > 1$ such that, if E_θ is a spherical cone satisfying (1.4) then $\beta/\alpha \geq \gamma_0$ and*

$$\delta_0 \leq \theta \leq \pi/2 - \delta_0.$$

As far as we know, this result is the first rigorous proof of the fact that Taylor cones may occur only for certain angles, and provided that the ratio β/α is sufficiently large. We remark that we are able to give explicit estimates of the constants δ_0 and γ_0. In particular, δ_0 and γ_0 are independent of the penalization factor ε, which is in accordance with the observations and theoretical results reported in the physical literature.

2. Regularity of elastic minima

In this section we study the regularity of the elastic minimum associated to a set E, i.e., solution of (1.2). In the main result of the section, Proposition 2.4, we prove that, if the density of E is close to 0 or 1 or the set E is asymptotically close to a hyperplane, then the elastic energy $\int_{B_\rho(x_0)} |Du|^2\,dx$ decays faster that $\rho^{n-1+2\sigma}$. We prove Proposition 2.4 with a direct argument and therefore we are able to provide explicit bounds for the relevant constants.

We begin by deriving the Caccioppoli's inequality for solutions of (1.2). Even though the argument is standard, we give the proof in order to keep track on the constants. We denote the cube, centred at x_0 and with side lenght $2R$, by $Q_R(x_0)$. In the case $x_0 = 0$ we simply write Q_R. We recall the Sobolev-Poincaré inequality, i.e., for every function $u \in W^{1,p}(Q_R)$, $1 \leq p < n$, it holds

$$||u - u_R||_{L^{p^*}(Q_R)} \leq c(n,p)||Du||_{L^p(Q_R)} \tag{2.1}$$

where $u_R = \fint_{Q_R} u\,dx$ and $p^* = \frac{pn}{n-p}$. In case of $p = \frac{2n}{n+2}$ we denote the constant $c(n,p)$ by $C_{S,n}$.

Lemma 2.1. *Let $u \in W^{1,2}(\Omega)$ be a solution of (1.2). Then for every cube $Q_{2R}(x_0) \subset\subset \Omega$ it holds*

$$\fint_{Q_R(x_0)} |Du|^2\,dx \leq C\left(\fint_{Q_{2R}(x_0)} |Du|^{2m}\,dx\right)^{\frac{1}{m}}, \tag{2.2}$$

where $m = \frac{n}{n+2}$, $C = C_{S,n}^2\,2^{n+8}\frac{\beta}{\alpha}$, and $C_{S,n}$ is the constant in the Sobolev-Poincaré inequality (2.1) with $p = \frac{2n}{n+2}$.

Proof. Without loss of generality we may assume that $x_0 = 0$. Let $\zeta \in C_0^\infty(Q_{2R})$ be a cut-off function such that $\zeta \equiv 1$ in Q_R and $|D\zeta| \leq 2/R$. We choose a test function $\varphi = (u - u_{2R})\zeta^2$ in (1.2), where $u_{2R} = \fint_{Q_{2R}} u\,dx$ and apply Young's inequality to obtain

$$\int_{Q_R} |Du|^2\,dx \leq \int_{Q_{2R}} |Du|^2\zeta^2\,dx \leq \frac{4\beta}{\alpha}\int_{Q_{2R}} |u-u_{2R}|^2|D\zeta|^2\,dx \leq \frac{16\beta}{R^2\alpha}\int_{Q_{2R}} |u-u_{2R}|^2\,dx.$$

We use the Sobolev-Poincaré inequality (2.1) to deduce

$$\int_{Q_{2R}} |u-u_{2R}|^2\,dx \leq C_{S,n}^2 \left(\int_{Q_{2R}} |Du|^{2m}\,dx\right)^{\frac{1}{m}}.$$

The result then follows from the two inequalities above. ☐

We apply Gehring's Lemma to obtain higher integrability for the gradient of u.

Lemma 2.2. *Let $u \in W^{1,2}(\Omega)$ solve (1.2). There exists $p > 1$ such that for any ball $B_{2R}(x_0) \subset\subset \Omega$ it holds*

$$\fint_{B_R(x_0)} |Du|^{2p}\,dx \leq C\left(\fint_{B_{2R}(x_0)} |Du|^2\,dx\right)^p.$$

The constants can be estimated explicitly as

$$p = \frac{2C_1 - m}{2C_1 - 1} \quad \text{for } C_1 = C_{S,n}^2\, 2^{10}\cdot 80^n \frac{\beta}{\alpha} \quad \text{and } C = 2^{2n+1}5^{np}n^{np/2}\omega_n^{p-1},$$

where ω_n is the volume of the unit ball and $m = \frac{n}{n+2}$.

The above result is well known but it is usually stated without estimates of the constants. At the end of Section 3 we will go through the proof of Lemma 2.2 from [4] and evaluate every constant explicitly.

In the next lemma we prove a monotonicity formula for the elastic minimum in the case when E is a half-space.

Lemma 2.3. *Let $E = \{x \in \mathbb{R}^n \mid \langle x - \bar{x}, e\rangle < 0\} \cap \Omega$ for some unit vector e and a point $\bar{x}$, and suppose u is a solution of (1.2). Let $x_0 \in \partial E \cap \Omega$ and $R > 0$ be such that $B_R(x_0) \subset\subset \Omega$. Then*

$$\rho \mapsto \fint_{B_\rho(x_0)} \sigma_E(x)|Du|^2\,dx$$

is increasing in $(0, R)$.

Proof. Without loss of generality we may assume that $E = \{x \in \mathbb{R}^n \mid x_n < 0\} \cap \Omega$ and $x_0 = 0$. Let us fix a radius R such that $B_R \subset \Omega$. From standard elliptic regularity theory we know that u is smooth in the upper and in the lower part of the ball B_R with respect to the hyperplane $\partial E = \{x_n = 0\}$. To be more precise, let us denote $\bar{B}_R^+ = \overline{B_R \setminus E}$ and $\bar{B}_R^- = \overline{B_R \cap E}$. Then $u_\alpha \in C^\infty(\bar{B}_R^+)$ and $u_\beta \in C^\infty(\bar{B}_R^-)$ and they are harmonic in the interior of $\bar{B}_R^+$ and $\bar{B}_R^-$, where u_α and u_β are the restrictions of u on $\Omega \setminus E$ and E.

The goal is to show that the function $\varphi : (0, R) \to \mathbb{R}$

$$\varphi(\rho) := \fint_{\partial B_\rho} \sigma_E(x)|Du(x)|^2\,d\mathcal{H}^{n-1}(x) = \fint_{\partial B_1} \sigma_E(\rho y)|Du(\rho y)|^2\,d\mathcal{H}^{n-1}(y)$$

is increasing. Notice that $\sigma_E(\rho y) = \sigma_E(y)$ since E is a half-space. Denote $v = |Du|^2$, $v_\alpha = |Du_\alpha|^2$ and $v_\beta = |Du_\beta|^2$. Since v_α and v_β are sub-harmonic in the interior of B_ρ^+ and B_ρ^- we deduce by the divergence theorem that

$$\begin{aligned}\varphi'(\rho) &= \fint_{\partial B_\rho} \sigma_E(x)\langle Dv(x), \frac{x}{\rho}\rangle \, d\mathcal{H}^{n-1}(x) \\ &= \frac{1}{\mathcal{H}^{n-1}(\partial B_\rho)}\left(\alpha \int_{\partial B_\rho^+} \partial_\nu v_\alpha \, d\mathcal{H}^{n-1} + \beta \int_{\partial B_\rho^-} \partial_\nu v_\beta \, d\mathcal{H}^{n-1} + \int_{\partial E \cap B_\rho} \alpha \, \partial_{x_n} v_\alpha - \beta \, \partial_{x_n} v_\beta \, d\mathcal{H}^{n-1}\right) \\ &= \frac{1}{\mathcal{H}^{n-1}(\partial B_\rho)}\left(\alpha \int_{B_\rho^+} \Delta v_\alpha \, d\mathcal{H}^{n-1} + \beta \int_{B_\rho^-} \Delta v_\beta \, d\mathcal{H}^{n-1} + \int_{\partial E \cap B_\rho} \alpha \, \partial_{x_n} v_\alpha - \beta \, \partial_{x_n} v_\beta \, d\mathcal{H}^{n-1}\right) \\ &\geq \frac{1}{\mathcal{H}^{n-1}(\partial B_\rho)} \int_{\partial E \cap B_\rho} \alpha \, \partial_{x_n} v_\alpha - \beta \, \partial_{x_n} v_\beta \, d\mathcal{H}^{n-1}.\end{aligned}$$

We will show that $\beta \partial_{x_n} v_\beta = \alpha \, \partial_{x_n} v_\alpha$ on ∂E, from which the claim follows.

Since $u_\alpha = u_\beta$ on ∂E we have

$$\partial_{x_i} u_\alpha = \partial_{x_i} u_\beta \quad \text{and} \quad \partial_{x_i x_i} u_\alpha = \partial_{x_i x_i} u_\beta \quad \text{for } i = 1, 2, \ldots, n-1. \tag{2.3}$$

The transition condition (1.3) reads as

$$\alpha \, \partial_{x_n} u_\alpha = \beta \, \partial_{x_n} u_\beta \quad \text{on } \partial E. \tag{2.4}$$

Differentiating (2.4) with respect to x_i, for $i = 1, 2, \ldots, n-1$, yields

$$\alpha \, \partial_{x_i x_n} u_\alpha = \beta \, \partial_{x_i x_n} u_\beta \quad \text{on } \partial E.$$

On the other hand, since u_α and u_β are harmonic, we have by (2.3) that

$$\partial_{x_n x_n} u_\alpha = -\sum_{i=1}^{n-1} \partial_{x_i x_i} u_\alpha = -\sum_{i=1}^{n-1} \partial_{x_i x_i} u_\beta = \partial_{x_n x_n} u_\beta \quad \text{on } \partial E.$$

Therefore on ∂E it holds

$$\alpha \, \partial_{x_n} v_\alpha = 2\sum_{i=1}^{n} \alpha \, \partial_{x_i} u_\alpha \, \partial_{x_i x_n} u_\alpha = 2\sum_{i=1}^{n} \beta \, \partial_{x_i} u_\beta \, \partial_{x_i x_n} u_\beta = \beta \, \partial_{x_n} v_\beta$$

which implies $\varphi'(\rho) \geq 0$.

□

The main result of this section is the following decay estimate for elastic minimum.

Proposition 2.4. *Let $u \in W^{1,2}(\Omega)$ be a solution of (1.2). Let $x_0 \in \Omega$ and $R > 0$ be such that $B_{2R}(x_0) \subset\subset \Omega$. There exist $\delta, c > 0$, $\sigma \in (0, 1/2]$, depending only on the dimension n and the ratio β/α, such that if one of the following conditions hold*

(i) $\frac{|E \cap B_R(x_0)|}{|B_R|} < \delta$,
(ii) $\frac{|B_R(x_0) \setminus E|}{|B_R|} < \delta$,
(iii) *there exists a half-space H such that* $\frac{|(E \Delta H) \cap B_R(x_0)|}{|B_R|} < \delta$,

then for every $\rho < 2R$

$$\int_{B_\rho(x_0)} |Du|^2\,dx \le c\left(\frac{\rho}{2R}\right)^{n-1+2\sigma} \int_{B_{2R}(x_0)} |Du|^2\,dx.$$

Proof. We first treat the cases (i) and (ii). We fix a ball $B_{2R}(x_0) \subset\subset \Omega$ and assume without loss of generality that $x_0 = 0$. Choose v to be the harmonic function in B_R with the boundary condition $v = u$ on ∂B_R. We choose a test function $\varphi = v - u \in W_0^{1,2}(B_R)$ in the equations

$$\int_{B_R} Dv \cdot D\varphi\,dx = 0$$

and

$$\alpha \int_{B_R \setminus E} Du \cdot D\varphi\,dx + \beta \int_{B_R \cap E} Du \cdot D\varphi\,dx = 0. \tag{2.5}$$

We write the latter equation as

$$\int_{B_R} Du \cdot (Dv - Du)\,dx = -\frac{\beta - \alpha}{\alpha} \int_{B_R \cap E} Du \cdot (Dv - Du)\,dx.$$

We substract to this $\int_{B_R} Dv \cdot (Dv - Du)\,dx = 0$ and use Hölder's inequality to deduce

$$\int_{B_R} |Dv - Du|^2\,dx \le \frac{(\beta-\alpha)^2}{\alpha^2} \int_{B_R \cap E} |Du|^2 dx.$$

By the higher integrability stated in Lemma 2.2 we have

$$\begin{aligned} \int_{B_\rho} |Dv - Du|^2\,dx &\le \frac{(\beta-\alpha)^2}{\alpha^2} \left(\frac{|E \cap B_R|}{|B_R|}\right)^{1-1/p} \left(\fint_{B_R \cap E} |Du|^{2p}\right)^{1/p} |B_R| \\ &\le \frac{(\beta-\alpha)^2}{\alpha^2} \left(\frac{|E \cap B_R|}{|B_R|}\right)^{1-1/p} \frac{C^{1/p}}{2^n} \int_{B_{2R}} |Du|^2\,dx \end{aligned} \tag{2.6}$$

for every $\rho \le R$, where C and $p > 1$ are from Lemma 2.2. Similarly we deduce

$$\int_{B_\rho} |Dv - Du|^2\,dx \le \frac{(\beta-\alpha)^2}{\beta^2} \left(\frac{|E \setminus B_R|}{|B_R|}\right)^{1-1/p} \frac{C^{1/p}}{2^n} \int_{B_{2R}} |Du|^2\,dx. \tag{2.7}$$

On the other hand, since v is harmonic, we have

$$\int_{B_\rho} |Dv|^2\,dx \le \left(\frac{\rho}{R}\right)^n \int_{B_R} |Dv|^2\,dx \le \left(\frac{\rho}{R}\right)^n \int_{B_R} |Du|^2\,dx.$$

Hence, for every $\rho \le R$ we may estimate

$$\begin{aligned} \int_{B_\rho} |Du|^2\,dx &\le 2\int_{B_\rho} |Dv - Du|^2\,dx + 2\int_{B_\rho} |Dv|^2\,dx \\ &\le 2\int_{B_\rho} |Dv - Du|^2\,dx + 2^{n+1}\left(\frac{\rho}{2R}\right)^n \int_{B_{2R}} |Du|^2\,dx. \end{aligned} \tag{2.8}$$

Find the largest $\chi > 0$ such that the equation

$$2\chi + 2^{n+1}\theta^n = \theta^{n-1}$$

holds for some $\theta > 0$. In other words $\chi = \frac{1}{2}\theta^{n-1} - 2^n\theta^n$. We may easily solve θ

$$\theta = \frac{n-1}{2^{n+1}n}. \tag{2.9}$$

In particular, both χ and θ are less than one. Then if

(2.10)
$$\frac{(\beta-\alpha)^2}{\alpha^2}\left(\frac{|E\cap B_R|}{|B_R|}\right)^{1-1/p}\frac{C^{1/p}}{2^n}<\chi \qquad \text{or} \qquad C^{1/p}\frac{(\beta-\alpha)^2}{\beta^2}\left(\frac{|E\setminus B_R|}{|B_R|}\right)^{1-1/p}\frac{C^{1/p}}{2^n}<\chi$$

we get from (2.6), (2.7) and (2.8) that

$$\int_{B_{2\theta R}}|Du|^2\,dx\le\theta^{n-1+2\sigma}\int_{B_{2R}}|Du|^2\,dx,$$

for some $\sigma>0$. A standard iteration argument then yields

$$\int_{B_\rho}|Du|^2\,dx\le c\left(\frac{\rho}{2R}\right)^{n-1+2\sigma}\int_{B_{2R}}|Du|^2\,dx$$

for every $\rho<R$. Thus we have proven the claim in the cases (i) and (ii).

We are left with the case (iii). Let H be the half-space from the assumption. We choose v which minimizes the energy $\int_{B_R}\sigma_H(x)|Dv|^2\,dx$ with the boundary condition $v=u$ on ∂B_R. Hence

$$\beta\int_{B_R\cap H}Dv\cdot D\varphi\,dx+\alpha\int_{B_R\setminus H}Dv\cdot D\varphi\,dx=0 \tag{2.11}$$

for every $\varphi\in W_0^{1,2}(B_R)$.

Lemma 2.3 yields

$$\int_{B_\rho}|Dv|^2\,dx\le\frac{\beta}{\alpha}\left(\frac{\rho}{R}\right)^n\int_{B_R}|Dv|^2\,dx.$$

Moreover, from the minimality of v it follows

$$\int_{B_R}|Dv|^2\,dx\le\frac{\beta}{\alpha}\int_{B_R}|Du|^2\,dx.$$

Hence, we have

$$\int_{B_\rho}|Du|^2\,dx\le 2\int_{B_\rho}|Dv-Du|^2\,dx+2^{n+1}\left(\frac{\beta}{\alpha}\right)^2\left(\frac{\rho}{2R}\right)^n\int_{B_{2R}}|Du|^2\,dx, \tag{2.12}$$

for every $\rho\le R$.

Let us now rewrite the equation (2.5) satisfied by u as

$$\begin{aligned}\beta\int_{B_R\cap H}&Du\cdot D\varphi\,dx+\alpha\int_{B_R\setminus H}Du\cdot D\varphi\,dx\\&=(\beta-\alpha)\int_{B_R\cap(H\setminus E)}Du\cdot D\varphi\,dx-(\beta-\alpha)\int_{B_R\cap(E\setminus H)}Du\cdot D\varphi\,dx\,.\end{aligned}$$

Then, subtracting (2.11) from this equation and choosing $\varphi=u-v$ we get at once

$$\int_{B_\rho}|Du-Dv|^2\,dx\le\left(\frac{\beta}{\alpha}-1\right)^2\int_{B_R\cap(E\Delta H)}|Du|^2\,dx$$

and from Lemma 2.2 we deduce

$$\int_{B_\rho}|Du-Dv|^2\,dx\le\left(\frac{\beta}{\alpha}\right)^2\left(\frac{|(E\Delta H)\cap B_R|}{|B_R|}\right)^{1-1/p}\frac{C^{1/p}}{2^n}\int_{B_{2R}}|Du|^2\,dx. \tag{2.13}$$

This time we choose $\chi > 0$ as the largest number such that the equation

$$2\chi + 2^{n+1}\left(\frac{\beta}{\alpha}\right)^2 \theta^n = \theta^{n-1}$$

has a solution $\theta > 0$. We may again solve θ

$$\theta = \frac{n-1}{2^{n+1}n}\left(\frac{\alpha}{\beta}\right)^2. \tag{2.14}$$

Arguing as before, the claim follows from (2.12) and (2.13) if

$$\left(\frac{\beta}{\alpha}\right)^2 \left(\frac{|(E\Delta H)\cap B_R|}{|B_R|}\right)^{1-1/p} \frac{C^{1/p}}{2^n} < \chi. \tag{2.15}$$

□

Remark 2.5. We may estimate the number δ by

$$\delta^{1-\frac{1}{p}} = 2^n \left(\frac{\alpha}{\beta}\right)^2 C^{-\frac{1}{p}}\chi,$$

where the constants C and p are from Lemma 2.2 and χ is given by $\chi = \frac{1}{2}\theta^{n-1} - 2^n\left(\frac{\beta}{\alpha}\right)^2\theta^n$ where θ is given by (2.14). This estimate easily follows by comparing the constants in (2.9), (2.10), (2.14) and (2.15).

We conclude the section by recalling the following result (see the proof of [3, Theorem 3.1])

Proposition 2.6. *Let $u \in W^{1,2}(\Omega)$ be a solution of (1.2). Let $x_0 \in \Omega$ and $R > 0$ such that $B_{2R}(x_0) \subset\subset \Omega$. There exist $\gamma_0 > 1$, $c > 0$, $\sigma \in (0, 1/2]$ depending only on n such that if*

$$\frac{\beta}{\alpha} < \gamma_0$$

then for every $\rho < 2R$

$$\int_{B_\rho(x_0)} |Du|^2\,dx \le c\left(\frac{\rho}{2R}\right)^{n-1+2\sigma} \int_{B_{2R}(x_0)} |Du|^2\,dx.$$

We remark that we may estimate the number γ_0 (see [3, (20)-(21)]) by

$$\gamma_0 = \frac{n^n + n(n-1)^{n-1} - (n-1)^n}{n^n - n(n-1)^{n-1} + (n-1)^n}. \tag{2.16}$$

3. Applications of Proposition 2.4

3.1. **Regularity of minimizers.**

Definition 3.1. Let $E \subset \mathbb{R}^n$ be a set of finite perimeter in an open set U. We say that E is a (σ, Λ, R)-*almost minimizer* of the perimeter if for any ball $B_r(x_0) \subset\subset U$ with $r < R$ and any set F such that $F\Delta E \subset B_r(x_0)$ it holds

$$P(E, U) \le P(F, U) + \Lambda r^{n-1+2\sigma}.$$

The next theorem can be found in [13, Sections 1.9 and 1.10].

Theorem 3.2. *Let $E \subset \mathbb{R}^n$ be a (σ, Λ, R)-almost minimizer in an open set U. Then*

(a) *the reduced boundary* $\partial^* E \cap U$ *is a* $C^{1,\sigma}$ *hypersurface,*
(b) $(\partial E \setminus \partial^* E) \cap U$ *is empty if* $n < 8$, *and if* $n \geq 8$ *it is relatively closed in* $\partial E \cap U$ *and* $\mathcal{H}^s\left((\partial E \setminus \partial^* E) \cap U\right) = 0$ *for every* $s > n - 8$.

In the above theorem the topological boundary ∂E must be understood by considering the correct representative of the set (see [10, Proposition 12.19]). We note that Tamanini does not state explicitly that the set $(\partial E \setminus \partial^* E) \cap U$ is relatively closed. However, this follows easily by the characterization of the singular set given in terms of area excess (see [13, Chapter 4] and [10, Theorem 26.5]).

As we mentioned in the introduction, Theorem 1.1 is a rather straightforward consequence of Proposition 2.4 and Theorem 3.2.

Proof of Theorem 1.1. By [3, Theorem 1] we conclude that, if (E, u) is a minimizer of problem (P_c), there is a constant Λ such that (E, u) is also a minimizer of the penalized functional

$$\mathcal{F}_\Lambda(F, v) = \varepsilon P(F, \Omega) + \int_\Omega \sigma_F(x) |Dv|^2 \, dx + \Lambda \big| |F| - |E| \big| \tag{3.1}$$

among all (F, v) such that $v = u_0$ on $\partial\Omega$.

Let $\delta > 0$ be as in Proposition 2.4. For $h > 0$ let $A_h \subset \Omega$ be a set such that $x \in A_h$ if and only if there exists $R_x \in \left(\frac{1}{h}, \frac{\mathrm{dist}(x, \partial\Omega)}{2}\right)$ such that one of the following holds

(i) $\frac{|E \cap B_{R_x}(x)|}{|B_{R_x}|} < \delta$,
(ii) $\frac{|B_{R_x}(x) \setminus E|}{|B_{R_x}|} < \delta$,
(iii) there exists a half-space H such that $\frac{|(E \Delta H) \cap B_{R_x}(x)|}{|B_{R_x}|} < \delta$.

We first note that for every $h > 0$, A_h is an open set in Ω. Moreover, if $x \in A_h$ and $\rho < 1/h$, Proposition 2.4 implies

$$\int_{B_\rho(x)} |Du|^2 \, dy \leq c \left(\frac{\rho}{2R_x}\right)^{n-1+2\sigma} \int_{B_{2R_x}(x)} |Du|^2 \, dy \leq c(h) \rho^{n-1+2\sigma}.$$

Since (E, u) is a minimizer of (3.1), we may use the above estimate to deduce that E is a $(\sigma, \Lambda_h, 1/h)$-almost minimizer in A_h for suitable Λ_h. Theorem 3.2 then implies that the reduced boundary $\partial^* E \cap A_h$ is a $C^{1,\sigma}$ hypersurface and $(\partial E \setminus \partial^* E) \cap A_h$ is relatively closed in $\partial E \cap A_h$. Hence, $\partial^* E \cap \Omega = \partial^* E \cap \bigcup_{h>0} A_h$ is a $C^{1,\sigma}$ hypersurface and $(\partial E \setminus \partial^* E) \cap \Omega$ is relatively closed in $\partial E \cap \Omega$.

We need yet to estimate the size of the singular set. To that aim we write it as

$$(\partial E \setminus \partial^* E) \cap \Omega = \left[\bigcup_{h>0} (\partial E \setminus \partial^* E) \cap A_h\right] \cup \left[(\partial E \cap \Omega) \setminus \bigcup_{h>0} A_h\right]. \tag{3.2}$$

From Theorem 3.2 we deduce that $\mathcal{H}^s\left((\partial E \setminus \partial^* E) \cap A_h\right) = 0$ for every $s > \max\{n-8, 0\}$ and every $h > 0$. In particular, this implies $\mathcal{H}^{n-1}\left(\bigcup_{h>0} (\partial E \setminus \partial^* E) \cap A_h\right) = 0$. For $t \in [0, 1]$ we denote by E^t the set

$$E^t = \left\{x \in \Omega \, : \, \lim_{\rho \to 0} \frac{|E \cap B_\rho(x)|}{|B_\rho|} = t\right\}.$$

It clearly holds $E^0 \cup E^1 \cup \partial^* E \subset \bigcup_{h>0} A_h$. Since E is a set of finite perimeter, it holds $\mathcal{H}^{n-1}\big(\Omega \setminus (E^0 \cup E^1 \cup \partial^* E)\big) = 0$ [2, Theorem 3.61]. Hence, from (3.2) it follows

$$\mathcal{H}^{n-1}\left((\partial E \setminus \partial^* E) \cap \Omega\right) = 0.$$

We conclude the proof by noting that since the definition of the set A_h is independent of ε also the characterization of the singular set, given by (3.2), is independent of ε. ☐

If the ratio β/α is close to 1, the regularity result of Theorem 1.1 can be improved. Indeed, in this case we have (see [3, Theorem 3.1]) that $\mathcal{H}^s\left((\partial E \setminus \partial^* E) \cap \Omega\right) = 0$ for all $s > n-8$.

We conclude this section with the following remark about the singular set.

Remark 3.3. If $n < 8$, the estimate (3.2) in the proof of Theorem 1.1 gives an interesting characterization of the singular set, namely

$$(\partial E \setminus \partial^* E) \cap \Omega = (\partial E \cap \Omega) \setminus \bigcup_{h>0} A_h.$$

In particular, this implies that if $x \in (\partial E \setminus \partial^* E) \cap \Omega$ then the following three conditions hold,

(a) $\liminf_{\rho \to 0} \frac{|E \cap B_\rho(x)|}{|B_\rho|} \geq \delta$,
(b) $\liminf_{\rho \to 0} \frac{|B_\rho(x) \setminus E|}{|B_\rho|} \geq \delta$,
(c) $\liminf_{\rho \to 0} \frac{|(E \Delta H) \cap B_\rho(x)|}{|B_\rho|} \geq \delta$ for every half-space H,

where $\delta > 0$ is the constant from Proposition 2.4.

3.2. **Taylor cones.** In this section we study conical sets which are critical points, i.e., they satisfy (1.4) outside the vertex. In particularly, we are interested in spherical cones, which satisfy (1.4) outside the vertex. It was shown in [7] and [11] that there exist spherical cones in $\mathbb{R}^3$

$$E_{\theta_0} = \big\{x \in \mathbb{R}^3 \mid x_3 > \frac{1}{\tan \theta_0}\sqrt{x_1^2 + x_2^2}\,\big\},$$

for $\theta_0 \in (0, \pi/2)$, which are critical. Indeed, one may find an associated elastic minimum given in spherical coordinates

$$u(\rho, \theta) = \sqrt{\rho} \cdot f(\theta),$$

where $\rho = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and θ is the angel formed by the vector $x \in \mathbb{R}^3$ with the positive x_3 semi-axis. Denote by $P_{\frac{1}{2}}$ the Legendre function of the first kind of order $1/2$ which solves the equation

$$P''(t)(1-t^2) - 2tP'(t) + \frac{3}{4}P(t) = 0, \quad t \in (-1, 1).$$

Then the function f is given by

$$f(\theta) = \begin{cases} P_{\frac{1}{2}}(-\cos\theta_0)P_{\frac{1}{2}}(\cos\theta), & \theta \in [0, \theta_0] \\ P_{\frac{1}{2}}(\cos\theta_0)P_{\frac{1}{2}}(-\cos\theta), & \theta \in [\theta_0, \pi]. \end{cases}$$

The transition condition (1.3) then reads as

$$\beta P_{\frac{1}{2}}(-\cos\theta_0)P'_{\frac{1}{2}}(\cos\theta_0) + \alpha P_{\frac{1}{2}}(\cos\theta_0)P'_{\frac{1}{2}}(-\cos\theta_0) = 0.$$

It can be proved that there exists a critical treshold $\gamma_1 \approx 17.59$ such that this equation has no solutions if $\frac{\beta}{\alpha} < \gamma_1$ and it has two solutions in $(0, \pi/2)$ if $\frac{\beta}{\alpha} > \gamma_1$.

In [12] estimates of the angles corresponding to critical cones are given by a different approach. Although the known results give sharp estimates for the angles which allow existence of critical spherical cones, they do not give any rigorous answer whether there exists a range of angles where no critical spherical cones appear.

We apply the regularity from the previous section to prove that only cones with certain angles are possible. This estimate is independent of ε, which reflects the fact that the perimeter has only regularizing effect. This result rigorously answers to the question connected to Taylor Cones, of why cones of certain angles do not appear. The result also generalizes to convex cones E, which base set is uniformly convex and C^2-regular. Since the result is purely local, with no loss of generality we set $\Omega = \mathbb{R}^n$.

Proof of Theorem 1.2. Let us recall the Euler-Lagrange equation for the critical set E_θ

$$\varepsilon H_{\partial E_\theta} + \beta |Du_\beta|^2 - \alpha |Du_\alpha|^2 = \lambda \quad \text{on } \partial E_\theta \setminus \{0\}.$$

This can be rewritten as

$$\varepsilon H_{\partial E_\theta} + \beta |\partial_\nu u_\beta|^2 - \alpha |\partial_\nu u_\alpha|^2 + (\beta - \alpha)|D_\tau u|^2 = \lambda \quad \text{on } \partial E_\theta \setminus \{0\}, \tag{3.3}$$

where $D_\tau u$ is the tangential gradient of u on $\partial E_\theta \setminus \{0\}$. From the transition condition (1.3) we deduce

$$\beta |\partial_\nu u_\beta|^2 < \alpha |\partial_\nu u_\alpha|^2 \quad \text{on } \partial E_\theta \setminus \{0\}. \tag{3.4}$$

Since $H_{\partial E_\theta}(x) = (n-1)\cos\theta \cdot |x|^{-1}$ for $x \in \partial E \setminus \{0\}$, we obtain from the Euler-Lagrange equation (3.3), from (3.4), and from the transition condition (1.3) that

$$|\partial_\nu u_\beta(x)| \geq \frac{c}{\sqrt{|x|}} \qquad \text{on } \partial E \setminus \{0\}, \tag{3.5}$$

for some constant $c > 0$. In particular, since the set $\partial E \setminus \{0\}$ is connected, this implies that $\partial_\nu u_\beta$ does not change sign on $\partial E \setminus \{0\}$, and we may thus assume it to be positive.

Let us fix $\rho > 0$ and choose a cut-off function $\zeta \in C_0^\infty(B_\rho)$ such that $\zeta \equiv 1$ in $B_{\rho/2}$ and $|D\zeta| \leq 4/\rho$. Since u_β is harmonic in E, we obtain from (3.5) and by integrating by parts that

$$\int_{E \cap B_\rho} \langle Du_\beta, D\zeta \rangle \, dx = \int_{\partial E \cap B_\rho} \partial_\nu u_\beta \, \zeta \, d\mathcal{H}^{n-1} \geq c \int_{\partial E \cap B_{\rho/2}} |x|^{-1/2} \, \mathcal{H}^{n-1} \geq \tilde{c} \rho^{n-3/2}.$$

On the other hand Hölder's inequality implies

$$\begin{aligned} \int_{E \cap B_\rho} \langle Du_\beta, D\zeta \rangle \, dx &\leq \left(\int_{E \cap B_\rho} |D\zeta|^2 \, dx \right)^{1/2} \left(\int_{E \cap B_\rho} |Du_\beta|^2 \, dx \right)^{1/2} \\ &\leq C \rho^{n/2-1} \left(\int_{E \cap B_\rho} |Du_\beta|^2 \, dx \right)^{1/2}. \end{aligned}$$

Therefore

$$\int_{E \cap B_\rho} |Du_\beta|^2 \, dx \geq c_0 \rho^{n-1}$$

for some constant $c_0 > 0$. The claim now follows from Proposition 2.4 (i) and (iii) and Proposition 2.6.

□

Remark 3.4. The constant δ_0 can be explicitly estimated in terms of the constant δ from Proposition 2.4, since the spherical sector has the volume

$$|E_\theta \cap B_1| = \omega_{n-1}\left(\int_0^\theta \sin^n t\,dt + \frac{\sin^{n-1}\theta\cos\theta}{n}\right),$$

where ω_{n-1} is the volume of the $(n-1)$-dimensional unit ball. The formula for δ is derived in Remark 2.5. The constant γ_0 is estimated in (2.16).

Note that in dimension 2 the Euler-Lagrange equation (1.4) reduces to

$$\beta|Du_\beta|^2 - \alpha|Du_\alpha|^2 = \lambda \quad \text{on } \partial E_\theta \setminus \{0\}$$

and it is not clear to us if this weaker information is enough to establish Theorem 1.2 also in this case. However, as we already mentioned earlier, for $n \geq 3$, the proof of that theorem can be easily generalized to more general cones.

Remark 3.5. Let $K \subset \mathbb{R}^{n-1}$ be an open, uniformly convex and C^2-regular set such that $0 \in K$. Theorem 1.2 can be generalized to conical sets of the form

$$E = \{\lambda(x',1) \in \mathbb{R}^n \mid x' \in K,\ \lambda \geq 0\}.$$

We conclude by going through the proof of Lemma 2.2 and estimate all the relevant constant in the statement.

Proof of Lemma 2.2. Without loss of generality we may assume that $Q_R(x_0)$ is the unit cube Q. Denote $d(x) = \operatorname{dist}(x, \partial Q)$ and define

$$\mathcal{C}_k = \{x \in Q \mid \frac{3}{4}2^{-k-1} \leq d(x) \leq \frac{3}{4}2^{-k}\}.$$

Each $\mathcal{C}_k$ can be divided into cubes of side $\frac{3}{4}2^{-k-1}$. We call this collection $\mathcal{G}_k$. By Lemma 2.1 we have for $F(x) = d(x)^n|Du(x)|^2$ that

$$\fint_P F\,dx \leq C_0\left(\fint_{\tilde{P}} F^m\,dx\right)^{\frac{1}{m}},$$

where $\tilde{P}$ is the concentric cube to $P \subset \mathcal{C}_k$ or $P \subset Q_{1/4}$, for a constant $C_0 = 4^nC$, where C is the constant from Lemma 2.1.

Denote next $\Phi_t = \{x \in Q \mid F(x) > t\}$, where $t > a := \fint_Q |Du|^2\,dx$. Applying Calderón-Zygmund decomposition we obtain (in the proof [4, Lemma 6.2] choose $\lambda = 2^{1/m}C_0$)

$$\int_{\Phi_t} F\,dx \leq C_1 t^{1-m}\int_{\Phi_t} F^m\,dx$$

for

$$C_1 = 5^n 2^n \lambda = 2^{1/m}10^n C_0 \leq 4\cdot 40^n\,C = C_{S,n}^2\,2^{10}\cdot 80^n\frac{\beta}{\alpha}.$$

The result of [4, Proposition 6.1] now follows with the constants $A = C_2$ and $r > 1$ such that

$$C_1(p-1) = \frac{p-m}{2}$$

that is

$$p = \frac{2C_1 - m}{2C_1 - 1}.$$

This leads to the inequality

$$\int_Q F^p\,dx \le 2a^{p-1}\int_Q F\,dx$$

for $a = \fint_Q |Du|^2\,dx$. Recalling the definition of F we finally obtain

$$\fint_{Q_{1/2}} |Du|^{2p}\,dx \le 2^{n+pn+1}\left(\fint_Q |Du|^2\,dx\right)^p. \tag{3.6}$$

Let $B_1 \subset\subset \Omega$. Observe that for any integer $h > 1$, $Q_{1/2}$ can be covered by h^n cubes of side length $1/h$. Hence, $B_{1/2}$ can be covered by N_h cubes $Q_{1/2h}(x_i)$ having non-empty intersection with $B_{1/2}$ and $N_h \le h^n$. Using the rescaled analogue of the inequality (3.6) we get

$$\begin{aligned}
\fint_{B_{1/2}} |Du|^{2p}\,dx &\le \frac{2^n}{\omega_n h^n}\sum_{i=1}^{N_h}\fint_{Q_{1/2h}(x_i)} |Du|^{2p}\,dx \\
&\le 2^{n+pn+1}\frac{2^n}{\omega_n h^n}\sum_{i=1}^{N_h}\left(\fint_{Q_{1/h}(x_i)} |Du|^2\,dx\right)^p \\
&\le 2^{n+pn+1}\frac{2^n}{\omega_n h^n}\sum_{i=1}^{N_h}\left(\frac{h^n\omega_n}{2^n}\fint_{B_1} |Du|^2\,dx,\right)^p
\end{aligned}$$

provided $h > 4\sqrt{n}$, in which case $Q_{1/h}(x_i) \subset B_1$ for every $i = 1,\dots,N_h$. We may choose $h \le 5\sqrt{n}$ and thus we get

$$\fint_{B_{1/2}} |Du|^{2p}\,dx \le 2^{2n+1}5^{np}n^{np/2}\omega_n^{p-1}\left(\fint_{B_1} |Du|^2\,dx,\right)^p.$$

□

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