REGULAR MINIMIZERS OF SOME FREE DISCONTINUITY PROBLEMS

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Abstract. We show that minimizers of free discontinuity problems with energy dependent on jump integrals and Dirichlet boundary conditions are smooth provided a smallness condition is imposed on data.
We examine several examples, including elastic-plastic beam and plate with free yield lines and deformable body with free damage. In all cases there is a gap between the condition for solvability (safe load condition) and this smallness condition (load regularity condition).
Such gap allows the existence of damaged/creased minimizers. Eventually we provide explicit examples of irregular solutions when the load stays in the gap.

1. Introduction

Free discontinuity problems related to image segmentation achieve minimum regardless to the size of the data, due to the structural growth of the forcing term ([22],[23],[12],[13]). Free discontinuity problems in continuum mechanics have minimizers only if the loads are small, say suitable safe load condition is satisfied ([10],[18],[38],[39],[40],[13],[21]).

Strong solutions of free discontinuity problems without jump integrals over the singular set were proven to exist under higher integrability assumptions on data in [24],[11],[14],[3].

In this paper we show that some functionals which allow free discontinuity and pay jump integrals over the singular set do have minimizers, provided the forcing term is sufficiently small say it fulfils an explicit safe load condition, which appears as a necessary condition for load with sign; actually we prove that such minimizers have empty discontinuity set when the load is smaller than required by safe load: e.g. admissible small load deform an elastic-plastic plate or beam in the elastic range without occurrence of plastic yield (see Sections 4 and 5). We call load regularity condition this more stringent inequality.

In several cases there is a gap between the safe load condition and the load regularity condition: in this situation the strong inequality in the safe load (sufficient condition for existence) allows the possibility of non regular (yielded) solutions for suitable load in between: we show explicit examples of cracked/creased solutions, when the load stays this gap. A detailed analysis of the structure of irregular solutions is given in 1D model problems (see Theorem 3.15, Theorem 4.10 and Theorems 4.13, 4.14).

More precisely we focus on four examples of functionals listed below. They are all related to continuum mechanics (involving deformations of elastic bodies with damage) with homogeneous or non homogeneous Dirichlet boundary conditions. The boundary condition is imposed by allowing variations (defined in the whole euclidian space) which are different from Dirichlet datum only in the reference bounded set.

Key words and phrases. Free discontinuity, bounded hessian, safe load, crack, plastic hinges, smooth minimizers.

The main focus of paper is a variational approach for detecting elastic-plastic yielding of beams and plates (Problems II, III listed below). Nevertheless the tools for the analysis are inspired by and tested on the simpler first order Problems I and IV: these ones are toy problems without ambition to grasp all the complexity of real phenomenon (see [17],[18],[19],[20],[25]-[30],[49],); anyway they try to describe with few macroscopic variables the effects of mesoscopic damage.

I. First order model problem (elastic rod with free damage under traction):

$$\mathcal{F}_1(w) \;=\; \int_{\mathbb{R}} \left(\frac{1}{2}\,|\dot{w}|^2 - f w \right) dx \,+\, \alpha\, \sharp(S_w) \,+\, \gamma \sum_{S_w} |[w]| \tag{1.1}$$

to be minimized among scalar functions w such that $w \in SBV(\mathbf{R})$ s.t. $\mathrm{spt}(w - w_0) \subset [0, L]$. Here $f \in L^1(\mathbb{R})$ with $\mathrm{spt}\, f \subset [0, L]$ is the traction load, $w_0 \in SBV(\mathbb{R})$ with $w_0 \in \mathrm{dom}\, \mathcal{F}_1$ is the boundary traction, $\sharp$ is the counting measure, $\alpha > 0$, $\gamma > 0$, and S_w is the singular set ([3]) of w. Here and in the following $\dot{w}$ denotes the absolutely continuous part of the distributional derivative w' , $[v]$ denotes the jump $v^+ - v_-$.

II Second order model problem (elastic plastic beam under transverse load):

$$\mathcal{F}_2(w) \;=\; \int_{\mathbb{R}} \left(\frac{1}{2}\,|\ddot{w}|^2 - f w \right) dx \,+\, \beta\, \sharp(S_{\dot{w}}) \,+\, \gamma \sum_{S_{\dot{w}}} |[\dot{w}]| \tag{1.2}$$

to be minimized among scalar functions w such that $w \in SBH(\mathbf{R})$ s.t. $\mathrm{spt}(w - w_0) \subset [0, L]$. Here $f \in \mathcal{M}(\mathbb{R})$ with $\mathrm{spt}\, f^s \subset (0, L)$ is the transverse load, $w_0 \in SBH(\mathbb{R})$ with $w_0 \in \mathrm{dom}\, \mathcal{F}_2$ provides the boundary condition, $\beta > 0$, $\gamma > 0$ and $S_{\dot{w}}$ is the singular set of $\dot{w}$ (see [38],[40]).

III. Clamped elastic plastic plate
(Kirchhoff-Love plate with plastic yield along free lines):

$$\begin{aligned} \mathcal{P}(w) \;=\;& \frac{2}{3}\mu \int_{\mathbb{R}^2} \left(\left|(D^2 w)^a\right|^2 + \frac{\lambda}{\lambda + 2\mu}\, |\Delta^a w|^2 - f w \right) d\mathbf{x} + \\ & + \beta\, \mathcal{H}^{n-1}(S_{Dw}) \,+\, \gamma \int_{S_{Dw}} |[Dw]| d\mathcal{H}^1 \end{aligned} \tag{1.3}$$

to be minimized among scalar functions $w \in SBH(\mathbb{R}^2)$ s.t.. $\mathrm{spt}\, w \subset \overline{\Sigma}$. In (1.3) $\Sigma \subset \mathbf{R}^2$ is a connected C^4 open set or an open convex polygon, $f \in L^p(\mathbb{R}^2)$ with $\mathrm{spt}\, f \subset \overline{\Sigma}$ is the transverse load, ∇ denotes the absolutely continuous part of the distributional gradient D, $\Delta^a w$ is the trace of $\nabla D w$, S_{Dw} is the singular set of Dw ([10]), $\alpha > 0$, $\gamma > 0$, $\mu > 0$, $\lambda + \mu > 0$, $p > 1$ and $\mathcal{H}^1$ is the length (1d Hausdorff measure).
Here Σ is the reference configuration of an elastic thin plate, w the transverse displacement of the plate. The functional $\mathcal{P}$ represents the mechanical energy of the deformed plate, subject to transverse dead load f, with free plastic yield lines whose pattern (the set S_{Dw}) is "a priori" unknown ([39],[10],[46]).

IV. Vector-valued deformations with cohesive damage along free surfaces:

$$\begin{aligned} \mathcal{F}(\mathbf{v}) \;=\;& \textstyle\int_{\mathbb{R}^3} \left(\mu\, |\mathcal{E}(\mathbf{v})|^2 + \frac{\lambda}{2}\, |\mathrm{Tr}\, \mathcal{E}(\mathbf{v})|^2 - \mathbf{f} \cdot \mathbf{v} \right) d\mathbf{x} + \\ & + \alpha\, \mathcal{H}^{n-1}(J_{\mathbf{v}}) \,+\, \gamma \textstyle\int_{J_{\mathbf{v}}} |[\mathbf{v}] \odot \nu_{\mathbf{v}}| d\mathcal{H}^{n-1} \end{aligned} \tag{1.4}$$

to be minimized over vector fields $\mathbf{v}$ with $\operatorname{spt}\mathbf{v} \subset \overline{\Omega}$ and special bounded deformation (say $\mathbf{v} \in SBD(\mathbb{R}^n)$). Here $\Omega \subset \mathbb{R}^n$ is a connected Lipschitz open set and $n = 2, 3$; $\mathbf{f} \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ is the given body force, with $p \geq n$, $\operatorname{spt}\mathbf{f} \subset \overline{\Omega}$.
The set Ω represents the un-stressed configuration of an elastic body, $\mathbf{v}$ is a vector field with special bounded deformation in Ω representing the displacement ([4],[18]), λ, μ are the Lamé coefficients satisfying $\mu > 0$, $2\mu + n\lambda > 0$, α, γ are constants related respectively to energy of crack surface formation and crack opening with $\alpha > 0, \gamma > 0$, $\mathcal{E}(\mathbf{v})$ is the absolutely continuous part of the linear strain tensor $\mathrm{e}(\mathbf{v}) = \frac{1}{2}\left(D\mathbf{v} + (D\mathbf{v})^T\right)$, $J_{\mathbf{v}}$ is the jump set of $\mathbf{v}$, $\nu_{\mathbf{v}}$ is the normal to $J_{\mathbf{v}}$, $[\mathbf{v}]$ is the jump of $\mathbf{v}$ in the $\nu_{\mathbf{v}}$ direction, $\mathcal{H}^{n-1}$ denotes the $(n-1)$ dimensional Hausdorff measure and $\odot$ denotes the symmetric tensor product.
The functional $\mathcal{F}$ represents a simplified description (inspired by Barenblatt approach [5]) of mechanical energy for a linear elastic body, with natural reference Ω, subject to dead load $\mathbf{f}$, with free small cohesive crack whose geometry (the set $J_{\mathbf{v}}$) is not "a priori" prescribed ([18],[38]).

In all four cases crack (respectively plastic yield) may be located also at the boundary $\partial\Sigma$. Actually in some particular 1-dimensional case we show in [42] that damage may take place only at the boundary.
For each one of the four problems above we give an explicit safe load condition and prove that it entails the existence of a finite minimum (Lemmas 3.1, 4.2, 5.1, 6.1); then we prove an excess estimate (Lemmas 3.4, 4.5, 5.5, 6.5) say a comparison with the energy of the solutions of the related elastic problems (minimizers of exactly the same functionals among competing functions which must be regular); hence we deduce regularity conditions which have an implicit form since they refer to the solution of the associated purely elastic problem (condition on stress: Theorems 3.5, 6.6; condition on bending moment for beam and plate: Theorems 4.6, 5.6); eventually we prove a load regularity condition explicitly dependent only on data (Theorems 3.6, 4.8, 5.7, 6.7). The usual method of calibrations ([2]) do not apply to the present context which admits dependence on second derivatives: this difficulty is circumvented by introducing a technique of calibration for minimizer based on *comparison with purely elastic solutions*, through suitable excess estimates and compliance identities.

The Euler equations are derived explicitly in 1-dimensional Problems I and II together with qualitative properties of free discontinuity set of minimizers (Theorems 3.3, 4.4). It is remarkable that some reminding of Weierstrass-Erdman corner condition hold true for functionals with free discontinuity: $\dot{w}$ is continuous in $(0, L)$ for minimizers of $\mathcal{F}_1$ (Theorem 3.3) and $\ddot{w}$, $\dddot{w}$ are continuous in $(0, L)$ for minimizers of $\mathcal{F}_2$ (Theorem 4.4).

An interesting issue about the consistence of these models is achieved in the 1D frame by analysis of minimizers structure: minimizers of $\mathcal{F}_1$ may crack not more than at a single point (Theorems 3.8, 3.15 describe the structure of $\mathcal{F}_1$ minimizers); minimizers of $\mathcal{F}_2$ may exhibit no more than two crease points (Theorem 4.10).

Explicit examples of load producing damaged minimizer of $\mathcal{F}_1$ and creased minimizers of $\mathcal{F}_2$ are shown when load belongs to the narrow gap between safe load condition and regularity load condition: Examples 3.17, 3.18, Theorem 4.11. In order to achieve these examples, a careful estimate of this gap is obtained by showing: first, sharp Poincaré inequalities (see (3.5), Lemma 4.1), then stress estimate (3.23) for rod and bending estimates (4.28),(5.25) for beam and plate by mean of Green functions (Theorems 3.12, 4.7) and elliptic regularity (Theorem 5.4). Suitable compliance identities for minimizers (Lemmas 3.11, 4.9, 5.3, 6.3) proved very useful in all the computations.
A sufficient condition (5.53) for development of plastic yield lines in a plate (functional $\mathcal{P}$) is shown by Theorem 5.11.

Some of the results proven here about plate (Problem III) and beam (Problem II) were announced in [41],[43].
We refer to [42] for a deeper analysis of the elastic-plastic beam and asymptotic analysis of Problem II as the parameter $\beta \to 0_+$, in the framework of L^∞ load.

We emphasize that the analysis of Problem I in the framework of L^∞ load would provide the same qualitative picture of rod deformation proven here for L^1 or measure load, since the constants in related safe load regularity conditions are the same (except for the different homogeneity in L) so that they coincide on constant load.
On the contrary the behavior of the beam (Problem II) do change a lot in the framework of L^∞ load since (in addition to different homogeneity in L) optimal constant in the L^1-BH Poincaré inequality (Lemma 2.1 in [42]) is quite different with respect to the one appearing in $L^\infty-BH$ Poincaré inequality (Lemma 4.1 in present paper): hence (see [42]) we can show that there are choices of constant load (fulfilling the appropriate L^∞ safe load condition but not the L^∞ regularity load condition (respectively (2.5),(3.13) in [42]) which produce plastic hinges at both endpoints of the beam. While in the present context we show that concentrated load with spt $\subset (0, L)$ of increasing intensity do not produce symmetric plastic hinges at endpoints before collapse (Theorem 4.13). Moreover Theorem 4.14 entails that symmetric load of constant sign and fulfilling the L^1 safe load condition (4.11) do not produce plastic hinges at all. About skew-symmetric load analysis for the elastic plastic beam we refer to [44].

Outline

2. Notation

We denote by $\|\mu\|_T$ the total variation in $\mathbb{R}$ of μ and by $\|\mu\|_{T(E)}$ the total variation in E for any $\mu \in \mathcal{M}(\mathbb{R})$ and any Borel set $E \subset \mathbb{R}$. For any $f \in \mathcal{M}(\mathbb{R})$ with $spt f^s \subset\subset (0,L)$ and any $v \in L^1_f(\mathbf{R})$ we write shortly $\displaystyle\int_0^L f(x)v(x)dx$ for $\displaystyle\int_{\mathbb{R}} v(x)df(x)$.
Any $\mu \in \mathcal{M}(\mathbb{R})$ can be split into three parts, say $\mu = \mu^a + \mu^s = \mu^a + \mu^j + \mu^c$ where μ^a is the absolutely continuous part, μ^s is the singular part, μ^j is the purely atomic part and μ^c is the diffuse singular one (the Cantor part of μ): the decomposition is unique.
Analogously, if I is an interval, then any $w \in BV(I)$ can be represented by $w = w_a + w_j + w_c$ where w_a has an absolutely continuous distributional derivative $(w_a)' = (w')^a \in L^1(I)$, w_j is a piece-wise constant function and $(w_j)' = (w')^j$ is purely atomic), w_c is a Cantor-type function (i.e. $(w_c)' = (w')^c$: for any $w \in BV(I)$ these three functions are uniquely defined up to additive constants ([3], Corollary 3.33), the constants are 0 when the support of w is a compact subset of I. We label $\dot{w} = (w_a)'$ the absolutely continuous part of distributional derivative w', hence we write as follows the unique decomposition of the derivative for a BV function with compact support: $w' = \dot{w} + (w_j)' + (w_c)'$. Approximate discontinuity sets of w and $\dot{w}$ (see [3]) are labeled by S_w, $S_{\dot{w}}$ and are shortly referred to as singular set of w, $\dot{w}$.
Symbols $\sharp$ and $\sharp\llcorner E$ respectively denote the counting measure and its restriction to $E \subset \mathbb{R}$.
Symbols [ ], $\otimes$ and $\odot$ denote respectively jumps, the tensor product and its symmetric part.
About the case of several variables we denote respectively by Dv and ∇v the distributional gradient and the approximate gradient of v. For any open set $\Omega \subset \mathbb{R}^n$ we denote:

$\mathcal{M}(\Omega) = \{\mu : \text{ real valued Radon measures in } \Omega\}$,
$BV(\Omega) = \{v \in L^1(\Omega) : Dv \in \mathcal{M}\}$,
$SBV(\Omega) = \{v \in BV(\Omega) : \ Dv \text{ has no Cantor part}\}$,
$BH(\Omega) = \{v \in W^{1,1}(\Omega) : \ D^2 v \in \mathcal{M}\}$,
$SBH(\Omega) = \{v \in BH(\Omega) : \ D^2 v \text{ has no Cantor part}\}$,
$SBD(\Omega) = \{\mathbf{v} : \Omega \to \mathbf{R}^n \ : \ \text{sym}\,(D\mathbf{v}) \text{ is a matrix-valued Radon measure}\}$.

For any Borel set $E \subset \Omega$ and $\mu \in \mathcal{M}(\Omega)$ we denote by $\|\mu\|_{T(E)}$ the total variation of μ in E; we will write shortly $\|\mu\|_T = \|\mu\|_{T(\mathbf{R}^n)}$ when $E = \Omega = \mathbf{R}^n$. The singular set of v (the set of points in Ω where v is not approximately continuous) is denoted S_v (see [3]).
The set of approximate jump of a vector valued function $\mathbf{v} \in SBD(\Omega)$ is denoted $J_{\mathbf{v}}$ (see [4]).
For definition and properties of the above function spaces we refer to [3],[4],[14],[18]).

3. (Pb I) Elastic rod with free damage under traction

We study the functional

$$\mathcal{F}_1(w) \ = \ \int_{\mathbb{R}} \left(\frac{1}{2}\, |\dot{w}|^2 - f w \right) dx \, + \, \alpha\, \sharp(S_w) \, + \, \gamma \sum_{S_w} \ |[w]| \tag{3.1}$$

to be minimized among scalar functions w such that $w \in SBV(\mathbf{R})$ s.t. $\text{spt}(w - w_0) \subset [0,L]$.
Here α, γ are given constants, w_0 is a given function and $\sharp$ is the counting measure.
All along this section we assume

$$\begin{cases} \alpha > 0, \ \gamma > 0, & \Sigma = (0,L), \qquad f \in L^1(\mathbb{R}), \qquad \text{spt}\, f \subset \overline{\Sigma}, \\ w_0 \in SBV(\mathbb{R}), & \sharp(S_{w_0}) \ < \ +\infty, \qquad S_{w_0} \subset \overline{\Sigma}, \qquad \mathcal{F}_1(w_0) \in \mathbb{R}\,. \end{cases} \tag{3.2}$$

Functional (3.1) describes the total energy of an elastic rod which may undergo damage at free locations and is subject to given traction body force f and given boundary traction

expressed by $w_0(0_-)$ and $w_0(L_+)$. The location of damage is a priori unknown and is given by the singular set of optimal w.
Actually functional (3.1) is a crude simplification of more realistic models involving a concave interface energy contribution in place of $\gamma \sum_{S_{\dot w}} |[\dot w]|$ ([**?**],[7],[19],[34]). Nevertheless (3.1) provides a simple framework in which we are able to describe completely the structure of minimizers, moreover (3.1) proved very helpful in suggesting the techniques to tackle the harder and more significant models of elastic plastic beams and plates faced in Sections 4,5.

We introduce a localization of the functional: for any Borel set $A \subset \mathbb{R}$ we set

$$\mathcal{F}_1(w,A) \;=\; \int_A \left(\frac{1}{2}\,|\dot w|^2 - fw\right) dx \,+\, \alpha\,\sharp(S_w\cap A) \,+\, \gamma \sum_{S_w\cap A} |[w]|.$$

At first we prove that a smallness condition (safe load condition) on f entails the existence of minimizers, while a violation of the safe load may lead to collapse.

Lemma 3.1. *Assume* (3.2) *and*

$$\|f\|_{L^1(\Sigma)} \;<\; 2\,\gamma \qquad (\mathcal{F}_1 \textbf{ safe load condition})\,. \tag{3.3}$$

Then $\mathcal{F}_1$ achieves a finite minimum among $w \in SBV(\mathbb{R})$ s.t. $\mathrm{spt}(w - w_0) \subset [0, L]$. *And any minimizer z verifies*

(3.4)

$$\|z'\|_{T(\overline{\Sigma})} \;\le\; \frac{1}{2\gamma - \|f\|_{L^1(\Sigma)}}\left(4\mathcal{F}_1(w_0) + L\gamma^2 + \|f\|_{L^1(\Sigma)}(|w_0(0_-)| + |w_0(L_+)|)\right).$$

Proof - If $\|f\|_{L^1(\Sigma)} \;<\; 2\,\gamma$, then we can apply the direct method since $\mathcal{F}_1$ is coercive in BV: in fact by fundamental theorem of calculus

$$w(x_-) \;=\; \frac{1}{2}\left(w_0(0_-) + \int_{[0,x)} w'\right) + \frac{1}{2}\left(w_0(L_+) - \int_{[x,L]} w'\right)$$

$$w(x_+) \;=\; \frac{1}{2}\left(w_0(0_-) + \int_{[0,x]} w'\right) + \frac{1}{2}\left(w_0(L_+) - \int_{(x,L]} w'\right)$$

$$\|w\|_{L^\infty(0,L)} \le \tfrac{1}{2}\left(\|w'\|_{T(\overline{\Sigma})} + |w_0(0_-)| + |w_0(L_+)|\right) \qquad \forall w \in BV(\mathbb{R}) : \,\mathrm{spt}(w - w_0) \subset [0, L]\,. \tag{3.5}$$

Hence for any admissible w

$$-\int_\Sigma fw \;\ge\; -\|f\|_{L^1(\Sigma)}\|w\|_{L^\infty(\Sigma)} \;\ge\; -\frac{1}{2}\|f\|_{L^1(\Sigma)}\left(\|w'\|_{T(\overline{\Sigma})} + w_0(0_-) + w_0(L_+)\right).$$

Moreover

$$\mathcal{F}_1(w) = \mathcal{F}_1(w,\mathbb{R}) = \mathcal{F}_1(w_0, \mathbb{R}\setminus\overline{\Sigma}) + \mathcal{F}_1(w,\overline{\Sigma})$$

Then by integrating over Σ the Young inequality $|\dot w|^2/2 \ge \gamma|\dot w| - \gamma^2/2$, we have

$$\mathcal{F}_1(w_0) \;\ge\; \mathcal{F}_1(w) \;\ge\; \alpha\,\sharp\,(S_w) + \left(\gamma - \frac{1}{2}\|f\|_{L^1(\Sigma)}\right)\|w'\|_{T(\overline{\Sigma})}$$

$$-\,\frac{L}{2}\gamma^2 - \frac{\|f\|_{L^1(\Sigma)}}{2}\left(|w_0(0-)| + |w_0(L+)|\right) + \mathcal{F}_1(w_0, \mathbb{R}\setminus\overline{\Sigma}).$$

Due to the inequality $2\gamma - \|f\|_{L^1} > 0$, the functional is bounded from below and, the elements of any minimizing sequence eventually fulfil the estimate (2.3). By w^*BV compactness and l.s.c properties ([3]) the existence of minimizers follows. □

The safe load (3.3) cannot be improved for generic L^1 load as shown by following Remark.

Remark 3.2. *If* $\|f\|_{L^1(\Sigma)} > 2\gamma$ *,* f *does not change sign and* $w_0 \equiv 0$*, then* $\inf \mathcal{F}_1 = -\infty$*. For instance, if* $f \geq 0$ *and* $\|f\|_{L^1(\Sigma)} \geq 2\gamma + \varepsilon$*,* $\varepsilon > 0$*, set* $v_t(x) = t > 0$ *if* $x \in \Sigma$*,* $v_t(x) = 0$ *if* $x \notin \Sigma$*; then* $J_{v_t} = \{0, L\}$*,* $\dot v \equiv 0$*,* $\int_{\mathbb{R}} f\, v_t\, dx \geq (2\gamma + \varepsilon)t$ *and* $\mathcal{F}_1(v_t) = 2\alpha - \varepsilon t \to -\infty$ *as* $t \to \infty$*.*

Theorem 3.3. *(***Euler equations for** $\mathcal{F}_1$*) Assume* (3.2) *and* w *is a minimizer of* $\mathcal{F}_1$ *among* v *s.t.* $v = w_0$ *on* $\mathbb{R} \setminus \overline{\Sigma}$*.*
Then $w' = \dot w \in AC(I)$ *for any interval* I *contained in* $\Sigma \setminus S_v$*. Hence* $w(x_\pm)$ *and* $\dot w(x_\pm)$ *are defined for all* $x \in \overline{\Sigma}$ *and, by setting* $w_\pm(x) = w(x_\pm)$*,* $\dot w_\pm(x) = \dot w(x_\pm)$*, the following equalities hold true*

$$\begin{array}{rrcll} (i) & -w'' & = & f & (0,L) \setminus S_w\,, \\ (ii) & \dot w_- & = & \gamma\,\mathrm{sign}([w]) & \text{in } S_w \cap (0,L]\,, \\ (iii) & \dot w_+ & = & \gamma\,\mathrm{sign}([w]) & \text{in } S_w \cap [0,L)\,, \end{array}$$

$$(iv)\ \int_0^L (\dot w(\dot z - \dot w) - f(z-w))\,dx + \gamma \sum_{S_{z-w}} |[z-w]| = 0,\ \forall z \in SBV(\mathbb{R}) \colon \mathrm{spt}(z-w) \subset \overline{\Sigma}.$$

Hence $-(\dot w)' = f$ *in* $\mathcal{D}'(0,L)$ *and* $\dot w \in AC(0,L)$ *even if* $S_w \cap (0,L)$ *is not empty. Nevertheless the continuity of* $\dot w$ *may fail at* 0 *and* L*.*

Proof - By choosing $\varphi \in C^\infty(\mathbb{R} \setminus S_v) \cap SBV(\mathbb{R})$ with $\mathrm{spt}\,\varphi \subset \overline{\Sigma}$, and with C^∞ limit from both sides at any point in S_v, we get $\mathcal{F}_1(w) \leq \mathcal{F}_1(w + \varepsilon\varphi)$. By convexity and taking into account that $w \in SBV$ entails $\dot w = w'$ in $(0,L) \setminus S_w$ and $\dot\varphi = \varphi' - \sum_{S_\varphi} [\varphi]\, \sharp \llcorner S_\varphi$, we get, for $0 < \varepsilon < \min_{S_w} |[w]|/\|\varphi\|_{L^\infty}$:

$$\begin{aligned} 0 \ &\leq \varepsilon \int_\Sigma (\dot w \dot\varphi - f\varphi) dx + \alpha\Big(\sharp(S_{w+\varepsilon\varphi}) - \sharp(S_w)\Big) + \\ &\qquad + \gamma \sum_{S_w} (|[w + \varepsilon\varphi]| - |[w]|) + o(\varepsilon) = \\ &= \varepsilon\left(\int_\Sigma (-w'' - f)\varphi dx + (\varphi(L_-)\dot w(L_-) - (\varphi(0_+)\dot w(0_+)) + \right. \\ &\qquad \left. + \sum_{S_w \cap (0,L)} ((\varphi_- \dot w_-) - (\varphi_+ \dot w_+)) + \gamma \sum_{S_w} [\varphi]\ \mathrm{sign}([w])\right) + o(\varepsilon) \\ &= \varepsilon\left(\int_\Sigma (-w'' - f)\varphi dx + (\varphi(L_-)\dot w(L_-) - (\varphi(0_+)\dot w(0_+)) + \right. \\ &\qquad \left. + \sum_{S_w \cap (0,L)} ((\varphi_- \dot w_-) - (\varphi_+ \dot w_+)) + \gamma \sum_{S_w} (\varphi_+ - \varphi_-)\ \mathrm{sign}([w])\right) + o(\varepsilon) \\ &= \varepsilon\left(\int_\Sigma (-w'' - f)\varphi dx \right. \\ &\qquad + \varphi(0_+)\Big(\gamma\,\mathrm{sign}([w](0)) - \dot w(0_+)\Big) - \varphi(L_-)\Big(\gamma\,\mathrm{sign}([w](L)) - \dot w(L_-)\Big) + \\ &\qquad \left. + \sum_{S_w \cap (0,L)} \Big(\varphi_+(\gamma\,\mathrm{sign}([w]) - \dot w_+) - \varphi_-(\gamma\,\mathrm{sign}[w] - \dot w_-)\Big)\right) + o(\varepsilon)\,. \end{aligned}$$

By choosing all φ with compact support in an interval contained in $(0,L) \setminus S_w$ we get the differential identity in $-w'' = f$ in $(0,L) \setminus S_w$. Then for any fixed $x_k \in S_w$ we can choose at

first (if $x_k < L$) all φ with compact support in $[x_k, x_{k+1})$ where x_{k+1} is the closest singular point bigger than x_k if any or L else, and then (if $0 < x_k$) all φ with compact support in $[x_k - 1, x_k)$ where x_{k-1} is the closest singular point smaller than x_k if any or 0 else: this provides the values of $\dot{w}_\pm$ in S_w. The derivation of (ii),(iii) at $0, L$ is analogous.
The statement about continuity of $\dot{w}$ is straightforward, since $\dot{w} = w'$ in open interval where w is continuous, then in such intervals w is AC $w' \in L^1$ and w minimizes the Dirichlet integral (hence w is C^1), $\dot{w}_\pm$ exist in these intervals and $\dot{w}_\pm = w'$; while $\dot{w}_+ - \dot{w}_- = (\gamma - \gamma)\,\mathrm{sign}[w] = 0$ in $S_w \cap (0, L)$.
Du Bois-Raymond equation (iv) follows in the same way by minimality of w with respect to variations $w + \varepsilon(z - w)$.

Lemma 3.4. *Assume* (3.2). *Let* u *be the solution of*

$$u \in H^1(\Sigma), \qquad -u'' = f \ \text{in } \Sigma, \qquad u(0_+) = w_0(0_-),\ u(L_-) = w_0(L_+) \tag{3.6}$$

then $u' = \dot{u} \in C^0(\overline{\Sigma})$ *and* u *has an extension, still denoted by* u, *s.t.* $u \in SBV(\mathbf{R}) \cap C^0(\mathbb{R})$ *and* $u \equiv w_0$ *in* $\mathbb{R} \backslash \overline{\Sigma}$. *Boundary values of* u *are always understood as interior traces Whenever Moreover*

Excess estimate for $\mathcal{F}_1$ *: If* u *solves* (3.6), *then for all* $v \in SBV(\mathbb{R})$ *s.t.* $\mathrm{spt}(v - w_0) \subset \overline{\Sigma}$

$$\mathcal{F}_1(v) - \mathcal{F}_1(u) \ \geq \ \alpha\sharp(S_v) + \sum_{S_v} (\gamma|[v]| - u'[v])\,. \tag{3.7}$$

Excess identity for minimizers of $\mathcal{F}_1$ *: If* v *minimize* $\mathcal{F}_1$ *among* $v \in SBV(\mathbb{R})$ *s.t.* $\mathrm{spt}(v - w_0) \subset \overline{\Sigma}$, *and* u *solves* (3.6), *then we have*

$$\mathcal{F}_1(v) - \mathcal{F}_1(u) \ = \ \alpha\sharp(S_v) + \frac{1}{2}\sum_{S_v} (\gamma|[v]| - u'[v])\,. \tag{3.8}$$

Necessary conditions for existence of discontinuous minimizers of $\mathcal{F}_1$ *: If If* v *minimize* $\mathcal{F}_1$ *among* $v \in SBV(\mathbb{R})$ *s.t.* $\mathrm{spt}(v - w_0) \subset \overline{\Sigma}$, $S_v \neq \emptyset$, *and* u *solves* (3.6), *then*

$$\|u'\|_{L^\infty(\Sigma)} > \gamma\,, \tag{3.9}$$

$$\sum_{S_v} (\gamma|[v]| - u'[v]) \leq -2\alpha\,\sharp\,(S_v) < 0\,. \tag{3.10}$$

Proof - u is the only minimizer of $\mathcal{F}_1$ over $w \in SBV(\mathbb{R}) \cap H^1(\Sigma)$ s.t. $\mathrm{spt}(w - w_0) \subset \overline{\Sigma}$. By exploiting $u' \in C(\overline{\Sigma})$, $\dot{v} = v' - [v]\, d\sharp \llcorner (S_v \cap (0, L))$ in $\mathcal{D}'(0, L)$, $u - w_0 \in H^1_0(\Sigma)$, $-u'' = f$ in Σ, convexity of $s \to s^2/2$ and

$$\int_0^L u'(v - u)'\,dx \ = \ -\int_0^L u''(v - u)\,dx \ - \ u'(L)\,[v](L) \ - \ u'(0)\,[v](0)$$

we have, for every $v \in SBV(\mathbb{R})$ s.t. $\mathrm{spt}(v - w_0) \subset \overline{\Sigma}$ we have

$$\mathcal{F}_1(v) \ \geq \ \mathcal{F}_1(u) + \int_0^L u'(\dot{v} - u')dx - \int_0^L f(v - u)dx + \alpha\sharp(S_v) + \gamma\sum_{S_v} |[v]| =$$

$$= \mathcal{F}_1(u) + \int_0^L u'(v' - u')dx - \int_0^L f(v - u)dx + \alpha\sharp(S_v) + \gamma\sum_{S_v} |[v]| - \sum_{S_v \cap (0,L)} u'[v] =$$

$$= \mathcal{F}_1(u) + \alpha\sharp(S_v) + \sum_{S_v} (\gamma|[v]| - u'[v])$$

Then (3.7) is proven.
If $v \in \operatorname{argmin} \mathcal{F}_1$ and u solves (3.6), then $\dot v$ is continuous in $(0, L)$ by Theorem 3.3 and $u = v$, and $u' = v' = \dot v$ hold true in $\mathbb{R} \setminus \overline{\Sigma}$ (assuming $S_w \subset \overline{\Sigma}$ and $\dot v$ continuous in $\mathbb{R}$ is not restrictive), and Du-Bois Raymond equation for $v_\varepsilon = v + \varepsilon(u - v)$ yields

$$\int_\Sigma \left(\dot v\,(u' - \dot v) - f(u - v)\right)\,dx - \gamma \sum_{S_v} |[v]| \;=\; 0\,. \tag{3.11}$$

Hence

$$\mathcal{F}_1(v) - \mathcal{F}_1(u) = \frac{1}{2}\int_\Sigma |\dot v|^2\,dx + \alpha\sharp(S_v) + \gamma \sum_{S_v} |[v]| - \int_\Sigma f v\,dx - \frac{1}{2}\int_\Sigma |u'|^2 + \int_\Sigma f u =$$

$$= \frac{1}{2}\int_\Sigma (\dot v + u')\,(\dot v - u') + \alpha\sharp(S_v) + \gamma \sum_{S_v} |[v]| - \int_\Sigma f v\,dx + \int_\Sigma f u =$$

$$= \frac{1}{2}\int_\Sigma f(v-u) - \int_\Sigma f(v-u) - \frac{\gamma}{2}\sum_{S_v} |[v]| + \gamma \sum_{S_v} |[v]| + \frac{1}{2}\int_\Sigma u'(\dot v - u') + \alpha\sharp(S_v)$$

$$= \frac{\gamma}{2}\sum_{S_v} |[v]| - \frac{1}{2}\int_\Sigma f(v-u) + \frac{1}{2}\int_\Sigma u'(\dot v - u') + \alpha\,\sharp\,(S_v)\,.$$

Since $u' \in C(\overline{\Sigma})$, $\dot v = v' - [v]\,d\,\sharp \llcorner \big(S_v \cap (0, L)\big)$ in $\mathcal{D}'(0, L)$, $u - w_0 \in H^1_0(0, L)$, $-u'' = f$ in $(0, L)$, $v(0_+) = u(0) + [v](0)$ and $v(L_-) = u(L) - [v](L)$ we get

$$\mathcal{F}_1(v) - \mathcal{F}_1(u) =$$

$$= \frac{1}{2}\int_0^L u'(v' - u') - \frac{1}{2}\int_0^L f(v-u) + \frac{\gamma}{2}\sum_{S_v} |[v]| - \frac{1}{2}\sum_{S_v \cap (0,L)} u'[v] + \alpha\sharp(S_v) =$$

$$= \alpha\sharp(S_v) \;+\; \frac{1}{2}\left(\sum_{S_v} \gamma|[v]| - \sum_{S_v \cap (0,L)} u'[v]\right) \;-\; \frac{1}{2}\,u'(0)\,[v](0) \;-\; \frac{1}{2}\,u'(L)\,[v](L) \;=$$

$$= \alpha\sharp(S_v) + \frac{1}{2}\sum_{S_v} \left(\gamma|[v]| - u'[v]\right).$$

The necessary conditions (3.9),(3.10) for minimizers with crack follow by substitution of $\sharp(S_v) \geq 1$ in (3.8). $\square$

We can restate the previous result in the form of a *calibration by comparison* as follows.

Theorem 3.5. *(***stress regularity condition for functional** $\mathcal{F}_1$*) If the solution u of* (3.6) *verifies*

$$\|u'\|_{L^\infty(\Sigma)} \;\leq\; \gamma \tag{3.12}$$

then $u \in \operatorname{argmin} \mathcal{F}_1$, u is the unique minimizer and, for all $v \in SBV(\mathbf{R})$ s.t. $\operatorname{spt}(v - w) \subset \overline{\Sigma}$,

$$\mathcal{F}_1(v) - \mathcal{F}_1(u) \;=\; \alpha\mathcal{H}^1(S_v) \;+\; \frac{1}{2}\sum_{S_v} \left(\gamma|[v]| - u' \cdot [v]\right) \;\geq\; 0\,.$$

Proof - Starting from the excess estimate (3.7) we find for any admissible v

$$\mathcal{F}_1(v) - \mathcal{F}_1(u) \geq \alpha \sharp (S_v) + \left(\sum_{S_v} \gamma |[v]| - u'[v] \right) \geq$$

$$= \alpha \sharp (S_v) + \left(\sum_{S_v} (\gamma - \|u'\|_\infty) |[v]| \right) \geq \alpha \sharp (S_v) \geq 0.$$

The last inequality is strict whenever $S_v \neq \emptyset$, since $\alpha > 0$. Hence the all the minimizer are regular. But there is only one regular minimizer, say u.

Theorem 3.6. *(***Load regularity condition for functional** $\mathcal{F}_1$*)*
Assume (3.2) *and*

$$\|f\|_{L^1(0,L)} \ + \ \frac{|w_0(L_+) - w_0(0_-)|}{L} \ \leq \ \gamma \tag{3.13}$$

then $\mathcal{F}_1$ achieves a unique regular minimizer: the solution u of (3.6), hence $S_u = \emptyset$.

Proof - By applying Lagrange Theorem and fundamental Theorem of calculus to u', where u is the minimizer among H^1 functions,

$$\|u'\|_{L^\infty(0,L)} \ \leq \ \|f\|_{L^1(0,L)} + \frac{|w_0(L_+) - w_0(0_-)|}{L} \ \leq \ \gamma$$

then Theorem 3.5 entails the thesis, since the regularity stress condition is fulfilled by u.

On the other hand we have a non-minimality test.

Theorem 3.7. *Assume* (3.2)*, u solves* (3.6)*, $v \in SBV(\mathbb{R})$ fulfills* $\mathrm{spt}(v - w_0) \subset \overline{\Sigma}$ *and the inequality*

$$\sum_{S_v} (\gamma |[v]| - u'[v]) \ > \ -2\alpha \sharp (S_v) \,. \tag{3.14}$$

Then $v \notin \operatorname{argmin} \mathcal{F}_1$.

Proof - Straightforward consequence of (3.10).

By summarizing: the safe load condition (3.3) entails existence, while the load regularity condition (3.13) entails existence, regularity and uniqueness of minimizer. Now we analyze what happen when the safe load condition (3.3) is fulfilled but the regularity load condition (3.13) fails.

Theorem 3.8. *Assume* (3.2)*, w is a minimizer of $\mathcal{F}_1$ with Dirichlet datum w_0, and*

$$\gamma \ < \ \|f\|_{L^1(0,L)} \ < \ 2\,\gamma \tag{3.15}$$

then there is at most one break-point, say $\sharp (S_w) \leq 1$.
The break point may be placed anywhere in $[0, L]$: (see Example 3.17 and Theorem 3.15).
Uniqueness of minimizer is not expected in general (in Example 3.17 are shown infinitely many solutions with exactly 1 crackpoint).

Proof - (Step I) if $w \in \operatorname{argmin} \mathcal{F}_1$ and $S_w \neq \emptyset$, then all jumps $[w]$ of w have the same sign. In fact, assuming by contradiction there are two jumps of different sign (we can always choose them consecutive) at $x_1, x_2 \in [0, L]$:

$$x_1 < x_2, \qquad [w](x_1)\,[w](x_2) \ < \ 0, \qquad w \in AC(x_1, x_2),$$

then the Euler equations in Theorem 3.3 imply

$$|\dot w_-(x_1)| = |\dot w_+(x_2)| = \gamma, \qquad w_+(x_1) = -\dot w_-(x_2)$$

would entail the following contradiction

$$2\gamma = |\dot w_-(x_2) - \dot w_+(x_1)| = \left|\int_{x_1}^{x_2} \ddot w \, dx\right| = \left|\int_{x_1}^{x_2} f\, dx\right| \le \int_{x_1}^{x_2} |f|\, dx = \|f\|_{L^1(\Sigma)} < 2\gamma.$$

(Step II) if $w \in \operatorname{argmin} \mathcal{F}_1$ and $S_w \neq \emptyset$, then there is exactly one jump point. In fact, assuming by contradiction w exhibits more than one, we know that they must have the same sign. By taking two consecutive jump points, one of the two can be eliminated as follows, by strictly reducing the functional energy at the same time: set

$$\widetilde w(x) = \begin{cases} w(x) \ + \ [w](x_2)\, \chi_{(x_1,x_2)}(x) & \text{if } \displaystyle\int_{x_1}^{x_2} f \ge 0,\ [w](x_1) > 0,\ [w](x_2) > 0 \\ w(x) \ - \ [w](x_1)\, \chi_{(x_1,x_2)}(x) & \text{if } \displaystyle\int_{x_1}^{x_2} f \ge 0,\ [w](x_1) < 0,\ [w](x_2) < 0 \\ w(x) \ + \ [w](x_2)\, \chi_{(x_1,x_2)}(x) & \text{if } \displaystyle\int_{x_1}^{x_2} f < 0,\ [w](x_1) < 0,\ [w](x_2) < 0 \\ w(x) \ - \ [w](x_1)\, \chi_{(x_1,x_2)}(x) & \text{if } \displaystyle\int_{x_1}^{x_2} f < 0,\ [w](x_1) > 0,\ [w](x_2) > 0 \end{cases}$$

then $x_2 \notin S_{\widetilde w}$ in first and third case, $x_1 \notin S_{\widetilde w}$ in second and fourth one. In any case

$$\mathcal{F}_1(\widetilde w) \ \le \ \mathcal{F}_1(w) - \alpha \ < \ \mathcal{F}_1(w)\ .$$

Remark 3.9. *We emphasize that* (3.15) *does not force minimizers to be discontinuous in any case. In fact: a constant load* $f = c\gamma/L$ *with* $1 < c < 2$ *fulfils* (3.15) *together with trivial Dirichlet condition* ($w_0 = 0$) *and leads to unique continuous minimizer :* $u(x) = \frac{c\gamma}{2L}\big((x - L/2)^2 - L^2/4\big)$*; notice that in this case* $\|u'\|_{L^\infty} = |u'(0)| = c\gamma/2$*, hence stress regularity condition* (3.12) *holds true (see also Theorem 3.13).*

Remark 3.10. *Statements analogous to the ones of Lemma 3.4 and Theorems 3.5, 3.6 hold true (with the same proof) when* $\int |\dot v|^2$ *is substituted by* $\int W(\dot v)dx$ *with* W *convex, proper and coercive with super-linear growth at* $\pm\infty$*.*
Precisely: if $\int_\Sigma |\dot v|^2 dx$ *is replaced in* $\mathcal{F}_1$ *by* $\int_\Sigma W(v)\, dx$ *where* W *is any strictly convex* C^2 *function with* $W(s) = W(|s|) \ge c_1 + c_2\,|s|^p,\ c_2 > 0,\ p > 1$*, by setting*

$$\mathcal{T}_1(v) \ = \ \int_{\mathbb{R}} (W(\dot v) - fw)\ dx \ + \ \alpha\, \sharp(S_v) \ + \ \gamma \sum_{S_v} \ |[v]|$$

we get the excess estimate (3.17) and identity (3.18) below (that are analogous to (3.7), (3.8) and are proven exactly in the same way).
Under assumptions (2.1), let u *be the solution of*

$$\text{(3.16)}\quad u \in H^1(\Sigma), \quad -\frac{d}{dx} W'(u') = f \ \text{ in } \Sigma, \quad u(0_+) = w_0(0_-),\ u(L_-) = w_0(L_+)$$

then $u' = \dot u \in C^0(\overline{\Sigma})$ *and* u *has an extension, still denoted by* u*, s.t.* $u \in SBV(\mathbf{R}) \cap C^0(\mathbb{R})$ *and* $u \equiv w_0$ *in* $\mathbb{R} \setminus \overline{\Sigma}$*. Then*
*(***Excess estimate for** $\mathcal{T}_1$*) For all* $v \in SBV(\mathbb{R})$ *with* $\operatorname{spt}(v - w_0) \subset \overline{\Sigma}$ *we have*

$$\text{(3.17)}\qquad \mathcal{T}_1(v) - \mathcal{T}_1(u) \ \ge \ \alpha\sharp(S_v) + \sum_{S_v} (\gamma|[v]| - W'(u')[v])\,.$$

(**Excess identity for** $\mathcal{T}_1$) *If v minimize $\mathcal{T}_1$ among $v \in SBV(\mathbb{R})$ s.t.* $\operatorname{spt}(v - w_0) \subset \overline{\Sigma}$, *then*

$$\mathcal{T}_1(v) - \mathcal{T}_1(u) \;=\; \alpha \sharp(S_v) + \frac{1}{2} \sum_{S_v} \left(\gamma |[v]| - W'(u')[v] \right). \tag{3.18}$$

(**Stress regularity condition for functional** $\mathcal{T}_1$) *If the solution u of* (3.16) *fulfils*

$$\| W'(u') \|_{L^\infty(\Sigma)} \;\leq\; \gamma \tag{3.19}$$

then $u \in \operatorname{argmin} \mathcal{T}_1$, u is the unique minimizer and, for all $v \in SBV(\mathbf{R})$ s.t. $\operatorname{spt}(v - w) \subset \overline{\Sigma}$,

$$\mathcal{T}_1(v) - \mathcal{T}_1(u) \;=\; \alpha \mathcal{H}^1(S_v) \;+\; \frac{1}{2} \sum_{S_v} \left(\gamma |[v]| - W'(u') \cdot [v] \right) \;\geq\; 0$$

(**Load regularity condition for** $\mathcal{T}_1$) *If*

$$W'\left(\|f\|_{L^1} + \frac{|w_0(L_+) - w_0(0_-)|}{L} \right) \;\leq\; \gamma \tag{3.20}$$

then $\mathcal{T}_1$ achieves a unique regular minimizer: the solution u of (3.16), *hence $S_u = \emptyset$.*

We prove a very helpful energy identity which simplifies the computations when testing wether an admissible function is a minimizer or when looking for examples of cracked minimizers.

Lemma 3.11. (**Compliance identity for** $\mathcal{F}_1$) *Assume w fulfils Euler equations* $(i), (ii)$ *and* (iii) *in theorem 3.3. Then*

$$\mathcal{F}_1(w) \;=\; -1/2 \int_0^L |\dot{w}|^2 \;+\; \alpha \,\sharp\,(S_v) \;-\; \dot{w}(0_+)\, w_0(0_-) \;+ \dot{w}(L_-)\, w_0(L_+).$$

Proof - By Euler equations $(\dot{w})' = -f$ in $\mathcal{D}'(0, L)$, then by taking into account the identity $w' = \dot{w} + \sum_{S_w} [w] \, d \,\sharp \llcorner S_w$ in $\mathcal{D}'(0, L)$, we get

$$\int_{\mathbb{R}} fw = \int_0^L fw = -\int_0^L (\dot{w})' w = \int_0^L (\dot{w}) w' + \; \dot{w}(0_+) w(0_+) - \dot{w}(L_-) w(L_-) \;=$$

$$= \int_0^L |\dot{w}|^2 dx + \; \dot{w}(0_+) w(0_+) - \dot{w}(L_-) w(L_-) + \sum_{S_w \cap (0,L)} \dot{w}[w].$$

By recalling $\dot{w} = \gamma \operatorname{sign}[w]$ in $S_w \cap (0, L)$, $\dot{w}(0_+) = \gamma \operatorname{sign}([w](0))$, $\dot{w}(L_-) = \gamma \operatorname{sign}([w](L))$, $w(0_+) = [w](0) + w_0(0_-)$, $w(L_-) = -[w](L) + w_0(L_+)$ and (by $S_{w_0} \subset [0, L]$) $S_w \subset [0, L]$ we get

$$\int_0^L fw = \int_0^L |\dot{w}|^2 + \gamma \sum_{S_w} |[w]| + \; \dot{w}(0_+) w_0(0_-) - \dot{w}(L_-) w_0(L_+)$$

and thesis follows by the definition of $\mathcal{F}_1$.

Lemma 3.12. *Let G be the Green function for the operator $+d^2/dx^2$ with homogeneous Dirichlet boundary condition in the open set $\Sigma = (0, L)$, say*
$G : (0, L) \times (0, L) \to \mathbb{R}$ *s.t* $G_{xx}(x, y) \;=\; \delta(x - y)$, $G(0, y) = G(L, y) = 0$. *Then*

$$G(x, y) \;=\; \begin{cases} \dfrac{(y - L)\, x}{L} & if\ 0 \leq x \leq y \leq L, . \\[2ex] \dfrac{(x - L)\, y}{L} & if\ 0 \leq y \leq x \leq L, \end{cases} \tag{3.21}$$

and, if $-v'' = f$ *on* Σ, *and* $v(0) = v(L) = 0$, *then*

$$v(x) = -\int_0^L G(x,y) f(y)\, dy \tag{3.22}$$

$$v'(x) = -\int_0^L G_x(x,y) f(y)\, dy = -\frac{1}{L}\int_0^L y f(y) dy + \int_x^L f(y)\, dy \tag{3.23}$$

By the representation formula (3.23) of Green function we deduce that there are no cracked minimizers if the load has vanishing resultant and/or moment (with respect to mid point $L/2$), say the presence of suitable symmetries weaken the regularity load condition (3.13).

Theorem 3.13. *Assume* (3.2),

$$\|f\|_{L^1} \;\le\; 2\,\gamma \;-\; 2\frac{|w_0(L_+) - w_0(0-)|}{L} \tag{3.24}$$

and either

$$\int_0^L f(y)\, dy = 0 \qquad \text{or} \qquad \int_0^L (y - L/2)\, f(y)\, dy = 0\,,$$

then $\mathcal{F}_1$ *has a unique regular minimizer: the solution* u *of* (3.6).
Notice that (3.24) *in general entails the safe load* (3.3) *(a weaker condition then the regularity load condition* (3.13) *), and* (3.24) *coincides with the safe load when* $w_0(L_+) = w_0(0-)$.

Proof - Let v be such that $-v'' = f$ in $(0, L)$ and $v(0_+) = v(L_-) = 0$, then, for $x \in (0, L)$,

$$u(x) = v(x) + w_0(0_-) + \frac{1}{L}(w_0(L_+) - w_0(0_-))\, x,$$
$$u'(x) = v'(x) + (w_0(L_+) - w_0(0_-))/L\,.$$

If $\int_0^L (y - L/2) f(y)\, dy = 0$ then by (3.23),(3.24) we get

$$\begin{aligned} |u'(x)| \quad &\le \left| -\frac{1}{2}\int_0^L f + \int_x^L f \right| + |w_0(L_+) - w_0(0_-)|/L \le \\ &\le \frac{1}{2}\|f\|_{L^1} + |w_0(L_+) - w_0(0_-)|/L \;\le\; \gamma \end{aligned}$$

and the stress regularity condition (3.12) is fulfilled and u minimizes $\mathcal{F}_1$.
If $\int_0^L f(y)\, dy = 0$ set $F(x) = \int_0^x f(y) dy$, then $F(0) = F(L) = 0$ and by (3.23)

$$v'(x) \;=\; -\frac{1}{L}\int_0^L y f(y) dy + \int_x^L f(y) dy = \frac{1}{L}\int_0^L F(y) dy - F(x)$$

hence, by taking into account (3.24) we get

$$\begin{aligned} |u'(x)| \;&\le\; |v'(x)| + |w_0(L_+) - w_0(0_-)|/L \;\le \\ \le &\left\| F - \frac{1}{L}\int_0^L F \right\|_{L^\infty} + |w_0(L_+) - w_0(0_-)|/L \;\le \\ &\frac{1}{2}\|f\|_{L^1} + |w_0(L_+) - w_0(0_-)|/L \;\le\; \gamma \end{aligned}$$

and the stress regularity condition (3.12) is fulfilled and u minimizes $\mathcal{F}_1$.
In the last inequality of the proof we used the following statement.

Lemma 3.14. *If $F \in AC(0,L)$ and $F(0)=F(L)=0$ then*

$$\left\| F - \fint_0^L F \right\|_{L^\infty(0,L)} \leq \frac{1}{2}\int_0^L |F'| \,.$$

Proof - Without loss of generality we can assume $\fint_0^L F \geq 0$; the other case can be dealt in the same way.
Then $A^0 = \{x \in (0,L) : \ 0 < F(x) < \fint_0^L F\}$ contains at least one pair of disjoint intervals E_l where $osc_{E_l}(F) > \fint_0^L F$, $l=1,2$, hence $|\fint_0^L F| < (1/2)\int_{A_0}|F'|$.
Set $A^+ = \{x \in (0,L) : \ F(x) > \fint_0^L F\}$, $A^- = \{x \in (0,L) : \ F(x) < 0\}$. Then $A^+ = \cup I_k$, $A^- = \cup J_h$, with I_k, J_h, E_l disjoint intervals and $F = \fint_0^L F$ on ∂I_k, $F=0$ on ∂J_h,

$$0 \leq \max_{I_k} F \leq \fint_0^L F + \frac{1}{2}\int_{I_k}|F'| \ \ \forall k\,, \qquad 0 \leq -\min_{J_h} F \leq \frac{1}{2}\int_{J_h}|F'| \ \ \forall l,$$

$$\left| F(x) - \fint_0^L F \right| \leq \fint_0^L F + \max_k \max_{I_k}|F| + \max_h \min_{J_h}|F| \leq \frac{1}{2}\int_0^L |F'|\,.$$

Now we show a general explicit method to construct cracked minimizers of $\mathcal{F}_1$ whenever the regularity stress condition (3.12) for functional $\mathcal{F}_1$ is violated

Theorem 3.15. Structure of $\mathcal{F}_1$ minimizers *Assume* (3.2),(3.3),(3.6) *and*

$$\|u'\|_{L^\infty(0,L)} \ > \ \gamma\ . \tag{3.25}$$

If

$$\alpha \ > \ \frac{L}{2}\big(\,\gamma - \|u'\|_{L^\infty}\,\big)^2 \tag{3.26}$$

hold true then

$$\operatorname*{argmin}_{\mathrm{spt}(\cdot - w_0)\subset[0,L]} \mathcal{F}_1 \ = \ \{u\}\,, \qquad \min_{\mathrm{spt}(v-w_0)\subset[0,L]} \mathcal{F}_1(v) \ = \ \mathcal{F}_1(u)\,. \tag{3.27}$$

If

$$\alpha \ < \ \frac{L}{2}\big(\,\gamma - \|u'\|_{L^\infty}\,\big)^2 \tag{3.28}$$

hold true, then

$$\min_{\mathrm{spt}(v-w_0)\subset[0,L]} \mathcal{F}_1(v) \ = \ \mathcal{F}_1(u) + \left(\alpha - \frac{L}{2}\big(\gamma - \|u'\|_{L^\infty}\big)^2\right) \tag{3.29}$$

and

$$\operatorname*{argmin}_{\mathrm{spt}(\cdot - w_0)\subset[0,L]} \mathcal{F}_1 \ = \ \{\, u + z_t \ : \ |u'(t)| = \|u'\|_{L^\infty} > \gamma\,\} \tag{3.30}$$

where the function $z_t \in SBV(\mathbf{R})$ is defined by

$$z_t(x) \ = \ \begin{cases} (\gamma \ \mathrm{sign}(u'(t)) - u'(t))\ x & 0 \leq x < t\,, \\ (\gamma \ \mathrm{sign}(u'(t)) - u'(t))\ (x-L) & t < x \leq L\,, \\ 0 & x \notin [0,L]\,. \end{cases} \tag{3.31}$$

More explicitly: (3.2),(3.3),(3.6), (3.25) *and* (3.28) *together imply that all the minimizers have exactly one crack. Both uniqueness and non uniqueness of minimizers are possible, depending on the cardinality of the set* $\{t \in [0,L] : \ |u'(t)| = \|u\|_{L^\infty} > \gamma\}$.

Eventually

$$\alpha \ = \ \frac{L}{2}\big(\, \gamma - \|u'\|_{L^\infty}\,\big)^2 \tag{3.32}$$

together with (3.2),(3.3),(3.6) *and* (3.25) *entail that both* u *and all the* $v_t = u + z_t$ *with* $|u'(t)| = \|u'\|_{L^\infty} = \gamma$ *are minimizers with* $\min\limits_{\mathrm{spt}(v-w_0)\subset[0,L]} \mathcal{F}_1(v) = \mathcal{F}_1(v_t) = \mathcal{F}_1(u)$. *The coexistence of continuous and cracked solutions is not a contradiction with the excess identity* (3.8) *since* $|u'(t)| > \gamma$ *say the regularity load condition* (3.13) *fails.*

Notice that (3.32) is the exact evaluation of the excess identity in this case, since $\sharp(S_{v_t}) = 1$ *and* $[v_t] = [z_t] = L(u'(t) - \gamma)$.

The weak inequality

$$\alpha \le \ \frac{L}{2}\big(\, \gamma - \|u'\|_{L^\infty}\,\big)^2 \tag{3.33}$$

is a necessary condition for the existence of minimizers with crack.

Proof - Define $J = \{\, t \in [0,L] \ : \ |u'(t)| > \gamma \,\}$ and, for every $t \in J$, the function $z_t \in SBV(\mathbf{R})$ as in (3.31).

Let S be the set of all $v \in SBV(\mathbf{R})$ that have at most one crack and fulfill the Euler conditions (i),(ii),(iii) of Theorem 3.3. We claim that

$$S \ = \ \{u + z_t : \ t \in J\} \ \cup \ \{u\}\,. \tag{3.34}$$

Indeed it is obvious that $u \in S$ and $u + z_t \in S$, $\forall t \in J$.

Conversely let $v \in S$, then either $v \equiv u$ or, by Theorem 3.8, $S_v = \{t\}$ for some $t \in [0,L]$. By Euler equations we get

$$\begin{cases} \dot v - u' \in AC(0,L), \quad (\dot v - u')' \ = \ 0 \ \text{ in (0,L)}, \qquad \text{hence:} \\ \dot v - u' \ \equiv \ \text{const} \ \equiv \ \dot v(t_\pm) - u'(t) \ = \ \gamma\,\mathrm{sign}([v])(t) - u'(t) \ \text{ in } (0,L) \\ S_{v-u} \ = \ S_v \ = \ \{t\} \end{cases} \tag{3.35}$$

and taking into account that $v = u = w_0$ in $\mathbf{R} \setminus [0,L]$ we get

$$v(x) - u(x) \ = \ \begin{cases} (\gamma\,\mathrm{sign}([v](t)) - u'(t))\ x & x \in [0,t) \\ (\gamma\,\mathrm{sign}([v](t)) - u'(t))\ (x-L) & x \in (t,L] \\ 0 & x \notin [0,L] \end{cases} \tag{3.36}$$

Since $[v](t) = [v-u](t) = -L\,(\gamma\,\mathrm{sign}([v])(t) - u'(t))$ we have

$$\begin{array}{c} [v](t) > 0 \ \Leftrightarrow \ u'(t) > \gamma > 0 \\ [v](t) < 0 \ \Leftrightarrow \ u'(t) < -\gamma < 0 \end{array} \tag{3.37}$$

hence $t \in J$ and $\mathrm{sign}(u'(t)) = \mathrm{sign}([v](t))$. Then

$$v(x) - u(x) \ = \ \begin{cases} (\gamma\,\mathrm{sign}(u'(t)) - u'(t))\ x & x \in [0,t) \\ (\gamma\,\mathrm{sign}(u'(t)) - u'(t))\ (x-L) & x \in (t,L] \\ 0 & x \notin [0,L] \end{cases} \tag{3.38}$$

say: $v(x) = u(x) + z_t(x)$ and

$$\min_{\mathrm{spt}(v-w_0)\subset[0,L]} \mathcal{F}_1 = \min_S \mathcal{F}_1 = \mathcal{F}_1(u) \bigwedge \left\{ \min_{t\in J} \mathcal{F}_1(u+z_t) \right\} \tag{3.39}$$

Now, by taking into account that, for all $t \in J$, $u + z_t$ satisfies the Euler equations, $\dot{z}_t = (\gamma \operatorname{sign}(u'(t)) - u'(t))$= constant in (0,L), by compliance identity (Theorem 3.11)

$$\begin{aligned}\mathcal{F}_1(u+z_t) &= \alpha - \frac{1}{2}\int_0^L |u' + \dot{z}_t|^2\,dx + (u'+\dot{z}_t)(L_-)w_0(L_+) - (u'+\dot{z}_t)(0_+)w_0(0_-)\\ &= \alpha - \frac{1}{2}\int_0^L |u'|^2\,dx + (u')(L_-)w_0(L_+) - (u')(0_+)w_0(0_-) - \frac{1}{2}\int_0^L |\dot{z}_t|^2\,dx\\ &= \mathcal{F}_1(u) + \alpha - \frac{L}{2}\big(\gamma \operatorname{sign}(u'(t)) - u'(t)\big)^2\\ &= \mathcal{F}_1(u) + \alpha - \frac{L}{2}\big(\gamma - |u'(t)|\big)^2\end{aligned}$$

hence (3.26) entails (3.27), and, if $|u'(t)| = \|u'\|_{L^\infty} > \gamma$, then both (3.28),(3.32) entail (3.29):

$$\begin{aligned}\min \mathcal{F}_1 &= \min_S \mathcal{F}_1(v) = \mathcal{F}_1(u) \wedge \min_{\tau\in J} \mathcal{F}_1(u+z_\tau) =\\ &= \mathcal{F}_1(u) \wedge \mathcal{F}_1(u+z_t) = \mathcal{F}_1(u) \wedge \Big(\mathcal{F}_1(u) + \alpha - \frac{L}{2}\big(\gamma - \|u'\|_{L^\infty}\big)^2\Big)\end{aligned} \tag{3.40}$$

Moreover, when $\alpha < \dfrac{L}{2}(\gamma - \|u'\|_{L^\infty})^2$ the above construction shows that $v \in \operatorname{argmin} \mathcal{F}_1$ if and only if: $t \in J$, $v = u + z_t$ and $|u'(t)| = \|u'\|_{L^\infty}$. Then (3.28) entail also (3.30).
If (3.26) holds true then u is the unique minimizer by (3.40).

With additional information on the load the location of cracks has less freedom.

Corollary 3.16. *Assume* (3.2)*,* (3.15)*,* (3.28) *and f does not change sign, then there are solutions with crack at the boundary. Interior cracks are not excluded in general. If in addition $f(x) \neq 0$ a.e.* $(0, L)$ *then the crack can be located only at the boundary and the solution is unique.*
In fact in this last case u' is strictly monotone and the statement follows by Theorem 3.15

We end this section by showing simple explicit examples of minimizers with cracks (coherent with structure Theorem 3.15), when the load regularity condition (3.13) fails. The first one with non-homogeneous Dirichlet datum and vanishing load, the second one with homogeneous Dirichlet datum and non trivial load. In the first one (Example 3.17) there are infinitely many cracked minimizers with one single discontinuity (which can be located anywhere in [0,L]). In the second one (Example 3.18) there is uniqueness of minimizer, and the discontinuity is at the boundary.

Example 3.17. Existence of infinitely many cracked minimizers of $\mathcal{F}_1$, all of them with one single crack, with null load and non-homogeneous Dirichlet datum.
Assume $f \equiv 0$, $w_0(x) = 0$ if $x < 0$ $w_0(x) = h > 0$ if $x > L$. Then $(\dot{z})' = \ddot{z} = 0$, $\ddot{z} = 0$ in $(0, L)$, for any $z \in \operatorname{argmin} \mathcal{F}_1$. Since S_z is at most a singleton then $z'' = 0$ in $(0, L) \setminus S_z$ say z is affine linear in $[0, L]$ and has at most one jump. If there is a minimizer z s.t $S_z \neq \emptyset$, then Euler conditions (Theorem 3.3) entail the graph of u has slope $\pm\gamma$: hence $z = \gamma x$ in $(0, L)$, and $\gamma \neq h/L$. By direct computation we get

$$\mathcal{F}_1(z) = \frac{1}{2} L\gamma^2 + \alpha + \gamma\,|h - \gamma L|\,,$$

$$\mathcal{F}_1(u) = \frac{1}{2}\frac{h^2}{L}\,,$$

where u is the continuous solution of (3.6)*, say $u(x) = (L/h)x$. By comparison of the two energies we get the complete description of minimizers as long as traction h at L increases:*

- *if* $h \le \gamma L$, *then the only solution is the continuous one,* $u(x) = (L/h)x$,
- *if* $\gamma L < h < 2\gamma L$ *and* $0 < \frac{1}{2L}(h-\gamma L)^2 < \alpha$, *again there is the unique solution* u,
- *if* $\gamma L < h < 2\gamma L$ *and* $0 < \alpha < \frac{1}{2L}(h-\gamma L)^2$, *then there are infinitely many solutions, all of them with a single crack-point and of the following type* z_t, *for* $t \in [0, L]$ *and*

$$z_t(x) \;=\; \begin{cases} 0 & \text{if } x < 0\,, \\ \gamma x + (h-\gamma L)\,\chi_{[t,L]}(x) & \text{if } 0 < x < L\,, \\ h & \text{if } L < x\,. \end{cases}$$

- *if* $\gamma L < h < 2\gamma L$ *and* $0 < \alpha = \frac{1}{2L}(h-\gamma L)^2$, *then the continuous solution* u *and all functions* z_t *above with a crack at* t *are minimizers.*

Eventually we notice that in this example (since $f \equiv 0$*) the safe load condition* (3.3) *is always (for any* h*) fulfilled, while the load regularity condition* (3.13) *reads* $h \le \gamma L$.

Example 3.18. - A cracked minimizer of $\mathcal{F}_1$ with non trivial f verifying the safe load (3.3) and homogeneous Dirichlet datum.
We choose the load $f(x) = 2cx$, $c > 0$, *s.t.* $3\gamma/2 < cL^2 < 2\gamma$, *and choose* $w_0 \equiv 0$.
In particular the safe load condition (3.3)*, which reads* $cL^2 \le 2\gamma$, *holds true while the regularity load condition* (3.13) *(which reads* $cL^2 \le \gamma$ *in this case) fails to be true.*
Then (by Euler equations, Theorem 3.15 and Corollary 3.16) the unique candidate regular minimizer and the unique candidate minimizer with crack for $\mathcal{F}_1$ *with support contained in* $[0, L]$ *are respectively* u *and* $u + z_L$, *say:*

$$u(x) \;=\; \frac{c}{3}\,x\,(L^2 - x^2)\,, \qquad x \in (0, L), \qquad u(x) = 0 \ \textit{elsewhere},$$

$$v(x) = (cL^2-\gamma)x - \frac{c}{3}x^3 = u(x) + \left(\frac{2}{3}cL^2 - \gamma\right)x,\ \ x \in (0, L),\ \ v(x) = 0\ \textit{elsewhere},$$

$$\|u'\|_\infty = |u'(L)| = \frac{2}{3}cL^2 > \gamma$$

and L *is the only point* t *where* $|u'(t)| = \|u'\|_{L^\infty}$,

$$S_v = \{L\}, \qquad [v](L) \;=\; -\,v(L_-) \;=\; \left(\gamma - \frac{2}{3}cL^2\right) L \;<\; 0, \qquad \dot v(L) = -\gamma$$

$v' \;=\; \dot v \;=\; (cL^2-\gamma) - cx^2$ *in* $(0, L)$ *and* $\dot v$ *vanishes at* $x = \sqrt{L^2 - \gamma/c} \in (0, L)$,
both v *and* u *are strictly concave in* $(0, L)$.
By using the compliance identity we evaluate the energy of both candidates

$$\mathcal{F}_1(u) = -\frac{2}{45}c^2L^5$$

$$\mathcal{F}_1(v) \;=\; \alpha \,-\, \frac{4}{15}c^2L^5 \,-\, \frac{\gamma^2 L}{2} \,+\, \frac{2}{3}c\gamma L^3\,.$$

By summarizing:

- *(i) if* $0 < \alpha < \dfrac{2}{9}c^2L^5 + \dfrac{\gamma^2 L}{2} - \dfrac{2}{3}c\gamma L^3 = \dfrac{L}{2}\left(\gamma - \dfrac{2}{3}cL^2\right)^2$
  then $\mathcal{F}_1(v) < \mathcal{F}_1(u)$ *and, by Theorem 3.15,* v *is the unique minimizer .*
- *(ii) if* $\alpha = \dfrac{L}{2}\left(\gamma - \dfrac{2}{3}cL^2\right)^2$
  then both v *and* u *are minimizers and, by Theorem 3.15, there are no more.*
- *(iii) if* $\alpha > \dfrac{L}{2}\left(\gamma - \dfrac{2}{3}cL^2\right)^2$
  then u *is the unique minimizer by Theorems 3.3, 3.15.*

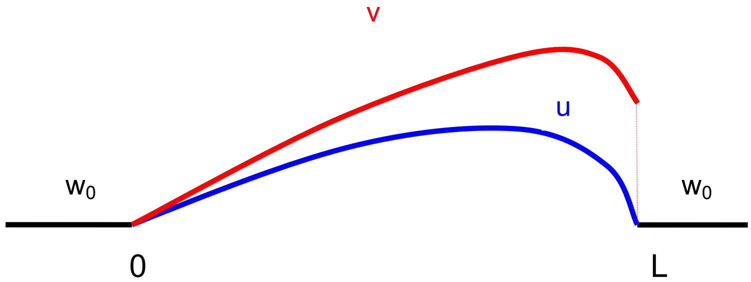

Figure - $\mathcal{F}_1$ minimizer v with damage under non trivial load: case *(i)* of Example 3.18, $\|u'\|_{L^\infty} > \gamma$.

4. **(Pb II) Elastic plastic beam under transverse load**

In this section we study the functional

$$\mathcal{F}_2(w) = \int_{\mathbb{R}} \left(\frac{1}{2}|\ddot{w}|^2 - fw\right) dx + \beta\, \sharp(S_{\dot{w}}) + \gamma \sum_{S_{\dot{w}}} |[\dot{w}]| \tag{4.1}$$

to be minimized among scalar functions w such that $w \in SBH(\mathbb{R})$ s.t. $\mathrm{spt}\, w \subset \overline{\Sigma}$. α, γ are given constants, $\sharp$ is the counting measure.
All along this section we assume

$$\beta > 0,\ \gamma > 0, \quad \Sigma = (0, L), \quad f \in \mathcal{M}(\mathbb{R}), \quad \mathrm{spt}\, f \subset \overline{\Sigma},\ \mathrm{spt}\, f^s \subset\subset \Sigma, \tag{4.2}$$

Functional (4.1) describes the total energy associated to deformation of an elastic-plastic beam which is clamped at both endpoints; w is the vertical displacement of the beam under the action of the transverse load f.

The crease points set $S_{\dot{w}}$ of a minimizer w may be interpreted as location of plastic hinges in the beam at equilibrium: functional (4.1) takes into account that the energy released in the deformation of an elastic plastic beam is the sum of elastic bending energy and of energy concentrated at plastic hinges. Jump points are not allowed (say $S_w = \emptyset$) for admissible displacements w which must be continuous since $SBH(\mathbb{R}) \subset C^0(\mathbb{R})$.

We introduce a localization of the functional: for any Borel set $A \subset \mathbb{R}$ we set

$$\mathcal{F}_2(w, A) = \int_A \left(\frac{1}{2}|\ddot{w}|^2 - fw\right) dx + \beta\, \sharp(S_{\dot{w}} \cap A) + \gamma \sum_{S_w \cap A} |[\dot{w}]|.$$

Lemma 4.1. *($L^\infty - BH$ Poincarè Inequality) Let $v \in BH(\mathbb{R})$ with $\operatorname{spt} v \subset [0, L]$. Then*

$$\|v\|_{L^\infty(0,L)} \;\le\; \frac{L}{8}\|v''\|_{T([0,L])}\,. \tag{4.3}$$

The equality in (4.3) holds true iff $v = r_s$ (roof-function), for some $s \in \mathbb{R}$:

$$r_s(x) \;=\; s\left(\frac{L}{2} - \left|x - \frac{L}{2}\right|\right)^+ . \tag{4.4}$$

Proof - Fix $v \in \mathcal{K}^* = \{v \in BH(\mathbf{R})\ s.t.\ \operatorname{spt} v \subset [0, L]\}$. Without loss of generality we assume $v \not\equiv 0$. Then define

$$\widetilde{v}(x) \;=\; \begin{cases} \text{convex envelope of } -|v| \text{ evaluated at } x & \text{if } x \in [0, L] \\ 0 & \text{if } x \notin [0, L]\,. \end{cases}$$

We claim that $\widetilde{v}$ fulfils

$$\begin{cases} \widetilde{v} \in BH(\mathbf{R})\,, \quad \operatorname{spt}\widetilde{v} \subset [0, L]\,, \quad v \le 0\,, \quad \widetilde{v} \text{ convex in } [0, L]\,, \\ \|\widetilde{v}\|_{L^\infty} \;=\; \|v\|_{L^\infty}\,, \qquad \|\widetilde{v}\,''\|_T \;\le\; \|v''\|_T\,. \end{cases} \tag{4.5}$$

Since [45] entail $\widetilde{v} \in BH(\mathbf{R})$, the only non trivial point in (4.5) is the estimate of total variation: $\|\widetilde{v}\,''\|_T \;\le\; \|v''\|_T$, which we prove below.
Set $\psi(s) = -|s|$, $z(x) = -|v(x)| = \psi \circ v$, so that $v \in BH(\mathbf{R})$, $\psi \in BH(\mathbf{R})$, ψ is Lipschitz and $\psi(0) = 0$. Hence, by Theorems 1 and 4 and Lemma 3.1 of [45], $-|v| = \psi \circ v$ belongs to $BH(\mathbf{R})$, and we can evaluate its second derivative by suitable chain-rule for superposition of BH functions (in the following $\operatorname{sign}(0) = 0$, $\operatorname{sign}(s) = s/|s|$, $s \neq 0$):

$$(-|v|)^{\ddot{}} \;=\; -\operatorname{sign}(v)\,\ddot{v} \tag{4.6}$$

$$((-|v|)'')^j \;=\; -\operatorname{sign}(v)\,(v'')^j \;-\; \sum_{t\,:\,v(t)=0} (\,|\dot{v}_+(t)| + |\dot{v}_-(t)|\,)\;\delta_t \tag{4.7}$$

$$((-|v|)'')^c \;=\; -\operatorname{sign}(v)\,(v'')^c \tag{4.8}$$

The three measures in (4.6)-(4.8) are mutually singular. Moreover the absolutely continuous (4.6) and Cantor part (4.8) obviously do not increase their total variation with respect to the corresponding part of v'', and the respective inequalities still hold true after taking the convex envelope:

$$\|(\widetilde{v}\,'')^{\ddot{}}\|_T \;\le\; \|\ddot{v}\|_T\,,$$
$$\|(\widetilde{v}\,'')^c\|_T \;\le\; \|(v)^c\|_T\,.$$

On the other hand, total variation of (4.7) could be bigger than total variation of $(v'')^j$ due to sign changes of v. Nevertheless, since $\widetilde{v}$ is strictly negative in $(0, L)$, the terms $(\,|\dot{v}_+(t)| + |\dot{v}_-(t)|\,)\;\delta_t$ disappear in the convex envelope for any $t \neq 0$ and any $t \neq L$. So

$$\|(\widetilde{v}\,'')^j\|_T \;\le\; \|(v'')^j\|_T \,+\, |\dot{v}_+(0)| \,+\, |\dot{v}_-(L)|\,,$$
$$\|\widetilde{v}\,''\|_{T(J)} \;\le\; \|v''\|_{T(J)} \qquad \forall \text{ open interval } J \subset\subset (0, L)\,.$$

In order to keep under control the total variation at the boundary of the interval we set $z(x) = -|v(x)|$ and we observe that, either

- $\dot{\widetilde{v}}_+(0) = \dot{z}_+(0)$, hence $\|\widetilde{v}''\|_{T(\{0\})} = \|v''\|_{T(\{0\})}$, $\|\widetilde{v}''\|_{T([0,L))} \le \|v''\|_{T([0,L))}$;

or

- $\dot{\widetilde{v}}_+(0) \neq \dot{z}_+(0)$, hence $\dot{\widetilde{v}}_+(0) < \dot{z}_+(0)$, $\widetilde{v}$ is strictly less than z in an open interval $(0, \bar{x})$

(where $\bar{x}$ is chosen such that the interval is the maximal one fulfilling this property), so $\widetilde{v}(\bar{x}) = z(\bar{x})$; then by convexity $\dot{z}_-(\bar{x}) \leq \dot{\widetilde{v}}_-(\bar{x}) \leq \dot{\widetilde{v}}_+(\bar{x}) \leq \dot{z}_+(\bar{x})$, $0 \leq [\dot{\widetilde{v}}](\bar{x}) \leq [\dot{z}](\bar{x})$ so that

$$\|\widetilde{v}\,''\|_{T(\{\bar{x}\})} \leq \|v''\|_{T(\{\bar{x}\})}\,,$$

moreover, by taking into account that $\dot{z}_\pm(\bar{x}) = -\operatorname{sign}(v(\bar{x}))\,\dot{v}_\pm$, $\operatorname{spt} v'' \subset [0,L]$ and $[\dot{\widetilde{v}}](0)$ is the slope of $\widetilde{v}$ in the interval $(0,\bar{x})$ we deduce

$$0 > -|\widetilde{v}\,''|_{T(\{0\})} = [\dot{\widetilde{v}}](0) = \dot{\widetilde{v}}_+(0) = \dot{\widetilde{v}}_-(\bar{x}) \geq \dot{z}_-(\bar{x}) = -\operatorname{sign}(v(\bar{x}))\,\dot{v}_-(\bar{x}) =$$

$$= -\operatorname{sign}(v(\bar{x}))\,(v'')\big([0,\bar{x})\big) \geq -|v''|\big([0,\bar{x})\big)\,,$$

and since $\widetilde{v}$ is affine linear in $(0,\bar{x})$

$$\|\widetilde{v}\,''\|_{T([0,\bar{x}))} < \|v''\|_{T([0,\bar{x}))}\,.$$

The behavior around L can be dealt exactly as the one around 0, so we achieve the inequality $\|\widetilde{v}''\|_{T([0,L])} < \|v''\|_{T([0,L])}$ involving total variations in second case too.
Then claim (4.5) is proven in any case. By (4.5) we get

$$\inf\left\{\frac{\|v''\|_T}{\|v\|_{L^\infty}} : v \in \mathcal{K}^*\right\} \;=\; \inf\left\{\frac{\|v''\|_T}{\|v\|_{L^\infty}} : v \in \mathcal{K}^*,\; v \text{ convex in } [0,L]\right\}. \tag{4.9}$$

If we take $v \in \mathcal{K}^*$, v convex in $[0,L]$ and $v \not\equiv 0$, then

$$-\infty < v'_+(0) \leq 0\,, \qquad 0 \leq v'_-(L) < +\infty$$

and we can define

$$\check{v}(x) \;=\; (v'_+(0)x) \vee \big((v'_-(L)(x-L)\big) \;\;\text{ if } x \in [0,L] \qquad \text{and } \;\check{v}(x) \equiv 0 \;\; \text{otherwise}\,.$$

Then $\check{v} \leq v$, $\|\check{v}\|_{L^\infty} \geq \|v\|_{L^\infty}$ and $\|\check{v}\,''\|_{T(\mathbb{R})} = 2(v_-(L) - v'_+(0)) = \|v''\|_{T([0,L])}$ by convexity.

$$\begin{aligned}
&\inf\,\{\,\|v''\|_T\,/\,\|v\|_{L^\infty} \,:\, v \in SBH,\ \operatorname{spt} v \subset [0,L],\ v \text{ convex in } [0,L]\,\} \geq \\
&\geq \inf\,\{\,\|v''\|_T\,/\,\|v\|_{L^\infty} \,:\, v(x) = (-ax) \vee (b(x-L)),\ a>0,\ b>0\,\} = \\
&= \min_{a>0,\,b>0} \frac{2(a+b)^2}{abL} \;=\; 8/L
\end{aligned} \tag{4.10}$$

Actually the infimum in (4.10) is a minimum since it is achieved at $a = b$ say when v is a roof function. By summarizing (4.9),(4.10) prove (4.3). About the fact that only roof functions (4.4) achieve the equality in (4.3) we emphasize that: the map $v \to \widetilde{v}$ strictly reduces the relevant quotient $\|v''\|_T\,/\,\|v\|_{L^\infty}$ whenever $|v| \not\equiv |\widetilde{v}|$, since in such case $\|(\cdot)''\|_T$ strictly decreases, while also the map $v \to \check{v}$ strictly reduces the relevant quotient for v convex in $[0,L]$ and $|v| \not\equiv |\check{v}|$, since in such case $\|\cdot\|_{L^\infty}$ strictly increases. □
For a different proof of (4.3) see [37].
Now we can prove that a smallness condition (safe load condition) on f entails existence of minimizers (for any boundary datum), while a violation of the safe load may lead to collapse.

Lemma 4.2. *Assume* (4.2), $w_0 \in SBH(\mathbb{R})$ *with* $\mathcal{F}_2(w_0) < +\infty$ *and*

$$\|f\|_{T(\overline{\Sigma})} \;<\; \frac{8\,\gamma}{L} \qquad (\mathcal{F}_2 \textbf{ safe load condition}) \tag{4.11}$$

then $\mathcal{F}_2$ *achieves a finite minimum among* $w \in SBH(\mathbb{R}^2)$ *with* $\mathrm{spt}(w - w_0) \subset \overline{\Sigma}$ *and any minimizer* z *fulfils*

$$\|z''\|_{T(\overline{\Sigma})} \le \frac{1}{\gamma - \frac{L}{8}\|f\|_{T(\overline{\Sigma})}}\left(\frac{1}{2}L\gamma^2 + \mathcal{F}_2(w_0, \overline{\Sigma}) + \left|\int_{\mathbb{R}} f w_0\, dx\right| + \frac{L}{8}\|f\|_{T(\overline{\Sigma})}\|w_0''\|_{T(\overline{\Sigma})}\right) \tag{4.12}$$

Proof - We use the direct method. First we show that $\mathcal{F}_1$ is coercive: by Lemma 4.1 ,

$$\|z\|_{L^\infty} \le \frac{L}{8}|z''|_{T(\overline{\Sigma})} \qquad \forall z \in SBH(\mathbb{R})\ s.t.\ \mathrm{spt} \subset \overline{\Sigma} \tag{4.13}$$

$$\begin{aligned}\left|\int_\Sigma f w\, dx\right| &\le \left|\int_0^L f(w - w_0)\, dx\right| + \left|\int_0^L f w_0\, dx\right| \le \\ &\le \|f\|_{T(\overline{\Sigma})}\, \|w - w_0\|_{L^\infty} + \left|\int_0^L f w_0\, dx\right| \le \\ &\le \frac{L}{8}\|f\|_{T(\overline{\Sigma})}\left(\|w''\|_{T(\overline{\Sigma})} + \|w_0''\|_{T(\overline{\Sigma})}\right) + \left|\int_\Sigma f w_0\, dx\right| = \\ &= \frac{L}{8}\|f\|_{T(\overline{\Sigma})}\left(\sum_{S_{\dot w}}[\dot w] + \int_0^L |\ddot w|\, dx + \|w_0''\|_{T(\overline{\Sigma})}\right) + \left|\int_0^L f w_0\, dx\right| \\ &\qquad \forall\, w \text{ s.t. } \mathrm{spt}(w - w_0) \subset [0, L],\end{aligned} \tag{4.14}$$

and by Young inequality

$$\frac{1}{2}\int_0^L |\ddot w|^2\, dx \ \ge\ \gamma \int_0^L |\ddot w|\, dx - \frac{L}{2}\gamma^2$$

hence, for w in a minimizing sequence for $\mathcal{F}_2$, we have ultimately

$$\begin{aligned}&\mathcal{F}_2(w_0, \overline{\Sigma}) + \mathcal{F}_2(w_0, \mathbb{R}\backslash\overline{\Sigma}) = \mathcal{F}_2(w_0) \ge \\ &\quad \ge \mathcal{F}_2(w) = \mathcal{F}_2(w, \overline{\Sigma}) + \mathcal{F}_2(w_0, \mathbb{R}\backslash\overline{\Sigma}) \ge \\ &\quad \ge \beta\sharp(S_{\dot w}) + \left(\gamma - \frac{L}{8}\|f\|_{T(\overline{\Sigma})}\right)\|w''\|_{T(\overline{\Sigma})} + \\ &\qquad -\frac{L}{2}\gamma^2 - \left|\int_{\mathbb{R}} f w_0\, dx\right| - \frac{L}{8}\|f\|_{T(\overline{\Sigma})}\|w_0''\|_{T(\overline{\Sigma})} + \mathcal{F}_2(w_0, \mathbb{R} \setminus \overline{\Sigma}).\end{aligned}$$

Hence by (4.11) the functional is bounded from below and (4.12) holds true. The existence of minimizers for $\mathcal{F}_2$ follows by sequential compactness of minimizing sequences and BH^* sequential lower semicontinuity of $\mathcal{F}_2$ ([38],[10],[11]). Moreover, by cancellation on both sides of $\mathcal{F}_2(w_0, \mathbb{R} \setminus \overline{\Sigma})$ we get (4.12) holds true, for any $z \in \mathrm{argmin}\, \mathcal{F}_2$. □

For sake of simplicity we study only the homogeneous case ($w_0 \equiv 0$) in the following.

The safe load (4.11) cannot improved for generic $\mathcal{M}$ or L^1 load as shown by the following Remark. Nevertheless for L^∞ load we refer to [42] where we prove a safe load condition which turns out to be less stringent on uniform load.

Remark 4.3. *There are examples with* $\|f\|_{T(\overline{\Sigma})} > 8\,\gamma/L$, *s.t.* $\inf \mathcal{F}_2 = -\infty$.
For instance, choose $f = \left(8\,\frac{\gamma}{L} + \varepsilon\right)\delta_{L/2}(x),\ \varepsilon > 0,$ *set* $w_t(x) = t\left(\frac{L}{2} - \left|x - \frac{L}{2}\right|\right)^+$. *Then* $J_{\dot w_t} = \{0, L/2, L\},\ \ddot w_t \equiv 0,\ \langle f\, w_t\rangle = t\left(4\gamma + \dfrac{L}{2}\varepsilon\right)$ *and* $\mathcal{F}_2(w_t) = 3\beta - \varepsilon\frac{L}{2}t \to -\infty$ *as* $t \to +\infty$.

Theorem 4.4. *($\mathcal{F}_2$ **Euler equations**) Assume* (4.1),(4.2) *and w minimizes $\mathcal{F}_2$ among v belongs to $SBH(\mathbb{R})$ s.t.* $\operatorname{spt} v \subset \overline{\Sigma}$.
Then $\dddot{w} = (\ddot{w})'' = f \in \mathcal{M}$ in $(0,L)$, $\dddot{w} = (\ddot{w})'$ belongs to $AC(I)$ for any interval $I \subset \Sigma \setminus S_{\dot{w}}$, so that $\dot{w}_\pm(x) = \dot{w}(x_\pm)$ $\ddot{w}_\pm(x) = \ddot{w}(x_\pm)$, $\dddot{w}_\pm(x) = \dddot{w}(x_\pm)$ are defined for all $x \in \overline{\Sigma}$ and

$$\begin{array}{llcll} (i) & w'''' & = & f & (0,L)\setminus S_{\dot{w}} \\ (ii) & \ddot{w}_- & = & \gamma\,\mathrm{sign}([\dot{w}]) & \text{in } S_{\dot{w}}\cap(0,L] \\ (iii) & \ddot{w}_+ & = & \gamma\,\mathrm{sign}([\dot{w}]) & \text{in } S_{\dot{w}}\cap[0,L) \\ (iv) & \dddot{w}_- & = & \dddot{w}_+ & \text{in } (0,L). \end{array}$$

$$(v)\qquad \int_0^L \big(\ddot{w}\,(\ddot{z}-\ddot{w})-f(z-w)\big)dx \;=\; \gamma \sum_{S_{\dot{z}-\dot{w}}} |[\dot{z}-\dot{w}]| \quad \forall z \in SBH(\mathbb{R}):\ \operatorname{spt}(z-w)\subset\overline{\Sigma}.$$

In particular $\ddot{w} \in BH(0,L)$, hence $\ddot{w}$ and $\dddot{w} = (\ddot{w})'$ are continuous in $(0,L)$ but may be discontinuous at 0 and L, even if these points do not belong to $S_{\dot{w}}$.

Proof - The inequality $\mathcal{F}_2(v) \le \mathcal{F}_2(v+\varepsilon\varphi)$ holds true for any $\varepsilon \in \mathbf{R}$ and $\varphi \in SBH(\mathbb{R})$. We choose φ with $\operatorname{spt}\varphi \subset \overline{\Sigma}$ and $\varphi \in C^\infty(\overline{J})$ on the closure of every interval $J \subset \overline{\Sigma}\setminus S_{\dot{w}}$, so that $S_{\dot{\varphi}} \subset S_{\dot{w}}$, $S_\varphi = \emptyset$, $\ddot{\varphi} = \varphi'' - [\dot{\varphi}]\sharp\llcorner S_{\dot{\varphi}}$ in $\mathcal{D}'(\mathbb{R})$.
Then taking into account that $w \in SBH$ entails $\ddot{w} = w''$ in $(0,L)\setminus S_{\dot{w}}$, by convexity property we get, for any ε s.t. $0 < \varepsilon < \min_{S_{\dot{w}}} [\dot{w}]/\|\dot{\varphi}\|_{L^\infty}$.

$$\begin{aligned} 0 \le \varepsilon &\int_{\mathbb{R}} (\ddot{w}\ddot{\varphi} - f\varphi)\,dx + \beta\big(\sharp(S_{\dot{w}+\varepsilon\dot{\varphi}}) - \sharp(S_{\dot{w}})\big) + \gamma\sum_{S_{\dot{w}}}(|[\dot{w}+\varepsilon\varphi']| - |[\dot{w}]|) + o(\varepsilon) = \\ &= \varepsilon\left(\int_{\Sigma\setminus S_{\dot{w}}} (-w'''\varphi' - f\varphi)\,dx + (\varphi'(L_-)\ddot{w}(L_-) - (\varphi'(0_+)\ddot{w}(0_+)) + \right. \\ &\qquad \left. + \sum_{S_{\dot{w}}\cap(0,L)} \big((\varphi'_-\ddot{w}_-) - (\varphi'_+\ddot{w}_+)\big) + \gamma\sum_{S_{\dot{w}}}(\varphi'_+ - \varphi'_-)\,\mathrm{sign}([\dot{w}])\right) + o(\varepsilon) \\ &= \varepsilon\left\{\int_{\Sigma\setminus S_{\dot{w}}} (w'''' - f)\varphi\,dx + \right. \\ &+ \left((\varphi'(L_-)\ddot{w}(L_-) - \varphi'(0_+)\ddot{w}(0_+)) + \sum_{S_{\dot{w}}\cap(0,L)} \big((\varphi'_-\ddot{w}_-) - (\varphi'_+\ddot{w}_+)\big) + \right. \\ &\qquad\qquad \left. + \gamma\sum_{S_{\dot{w}}}[\varphi']\,\mathrm{sign}([\dot{w}])\right) + \\ &+ \left.\left((\varphi(L_-)\dddot{w}(L_-) - (\varphi(0_+)\dddot{w}(0_+)) + \sum_{S_{\dot{w}}\cap(0,L)} ((\varphi_-\dddot{w}_-) - (\varphi_+\dddot{w}_+))\right)\right\} + o(\varepsilon). \end{aligned}$$

By choosing all φ with compact support in an interval contained in $(0,L)\setminus S_{\dot{w}}$ we get (i), hence $\dddot{w} \in AC(J)$ for all interval $J \subset \overline{\Sigma}\setminus S_{Dw}$ say $\dddot{w}_\pm$ is defined .
Then for any fixed $x_k \in S_{\dot{w}}$ we can choose at first (if $x_k < L$) all φ with compact support in $[x_k, x_{k+1})$ where x_{k+1} is the closest singular point bigger than x_k if any or L else, and then (if $0 < x_k$) all φ with compact support in $[x_k - 1, x_k)$ where x_{k-1} is the closest singular point smaller than x_k if any or 0 else. Both φ and φ' can be chosen independently on the singular set. Hence the four identities in $S_w \cup \{0,L\}$ follow. The last statement summarizes identities $(i)-(iv)$.

Du Bois-Raymond identity (v) is achieved by starting from minimality of w with respect to variations $w+\varepsilon(z-w)$ and repeating the above computation. $\square$

Lemma 4.5. *Assume* (4.1),(4.2). *Let u be the unique solution of*

$$\begin{cases} u \in H^2_0(0,L) \\ u'''' = f \qquad \text{in } (0,L)\,. \end{cases} \tag{4.15}$$

Then $u''' = \dddot{u} \in C^0(\overline{\Sigma})$ and there is a unique extension still denoted by u s.t. $u \in SBH(\mathbf{R})$ and $u \equiv 0$ in $\mathbb{R}\setminus\Sigma$.
Moreover
Excess estimate for $\mathcal{F}_2$ *: If u solves* (4.15) *then for all $v \in SBH(\mathbb{R})$ s.t.* $\operatorname{spt} v \subset \overline{\Sigma}$

$$\mathcal{F}_2(v) - \mathcal{F}_2(u) \;\geq\; \beta\,\sharp\,(S_{\dot v}) + \left(\sum_{S_{\dot v}} \gamma\,|[\dot v]| - u''\,[\dot v]\right). \tag{4.16}$$

Excess identity for minimizers of $\mathcal{F}_2$ *: If v minimize $\mathcal{F}_2$ among $v \in SBH(\mathbb{R})$, s.t.* $\operatorname{spt} v \subset \overline{\Sigma}$ *and u solves* (4.15) *then*

$$\mathcal{F}_2(v) - \mathcal{F}_2(u) \;=\; \beta\,\sharp\,(S_{\dot v}) + \frac{1}{2}\left(\sum_{S_{\dot v}} \gamma\,|[\dot v]| - u''\,[\dot v]\right). \tag{4.17}$$

Necessary conditions for existence of creased minimizers of $\mathcal{F}_2$ *: If v minimize $\mathcal{F}_2$ among $v \in SBH(\mathbb{R})$ s.t.* $\operatorname{spt} v \subset \overline{\Sigma}$, $S_{\dot v} \neq \emptyset$, *and u solves* (4.15), *then*

$$\|u''\|_{L^\infty(\Sigma)} > \gamma\,, \tag{4.18}$$

$$\sum_{S_{\dot v}} [\dot v]\;(\gamma\ \mathrm{sign}[\dot v] - u'') = \sum_{S_{\dot v}} (\gamma\,|[\dot v]| - u''\,[\dot v]) \leq -2\beta\,\sharp\,(S_{\dot v}) < 0\,, \tag{4.19}$$

$$\beta \;\leq\; \frac{1}{2}\,L\,\gamma^2\,. \tag{4.20}$$

By (4.19), *if $S_{\dot v}$ consists exactly in one point $\overline{x}$ then $|u''(\overline{x})| > \gamma$.*

Proof - u is the only minimizer of $\mathcal{F}_2$ over $w \in SBH(\mathbb{R}) \cap H^2(\Sigma)$ s.t. $\operatorname{spt} w \subset \overline{\Sigma}$, hence $w \in H^2_0(\Sigma)$.
By exploiting $u'' \in C(\overline{\Sigma})$, $\ddot v = v'' - [\dot v]\, d\sharp\llcorner S_{\dot v} \cap (0,L)$ in $\mathcal{D}'(0,L)$, $u \in H^2_0(\Sigma)$, $u'''' = f$ in Σ and $u = 0$ on $\mathbb{R}\setminus\Sigma$, convexity of $s \to s^2/2$, and

$$\int_0^L u''(v-u)''\,dx \;=\; \int_0^L u''''(v-u)\,dx \;-u''(L)\,[\dot v](L) \;-\; u''(0)\,[\dot v](0)$$

we have, for every $v \in SBH(\mathbb{R})$ s.t. $\operatorname{spt} v \subset \overline{\Sigma}$,

$$\mathcal{F}_2(v) \;\geq\; \mathcal{F}_2(u) + \int_0^L u''(\ddot v - u'')dx - \int_0^L f(v-u)dx + \beta\sharp(S_{\dot v}) + \gamma \sum_{S_{\dot v}} |[\dot v]| =$$

$$= \mathcal{F}_2(u) + \int_0^L u''(v''-u'')dx - \int_0^L f(v-u)dx + \beta\sharp(S_{\dot v}) + \gamma\sum_{S_{\dot v}}|[\dot v]| - \sum_{S_{\dot v}\cap(0,L)} u''[\dot v] =$$

$$= \mathcal{F}_2(u) + \beta\sharp(S_{\dot v}) + \sum_{S_{\dot v}} (\gamma|[\dot v]| - u''[\dot v])$$

Then (4.16) is proved.

If $v \in \operatorname{argmin} \mathcal{F}_2$ and u solves (4.15), then $\ddot{v}$ is continuous in $(0,L)$ by Theorem 4.4 and $u = v$, and $u'' = v'' = \ddot{v}$ hold true in $\mathbb{R} \setminus \overline{\Sigma}$ while Du-Bois Raymond equation (v) relative to variations $v_\varepsilon = v + \varepsilon(u - v)$ yields

$$\int_\Sigma \left(\ddot{v}\,(u'' - \ddot{v}) - f(u - v)\right)\,dx - \gamma \sum_{S_{\dot v}} |[\dot{v}]| \,=\, 0\,. \tag{4.21}$$

Hence

$$\mathcal{F}_2(v) - \mathcal{F}_2(u) = \frac{1}{2}\int_\Sigma |\ddot{v}|^2\,dx + \beta\,\sharp(S_{\dot v}) + \gamma \sum_{S_{\dot v}} |[\dot{v}]| - \int_\Sigma f v\,dx - \frac{1}{2}\int_\Sigma |u''|^2 + \int_\Sigma f u =$$

$$= \frac{1}{2}\int_\Sigma (\ddot{v} + u'')\,(\ddot{v} - u'') + \beta\,\sharp(S_{\dot v}) + \gamma \sum_{S_{\dot v}} |[\dot{v}]| - \int_\Sigma f v\,dx + \int_\Sigma f u =$$

$$= \frac{1}{2}\int_\Sigma f(v-u) - \int_\Sigma f(v-u) - \frac{\gamma}{2}\sum_{S_{\dot v}} |[\dot{v}]| + \gamma \sum_{S_{\dot v}} |[\dot{v}]| + \frac{1}{2}\int_\Sigma u''(\ddot{v} - u'') + \beta\,\sharp(S_{\dot v}) =$$

$$= \frac{\gamma}{2}\sum_{S_{\dot v}} |[\dot{v}]| - \frac{1}{2}\int_\Sigma f(v - u) + \frac{1}{2}\int_\Sigma u''(\ddot{v} - u'') + \beta\,\sharp\,(S_{\dot v})\,.$$

Since $u'' \in C(\overline{\Sigma})$, $\ddot{v} \,=\, v'' \,-\, [\dot{v}]\,d\,\sharp \llcorner \big(S_{\dot v} \cap (0,L)\big)$ in $\mathcal{D}'(0,L)$, $u \in H^2_0(\Sigma)$, $u'''' = f$ in Σ and $u = v$, on $\partial\Sigma$, $\dot{v}_+(0) - u'(0) = [\dot{v}(0)]$, $\dot{v}_-(L) - u'(L) = -[\dot{v}(L)]$, we get

$$\mathcal{F}_2(v) - \mathcal{F}_2(u) =$$

$$= \frac{1}{2}\int_\Sigma u''(v'' - u'') - \frac{1}{2}\int_{S_{\dot v}} f(v - u) + \frac{\gamma}{2}\sum_{S_{\dot v}} |[\dot{v}]| - \frac{1}{2}\sum_{S_{\dot v} \cap (0,L)} u''[\dot{v}] + \beta\sharp(S_{\dot v}) =$$

$$= \beta\,\sharp(S_{\dot v}) \,+\, \frac{1}{2}\left(\sum_{S_{\dot v}} \gamma|[\dot{v}]| - \sum_{S_{\dot v} \cap (0,L)} u''[\dot{v}]\right) \,-\, \frac{1}{2}\,u''(0)\,[\dot{v}](0) \,-\, \frac{1}{2}\,u''(L)\,[\dot{v}](L)$$

$$= \beta\,\sharp(S_{\dot v}) \,+\, \frac{1}{2}\sum_{S_{\dot v}} \left(\gamma|[\dot{v}]| - u''[\dot{v}]\right).$$

The necessary conditions for creased minimizers (4.19),(4.18) follows by substituting $\sharp(S_v) \geq 1$ in (4.17).

By $\mathcal{F}_2(v) \leq \mathcal{F}_2(0) = 0\,,$ $L^\infty - BV$ Poincaré inequality (4.3), safe load (4.11) and Young inequality

$$\begin{aligned} &\frac{1}{2}\int_0^L |\ddot{v}|^2 dx + \gamma \sum_{S_{\dot v}} |[\dot{v}]| + \beta\sharp(S_{\dot v}) \leq \int_0^L f v dx \leq \|f\|_{L^1}\|v\|_{L^\infty} \leq \\ &\leq \frac{L}{8}\|f\|_{L^1}\left(\int_0^L |\ddot{v}| + \sum_{S_{\dot v}} |[\dot{v}]|\right) \leq \gamma\left(\int_0^L |\ddot{v}| + \sum_{S_{\dot v}} |[\dot{v}]|\right) \leq \\ &\leq \frac{1}{2}\int_0^L |\ddot{v}|^2 dx + \frac{1}{2}L\gamma^2 dx + \gamma \sum_{S_{\dot v}} |[\dot{v}]| \end{aligned} \tag{4.22}$$

if $\sharp(S_{\dot v}) \geq 1$ then $\beta \,\leq\, \beta\,\sharp(S_{\dot v}) \,\leq\, \frac{1}{2}\,L\,\gamma^2$ say (4.20). $\square$

Theorem 4.6. *(*L^∞ **bending moment regularity condition for clamped beam***)* *Assume* (4.1),(4.2) *and the unique solution* u *of* (4.15) *fulfils*

$$\|u''\|_{L^\infty(0,L)} \,\leq\, \gamma\,. \tag{4.23}$$

Then u *is also a minimizer of* $\mathcal{F}_2$. *Moreover* u *is the unique minimizer of* $\mathcal{F}_2$.

Proof - By excess estimate (4.16) and (4.23) we get

$$\mathcal{F}(v) \ \geq \ \mathcal{F}(u) + \beta\sharp(S_{\dot v}) + \sum_{S_{\dot v}} (\gamma|[\dot v]| - u''[\dot v]) \geq \mathcal{F}(u) + \beta\sharp(S_{\dot v}) \geq \mathcal{F}(u)\,, \tag{4.24}$$

hence u is a minimizer. If in addition $\sharp(S_{\dot v}) > 0$ then the last inequality is strict, so that no minimizer can have creases. ■

Lemma 4.7. *(***Green representation formulae***) Assume u solves* (4.15)*. Then*

$$u''(x) \ = \ \int_0^L K(x,y)\, f(y)\, dy \tag{4.25}$$

where

$$K(x,y) = \frac{1}{2\,L^3}(2\,x - L)\ y^2\,(3L-2y) \ - \ \frac{1}{2\,L}\,y^2 \ + \ (y-x)^+\,. \tag{4.26}$$

Moreover

$$\max_{x,y\in[0,L]} |K(x,y)| \ = \ \frac{4}{27}\,L\,, \tag{4.27}$$

hence

$$\|u''\|_{L^\infty} \ \leq \ \frac{4}{27}\,L\,\|f\|_{T(\overline{\Sigma})} \tag{4.28}$$

and the equality in (4.28) *can be achieved, hence the constant* $4L/27$ *is the best possible.* □

Proof - We perform the computations by assuming $f \in L^1$; the general case can be handled exctly in the same way, since spt $f^s \subset\subset \Sigma$.
The classical Green formula provides the standard representation

$$u(x) \ = \ \int_0^L \mathcal{G}(x,y)\, f(y)\, dy \tag{4.29}$$

where we denote by $\mathcal{G}(x,y)$ the Green function associated to the operator $(d/dx)^4$ in$(0,L)$ with homogeneous boundary conditions:

$$\mathcal{G}_{xxxx}(x,y) = \delta(x-y) \ \ (0,L), \qquad \mathcal{G}(0,y) = \mathcal{G}_x(0,y) = \mathcal{G}(L,y) = \mathcal{G}_x(L,y) = 0$$

Moreover, by setting $P_3(y) = \ \frac{1}{L^3}\,(3L-2y)\,y^2, \quad P_1(y) = y/L$, we get

$$P_3(y) + P_3(L-y) = 1 \tag{4.30}$$

$$J_3(x,y) = \begin{cases} P_3(y) & \text{if } 0 \leq y \leq x \leq L, \\ -P_3(L-y) & \text{if } 0 \leq x < y \leq L, \end{cases} \tag{4.31}$$

$$J_1(x,y) = \begin{cases} P_1(y) & \text{if } 0 \leq y \leq x \leq L, \\ -P_1(L-y) & \text{if } 0 \leq x < y \leq L, \end{cases} \tag{4.32}$$

$$u'''(x) \ = \ \int_0^L J_3(x,y)\, df(y) \ = \ \int_0^x P_3(\tau)\, df(\tau) - \int_x^L P_3(L-\tau)\, f(\tau)\, d\tau \tag{4.33}$$

(4.34)

$$
\begin{aligned}
u''(x) \;&=\; \int_0^L J_1(x,y)\, u'''(y)\, dy \;= \\
&= \int_0^L J_1(x,y) \left(\int_0^y P_3(\tau)\, f(\tau)\, d\tau - \int_y^L P_3(L-\tau)\, f(\tau)\, d\tau \right) dy = \\
&= \int_0^L K(x,y)\, f(y)\, dy
\end{aligned}
$$

Hence (4.25). Moreover the Green function G for the operator $(d/dx)^2$ fulfils

$$
\mathcal{G}(x,y) \;=\; \int_0^x \left(\int_0^L G(s,\tau)\, J_3(\tau,y)\, d\tau \right) \; dy\,. \tag{4.35}
$$

Then $\mathcal{G}(0,y)=0$ and

$$
\mathcal{G}_x(x,y) \;=\; \int_0^L G(x,\tau)\, J_3(\tau,y)\, d\tau \tag{4.36}
$$

hence

$$
\begin{cases}
\mathcal{G}_{xxx}(x,y) \;=\; J_3(x,y) \\
\mathcal{G}_x(0,y) \;=\; \mathcal{G}_x(L,y) \;=\; 0 \\
\mathcal{G}_{xxxx}(x,y) \;=\; \delta(x-y)\,.
\end{cases}
$$

Eventually

$$
\begin{aligned}
\mathcal{G}(L,y) \;&=\; \int_0^L \int_0^L G(s,\tau) J_3(\tau,y) ds\, d\tau = \\
= \int_0^L J_3(\tau,y)\, d\tau \int_0^L G(s,\tau)\, ds &= \int_0^L J_3(\tau,y)\, d\tau \left(\int_0^\tau \frac{s(\tau-L)}{L}\, ds + \int_\tau^L \frac{\tau(s-L)}{L} \right) \\
= \int_0^L \tau\,(\tau-L)\, J_3(\tau,y)\, d\tau &= -\frac{1}{2} P_3(L-y) \int_0^y \tau\,(\tau-L)\, d\tau + \frac{1}{2} P_3(y) \int_y^L \tau\,(\tau-L)\, d\tau = 0\,.
\end{aligned}
$$

By (4.26)

$$
K_x \;=\; \frac{y^2}{L^3}\,(3L-2y) \;-\; \mathbf{1}_{\{y>x\}}\,, \qquad K_y \;=\; \frac{3y}{L^3}\,(2x-L)\,(L-y) \;-\; \frac{y}{L} \;+\; \mathbf{1}_{\{y>x\}}
$$

hence $K_x \neq 0$ if $y<x$, if $y>x$ then $K_x<0$ since $K_x(L,0)=0$ and K_x is increasing in y. So $\nabla K \neq (0,0)$ in $(0,L)^2 \setminus \{y=x\}$. Since $K(x,0)=K(x,L)=0$ we get

$$
\max_{x,y\in[0,L]^2} |K| \;=\; \max\{ \max_y |K(y,y)|\,,\, \max_y |K(0,y)|\,,\, \max_y |K(L,y)| \}\,.
$$

By computations:

$$
\begin{aligned}
K(y,y) \;&=\; -\frac{2\,y^2}{L^3}\,(L-y)^2 \\
K(0,y) \;&=\; y\left(1 \;-\; \frac{y}{L}\right)^2 \\
K(L,y) \;&=\; \frac{y^2}{L} \;-\; \frac{y^3}{L^2}
\end{aligned}
$$

$$
\max_y |K(y,y)| \;=\; \frac{L}{8}\,, \qquad \max_y |K(0,y)| \;=\; \max_y |K(L,y)| \;=\; \frac{4}{27}\, L
$$

and (4.27) follows. Estimate (4.28) follows by (4.34),(4.27). The equality is achieved in (4.28) since $f(y) = \delta(y - 2L/3)$ entails $\|u''\| = 4L/27$. □

Theorem 4.8. *(***Load regularity condition for functional** $\mathcal{F}_2$*)*
Assume (4.1),(4.2) *and*

$$\|f\|_{T(\overline{\Sigma})} \;\leq\; \frac{27}{4}\,\frac{\gamma}{L}\,. \tag{4.37}$$

Then $\mathcal{F}_2$ *has unique regular minimizer among* w *s.t.* $\operatorname{spt} w \subset \overline{\Sigma}$*: the solution* u *of* (4.15).

Proof - Assume u is the minimizer among H^2 functions assuming the boundary data. Hence by (4.28)

$$\|u''\|_{L^\infty} \;\leq \frac{4\,L}{27}\,\|f\|_{T(\overline{\Sigma})} \;\leq\; \gamma$$

and Theorem 4.6 give the conclusion, since the bending moment condition is fulfilled by u.

Lemma 4.9. *(***Compliance identity** $\mathcal{F}_2$*)* *Assume* (4.1),(4.2)*,* w *satisfies Euler conditions (i),(ii),(iii) of Theorem 4.4 and* $\operatorname{spt} w \subset \overline{\Sigma}$*. Then*

$$\mathcal{F}_2(w) \;=\; -\frac{1}{2}\int_0^L |\ddot{w}|^2\,dx \,+\, \beta\,\sharp\,(S_{\dot{w}}). \tag{4.38}$$

Proof - By (i) we have $(\ddot{w})'' = f$ in $\mathcal{D}'(0,L)$. Then, by $w(0) = w(L) = 0$ and

$$w'' = \ddot{w} + \sum_{S_{\dot{w}}\cap(0,L)} [\dot{w}]\,d\,\sharp\llcorner S_{\dot{w}} \qquad \text{in } \mathcal{D}'(0,L)\,,$$

we get

$$\int_{\mathbb{R}} f w\,dx \;=\; \int_0^L f w\,dx \;=\; \int_0^L (\ddot{w})'' w\,dx \;=\; -\int_0^L (\ddot{w})' w'\,dx \;=$$

$$= \int_0^L \ddot{w}\,d(w'') -\, \ddot{w}_-(L)\,\dot{w}_-(L) \,+\, \ddot{w}_+(0)\,\dot{w}_+(0) \;=$$

$$= \int_0^L |\ddot{w}|^2\,dx + \sum_{S_{\dot{w}}\cap(0,L)} \ddot{w}[\dot{w}] \,+\, \ddot{w}_-(L)\,[\dot{w}](L) \,+\, \ddot{w}_+(0)\,[\dot{w}](0)\,.$$

By substitution of (ii),(iii): $\displaystyle\int_0^L f w\,dx = \int_0^L |\ddot{w}|^2 + \gamma \sum_{S_{\dot{w}}} |[\dot{w}]|$

and thesis follows by the definition of $\mathcal{F}_2$. □

Theorem 4.10. *Assume* (4.1),(4.2) *and* v *minimizes* $\mathcal{F}_2$ *among function with* $\operatorname{spt} \subset \overline{\Sigma}$*. Then* $\sharp(S_{\dot{v}}) \leq 2$*.*

Proof - See Theorem 4.1 in [41].

We show an example of creased minimizer of $\mathcal{F}_2$ with homogeneous boundary condition (*spt* contained in $\overline{\Sigma}$) and load f fulfilling (4.11).

Theorem 4.11. (Example of load which produces creased minimizers)
It is possible to choose the parameters δ*,* k *and the load* f *such that*

$$f \;=\; \frac{k\,\gamma}{\delta\,L}\,\chi_{[2L/3-\delta,2L/3+\delta]}\,, \qquad 0 < \delta < L/3\,, \qquad 27 < k < 28\,, \tag{4.39}$$

$f \in L^1$, *fulfills the safe load condition* (4.11)*, violates the stress regularity condition (e.g* $\|u''\|_{L^\infty} > \gamma$*) for the related non creased solution and the the minimizers of* $\mathcal{F}_2$ *among functions with* $\text{spt} \subset \overline{\Sigma}$*, have a crease whenever*

$$0 \ < \ \beta \ < \ \frac{L}{4}(\ddot{u}(L) - \gamma)^2 \,.$$

Proof - We show that the solution u of (4.15) verifies $u''(L) > \gamma$, and we construct w with a crease at L, fulfilling all Euler conditions in Theorem 4.4 and $\mathcal{F}_2(w) \ < \ \mathcal{F}_2(u)$.
Let v be solution of

$$\begin{cases} v'''' \ = \ 0 \qquad \text{in } (0,L) \\ v(0) = v'(0) = v(L) = 0, \qquad v''(L) = \gamma - u''(L), \end{cases}$$

explicitly

$$v(x) = \frac{\gamma - u''(L)}{4L}\, x^2 (x - L)\,, \qquad v'(L) = \frac{L}{4}(\gamma - u''(L)).$$

Then $w = u + v$ is a solution of

$$\begin{cases} w'''' \ = \ f \qquad (0,L) \\ w(0) = w'(0) = w(L) = w'(L) = 0, \end{cases}$$

If we show that $u''(L) > \gamma$ for a suitable choice of f, then
$S_{\dot{w}} = \{L\}$, $\dot{w}(L) = \dfrac{L}{4}(\gamma - u''(L))$, $[\dot{w}](L) = \dfrac{L}{4}(\gamma - u''(L)) > 0$, $\ddot{w}(L) = \gamma = \gamma\, \text{sign}([w](L))$.
Hence all the Euler conditions of Th. 4.4 are fulfilled and we can evaluate both energies of w and u by mean of compliance identity (Lemma 4.9):

$$\begin{aligned} \mathcal{F}_2(w) = \beta - \frac{1}{2}\int_0^L |\ddot{w}|^2 = \beta - \frac{1}{2}\int_0^L |\ddot{u}|^2 - \int_0^L \ddot{u}\,\ddot{v} - \frac{1}{2}\int_0^L |\ddot{v}|^2 = \\ = \beta - \frac{1}{2}\int_0^L |\ddot{u}|^2 - (\dot{u}\,\ddot{v})\big|_0^L + \int_0^L \dot{u}\,\dddot{v} - \frac{1}{2}(\dot{v}\,\ddot{v})\big|_0^L + \frac{1}{2}\int_0^L \dot{v}\,\dddot{v} = \\ = \beta - \frac{1}{2}\int_0^L |\ddot{u}|^2 - \frac{1}{2}\dot{v}(L)\,\ddot{v}(L) = \\ = -\frac{1}{2}\int_0^L |\ddot{u}|^2 + \beta - \frac{L}{4}(\ddot{u}(L) - \gamma)^2 < -\frac{1}{2}\int_0^L |\ddot{u}|^2 = \mathcal{F}_2(u) \end{aligned}$$

if $0 \ < \ \beta \ < \ \frac{L}{4}\,(\ddot{u}(L) - \gamma)^2$. In the above computation of the energy we took into account $u(0) = u(L) = \dot{u}(0) = \dot{u}(L) = 0 = v(0) = \dot{v}(0) = v(L)$ and $\dddot{v} \equiv$ to a constant.
We show that (4.39) entails $u''(L) > \gamma$. By the explicit representation in term of Green function (Lemma 4.7), taking into account $P_3(\tau) + P_3(L - \tau) = 1$, we get

$$\ddot{u}(L) = \int_0^L K(L,y)\, f(y)\, dy \ = \ \frac{1}{L^2}\int_0^L (L - y)\, y^2\, f(y)\, dy \ = \ \frac{1}{L^2}\int_0^L h(y)\, f(y)\, dy$$

where we set $h(\tau) = \tau^2(L - \tau)$; hence $\max_{[0,L]} h(\tau) = h(2L/3) = 4L^3/27$.
We are left only to show that it is possible to choose $\delta \in (0, L/3)$ and $f \in L^1$ s.t. the safe load condition (4.11) is fulfilled and the regularity stress condition ($|u''| \le \gamma$)) is violated:

$$\|f\|_{L^1} = 2k\gamma/L < 8\gamma/L$$

$$\ddot{u}(L) \ = \ \frac{1}{L^2}\int_0^L h(\tau)\, f(\tau)\, d\tau \ \geq \ \frac{1}{L^2}\frac{7}{54}L^3 \frac{k\,\gamma}{\delta\, L}\, 2\,\delta \ = \ \frac{7\,k}{27}\gamma \ > \ \gamma$$

the last one, by the continuity of γ is achieved as soon as $h(\tau) \geq \dfrac{7}{54}\, L^3$ for $\tau \in [\frac{2}{3}L - \delta, \frac{2}{3}L + \delta]$.

Remark 4.12. *Referring to previous example (Thm 4.11), notice that the necessary condition* (4.20) *(say* $2\beta \le L\gamma^2$ *) for the existence of a crease is fulfilled if* $0<\beta<(\ddot u(L)-\gamma)^2$*. In fact:*

$$\gamma < \ddot u(L) = \frac{1}{L^2}\int_0^L h(\tau)f(\tau)d\tau < \frac{1}{L^2}\,\|h\|_{L^\infty}\|f\|_{L^1} < \frac{1}{L^2}\,\frac{4}{27}\,L^3\,\frac{2k\gamma}{L} = \frac{8\,k\,\gamma}{27} \;<\; \frac{32}{27}\,\gamma\,.$$

hence $\ddot u(L)-\gamma < \frac{5}{27}\gamma$ *so that* $\beta \;<\; \frac{L}{4}\,(\ddot u(L)-\gamma)^2$ *entails*

$$\beta \;<\; \frac{25}{729}\,\frac{L}{4}\,\gamma^2 \;<\; \frac{1}{2}\,L\,\gamma^2\,.$$

We emphasize also that Theorem 4.11 do not prove that w *is a minimizer, but shows only that any minimizer must have a crease and energy not bigger than* w*.*

Theorem 4.13. *Assume* (4.1),(4.2),(4.11)*,* v *minimizes* $\mathcal{F}_2$ *among functions with* $\mathrm{spt}\subset\overline{\Sigma}$ *and either* $f\ge 0$ *or* $f\le 0$*. Then* $S_{\dot v}\ne\{0,L\}$

Proof - Assume by contradiction $S_{\dot v}=\{0,L\}$. Then by Euler equations

$$\begin{cases} (\ddot v)'' = f & (0,L)\\ \ddot v(0_+) = \gamma\ \mathrm{sign}[\dot v(0)] \\ \ddot v(L_-) = \gamma\ \mathrm{sign}[\dot v(L)] \\ v(0)=v(L)=0 \end{cases}$$

and (up to interchanging boundary vaues at 0 and L, or changing sign in both boundary values) only two cases may occur: either

$$\text{(4.40)}\qquad \begin{cases} (\ddot v)'' = f & (0,L)\\ \ddot v(0_+) = +\gamma\,, \quad \dot v(0_+)>0 \\ \ddot v(L_-) = -\gamma\,, \quad \dot v(L_-)>0 \\ v(0)=v(L)=0 \end{cases}$$

or

$$\text{(4.41)}\qquad \begin{cases} (\ddot v)'' = f & (0,L)\\ \ddot v(0_+) = \ddot v(L_-) = +\gamma\,, \quad \dot v(0_+)>0\,,\ \dot v(L_-)<0 \\ v(0)=v(L)=0\,. \end{cases}$$

We show that both cases lead to a contradiction.
In case (4.40), we claim

$$\text{(4.42)}\qquad [\dot v](0)\,[\dot v](L) \;<\; 0 \;\Rightarrow\; \begin{cases} [\dot v](0)\,\big(\gamma\,\mathrm{sign}[\dot v](0) - u''(0)\big) \;<\; 0 \\ [\dot v](L)\,\big(\gamma\,\mathrm{sign}[\dot v](L) - u''(L)\big) \;<\; 0 \end{cases}$$

We set $s_0=[\dot v](0),\ s_L=[\dot v](L)$. Since $\big(\ddot v - u''\big)''\equiv 0\ (0,L)$ and $v''=\ddot v + s_0\delta_0 + s_L\delta_L$, there exist c,d s.t.

$$\text{(4.43)}\qquad \ddot v(x) - u''(x) = (cx+d)\,\chi_{[0,L]}(x).$$

By integration over $\mathbb{R}$ we get

$$\text{(4.44)}\qquad \frac{c}{2}\,L^2 + d\,L + s_0 + s_L \;=\; 0$$

by integrating (4.43) twice we get

$$\text{(4.45)}\qquad c\,\frac{L^3}{6} + d\,\frac{L^2}{2} + L\,s_0 \;=\; 0\,.$$

Euler equations

$$\text{(4.46)}\qquad \ddot v(0) \;=\; \gamma\ \mathrm{sign}\,s_0 \qquad v''(L)\ddot v(L) \;=\; \gamma\ \mathrm{sign}\,s_L$$

entail

$$\ddot v(0) - u''(0) \;=\; \gamma \,\mathrm{sign}\, s_0 - u''(0) \;=\; d \tag{4.47}$$

$$\ddot v(L) - u''(L) \;=\; \gamma \,\mathrm{sign}\, s_L - u''(L) \;=\; c\,L + d\,. \tag{4.48}$$

By solving (4.44),(4.45) we get

$$\begin{cases} c \;=\; \dfrac{6}{L^2}\,(s_0 - s_L) \\[2ex] d \;=\; \dfrac{2}{L}(s_L - 2\,s_0)\,. \end{cases} \tag{4.49}$$

By setting $a = [\dot v](0)\,\big(\gamma\,\mathrm{sign}[\dot v](0) - u''(0)\big) = s_0(\gamma\,\mathrm{sign}\, s_0 - u''(0))$,
$b = [\dot v](L)\,\big(\gamma\,\mathrm{sign}[\dot v](L) - u''(L)\big) = s_L(\gamma\,\mathrm{sign}\, s_L - u''(L))\,,$
the thesis of claim (4.42) reads

$$a = s_0 d \;<\; 0\,, \qquad b = s_L(cL + d) \;<\; 0\,, \tag{4.50}$$

and since (4.40) entails $s_L < 0 < s_0$ we get

$$a \;=\; 2s_0(s_L - 2s_0)\,/\,L \;<\; 0\,, \qquad b \;=\; 2s_L(s_0 - 2s_L)\,/\,L < 0 \tag{4.51}$$

(notice that also in the other case with $s_0 s_L < 0$, e.g. $s_0 < 0 < s_L$, we get (4.51)) hence (4.50) and the claim (4.42) is proven. By using claim (4.42) we get

$$u''(L) - u''(0) \;<\; \gamma\,(s_L - s_0) \;=\; -\,2\,\gamma, \tag{4.52}$$

notice that also in the other case with $s_0 s_L < 0$, e.g. $s_0 < 0 < s_L$, we get

$$u''(L) - u''(0) \;>\; \gamma\,(s_L - s_0) \;=\; +\,2\,\gamma. \tag{4.53}$$

In any case by (4.25),(4.26)

$$2\gamma \;<\; |u''(L) - u''(0)| \;<\; \left|\int_0^L (K(L,y) - K(0,y))\; f(y)\,dy\right| \tag{4.54}$$

$$K(L,y) - K(0,y) \;=\; \frac{1}{L^2}\,\big(3Ly^2 - 2y^3 - L^2 y\big) \tag{4.55}$$

$$\begin{aligned} |K(L,y) - K(0,y)| \;\le\; & |K(L, L(1-1/\sqrt{3}) - K(0, L(1-1/\sqrt{3})| \;= \\ =\; & \frac{L}{3}\,\Big(3 - \sqrt{3} - 2/\sqrt{3}\Big) \end{aligned} \tag{4.56}$$

By (4.54),(4.56),(4.11) we get the contradiction

$$2\gamma < |u''(L) - u''(0)| < \frac{L}{3}\Big(3 - \sqrt{3} - 2/\sqrt{3}\Big)\,\|f\|_{L^1} \;<\; \frac{8}{3}\Big(3 - \sqrt{3} - 2/\sqrt{3}\Big)\,\gamma < 2\gamma\,.$$

In case (4.41), by using Green function G for d^2/dx^2 with homogeneous boundary conditions (3.21), we get $\ddot v(x) = \gamma + \int_0^L G(x,y) f(y)\,dy$. Hence $\dot v(L) - \dot v(0) = \gamma L + \int_0^L \int_0^L G(x,y)dy\,dx\,.$
Since

$$\max_{y\in[0,L]} \left|\int_0^L G(x,y)\,dx\right| \;=\; \frac{L^2}{8} \tag{4.57}$$

by Fubini-Toneli and (4.11) we get

$$\left|\int_0^L \int_0^L G(x,y)\,f(y)\,dy\,dx\right| \;\le\; \frac{L^2}{8}\|f\|_{L^1} \;<\; \frac{L^2}{8}\,\frac{8\gamma}{L} \;<\; \gamma\,L \tag{4.58}$$

hence the contradiction $\dot v(L_-) - \dot v(0_+) > 0$. □

Theorem 4.14. *Assume* (4.1),(4.2),(4.11), *f does not change sign and is symmetric:*

$$f(x) = f(L-x)\ \ x \in (0,L) \quad \text{and either } f \geq 0 \text{ or } f \leq 0\,. \tag{4.59}$$

Then there is a unique minimizer of $\mathcal{F}_1$ *among v s.t.* $\operatorname{spt} v \subset [0,L]$, *moreover such minimizer is the regular regular solution of* (4.15).

Proof - By taking into account the symmetry of f and K

$$f(y) \;=\; f(L-y) \qquad K(x,y) \;=\; K(L-x, L-y) \tag{4.60}$$

in the Green representation (4.33) we get, for $x \in (0,L)$,

$$\begin{aligned} u''(x) \qquad &= \int_0^L K(x,y)\, f(y)\, dy \;= \\ &= \int_0^L K(L-x, L-y)\, f(y)\, dy \;= \\ &= \int_0^L K(L-x, y)\, f(y)\, dy \;=\; u''(L-x)\,. \end{aligned} \tag{4.61}$$

then u'' is even with respect to $L/2$.
Therefore $\|u''\|_{L^\infty} = \max\{|u''(L)|, |u''(L/2)|\}$ and by (4.25),(4.26),(4.11) we get
(4.62)

$$\begin{aligned} |\,u''(L)\,| \;&=\; \left| \int_0^L K(L,y)\, f(y)\, dy \right| \;= \\ &=\; \left| \frac{1}{L^2} \int_0^L y^2 (L-y) f(y), dy \right| \;= \\ &=\; \left| \frac{1}{L^2} \int_0^{L/2} y^2 (L-y) f(y)\, dy \;+\; \frac{1}{L^2} \int_{L/2}^L y (L-y)^2 f(y)\, dy \right| = \\ &=\; \frac{1}{L} \left| \int_0^{L/2} y(L-y) f(y)\, dy \right| \;\leq \\ &\leq\; \frac{1}{L}\, \frac{L^2}{4}\, \frac{1}{2}\, \|f\|_{L^1(0,L)} \;=\; \frac{L}{8}\, \|f\|_{L^1(0,L)} \;<\; \gamma\,. \end{aligned}$$

$$\begin{aligned} &\left| u''\left(\frac{L}{2}\right) \right| = \\ &= \left| \int_0^L \left((y - L/2)^+ \;-\; \frac{y^2}{2L} \right) \right| \;= \\ &= \frac{1}{2L} \left| \int_0^{L/2} y^2\, f(y)\, dy \;+\; \int_{L/2}^L (L-y)^2\, f(y)\, dy \right| \\ &= \frac{1}{L} \left| \int_0^{L/2} y^2\, f(y)\, dy \right| = \\ &= \leq\; \frac{1}{L}\, \frac{L^2}{4}\, \frac{1}{2}\, \|f\|_{L^1(0,L)} \;=\; \frac{L}{8}\, \|f\|_{L^1(0,L)} \;<\; \gamma\,. \end{aligned} \tag{4.63}$$

hence

$$\|u''\|_{L^\infty} \;<\; \gamma \tag{4.64}$$

and the thesis follows by Theorem 4.6.

5. (Pb III) Clamped Kirchhoff-Love plate with plastic yield along free lines

In this Section we look for minimizers of functional

$$\mathcal{P}_{KL}(w) = \\ = \frac{2}{3}\mu\int_\Sigma (|\nabla^2 w|^2 + \frac{\lambda}{\lambda+2\mu}|\Delta^a w|^2)d\mathbf{x} + \beta\mathcal{H}^1(S_{\nabla w}) + \gamma\int_{S_{Dw}} |[Dw]|\, d\mathcal{H}^1 - \int_\Sigma f w d\mathbf{x}$$

among scalar functions $w \in SBH(\mathbb{R}^2)$ s.t. $\operatorname{spt} w \subset \overline{\Sigma}$.
Here Δ^a denotes the absolutely continuous part of the distributional Laplace operator, say $\Delta^a w = \operatorname{Tr}\big(\nabla Dw\big) = (\Delta w)^a$, β, γ are given constants, $\mathcal{H}^1$ is the 1dimensional Hausdorff measure and f is a given transverse load.
All along this section we assume

$$\Sigma \subset \mathbb{R}^2 \text{ connected Lipschitz open set }, \tag{5.1}$$

$$\beta > 0,\ \gamma > 0, \tag{5.2}$$

$$\mu > 0,\ 2\mu + 3\lambda > 0,\ . \tag{5.3}$$

$$f \in \mathcal{M}(\mathbb{R}),\ \operatorname{spt} f \subset \overline{\Sigma}, \tag{5.4}$$

Notice that for any $w \in SBH$ we have (see [10],[29],[3]):

$$\nabla w = Dw\,, \qquad S_{\nabla v} = S_{Dv}\,, \tag{5.5}$$

$$S_{Dv} \text{ is a countably } \mathcal{H}^1 \text{ rectifiable set} \tag{5.6}$$

$$\begin{cases} S_{Dv} \text{ has an approximate normal vector } \nu_{S_{Dv}}\ \ \mathcal{H}^1\text{a.e. in } S_{Dv} \text{ and} \\ \nu_{S_{Dv}} \text{ is unique up to the orientation at any point where it is defined,} \end{cases} \tag{5.7}$$

$$[Dv] = [\nabla v] = \left[\frac{\partial v}{\partial \nu_{S_{Dv}}}\right] \nu_{S_{Dv}}\,, \qquad |[Dv]| = \left|\left[\frac{\partial v}{\partial \nu_{S_{Dv}}}\right]\right|, \tag{5.8}$$

$$\text{both } D^2 w \text{ and } \nabla^2 w \text{ are symmetric}\,. \tag{5.9}$$

We recall the following statement for the quadratic form associated to the Kirchhoff-Love plate energy.

Lemma 5.1. *Assume* (5.1),(5.2),(5.3),(5.4) *and*

$$\|f\|_T(\overline{\Sigma}) < 4\gamma \qquad \textbf{(safe load condition for clamped plate)} \tag{5.10}$$

Then $\mathcal{P}_{KL}$ achieves a finite minimum over $w \in SBH(\mathbb{R}^2)$ with $\operatorname{spt} w \subset \overline{\Sigma}$.

Proof - It is a particular case of Theorem 8.3 in [13]. □

Obviously the above existence result holds true not only for the quadratic form

$$\mathrm{Q}_{KL}(\mathbb{M}) = \frac{2}{3}\left(\mu|\mathbb{M}|^2 + \frac{\mu\lambda}{\lambda+2\mu}|\operatorname{Tr}\mathbb{M}|^2\right) \tag{5.11}$$

associated to Kirchhoff-Love plate energy $\mathcal{P}_{KL}$, but also for any other positive definite quadratic Q form evaluated on $\nabla^2\mathbf{v}$ ([13]). For this reason we study the whole class of functionals $\mathcal{P}$ (including $\mathcal{P}_{KL}$)

$$\mathcal{P}(w) = \int_\Sigma \big(\mathrm{Q}\big(\nabla^2 w\big) - f w\big)\, d\mathbf{x} + \beta\mathcal{H}^1(S_{\nabla w}) + \gamma\int_{S_{Dw}} |[Dw]|\, d\mathcal{H}^1\,, \tag{5.12}$$

to be minimized among $w \in SBH(\mathbf{R}^2)$ s.t. $\operatorname{spt} w \subset \overline{\Sigma}$.
We assume that the quadratic form Q fulfils

$$(5.13)\qquad \begin{cases} \exists\, q_{ijhk} \in \mathbb{R},\ q_{ijhk} = q_{hkij}\ :\ \mathrm{Q}(\mathbb{M}) = \displaystyle\sum_{i,j,h,k=1}^{2} q_{ijhk}\,\mathbb{M}_{ij}\,\mathbb{M}_{hk} \quad \forall\, \mathbb{M}\,, \\ \exists\, a,\ A\,,\ 0 < a \le A < +\infty\ :\quad a\,\|\mathbb{M}\|_2^2 \ \le\ \mathrm{Q}(\mathbb{M})\ \le\ A\,\|\mathbb{M}\|_2^2 \quad \forall\, \mathbb{M}\,, \end{cases}$$

here and in the following the summation convention over repeated indexes is understood, $\mathbb{M}, \mathbb{A}, \mathbb{B}$ are 2×2 real symmetric matrices, $\mathbb{A}\!:\!\mathbb{B} = \mathbb{A}_{ij}\!:\!\mathbb{B}_{ij}$ and $\|\cdot\|_p$ denotes the l^p norm.
We denote $\mathrm{Q}' = \partial \mathrm{Q}/\partial \mathbb{M}$ so that, by (5.13) we get

$$(5.14)\qquad (\mathrm{Q}'(\mathbb{M}))_{hk} = 2\sum_{ij=1}^{2} q_{ijhk}\,\mathbb{M}_{ij}\,, \qquad \frac{\|\mathrm{Q}'(\mathbb{M})\|_\infty}{\|\mathbb{M}\|_\infty} \ \le\ 2A\ \forall\, \mathbb{M}\,.$$

In the particular case of $\mathrm{Q} = \mathrm{Q}_{KL}$ we have

$$(5.15)\qquad \mathrm{Q}'_{KL}(\mathbb{M}) = \frac{4}{3}\left(\mu\mathbb{M} + \frac{\mu\lambda}{\lambda+2\mu}(\mathrm{Tr}\,\mathbb{M})\mathbb{I}\right)\,.$$

We are not able to prove (and even to write) the complete system of Euler equations or even Du Bois-Raymond equation for functional (5.12) since we cannot hope to have enough regularity of minimizers v to give meaning to the product $(\nabla^2 v : \mu)$ when μ is a measure; moreover for a general minimizer v the set S_{Dv} is not smooth enough to perform integration by parts. The difference with respect to beam problem faced in Section 3 is that weak and strong formulation of free gradient discontinuity problems coincide only in dimension $n = 1$. Nevertheless we can prove something similar to Du Bois-Raymond equation, by considering particular variations $\varepsilon(w - v)$, where $w \in C^2(\overline{\Sigma}) \cap SBH(\mathbb{R}^2)$, $\operatorname{spt} w \subset \overline{\Sigma}$, $v \in \operatorname{argmin}\mathcal{P}$ and $\varepsilon \in \mathbb{R}$, as stated by the following Lemma 5.2. So we get Euler equation (5.48) only in the set $\Sigma \setminus \overline{S_{Dv}}$ and the compliance identity as stated in Lemma 5.3. Moreover additional assumptions on f allow proof of basic relationship: a sufficiently small load f in $L^p(\Sigma)$ with $p > 1$ entails excess identity (5.31) and the regularity Theorem 5.7. In a different perspective any $f \in L^p(\Sigma)$ with $p > 2$ leads to partial regularity result stated in Theorem 5.9.

Lemma 5.2. *Assume* (5.1),(5.2),(5.4),(5.12),(5.13). *Then, for any* $w \in C^2(\overline{\Sigma}) \cap SBH(\mathbb{R}^2)$ *with* $\operatorname{spt} w \subset \overline{\Sigma}$, *and* $v \in \operatorname{argmin}\mathcal{P}$

$$(5.16)\qquad \int_\Sigma \big(Q'(\nabla^2 v) : \big(D^2 w - \nabla^2 v\big) \ -\ f(w - v)\big)\, d\mathbf{x} \ -\ \gamma \int_{S_{Dv}} |[Dv]|\, d\mathcal{H}^1 \ =\ 0\,.$$

Proof - By exploiting minimality of v, convexity of Q, $S_{\nabla(w-v)} = S_{\nabla(v)}$, small positive and negative ε, we get the thesis.

Lemma 5.3. ***(Compliance identity for elastic-plastic plate)***
Assume (5.1),(5.2),(5.4),(5.12),(5.13). *Then for any* $v \in \operatorname{argmin}\mathcal{P}$

$$(5.17)\qquad \int_\Sigma \big(\, Q'(\nabla^2 v) : \nabla^2 v \ -\ f\,v\,\big)\, d\mathbf{x} \ +\ \gamma \int_{S_{Dv}} |[Dv]|\, d\mathcal{H}^1 \ =\ 0\,,$$

say

$$(5.18)\qquad 2\int_\Sigma Q(\nabla^2 v)\, d\mathbf{x} \ =\ \int_\Sigma f\,v\, d\mathbf{x} - \gamma \int_{S_{Dv}} |[Dv]|\, d\mathcal{H}^1\,.$$

Hence the following compliance identity holds true

$$(5.19)\qquad \mathcal{P}(v) \ =\ -\int_\Sigma Q(\nabla^2 v)\, d\mathbf{x} + \beta\,\mathcal{H}^1(S_{Dv}) \qquad \forall v \in \operatorname{argmin}\mathcal{P}\,.$$

Proof - Choose $w = 0$ in (5.16). □

From now on, in order to perform a deeper analysis of $\mathcal{P}$, we enforce regularity assumptions (5.1),(5.4) about load and plate boundary.

Lemma 5.4. ***(Elliptic regularity)*** *Assume* (5.13) *and*

$$f \in L^p\,, \quad 1 < p < +\infty\,, \tag{5.20}$$

$$\partial\Sigma \text{ is either a convex polygonal or a } C^4 \text{ simple curve.} \tag{5.21}$$

Then the elliptic problem of fourth order

$$u \in H_0^2(\Sigma)\,, \qquad \text{div div } Q'(D^2u) \;=\; f \;\text{ in } \Sigma\,, \tag{5.22}$$

has unique solution u which is also the unique minimizer of $\int_\Sigma \big(Q(D^2v) - fv\big)\,d\mathbf{x}$ over $v \in H_0^2(\Sigma)$ and fulfils the associate compliance inequality

$$\int_\Sigma Q(D^2u)\,d\mathbf{x} \;=\; \frac{1}{2}\int_\Sigma f u\,d\mathbf{x}\,, \tag{5.23}$$

moreover u belongs to $W^{4,p}(\Sigma)$ and there are two constants $C_1,\ C_2$, with $C_1 = C_1(\Sigma, p, a, A)$ and $C_2 = C_2(\Sigma, p, a, A)$ s.t.

$$\|u\|_{W^{4,p}(\Sigma)} \;\le\; C_1\,\|f\|_{L^p(\Sigma)}\,, \tag{5.24}$$

$$\|D^2u\|_{C^0(\overline{\Sigma})} \;\le\; C_2\,\|f\|_{L^p(\Sigma)}\,. \tag{5.25}$$

If $Q = Q_{KL}$ (Kirchhoff-Love elastic plate) then problem (5.22) *reads as follows:*

$$u \in H_0^2(\Sigma)\,, \qquad \Delta^2 u \;=\; \frac{3\,(\lambda + 2\mu)}{8\,\mu\,(\lambda+\mu)}\; f \;\text{ in } \Sigma\,. \tag{5.26}$$

Proof - Since $L^p(\Sigma) \subset H^{-2}(\Sigma)$, by denoting C_3 the related embedding constant and applying standard Hilbert technique for elliptic equations, we get existence and uniqueness of solution for (5.22), minimizing the purely elastic energy $\int_\Sigma \big(\mathrm{Q}(D^2v) - fv\big)\,d\mathbf{x}$ and fulfilling (5.23) together with the following estimates (due to (5.13), $\Sigma \subset \mathbf{R}^2$, $\operatorname{spt} u \subset \overline{\Sigma}$):

$$\begin{aligned}
\|u\|_{L^\infty(\Sigma)} \le\ & \frac{1}{4}\|Du^2\|_{T(\overline{\Sigma})} \;\le\; \frac{1}{4}|\Sigma|^{1/2}\|D^2u\|_{L^2(\overline{\Sigma})} \;\le \\
\le\ & \frac{|\Sigma|^{1/2}}{4\sqrt{a}}\left(\int_\Sigma \mathrm{Q}(D^2u)\,d\mathbf{x}\right)^{1/2} \;=\; \frac{|\Sigma|^{1/2}}{4\sqrt{a}}\left(\frac{1}{2}\int_\Sigma f u\,d\mathbf{x}\right)^{1/2} \;\le \\
\le\ & \frac{|\Sigma|^{1/2}}{4\sqrt{2a}}\;\|f\|^{1/2}_{L^p(\Sigma)}\,\|u\|^{1/2}_{L^{p'}(\Sigma)} \;\le\; \frac{|\Sigma|^{\frac{1}{2}+\frac{1}{2p'}}}{4\sqrt{2a}}\;\|f\|^{1/2}_{L^p(\Sigma)}\,\|u\|^{1/2}_{L^\infty(\Sigma)}\,,
\end{aligned}$$

hence, by $H_0^2(\Sigma) \subset L^\infty(\Sigma)$,

$$\|u\|_{L^\infty(\Sigma)} \;\le\; \frac{|\Sigma|^{2-1/p}}{32\,a}\|f\|_{L^p(\Sigma)}\,,$$

$$\|u\|_{L^p(\Sigma)} \;\le\; |\Sigma|^{1/p}\,\|u\|_{L^\infty(\Sigma)} \;\le\; \frac{|\Sigma|^2}{32\,a}\|f\|_{L^p(\Sigma)}\,. \tag{5.27}$$

The fact that the solution u of (5.22) belongs to $W^{4,p}$ follows by standard interior regularity and use of Lemma 4.2 p.414 of [1] (with $m = j = 2$) on a finite atlas of the boundary $\partial\Sigma$ in the convex polygon case, and by Theorem 8.1 p.443 of [1] in the C^4 boundary case: hence in both cases:

$$\|u\|_{W^{4,p}(\Sigma)} \;\le\; C_0\,\big(\|f\|_{L^p(\Sigma)} + \|u\|_{L^p(\Sigma)}\big)\,. \tag{5.28}$$

Then (5.27), (5.28) entail (5.24) with $C_1 = \big(1+|\Sigma|^2/(32a)\big)C_0$.
Estimate (5.24) together with Sobolev inequality entail estimate (5.25) with $C_2 = C_1C_3 = \big(1+|\Sigma|^2/(32a)\big)C_0C_3$ where C_3 is the embedding constant: $\|D^2u\|_{C^0(\Sigma)} \le C_3\|u\|_{W^{4,p}(\Sigma)}$. □

Lemma 5.5. *Assume* (5.2),(5.12),(5.13),(5.20),(5.21) *and u is the unique solution of*

$$u \in H^2_0(\Sigma)\,, \quad \operatorname{spt} w \subset \overline{\Sigma}\,, \quad \operatorname{div}\operatorname{div}\, Q'(D^2u) \;=\; f \;\; in\;\; \Sigma\,, \tag{5.29}$$

Then trivial extension of u belongs to $C^2(\overline{\Sigma}) \cap C^1(\mathbb{R}^n)$ and the following statements hold true.
Excess estimate for $\mathcal{P}$ *: If u solves* (5.29) *then for all $v \in SBH(\mathbb{R}^2)$ s.t.* $\operatorname{spt} v \subset \overline{\Sigma}$

$$\begin{aligned} \mathcal{P}(v) - \mathcal{P}(u) \;&\ge\; \beta\,\mathcal{H}^1(S_{Dv}) + \\ &+ \int_{S_{Dv}} \Big(\gamma\,|[Dv]| - Q'(D^2u) : ([Dv]\otimes \nu_{S_{Dv}})\Big)\, d\mathcal{H}^1\,. \end{aligned} \tag{5.30}$$

Excess identity for minimizers of $\mathcal{P}$ *: If v minimize $\mathcal{P}$ among $v \in SBH(\mathbb{R}^2)$, s.t.* $\operatorname{spt} v \subset \overline{\Sigma}$ *and u solves* (5.29) *then*

$$\mathcal{P}(v) - \mathcal{P}(u) \;=\; \beta\,\sharp(S_{\dot v}) + \frac{1}{2}\int_{S_{Dv}} \Big(\gamma\,|[Dv]| - Q'(D^2u) : ([Dv]\otimes \nu_{S_{Dv}})\Big)\, d\mathcal{H}^1. \tag{5.31}$$

Necessary conditions for existence of creased minimizers of $\mathcal{P}$ *: If v minimize $\mathcal{P}$ among $v \in SBH(\mathbb{R}^2)$ s.t. $\operatorname{spt} v \subset \overline{\Sigma}$, $S_{Dv} \ne \emptyset$, and u solves* (5.29)*, then*

$$\|\, Q'(D^2u)\|_{L^\infty(\Sigma, \mathcal{L}(\mathbb{R}^2,\mathbb{R}^2))} > \gamma\,, \tag{5.32}$$

$$\int_{S_{Dv}} \Big(\gamma\,|[Dv]| - Q'(D^2u) : ([Dv]\otimes \nu_{S_{Dv}})\Big)\, d\mathcal{H}^1 \le -2\beta\,\mathcal{H}^1(S_{Dv}) < 0\,. \tag{5.33}$$

We emphasize that the excess estimate (5.30) *holds true under weaker assumptions: Q convex and C^2; while the excess identity* (5.31) *and its consequence, say the fact that* (5.32),(5.33) *are necessary conditions for creased minimizers, require the quadratic structure* (5.13) *of Q.*

Proof - By (5.20),(5.21),(5.29) and Lemma 5.4 we know: $D^2u \in C(\overline{\Sigma})$, $u \in H^2_0(\Sigma)$; hence $\mathbf{u} \in C^2(\overline{\Sigma}) \cap C^1(\mathbb{R}^n)$. For simplicity, we will write shortly ν instead of $\nu_{S_{Dv}}$ in the proof.
By convexity of Q we get

$$\begin{aligned} &\mathcal{P}(v) - \mathcal{P}(u) = \\ &\quad = \beta\mathcal{H}^1(S_{\nabla v}) + \int_\Sigma \mathrm{Q}(\nabla^2 v)\,dx + \gamma\int_{S_{Dv}} \left|\left[\frac{\partial v}{\partial \nu}\right]\right| d\mathcal{H}^1 + \\ &\quad - \int_\Sigma f v\,dx - \int_\Sigma \mathrm{Q}(D^2u) + \int_\Sigma f u \ge \\ &\quad \ge \beta\mathcal{H}^1(S_{Dv}) + \gamma\int_{S_{\nabla v}} \left|\left[\frac{\partial v}{\partial \nu}\right]\right| d\mathcal{H}^1 - \int_\Sigma f(v-u) + \\ &\quad + \int_\Sigma \mathrm{Q}'(D^2u) : (\nabla^2 v - D^2u)dx\,. \end{aligned} \tag{5.34}$$

Thanks to Lemma 5.4

$$D^2u \in C^0(\overline{\Sigma}) \tag{5.35}$$

so that we can apply Lemma 5.2 with $w = u$ and $v \in \operatorname{argmin}\mathcal{P}$.
By (5.8),(5.22),(5.29),(5.35) and Theorems 2.15, 6.3, 6.4 of [10] we have:

$$\begin{aligned} \nabla^2 v \;&=\; D^2 v - [Dv]\otimes\nu\, d\mathcal{H}^1 \llcorner S_{Dv}\cap\Sigma \;= \\ &=\; D^2 v - \left[\frac{\partial v}{\partial \nu}\right]\otimes\nu\, d\mathcal{H}^1 \llcorner S_{Dv}\cap\Sigma \qquad \text{in } \mathcal{D}'(\Sigma), \end{aligned} \tag{5.36}$$

$$[Dv]\otimes\nu\, d\mathcal{H}^1 \llcorner S_{Dv}\cap\Sigma = \left[\frac{\partial v}{\partial\nu}\right]\otimes\nu\, d\mathcal{H}^1 \llcorner S_{Dv}\cap\Sigma \tag{5.37}$$

$$f = div\, div\, \mathrm{Q}'(D^2u) \qquad \text{say} \qquad f = \frac{8\mu(\lambda+\mu)}{3(\lambda+2\mu)}\,\Delta^2 u \quad \text{if } \mathrm{Q}=\mathrm{Q}_{KL}\,.$$

Hence, integrating by parts twice and taking into account $u = v = 0$ on $\partial\Sigma$, we get

$$\begin{aligned}
&\int_\Sigma \mathrm{Q}'(D^2u):(D^2v - D^2u)\,d\mathbf{x} \;=\\
&= -\int_\Sigma \mathrm{div}\;\mathrm{Q}'(D^2u)\cdot D(v-u)\,d\mathbf{x} + \int_{\partial\Sigma}\mathrm{Q}'(D^2u):\left(\frac{\partial v}{\partial\nu_\Sigma}\nu_\Sigma\otimes\nu_\Sigma\right) \;=\\
&= \int_\Sigma \mathrm{div}\;\mathrm{div}\;\mathrm{Q}'(D^2u)\cdot(v-u)\,d\mathbf{x} + \int_{\partial\Sigma}\mathrm{Q}'(D^2u):\left(\frac{\partial v}{\partial\nu_\Sigma}\nu_\Sigma\otimes\nu_\Sigma\right) \;=\\
&= \int_\Sigma f(v-u)\,d\mathbf{x} + \int_{\partial\Sigma}\mathrm{Q}'(D^2u):\left(\frac{\partial v}{\partial\nu_\Sigma}\,\nu_\Sigma\otimes\nu_\Sigma\right)
\end{aligned} \tag{5.38}$$

where ν_Σ is the outward normal to $\partial\Sigma$. We choose $\nu = \nu_{S_{Dv}} = \nu_\Sigma$ on $\partial\Sigma\cap S_{Dv}$ and, abusing notation we define $\left(\frac{\partial v}{\partial\nu}\nu\otimes\nu\right) = \mathbb{O}$ on $\partial\Sigma\setminus S_{Dv}$; with this convention, by denoting $|_{out}$ and $|_{in}$ respectively the outer and inner traces at $\partial\Omega$ and taking into account that $\partial/\partial\nu_\Sigma$ stands for the inner trace of the derivative in the direction of outer normal, we get

$$\left[\frac{\partial v}{\partial\nu}\right] \;=\; \left.\frac{\partial v}{\partial\nu}\right|_{out} - \left.\frac{\partial v}{\partial\nu}\right|_{in} = -\left.\frac{\partial v}{\partial\nu}\right|_{in} = -\frac{\partial v}{\partial\nu_\Sigma} \tag{5.39}$$

$$\frac{\partial v}{\partial\nu_\Sigma}\,\nu_\Sigma\otimes\nu_\Sigma \;=\; -\left[\frac{\partial v}{\partial\nu}\right]\nu\otimes\nu \tag{5.40}$$

so that (5.38) reads as follows

$$\begin{aligned}
&\int_\Sigma \mathrm{Q}'(D^2u):(D^2v - D^2u)\,d\mathbf{x} \;=\\
&\qquad = +\int_\Sigma f(v-u)\,d\mathbf{x} - \int_{\partial\Sigma}\mathrm{Q}'(D^2u):\left(\left[\frac{\partial v}{\partial\nu}\right]\nu\otimes\nu\right)
\end{aligned} \tag{5.41}$$

by substituting (5.36) in (5.34) and taking into account (5.37),(5.41) and $Dv = \nabla v$ we get

$$\begin{aligned}
\mathcal{P}(v)-\mathcal{P}(u) \;\geq\;& \beta\mathcal{H}^1(S_{Dv}) + \int_\Sigma \mathrm{Q}'(D^2u):(D^2v-D^2u) +\\
& -\int_\Sigma f(v-u) + \gamma\int_{S_{Dv}\cap\Sigma}\left|\left[\frac{\partial v}{\partial\nu}\right]\right| d\mathcal{H}^1 - \int_{S_{Dv}\cap\Sigma}\mathrm{Q}'(D^2u):[\nabla v]\otimes\nu d\mathcal{H}^1 =\\
=\;& \beta\mathcal{H}^1(S_{Dv}) + \int_\Sigma f(v-u)\,d\mathbf{x} - \int_{\partial\Sigma}\mathrm{Q}'(D^2u):\left(\left[\frac{\partial v}{\partial\nu}\right]\nu\otimes\nu\right) +\\
& -\int_\Sigma f(v-u) + \gamma\int_{S_{Dv}\cap\Sigma}\left|\left[\frac{\partial v}{\partial\nu}\right]\right| d\mathcal{H}^1 - \int_{S_{Dv}\cap\Sigma}\mathrm{Q}'(D^2u):[\nabla v]\otimes\nu d\mathcal{H}^1 =\\
=\;& \beta\mathcal{H}^1(S_{Dv}) + \left(\int_{S_{Dv}}\gamma\left|\left[\frac{\partial v}{\partial\nu}\right]\right| d\mathcal{H}^1 - \int_{S_{Dv}}\mathrm{Q}'(D^2u):[\nabla v]\otimes\nu\, d\mathcal{H}^1\right)\\
=\;& \beta\mathcal{H}^1(S_{Dv}) + \left(\int_{S_{Dv}}\gamma\left|\left[\frac{\partial v}{\partial\nu}\right]\right| - \mathrm{Q}'(D^2u):\left[\frac{\partial v}{\partial\nu}\right]\nu\otimes\nu\, d\mathcal{H}^1\right)
\end{aligned} \tag{5.42}$$

so that (5.30) follows by(5.37). Since Q is a symmetric quadratic form we get

$$\mathrm{Q}(\mathbb{A})-\mathrm{Q}(\mathbb{B})=\frac{1}{2}(\mathrm{Q}'(\mathbb{A})+\mathrm{Q}'(\mathbb{B})):(\mathbb{A}-\mathbb{B}) \tag{5.43}$$

hence by using (5.36),(5.37),(5.41),(5.43) and eventually (5.16) we get (5.31) as follows:

$$\begin{aligned}
&\mathcal{P}(v)-\mathcal{P}(u)=\\
&=\int_\Sigma(\mathrm{Q}(\nabla^2 v)-\mathrm{Q}(D^2u))\,d\mathbf{x}-\int_\Sigma f(v-u)\,d\mathbf{x}+\\
&+\gamma\int_{S_{Dv}}\left|\left[\frac{\partial v}{\partial\nu}\right]\right|\,d\mathcal{H}^1+\beta\mathcal{H}^1(S_{Dv})=\\
&\frac{1}{2}\int_\Sigma(\mathrm{Q}'(\nabla^2 v)+\mathrm{Q}'(D^2u)):(\nabla^2 v-D^2u)+\\
&-\int_\Sigma f(v-u)\,d\mathbf{x}+\gamma\int_{S_{Dv}}\left|\left[\frac{\partial v}{\partial\nu}\right]\right|\,d\mathcal{H}^1+\beta\mathcal{H}^1(S_{Dv})=\\
&=\frac{1}{2}\gamma\int_{S_{Dv}}\left|\left[\frac{\partial v}{\partial\nu}\right]\right|\,d\mathcal{H}^1-\frac{1}{2}\int_\Sigma f(v-u)\,d\mathbf{x}+\\
&\beta\mathcal{H}^1(S_{Dv})+\frac{1}{2}\int_\Sigma \mathrm{Q}'(D^2u):(D^2v-D^2u)+\\
&-\frac{1}{2}\int_{S_{Dv}\cap\Sigma}\mathrm{Q}'(D^2u):([Dv]\otimes\nu)\,d\mathcal{H}^1=\\
&=\beta\mathcal{H}^1(S_{Dv})+\frac{1}{2}\int_{S_{Dv}}\left\{\gamma\left|\left[\frac{\partial v}{\partial\nu}\right]\right|-\mathrm{Q}'(D^2u):([Dv]\otimes\nu)\right\}d\mathcal{H}^1\,.
\end{aligned} \tag{5.44}$$

Thesis (5.32) follows by (5.2),(5.31) and minimality of v.
Thesis (5.33) follow from (5.31) and (5.32). □
As a consequence of Lemma 5.5 we can prove the following result (which was announced in [41], Th.2.2) which states that the minimizers of (5.12) do not exhibit any plastic yield whenever the purely elastic solution has small second derivatives .

Theorem 5.6. *(***Bending moment regularity condition for clamped plate***)*
Assume (5.1),(5.2),(5.12),(5.13),(5.20),(5.21) *and the solution* u *of purely elastic problem* (5.22) *fulfils*

$$\|Q'(D^2u)\|_{L^\infty(\Sigma)}\le\gamma\,. \tag{5.45}$$

Then $u\in\operatorname{argmin}\mathcal{P}(w)$ *and* u *is the unique minimizer of* $\mathcal{P}$ *.*
More explicitly, in the case of Kirchhoff-Love plate (say $Q=Q_{KL}$*) condition* (5.45) *reads*

$$\left\|\mu\,D^2u\;+\;\frac{\mu\,\lambda}{\lambda+2\mu}\,(\mathrm{Tr}\,D^2u)\,I\right\|_{L^\infty(\Sigma)}\;\le\;\frac{3}{4}\,\gamma\,. \tag{5.46}$$

Proof - By (5.30) and (5.45) we get

$$\mathcal{P}(v)-\mathcal{P}(u) \;\geq\; \beta\mathcal{H}^{1}(S_{Dv})+\left(\int_{S_{Dv}}\left(\gamma\,|[Dv]|-\mathrm{Q}'(D^2u):([Dv]\otimes\nu_{S_{Dv}})\right)d\mathcal{H}^{1}\right)\geq$$

$$=\beta\mathcal{H}^{1}(S_{Dv})+\left(\int_{S_{Dv}}(\gamma-\|\mathrm{Q}'(D^2u)\|_\infty)\,|[Dv]|\;d\mathcal{H}^{1}\right)\geq\beta\mathcal{H}^{1}(S_{Dv})\geq 0\,.$$

and the last inequality is strict if $\mathcal{H}^1(S_{Dv})>0$. □

Theorem 5.7. ***Load regularity condition for clamped plate*** $\mathcal{P}$
Assume (5.2),(5.12),(5.13),(5.20),(5.21) *and*

$$\|f\|_{L^p}\;\leq\;\frac{\gamma}{2\,A\,C_2} \tag{5.47}$$

where $C_2=C_2(\Sigma,p,a,A)$ *is the constant appearing in the estimate* (5.25).
Then u *minimizes energy* $\mathcal{P}$ *among scalar functions in* $SBH(\mathbf{R}^2)$ *with support in* Σ.
Moreover u *is the unique minimizer of* $\mathcal{P}$ *in this class.*

Proof - Inequalities (5.14),(5.25),(5.47) entail (5.45), hence thesis follows by Theorem 5.6.

Theorem 5.8. *(* **Euler equation** *for* $\mathcal{P}$ *) Assume* (5.2),(5.12),(5.13),(5.20),(5.21) *and* w *minimizes* $\mathcal{P}$ *among* v *in* $SBH(\mathbb{R}^2)$ *s.t.* $\mathrm{spt}\,v\subset\overline{\Sigma}$. *Then*

$$\mathrm{div}\,\mathrm{div}\;Q'(D^2w)\;=\;f \qquad\qquad \Sigma\setminus\overline{S_{Dw}}\,. \tag{5.48}$$

Proof - Perform smooth variations with support in $\Sigma\setminus\overline{S_{Dw}}$. □

Theorem 5.9. ***(Partial regularity for elastic-plastic clamped plate)***
Assume (5.1),(5.2),(5.10),(5.12), $Q(\mathbb{M})=\mathbb{M}:\mathbb{M}$ *and* $f\in L^p(\Sigma)$ *with* $p>2$.
Then the set of w *minimizing* $\mathcal{P}$ *among* $v\in SBH(\mathbf{R}^2)$ *s.t.* $\mathrm{spt}\,v\subset\overline{\Sigma}$ *is not empty and any* w *among these minimizers is a strong solution, say:*

$$w\in C^0(\overline{\Sigma})\cap C^2(\Sigma\setminus\overline{S_{Dw}})\,, \tag{5.49}$$

$$\mathcal{H}^{1}\left(\overline{S_{Dw}}\setminus S_{Dw}\right)\;=\;0 \tag{5.50}$$

and the pair $(\overline{S_{Dw}},w)$ *minimizes the functional*

$$P(K,v)\;=\;\int_{\Sigma\setminus K}\left(|D^2v|^2-fv\right)d\mathbf{x}\,+\,\beta\mathcal{H}^{1}(K\cap\overline{\Sigma})\,+\,\gamma\int_{K\cap\overline{\Sigma}}|[Dv]|\;d\mathcal{H}^{1} \tag{5.51}$$

among pairs (K,v) *such that* $K\subset\mathbf{R}^2$ *is a closed sets and* $v\in C^0(\overline{\Sigma})\cap C^2(\Sigma\setminus K)$.

Proof - Safe load condition (5.10) together with Lemma 5.1 entail the existence of minimizers for $\mathcal{P}$.
So we can apply Corollary 4.14 and Theorem 4.15 in [11] to any minimizer of $\mathcal{P}$ and get interior regularity in Σ, then repeat the technique of [15] in this simpler case (homogeneous Dirichlet datum, free discontinuity allowed only for derivatives) to prove partial regularity up to the boundary $\partial\Sigma$. □

Remark 5.10. *About analysis of plastic yield lines (regularity and geometric properties of crease set, squared-hessian jump, stress concentration and asymptotic expansion around crease-tip of a minimizer) we refer to a forthcoming research. We emphasize the analogy of properties between yield lines and free discontinuity set in Blake & Zisserman functional for optimal segmentation of an image (*[14],[16]*).*

Theorem 5.11. Sufficient conditions for existence of creased minimizers of $\mathcal{P}$: *Assume structural assumptions* (5.2),(5.10),(5.12),(5.13),(5.20),(5.21) *and u solves* (5.29). *If there exists $v \in SBH(\mathbb{R}^2)$ s.t.* $\operatorname{spt} v \subset \overline{\Sigma}$, *and v fulfils*

$$\int_\Sigma \big(Q'(\nabla^2 v) : \big(D^2 u - \nabla^2 v\big) \; - \; f(u-v)\big) \, d\mathbf{x} \; - \; \gamma \int_{S_{Dv}} |[Dv]| \, d\mathcal{H}^1 \; = \; 0\,, \tag{5.52}$$

$$\int_{S_{Dv}} \Big(\gamma|[Dv]| - \; Q'(D^2u) : ([Dv] \otimes \nu_{S_{Dv}})\Big) \, d\mathcal{H}^1 \; < \; -2\,\beta\, \mathcal{H}^1(S_{Dv})\,. \tag{5.53}$$

Then v has non empty crease set and lower energy than u:

$$\mathcal{H}^1\,(S_{Dv}) > 0; \quad \mathcal{P}(v) \le \mathcal{P}(u). \tag{5.54}$$

Hence the set of minimizers is non empty, all minimizer z exhibits a non empty crease set and lower energy than v : say $\mathcal{P}(z) \le \mathcal{P}(v)$ (the inequality may be strict).

Proof - Inequality (5.53) entails $\mathcal{H}^1\,(S_{Dv}) > 0$. Assumption (5.52) allows to repeat exactly the same computations in (5.44), so that v fulfills the excess identity (5.31) too. Then (5.53),(5.31) together entail the thesis. □

Remark 5.12. *Notice that Theorem 5.11 does not state neither that v itself is a minimizer nor the existence of creased minimizers.*

6. **(Pb.IV) Vector-valued deformations with cohesive damage along free surfaces**

In this Section we study the functional

$$\begin{aligned} \mathcal{F}(\mathbf{v}) \; = \; & \int_\Omega \left(\mu\,|\mathcal{E}(\mathbf{v})|^2 + \frac{\lambda}{2}\,|\mathrm{Tr}\,\mathcal{E}(\mathbf{v})|^2 - \mathbf{f}\cdot\mathbf{v}\right) d\mathbf{x} \; + \\ & + \alpha\,\mathcal{H}^{n-1}(J_{\mathbf{v}}) \; + \; \gamma \int_{J_{\mathbf{v}}} |[\mathbf{v}] \odot \nu_{\mathbf{v}}| d\mathcal{H}^{n-1} \end{aligned} \tag{6.1}$$

to be minimized among displacement vector fields $\mathbf{v}$ such that $\mathbf{v} \in SBD(\mathbb{R}^n)$ such that $\operatorname{spt}(\mathbf{v}) \subset \overline{\Omega}$. Here $\alpha, \gamma, \lambda, \mu$ are given constants, $\odot$ denotes the symmetric tensor product, $\mathcal{H}^{n-1}$ is the $n-1$ dimensional Hausdorff measure. $\mathcal{E}(\mathbf{v})$ is the absolutely continuous part of the linear strain tensor $\mathrm{e}(\mathbf{v}) \; = \; \frac{1}{2}\left(D\mathbf{v} + (D\mathbf{v})^T\right)$, $J_{\mathbf{v}}$ is the jump set of $\mathbf{v}$, $\nu_{\mathbf{v}}$ is the normal to $J_{\mathbf{v}}$ and $[\mathbf{v}]$ is the jump of $\mathbf{v}$ in the direction of $\nu_{\mathbf{v}}$.
The non convex stored energy functional $\mathcal{F}$ is a naïf description of mechanical energy for a deformable body with natural reference Ω, subject to prescribed volume dead load $\mathbf{f}$, homogeneous Dirichlet boundary conditions and free small cohesive damage whose geometry (the set $J_{\mathbf{v}}$) is not "a priori" prescribed.
The space of vector fields with bounded deformation *BD* is the natural framework for the study of functionals with linear growth in the symmetrized gradient and its subspace *SBD* allows only jump-type discontinuity ([4],[6],[18]).
All along this section we assume

$$\begin{cases} \mu > 0, \; 2\mu + n\lambda > 0, \; \alpha > 0, \; \gamma > 0, \\ \Omega \subset \mathbb{R}^n \text{ connected lipschitz open set, n=2,3,} \\ \mathbf{f} \in L^p(\mathbb{R}^n, \mathbb{R}^n), \; p > n\,, \; \operatorname{spt} f \subset \overline{\Omega}\,. \end{cases} \tag{6.2}$$

If Y is a finite dimensional space and $A \subset \mathbb{R}^n$ is an open set, we denote by $L^p(A, Y)$ the space of Yvalued, p integrable functions with respect to the Lebesgue measure $\mathcal{L}^n$. Let $\mathcal{M}(A, Y)$ be the space of the bounded measures on A with values in Y ($\mathcal{M}(A)$ when $Y = \mathbf{R}$) and let $|\cdot|_{T(A)}$ be the total variation of a measure in $\mathcal{M}(A, Y)$, i.e.

$$|\mu|_{T(A)} = \int_A |\mu| = \sup\Big\{ \int_A \sum_{ij} \phi_{ij} d\mu_{ij} \ : \ \phi_{ij} \in C_0^0(A), \ \ \sum_{ij} \phi_{ij}^2 \le 1, \ \text{ in } \ A \Big\}.$$

We define a Borel measure $|\mu|$, by setting for every Borel set $B \subset \mathbb{R}^n$

$$|\mu|(B) = |\mu|_{T(B)} = \inf \left\{ |\mu|_{T(A)};\ B \subset A,\ A \text{ open } \right\}.$$

For any $\mathbf{v} \in L^1(\mathbf{R}^n, \mathbf{R}^n)$ the set of *Lebesgue points* is the set of $\mathbf{x} \in \mathbf{R}^n$ s.t. there is $\widetilde{\mathbf{v}}(\mathbf{x}) \in \mathbf{R}^n$ with $\lim_{\varrho \to 0+} \int_{B_\varrho(\mathbf{x})} |\mathbf{v}(\mathbf{y}) - \widetilde{\mathbf{v}}(\mathbf{y})| dy \,/\, |B_\varrho| = 0$.
The *Lebesgue discontinuity set* $S_\mathbf{v}$ is the complement of Lebesgue points.
We say that $\mathbf{v}$ has *one-sided limits* $\mathbf{v}^+(\mathbf{x}), \mathbf{v}^-(\mathbf{x})$ at $\mathbf{x} \in \Omega$ with respect to a suitable direction $\nu_\mathbf{v}(\mathbf{x}) \in \{\mathbf{x} \in \mathbf{R}^n : |\mathbf{x}| = 1\}$ if

$$\lim_{\rho \to 0^+} \rho^{-n} \int_{\{\mathbf{y} \in B_\rho(\mathbf{x});\ (\mathbf{y}-\mathbf{x})\cdot \nu_\mathbf{v} > 0\}} |\mathbf{v}(\mathbf{y}) - \mathbf{v}^+(\mathbf{x})| \, d\mathbf{y} = 0,$$

$$\lim_{\rho \to 0^+} \rho^{-n} \int_{\{\mathbf{y} \in B_\rho(\mathbf{x});\ (\mathbf{y}-\mathbf{x})\cdot \nu_\mathbf{v} < 0\}} |\mathbf{v}(\mathbf{y}) - \mathbf{v}^-(\mathbf{x})| \, d\mathbf{y} = 0.$$

The *jump set* $J_\mathbf{v}$ of $\mathbf{v}$ is the subset of points $\mathbf{x}$ in $S_\mathbf{v}$ where $\mathbf{v}$ has one-sided limits $\mathbf{v}^+(\mathbf{x}), \mathbf{v}^+(\mathbf{x})$ with respect to $\nu_\mathbf{v}(\mathbf{x})$ and $\mathbf{v}^+(\mathbf{x}) \neq \mathbf{v}^+(\mathbf{x})$.
We emphasize that $\operatorname{spt} \mathbf{v} \subset \overline{\Omega}$ entails: $J_\mathbf{v} \cap \partial\Omega$ may be nonempty, while $J_\mathbf{v} \setminus \overline{\Omega} = \emptyset$.
We denote by $\mathbf{v}$, $\mathcal{E}(\mathbf{v})$, and $\mathcal{E}(\mathbf{v})$, respectively, the admissible displacement vector field, the linearized strain tensor and its absolutely continuous part:

$$\mathbf{v} \ : \ \mathbf{R}^n \ \to \ \mathbf{R}^n\,, \qquad \operatorname{spt} \mathbf{v} \subset \overline{\Omega}\,, \qquad \mathrm{e}(\mathbf{v}) = \frac{1}{2}(D\mathbf{v} + (D\mathbf{v})^T),$$

$$\mathcal{E}(\mathbf{v}) = \frac{d\mathcal{E}(\mathbf{v})}{d\mathcal{L}^n}, \qquad \mathcal{E}^a(\mathbf{v}) = \mathcal{E}(\mathbf{v}) d\mathcal{L}^n, \qquad \operatorname{div} \ \mathbf{v} = \operatorname{Tr} \ \mathcal{E}(\mathbf{v}) = \nabla \cdot \mathbf{v}\,,$$

here $D\mathbf{v} = \{D_j v_i\}$, $(i = 1, ..., k, \ j = 1, ..., m)$ denotes the distributional derivatives of $\mathbf{v}$; $\nabla \mathbf{v} = \frac{dD\mathbf{v}}{d\mathcal{L}^n}$ denotes its absolutely continuous part and $[\mathbf{v}] \odot \nu_\mathbf{v} = \operatorname{sym}(\mathbf{v} \otimes \nu_\mathbf{v})$; the absolutely continuous part div^a of distributional divergence is defined as follows

$$\operatorname{div}^a \mathbf{v} := (d/d\mathcal{L}^n) \operatorname{div} \mathbf{v} = \operatorname{Tr} \mathcal{E}(\mathbf{v}) = \nabla \cdot \mathbf{v}\ .$$

We set $|\mathcal{E}(\mathbf{v})|^2 = \mathcal{E}(\mathbf{v})_{ij} \mathcal{E}(\mathbf{v})_{ij}$ and

$$\text{(6.3)} \qquad \mathcal{Q}(\mathbb{E}) = \mu |\mathbb{E}|^2 + \frac{\lambda}{2} (\operatorname{Tr} \mathbb{E})^2\,, \ \text{ for any } n \times n \text{ symmetric matrix } \mathbb{E}\,,$$

$$\mathcal{Q}\big(\mathcal{E}(\mathbf{v})\big) = \mu \, |\mathcal{E}(\mathbf{v})|^2 + \frac{\lambda}{2} \, (\operatorname{Tr} \mathcal{E}(\mathbf{v}))^2 = \mu \, \mathrm{e}_{ij}(\mathbf{v}) \, \delta_{ih} \delta_{jk} \, \mathcal{E}_{hk}(\mathbf{v}) + \frac{\lambda}{2} \, \delta_{ij} \, \delta_{hk} \, \mathcal{E}_{ij}(\mathbf{v}) \, \mathcal{E}_{hk}(\mathbf{v}).$$

Hence

$$\text{(6.4)} \qquad \mathcal{Q}'(\mathbb{E}) = 2\mu \mathbb{E} + \lambda (\operatorname{Tr} \mathbb{E}) I \ \text{ for any } n \times n \text{ symmetric matrix } \mathbb{E}.$$

$$\text{(6.5)} \ \ \mathcal{F}(\mathbf{v}) \ = \int_{\Omega^n} \left(\mathcal{Q}\left(\mathcal{E}(\mathbf{v})\right) - \mathbf{f} \cdot \mathbf{v} \right) d\mathbf{x} \ + \ \alpha \mathcal{H}^{n-1}(J_\mathbf{v}) \ + \ \gamma \int_{J_\mathbf{v}} |[\mathbf{v}] \odot \nu_\mathbf{v}| d\mathcal{H}^{n-1}$$

Space BD_Ω of functions with bounded deformation and support contained in $\overline{\Omega}$: we define

$$\text{(6.6)} \qquad BD_\Omega = \Big\{ \mathbf{v} \in L^1(\mathbb{R}^n, \mathbb{R}^n) \ : \ \mathrm{e}(\mathbf{v}) \in \mathcal{M}(\mathbb{R}^n, M_{n,n})\,, \ \operatorname{spt} \mathbf{v} \subset \overline{\Omega} \Big\}\,,$$

the space BD_Ω is endowed with the norm

$$\|\mathbf{v}\|_{BD_\Omega} := \|\mathbf{v}\|_{L^1(\mathbf{R}^n, \mathbf{R}^n)} + \int_{\mathbb{R}^n} |\mathrm{e}(\mathbf{v})| \ = \|\mathbf{v}\|_{L^1(\mathbf{R}^n, \mathbf{R}^n)} + |\mathrm{e}(\mathbf{v})|_{T(\overline{\Omega})}\,.$$

We list the main properties of functions with bounded deformation (see [4],[6],[18]).
The linear strain tensor $\mathrm{e}(\mathbf{v})$ has the following decomposition

$$\mathrm{e}(\mathbf{v}) = \mathrm{e}^a(\mathbf{v}) + \mathrm{e}^s(\mathbf{v}) = \mathrm{e}^a(\mathbf{v}) + \mathrm{e}^j(\mathbf{v}) + \mathrm{e}^c(\mathbf{v}), \tag{6.7}$$

where $\mathrm{e}^a(\mathbf{v}) = \mathcal{E}(\mathbf{v})d\mathbf{x}$ and $\mathrm{e}^s(\mathbf{v})$ are, respectively, the absolutely continuous and the singular part of $\mathrm{e}(\mathbf{v})$ with respect to $\mathcal{L}^n$, while $\mathrm{e}^j(\mathbf{v})$ and $\mathrm{e}^c(\mathbf{v})$ are respectively the restriction of $\mathrm{e}^s(\mathbf{v})$ to $J_\mathbf{v}$ and to its complement; $\mathrm{e}^j(\mathbf{v})$ and $\mathrm{e}^c(\mathbf{v})$ are called the *jump part* and the *Cantor part* of $\mathrm{e}^s(\mathbf{v})$.
For any $\mathbf{v} \in BD_\Omega$, the jump set $J_\mathbf{v}$ is $\mathcal{L}^n$ negligible, countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable, and

$$\mathrm{e}^j(\mathbf{v}) \,=\, (\mathbf{v}^+(x) - \mathbf{v}^-(x)) \odot \nu_\mathbf{v}\, \mathcal{H}^{n-1} \llcorner J_\mathbf{v} \qquad \mathcal{H}^{n-1}\text{a.e. in } J_\mathbf{v}\,. \tag{6.8}$$

For every connected Lipschitz open set Ω there is a constant C_Ω, dependent only on Ω s.t. ([47]) a *Korn-Poincaré inequality* holds:

$$\|\mathbf{v}\|_{L^{n/(n-1)}(\Omega)} \;\le\; C_\Omega\, |\mathrm{e}(\mathbf{v})|_{T(\overline{\Omega})} \qquad \forall \mathbf{v} \in\; BD(\mathbf{R}^n) :\, \mathrm{spt}\,\mathbf{v} \subset \overline{\Omega}. \tag{6.9}$$

The *Space* SBD_Ω *of functions with special bounded deformation and support contained in* $\overline{\Omega}$ is defined as follows:

$$SBD_\Omega = \Big\{ \mathbf{v} \in BD_\Omega \,:\quad \mathrm{e}^s(\mathbf{v}) = \mathrm{e}^j(\mathbf{v}) \Big\}. \tag{6.10}$$

Lemma 6.1. *(***safe load condition for free cohesive damage***)*
Assume (6.1),(6.2),(6.3) *and*

$$\|\mathbf{f}\|_{L^p(\Omega)} \;<\; \frac{\gamma}{C_\Omega\, |\Omega|^{\frac{1}{n}-\frac{1}{p}}} \qquad\qquad (\textbf{safe load})\,, \tag{6.11}$$

where C_Ω is the constant in the Korn-Poincaré inequality (6.9).
Then the functional $\mathcal{F}$ achieves a finite minimum over the space SBD_Ω (see (6.6),(6.10)*).*

Proof - See Theorem 3.1 in [18]. □

So far we have existence of minimizers. Studying regularity of $\mathcal{F}$ minimizers is even more delicate than the minimizers of Problem II about plates: here not only we cannot achieve a complete system of Euler equations analogous to the ones in Theorem 3.3 about 1d model problem (besides the standard PDE system of linear elasticity outside the jump set, say: $-\mu\Delta\mathbf{u} - (\lambda+\mu)D(\mathrm{div}\,\mathbf{u}) = \mathbf{f}$ in $\Omega\backslash J_\mathbf{v}$), but also we lack a partial regularity result analogous to Theorem 5.9. Nevertheless we can show again of Du Bois-Raymond equation (see (6.12)) and an excess estimate (see (6.21)) which allows the proof of regularity for minimizers under an explicit smallness condition of L^p norm of load (Theorem 6.7).

Lemma 6.2. *Assume* (6.1),(6.2),(6.3)(6.11)*. Then, for any* $\mathbf{w} \in C^1(\overline{\Omega}) \cap SBD(\mathbb{R}^n)$ *with* $\mathrm{spt}\,\mathbf{w} \subset \overline{\Omega}$ *and* $\mathbf{v} \in \mathrm{argmin}\,\mathcal{F}$

$$\int_\Omega (\mathcal{Q}'(\mathcal{E}(\mathbf{v})) : (\mathrm{e}(\mathbf{w}) - \mathcal{E}(\mathbf{v})) - \mathbf{f}\cdot(\mathbf{w} - \mathbf{v}))\, d\mathbf{x} - \gamma \int_{J_\mathbf{v}} |[\mathbf{v}] \odot \nu_\mathbf{v}|\; d\mathcal{H}^{n-1} \,=\, 0\,. \tag{6.12}$$

Proof - By exploiting representation (6.5) of $\mathcal{F}$, minimality of v, convexity of $\mathcal{Q}$, jump sets coincidence $J_{\mathbf{w}-\mathbf{v}} = J_\mathbf{v}$, small positive and negative ε, we get the thesis. □

Lemma 6.3. ***(Compliance identity for cohesive damage)***
Assume (6.1),(6.2),(6.3)(6.11)*. Then for any* $\mathbf{v} \in \mathrm{argmin}\,\mathcal{F}$

$$\int_\Omega (\, \mathcal{Q}'(\mathcal{E}(\mathbf{v})) : \mathcal{E}(\mathbf{v}) \,-\, \mathbf{f}\cdot\mathbf{v}\,)\; d\mathbf{x} \;+\; \gamma \int_{J_\mathbf{v}} |[\mathbf{v}] \odot \nu_\mathbf{v}|\; d\mathcal{H}^{n-1} \;=\; 0\,, \tag{6.13}$$

say, by (6.4),

$$2\int_{\Sigma}\mathcal{Q}(\mathcal{E}(\mathbf{v}))\,d\mathbf{x} \;=\; \int_{\Sigma}\mathbf{f}\cdot\mathbf{v}\,d\mathbf{x}\,-\,\gamma\int_{J_{\mathbf{v}}}|[\mathbf{v}]\odot\nu_{\mathbf{v}}|\;d\mathcal{H}^{\,n-1}\,. \tag{6.14}$$

Hence the following compliance identity holds true

$$\mathcal{P}(\mathbf{v}) \;=\; -\int_{\Sigma}\mathcal{Q}(\mathcal{E}(\mathbf{v}))\,d\mathbf{x}\,+\,\alpha\,\mathcal{H}^{n-1}(S_{\mathbf{v}}) \qquad \forall\mathbf{v}\in\operatorname{argmin}\mathcal{F}\,. \tag{6.15}$$

Proof - Choose $\mathbf{w}=0$ in (6.12). □

Lemma 6.4. ***(Elliptic regularity)*** *Assume* (6.2) *and*

(6.16) Ω *bilipschitz-homeomeorphic to* $(0,1)^n$ *with boundary* $\partial\Omega$ *of class* C^2.

Then the system of elasticity with constant coefficients

$$\mathbf{u}\in H^1_0(\Omega),\quad \mathcal{L}(\mathbf{u})\;:=-\mu\Delta\mathbf{u}-(\lambda+\mu)D(\operatorname{div}\mathbf{u})=\mathbf{f}\ \textit{in}\ \Omega, \tag{6.17}$$

has unique solution $\mathbf{u}$ *which is also the unique minimizer over* $\mathbf{v}\in H^1_0(\Omega)$ *of purely elastic energy* $\int_\Omega(\mathcal{Q}(\mathbf{e}\mathbf{v})-\mathbf{f}\cdot\mathbf{v})\;d\mathbf{x}$. *Such* $\mathbf{u}$ *fulfils the associate compliance inequality:*

$$\int_{\Omega}\mathcal{Q}(\mathbf{e}(\mathbf{u}))\,d\mathbf{x}\;=\;\frac{1}{2}\int_{\Omega}\mathbf{f}\cdot\mathbf{u}\,d\mathbf{x}\,.\,, \tag{6.18}$$

Moreover $\mathbf{u}$ *belongs to* $W^{2,p}(\Omega)$ *and and there are constants* $K_1,\,K_2$, *with* $K_1=K_1(\Omega,p,\lambda,\mu)$ *and* $K_2=K_2(\Omega,p,\lambda,\mu)$ *s.t.*

$$\|\mathbf{u}\|_{W^{2,p}(\Omega)}\;\leq\;K_1\,\|\mathbf{f}\|_{L^p(\Omega)}\,, \tag{6.19}$$

$$\|D\mathbf{u}\|_{C^0(\overline{\Omega})}\;\leq\;K_2\,\|\mathbf{f}\|_{L^p(\Sigma)}\,. \tag{6.20}$$

Proof - The system of elasticity (6.17) is a Dirichlet problem with null boundary data for an equation of the type $a^{ij}_{\alpha\beta}D_{ij}u^\beta=f_\alpha$ with constant coefficients satisfying Legendre-Hadamard condition ((10.60) in [31] pag.381). Hence (by estimates (10.62),(10.63) in [31], page 381) we get the existence of $K_1=K_1(\Omega,p,\lambda,\mu)>0$ such that

$$\|D^2\mathbf{u}\|_{L^p(\Omega)}\leq K_1\|\mathbf{f}\|_{L^p(\Omega)}\,.$$

Hence $\mathbf{u}\in H^1_0$, $\fint_\Omega D\mathbf{u}\,d\mathbf{x}=\mathbf{0}$ entail (6.19). Inequality (6.20) follows by Sobolev embedding.

Lemma 6.5. *Assume* (6.2) (5.21) *and* $\mathbf{u}$ *is the trivial extension of the unique solution of* (6.17). *Then* $\mathbf{u}$ *admits a trivial extension (still denoted* $\mathbf{u}$*) in* $C^0(\mathbb{R}^n)\cap C^1(\overline{\Omega})$ *and the following statements hold true.*

Excess estimate for minimizers of $\mathcal{F}$ **:** *for all* $\mathbf{v}\in SBD_\Omega$

$$\mathcal{F}(\mathbf{v})-\mathcal{F}(\mathbf{u})\;\geq\;\alpha\mathcal{H}^{n-1}(J_{\mathbf{v}})+\int_{J_{\mathbf{v}}}\Big(\gamma|[\mathbf{v}]\odot\nu_{\mathbf{v}}|-\mathcal{Q}'(\mathbf{e}(\mathbf{u})):([\mathbf{v}]\odot\nu_{\mathbf{v}})\Big)\,d\mathcal{H}^{\,n-1}\,. \tag{6.21}$$

Excess identity for minimizers of $\mathcal{F}$**:** *for all* $\mathbf{v}$ *minimizinf* $\mathcal{F}$ *over* SBD_Ω

$$\mathcal{F}(\mathbf{v})-\mathcal{F}(\mathbf{u})\;=\;\alpha\mathcal{H}^{n-1}(J_{\mathbf{v}})+\frac{1}{2}\int_{J_{\mathbf{v}}}\Big(\gamma|[\mathbf{v}]\odot\nu_{\mathbf{v}}|-\mathcal{Q}'(\mathbf{e}(\mathbf{u})):([\mathbf{v}]\odot\nu_{\mathbf{v}})\Big)\,d\mathcal{H}^{\,n-1}\,. \tag{6.22}$$

Proof - We write shortly ν instead of $\nu_{\mathbf{v}}$ and we omit $d\mathbf{x}$ and $d\mathcal{H}^{\,n-1}$ in the proof. Lemma (6.4) entails that trivial extension of $\mathbf{u}$ belongs to $H^1_0(\Omega,\mathbb{R}^3)\cap C^1(\overline{\Omega})$, hence

$$D\mathbf{u}\in C^0(\overline{\Omega})\,,\qquad \mathbf{u}\in C^0(\mathbb{R}^n) \tag{6.23}$$

By convexity, for every $\mathbf{v} \in SBD_\Omega$ we have

$$\begin{aligned}\mathcal{F}(\mathbf{v}) - \mathcal{F}(\mathbf{u}) \;&\geq \\ &\geq \int_{\mathbb{R}} \big(\mathcal{Q}'(\mathrm{e}(\mathbf{u})) : \big(\mathcal{E}(\mathbf{v}) - \mathrm{e}(\mathbf{u})\big) - \mathbf{f}\cdot(\mathbf{v}-\mathbf{u})\big) + \\ &\quad + \gamma \int_{J_\mathbf{v}} |[\mathbf{v}] \odot \nu_\mathbf{v}| + \alpha\, \mathcal{H}^{n-1}(J_\mathbf{v})\,.\end{aligned} \tag{6.24}$$

By (6.23) we can apply Lemma 6.2 with $\mathbf{w} = \mathbf{u}$ and $\mathbf{v}$ minimizer of $\mathcal{F}$ over SBD_Ω.
By (6.8),(6.17) and (6.24) we have:

$$\mathcal{E}(\mathbf{v}) \;=\; \mathrm{e}(\mathbf{v}) - [\mathbf{v}] \odot \nu\, d\mathcal{H}^{n-1} \llcorner J_\mathbf{v} \cap \Omega \qquad \text{in } \mathcal{D}'(\Omega)\,. \tag{6.25}$$

Hence, via integration by parts and taking into account (6.17), spt $\mathbf{v} \subset \overline{\Omega}$ and $\mathbf{u} = \mathbf{0}$ on $\partial\Omega$, we get

$$\begin{aligned}&\int_\Omega \mathcal{Q}'(\mathrm{e}(\mathbf{u})) : (\mathrm{e}(\mathbf{v}) - \mathrm{e}(\mathbf{u})) \;= \\ &= \int_\Omega \mathcal{L}(\mathbf{u})\cdot(\mathbf{v}-\mathbf{u}) \;+\; \int_{\partial\Omega} \mathcal{Q}'(\mathrm{e}(\mathbf{u})) : \big(\mathbf{v}_{in} \odot \nu_\Omega\big) \;= \\ &= \int_\Omega \mathbf{f}(\mathbf{v}-\mathbf{u}) \;+\; \int_{\partial\Omega} \mathcal{Q}'(\mathrm{e}(\mathbf{u})) : \big(\mathbf{v}_{in} \odot \nu_\Omega\big)\end{aligned} \tag{6.26}$$

where ν_Ω is the outward normal to $\partial\Omega$ and $\mathbf{v}_{in}$, $\mathbf{v}_{out}$ denote the inner and outer traces of $\mathbf{v}$ in Ω. We choose $\nu = \nu_{J_\mathbf{v}} = \nu_\Omega$ on $\partial\Omega \cap J_\mathbf{v}$ and, abusing notation we define $\big(\mathbf{v}_{in} \odot \nu\big) = \mathbb{O}$ on $\partial\Omega \setminus J_\mathbf{v}$; with this convention we get

$$[\mathbf{v}] \;=\; \mathbf{v}_{out} - \mathbf{v}_{in} = -\mathbf{v}_{in} \tag{6.27}$$

$$\mathbf{v}_{in} \odot \nu_\Omega \;=\; -\,[\mathbf{v}] \odot \nu \tag{6.28}$$

so that (6.26) reads as follows

$$\begin{aligned}&\int_\Omega \mathcal{Q}'(\mathrm{e}(\mathbf{u})) : (\mathrm{e}(\mathbf{v}) - \mathrm{e}(\mathbf{u})) \;= \\ &\qquad = + \int_\Omega \mathbf{f}(\mathbf{v}-\mathbf{u}) \;-\; \int_{\partial\Omega} \mathcal{Q}'(\mathrm{e}(\mathbf{u})) : ([\mathbf{v}] \odot \nu)\end{aligned} \tag{6.29}$$

by substituting (6.25) in (6.24) and taking into account (6.29) we get

$$\begin{aligned}\mathcal{F}(\mathbf{v}) - \mathcal{F}(\mathbf{u}) \;&\geq\; \alpha\mathcal{H}^{n-1}(J_\mathbf{v}) + \int_\Omega \mathcal{Q}'(\mathrm{e}(\mathbf{u})) : (\mathrm{e}(\mathbf{v}) - \mathrm{e}(\mathbf{u})) + \\ &\quad - \int_\Sigma \mathbf{f}(\mathbf{v}-\mathbf{u}) + \gamma \int_{J_\mathbf{v}\cap\Omega} |[\mathbf{v}] \odot \nu| \;- \int_{J_\mathbf{v}\cap\Omega} \mathcal{Q}'(\mathrm{e}(\mathbf{u})) : ([\mathbf{v}] \odot \nu) = \\ &= \alpha\mathcal{H}^1(J_\mathbf{v}) + \int_\Omega \mathbf{f}(\mathbf{v}-\mathbf{u}) \;-\; \int_{\partial\Omega} \mathcal{Q}'(\mathrm{e}(\mathbf{u})) : ([\mathbf{v}] \odot \nu) + \\ &\quad - \int_\Sigma \mathbf{f}(\mathbf{v}-\mathbf{u}) + \gamma \int_{J_\mathbf{v}\cap\Omega} |[\mathbf{v}] \odot \nu| - \int_{J_\mathbf{v}\cap\Omega} \mathcal{Q}'(\mathrm{e}(\mathbf{u})) : ([\mathbf{v}] \odot \nu) = \\ &= \alpha\mathcal{H}^{n-1}(J_\mathbf{v}) + \left(\int_{J_\mathbf{v}} \gamma\, |[\mathbf{v}] \odot \nu| - \int_{J_\mathbf{v}} \mathcal{Q}'(\mathrm{e}(\mathbf{u})) : ([\mathbf{v}] \odot \nu)\right) \;= \\ &= \alpha\mathcal{H}^{n-1}(J_\mathbf{v}) + \int_{J_\mathbf{v}} \big(\gamma\, |[\mathbf{v}] \odot \nu| - \mathcal{Q}'(\mathrm{e}(\mathbf{u})) : \big([\mathbf{v}] \odot \nu\big)\big)\end{aligned} \tag{6.30}$$

hence (6.21). Since $\mathcal{Q}$ is a symmetric quadratic form we get

$$\mathcal{Q}(\mathbb{A}) - \mathcal{Q}(\mathbb{B}) = \frac{1}{2}(\mathcal{Q}'(\mathbb{A}) + \mathcal{Q}'(\mathbb{B})) : (\mathbb{A} - \mathbb{B}) \tag{6.31}$$

hence by using (6.25),(6.29),(6.31) and eventually (6.12) we get (6.22) as follows:

$$\begin{aligned}
&\mathcal{F}(\mathbf{v}) - \mathcal{F}(\mathbf{u}) = \\
&= \int_\Omega \Big(\big(\mathcal{Q}(\mathcal{E}(\mathbf{v})) - \mathcal{Q}(\mathrm{e}(\mathbf{u}))\big) - \mathbf{f}(\mathbf{v}-\mathbf{u}) \Big)\, d\mathbf{x} + \\
&\quad +\gamma \int_{J_\mathbf{v}} |[\mathbf{v}]\odot \nu|\; d\mathcal{H}^{n-1} + \alpha \mathcal{H}^{n-1}(J_\mathbf{v}) = \\
&\quad \frac{1}{2}\int_\Omega \big(\mathcal{Q}'(\mathcal{E}(\mathbf{v})) + \mathcal{Q}'(\mathrm{e}(\mathbf{u}))\big) : \big(\mathcal{E}(\mathbf{v}) - \mathrm{e}(\mathbf{u})\big)\, d\mathbf{x} + \\
&\quad - \int_\Omega \mathbf{f}(\mathbf{v}-\mathbf{u})\, d\mathbf{x} + \gamma \int_{J_\mathbf{v}} |[\mathbf{v}]\odot \nu|\; d\mathcal{H}^{n-1} + \alpha \mathcal{H}^{n-1}(J_\mathbf{v}) = \\
&= \frac{1}{2}\gamma \int_{J_\mathbf{v}} |[\mathbf{v}]\odot \nu|\; d\mathcal{H}^{n-1} - \frac{1}{2}\int_\Omega \mathbf{f}(\mathbf{v}-\mathbf{u})\, d\mathbf{x} + \\
&\quad \alpha \mathcal{H}^{n-1}(J_\mathbf{v}) + \frac{1}{2}\int_\Omega \mathcal{Q}'(\mathrm{e}(\mathbf{u})) : (\mathrm{e}(\mathbf{v}) - \mathrm{e}(\mathbf{u}))\, d\mathbf{x} + \\
&\quad -\frac{1}{2}\int_{J_\mathbf{v}\cap\Omega} \mathcal{Q}'(\mathrm{e}(\mathbf{u})) : ([\mathbf{v}]\odot\nu)\, d\mathcal{H}^{n-1} = \\
&\quad = \alpha\mathcal{H}^{n-1}(J_\mathbf{v}) + \frac{1}{2}\int_{J_\mathbf{v}} \Big(|[\mathbf{v}]\odot\nu| - \mathcal{Q}'(\mathrm{e}(\mathbf{u})) : ([\mathbf{v}]\odot\nu) \Big)\, d\mathcal{H}^{n-1}. \ \square
\end{aligned} \tag{6.32}$$

We can restate excess estimate (6.21) in the form of a calibration by comparison as follows.

Theorem 6.6. *(***stress regularity condition in elasticity with free damage***)* *Assume* (6.2) (6.16) *and* $\mathbf{u}$ *is the trivial extension of the unique solution of* (6.17) *and*

$$\| \, \mathcal{Q}'(\mathrm{e}(\mathbf{u})) \, \|_{L^\infty(\Omega, M^3)} \;\le\; \gamma \tag{6.33}$$

Then $\mathbf{u}$ *is the unique minimizer of* $\mathcal{F}$ *in* SBD_Ω.
Explicitly the above stress regularity condition (6.33) *reads:*

$$\| \, 2\,\mu\,\mathrm{e}(\mathbf{u}) \,+\, \lambda\,(\mathrm{Tr}\,\mathrm{e}(\mathbf{u}))\, I \, \|_{L^\infty(\Omega, M^3)} \;\le\; \gamma\,. \tag{6.34}$$

Proof - By excess estimate (6.21) we get, for any $\mathbf{v} \in SBD_\Omega$,

$$\mathcal{F}(\mathbf{v}) - \mathcal{F}(\mathbf{u}) \;\ge\; \alpha\mathcal{H}^1(J_\mathbf{v}) \;+\; \int_{J_\mathbf{v}} \big(\gamma\,|[\mathbf{v}]\odot\nu_\mathbf{v}| \;-\; \mathcal{Q}'\big(\mathcal{E}(\mathbf{u})\big) : [\mathbf{v}]\odot\nu_\mathbf{v}\big) \;\ge\; 0\,.$$

The last inequality is strict if $J_\mathbf{v} \neq \emptyset$, hence the all the minimizer of $\mathcal{F}$ H^1 are regular. But there is only one H^1 regular minimizer: the solution u of (6.17).

Theorem 6.7. ***Load regularity condition in elasticity with damage***
Assume (6.2),(6.16) *and*

$$\|\mathbf{f}\|_{L^p} \ \leq \ \frac{\gamma}{(2\mu + 3\lambda)\, K_2} \tag{6.35}$$

where $K_2 = K_2(\Omega, p, a, A)$ *is the constant appearing in the estimate* (6.20).
Then u *minimizes* $\mathcal{F}$ *in* SBD_Ω. *Moreover* u *is the unique minimizer of* $\mathcal{F}$ *in this class.*

Proof - Inequalities (6.4),(6.20),(6.35) entail (6.33), hence thesis follows by Theorem 6.6.

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