Semmes family of curves and a characterization of functions of bounded variation in terms of curves *

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Abstract

On metric spaces supporting a geometric version of a Semmes family of curves, we provide a Reshetnyak-type characterization of functions of bounded variation in terms of the total variation on such a family of curves. We then use this characterization to obtain a Federer-type characterization of sets of finite perimeter, that is, we show that a measurable set is of finite perimeter if and only if the Hausdorff measure of its measure theoretic boundary is finite. We present a construction of a geometric Semmes family of curves in the first Heisenberg group.

1 Introduction

Research on analysis in metric measure spaces, a field that has seen much recent activity, has produced various analogs of the Sobolev spaces of functions $W^{1,p}(\Omega)$, where Ω is a Euclidean domain. As expected from the behavior of the classical Sobolev spaces, the behavior of these spaces is significantly different for $p = 1$ than for $p > 1$. It is well-known that the relaxation of the space $W^{1,1}(\Omega)$ is associated with the more geometric objects called functions of bounded variation, denoted $BV(\Omega)$. Using this observation, a theory of functions of bounded variation was developed in [Mir], [Am1], [Am2] and [AMP] in the setting of metric spaces equipped with a doubling measure.

Surprisingly, the theory of functions of bounded variation plays a role in the study of the analogs of the Sobolev classes $W_0^{1,p}(\Omega)$, $p > 1$, of functions with zero boundary values in such a metric setting. In [KKST] it was first shown that a so-called strong relative isoperimetric inequality is equivalent with a Federer-type characterization of sets of finite perimeter, that is, sets whose characteristic functions are functions of bounded variation. According to

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the Federer-type characterization, a set has finite perimeter if and only if the codimension 1 Hausdorff measure of its measure theoretic boundary is finite. In verifying a characterization of the analogs of Sobolev functions with zero boundary values on a domain in the metric measure space, it was shown in [KKST] that this Federer-type characterization allows us to eliminate the overly restrictive assumption of finiteness of a codimension 1 Hausdorff measure of the topological boundary of the domain. In the Euclidean setting it is well-known that sets whose measure theoretic boundaries are of finite codimension 1 Hausdorff measure are of finite perimeter. This fact was first proved by Federer in [Fed], and a simplified version of the proof can be found in [EvGa]. A weaker version of the result is proved in [KL] in the setting of a metric space equipped with a doubling measure and supporting a Poincaré inequality.

The proof of the Federer-type characterization in the Euclidean setting uses a Reshetnyak-type characterization of functions of bounded variation in terms of the total variation of the function restricted to lines parallel to a coordinate axis. An analogous characterization of Sobolev functions, involving absolute continuity on lines, holds in the Euclidean setting, see for example [Zie], [Vai] and [Oht]. This characterization has been used in [Shan] and [HKST] to construct an analog of Sobolev classes of functions in the metric measure space setting; such classes of functions, called the Newtonian spaces $N^{1,p}$, have been used extensively to study potential theory and quasiconformal mappings. In the metric setting, a Reshetnyak-type characterization involving *all* curves in the space has been established for BV functions, see [AmDi]. However, it is unclear how to proceed from this to the Federer-type characterization. The point of the present work is to show that if a doubling metric measure space supports a geometric version of a *Semmes family of curves* originally presented in [Sem], then measurable sets whose measure theoretic boundaries have finite codimension 1 Hausdorff measure are necessarily of finite perimeter in the sense of [Mir] and [Am2]. One of the simplest non-Euclidean metric measure spaces, the first Heisenberg group, appears to support a geometric Semmes family of curves.

This paper is organized as follows. Section 2 will contain the basic definitions and notation used throughout the paper, and in Section 3 we discuss the notion of a Semmes family of curves and its additional geometric properties. In Section 4 we provide a Reshetnyak-type characterization of functions of bounded variation in terms of the total variation of a function on curves belonging to the geometric Semmes family. In Section 5 we proceed to give a proof of the Federer-type characterization of sets of finite perimeter, by means of the Reshetnyak-type characterization. Section 6 contains examples of metric measure spaces that support a geometric Semmes family of curves, including Euclidean spaces and Fred Gehring's bow-tie. In particular, we present a construction of the geometric Semmes family in the first Heisenberg group.

2 Preliminaries

2.1 Notation and assumptions

We assume throughout that $X = (X, d, \mu)$ is a metric measure space equipped with a metric d and a Borel regular, *doubling* outer measure μ. The doubling property means that there is a fixed constant $c_d \geq 1$, called the *doubling constant* of μ, such that

$$\mu(2B) \leq c_d\, \mu(B) \tag{2.1}$$

for every ball $B = B(x,r) := \{y \in X : d(y,x) < r\}$. Here $tB = B(x,tr)$, and we also denote $r =: \mathrm{rad}(B)$. We note that in a metric space, balls do not necessarily have unique centers or radii, but we assume each ball to have a prescribed center point and radius. We assume that the measure of every open set is positive and that the measure of every bounded set is finite. The doubling condition implies that for every $x \in X$, every $0 < r \leq R < \infty$ and every $y \in B(x,R)$, we have

$$\frac{\mu(B(y,r))}{\mu(B(x,R))} \geq c \left(\frac{r}{R}\right)^Q \tag{2.2}$$

for some constants $c > 0$ and $Q > 0$, which only depend on the doubling constant c_d. We also assume the space to be connected, which is implied by the Poincaré inequality (see (2.10)), and which in turn implies that $\mu(\{x\}) = 0$ for every $x \in X$ (we assume that X is not a one-point space). The Poincaré inequality also implies that necessarily $Q \geq 1$.

We further assume that X is complete; recall that a metric space with a doubling measure is complete if and only if the space is proper, that is, closed and bounded sets are compact. Since X is proper, for any open set $\Omega \subset X$ we define local spaces as follows: e.g. $\mathrm{Lip}_{\mathrm{loc}}(\Omega)$ is the space of functions that are Lipschitz in every open $\Omega' \Subset \Omega$. Here $\Omega' \Subset \Omega$ means that $\overline{\Omega'}$ is a compact subset of Ω. The support of a function u on X is denoted $\mathrm{supp}(u)$.

The integral average of a function $u \in L^1(A)$ over a μ-measurable set A with finite and positive measure is $u_A := \fint_A u\, d\mu := \mu(A)^{-1} \int_A u\, d\mu$. The characteristic function of a set $E \subset X$ is denoted χ_E. In general, C will denote a positive constant whose value is not necessarily the same at each occurrence. When a constant C depends on e.g. the numbers a and b, it will be denoted $C(a,b)$. When we say that a property holds for μ-almost every $x, y \in X$, we mean that there is a set $E \subset X$ such that $\mu(E) = 0$ and the property holds for every $x, y \in X \setminus E$. We use the abbreviation "a.e." for "almost every" or "almost everywhere".

We recall that because μ is doubling, for every locally integrable function $u \in L^1_{\mathrm{loc}}(X)$, μ-a.e. point $x \in X$ is a Lebesgue point, meaning that

$$\lim_{r \to 0} \fint_{B(x,r)} |u - u(x)|\, d\mu = 0.$$

See [Hei, Theorem 1.8] for a proof of this fact. Furthermore, for every μ-measurable function u that is finite valued μ-a.e., μ-a.e. point is a point of approximate continuity (see e.g. [EvGa, Section 1.7.2]). This means that for every $\varepsilon > 0$,

$$\lim_{r \to 0} \frac{\mu(B(x,r) \cap \{|u - u(x)| \geq \varepsilon\})}{\mu(B(x,r))} = 0.$$

Each Lebesgue point of u is clearly a point of approximate continuity, and the converse holds when u is essentially bounded.

Given two sets $A_1, A_2 \subset X$, the distance $\mathrm{dist}(A_1, A_2)$ is defined by

$$\mathrm{dist}(A_1, A_2) := \inf\{d(x,y) \,:\, x \in A_1, y \in A_2\},$$

whereas the Hausdorff distance is defined by

$$d_H(A_1,A_2) := \inf\left\{\varepsilon>0\,:\, A_1\subset\bigcup_{x\in A_2}B(x,\varepsilon)\text{ and }A_2\subset\bigcup_{x\in A_1}B(x,\varepsilon)\right\}.$$

A curve is a rectifiable continuous mapping from a compact interval into X. The length of a curve γ is denoted ℓ_γ or $\ell(\gamma)$. We will assume that all curves are parametrized by arc length; such re-parametrization can always be done because γ is rectifiable (see e.g. [Haj] or [AT]). The image of a curve γ in X is also denoted γ. A curve γ' is a subcurve of a curve γ if γ' is, after reparameterization, equal to $\gamma|_{[a,b]}$ for some $0\le a<b\le \ell_\gamma$.

A nonnegative Borel function g on X is an upper gradient of an extended real valued function u on X, if for all curves γ in X, we have

$$|u(x)-u(y)|\le\int_\gamma g\,ds \tag{2.3}$$

whenever both $u(x)$ and $u(y)$ are finite, and $\int_\gamma g\,ds=\infty$ otherwise. Here x and y denote the end points of γ.

Next we recall the definition of functions of bounded variation on metric spaces, given by Miranda in [Mir].

Definition 2.4. For $u\in L^1_{\mathrm{loc}}(X)$, we define the total variation of u as

$$\|Du\|(X):=\inf\left\{\liminf_{i\to\infty}\inf_{g_{u_i}}\int_X g_{u_i}\,d\mu : u_i\in \mathrm{Lip}_{\mathrm{loc}}(X), u_i\to u\text{ in }L^1_{\mathrm{loc}}(X)\right\},$$

where the second infimum in the above is over all upper gradients g_{u_i} of u_i. We say that a function $u\in L^1_{\mathrm{loc}}(X)$ is of bounded variation, $u\in BV(X)$, if $\|Du\|(X)<\infty$. If the function u is the characteristic function of a set $E\subset X$ and $\|D\chi_E\|(X)<\infty$, we say that the set E has finite perimeter. Similarly, we define $\|Du\|(\Omega)$ for any open set $\Omega\subset X$. If $u\in BV(X)$, given an arbitrary set $A\subset X$ (not necessarily open) we define

$$\|Du\|(A):=\inf\{\|Du\|(\Omega): A\subset\Omega, \Omega\text{ is open}\}.$$

According to [Mir], $\|Du\|(\cdot)$ is then a finite Borel outer measure. For a set E, we also define

$$P(E,A):=\|D\chi_E\|(A),$$

which we call the perimeter of E in A.

When the space is taken to be the real line $\mathbb{R}$ and Ω is an interval (a,b), with $-\infty\le a<b\le\infty$, we have a useful equivalent formulation for the variation measure. Namely, for a Lebesgue measurable function u, we have

$$\|Du\|((a,b))=\sup\left\{\sum_{n=1}^{l}|u(t_{n-1})-u(t_n)|\right\}, \tag{2.5}$$

where the supremum is taken over all finite partitions $a<t_0<\ldots<t_l<b$ such that each t_n is a point of approximate continuity of u, see e.g. [EvGa, Section 5.10.1]. For a Borel function u on X and curve γ, we use the notation

$$\|D_\gamma u\|((0,\ell_\gamma)):=\|D(u\circ\gamma)\|((0,\ell_\gamma)).$$

Remark 2.6. In standard texts on real analysis (see for example [Roy] or [Rud]), the definition of functions of bounded variation on an interval does not require the partitions $\{t_n\}$ to consist of points of approximate continuity. This type of definition is suitable if one works with a particular L^1-representative of each BV function. However, in our setting one can always perturb a function on a set of measure zero without changing the total variation.

For any set $A \subset X$, the restricted spherical Hausdorff content of codimension 1 is defined as

$$\mathcal{H}_R^h(A) := \inf\left\{\sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i} : A \subset \bigcup_{i=1}^{\infty} B(x_i, r_i), r_i \leq R\right\},$$

where $0 < R < \infty$. The Hausdorff measure of codimension 1 of a set $A \subset X$ is

$$\mathcal{H}^h(A) := \lim_{R\to 0} \mathcal{H}_R^h(A). \tag{2.7}$$

The (topological) boundary ∂E of a set $E \subset X$ is defined as usual. The measure theoretic boundary $\partial^* E$ of a set $E \subset X$ is defined as the set of points $x \in X$ where both E and its complement have positive upper density, i.e.

$$\limsup_{r\to 0} \frac{\mu(B(x,r)\cap E)}{\mu(B(x,r))} > 0 \quad \text{and} \quad \limsup_{r\to 0} \frac{\mu(B(x,r)\setminus E)}{\mu(B(x,r))} > 0.$$

Note that $\partial^* E \subset \partial E$. Further, we define the measure theoretic interior of E to be

$$I := \left\{x \in X : \lim_{r\to 0} \frac{\mu(B(x,r)\setminus E)}{\mu(B(x,r))} = 0\right\}, \tag{2.8}$$

and its measure theoretic exterior to be

$$O := \left\{x \in X : \lim_{r\to 0} \frac{\mu(B(x,r)\cap E)}{\mu(B(x,r))} = 0\right\}. \tag{2.9}$$

Note that the measure theoretic boundary of E satisfies $\partial^* E = X \setminus (I \cup O)$, and that for any μ-measurable set E we have

$$\mu(E\Delta I) = \mu(O\Delta(X\setminus E)) = 0,$$

where Δ is the symmetric difference operator.

We say that X supports a $(1,1)$-Poincaré inequality if there exist constants $c_P > 0$ and $\tau \geq 1$ such that for all balls $B = B(x,r)$, all locally integrable functions u, and all upper gradients g of u, we have

$$⨍_B |u - u_B|\, d\mu \leq c_P r ⨍_{\tau B} g\, d\mu. \tag{2.10}$$

In this paper, with the exception of the last section, we assume that the metric space X supports a $(1,1)$-Poincaré inequality — see also the discussion in Section 3.1. Given any $u \in L^1_{\mathrm{loc}}(X)$ and any ball B, by applying (2.10) to approximating locally Lipschitz functions in the definition of $BV(\tau B)$, we get

$$⨍_B |u - u_B|\, d\mu \leq c_P r \frac{\|Du\|(\tau B)}{\mu(\tau B)}, \tag{2.11}$$

where the constant c_P and the dilation factor τ are the same as in (2.10).

2.2 Characterizations of BV functions

Let $\Omega \subset X$ be an open set. For $r > 0$ and $\widehat{\tau} \geq 1$, let $\mathcal{B}_{\widehat{\tau},r}(\Omega)$ denote the collection of all families $\{B_i\}$ of balls B_i with radii no more than r such that the family $\{\widehat{\tau}B_i\}$ is pairwise disjoint and contained in Ω. The following result can be found in [HKT], and is originally derived in [Mir].

Theorem 2.12 ([Mir], also see Theorem 1.1(1) of [HKT]). *If $u \in L^1_{loc}(\Omega)$ such that*

$$\|u\|_{A^{1,1}_{\widehat{\tau}}(\Omega)} := \lim_{r\to 0} \sup_{\mathcal{F}\in\mathcal{B}_{\widehat{\tau},r}} \left\| \sum_{B\in\mathcal{F}} \left(\operatorname{rad}(B)^{-1} \fint_B |u-u_B|\, d\mu \right) \chi_B \right\|_{L^1(\Omega)}$$

is finite for some $\widehat{\tau} \geq 1$, then $u \in BV(\Omega)$ with $\|Du\|(\Omega) \leq C(c_d, \widehat{\tau})\|u\|_{A^{1,1}_{\widehat{\tau}}(\Omega)}$.

In particular, the result tells us that if there is a Radon measure ν of finite mass on Ω (that is, $\nu(\Omega) < \infty$) and constants $C_0 > 0$, $\widehat{\tau} \geq 1$ such that for all balls $B \subset \widehat{\tau}B \subset \Omega$,

$$\int_B |u - u_B|\, d\mu \leq C_0 \operatorname{rad}(B)\, \nu(\widehat{\tau}B),$$

then $u \in L^1_{\mathrm{loc}}(\Omega)$ is a function of bounded variation with $\|Du\|(\Omega) \leq C(C_0, c_d, \widehat{\tau})\nu(\Omega)$. This formulation is given in [Mir]. Conversely, if $u \in BV(\Omega)$, then u satisfies (2.11) for all balls $B \subset \tau B \subset \Omega$ and thus $\|u\|_{A^{1,1}_{\tau}(\Omega)} < \infty$.

Next we give another characterization of BV functions in terms of a Hajłasz-type inequality. This result can essentially be found in [LT], but here we provide a slightly different proof that relies heavily on results found in [Haj]. This result will be needed in Section 4.

Proposition 2.13. *Suppose that $\Omega \subset X$ is open, $u \in L^1_{loc}(\Omega)$, ν is a finite Radon measure on Ω, $\widehat{\tau} \geq 1$ is a constant, and that for all balls B for which $\widehat{\tau}B \subset \Omega$ there is a nonnegative μ-measurable function h_B on B such that for μ-a.e. $x, y \in B$,*

$$|u(x) - u(y)| \leq d(x,y)\big(h_B(x) + h_B(y)\big). \tag{2.14}$$

Suppose in addition that for some constant $c_w > 0$ and all $t > 0$, we have

$$\mu(\{x \in B : h_B(x) > t\}) \leq \frac{c_w}{t}\, \nu(\widehat{\tau}B). \tag{2.15}$$

Then for each ball B with $2\widehat{\tau}B \subset \Omega$, we have

$$\int_B |u - u_B|\, d\mu \leq C(c_d, c_w)\operatorname{rad}(B)\, \nu(2\widehat{\tau}B),$$

and hence $u \in A^{1,1}_{2\widehat{\tau}}(\Omega) \subset BV(\Omega)$, with $\|Du\|(\Omega) \leq C(c_d, c_w, \widehat{\tau})\nu(\Omega)$.

Proof. Pick an arbitrary ball $B \subset 2\widehat{\tau}B \subset \Omega$. By (2.15) we know that $h_{2B} \in \text{weak-}L^1(2B)$. Suppose $0 < q < 1$. Then by Lemma 9.6 of [Haj] (with $m = c_w\, \nu(2\widehat{\tau}B)$) we see that $h \in L^q(2B)$ with

$$\left(\fint_{2B} h_{2B}^q\, d\mu \right)^{1/q} \leq 2^{1/q}\, \frac{q}{1-q}\, \frac{c_w\, \nu(2\widehat{\tau}B)}{\mu(2B)}. \tag{2.16}$$

Hence u belongs to the Hajłasz-Sobolev space $M^{1,q}(2B)$ with h_{2B} a Hajłasz-gradient of u in $L^q(2B)$. Recalling the lower mass bound (2.2) for the measure, we choose $q = Q/(Q+1) < 1$. Then by Corollary 8.9 of [Haj] (a corollary of Theorem 8.7, which has a complicated proof using level-set estimates for the Hajłasz-gradients) and (2.16), we have with $q^* = Qq/(Q-q) = 1$,

$$\inf_{c\in\mathbb{R}} \fint_B |u-c|\,d\mu = \inf_{c\in\mathbb{R}} \left(\fint_B |u-c|^{q^*}\,d\mu\right)^{1/q^*} \le C(c_d)\,\mathrm{rad}(B)\,\left(\fint_{2B} h_{2B}^q\,d\mu\right)^{1/q}$$
$$\le C(c_d,c_w)\,\mathrm{rad}(B)\,\frac{\nu(2\widehat{\tau}B)}{\mu(2B)} \le C(c_d,c_w)\mathrm{rad}(B)\,\frac{\nu(2\widehat{\tau}B)}{\mu(B)}.$$

To complete the proof, we simply note that

$$\fint_B |u-u_B|\,d\mu \le 2\inf_{c\in\mathbb{R}} \fint_B |u-c|\,d\mu. \qquad \square$$

The pointwise condition given in the proposition is indeed a characterization, since a converse holds as well. Namely, any $u \in BV(\Omega)$ satisfies (2.14) and (2.15) with $\nu = \|Du\|$, and $\widehat{\tau}$ depending only on the dilation factor of the Poincaré inequality (see e.g. the proof of [HajKo, Theorem 3.2] as well as Lemma 4.1 of this paper).

3 Semmes family of curves

In this section, we introduce a way to define a "thick" family of curves between any two points in the space, and prove an analogue of Fuglede's lemma. We then consider additional geometric uniformity conditions for the curve family.

3.1 Basic properties

Definition 3.1. We say that the space X supports a *Weiss-David-Semmes family of curves* (from now on, referred to as the Semmes family of curves)[1] if there exist constants $\lambda > 1$ and $c_S > 0$ such that for every $x, y \in X$ with $x \neq y$ there is a family $\Gamma_{x,y}$ of curves γ in $B_{xy} := B(x, \lambda d(x,y))$ with $\gamma(0) = x$ and $\gamma(\ell_\gamma) = y$, and a probability measure $\alpha_{x,y}$ on $\Gamma_{x,y}$ with the property that for all Borel sets $A \subset X$,

$$\int_{\Gamma_{x,y}} \ell(\gamma\cap A)\,d\alpha_{x,y}(\gamma) \le c_S \int_{A\cap B_{xy}} R_{x,y}(z)\,d\mu(z), \tag{3.2}$$

where we first define

$$\widetilde{R}_{x,y}(z) := \widetilde{R}^1_{x,y}(z) + \widetilde{R}^2_{x,y}(z) := \frac{d(z,x)}{\mu(B(x,d(z,x)))} + \frac{d(z,y)}{\mu(B(y,d(z,y)))},$$

and then we let $R_{x,y}$ be a function that is continuous outside $\{x,y\}$ and satisfies $\widetilde{R}_{x,y}/c_d \le R_{x,y} \le c_d\widetilde{R}_{x,y}$. At the points x and y we let $R_{x,y}$ take an arbitrary value.

[1] We thank Stephen Semmes for pointing out that the initial idea of such a family of curves can be found in the works of Mary Weiss on lacunary series.

The function $R_{x,y}$ can be constructed as follows. First we define $R^1_{x,y}(z) = \widetilde{R}^1_{x,y}(z)$ on the spheres $\{z \in X : d(z,x) = 2^i\}$, $i \in \mathbb{Z}$, and similarly $R^2_{x,y}(z) = \widetilde{R}^2_{x,y}(z)$ on the spheres $\{z \in X : d(z,y) = 2^i\}$, $i \in \mathbb{Z}$. Then we extend $R^1_{x,y}$ (respectively $R^2_{x,y}$) to the whole space X by linear interpolation with respect to $d(z,x)$ (respectively $d(z,y)$). Finally we let $R_{x,y} := R^1_{x,y} + R^2_{x,y}$.

Remark 3.3. If the space X is geodesic, by [Buc] we know that the measure μ satisfies the so-called annular decay property, implying that the map $r \mapsto \mu(B(x,r))$ is continuous on $(0,\infty)$ for each $x \in X$. In such geodesic setting, the function $\widetilde{R}_{x,y}$ is itself automatically continuous outside $\{x,y\}$. By the fact that X is a complete metric space with a doubling measure and a Poincaré inequality, we know that there is a bi-Lipschitz change in the metric that results in a geodesic metric space, see e.g. [HajKo, Proposition 4.4]. The invariance of the Semmes family of curves under a bi-Lipschitz change in the metric is discussed in Remark 3.12.

By a probability measure we mean a Radon measure with total mass 1. An implicit requirement as part of the definition of a Semmes family of curves is that the function

$$\Gamma_{x,y} \ni \gamma \mapsto \int_\gamma \chi_A \, ds = \ell(\gamma \cap A)$$

is $\alpha_{x,y}$-measurable for every Borel set $A \subset X$. We also assume that the function

$$\Gamma_{x,y} \ni \gamma \mapsto \#(\gamma \cap A)$$

is $\alpha_{x,y}$-measurable for every Borel set $A \subset X$; see Lemma 5.2.

To give an idea what a Semmes family of curves "looks like", we refer to the beginning of Section 6, where a Semmes family is constructed in the Euclidean setting. We also point out that we always assume $\Gamma_{y,x}$ to consist of the curves in $\Gamma_{x,y}$ with direction reversed.

Assume that X supports a Semmes family of curves. Presenting any nonnegative Borel function ρ on X as $\rho = \sum_{i=1}^\infty \frac{1}{i}\chi_{A_i}$, where A_i are Borel sets (see e.g. [EvGa, Section 1.1.2]) and using the monotone convergence theorem, we get

$$\int_{\Gamma_{x,y}} \int_\gamma \rho \, ds \, d\alpha_{x,y}(\gamma) \le c_S \int_{B_{xy}} \rho(z) R_{x,y}(z) \, d\mu(z) \tag{3.4}$$

for every $x, y \in X$, $x \neq y$. This can as well be taken to be the definition of the Semmes family of curves, as the two formulations are equivalent. From now on, we usually drop the variable "z" from the notation for brevity. If $u \in L^1_{\mathrm{loc}}(X)$ and g is an upper gradient of u, by the above inequality we have for all $x, y \in X$, $x \neq y$,

$$|u(x) - u(y)| \le c_S \int_{B_{xy}} g \, R_{x,y} \, d\mu.$$

This implies a $(1,1)$-Poincaré-inequality for the pair u, g [Hei, Theorem 9.5], with the constants c_P and τ depending only on c_S and λ. We will assume throughout that X supports a Semmes family of curves, so we are assuming the space to support a $(1,1)$-Poincaré inequality as well.

When $x, y \in X$ with $x \neq y$ are fixed and $A \subset X$, we denote by $\Gamma(A)$ the subfamily of curves $\gamma \in \Gamma_{x,y}$ that intersect A. With this notation, let us consider a ball $B(z, 2r)$ that intersects neither x nor y. If a curve $\gamma \in \Gamma_{x,y}$ intersects $B(z, r)$, then γ travels more than the length r inside $B(z, 2r)$, that is, $\ell(\gamma \cap B(z, 2r)) \geq r$. From this,

$$\begin{aligned}\alpha_{x,y}(\Gamma(B(z,r))) = \int_{\Gamma(B(z,r))} 1\, d\alpha_{x,y}(\gamma) &\leq \int_{\Gamma(B(z,r))} \frac{1}{r} \int_\gamma \chi_{B(z,2r)}\, ds\, d\alpha_{x,y}(\gamma) \\ &\leq \frac{1}{r} \int_{\Gamma_{x,y}} \int_\gamma \chi_{B(z,2r)}\, ds\, d\alpha_{x,y}(\gamma) \\ &\leq \frac{c_S}{r} \int_{B(z,2r)} R_{x,y}\, d\mu.\end{aligned} \tag{3.5}$$

Thus we have an upper bound for the number of curves intersecting a small ball.

The following auxiliary result will be needed on a few occasions in this paper.

Lemma 3.6. *For every $x, y, z \in X$ with $x \neq y$ and $R > 0$, it is true that*

$$\int_{B(z,R)} R_{x,y}\, d\mu < \infty.$$

That is, the constant function $1 \in L^1_{\mathrm{loc}}(X, R_{x,y}\, d\mu)$.

Proof. Without loss of generality, we may assume that R is large enough so that x and y belong to $B(z, R)$. Let $B_i := B(x, 2^{2-i}R)$ for $i = 1, 2, \ldots$. Then $B(z, R) \subset B_1$ and, by the doubling property of the measure μ, we have

$$\begin{aligned}\int_{B(z,R)} \widetilde{R}^1_{x,y}\, d\mu &\leq \sum_{i=1}^\infty \int_{B_i \setminus B_{i+1}} \widetilde{R}^1_{x,y}\, d\mu \\ &\leq 4c_d \sum_{i=1}^\infty 2^{-i} R < \infty.\end{aligned}$$

A similar argument with the choice of $B_i = B(y, 2^{2-i}R)$ yields

$$\int_{B(z,R)} \widetilde{R}^2_{x,y}\, d\mu \leq 4c_d \sum_{i=1}^\infty 2^{-i} R < \infty,$$

from which the result follows. □

Observe that by condition (3.4), altering a Borel function on a set of μ-measure zero changes it in the L^1-sense only on a family of curves of $\alpha_{x,y}$-measure zero. Furthermore, since μ is a Borel regular outer measure, every μ-measurable function can be modified on a set of μ-measure zero to obtain a Borel function (see e.g. [BjBj, Proposition 1.2]). Thus a μ-measurable function is well-defined in the L^1-sense on $\alpha_{x,y}$-a.e. curve.

The following is an analogue of Fuglede's lemma, with the p-modulus (see e.g. [Fug], [BjBj] or [Shan]) replaced by the measure $\alpha_{x,y}$.

Lemma 3.7. *Suppose that X supports a Semmes family of curves. Let u, $\{u_i\}_{i=1}^{\infty}$ be μ-measurable functions on X with $0 \le u \le 1$ and $0 \le u_i \le 1$, such that $u_i \to u$ in $L^1(X)$. Let $x, y \in X$ with $x \neq y$. Then, by passing to a subsequence if necessary, with the subsequence perhaps dependent on x, y, we have for $\alpha_{x,y}$-almost every $\gamma \in \Gamma_{x,y}$,*

$$\lim_{i\to\infty} \int_\gamma |u_i - u|\, ds = 0,$$

and so in particular, for such γ, we have

$$\lim_{i\to\infty} \int_\gamma u_i\, ds = \int_\gamma u\, ds.$$

Proof. By passing to a subsequence if necessary, we may assume that

$$\int_X |u_i - u|\, d\mu \le 2^{-i}$$

for every $i \in \mathbb{N}$. By the standard measure theoretic proof, we then also have that $u_i \to u$ pointwise μ-almost everywhere and hence pointwise $R_{x,y}\, d\mu$-almost everywhere. By Lemma 3.6 and Lebesgue's dominated convergence theorem, we further have $u_i \to u$ in $L^1(B_{xy}, R_{x,y}\, d\mu)$, and so we can, by passing to a further subsequence if necessary, assume that

$$\int_{B_{xy}} |u_i - u| R_{x,y}\, d\mu \le 2^{-i}$$

for every $i \in \mathbb{N}$. Let

$$\Gamma^+ := \left\{ \gamma \in \Gamma_{x,y} :\, \limsup_{i\to\infty} \int_\gamma |u_i - u|\, ds > 0 \right\},$$

and for all $n, k \in \mathbb{N}$, let

$$\Gamma^n := \left\{ \gamma \in \Gamma_{x,y} :\, \limsup_{i\to\infty} \int_\gamma |u_i - u|\, ds > 1/n \right\}$$

and

$$\Gamma^{k,n} := \left\{ \gamma \in \Gamma_{x,y} :\, \int_\gamma |u_k - u|\, ds > 1/n \right\}.$$

We have $\Gamma^+ = \bigcup_{n\in\mathbb{N}} \Gamma^n$ and $\Gamma^n \subset \bigcap_{j\in\mathbb{N}} \bigcup_{k\ge j} \Gamma^{k,n}$. By (3.4), we have

$$\begin{aligned}
\alpha_{x,y}(\Gamma^{k,n}) = \int_{\Gamma^{k,n}} 1\, d\alpha_{x,y}(\gamma) &\le n \int_{\Gamma^{k,n}} \int_\gamma |u_k - u|\, ds\, d\alpha_{x,y}(\gamma) \\
&\le n \int_{\Gamma_{x,y}} \int_\gamma |u_k - u|\, ds\, d\alpha_{x,y}(\gamma) \\
&\le n c_S \int_{B_{xy}} |u_k - u| R_{x,y}\, d\mu \\
&\le n c_S 2^{-k}
\end{aligned}$$

for every $n, k \in \mathbb{N}$. It follows that for each $j \in \mathbb{N}$,

$$\alpha_{x,y}\left(\bigcup_{k\geq j}\Gamma^{k,n}\right) \leq nc_S\sum_{k=j}^{\infty}2^{-k} = nc_S\, 2^{1-j}.$$

Therefore $\alpha_{x,y}(\Gamma^n) = 0$ for each $n \in \mathbb{N}$, and hence $\alpha_{x,y}(\Gamma^+) = 0$, which is the desired result. □

3.2 Geometric Semmes family of curves

To prove the main result of this paper, Theorem 5.1, we need a Semmes family of curves with some additional properties. The standard construction of a Semmes family in the Euclidean setting has these additional properties, and non-Euclidean examples such as Fred Gehring's bow-tie and the Heisenberg group are also discussed in Section 6.

For $x, y \in X$, $x \neq y$, let us divide the ball $B_{xy} = B(x, \lambda d(x, y))$ into sets that are almost annuli close to the points x and y. First we let

$$f(z) := \frac{d(z,x)}{d(z,x)+d(z,y)}. \tag{3.8}$$

In most cases, we can then fix $0 < \delta \leq \frac{1}{2}$ and set

$$A_0 := \{z \in B_{xy} :\ \tfrac{1}{2}(1-\delta) \leq f(z) < \tfrac{1}{2}(1+\delta)\}$$

and for $j = 1, 2, \ldots$,

$$\begin{aligned} A_j &:= \{z \in B_{xy} :\ \tfrac{1}{2}(1-\delta)^{j+1} \leq f(z) < \tfrac{1}{2}(1-\delta)^{j}\},\\ A_{-j} &:= \{z \in B_{xy} :\ 1-\tfrac{1}{2}(1-\delta)^{j} \leq f(z) < 1-\tfrac{1}{2}(1-\delta)^{j+1}\}. \end{aligned} \tag{3.9}$$

However, this type of division is not suitable in some cases (see for example the description of Fred Gehring's bow-tie in Section 6), motivating the following more general construction. For any given $\delta > 0$, let $\mathcal{A}_\delta = \{a_j\}_{j=-\infty}^{\infty}$ be a countable collection of distinct numbers $a_j \in (0,1)$ satisfying the following requirements: $\inf_j a_j = 0$, $\sup_j a_j = 1$, and for any a_j we can find the "subsequent" number $\widetilde{a}_j \in \mathcal{A}_\delta$ with $\widetilde{a}_j > a_j$, such that there is no $a \in \mathcal{A}_\delta$ for which $a_j < a < \widetilde{a}_j$ (however, it may not be possible to present $\mathcal{A}_\delta$ as an increasing sequence). Given such a collection $\mathcal{A}_\delta$, for each $j \in \mathbb{Z}$ we let

$$A_j := \{z \in B_{xy} : a_j \leq f(z) < \widetilde{a}_j\}. \tag{3.10}$$

Furthermore, we require $\sup_j |\widetilde{a}_j - a_j| \leq \delta$, and

$$\widetilde{a}_j \leq 2a_j \quad \text{and} \quad 1-a_j \leq 2(1-\widetilde{a}_j) \quad \text{for each } j \in \mathbb{Z}.$$

These conditions ensure that the sets A_j are sufficiently "thin" and that the function $R_{x,y}$ is approximately constant on each set A_j — more precisely, constant up to a factor $C = C(c_d, \lambda)$.

Next we give the definition of the geometric Semmes family of curves. Once again we refer to the beginning of Section 6 for a concrete example of this type of curve family.

Definition 3.11. We say that X supports a *geometric Semmes family of curves* if there exist constants $c_S \geq 1$, $\lambda > 1$ and $0 < w \leq 1$ such that for every $x, y \in X$ with $x \neq y$ and every $\delta > 0$ there is a set $\mathcal{A}_\delta$ as above and a Semmes family of curves $\Gamma_{x,y}$ (lying, by definition, in $B_{xy} = B(x, \lambda d(x,y))$) with constants λ and c_S, satisfying the following conditions:

(1) For all balls $B = B(z,r) \subset B_{xy} \setminus \{x,y\}$ for which $c_S r < \operatorname{dist}(\{z\}, \{x,y\})$, the following holds: if there is a curve $\gamma \in \Gamma_{x,y}$ intersecting $c_S^{-1} B$, then

$$\alpha_{x,y}(\Gamma(B)) \geq \frac{1}{c_S r} \int_{c_S^{-1} B} R_{x,y} \, d\mu,$$

where $\Gamma(B)$ is the subfamily of curves $\gamma \in \Gamma_{x,y}$ that intersect B.

(2) For every pair of curves $\gamma_1, \gamma_2 \in \Gamma_{x,y}$ and every $j \in \mathbb{Z}$, the Hausdorff distance between $\gamma_1 \cap A_j$ and $\gamma_2 \cap A_j$ is at most c_S times the distance between $\gamma_1 \cap A_j$ and $\gamma_2 \cap A_j$, i.e.

$$d_H(\gamma_1 \cap A_j, \gamma_2 \cap A_j) \leq c_S \operatorname{dist}(\gamma_1 \cap A_j, \gamma_2 \cap A_j).$$

(3) For every $\gamma \in \Gamma_{x,y}$, $t \mapsto f(\gamma(t))$ is an increasing function with a lower limit on the growth pace:

$$\frac{f(\gamma(t_2)) - f(\gamma(t_1))}{t_2 - t_1} \geq \frac{w}{d(x,y)} > 0 \quad \text{for each pair } t_1, t_2 \in [0, \ell_\gamma] \text{ with } t_2 > t_1.$$

Here, f is given by (3.8).

(4) For some $n \in \mathbb{N}$ and values $b_1, \dots, b_n \in (0,1)$ that are independent of δ, we have

$$B_{xy} = \bigcup_{j=-\infty}^{\infty} A_j \cup \bigcup_{i=1}^{n} \{z \in B_{xy} : f(z) = b_i\}$$

such that the following weak continuity property holds for every μ-measurable set E and $\alpha_{x,y}$-a.e. curve $\gamma \in \Gamma_{x,y}$: if $t \in [0, \ell_\gamma]$ such that $f(\gamma(t)) = b_i$ for some $i = 1, \dots, n$, and $\gamma(t) \in I$ (respectively, $\gamma(t) \in O$), then there exist sequences $u_k \nearrow t$ and $v_k \searrow t$ such that $\gamma(u_k) \in I$ and $\gamma(v_k) \in I$ ($\gamma(u_k) \in O$ and $\gamma(v_k) \in O$) for every $k \in \mathbb{N}$. Here the symbols I and O denote the measure theoretic interior and exterior of the set E as in (2.8) and (2.9).

Let us consider the motivation behind the above conditions. Condition (1) is simply a converse of (3.5). Hence, in some sense the distribution of curves in a geometric Semmes family spreads out uniformly (up to a constant), as determined by the Riesz kernel $R_{x,y}$. In the Euclidean case we see that $R_{x,y}$ has precisely the behavior that gives an essentially uniform distribution, see Example 6.1. The Riesz kernel $\widetilde{R}_{x,y}$ also has the property that $\widetilde{R}^1_{x,y}(z)$ is proportional to the normalized perimeter measure of $B(z, d(z,x))$; see for example [AMP, Theorem 4.3].

Condition (2) means that all curves in $\Gamma_{x,y}$ that come within the distance r of a curve $\gamma \in \Gamma_{x,y}$ inside A_j must stay within the distance $c_S r$ of γ while inside A_j. For example, this prohibits the curves in $\Gamma_{x,y}$ from "fanning out" too much as they travel from the inner boundary of A_j towards the outer boundary. Notice that our definition of $\mathcal{A}_\delta$ allows us to make the sets A_j thinner in some areas, as necessary, in order to ensure that condition (2) is satisfied — see Example 6.2.

Condition (3) simply requires the curves to travel at an essentially uniform speed from x to y. Condition (4) gives us sufficient control over the behavior of the level sets that are left outside the sets A_j. In the model case (3.9) we simply have $B_{xy} = \cup_{j=-\infty}^{\infty} A_j$, and condition (4) can be disregarded.

Remark 3.12. The definition (3.8) of the function f is not the only possible one we could use. Apart from conditions (2)–(4) listed above, we will only need the local Lipschitz continuity of f and the property that $R_{x,y}$ is approximately constant on each set A_j. We can see that the geometric Semmes family of curves survives a bi-Lipschitz change in the metric as long as the function f is kept the same, so in particular it is enough to use in (3.8) any bi-Lipschitz equivalent metric d' for which the space does support the geometric Semmes family.

4 Characterizing BV functions in terms of curves

In this section we provide a Reshetnyak-type characterization of bounded BV functions in terms of curves. We first prove a few preliminary lemmas, and the characterization is then given in Theorem 4.9.

We start by recalling that the following weak-type $(1,1)$-inequality holds for Radon measures.

Lemma 4.1. *For $R > 0$ and a Radon measure ν, let $M_R\nu$ be the maximal function given by*

$$\mathcal{M}_R\,\nu(x) := \sup_{0<s\le R} \frac{\nu(B(x,s))}{\mu(B(x,s))}, \quad \text{for } x \in X.$$

Then $\mathcal{M}_R\,\nu$ satisfies the following weak-type $(1,1)$-inequality. There is a constant $C > 0$, depending only on the doubling constant c_d, such that for any $z \in X$ and $r, t > 0$, setting $E_t := \{x \in B(z,r) : \mathcal{M}_R\,\nu(x) > t\}$, we have

$$\mu(E_t) \le \frac{C}{t}\ \nu(B(z, r+R)).$$

The fact that $M_R\nu$ is μ-measurable follows along the same lines as the classical proof that the Hardy-Littlewood maximal functions are measurable; we therefore omit the proof here.

Proof of Lemma 4.1. For each $x \in E_t$ we can find $0 < r_x \le R$ satisfying

$$\nu(B(x,r_x)) > t\,\mu(B(x,r_x)).$$

By the standard 5-covering theorem, we can select a countable family of pairwise disjoint balls $\{B_j = B(x_j, r_{x_j})\}_{j\in\mathbb{N}}$ such that $\{5B_j\}_{j\in\mathbb{N}}$ is a cover of E_t. Since μ is doubling, we have

$$\mu(E_t) \le \sum_{j\in\mathbb{N}} \mu(5B_j) \le c_d^3 \sum_{j\in\mathbb{N}} \mu(B_j) \le \frac{c_d^3}{t} \sum_{j\in\mathbb{N}} \nu(B_j) \le \frac{c_d^3}{t} \nu(B(z, r+R)). \qquad \square$$

In the following lemma, given a Radon measure ν and a number $R>0$, the function $M_R\nu$ is defined as in the previous lemma.

Lemma 4.2. *If $x,y\in X$ with $x\neq y$, ν is a Radon measure, $\kappa\geq 1$, and $R\geq 2\kappa d(x,y)$, then there is a constant $C=C(c_d,\kappa)$ such that*

$$\int_{B(x,\kappa d(x,y))} R_{x,y}\,d\nu \leq Cd(x,y)[\mathcal{M}_R\,\nu(x)+\mathcal{M}_R\,\nu(y)].$$

From Lemmas 4.1 and 4.2 it follows that for μ-almost every $x,y\in X$ the integral on the left-hand side of the above inequality is finite.

Proof of Lemma 4.2. If $\nu(\{x\})>0$ or $\nu(\{y\})>0$, the right-hand side is infinity, so the inequality holds. Otherwise recall that $R_{x,y}\leq c_d(\widetilde{R}^1_{x,y}+\widetilde{R}^2_{x,y})$, as given in the definition of the Semmes family of curves. With $B_i:=B(x,2^{1-i}\kappa d(x,y))$, $i=1,2,\ldots$, we see as in the proof of Lemma 3.6 that

$$\begin{aligned}\int_{B(x,\kappa d(x,y))}\widetilde{R}^1_{x,y}\,d\nu &\leq 2\kappa\sum_{i\in\mathbb{N}}2^{-i}d(x,y)c_d\frac{\nu(B_i)}{\mu(B_i)}\\ &\leq Cd(x,y)\sum_{i\in\mathbb{N}}2^{-i}\,\mathcal{M}_R\,\nu(x)\\ &\leq Cd(x,y)\,\mathcal{M}_R\,\nu(x).\end{aligned}$$

Similarly, we see that

$$\int_{B(x,\kappa d(x,y))}\widetilde{R}^2_{x,y}\,d\nu\leq\int_{B(y,2\kappa d(x,y))}\widetilde{R}^2_{x,y}\,d\nu\leq Cd(x,y)\,\mathcal{M}_R\,\nu(y).$$

Thus the proof is complete. □

The following approximation result will also be needed.

Lemma 4.3. *Let $\Omega\subset X$ be open and let $u\in BV(\Omega)$ with $0\leq u\leq 1$. Then there exist sequences of locally Lipschitz functions u_i,g_i on Ω such that $0\leq u_i\leq 1$ for each $i\in\mathbb{N}$, each g_i is an upper gradient of u_i, $u_i\to u$ in $L^1(\Omega)$, and for μ-a.e. $x,y\in\Omega$ with $2B_{xy}\subset\Omega$, we have*

$$\limsup_{i\to\infty}\int_{B_{xy}}R_{x,y}\,g_i\,d\mu\leq C\int_{2B_{xy}}R_{x,y}\,d\|Du\|,$$

where $C=C(c_d,c_P,\tau)$.

Proof. For a given $\varepsilon>0$, pick a Whitney-type covering of Ω, consisting of balls $\{B^\varepsilon_j\}_{j=1}^\infty$ that cover Ω and whose radii $\operatorname{rad}(B^\varepsilon_j)$ are comparable to $\min\{\varepsilon,\operatorname{dist}(B^\varepsilon_j,X\setminus\Omega)\}$. For the construction of this type of covering, see e.g. [KST] or [AK, Lemma 4.1]. Furthermore, we require that the balls $5\tau B^\varepsilon_j$ are contained in Ω, and that each ball $5\tau B^\varepsilon_j$ meets at most $C(c_d,\tau)$ other balls $5\tau B^\varepsilon_k$. Here τ is the dilation constant of the Poincaré inequality. Thanks to the doubling property of μ such a cover is always possible to arrange. Let $\{\varphi^\varepsilon_j\}_{j=1}^\infty$ be a partition of unity with $\operatorname{supp}(\varphi_j^\varepsilon) \subset 2B_j^\varepsilon$ for each $j \in \mathbb{N}$, and each φ_j^ε is $C(c_d)/\operatorname{rad}(B_j^\varepsilon)$-Lipschitz. Because of the doubling property of μ and the $(1,1)$-Poincaré inequality, we can approximate the function u by

$$u_\varepsilon := \sum_j u_{B_j^\varepsilon} \varphi_j^\varepsilon.$$

The function u_ε takes values between 0 and 1, and $u_\varepsilon \to u$ in $L^1(\Omega)$, see e.g. [HKT, Lemma 5.3]. Furthermore, u has the following upper gradient:

$$g_\varepsilon := C \sum_j \frac{\|Du\|(5\tau B_j^\varepsilon)}{\mu(5B_j^\varepsilon)} \varphi_j^\varepsilon,$$

where $C = C(c_d, c_P)$ [HKT, Lemma 5.3]. Since $\|Du\|(\Omega) < \infty$, we have

$$\limsup_{r\to 0} \frac{\|Du\|(B(x,r))}{\mu(B(x,r))} < \infty \tag{4.4}$$

for μ-a.e. $x \in \Omega$. For any $x, y \in \Omega$ satisfying this condition and the condition $2B_{xy} \subset \Omega$, we let $\varepsilon > 0$ be sufficiently small, e.g. $\varepsilon < d(x,y)/(20\tau)$, and compute

$$\begin{aligned}
\int_{B_{xy}} R_{x,y} g_\varepsilon \, d\mu &= C \int_{B_{xy}} R_{x,y} \sum_{j:\, 2B_j^\varepsilon \cap B_{xy} \neq \emptyset} \frac{\|Du\|(5\tau B_j^\varepsilon)}{\mu(5B_j^\varepsilon)} \varphi_j^\varepsilon \, d\mu \\
&\le C \sum_{j:\, 2B_j^\varepsilon \cap B_{xy} \neq \emptyset} \int_{2B_j^\varepsilon} R_{x,y} \frac{\|Du\|(5\tau B_j^\varepsilon)}{\mu(5B_j^\varepsilon)} \, d\mu \\
&\le C \sum_{j:\, 2B_j^\varepsilon \cap B_{xy} \neq \emptyset,\, x,y \notin 6\tau B_j^\varepsilon} \int_{2B_j^\varepsilon} R_{x,y} \frac{\|Du\|(5\tau B_j^\varepsilon)}{\mu(5B_j^\varepsilon)} \, d\mu \\
&\quad + C \frac{\|Du\|(B(x, 11\tau\varepsilon))}{\mu(B(x, 11\tau\varepsilon))} \int_{B(x,11\tau\varepsilon)} R_{x,y} \, d\mu + C \frac{\|Du\|(B(y, 11\tau\varepsilon))}{\mu(B(y, 11\tau\varepsilon))} \int_{B(y,11\tau\varepsilon)} R_{x,y} \, d\mu.
\end{aligned}$$

In the last inequality we used the bounded overlap property of the balls $2B_j^\varepsilon$ as well as the fact that $\mu(5B_j^\varepsilon) \ge C(c_d, \tau)\mu(B(x, 11\tau\varepsilon))$ for each $j \in \mathbb{N}$, since $\operatorname{rad}(B_j^\varepsilon)$ is comparable to $\min\{\varepsilon, \operatorname{dist}(B_j^\varepsilon, X \setminus \Omega)\}$. Now, in the last two terms the integrals converge to zero as $\varepsilon \to 0$ by Lemma 3.6, while the coefficients remain bounded due to (4.4). Thus these terms converge to zero. In the remaining term, we know that $\sup_{5\tau B_j^\varepsilon} R_{x,y} \le C(c_d) \inf_{5\tau B_j^\varepsilon} R_{x,y}$ for every j in the sum. Thus we have

$$\sum_{j:\, 2B_j^\varepsilon \cap B_{xy} \neq \emptyset,\, x,y \notin 6\tau B_j^\varepsilon} \int_{2B_j^\varepsilon} R_{x,y} \frac{\|Du\|(5\tau B_j^\varepsilon)}{\mu(5B_j^\varepsilon)} \, d\mu \le C \sum_{j:\, 2B_j^\varepsilon \cap B_{xy} \neq \emptyset,\, x,y \notin 6\tau B_j^\varepsilon} R_{x,y}(z_j^\varepsilon) \|Du\|(5\tau B_j^\varepsilon),$$

where z_j^ε is the center of the ball B_j^ε. Now, due to the bounded overlap property of the balls $5\tau B_j^\varepsilon$, the last sum can be broken up into a finite number (depending only on c_d and τ) of sums such that in each sum, the balls $5\tau B_j^\varepsilon$ are disjoint. Each of these sums is bounded from above by

$$C \int_{2B_{xy}} R_{x,y} \, d\|Du\|.$$

Now we get the result simply by defining $u_i := u_{1/i}$, $g_i := g_{1/i}$, for $i \in \mathbb{N}$. □

As mentioned before, given any points $x, y \in X$ and any μ-measurable function u that is finite μ-almost everywhere, for $\alpha_{x,y}$-a.e. curve $\gamma \in \Gamma_{x,y}$ we can assume $u \circ \gamma$ to be a Borel function that is also finite $\mathcal{L}^1$-almost everywhere, where $\mathcal{L}^1$ is the 1-dimensional Lebesgue measure. Thus we know that $u \circ \gamma$ is approximately continuous at $\mathcal{L}^1$-a.e. $t \in [0, \ell_\gamma]$, see e.g. [EvGa, Section 1.7.2]. However, with the Semmes family of curves, the end points x and y are of special interest, motivating the following result.

Lemma 4.5. *Suppose that X supports a geometric Semmes family of curves. Let u be a μ-measurable function, and let $x, y \in X$, $x \neq y$ be such that x is a Lebesgue point of u. Let $\Gamma^x \subset \Gamma_{x,y}$ be the curve family*

$$\Gamma^x := \left\{ \gamma \in \Gamma_{x,y} : \exists \varepsilon > 0 \text{ s.t. } \liminf_{t \to 0} \frac{\mathcal{L}^1(\{s \in [0,t] : |u \circ \gamma(s) - u \circ \gamma(0)| > \varepsilon\})}{t} > 0 \right\}.$$

Then $\alpha_{x,y}(\Gamma^x) = 0$.

If we had "lim sup" instead of "lim inf" in the definition of Γ^x, we would be showing that $u \circ \gamma$ is approximately continuous at 0 (corresponding to x) for $\alpha_{x,y}$-a.e. $\gamma \in \Gamma_{x,y}$. With "lim inf", the curve family Γ^x is more restricted and thus the claim is weaker, but it will be more than enough for our purposes.

Proof of Lemma 4.5. If $\gamma \in \Gamma^x$, then there exist numbers $\varepsilon > 0$, $\delta > 0$ and $T > 0$ such that

$$\inf_{0<t<T} \frac{\mathcal{L}^1(\{s \in [0,t] : |u \circ \gamma(s) - u \circ \gamma(0)| > \varepsilon\})}{t} > \delta.$$

Thus we can write

$$\Gamma^x = \bigcup_{n=1}^{\infty} \Gamma_n,$$

where

$$\Gamma_n := \left\{ \gamma \in \Gamma_{x,y} : \inf_{0<t<1/n} \frac{\mathcal{L}^1(\{s \in [0,t] : |u \circ \gamma(s) - u \circ \gamma(0)| > 1/n\})}{t} > \frac{1}{n} \right\}.$$

Now, if $\gamma \in \Gamma_n$, we have $\gamma \in \Gamma_{n,m}$ for every $m \in \mathbb{N}$, where

$$\Gamma_{n,m} := \left\{ \gamma \in \Gamma_{x,y} : \frac{\mathcal{L}^1(\{s \in [a^{m+1}, a^m] : |u \circ \gamma(s) - u \circ \gamma(0)| > 1/n\})}{a^m} > \frac{1}{3n} \right\},$$

where $a = 1/(3n)$. We now have

$$\Gamma_n \subset \bigcap_{m=1}^{\infty} \Gamma_{n,m},$$

and it is enough to show that $\alpha_{x,y}(\Gamma_{n,m}) \to 0$ as $m \to \infty$, for every $n \in \mathbb{N}$. Take a curve $\gamma \in \Gamma_{n,m}$. By definition, we have

$$\int_{\gamma|_{[a^{m+1}, a^m]}} |u - u(x)| \, ds \geq \frac{a^m}{3n^2}.$$

By condition (3) of the geometric Semmes family of curves, we have

$$\gamma|_{[a^{m+1},a^m]} \subset B(x,a^m) \setminus B(x,a^{m+1}/C)$$

for large enough $m \in \mathbb{N}$, with $C = C(w)$. Now we can calculate for any $n \in \mathbb{N}$ and large enough $m \in \mathbb{N}$

$$\begin{aligned}
\alpha_{x,y}(\Gamma_{n,m}) &\le \frac{3n^2}{a^m}\int_{\Gamma_{x,y}}\int_{\gamma|_{[a^{m+1},a^m]}} |u-u(x)|\,ds\,d\alpha_{x,y}(\gamma)\\
&\le \frac{3n^2}{a^m}\int_{\Gamma_{x,y}}\int_{\gamma} |u-u(x)|\chi_{B(x,a^m)\setminus B(x,a^{m+1}/C)}\,ds\,d\alpha_{x,y}(\gamma)\\
&\le \frac{3n^2 c_S}{a^m}\int_{B(x,a^m)\setminus B(x,a^{m+1}/C)} |u-u(x)|R_{x,y}\,d\mu\\
&\le \frac{3n^2 c_S}{a^m}\frac{a^m}{\mu(B(x,a^{m+1}/C))}\int_{B(x,a^m)} |u-u(x)|\,d\mu\\
&\le 3n^2 c_S C(c_d,w,n)\fint_{B(x,a^m)} |u-u(x)|\,d\mu \overset{m\to\infty}{\longrightarrow} 0.
\end{aligned}$$

The convergence on the last line follows from the assumption that x is a Lebesgue point. Thus we have the result. □

The next lemma is a consequence of the above lemma.

Lemma 4.6. *If X supports a geometric Semmes family of curves and $u \in L^1_{loc}(X)$, then for Lebesgue points $x,y \in X$ of u with $x \ne y$ we have*

$$|u(x)-u(y)| \le \|D_\gamma u\|((0,\ell_\gamma))$$

for $\alpha_{x,y}$-a.e. curve $\gamma \in \Gamma_{x,y}$. We recall that $\|D_\gamma u\|((0,\ell_\gamma)) := \|D(u\circ\gamma)\|((0,\ell_\gamma))$.

Recall that in order to compute $\|D_\gamma u\|((0,\ell_\gamma))$, we divide the interval $(0,\ell_\gamma)$ at the points of approximate continuity of $u\circ\gamma$; see (2.5) and the remark following it. Essentially this lemma tells us that on almost every curve the function u is sufficiently "continuous" at the end points x and y to enable the difference $|u(x)-u(y)|$ to be controlled by the total variation on the curve, even if we do not know that 0 and ℓ_γ are points of approximate continuity of $u\circ\gamma$.

Proof of Lemma 4.6. Let $x,y \in X$ be two distinct Lebesgue points of u. Using the notation of the previous lemma, pick a curve $\gamma \in \Gamma_{x,y} \setminus (\Gamma^x \cup \Gamma^y)$. Again, we can assume that $u\circ\gamma$ is a Borel function on $[0,\ell_\gamma]$, and that it is finite $\mathcal{L}^1$-a.e. This implies that $\mathcal{L}^1$-a.e. point $s \in [0,\ell_\gamma]$ is a point of approximate continuity of u.

Let $\varepsilon > 0$. By the definition of the family Γ^x, there exists $t \in (0,\varepsilon)$ such that

$$\frac{\mathcal{L}^1(\{s\in[0,t] : |u\circ\gamma(s)-u\circ\gamma(0)|>\varepsilon\})}{t} < 1.$$

Thus there exists a point $s_x \in (0, \varepsilon)$ which is a point of approximate continuity of $u \circ \gamma$ and which satisfies $|u \circ \gamma(s_x) - u \circ \gamma(0)| \le \varepsilon$. Similarly we can define $s_y \in (\ell_\gamma - \varepsilon, \ell_\gamma)$. Now we can compute

$$\begin{aligned}|u(x) - u(y)| = |u \circ \gamma(0) - u \circ \gamma(\ell_\gamma)| &\le |u \circ \gamma(s_x) - u \circ \gamma(s_y)| + 2\varepsilon \\ &\le \|D_\gamma u\|((0, \ell_\gamma)) + 2\varepsilon.\end{aligned}$$

The last inequality follows simply from the definition of the total variation on a curve. By letting $\varepsilon \to 0$, we get the result. □

Recently it was shown by Ambrosio and Di Marino [AmDi] that in a complete separable metric space equipped with a locally finite Borel measure, a function is in the BV class if and only if there is a Borel measure with finite mass on the metric space such that for each (probability) test plan on the metric space, the function is in the BV class for almost every (with respect to this test plan measure) curve in the metric space, and the integral of the path-BV norm of the function with respect to the test plan is majorized by the finite-mass Borel measure. Their result does not require that the underlying measure on the space is doubling nor that it support a Poincaré inequality.

In this section we show that if the metric measure space supports a geometric Semmes family of curves, then instead of considering all test plans and all curves in the metric space, we can consider the families of curves that comprise the geometric Semmes family. The following lemma gives one direction of our characterization.

Lemma 4.7. *Suppose that X supports a geometric Semmes family of curves, and let $\Omega \subset X$ be an open set. Let $u \in BV(\Omega)$ with $0 \le u \le 1$. Then for μ-a.e. $x, y \in B \subset 9\lambda B \subset \Omega$ with $x \neq y$ and for $\alpha_{x,y}$-a.e. $\gamma \in \Gamma_{x,y}$, we have $u \circ \gamma \in BV((0, \ell_\gamma))$ and*

$$|u(x) - u(y)| \le \int_{\Gamma_{x,y}} \|D_\gamma u\|((0, \ell_\gamma))\, d\alpha_{x,y}(\gamma) \le C \int_{2B_{xy}} R_{x,y}\, d\|Du\|,$$

with $C = C(c_d, c_P, \tau, c_S)$.

Proof. Let $x, y \in B \subset 9\lambda B \subset \Omega$ be Lebesgue points of u. For $\alpha_{x,y}$-a.e. curve $\gamma \in \Gamma_{x,y}$ we have by the previous lemma

$$|u(x) - u(y)| \le \|D_\gamma u\|((0, \ell_\gamma)). \tag{4.8}$$

We pick sequences of locally Lipschitz functions u_i and g_i guaranteed by Lemma 4.3 such that $u_i \to u$ in $L^1(\Omega)$, $0 \le u_i \le 1$, and each g_i is an upper gradient of u_i. By Lemma 3.7 we may also assume that for $\alpha_{x,y}$-a.e. $\gamma \in \Gamma_{x,y}$, $u_i \circ \gamma \to u \circ \gamma$ in $L^1((0, \ell_\gamma))$. By the lower semicontinuity of the total variation, for such curves γ we have

$$\|D_\gamma u\|((0, \ell_\gamma)) \le \liminf_{i \to \infty} \int_\gamma g_i\, ds,$$

and so by Fatou's lemma and the definition of the Semmes family of curves (3.4),

$$\begin{aligned}\int_{\Gamma_{x,y}} \|D_\gamma u\|((0,\ell_\gamma))\, d\alpha_{x,y}(\gamma) &\le \int_{\Gamma_{x,y}} \left(\liminf_{i\to\infty} \int_\gamma g_i\, ds \right) d\alpha_{x,y}(\gamma) \\ &\le \liminf_{i\to\infty} \int_{\Gamma_{x,y}} \int_\gamma g_i\, ds\ d\alpha_{x,y}(\gamma) \\ &\le \liminf_{i\to\infty} c_S \int_{B_{xy}} g_i R_{x,y}\, d\mu.\end{aligned}$$

By the choice of u_i, g_i from Lemma 4.3, we have

$$\liminf_{i\to\infty} \int_{B_{xy}} g_i\, R_{x,y}\, d\mu \le C \int_{2B_{xy}} R_{x,y}\, d\|Du\|.$$

By combining (4.8) with the last two inequalities, we get

$$|u(x) - u(y)| \le \int_{\Gamma_{x,y}} \|D_\gamma u\|((0,\ell_\gamma))\ d\alpha_{x,y}(\gamma) \le C \int_{2B_{xy}} R_{x,y}\, d\|Du\|,$$

where $C = C(c_d, c_P, \tau, c_S)$. By Lemma 4.2 and the subsequent comment, the integral on the right-hand side is finite for μ-a.e. $x, y \in B \subset 9\lambda B \subset \Omega$, so for these points $u\circ\gamma \in BV((0,\ell_\gamma))$ for $\alpha_{x,y}$-almost every γ. This completes the proof. □

With the help of Lemma 4.7, we can now prove the Reshetnyak-type characterization of bounded BV functions on X in terms of the total variation on the curves in the geometric Semmes family.

Theorem 4.9. *Suppose that X supports a geometric Semmes family of curves, and let $\Omega \subset X$ be open. Suppose also that u is a μ-measurable function on Ω with $0 \le u \le 1$. Then $u \in BV(\Omega)$ if and only if there exist constants $\kappa > 1$, $C_0 > 0$ and a Radon measure ν of finite mass on Ω such that for μ-a.e. $x, y \in \Omega$ with $x \ne y$ and $B(x, \kappa d(x,y)) \subset \Omega$ the following condition is satisfied:*

$$\int_{\Gamma_{x,y}} \|D_\gamma u\|((0,\ell_\gamma))\, d\alpha_{x,y}(\gamma) \le C_0 \int_{B(x,\kappa d(x,y))} R_{x,y}\, d\nu. \tag{4.10}$$

In particular, for $\alpha_{x,y}$-almost every $\gamma \in \Gamma_{x,y}$ we have $u \circ \gamma \in BV((0,\ell_\gamma))$. Furthermore, $\|Du\|(\Omega) \le C(C_0, c_d, \kappa)\nu(\Omega)$.

Implicit in the assumptions for the converse part of the statement given in the above theorem is the requirement that $\gamma \mapsto \|D_\gamma u\|((0,\ell_\gamma))$ is $\alpha_{x,y}$-measurable, so that the integral on the left-hand side makes sense.

Proof of Theorem 4.9. The "only if" part is clear by the previous lemma, for we can choose $\nu = \|Du\|$ and $\kappa = 9\lambda$. For the "if" part, we choose Lebesgue points $x, y \in B \subset 5\kappa B \subset \Omega$

of u such that (4.10) holds. By using, in order, Lemma 4.6, (4.10), and Lemma 4.2, we get

$$\begin{aligned}|u(x)-u(y)| &\le \int_{\Gamma_{x,y}} \|D_\gamma u\|((0,\ell_\gamma))\,d\alpha_{x,y}(\gamma)\\ &\le C_0\int_{B(x,\kappa d(x,y))} R_{x,y}\,d\nu\\ &\le C(C_0,c_d,\kappa)d(x,y)[\mathcal{M}_{2\kappa d(x,y),\nu}(x)+\mathcal{M}_{2\kappa d(x,y),\nu}(y)].\end{aligned}$$

Now we get $u\in BV(\Omega)$ as well as $\|Du\|(\Omega)\le C(C_0,c_d,\kappa)\nu(\Omega)$ by Proposition 2.13. The weak-type estimate required in the proposition is given by Lemma 4.1, and in particular it guarantees the finiteness of all the quantities above for μ-almost all $x,y\in\Omega$ such that $x,y\in B\subset 5\kappa B\subset\Omega$. □

5 Federer-type characterization

In this section we prove that if the metric measure space supports a geometric Semmes family of curves, then a set has finite perimeter if and only if the codimension 1 Hausdorff measure of its measure theoretic boundary is finite. Recall that the definition of the codimension 1 Hausdorff measure $\mathcal{H}^h$ is given in (2.7).

One direction is well-known and requires only the weaker assumption of a $(1,1)$-Poincaré inequality: if the set $E\subset X$ has finite perimeter, then for any $A\subset X$ we have

$$\frac{1}{C}\mathcal{H}^h(\partial^*E\cap A)\le P(E,A)\le C\mathcal{H}^h(\partial^*E\cap A),$$

where the constant C only depends on the doubling constant and the constants in the Poincaré inequality. The first inequality is proved in [Am1, Theorem 4.2] and [Am2, Theorem 5.3], whereas the second inequality is proved in [AMP, Theorem 4.6]. Hence our main result is the following.

Theorem 5.1. *If X supports a geometric Semmes family of curves, $\Omega\subset X$ is an open set, and $E\subset X$ is a μ-measurable set, then*

$$P(E,\Omega)\le C(c_d,c_S,\lambda)\mathcal{H}^h(\partial^*E\cap\Omega).$$

In particular, E has finite perimeter in Ω if $\mathcal{H}^h(\partial^*E\cap\Omega)$ is finite.

The following two lemmas are needed in the proof of Theorem 5.1.

Lemma 5.2. *Suppose that X supports a Semmes family of curves. Let $A\subset X$ with $\mathcal{H}^h(A)$ finite. Then for μ-a.e. $x,y\in X$ with $x\ne y$ we have*

$$\int_{\Gamma_{x,y}} \#(\gamma\cap A)\,d\alpha_{x,y}(\gamma)\le C(c_d,c_S)\int_{3B_{xy}} R_{x,y}\,d\mathcal{H}^h|_A.$$

Proof. Fix two distinct points $x, y \in X \setminus A$. First let us also assume that there is a positive number $\delta < d(x,y)$ so that $A \cap \big(B(x,\delta) \cup B(y,\delta)\big) = \emptyset$. Take any $0 < \varepsilon < \min\{\delta/6, 1\}$ and let $\{B_j\}_{j=1}^\infty$ be a cover of A by balls of radii $r_j = \operatorname{rad}(B_j) < \varepsilon$ such that $A \cap B_j \neq \emptyset$ for each $j \in \mathbb{N}$, and

$$\sum_j \frac{\mu(B_j)}{r_j} \le \mathcal{H}^h(A) + \varepsilon.$$

For any $\gamma \in \Gamma_{x,y}$,

$$\#(\gamma \cap A) = \mathcal{H}^0(\gamma \cap A) = \lim_{\varepsilon \to 0} \mathcal{H}^0_\varepsilon(\gamma \cap A).$$

Here $\mathcal{H}^0$ is the usual 0-dimensional Hausdorff measure (counting measure): for any set $A \subset X$,

$$\mathcal{H}^0_\varepsilon(A) := \inf\left\{ n \in \mathbb{N} :\, A \subset \bigcup_{i=1}^n B_i,\, \operatorname{rad}(B_i) \le \varepsilon \right\}.$$

Observe that for $\gamma \in \Gamma_{x,y}$,

$$\gamma \cap A \subset \gamma \cap \bigcup_j B_j,$$

and if $\gamma \cap B_j \neq \emptyset$, it follows that $\ell(\gamma \cap 2B_j) \ge r_j$ — here we used the fact that $r_j < \varepsilon < d(x,y)$. Thus

$$\mathcal{H}^0_\varepsilon(\gamma \cap A) \le \sum_j \frac{\ell(\gamma \cap 2B_j)}{r_j} = \int_\gamma \sum_j \frac{1}{r_j} \chi_{2B_j}\, ds.$$

By the above estimates, Fatou's lemma, and the definition of the Semmes family of curves, we obtain

$$\begin{aligned}
\int_{\Gamma_{x,y}} \#(\gamma \cap A)\, d\alpha_{x,y}(\gamma) &\le \int_{\Gamma_{x,y}} \liminf_{\varepsilon \to 0} \int_\gamma \left(\sum_j \frac{1}{r_j} \chi_{2B_j} \right) ds\, d\alpha_{x,y}(\gamma) \\
&\le \liminf_{\varepsilon \to 0} \int_{\Gamma_{x,y}} \int_\gamma \left(\sum_j \frac{1}{r_j} \chi_{2B_j} \right) ds\, d\alpha_{x,y}(\gamma) \\
&\le c_S \liminf_{\varepsilon \to 0} \int_{B_{xy}} R_{x,y} \left(\sum_j \frac{1}{r_j} \chi_{2B_j} \right) d\mu \\
&\le C(c_S, c_d) \liminf_{\varepsilon \to 0} \int_{2B_{xy}} R_{x,y} \left(\sum_j \frac{1}{r_j} \chi_{B_j} \right) d\mu.
\end{aligned} \tag{5.3}$$

The last inequality follows from the facts that each B_j has radius no more than $\delta/6$ and that the balls $2B_j$ do not intersect $B(x,\delta/2) \cup B(y,\delta/2)$, and so $\sup_{2B_j} R_{x,y} \le c_d^3 \inf_{2B_j} R_{x,y}$ for each $j \in \mathbb{N}$. Let

$$\varphi_\varepsilon := \sum_j \frac{1}{r_j} \chi_{B_j}.$$

We have

$$\int_X \varphi_\varepsilon\, d\mu \le \mathcal{H}^h(A) + \varepsilon \le \mathcal{H}^h(A) + 1.$$

Hence the sequence of measures $\varphi_\varepsilon\, d\mu$ has a subsequence $\varphi_{\varepsilon_i}\, d\mu$, with $\varepsilon_i \to 0$, that converges weakly* to a Radon measure ν. We now show that $\nu \le \mathcal{H}^h|_A$. It is enough to show that $\nu(K) \le \mathcal{H}^h|_A(U)$ for every bounded open set $U \subset X$ and every compact set $K \subset U$ (see [AFP, Proposition 1.43] or [AT, Theorem 1.1.12]). If we pick any such pair of sets, we can then pick an open set U' such that $K \subset U' \Subset U$. By the properties of the weak convergence of measures, we can calculate

$$\nu(K) \le \nu(U') \le \liminf_{i\to\infty} \int_{U'} \varphi_{\varepsilon_i}\, d\mu,$$

see [AFP, Proposition 1.62] or [AT, Proposition 1.32]. This gives us the result, as long as the last quantity is at most $\mathcal{H}^h|_A(U)$. Assume on the contrary that

$$\liminf_{i\to\infty} \int_{U'} \varphi_{\varepsilon_i}\, d\mu > \mathcal{H}^h|_A(U).$$

Since we have

$$\lim_{i\to\infty} \int_X \varphi_{\varepsilon_i}\, d\mu = \mathcal{H}^h|_A(X),$$

we would now have

$$\liminf_{i\to\infty} \int_{X\setminus U'} \varphi_{\varepsilon_i}\, d\mu < \mathcal{H}^h|_A(X \setminus U).$$

However, for large enough $i \in \mathbb{N}$ (that is, for small enough ε_i) this would mean that

$$\sum_j \frac{\mu(B_j)}{r_j} < \mathcal{H}^h|_A(X \setminus U),$$

where the sum is taken over balls that cover $A \setminus U$. This is a contradiction by the definition of $\mathcal{H}^h$. Thus $\nu \le \mathcal{H}^h\big|_A$.

As mentioned earlier, the balls B_j do not intersect $B(x, \delta/2) \cup B(y, \delta/2)$. Let us take a Lipschitz function $0 \le \psi \le 1$ that takes the value 1 in $2B_{xy} \setminus \big(B(x, \delta/2) \cup B(y, \delta/2)\big)$, and the value 0 in neighborhoods of x and y and outside $3B_{xy}$. Then the function $\psi R_{x,y}$ is continuous and has compact support. Now we can estimate the last line of (5.3) by using the definition of the weak convergence of measures:

$$\liminf_{i\to\infty} \int_{2B_{xy}} R_{x,y}\varphi_{\varepsilon_i}\, d\mu \le \liminf_{i\to\infty} \int_X \psi R_{x,y}\varphi_{\varepsilon_i}\, d\mu = \int_X \psi R_{x,y}\, d\mathcal{H}^h|_A \le \int_{3B_{xy}} R_{x,y}\, d\mathcal{H}^h|_A.$$

In conclusion, we have

$$\int_{\Gamma_{x,y}} \#(\gamma \cap A)\, d\alpha_{x,y}(\gamma) \le C(c_S, c_d) \int_{3B_{xy}} R_{x,y}\, d\mathcal{H}^h|_A. \tag{5.4}$$

Finally, take a general set A that does not contain x or y. For any $0 < \delta < d(x, y)$, define

$A_\delta := A \setminus \big(B(x,\delta) \cup B(y,\delta)\big)$. Then we have by Fatou's lemma and by (5.4),

$$\begin{aligned}\int_{\Gamma_{x,y}} \#(\gamma \cap A)\, d\alpha_{x,y}(\gamma) &= \int_{\Gamma_{x,y}} \liminf_{\delta \to 0} \#(\gamma \cap A_\delta)\, d\alpha_{x,y}(\gamma)\\ &\le \liminf_{\delta\to 0} \int_{\Gamma_{x,y}} \#(\gamma \cap A_\delta)\, d\alpha_{x,y}(\gamma)\\ &\le C(c_S, c_d) \liminf_{\delta\to 0} \int_{3B_{xy}} R_{x,y}\, d\mathcal{H}^h|_{A_\delta}\\ &\le C(c_S, c_d) \int_{3B_{xy}} R_{x,y}\, d\mathcal{H}^h|_{A}.\end{aligned}$$

This concludes the proof. □

The following lemma can be considered the crux of the proof of Theorem 5.1.

Lemma 5.5. *Let X support a geometric Semmes family of curves, let $E \subset X$ be a μ-measurable set, and let $x, y \in X$ such that $x \neq y$. Let $\Gamma \subset \Gamma_{x,y}$ be the subfamily of curves γ with a subcurve $\tilde\gamma$ such that both $\tilde\gamma \cap I$ and $\tilde\gamma \cap O$ are nonempty, but $\tilde\gamma$ does not intersect $\partial^* E$. Then $\alpha_{x,y}(\Gamma) = 0$.*

Proof. Fix $\varepsilon > 0$ to be determined later. For $k \in \mathbb{N}$, let

$$I(k) := \left\{ w \in X \setminus \{x,y\} \,:\, \sup_{0<r<1/k} \frac{\mu(B(w,r)\cap O)}{\mu(B(w,r))} \le \varepsilon \right\},$$

$$O(k) := \left\{ w \in X \setminus \{x,y\} \,:\, \sup_{0<r<1/k} \frac{\mu(B(w,r)\cap I)}{\mu(B(w,r))} \le \varepsilon \right\}.$$

Notice that

$$I \subset \bigcup_{k=1}^\infty I(k) \quad \text{and} \quad O \subset \bigcup_{k=1}^\infty O(k). \tag{5.6}$$

For $k, m \in \mathbb{N}$, we also define (recall from Definition 3.11 the numbers b_i from condition (4) of the geometric Semmes family of curves)

$$\Gamma_m(I(k) \to O) := \{\gamma \in \Gamma_{x,y} \,:\, \exists t \text{ with } \gamma(t) \in I(k) \setminus \bigcup_{i=1}^n \{f = b_i\} \text{ and } \gamma((t, t+\tfrac{1}{m})) \subset O\},$$

$$\Gamma_m(O \to I(k)) := \{\gamma \in \Gamma_{x,y} \,:\, \exists t \text{ with } \gamma(t) \in I(k) \setminus \bigcup_{i=1}^n \{f = b_i\} \text{ and } \gamma((t-\tfrac{1}{m}, t)) \subset O\},$$

$$\Gamma_m(O(k) \to I) := \{\gamma \in \Gamma_{x,y} \,:\, \exists t \text{ with } \gamma(t) \in O(k) \setminus \bigcup_{i=1}^n \{f = b_i\} \text{ and } \gamma((t, t+\tfrac{1}{m})) \subset I\},$$

$$\Gamma_m(I \to O(k)) := \{\gamma \in \Gamma_{x,y} \,:\, \exists t \text{ with } \gamma(t) \in O(k) \setminus \bigcup_{i=1}^n \{f = b_i\} \text{ and } \gamma((t-\tfrac{1}{m}, t)) \subset I\}.$$

We wish to establish that the $\alpha_{x,y}$-measure of all of these curve families is zero. For $k, m \in \mathbb{N}$ fixed, we will show that $\alpha_{x,y}(\Gamma_m(I(k) \to O)) = 0$. Analogous arguments hold for the other terms.

Let $\delta < w/(m\, d(x,y))$, where w is the "growth speed" from condition (3) of the geometric Semmes family of curves. Next, we take the collection of numbers $\mathcal{A}_\delta$ and the corresponding sets A_j, $j \in \mathbb{Z}$, guaranteed by Definition 3.11.

For $j \in \mathbb{Z}$, let $\Gamma(j)$ denote the collection of curves $\gamma \in \Gamma_m(I(k) \to O)$ such that there exists t_γ with the property that

$$\gamma(t_\gamma) \in I(k) \cap A_j \quad \text{and} \quad \gamma((t_\gamma, t_\gamma + 1/m)) \subset O. \tag{5.7}$$

Since

$$\Gamma_m(I(k) \to O) = \bigcup_{j \in \mathbb{Z}} \Gamma(j),$$

it suffices to show that $\alpha_{x,y}(\Gamma(j)) = 0$ for each $j \in \mathbb{N}$.

For $j \in \mathbb{N}$ fixed, equip the space $\Gamma_{x,y}$ with the metric

$$d_j(\gamma_1, \gamma_2) := d_H(\gamma_1 \cap A_j, \gamma_2 \cap A_j),$$

where γ_1, γ_2 are any curves in $\Gamma_{x,y}$ and d_H is the Hausdorff distance. Now, for $\gamma_0 \in \Gamma_{x,y}$, let

$$D(\gamma_0, r) := \{\gamma \in \Gamma_{x,y} : d_j(\gamma, \gamma_0) < r\}.$$

The set $D(\gamma_0, r)$ is thus a ball in the metric space $\Gamma_{x,y}$. Due to the conditions satisfied by the geometric Semmes family of curves, we can now show that $\alpha_{x,y}$ is doubling with respect to the metric d_j for small enough balls. More precisely, let us require

$$r < \min\left\{ \frac{\operatorname{dist}(\{x,y\}, A_j)}{9}, \frac{c_S d(x,y)(\widetilde{a}_j - a_j)}{6} \right\}. \tag{5.8}$$

Recall that $a_j \le f < \widetilde{a}_j$ in the set A_j. It is easy to verify that the function f is $3/d(x,y)$-Lipschitz continuous. Take any $\gamma \in \Gamma_{x,y}$. We can, by continuity, pick t such that $f(\gamma(t)) = (a_j + \widetilde{a}_j)/2$, and then

$$B\left(\gamma(t), \frac{d(x,y)(\widetilde{a}_j - a_j)}{6}\right) \subset A_j. \tag{5.9}$$

The curves that intersect $B(\gamma(t), 2r)$ travel at least the length r inside the ball $B(\gamma(t), 3r)$. Thus

$$\begin{aligned}
\alpha_{x,y}(D(\gamma, 2r)) &\le \alpha_{x,y}(\Gamma(B(\gamma(t), 2r))) \\
&\le \int_{\Gamma(B(\gamma(t),2r))} \frac{1}{r} \int_\gamma \chi_{B(\gamma(t),3r)}\, ds\, d\alpha_{x,y}(\gamma) \\
&\le \int_{\Gamma_{x,y}} \frac{1}{r} \int_\gamma \chi_{B(\gamma(t),3r)}\, ds\, d\alpha_{x,y}(\gamma) \\
&\le \frac{c_S}{r} \int_{B(\gamma(t),3r)} R_{x,y}\, d\mu \\
&\le \frac{C(c_S, c_d)}{r} \int_{B(\gamma(t), c_S^{-2} r)} R_{x,y}\, d\mu \\
&\le C(c_S, c_d)\, \alpha_{x,y}(\Gamma(B(\gamma(t), c_S^{-1} r))) \\
&\le C(c_S, c_d)\, \alpha_{x,y}(D(\gamma, r)).
\end{aligned} \tag{5.10}$$

In the fourth inequality we used the definition of Semmes family of curves, and in the fifth inequality we used the fact that for small enough radii the measure $d\nu := R_{x,y}\, d\mu$ is doubling at $\gamma(t)$, with doubling constant c_d^4. In the last two inequalities we used conditions (1) and (2) of the geometric Semmes family of curves. Note that in order to use condition (2) to obtain the last inequality, we need $B(\gamma(t), c_S^{-1} r) \subset A_j$, which is guaranteed by (5.8) and (5.9).

Now fix $\gamma_0 \in \Gamma(j)$. Let

$$r < \min\left\{ \frac{w}{100c_S^2} \operatorname{dist}(\{x, y\}, A_j), \frac{w(\widetilde{a}_j - a_j)d(x, y)}{100c_S}, \frac{w}{10k}, \frac{1}{m} \right\}, \tag{5.11}$$

and further let

$$D_*(\gamma_0, r) := D(\gamma_0, r) \cap \Gamma(j).$$

Let $\gamma_1 \in D_*(\gamma_0, r)$ be such that there is a choice of t_{γ_1} satisfying the condition that for all $\gamma \in D_*(\gamma_0, r)$,

$$f(\gamma_1(t_{\gamma_1})) + r/d(x, y) > f(\gamma(t_\gamma)); \tag{5.12}$$

recall the definition of "t_γ" from (5.7). This condition implies that of all the curves $\gamma \in D_*(\gamma_0, r)$, γ_1 has the "bad point" $\gamma_1(t_{\gamma_1})$ almost the furthest away from the inner boundary of the "annulus" A_j.

Next we show that for all $\gamma \in D_*(\gamma_0, r)$, by the choice of γ_1 and by the fact that $\gamma((t_\gamma, t_\gamma + 1/m)) \subset O$, we have

$$\ell(\gamma \cap O \cap B(\gamma_1(t_{\gamma_1}), 10r/w)) \geq r. \tag{5.13}$$

This is seen as follows. By the condition $\gamma \in D_*(\gamma_0, r)$ we know that for some t we have $\gamma(t) \in A_j$ and $d(\gamma(t), \gamma_1(t_{\gamma_1})) < 2r$. If γ has its "bad point" $\gamma(t_\gamma)$ "earlier" on the curve, i.e. $t_\gamma \leq t$, then condition (5.13) is easily satisfied. Let us assume that $t_\gamma > t$ instead. Since f is a $3/d(x, y)$-Lipschitz function, we have

$$f(\gamma(t)) > f(\gamma_1(t_{\gamma_1})) - 6r/d(x, y). \tag{5.14}$$

Hence we know that $t_\gamma - t < 7r/w$, for otherwise the growth condition (3) of the geometric Semmes family of curves and (5.14) would imply that $f(\gamma(t_\gamma))$ is too big, violating (5.12). Now we have of course $d(\gamma(t_\gamma), \gamma(t)) < 7r/w$, further implying that $d(\gamma(t_\gamma), \gamma_1(t_{\gamma_1})) < 9r/w$. Thus γ travels at least the length r in $O \cap B(\gamma_1(t_{\gamma_1}), 10r/w)$, since we had $r < 1/m$. This is condition (5.13).

Now we can use the definition of the (ordinary) Semmes family of curves to calculate

$$\begin{aligned} \alpha_{x,y}(D_*(\gamma_0, r)) &\leq \frac{1}{r} \int_{D_*(\gamma_0,r)} \int_\gamma \chi_{O \cap B(\gamma_1(t_{\gamma_1}),10r/w)}\, ds\, d\alpha_{x,y}(\gamma) \\ &\leq \frac{1}{r} \int_{\Gamma_{x,y}} \int_\gamma \chi_{O \cap B(\gamma_1(t_{\gamma_1}),10r/w)}\, ds\, d\alpha_{x,y}(\gamma) \\ &\leq \frac{c_S}{r} \int_{O \cap B(\gamma_1(t_{\gamma_1}),10r/w)} R_{x,y}\, d\mu \\ &\leq \frac{c_S c_d^3 \varepsilon}{r} \int_{B(\gamma_1(t_{\gamma_1}),10r/w)} R_{x,y}\, d\mu. \end{aligned} \tag{5.15}$$

In the last inequality, we used the fact that by the doubling property of μ and by (5.11), $R_{x,y}$ is approximately constant in the ball $B = B(\gamma_1(t_{\gamma_1}), 10r/w)$ — more precisely $\sup_B R_{x,y} \le c_d^3 \inf_B R_{x,y}$ — and so, since $\gamma_1(t_{\gamma_1}) \in I(k)$,

$$\frac{\int_{O\cap B(\gamma_1(t_{\gamma_1}),10r/w)} R_{x,y}\,d\mu}{\int_{B(\gamma_1(t_{\gamma_1}),10r/w)} R_{x,y}\,d\mu} \le c_d^3\,\frac{\mu(O\cap B(\gamma_1(t_{\gamma_1}),10r/w))}{\mu(B(\gamma_1(t_{\gamma_1}),10r/w))} \le c_d^3\varepsilon.$$

Now pick $\tilde{t}$ such that $\gamma_0(\tilde{t}) \in A_j$ and $d(\gamma_0(\tilde{t}), \gamma_1(t_{\gamma_1})) < r$. Then pick another number t such that

$$a_j + \frac{10c_S r}{w}\frac{3}{d(x,y)} \le f(\gamma_0(t)) \le \widetilde{a}_j - \frac{10c_S r}{w}\frac{3}{d(x,y)}. \tag{5.16}$$

By the $3/d(x,y)$-Lipschitz-continuity of the function f we have $B(\gamma_0(t), 10c_S r/w) \subset A_j$. By the growth condition (3) of the geometric Semmes family of curves and condition (5.16), we can pick the number t so that it satisfies

$$\frac{w}{d(x,y)}|\widetilde{t} - t| \le \frac{10c_S r}{w}\frac{3}{d(x,y)},$$

and consequently $d(\gamma_0(\widetilde{t}), \gamma_0(t)) \le 30c_S r/w^2$. Thus we have $d(\gamma_0(t), \gamma_1(t_{\gamma_1})) \le 31c_S r/w^2$. This implies that $\mu(B(\gamma_1(t_{\gamma_1}), 10r/w))$ and $\mu(B(\gamma_0(t), 10r/w))$ are comparable by a factor $C = C(c_d, c_S, w)$ [BjBj, Lemma 3.6].

Additionally, recall from Section 3.2 that $R_{x,y}$ is, up to a factor $C(c_d, \lambda)$, constant on the set A_j, and even in a $10r/w$-neighborhood of it. This enables us to switch to a different ball in (5.15):

$$\alpha_{x,y}(D_*(\gamma_0, r)) \le \frac{C(c_S, c_d, w, \lambda)\varepsilon}{r} \int_{B(\gamma_0(t),10r/w)} R_{x,y}\,d\mu.$$

On the other hand, by condition (1) of the geometric Semmes family of curves,

$$\alpha_{x,y}(\Gamma(B(\gamma_0(t), 10c_S r/w))) \ge \frac{1}{c_S}\frac{1}{10c_S r/w}\int_{B(\gamma_0(t),10r/w)} R_{x,y}\,d\mu.$$

Furthermore, by condition (2) and the fact that $B(\gamma_0(t), 10c_S r/w) \subset A_j$, we have

$$\Gamma(B(\gamma_0(t), 10c_S r/w)) \subset D(\gamma_0, 10c_S^2 r/w).$$

It follows that

$$\alpha_{x,y}(D_*(\gamma_0, r)) \le C(c_S, c_d, \lambda, w)\varepsilon\alpha_{x,y}(D(\gamma_0, 10c_S^2 r/w)).$$

By the doubling property (5.10) of $\alpha_{x,y}$ with respect to the metric d_j, where again we need r to be small enough, we now have

$$\alpha_{x,y}(D_*(\gamma_0, r)) \le C(c_S, c_d, \lambda, w)\varepsilon\alpha_{x,y}(D(\gamma_0, r)).$$

Thus taking ε small enough so that $C\varepsilon < \frac{2}{3}$, we get

$$\alpha_{x,y}(D_*(\gamma_0, r)) \le \tfrac{2}{3}\alpha_{x,y}(D(\gamma_0, r)).$$

We recall that $D(\gamma, r)$ denotes a ball with respect to the metric d_j. Now the density of $\Gamma(j)$ at γ_0 is

$$\limsup_{r\to 0^+} \frac{\alpha_{x,y}(D(\gamma_0, r)\cap\Gamma(j))}{\alpha_{x,y}(D(\gamma_0, r))} = \limsup_{r\to 0^+} \frac{\alpha_{x,y}(D_*(\gamma_0, r))}{\alpha_{x,y}(D(\gamma_0, r))} \leq 2/3 < 1.$$

Furthermore, $\gamma_0 \in \Gamma(j)$ was arbitrary. On the other hand, for a Borel regular outer measure that is doubling for sufficiently small radii, such as $\alpha_{x,y}$, we know that Lebesgue's differentiation theorem holds, see e.g. [Hei, Theorem 1.8]. In particular, almost every point of a set has density 1, so now we must have $\alpha_{x,y}(\Gamma(j)) = 0$. This is the desired result.

For the remainder of the proof, fix a curve

$$\gamma \in \Gamma_{x,y} \setminus \bigcup_{k,m\in\mathbb{N}} \big(\Gamma_m(I(k)\to O)\cup\Gamma_m(O\to I(k))\cup\Gamma_m(O(k)\to I)\cup\Gamma_m(I\to O(k))\big); \quad (5.17)$$

by the above result, this holds for $\alpha_{x,y}$-a.e. curve. We wish to show that $\gamma \in \Gamma_{x,y} \setminus \Gamma$. First let us assume that $\gamma(u) \in I$ and $\gamma(v) \in O$ for some $0 \leq u < v \leq \ell_\gamma$ so that there is no b_i (defined in condition (4) of the geometric Semmes family of curves) in the interval $(f(\gamma(u)), f(\gamma(v)))$.

We note that the sets $I(k)$ and $O(k)$ are closed, which can be seen as follows. Let $z_i \in I(k)$, $i = 1, 2, \ldots$, and $z_i \to z$. Now, for any given $r < 1/k$ we simply calculate

$$\frac{\mu(B(z, r)\cap O)}{\mu(B(z, r))} = \lim_{i\to\infty} \frac{\mu(B(z_i, r - d(z_i, z))\cap O)}{\mu(B(z_i, r - d(z_i, z)))} \leq \varepsilon,$$

which gives the result. Now we recall (5.6), and note that $\{I(k)\}_{k=1}^\infty$ and $\{O(k)\}_{k=1}^\infty$ are increasing sequences of closed sets. Thus there exists some $k_0 \in \mathbb{N}$ such that $\gamma(u) \in I(k_0)$ and $\gamma(v) \in O(k_0)$. We define

$$u_0 := \sup\{t \in [u, v) : \gamma(t) \in I(k_0)\},$$

and note that $I(k_0)$ is closed and $I(k_0) \cap O(k_0) = \emptyset$, as we can choose $\varepsilon < \frac{1}{2}$. Thus we have $\gamma(u_0) \in I(k_0)$ and $u_0 < v$. Next define

$$v_0 := \inf\{t \in (u_0, v] : \gamma(t) \in O(k_0)\}.$$

By the same reasoning as above, we can conclude that $\gamma(v_0) \in O(k_0)$, and $u \leq u_0 < v_0 \leq v$. Clearly we also have

$$\gamma(t) \notin I(k_0) \cup O(k_0) \quad \text{for } t \in (u_0, v_0).$$

Now, by (5.17) we know that there exist $u_0 < \tilde{u}_1 < \tilde{v}_1 < v_0$ such that $\gamma(\tilde{u}_1) \in I$ and $\gamma(\tilde{v}_1) \in O$. By the same reasoning as above, we find $k_1 > k_0$ and $u_0 < u_1 < v_1 < v_0$ such that $\gamma(u_1) \in I(k_1)$, $\gamma(v_1) \in O(k_1)$, and

$$\gamma(t) \notin I(k_1) \cup O(k_1) \text{ for } t \in (u_1, v_1).$$

Continuing like this, we get a sequence $k_i \to \infty$ and sequences u_i, v_i such that

$$\begin{cases} u_0 < u_1 < u_2 < \ldots,\ v_0 > v_1 > \ldots, \\ u_i < v_i, \\ \gamma(u_i) \in I(k_i),\ \gamma(v_i) \in O(k_i), \\ \gamma(t) \notin I(k_i) \cup O(k_i) \text{ for } t \in (u_i, v_i),\ \text{for all } i \in \mathbb{N}. \end{cases}$$

Now we can choose

$$\lim_{i\to\infty} u_i \le t \le \lim_{i\to\infty} v_i,$$

and clearly

$$\gamma(t) \notin \bigcup_{i=1}^{\infty} I(k_i) \cup O(k_i).$$

By definition of the sets $I(k)$ and $O(k)$, we get

$$\limsup_{r\to 0} \frac{\mu(B(\gamma(t),r)\cap O)}{\mu(B(\gamma(t),r))} > \varepsilon \quad \text{and} \quad \limsup_{r\to 0} \frac{\mu(B(\gamma(t),r)\cap I)}{\mu(B(\gamma(t),r))} > \varepsilon,$$

which implies that $\gamma(t) \in \partial^* E$. We conclude that if $\gamma(u) \in I$ and $\gamma(v) \in O$ for some $0 \le u < v \le \ell_\gamma$, and there is no b_i in the interval $(f(\gamma(u)), f(\gamma(v)))$, then $\gamma(t) \in \partial^* E$ for some $t \in (u,v)$. The same is true for I and O reversed.

Now let $\widetilde{\gamma}$ be a subcurve of γ that does not intersect $\partial^* E$. Between any two level sets $\{f = b_i\}$ and $\{f = b_{i+1}\}$, $\widetilde{\gamma}$ intersects only I or only O, by the above result. For definiteness, assume that $\widetilde{\gamma}$ intersects I at some point. Then $\widetilde{\gamma}$ only intersects I between some level sets $\{f = b_i\}$ and $\{f = b_{i+1}\}$, and by the weak continuity condition (4) in the geometric Semmes family of curves, we can conclude that if $f(\gamma(t)) = b_i$ or $f(\gamma(t)) = b_{i+1}$, then $\gamma(t) \in I$ as well. Using the same condition several times, we conclude that $\widetilde{\gamma}$ travels entirely in I. Similarly we can deduce that if $\widetilde{\gamma}$ intersects O at any point, it travels entirely in O. This gives us the result. □

Now we are ready to prove Theorem 5.1.

Proof of Theorem 5.1. We can assume that $\mathcal{H}^h(\partial^* E \cap \Omega) < \infty$. Let $x, y \in \Omega$ such that $3B_{xy} \subset \Omega$. Let us consider a curve $\gamma \in \Gamma_{x,y} \setminus \Gamma$, where Γ is the curve family defined in Lemma 5.5. Then

$$\|D_\gamma \chi_I\|((0,\ell_\gamma)) \le \sup_{0<t_0<\ldots<t_l<\ell_\gamma} \sum_{n=1}^{l} |\chi_I(\gamma(t_{n-1})) - \chi_I(\gamma(t_n))|,$$

and by Lemma 5.5 it is possible to have $|\chi_I(\gamma(t_{n-1})) - \chi_I(\gamma(t_n))| = 1$ only if $\gamma\big|_{[t_{n-1},t_n]}$ intersects $\partial^* E$. Otherwise $|\chi_I(\gamma(t_{n-1})) - \chi_I(\gamma(t_n))| = 0$. Consequently, we have

$$\|D_\gamma \chi_I\|((0,\ell_\gamma)) \le \sup_{0<t_0<\ldots<t_l<\ell_\gamma} \sum_{n=1}^{l} \#(\gamma\big|_{[t_{n-1},t_n]} \cap \partial^* E) \le 2\#(\gamma \cap \partial^* E).$$

Thus by Lemma 5.2, we have

$$\int_{\Gamma_{x,y}} \|D_\gamma \chi_I\|((0,\ell_\gamma))\, d\alpha_{x,y}(\gamma) \le C(c_d, c_S) \int_{3B_{xy}} R_{x,y}\, d\mathcal{H}^h\big|_{\partial^* E}.$$

Hence by Theorem 4.9, we have that E has finite perimeter in Ω, with

$$P(E,\Omega) \le C(c_d, c_S, \lambda)\mathcal{H}^h(\partial^* E \cap \Omega). \qquad \square$$

Remark 5.18. By combining Theorem 5.1 with the discussion in the beginning of this section, we conclude that always $\mathcal{H}^h(\partial^* E \cap \Omega) \approx P(E, \Omega)$, with constants of comparison $C = C(c_d, c_P, \tau)$. In particular, from the relative isoperimetric inequality (2.11) we get the following *strong relative isoperimetric inequality* as promised in [KKST]:

$$\min\{\mu(B \cap E), \mu(B \setminus E)\} \leq C \operatorname{rad}(B) \mathcal{H}^h(\partial^* E \cap \lambda B)$$

for any μ-measurable $E \subset X$ and any ball B.

6 Examples of spaces supporting a geometric Semmes family of curves

The geometric version of the Semmes family of curves proposed in this article would not be compelling if we could not show that the classical metric space, the Euclidean space equipped with the Lebesgue measure, supports such a geometric family. In this section we will show that the Euclidean space as well as certain examples of non-Euclidean spaces support the geometric Semmes family of curves.

It was pointed out to us by Riikka Kangaslampi that the boundaries of certain hyperbolic buildings, constructed by Bourdon and Pajot in [BoPaj], are unlikely to support such a geometric version of a Semmes family of curves (although they do support the standard Semmes family of curves and hence the $(1,1)$-Poincaré inequality) because geodesics between two distinct points in these spaces cross each other at all scales.

Example 6.1 (Euclidean spaces)**.** In the Euclidean space $\mathbb{R}^n$ a geometric Semmes family of curves is simple to construct. For any points x, y, consider the hyperplane orthogonal to the line segment $[x, y]$, and lying at an equal distance from both points. Take an $(n-1)$-dimensional disk B_{n-1} in the hyperplane, centered at the intersection point and with radius $|x - y|/2$. For any $z \in B_{n-1}$, let γ_z be the curve consisting of two line segments: $[x, z]$ and $[z, y]$. Let $\Gamma_{x,y} := \{\gamma_z : z \in B_{n-1}\}$. Due to the shape of this type of curve family, it is often called a Semmes *pencil* of curves.

We define the measure $\alpha_{x,y}$ as follows: if $\Gamma \subset \Gamma_{x,y}$,

$$\alpha_{x,y}(\Gamma) := \mathcal{H}^{n-1}(\{z \in B_{n-1} : \gamma_z \in \Gamma\})/\mathcal{H}^{n-1}(B_{n-1}).$$

The "annuli" A_j can be defined as in (3.9). Now all the conditions are easy to verify — for example, for a Borel set $A \subset \mathbb{R}^n$, the $\alpha_{x,y}$-measurability of $\Gamma_{x,y} \ni \gamma \mapsto \ell(\gamma \cap A)$ follows as in the classical coarea formula, see e.g. [AFP, Lemma 2.99]. The $\alpha_{x,y}$-measurability of $\Gamma_{x,y} \ni \gamma \mapsto \#(\gamma \cap A)$ can be deduced from the same lemma. Perhaps the most non-trivial condition to check is that of the (ordinary) Semmes family of curves. For example, if A consists of points that are closer to x than to y, we can define $A_j := A \cap [B(x, 2^{-j} d(x, y)) \setminus B(x, 2^{-j-1} d(x, y))]$,

$j = 0, 1, 2, \ldots$. Then we can use the classical coarea formula to calculate

$$\begin{aligned}\int_{B_{n-1}} \ell(\gamma_z \cap A)\, d\mathcal{H}^{n-1}(z) &= \sum_{j=0}^{\infty} \int_{B_{n-1}} \ell(\gamma_z \cap A_j)\, d\mathcal{H}^{n-1}(z) \\ &\le \sum_{j=0}^{\infty} 2^{(j+2)(n-1)} \mathcal{L}^n(A_j) \\ &\le 4^{n-1} \int_{A \cap B(x, d(x,y))} R_{x,y}\, d\mathcal{L}^n.\end{aligned}$$

For a general A, the result follows similarly.

6.1 Fred Gehring's bow-tie example

As a first example of a non-Euclidean space supporting the geometric Semmes family of curves, we consider the following.

Example 6.2 (Fred Gehring's bow-tie)**.** Let $X = X_+ \cup X_-$, where

$$\begin{aligned} X_+ &:= \{z \in \mathbb{R}^n : 0 \le z_n \le 1, |z_j| \le z_n, j = 1, \ldots, n-1\}, \\ X_- &:= \{z \in \mathbb{R}^n : -1 \le z_n \le 0, |z_j| \le |z_n|, j = 1, \ldots, n-1\}. \end{aligned}$$

We endow X with the metric inherited from $\mathbb{R}^n$ and a weighted Lebesgue measure with the weight $\omega(z) = |z|^\alpha$. It is known that the measure is doubling for $\alpha > -n$, and that X supports a $(1,1)$-Poincaré inequality precisely when $\alpha \le -n+1$ [BjBj, Example 3.5, Example 5.7], so the latter condition is necessary for the existence of a Semmes family of curves. In the following we will show that it is also sufficient, while the existence of a *geometric* Semmes family of curves requires that α is precisely $-n+1$.

Let O denote the origin. Fix two points $a, b \in X$. If a and b are both in X_+ or both in X_-, we can choose the Semmes family of curves in the same way as in $\mathbb{R}^n$. If not all the curves are in X, we can remove those curves that leave X, and rescale the measure $\alpha_{a,b}$ so that the total measure of all remaining curves in the family is one.

Now assume that $a_n < 0 < b_n$. Here we simply take a Semmes family of curves between a and O, and another one between O and b. Precisely speaking, first we let B_{n-1} be the $n-1$-dimensional unit disk, and we use the notation $z = (\widetilde{z}, z_n)$ for a point $z \in X$. Then we define for any $t \in \frac{1}{2}B_{n-1}$,

$$A_t := (|a_n| t, a_n/2) \in X \quad \text{and} \quad B_t := (|b_n| t, b_n/2) \in X.$$

Let γ^t be a curve consisting of four line segments: $[a, A_t]$, $[A_t, O]$, $[O, B_t]$ and $[B_t, b]$. We set $\Gamma_{a,b} := \{\gamma^t : t \in \frac{1}{2}B_{n-1}\}$. For $\Gamma \subset \Gamma_{a,b}$, we set

$$\alpha_{a,b}(\Gamma) := \mathcal{H}^{n-1}\left(\left\{t \in \frac{1}{2}B_{n-1} : \gamma^t \in \Gamma\right\}\right) / \mathcal{H}^{n-1}\left(\frac{1}{2}B_{n-1}\right).$$

We pick $\mathcal{A}_\delta$ so that the "annuli" A_j become geometrically thinner not only near a and b, but also near O, ensuring that condition (2) in the definition of the geometric Semmes family of curves is satisfied. Note that by this type of choice, the origin O does not belong to any of the sets A_j. However, since the Semmes family consists of two Semmes families that both have the origin as an end point, we can use Lemma 4.5 to deduce that the weak continuity condition (4) of the geometric Semmes family of curves is satisfied.

Condition (3) is again easy to check, while the condition for the ordinary Semmes family of curves is more interesting. We note that the family of curves becomes highly concentrated near O, but the measure $R_{a,b}\,d\mathcal{L}^n$ does not. However, for $\alpha \le -n+1$ the weight $\omega = |x|^\alpha$ blows up near O at least as rapidly as $R_{a,b}$ blows up near the points a and b, eliminating this problem. More precisely, for any $x, y \in X$, let $B_{xy} := B(x, 2\sqrt{n}d(x,y))$ so that it encompasses the whole family $\Gamma_{x,y}$. Keeping in mind that $\Gamma_{a,b}$ is built from two geometric Semmes families of curves in the Euclidean space, we can calculate for any Borel function ρ on X (here we use the notation $\lesssim$ to signify that the left-hand side is at most a constant $C = C(n,\alpha)$ times the right-hand side)

$$\int_{\Gamma_{a,b}} \int_\gamma \rho\, ds\, d\alpha_{a,b}(\gamma) \lesssim \int_{B(a,3|a|/4)} \rho(z)|a-z|^{-n+1}\, d\mathcal{L}^n(z) + \int_{B_{aO}} \rho(z)|z|^{-n+1}\, d\mathcal{L}^n(z)$$
$$+ \int_{B_{Ob}} \rho(z)|z|^{-n+1}\, d\mathcal{L}^n(z) + \int_{B(b,3|b|/4|)} \rho(z)|b-z|^{-n+1}\, d\mathcal{L}^n(z) =: \Diamond.$$

Here we used the fact that based on the behavior of the function $|z|^{-n+1}$, on the right-hand side it is enough to integrate over the smaller balls centered at the end points a and b. Now we can further estimate the above by

$$\begin{aligned}
\Diamond &\le \int_{B(a,3|a|/4)} \rho(z) R^1_{a,O}(z) \left(\frac{|a|}{4}\right)^\alpha d\mathcal{L}^n(z) + \int_{B_{aO}} \rho(z)|z|^\alpha\, d\mathcal{L}^n(z) \\
&\quad + \int_{B_{Ob}} \rho(z)|z|^\alpha\, d\mathcal{L}^n(z) + \int_{B(b,3|b|/4|)} \rho(z) R^2_{O,b}(z) \left(\frac{|b|}{4}\right)^\alpha d\mathcal{L}^n(z) \\
&\lesssim \int_{B_{aO}} \rho(z) R^1_{a,O}(z)|z|^\alpha\, d\mathcal{L}^n(z) + \int_{B_{Ob}} \rho(z) R^2_{O,b}(z)|z|^\alpha\, d\mathcal{L}^n(z) \\
&\le \int_{B_{ab}} \rho(z) R_{a,b}(z)\, |z|^\alpha d\mathcal{L}^n(z).
\end{aligned}$$

Condition (1) for the geometric Semmes family of curves follows as in the Euclidean case, as long as $\alpha \ge -n+1$.

Remark 6.3. Note that we can also take X_+ and X_- to be of two different dimensions n_1 and n_2, if we require that $\alpha_i + n_i = 1$ for $i = 1, 2$. The space $X = X_+ \cup X_-$ will still be doubling and support a $(1,1)$-Poincaré inequality, see [BjBj, Example A.24], and we can show that X supports a geometric Semmes family of curves as above.

6.2 Heisenberg group

Finally, we present a construction of the geometric Semmes family of curves in the Heisenberg group. Unlike the earlier examples, in this case we had to use some numerical calculations in the verification of the conditions. Let us model the Heisenberg group as $\mathbb{H} = (\mathbb{R}^3, *)$, where the multiplication of $p, q \in \mathbb{H}$ is given by

$$(p_1, p_2, p_3) * (q_1, q_2, q_3) = \left(p_1 + q_1, p_2 + q_2, p_3 + q_3 + \tfrac{1}{2}(p_1 q_2 - p_2 q_1)\right).$$

The group law respects the dilations $\delta_r(p_1, p_2, p_3) := (rp_1, rp_2, r^2 p_3)$, $r > 0$. We have the following left-invariant vector fields in $\mathbb{H}$:

$$\begin{aligned} X_p &= \partial_{p_1} - \tfrac{p_2}{2}\partial_{p_3}, \\ Y_p &= \partial_{p_2} + \tfrac{p_1}{2}\partial_{p_3}, \\ Z_p &= \partial_{p_3}, \end{aligned}$$

which satisfy $[X, Y] = Z$. At each point $p \in \mathbb{H}$, the horizontal space $H_p\mathbb{H} \subset T_p\mathbb{H}$ is spanned by X_p and Y_p. For an interval $I \subset \mathbb{R}$, a piecewise C^1 curve $p : I \to \mathbb{H}$ is horizontal if $\dot{p}(t) \in H_{p(t)}\mathbb{H}$ whenever it exists.

We endow $\mathbb{H}$ with a sub-Riemannian metric, defined on the horizontal spaces, so that X and Y are everywhere orthonormal. Let $\|\cdot\|$ be the norm induced by this inner product. Let $\gamma : I \to \mathbb{H}$ be a horizontal curve. We define the length of γ by

$$\ell(\gamma) := \int_I \|\dot{\gamma}(t)\|\, dt.$$

The Carnot-Carathéodory distance between two points $p, q \in \mathbb{H}$ is

$$d_c(p, q) := \min\{\ell(\gamma) : \gamma \text{ is horizontal and connects } p \text{ and } q\}$$

The Carnot-Carathéodory metric is bi-Lipschitz equivalent to the metric $d(p, q) := \|p^{-1} * q\|_{\mathbb{H}}$, where $\|p\|_{\mathbb{H}} := (p_1^4 + p_2^4 + p_3^2)^{1/4}$.

Next, we construct a geometric Semmes family of curves in the Heisenberg group.

Remark 6.4. As we have seen previously, in $\mathbb{R}^n$ the natural way to construct a (geometric) Semmes family of curves is to choose some hypersurface S between the points x and y, and then to connect the points of S by geodesics to x and y. However, in the Heisenberg group, this construction does not work. This can be seen by considering geodesics starting from the origin, presented in (6.5) (parametrized by arc-length). The z_3-component is $z_3(t) = \frac{a}{12}t^3 + O(t^5)$. The condition (3.4) can hold only if the curves cover approximately the same portion of each sphere around x (in this case the origin), but this would require that $z_3(t)$ decrease at the rate t^2 as $t \to 0$.

We will first construct a family of curves Γ_0 starting from the origin. We will then use this curve family to construct the geometric Semmes family. Let $S := \{s \in \mathbb{H} : d_c(s, \delta_2(s)) = 1\}$. For $s \in S$, let $\widetilde{\gamma}_s$ be a geodesic joining the points s and $\delta_2(s)$, and let

$$\gamma_s := \bigcup_{k \in \mathbb{Z}} \delta_{2^k}(\widetilde{\gamma}_s).$$

The family Γ_0 will consist of the curves $\gamma_s, s \in S$, that are not "too vertical" — we will specify this later. This construction forces the curves to stay far enough from each other as they get close to the origin, as described in Remark 6.4. The distance between two curves at the end of one geodesic subcurve $\delta_{2^k}(\widetilde{\gamma}_s)$ is exactly half of the distance at the other end, and therefore the distance is well under control at the end points of each geodesic subcurve. Next we will consider the behavior of the distance between the end points.

Geodesics $\gamma(t) = (z_1(t), z_2(t), z_3(t))$ that start from the origin at unit speed can be expressed as

$$
\begin{aligned}
z_1(t) &= \frac{\cos\theta \sin(at) - \sin\theta(1-\cos(at))}{a}, \\
z_2(t) &= \frac{\sin\theta \sin(at) + \cos\theta(1-\cos(at))}{a}, \\
z_3(t) &= \frac{at - \sin(at)}{2a^2},
\end{aligned}
\tag{6.5}
$$

where $|a| \leq 2\pi$, $0 \leq \theta < 2\pi$ and $t \in [0,1]$, see for example [BelRis]. The initial velocity of a curve is $X\cos\theta + Y\sin\theta$.

We want to construct the geodesic $\widetilde{\gamma}_s$, $s \in S$, with length 1 and connecting the points $s = (s_1, s_2, s_3)$ and $\delta_2(s) = (2s_1, 2s_2, 4s_3)$. If we left-translate the geodesic $\widetilde{\gamma}_s$ so that it starts from the origin, then the endpoint is $(s_1, s_2, s_3)^{-1} * (2s_1, 2s_2, 4s_3) = (s_1, s_2, 3s_3)$. Thus we have $s_1 = z_1(1), s_2 = z_2(1)$ and $s_3 = \frac{1}{3}z_3(1)$, and the geodesic is

$$
\begin{aligned}
\widetilde{\gamma}_s(t) &= \big(z_1(1), z_2(1), \tfrac{1}{3}z_3(1)\big) * (z_1(t), z_2(t), z_3(t)) \\
&= \Bigg(\frac{\cos\theta(\sin a + \sin(at)) - \sin\theta(2 - \cos a - \cos(at))}{a}, \\
&\qquad \frac{\sin\theta(\sin a + \sin(at)) + \cos\theta(2 - \cos a - \cos(at))}{a}, \\
&\qquad \frac{a + 3at + 2\sin a - 6\sin(at) - 3\sin(a - at)}{6a^2}\Bigg), \quad t \in (0,1).
\end{aligned}
\tag{6.6}
$$

We choose Γ_0 to consist of all geodesics γ_s such that $|a| \leq 1$.

Now we are ready to construct the geometric Semmes family of curves. We associate to each point $x \in \mathbb{H}$ the left-translated curve family $\Gamma_x = \{x * \gamma : \gamma \in \Gamma_0\}$. For $x, y \in \mathbb{H}$, let $B_{xy} := B(x, 100d_c(x,y))$ and let $M := \{m \in B_{xy} : d_c(m,x) = d_c(m,y)\}$.

If a point $m \in M$ can be connected to x by a subcurve of a curve in Γ_x and to y by a subcurve of a curve in Γ_y, then we will call the union of these two curves γ^m. The Semmes family of curves $\Gamma_{x,y}$ connecting x and y consists of all such curves. We endow the curve family with the measure

$$
\alpha_{xy}(\Gamma) = \frac{\mathcal{H}^3(\Gamma \cap M)}{\mathcal{H}^3(\Gamma_{x,y} \cap M)},
$$

where $\mathcal{H}^3$ denotes the 3-dimensional Hausdorff measure.

If $x^{-1} * y$ is close to horizontal, it is easy to see that there are plenty of curves in $\Gamma_{x,y}$. The "worst" situation occurs when $x^{-1} * y = (0,0,t)$ for some $t \in \mathbb{R}$, as the shortest possible curves connecting x and y are not included in the curve family $\Gamma_{x,y}$, since we require that $|a| \leq 1$. Therefore we had to choose the reference set B_{xy} to be large enough so that the curve family is not empty even in this case. The main difficulty in verifying that this family of curves satisfies the conditions of the geometric Semmes family of curves is in confirming that condition (2) is satisfied. More precisely, we need to show that the distance between two geodesic subcurves $\widetilde{\gamma}_s$ remains comparable to the distance of the end points. This was done by numerical computations with the help of Mathematica when $|a| \leq 1$, using (6.6).

Remark 6.7. Instead of the set S, we could use for example the unit sphere to construct the curves. One should get an equally well behaving family of curves this way, but our choice of S makes it easier to find an explicit representation for the curves and thus to check that the distance between two curves behaves well.

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