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# NONLINEAR PARABOLIC EQUATIONS WITH MORREY DATA

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ABSTRACT. We make a short survey of how the heuristic principle

*"the less the measure concentrates, the better the gradient is"*

about measure data problems can be implemented for elliptic and parabolic equations of $p$-Laplacian type, both in terms of integrability and differentiability properties. Moreover we prove improved fractional differentiability for the gradient to solution to parabolic equations with linear growth, in the case of Morrey measure data.

## 1. INTRODUCTION

The purpose of this note is the study of the regularity of solutions to equations having measures as data. In particular we will show how recent results of differentiability and integrability type for the gradient of *very weak solutions* can be improved when considering measures which *do not concentrate too much*. We shall consider the following null boundary data Cauchy-Dirichlet problem

$$\begin{cases} u_t - \operatorname{div} a(x,t,Du) = \mu & \text{in } \Omega_T, \\ u = 0 & \text{on } \partial_{\mathcal{P}}\Omega_T, \end{cases} \tag{1.1}$$

where the right-hand side is *a measure*, in the sense that it *does not belong* to the dual of the energy space naturally associated with the parabolic operator on the left-hand side, and furthermore it satisfies certain density properties of Morrey type. We are going to consider equation (1.1) in the cylindrical domain $\Omega_T := \Omega \times (-T, 0)$, being $\Omega \subset \mathbb{R}^n$, $n \geq 2$, a bounded connected open set, $T > 0$; we recall that its parabolic boundary is defined by $\partial_{\mathcal{P}}\Omega_T := \Omega \times \{-T\} \cup \partial\Omega \times (-T, 0)$. We shall moreover impose a certain degree of regularity on the vector field $a$, see (3.1).

We shall henceforth deal with right-hand sides being signed Borel measures with finite total mass $|\mu|(\Omega_T) < \infty$ and also we shall briefly discuss the case when $\mu$ is a Lebesgue functions in $L^\gamma(\Omega_T)$, where $\gamma$ will be such that parabolic Sobolev's embedding cannot ensure that $u \in L^2(-T, 0; H^{-1}(\Omega))$; here we are considering vector field with linear growth. Moreover we want in particular to study the cases where the singular part of the measure has "small dimension" (respectively, where the integral of the datum does not concentrate too much around certain points). A way to encode this condition is to consider

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measures such that the condition

$$|\mu|(Q_R) \le c\, R^{N-\vartheta} \qquad \text{for all } Q_R \subset \Omega_T \tag{1.2}$$

holds for some constant $c \ge 0$ and for some exponent $2 \le \vartheta \le N$, where we denote by $N := n+2$ the parabolic dimension. The space of all such measures will be denoted, with a slight abuse of notation, $L^{1,\vartheta}(\Omega_T)$ and the smallest constant $c$ such that (1.2) holds will be $[\mu]_{L^{1,\vartheta}(\Omega_T)}$. Moreover we shall set

$$\|\mu\|_{L^{1,\vartheta}(\Omega_T)} := |\mu|(\Omega_T) + [\mu]_{L^{1,\vartheta}(\Omega_T)}.$$

Note that assuming (1.2) does not allow $\mu$ to concentrate on sets with parabolic Hausdorff dimension less than $N - \vartheta$. A fine analysis in Morrey spaces is often decisive to establish higher regularity, existence and uniqueness results, as it has been shown, for instance, via by-now classic applications to the theory of Navier-Stokes and other evolution equations (see, for instance, the papers by Kato [23] and Taylor [34] and the references therein). More in general, an investigation in the Morrey framework permits to obtain deeper informations that would be not detected in the Lebesgue scale; it typically arises indeed in contexts when various informations are minimal in terms of Lebesgue spaces, but better density informations are available, like, e. g., in the case of systems involving geometrically constrained problems and construction of related Coulomb gauges, in fluid-dynamics, in the analysis of Schrödinger operators and in convergence issues related to Euler's equations. In particular as regarding the regularity, existence, uniqueness and regularity of functions under Morrey density conditions, very early results are due to Campanato [11, 12, 13] and Caffarelli [10], up to Lieberman's recent ones [22, 21]. In this respect, the interest of the community in such norms is not decreasing, as seen, for instance, in the recent improved Sobolev embeddings by Palatucci and Pisante [32], in turn implying, among other things, the celebrated profile decomposition in the fractional Sobolev spaces.

## 2. Measure data problems

Note that already the notion of solution for problem (1.1) in the general case requires some care. Indeed usual monotonicity methods do not apply, due to the fact that the right-hand side does not belong to the dual of the natural energy space. This gives origin to a concept of solution which *does not belong to the energy space*, which is hence usually called *very weak solution*; notice moreover that, due to this fact, uniqueness for these equations does not hold in general. In both elliptic and parabolic cases a way to overcome this difficulty and to find a *particular* distributional solution to a measure data problem is to consider solutions obtained by regularization methods as showed in [7, 8, 9]; this generates the notion of solution called SOLA (Solution Obtained as Limit of Approximations). Let us briefly outline the strategy, which is on the other hand very natural.

2.1. **Elliptic SOLAs.** In the elliptic case

$$\begin{cases} \operatorname{div} a(x, Du) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

$\Omega$ bounded domain of $\mathbb{R}^n$, $n \geq 2$, where $\mu$ is a measure and the vector field is Carathéodory regular and satisfies only natural monotonicity and growth conditions

$$\begin{cases} \langle a(x,\xi_1) - a(x,\xi_2), \xi_1 - \xi_2 \rangle \geq \nu(s^2 + |\xi_1|^2 + |\xi_2|^2)^{\frac{p-2}{2}} |\xi_1 - \xi_2|^2, \\ |a(x,\xi)| \leq L\big(s^2 + |\xi|^2\big)^{\frac{p-1}{2}}, \end{cases} \tag{2.2}$$

and for all $x \in \Omega$, any $\xi, \xi_1, \xi_2 \in \mathbb{R}^n$, with $s \in [0,1]$ and $0 < \nu \leq 1 \leq L < \infty$, with $2 - 1/n < p \leq n$, one considers smooth, $L^\infty$ functions $f_k$ converging to $\mu$ in the weak-$*$ topology of measures, such that

$$\begin{aligned} \|f_k\|_{L^1(\Omega)} &\leq |\mu|(\Omega), & \|f_k\|_{L^1(B_R)} &\leq |\mu|(B_{R+1/k}), \\ \|f_k\|_{L^{1,\vartheta}(\Omega)} &\leq \|\mu\|_{L^{1,\vartheta}(\Omega)}, & \|f_k\|_{L^{1,\vartheta}(B_R)} &\leq \|\mu\|_{L^{1,\vartheta}(B_{R+1/k})}, \end{aligned}$$

see [31], and the regularized problems

$$\begin{cases} \operatorname{div} a(x, Du_k) = f_k & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Solutions of such problems are found by classic monotonicity methods, while in [7, 9] it is shown a suitable strong convergence $u_k \to u$ in $W^{1,p-1}(\Omega)$; such limit function $u \in W_0^{1,p-1}(\Omega)$ is called a SOLA, since it solves the distributional formulation of (1.1). This scheme involves a priori estimates and therefore implicitly carries on regularity results. Summarizing, we have the following, that contains *the lower order Calderón-Zygmund theory for measure data problems*.

**Theorem 2.1** ([8, 5, 14, 6, 35, 19])**.** *There exist a SOLA $u \in W_0^{1,p-1}(\Omega)$ to (2.1). Moreover*

$$u \in W_0^{1,q}(\Omega) \quad \text{for every } q < \frac{n(p-1)}{n-1} \quad \text{and} \quad |Du|^{p-1} \in \mathcal{M}^{\frac{n}{n-1}}(\Omega). \tag{2.3}$$

*Finally, there exists a unique SOLA when $\mu \in L^1(\Omega)$, $p = 2$ or $p = n$.*

We recall the reader that the Marcinkiewicz space $\mathcal{M}^t(\Omega)$, for $t \geq 1$, is defined by requiring the following decay rate for the measure of the measure of the super-level sets: $f \in \mathcal{M}^t(\Omega)$ if

$$\sup_{\lambda > 0} \lambda^t \big|\{x \in \Omega : |f(x)| > \lambda\}\big| < \infty.$$

Uniqueness in Theorem 2.1 is in the sense that by considering a different approximating sequence $\{\bar{f}_k\}$ converging to $\mu$ in $L^1(\Omega)$, we still get the same limiting solution $u$. On the other hand, one of the very few cases uniqueness of SOLA for measures is given when $\mu$ concentrates at one point; in this case we have a Dirac measure. Indeed the only SOLA to

$$\begin{cases} -\operatorname{div}(|Du|^{p-2}Du) = \delta_0 & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases}$$

is given by the following nonlinear fundamental solution, or nonlinear Green's function:

$$G_p(x) \equiv G_p(|x|) \approx \begin{cases} |x|^{\frac{p-n}{p-1}} - 1, & 1 < p < n, \\ \log|x|, & p = n. \end{cases} \tag{2.4}$$

Here the symbol $\approx$ means that this is the definition of $G_p$ modulo a non-essential renormalization constant. In turn, such uniqueness result allows to test the optimality of the regularity result, such as for instance Theorem 2.1, which is in fact optimal.

We now want to recall a few recent results on one hand aimed at obtaining what can be called *the maximal Calderón-Zygmund theory for measure data problems*, and on the other one at outlining a few results aim at going beyond Theorem 2.1 when certain more special measures are considered. We shall consider the general case of $p$-Laplacian type operators, however restricting mainly our attention to the case

$$p \geq 2$$

for ease of exposition; we stress however that a completely parallel theory has been developed also in the subquadratic case $2 - 1/n < p \leq 2$ and we refer to the mentioned papers for the appropriate statements.

The starting idea is very basic: since equations as in (2.1) formally involve second order operators, then it is natural to expect for the gradient of solutions a degree of regularity that goes beyond the integrability one considered in Theorem 2.1; more precisely, we want to consider differentiability rather that integrability properties of the gradient. For this we need to consider assumptions which are stronger than those considered in Theorem 2.1, but that are nevertheless natural towards the forthcoming result: we shall consider differentiable and Carathéodory vector fields $a$ such that $\partial_\xi a$ is a Carathéodory function and such that

$$\begin{cases} \langle \partial_\xi a(x,\xi)\lambda, \lambda\rangle \geq \nu\big(s^2 + |\xi|^2\big)^{\frac{p-2}{2}}|\lambda|^2, \\ |a(x,\xi)| + |\partial_\xi a(x,\xi)|\big(s^2+|\xi|^2\big)^{\frac{1}{2}} \leq L\big(s^2+|\xi|^2\big)^{\frac{p-1}{2}}, \\ |a(x_1,\xi) - a(x_2,\xi)| \leq L\,\omega(|x_1-x_2|)\big(s^2+|\xi|^2\big)^{\frac{p-1}{2}}, \end{cases} \tag{2.5}$$

for almost every $x, x_1, x_2 \in \Omega$ and all $\xi, \xi_1, \xi_2, \lambda \in \mathbb{R}^n$. Again $0 < \nu \leq 1 \leq L$, $s \in [0,1]$. Here the (Lipschitz) regularity with respect to $x$ is encoded by

$$\omega(R) \leq R. \tag{2.6}$$

Note that the previous differentiability of the vector field with respect to the gradient variable can be weakened in Lipschitz continuity as in the forthcoming (3.1), but we prefer to treat these assumptions for compactness of exposition. It turns out that the right scale where to look for differentiability results for solutions to measure data problems is the one of *fractional Sobolev spaces*: the space $W^{\alpha,q}(\Omega)$, $\alpha \in (0,1)$, $1 \leq q < \infty$ is defined as the space of integrable functions $g$ such that

$$\int_\Omega \int_\Omega \frac{|g(x)-g(y)|^q}{|x-y|^{n+\alpha q}}\,dx\,dy < \infty;$$

this is to prescribe the quantity $\frac{|g(x)-g(y)|^q}{|x-y|^{\alpha q}}$ to compensate the blow-up of the singular kernel $|x-y|^{-n}$ on the diagonal, in an integral sense.

The first results in the direction of fractional differentiability are due to Mingione, and can be summarized in the following:

**Theorem 2.2** ([28, 30])**.** *Let $u$ be as in previous Theorem 2.1, except from the fact that the vector field now satisfies (2.5)-(2.6) and $p \geq 2$, and let $\mu$ be a signed Borel measure with finite total mass. Then*

$$Du \in W^{\frac{1-\varepsilon}{p-1},p-1}_{\mathrm{loc}}(\Omega) \qquad \text{for } \varepsilon \in (0,1).$$

*In particular*

$$Du \in W^{1-\varepsilon,1}_{\mathrm{loc}}(\Omega) \qquad \text{for } \varepsilon \in (0,1), \qquad \text{when} \quad p = 2. \tag{2.7}$$

The previous result is optimal in the sense that we cannot allow for $\varepsilon = 0$, as easily shown by considering the fundamental solution displayed in (2.4) together with the following well-known fractional versions of Sobolev embedding, see [17, Theorem 6.7]:

$$W^{\alpha,\gamma}(\Omega) \hookrightarrow L^{\frac{n\gamma}{n-\alpha\gamma}}(\Omega), \qquad \text{for} \qquad 1 \leq \gamma < \infty, \quad \alpha\gamma < n.$$

The previous theorem has been generalized by Di Castro and Palatucci, who relaxed the regularity with respect to the spatial variable considered a merely Hölder continuous dependence (moreover coupled with an integral differentiability assumption in the flavor of DeVore and Sharpley [15], which we shall not report here):

**Theorem 2.3** ([16])**.** *Let $u$ be a SOLA to (2.1), where the vector field satisfies (2.5) for $p \geq 2$ and with $\omega(R) \leq R^\alpha$ for some $\alpha \in (0,1]$ and let $\mu$ be a signed Borel measure with finite total mass. Then*

$$Du \in W^{\frac{2}{p}\cdot\min\{\alpha,\frac{p}{2(p-1)}\}-\varepsilon,p-1}_{\mathrm{loc}}(\Omega)$$

*for $\varepsilon > 0$ small enough.*

Notice that for $\alpha = 1$ the previous results gives back the result of Theorem 2.2.

In the case when $\mu$ is a function in $L^\gamma(\Omega)$, with

$$1 < \gamma < (p^*)' = \frac{np}{np-(n-p)}, \qquad p < n, \tag{2.8}$$

similar results hold. We restrict to this range since if $\gamma \geq (p^*)'$ then $\mu$ belongs to the dual of $W^{1,p}_0$ by Sobolev's embedding. The same approximation approach described in the lines above apply also in this case, and moreover leads to a unique solution. Therefore without loss of generality here we shall consider the approximating sequence given by the truncations of $\mu$:

$$f_k := \min\big\{k, \max\{\mu, -k\}\big\} \qquad\qquad k \in \mathbb{N}.$$

Moreover we shall focus from now on on the case

$$2 \leq p < n;$$

note that forcing $p = n$ would give in the limit as $p \nearrow n$, $\gamma \searrow 1$, a case we are not going to consider since it falls in the realm of the previously considered measure data problems.

We begin with the analog of Theorem 2.1; here again for existence and integrability results the vector field has only to satisfy assumptions (2.2). We are therefore allowing just for measurable coefficients. In this case we have the following classic result:

**Theorem 2.4** ([7])**.** *Let $u \in W_0^{1,p-1}(\Omega)$ be the SOLA to equation (2.1), where $\mu$ belongs to $L^\gamma(\Omega)$, with $\gamma$ as in (2.8). Then*

$$|Du|^{p-1} \in L^{\frac{n\gamma}{n-\gamma}}(\Omega).$$

Again, also in this case Mingione provided a maximal analogue of this theorem in terms of fractional Sobolev spaces:

**Theorem 2.5** ([29])**.** *Let $u$ be as in Theorem 2.2 and let $\mu \in L^\gamma(\Omega)$, with $\gamma$ as in (2.8). Then*

$$Du \in W^{\frac{1-\varepsilon}{p-1},\gamma(p-1)}_{\mathrm{loc}}(\Omega) \qquad \textit{for } \varepsilon \in (0,1);$$

*in particular*

$$Du \in W^{1-\varepsilon,\gamma}_{\mathrm{loc}}(\Omega) \qquad \textit{for } \varepsilon \in (0,1), \qquad \textit{when} \quad p = 2.$$

2.2. **Morrey data.** As we already pointed out, the sharpness of some of the previous results can be tested with the fundamental solution (2.4); this, in some sense, is the "worst possible case" since in this case the Dirac measure is concentrated at one point. The question is now: What happens if we consider measures that do not concentrate on sets with small Hausdorff dimension? It turns out that Theorem 2.1 can be upgraded in a different, in some sense orthogonal direction. Measure data problems obey the heuristic principle

(2.9) *"the less the measure concentrates, the better the gradient is".*

A natural way to quantify this can be, for a given signed finite Borel measures $\mu$, to consider the Morrey type density conditions

$$R^{\vartheta-n}|\mu|(B_R(x)) < \infty, \qquad 0 \le \vartheta \le n, \tag{2.10}$$

the inequality being valid for all the balls $B_R(x) \subset \Omega$, which is the analogue of our parabolic condition (1.2). We shall divide the range $0 \le \vartheta \le n$ into two separate subranges: a classic Harmonic Analysis result [1] indeed asserts that if $0 \le \vartheta < p$, then $L^{1,\vartheta}(\Omega) \subset W^{-1,p}(\Omega)$. Note also that this obviously occurs when $p \le n$. We shall ignore this case, and focus on the range $p \le \vartheta \le n$. The principle in (2.9) finds now the following "quantified form", in the setting of integrability properties, due to Mingione:

**Theorem 2.6** ([28, 30])**.** *Let $u \in W^{1,p-1}(\Omega)$ be a SOLA of problem (2.1), as in Theorem 2.1, where the measure $\mu$ moreover satisfies the density condition (2.10) for $p \le \vartheta \le n$. Then*

$$|Du|^{p-1} \in \mathcal{M}^{\frac{\vartheta}{\vartheta-1}}_{\mathrm{loc}}(\Omega).$$

*Moreover, it holds that*

$$R^{\vartheta-n}\big\||Du| + s\big\|^{\frac{\vartheta}{\vartheta-1}}_{\mathcal{M}^{\frac{\vartheta}{\vartheta-1}}_{\mathrm{loc}}(B_R(x))} < \infty \tag{2.11}$$

*for all $B_R \subset \Omega$.*

Note that the inequality in (2.11) means that in the information on the density of the measure is inherited by the Marcinkiewicz norm of the gradient of $u$; that is to say that

$$|Du|^{p-1} \in \mathcal{M}^{\frac{\vartheta}{\vartheta-1},\vartheta}_{\rm loc}(\Omega),$$

following our convention on Morrey spaces, see Section 3.2. Note also that the previous result reduces to the one in Theorem 2.1 in the case of general measures $\vartheta = n$, and claims a better integrability of the gradient when $\vartheta < n$, i.e. when the measure diffuses. The principle (2.9) naturally applies also at level of gradient differentiability:

**Theorem 2.7** ([28, 30])**.** *Let $u$ be a SOLA to (2.1), where the vector field satisfies (2.5)-(2.6) and $\mu$ satisfies the density condition (2.10) for $p \le \vartheta \le n$. Then*

$$Du \in W^{\frac{1-\varepsilon}{p-1},p-1;\vartheta}_{\rm loc}(\Omega) \qquad \textit{for } \varepsilon \in (0,1).$$

Notice a difference: here one don't see the improvement of differentiability. On the other hand, to simplify the exposition, we preferred to reduce ourselves to the integrability exponents $p-1$, but similar estimates are also available for all exponents $q$ as in (2.3); see moreover the parabolic result in Theorem 3.1 for $\vartheta = N$. In this scale of integrability one can indeed see that fractional estimates are available for all exponent smaller than the natural one $\vartheta(p-1)/(\vartheta-1)$. For example we have

$$(2.10) \text{ holds} \qquad \Longrightarrow \qquad Du \in W^{\frac{\delta-\varepsilon}{q},q;\vartheta}_{\rm loc}(\Omega) \qquad \text{for } \varepsilon \in (0,\delta),$$

where

$$p-1 \le q < \frac{\vartheta(p-1)}{\vartheta-1} \qquad \text{and} \qquad \delta \equiv \delta(q,\vartheta) := \vartheta - \frac{q(\vartheta-1)}{p-1}.$$

Note that the range of exponent in the last display is in clear accordance with the result of Theorem 2.6, once recalling the embedding properties for fractional Sobolev spaces. Clearly one can deduce the full result in the general case by simply imposing $\vartheta = n$.

Perfect analogues of the previous results hold in Lebesgue setting, when an information such

$$R^{\vartheta-n}\int_{B_R(x)} |\mu|^{\gamma} < \infty, \qquad p < \vartheta \le n$$

is available, i.e. $\mu \in L^{\gamma,\vartheta}(\Omega_T)$. We resume them in the following

**Theorem 2.8** ([29])**.** *Let $u \in W^{1,p-1}_0(\Omega)$ be the SOLA of problem (2.1), where the vector field satisfies the monotonicity and growth conditions (2.2), and $\mu \in L^{\gamma,\vartheta}(\Omega)$, for $p < \vartheta \le n$ and*

$$1 < \gamma \le \frac{\vartheta p}{\vartheta p - (\vartheta - p)}; \tag{2.12}$$

*then*

$$|Du|^{p-1} \in L^{\frac{\vartheta\gamma}{\vartheta-\gamma},\vartheta}_{\rm loc}(\Omega).$$

*If moreover the vector field satisfies (2.5)-(2.6), then*

$$Du \in W^{\frac{1-\varepsilon}{p-1},\gamma(p-1);\vartheta}_{\rm loc}(\Omega) \qquad \textit{for } \varepsilon \in (0,1).$$

It is interesting here to compare the critical exponent in (2.12) with the duality one in (2.8), and also to examine what happens when one of the parameters $\gamma, \vartheta$ approaches the borderline cases, i.e. $\vartheta \searrow p$ which implies $\gamma \searrow 1$ and $\vartheta = n$.

2.3. **The parabolic world.** When trying to extend the results of the previous section to the parabolic analogue (1.1), one immediately focuses that, despite proving existence and some kind of integrability results is not a problem in the general case $p > 2 - 1/(N-1)$ (this lower bound is natural in parabolic measure data problems), the situation becomes much more difficult for other kind of results. For instance, when $p \neq 2$, all the fractional differentiability results of the previous section are currently *open questions* since it is not clear how to adapt the technique of the proof to *degenerate and singular* evolutionary equations. Indeed, since the $p$-Laplacian equation is not invariant under multiplication of solution by a (nontrivial) constant, estimates in general have a *non homogeneous character*. Hence, all the iteration and covering techniques which apply in the elliptic case and can be extended to the parabolic with $p = 2$ are ruled out. However, as we shall see, still some of the integrability results in the case of Morrey measure data can be proved. The natural extension of assumptions (2.2) to the parabolic case is

$$\begin{cases} \langle a(x,t,\xi_1) - a(x,t,\xi_2), \xi_1 - \xi_2 \rangle \geq \nu (s^2 + |\xi_1|^2 + |\xi_2|^2)^{\frac{p-2}{2}} |\xi_1 - \xi_2|^2, \\ |a(x,t,\xi)| \leq L\big(s^2 + |\xi|^2\big)^{\frac{p-1}{2}}, \end{cases} \tag{2.13}$$

for almost every $(x,t) \in \Omega_T$, $\xi, \xi_1, \xi_2 \in \mathbb{R}^n$, with $s, \nu, L$ as in (2.2). With these assumption the following existence theorem can be proved:

**Theorem 2.9** ([7, 9, 14])**.** *There exists a SOLA $u \in L^{p-1}(-T, 0; W_0^{1,p-1}(\Omega))$ to (1.1). Moreover*

$$Du \in L^q(\Omega_T) \quad \textit{for every } \; q < \frac{N(p-1)}{N-1}$$

*and such SOLA is unique, in the sense explained after Theorem 2.1, when $\mu \in L^1(\Omega_T)$ or $p = 2$.*

Clearly the concept of SOLA has to be slightly, but appropriately, modified, as explained in Section 3. Moreover in [9] anisotropic integrability properties for $Du$ are also proved, that is

$$\int_{-T}^{0} \left( \int_{\Omega} |Du(x,t)|^q \, dx \right)^{\frac{r}{q}} dt < \infty$$

for appropriate couples of exponents $(q, r)$, and some of the following results can be extended in this direction, but we shall not focus on this point here.

In [3] Habermann and the author extended Theorem 2.6 and (the first, integrability, part of) Theorem 2.8 to this parabolic setting, in the quadratic case:

**Theorem 2.10** ([3])**.** *Let $u$ be a SOLA to (1.1) where the monotonicity and growth assumptions (2.13), for $p = 2$, are in force. Then if $\mu \in L^{1,\vartheta}(\Omega_T)$ with $2 \leq \vartheta \leq N$, then*

$$Du \in \mathcal{M}_{\mathrm{loc}}^{\frac{\vartheta}{\vartheta-1},\vartheta}(\Omega_T).$$

*On the other hand, if $\mu \in L^{\gamma,\vartheta}(\Omega_T)$, with*

$$2 < \vartheta \le N \qquad \textit{and} \qquad 1 < \gamma \le \frac{2\vartheta}{\vartheta - 2},$$

*then*

$$Du \in L^{\frac{\vartheta\gamma}{\vartheta-\gamma},\vartheta}_{\mathrm{loc}}(\Omega_T).$$

Notice that, up to changing the elliptic dimension $n$ with the parabolic one $N = n+2$, these estimates look exactly as the corresponding stationary ones, mentioned few lines above. Also higher differentiability results as in Theorem 2.2 extend to the parabolic setting: for general measures, this has been proved in [2], for $p = 2$; the aim of the second part of this note is to extend the following result to the case when the measure $\mu$ satisfies (1.2). Also here we require more stringent assumptions on the vector field, when looking for higher order properties: we consider for simplicity a Carathéodory vector fields $a$ such that $\partial_\xi a$ is Carathéodory and such that

$$\begin{cases} \langle \partial_\xi a(x,t,\xi)\lambda, \lambda\rangle \ge \nu\big(s^2 + |\xi|^2\big)^{\frac{p-2}{2}}|\lambda|^2, \\ |a(x,t,\xi)| + |\partial_\xi a(x,t,\xi)|\big(s^2+|\xi|^2\big)^{\frac{1}{2}} \le L\big(s^2+|\xi|^2\big)^{\frac{p-1}{2}}, \\ |a(x_1,t,\xi) - a(x_2,t,\xi)| \le L\,\omega(|x_1-x_2|)\big(s^2+|\xi|^2\big)^{\frac{p-1}{2}}, \end{cases} \tag{2.14}$$

for almost every $x, x_1, x_2 \in \Omega$, $t \in (-T, 0)$ and all $\xi, \xi_1, \xi_2, \lambda \in \mathbb{R}^n$; $\nu, L, s$ as in (2.5). Assumptions (2.14) can be slightly weakened (see (3.1)); notice that we only assume regularity (dictated by the behavior of $\omega(\cdot)$) with respect to the spatial variable, accordingly with the results in [18]; hence, sole measurability and boundedness is assumed with respect to the time variable, i.e. for the map $t \to a(x,t,\xi)$.

**Theorem 2.11** ([2]). *Let $u$ be a SOLA to (1.1), where the vector field satisfies (2.14) for $p = 2$ and with $\omega(R) \le R$ (or (3.1)) with (2.6) and $\mu$ is a signed Borel measure with finite total mass. Then, setting*

$$\delta \equiv \delta(q) = N - q(N-1) \qquad \textit{for} \qquad 1 \le q < \frac{N}{N-1}, \tag{2.15}$$

*we have*

$$Du \in W^{\frac{\delta-\varepsilon}{q},\frac{\delta-\varepsilon}{2q};q}_{\mathrm{loc}}(\Omega_T) \qquad \textit{for } \varepsilon \in (0,\delta). \tag{2.16}$$

*In particular*

$$Du \in W^{1-\varepsilon,\frac{1-\varepsilon}{2};1}_{\mathrm{loc}}(\Omega_T) \qquad \textit{for } \varepsilon \in (0,1).$$

As the reader might see, also here there is a perfect analogy between the elliptic result (2.7) and the parabolic one; moreover, as expected by the structure of the equation, one gets that the differentiability in time is half the differentiability in space. Indeed the *parabolic fractional Sobolev spaces* are the appropriate spaces to deal with parabolic regularity: one naturally asks, referring to (2.16), that

$$\int_{-T}^{0}\int_{\Omega}\int_{\Omega}\frac{|Du(x,t)-Du(y,t)|^q}{|x-y|^{n+\delta-\varepsilon}}\,dx\,dy\,dt$$
$$+\int_{\Omega}\int_{-T}^{0}\int_{-T}^{0}\frac{|Du(x,t)-Du(x,s)|^q}{|t-s|^{1+\frac{\delta-\varepsilon}{2}}}\,ds\,dt\,dx<\infty.$$

Let us conclude this section with the unique result, when the datum is in a Morrey space, for degenerate parabolic equations; in this case we have the following

**Theorem 2.12** ([4]). *Let $u$ be a SOLA to (1.1) where the vector field satisfies (2.13), for $p \geq 2$, and $\mu \in L^{1,\vartheta}(\Omega_T)$ for $2 \leq \vartheta \leq N$. Then*

$$Du \in \mathcal{M}_{\mathrm{loc}}^{p-1+\frac{1}{\vartheta-1}}(\Omega_T).$$

We stress that in the general case $\vartheta = N$ the exponent appearing in the previous theorem perfectly fits the existence theory of Theorem 2.9 and (at least locally) sharpens this result. Finally we can get the following integrability result in the case the dependence on the coefficient is more regular, say (uniformly) continuous: in this case we can allow the Morrey parameter $\vartheta$ to take values less than two and in the particular case $\vartheta = 1$ we proved that

**Theorem 2.13** ([4]). *Let $u$ be a SOLA to (1.1) where the vector field satisfies (2.14) for $p \geq 2$ and with $\omega(R) \to 0$ as $R \to 0$ and $\mu \in L^{1,\vartheta}(\Omega_T)$, i.e. $R^{1-N}|\mu|(Q_R) < \infty$. Then*

$$|Du| \in L^q_{\mathrm{loc}}(\Omega_T) \qquad \textit{for all } q \geq 1.$$

2.4. **Continuities.** The last part of this introduction is dedicated to a brief review of continuity results following from densities condition; for simplicity we will drop any $x$- (or $(x,t)$-) dependence of the vector field, due to the different level of regularity we are going to consider. One of the prototypes of such results is the result of Lieberman [21], where the density information

$$|\mu|(B_R) \leq c\,R^{n-1+\epsilon}, \qquad \text{for some } \epsilon \in (0,1)$$

for balls $B_{2R} \subset \Omega$, implies the local Hölder continuity of the gradient, $u \in C^{1,\beta}_{\mathrm{loc}}(\Omega)$, for problems $-\mathrm{div}\, a(Du) = \mu$ in $\Omega$, where $a$ has smooth $p$-Laplacian structure, in the sense that it satisfies assumptions (2.5), obviously recast without $x$-dependence; $\beta$ depends on $n, p, \nu, L, \epsilon$. The zero-order analogue has been proved by Kilpeläinen and Zhong in [24] (actually just for positive measures):

$$\mu(B_R) \leq c\,R^{n-p+\tilde{\epsilon}(p-1)} \qquad \Longrightarrow \qquad u \in C^{0,\tilde{\epsilon}}_{\mathrm{loc}}(\Omega)$$

for $\tilde{\epsilon}$ small, smaller than a certain parameter depending on $n, p, \nu, L$ (here we could refer to assumptions (2.2)) and for balls as above. Notice that in both these cases the measure *belongs to the dual of the energy space*, since $\vartheta < p$. The parabolic version of the gradient result by Lieberman has been settled by Kuusi and Mingione in [26], for solutions to $u_t - \operatorname{div} a(Du) = \mu$, $a$ as above:

$$|\mu|(Q_R) \le c\, R^{n-1+\epsilon}, \text{ for some } \epsilon \in (0,1) \qquad \Longrightarrow \qquad Du \in C^{0,\beta}_{\mathrm{loc}}(\Omega_T, \mathbb{R}^n),$$

$\beta$ with the same dependencies as above, and they also caught the borderline case:

$$|\mu|(Q_R) \le c\, R^{n-1} h(R) \qquad \text{with} \qquad \int_0^1 h(\rho) \frac{d\rho}{\rho} < \infty,$$

that is $h$ Dini continuous, implies the continuity of the gradient: $Du \in C^0_{\mathrm{loc}}(\Omega_T, \mathbb{R}^n)$. It is interesting to compare this result with Theorem 2.13 and to examine the delicate interplay between regularity of the vector field, density properties of the measure and regularity of the solution.

## 3. Results, notations and relevant spaces

As we said, since we are interested in differentiability results, we are going to consider quite regular vector field in the sense that we take a vector field $a : \Omega_T \times \mathbb{R}^n \to \mathbb{R}^n$ Carathéodory regular and satisfying the following assumptions:

$$\begin{cases} \langle a(x,t,\xi_1) - a(x,t,\xi_2),\, \xi_1 - \xi_2 \rangle \ge \nu\, |\xi_1 - \xi_2|^2, \\ \big| a(x,t,\xi_1) - a(x,t,\xi_2) \big| \le L |\xi_1 - \xi_2|, \\ |a(x,t,0)| \le Ls, \\ \big| a(x_1,t,\xi) - a(x_2,t,\xi) \big| \le L\, |x_1 - x_2| (s + |\xi|), \end{cases} \tag{3.1}$$

for all $(x,t), (x_1,t), (x_2,t) \in \Omega_T$, all $\xi, \xi_1, \xi_2$ and with $s \ge 0$ and $0 < \nu \le 1 \le L$. Note that here we just impose differentiability of the vector field with respect to the $x$ variable, while just measurability is assumed on the map $t \to a(x,t,\xi)$ for $x, \xi$ fixed.

**Solvability.** By a solution to (1.1) we mean a function $u \in L^1(-T, 0; W^{1,1}_0(\Omega))$ solving $(1.1)_1$ in the distributional sense:

$$\int_{\Omega_T} \big( u\varphi_t - \langle a(x,t,Du), D\varphi \rangle \big)\, dz = \int_{\Omega_T} \varphi\, d\mu \qquad \text{for all } \varphi \in C^\infty_0(\Omega_T), \tag{3.2}$$

where in the case $\mu$ is a function we clearly mean $d\mu := \mu\, dz$. As already mentioned, the existence of such a solution is obtained in [7, 9] by an approximation argument, similar to that described in Paragraph 2.1. In particular one considers a sequence $f_k \in L^\infty(\Omega_T)$, $k \in \mathbb{N}$, such that $f_k \to \mu$ in the weak-$*$ topology of measures (or simply in $L^1(\Omega_T)$, when considering Lebesgue data) when $k \to \infty$, finds approximating solutions $u_k \in C^0([-T,0]; L^2(\Omega)) \cap L^2(-T,0; W^{1,2}_0(\Omega))$ to (1.1) when in place of $\mu$ one has $f_k$ and proves, as done in the mentioned papers, that

$$u_k \to u \quad \text{strongly in } L^1(-T,0; W^{1,1}_0(\Omega)) \quad \text{and} \quad Du_k \to Du \quad \text{a.e.};$$

therefore one is allowed to pass to the limit in (3.2). See also the beginning of Section 4 for a further discussion about the approximation of the measure $\mu$.

Note that the fact that $u$ vanishes on the lateral boundary is prescribed by denoting $u(\cdot,t) \in W_0^{1,1}(\Omega)$ for a.e. $t$, while the initial boundary value $u(x,-T)=0$ should be understood in the $L^1$ sense, which means that

$$\lim_{h\searrow 0}\frac{1}{h}\int_{-T}^{-T+h}\int_{\Omega}|u|\,dx\,dt = 0.$$

Note that also for the approximating problems, using as test function the solution itself can be problematic, since it enjoys a few regularity properties with respect to the time variable. However, this difficulty can be overcome in a standard way by using regularizing procedures as Steklov averaging. We refer for instance to [3] for precise definitions and detailed computations.

The result we want to prove in this paper wants to be the full parabolic analog of Theorem 2.7; indeed we shall show that

**Theorem 3.1.** *Let $u$ be a SOLA to (1.1), where the vector field satisfies (3.1) and $\mu \in L^{1,\vartheta}(\Omega_T)$, for $2\le\vartheta\le n$, in the sense of (1.2). Then*

$$Du \in W^{\delta-\varepsilon,\frac{\delta-\varepsilon}{2};q;\vartheta}_{\mathrm{loc}}(\Omega_T) \qquad \textit{for } \varepsilon\in(0,\delta), \qquad \textit{if} \quad p\ge 2,$$

*for*

$$1\le q<\frac{\vartheta}{\vartheta-1} \qquad \textit{and where} \qquad \delta\equiv\delta(q,\vartheta)=\frac{\vartheta-1}{q}-\vartheta \tag{3.3}$$

*and in particular*

$$Du\in W^{1-\varepsilon,\frac{1-\varepsilon}{2};1;\vartheta}_{\mathrm{loc}}(\Omega_T) \qquad \textit{for } \varepsilon\in(0,1).$$

It could be useful to compare the previous theorem with Theorem 2.11, observing that in the case $\vartheta = N$ they collapse in the same statement. Moreover we have the following local estimates

**Proposition 3.2** (Local estimates)**.** *With $u$, $\mu$, $q$ and $\delta$ as in the previous theorem, we have the local estimates*

$$R^{\vartheta-N}[Du]^q_{W^{\delta-\varepsilon,\frac{\delta-\varepsilon}{2};q}(Q_R(z_0))} \;\le\; c\left[\big\||Du|+s\big\|_{L^{1,\vartheta}(Q_{2R})}+\|f\|_{L^{1,\vartheta}(Q_{2R})}\right]^q \tag{3.4}$$

*for $\varepsilon\in(0,\delta)$ and for any $Q_R(z_0)\Subset\Omega_T$, with a constant depending on $n,\nu,L,q,\varepsilon$. Moreover for $\mathcal{K}\Subset\Omega_T$*

$$[Du]_{W^{\delta-\varepsilon,\frac{\delta-\varepsilon}{2};q;\vartheta}(\mathcal{K})}\le c\left[s+\|f\|_{L^{1,\vartheta}(\Omega_T)}\right] \tag{3.5}$$

*for a constant depending on $n,\nu,L,q,\varepsilon,\mathrm{dist}(\mathcal{K},\partial\Omega_T),|\Omega|,T$.*

3.1. **Notation.** We denote by $c$ a constant, *greater than one*, that may vary from line to line. Peculiar dependencies on parameters will be emphasized in parentheses when needed and special constants will be denoted by $\tilde{c}$, $c_0$, $c_1 \ldots$ Euclidean space $\mathbb{R}^{n+1}$ will be thought as $\mathbb{R}^n \times \mathbb{R}$, so points in $\mathbb{R}^{n+1}$ will be $z = (x,t)$, $z_0 = (x_0, t_0)$ and so on. We denote by $B_R(x_0) = B(x_0, R) := \{x \in \mathbb{R}^n : |x - x_0| < R\}$ the ball with center $x_0$ and radius $R$, and with $Q_R(z_0)$ the *symmetric* parabolic cylinder $B(x_0,R) \times (t_0 - R^2, t_0 + R^2)$. When suppressing the center, unless expressly indicated, we shall mean that is the origin: so *in general* $B_1$ is $B_1(0)$, $Q_2$ is $Q_2(0)$, etc. With $B$ and $Q$ being balls and cylinders respectively, by $\gamma B$, $\gamma Q$ we shall denote the concentric balls and cylinder with radius scaled by a non-negative factor $\gamma > 0$.

3.2. **Functional spaces.** First of all a general notation, slightly different from the classical one: whereas $E = E(\Omega)$ is an "elliptic" space of integrable functions over $\Omega$, its local variant $E_{\text{loc}}$ is defined in the usual way, that is $f \in E_{\text{loc}}(\Omega)$ if $f \in E(\Omega')$ whenever $\Omega' \Subset \Omega$, that is whenever $\Omega'$ is compactly contained in $\Omega$. When dealing with parabolic function spaces, being $A = C \times I \subset \Omega_T = \Omega \times (-T, 0)$, we shall write $A \Subset \Omega_T$ if $C \Subset \Omega$ and $I \Subset (-T,0)$. Notice that is not the usual concept of parabolic compact inclusion. The localized version of parabolic spaces now it is defined starting from this definition of compact inclusion. Note that in the following we will define spaces just for function real-valued; however the adaptions needed to extend such spaces to vector-valued functions are straightforward. Moreover we shall use the same notation in both cases, since this will never yield any confusion.

For a domain $\Omega \subset \mathbb{R}^n$, we recall that the elliptic fractional Sobolev space $W^{\alpha,q}(\Omega)$, $\alpha \in (0,1)$, $1 \le q < \infty$, is the subspace of $L^q(\Omega)$ made up of all the functions $g$ whose Gagliardo seminorm

$$[g]^q_{W^{\alpha,q}(\Omega)} := \int_\Omega \int_\Omega \frac{|g(x) - g(y)|^q}{|x-y|^{n+\alpha q}}\, dx\, dy$$

is finite. It is endowed with the norm $\|g\|_{W^{\alpha,q}(A)} := \|g\|_{L^q(A)} + [g]_{W^{\alpha,q}(A)}$. For a function $g : \Omega \to \mathbb{R}$, any "small" real number $h \in \mathbb{R}$ and $i \in \{1, \ldots, n\}$, we define the spatial finite difference operator $\tau_{i,h}$ as

$$\big[\tau_{i,h} g\big](x) = \tau_{i,h} g(x) := g(x + h\, e_i) - g(x),$$

being $e_i$ the $i$-th vector of the standard orthonormal basis of $\mathbb{R}^n$. This will make sense, for example, whenever $x \in A \Subset \Omega$, $A$ an open set and $0 < |h| < \text{dist}(A, \partial\Omega)$, an assumption that will be always satisfied whenever we shall use this operator. Analogously, we define also the finite difference operator in time $\tau_h$ as

$$\big[\tau_h \tilde{g}\big](t) = \tau_h \tilde{g}(t) := \tilde{g}(t+h) - \tilde{g}(t),$$

again for $|h| > 0$ sufficiently small such that the definition makes sense.

These operators are useful when dealing with these nonlocal seminorms through the following spaces. For a set $A \Subset \Omega$, we define the Nikolsky space $\mathcal{N}^{\alpha,q}(A)$ as the space of the $L^q(\Omega)$ functions $g$ such that their $\mathcal{N}^{\alpha,q}$ norm $\|g\|_{\mathcal{N}^{\alpha,q}(A)} := \|g\|_{L^q(A)} + [g]_{\mathcal{N}^{\alpha,q}(A)}$, with

$$[g]_{\mathcal{N}^{\alpha,q}(A)} := \sum_{i=1}^{n} \sup_{0<h<\mathrm{dist}(A,\partial\Omega)} |h|^{-\alpha} \|\tau_{i,h} g\|_{L^q(A)},$$

is finite. It is well known that there exists a precise chain of inclusions between fractional Sobolev and Nikolsky spaces (see, among the others, [25, Lemma 2.3] or [17]), which reads as

$$W^{\alpha,q}(A) \subset \mathcal{N}^{\alpha,q}(A) \subset W^{\alpha-\varepsilon,q}(A) \qquad \text{for all } \varepsilon \in (0,\alpha). \tag{3.6}$$

*Parabolic spaces.* We say that a function $g \in L^q(\Omega_T)$ belongs to the parabolic fractional Sobolev space $W^{\alpha,\tilde{\alpha};q}(\Omega_T)$, with $\alpha, \tilde{\alpha} \in (0,1)$ and $1 \le q < \infty$, if the seminorm

$$[g]^q_{W^{\alpha,\tilde{\alpha};q}(\Omega_T)} := \int_{-T}^{0} \int_{\Omega} \int_{\Omega} \frac{|g(x,t)-g(y,t)|^q}{|x-y|^{n+\alpha q}}\, dx\, dy\, dt + \int_{\Omega} \int_{-T}^{0} \int_{-T}^{0} \frac{|g(x,t)-g(x,s)|^q}{|t-s|^{1+\tilde{\alpha} q}}\, ds\, dt\, dx$$

is finite. It is a Banach space if it is endowed with the norm, see [20], $\|g\|^q_{W^{\theta,\tilde{\theta};q}(\Omega_T)} := \|g\|^q_{L^q(\Omega_T)} + [g]^q_{W^{\theta,\tilde{\theta};q}(\Omega_T)}$.

Also Nikolsky spaces have a natural generalization when considered in parabolic setting: precisely, we call the parabolic Nikolsky space $\mathcal{N}^{\alpha,\tilde{\alpha};q}(\Omega_{T'})$, for $\mathcal{A} := C \times I$, $C \Subset \Omega$, $I \Subset (-T,0)$ and $\alpha, \tilde{\alpha} \in (0,1]$, the space of functions $\tilde{g} \in L^q(\Omega_T)$ such that

$$[\tilde{g}]_{\mathcal{N}^{\alpha,\tilde{\alpha},q}(\mathcal{A})} := \sup_{0<|h|<\mathrm{dist}(I,\partial(-T,0))} |h|^{-\tilde{\alpha}} \|\tau_h g\|_{L^q(\mathcal{A})} + \sum_{i=1}^{n} \sup_{0<h<\mathrm{dist}(C,\partial\Omega)} |h|^{-\alpha} \|\tau_{i,h} g\|_{L^q(\mathcal{A})} < \infty.$$

Obviously there is a chain of inclusion similar to (3.6) between the $W^{\alpha,\tilde{\alpha};q}_{\mathrm{loc}}$ and the $\mathcal{N}^{\alpha,\tilde{\alpha};q}$ spaces; we shall deduce the parabolic version of the second inclusion in (3.6) directly for our solutions, see (5.7).

*"Morreyzations".* Given a space of functions as above, we can define its "Morreyzation" by requiring that the norm of functions satisfies a density condition like (1.2). Hence for a Lebesgue space $L^\gamma(\Omega)$, $\gamma < \infty$, its "Morreyzation" turns out to be the Morrey space $L^{\gamma,\vartheta}(\Omega)$ defined as the subset of $L^\gamma(\Omega)$, we shall test the condition

$$\int_{B_R} |\mu|^\gamma\, dz \le c\, R^{n-\vartheta} \qquad \text{for all } B_R \subset \Omega; \tag{3.7}$$

functions satisfying (3.7) belong to the Morrey space $L^{\gamma,\vartheta}(\Omega)$ and its norm $\|\mu\|^\gamma_{L^{\gamma,\vartheta}(\Omega)}$ is given by $\|f\|_{L^\gamma(\Omega)}$ plus the infimum of the constants such that (3.7) holds. For the

Marcinkiewicz space $\mathcal{M}^m(\Omega)$, $m \geq 1$, one requires that

$$\begin{aligned}\|g\|^m_{\mathcal{M}^{m,\vartheta}(\Omega)} &:= \|g\|^m_{\mathcal{M}^m(\Omega)} + \sup_{B_R} R^{\vartheta-n}\|g\|^m_{\mathcal{M}^m(Q_R)} \\ &= \|g\|^m_{\mathcal{M}^m(\Omega)} + \sup_{B_R} R^{\vartheta-n} \sup_{\lambda>0} \lambda^m \big|\{x \in B_R : |g(x)| > \lambda\}\big| < \infty,\end{aligned} \tag{3.8}$$

where $b_R \subset \Omega$. For the fractional parabolic Sobolev spaces, with $q \geq 1$, $\alpha \in (0,1)$, one naturally requires

$$[g]^q_{W^{\alpha,q;\vartheta}(\Omega)} := \sup_{B_R} R^{\vartheta-n}[g]^q_{W^{\alpha,\tilde{\alpha};q}(\Omega_T)}$$

$Q_R$ as above, to be finite, and the norm of the space is given by adding the $L^q(\Omega)$ norm of $g$ to the seminorm defined in the display above. "Morreyzations" of parabolic functional spaces are naturally defined starting from the corresponding parabolic spaces and adding a density condition of the type defined above, replacing the balls $B_R$ with the parabolic cylinders $Q_R$ and the dimension $n$ with $N$. Finally a useful properties regarding the scaling property of the $L^{\gamma,\vartheta}$ seminorm.

**Lemma 3.3.** *Let $g \in L^{\gamma,\vartheta}(Q_R(z_0))$ with $1 \leq \gamma < \infty$. Then the map $\tilde{g}(y,\tau) := g(x_0 + Ry, t_0 + R^2\tau)$, $(y,\tau) \in Q_1$, belongs to $L^{\gamma,\vartheta}(Q_1)$ and*

$$\|\tilde{g}\|_{L^{\gamma,\vartheta}(Q_1)} = R^{-\frac{\vartheta}{\gamma}}\|g\|_{L^{\gamma,\vartheta}(Q_R(z_0))}.$$

## 4. Setting of the proof and Morrey estimates

We *initially stress* that all the computation we shall perform will be done for the approximating functions in $f_k$ (and for ease of notation we will drop the subscript $k$); therefore from now on we will consider (and prove results for) energy solutions

$$u \in L^2(-T,0;W^{1,2}_0(\Omega)) \cap C(-T,0;L^2(\Omega)) \tag{4.1}$$

to the problem

$$\begin{cases} u_t - \operatorname{div} a(x,t,Du) = f & \text{in } \Omega_T, \\ u = 0 & \text{on } \partial_{\mathcal{P}}\Omega_T, \end{cases} \tag{4.2}$$

where $f$ will be a function in $L^\infty$, satisfying also

$$\|f_k\|_{L^1(\Omega_T)} \leq |\mu|(\Omega_T), \qquad \|f_k\|_{L^{1,\vartheta}(\Omega_T)} \leq \|\mu\|_{L^{1,\vartheta}(\Omega_T)},$$

see again [31]. Estimates will pass to the limit, therefore applying to the SOLAs, in a standard manner, using strong convergence of the gradients (which, on the other hand, can be deduced directly by our *a priori estimates*, using a compactness argument which can be found in [33], as in [2]) and weak convergence of the right-hand sides. Note moreover that a backward mollification in time gives

$$\limsup_{k\to\infty} |f_k|(Q) \leq |\mu|\big(\lfloor Q\rfloor_{\mathcal{P}}\big),$$

where the parabolic closure of the cylinder $Q$ is given by $\lfloor Q\rfloor_{\mathcal{P}} := Q \cup \partial_{\mathcal{P}}Q$, see [27]; it has a more "parabolic flavor" than the elliptic closure $\overline{Q} = Q \cup \partial Q$. This can be used when passing to the limit in local estimates. We start with the global estimate for solutions to (4.2), see [2, Section 5]:

**Lemma 4.1.** *Let $u$ as in (4.1) the solution to (4.2); then*

$$\|Du\|_{L^q(\Omega_T)} \leq c\left[s + \|f\|_{L^1(\Omega_T)}\right],$$

*for any $q$ as in (2.15). The constant depends on $n, \nu, q, |\Omega|, T$.*

*Remark* 4.2. In the rest of the paper we shall need to perform two different scaling procedures, which will give the exact dependence certain constant we are interested in. If we consider a solution to

$$u_t - \operatorname{div} a(x,t,Du) = f \qquad \text{in } \mathcal{K}, \tag{4.3}$$

with $f$ in the beginning of the section and $a$ satisfying (3.1). Considering, for $A > 0$, $\tilde{u} := u/A$, $\tilde{f} = f/A$, $\tilde{a}(x,t,\xi) := a(x,t,A\xi)/A$,with $(x,t) \in \mathcal{K}$ and $\xi \in \mathbb{R}^n$, we have that these functions satisfy $\partial_t \tilde{u} - \operatorname{div} \tilde{a}(x,t,D\tilde{u}) = \tilde{f}$ and $\tilde{a}$ has the same structure of $a$, in the sense that $\tilde{a}$ also satisfies (3.1), with $s$ replaced by $\tilde{s} = s/A$.

Another useful scaling allowed by the structure of the equation can be done if we consider (4.3) on some parabolic cylinder: $\mathcal{K} = Q_r(z_0) = Q_r(x_0,t_0)$. Here scaling

$$\begin{cases} \tilde{u}(x,t) := \dfrac{u(x_0 + Rx, t_0 + R^2 t)}{R}, \\ \tilde{a}(x,t,\xi) := a(x_0 + Rx, t_0 + R^2 t, \xi), \\ \tilde{f}(x,t) := Rf(x_0 + Rx, t_0 + R^2 t). \end{cases}$$

for $(x,t) \in Q_1$, it is easy to verify that $\partial \tilde{u} - \operatorname{div} \tilde{a}(x,t,D\tilde{u}) = \tilde{f}$ in $Q_1$.

In this first part of this section, we want to describe the general setting of the proof, together with some related regularity results and estimates we shall need in the following. Moreover, we shall prove some preliminary lemmata we are going to use in the proof of the fractional differentiability results. A basic tool in our approach will be comparison functions. These functions are intended to be more regular functions, defined on cylinders and sharing boundary data with our solution $u$ to (4.2). Properties of the function $u$ will be therefore deduced by "transfering" regularity results for the reference problem *via* comparison estimates. We define the first of our comparison function, for a cylinder $Q_{2R} \equiv Q_{2R}(z_0) \subset \Omega_T$, as the solution of the Cauchy-Dirichlet problem

$$\begin{cases} v_t - \operatorname{div} a(x,t,Dv) = 0 & \text{in } Q_{2R}, \\ v = u & \text{on } \partial_{\mathcal{P}} Q_{2R}, \end{cases} \tag{4.4}$$

Existence of such a function is a well-known fact, since we are assuming $u \equiv u_k$ regular, and also the fact that

$$v \in u + L^2(t_0 - 4R^2, t_0 + 4R^2; W_0^{1,2}(B_{2R}(x_0))).$$

The following comparison estimate has been proved in [2, Lemma 6.4].

**Lemma 4.3.** *Let $u$ be the solution to (4.2) and let $v$ be the solution to the Cauchy-Dirichlet problem (4.4). Then for every $q$ as in (2.15) there exists a constant $c \equiv c(n,\nu,q)$ such that*

$$\int_{Q_{2R}} |Du - Dv|^q \, dz \le c\, R^{N-(N-1)q} \bigg( \int_{Q_{2R}} |f| \, dz \bigg)^q .$$

The proof of the following corollary is straightforward once recalling (1.2).

**Corollary 4.4.** *Let $u$ and $v$ as in Lemma 4.3, and $q$ as in (2.15). If $f \in L^{1,\vartheta}(\Omega_T)$, for $\vartheta \in [0,N]$, then*

$$\int_{Q_{2R}} |Du - Dv|^q \, dz \le c\, R^{N-(\vartheta-1)q} \|f\|^q_{L^{1,\vartheta}(Q_{2R})} \tag{4.5}$$

*for a constant depending on $n,\nu,q$.*

The following lemma is a first step towards the proof of Theorem 3.1, since it encodes the density properties of the gradient of the solution $u$, when also the right-hand side has some density properties. Note that in particular this result yields

$$\mu \in L^{1,\vartheta}(\Omega) \qquad \Longrightarrow \qquad Du \in L^{1,\vartheta-1}_{\text{loc}}(\Omega_T) \qquad \text{for} \quad 2 \le \vartheta \le N.$$

**Lemma 4.5.** *Let $u$ be a solution to (4.2) with $f \in L^{1,\vartheta}(\Omega_T)$, for $\vartheta \in [2,N]$; then $Du \in L^{q,(\vartheta-1)q}_{\text{loc}}(\Omega_T)$ for every $q$ as in (3.3). Moreover the following estimates hold true, for every compact sets $\mathcal{K}_1 \Subset \mathcal{K}_2 \subset \Omega_T$:*

$$\big\| |Du| + s \big\|_{L^{q,(\vartheta-1)q}(\mathcal{K}_1)} \le c \, \big\| |Du| + s \big\|_{L^q(\mathcal{K}_2)} + c\, \|f\|_{L^{1,\vartheta}(\mathcal{K}_2)}, \tag{4.6}$$

*for $c \equiv c(n,\nu,L,q,\text{dist}(\mathcal{K}_1,\partial\mathcal{K}_2))$, and*

$$\big\| |Du| + s \big\|_{L^{q,(\vartheta-1)q}(\mathcal{K}_1)} \le c \Big[ s + \|f\|_{L^{1,\vartheta}(\Omega_T)} \Big], \tag{4.7}$$

*for a constant depending on $n,\nu,L,q,\text{dist}(\mathcal{K}_1,\partial\Omega_T),|\Omega|,T$.*

*Proof.* To prove estimate (4.6) we follow the proof of [28, Lemma 8.1], using a standard iteration technique; we will be a bit sketchy. Take a cylinder $Q_{2R} \equiv Q_{2R}(z_0) \subset \mathcal{K}_2$ and define there the comparison function $v$ as in (4.4). Note that the theory of De Giorgi applies to $v$, see [20], and therefore, for radii $0 < \varrho \le 2R$, we have the estimate

$$\int_{Q_\varrho(z_0)} \big(|Dv| + s\big)^q \, dz \le c \left( \frac{\varrho}{R} \right)^{N-q+\alpha q} \int_{Q_{2R}(z_0)} \big(|Dv| + s\big)^q \, dz,$$

for any $q > 0$, a constant depending on $n,\nu,L,q$ and an exponent $\alpha \in (0,1)$ depending on $n,\nu,L$, see for instance [3, Theorem 5.5]. Using the triangle inequality and comparison estimate (4.5) we get

$$
\begin{aligned}
\int\limits_{Q_\varrho(z_0)} \big(|Du|+s\big)^q\,dz &\le c\left(\frac{\varrho}{R}\right)^{N-q+\alpha q} \int\limits_{Q_{2R}(z_0)} \big(|Du|+s\big)^q\,dz \\
&\qquad\qquad + c\int\limits_{Q_{2R}(z_0)} |Du-Dv|^q\,dz \\
&\le c\left(\frac{\varrho}{R}\right)^{N-q+\alpha q} \int\limits_{Q_{2R}(z_0)} \big(|Du|+s\big)^q\,dz + c\,R^{N-(\vartheta-1)q}\|f\|^q_{L^{1,\vartheta}(Q_{2R})};
\end{aligned}
$$

note that since $2\le\vartheta$, then $N-q\ge N-q\vartheta+q$. Using hence a standard iteration lemma, see [28, Lemma 2.11], we infer that

$$
\begin{aligned}
&\varrho^{-N+(\vartheta-1)q}\int\limits_{Q_\varrho(z_0)} \big(|Du|+s\big)^q\,dz \\
&\qquad\le c\left[R^{-N+(\vartheta-1)q}\left(\frac{\varrho}{R}\right)^{\alpha q/2}\int\limits_{Q_{2R}(z_0)} \big(|Du|+s\big)^q\,dz + \|f\|^q_{L^{1,\vartheta}(Q_{2R})}\right]
\end{aligned}
$$

for every $0<\varrho\le 2R$ and with a constant depending on $n,\nu,L,q$. Now for $\mathcal{K}_1$ and $\mathcal{K}_2$ as in the statement, make the following choice for $Q_{2R}(z_0)$: $z_0\in\mathcal{K}_1$ and $R=\mathrm{dist}(\mathcal{K}_1,\partial\mathcal{K}_2)/4$. Hence $Q_{2R}(z_0)\subset\mathcal{K}_2$ and for any $0<\varrho\le\mathrm{dist}(\mathcal{K}_1,\partial\mathcal{K}_2)/2$ and for any $z_0\in\mathcal{K}_1$,

$$
\varrho^{-N+(\vartheta-1)q}\int\limits_{Q_\varrho(z_0)} \big(|Du|+s\big)^q\,dz \le c\left[\int\limits_{\mathcal{K}_2}\big(|Du|+s\big)^q\,dz + \|f\|^q_{L^{1,\vartheta}(\mathcal{K}_2)}\right],
$$

where now $c$ depends also on $\mathrm{dist}(\mathcal{K}_1,\partial\mathcal{K}_2)$. The estimate in (4.6) follows taking the supremum over all the cylinders involved in definition (1.2) and at the same time noting that in the case $\mathrm{dist}(\mathcal{K}_1,\partial\mathcal{K}_2)<1$, for the radii $\varrho>\mathrm{dist}(\mathcal{K}_1,\partial\mathcal{K}_2)$, (4.6) follows just by enlarging the constant by a factor $[\mathrm{dist}(\mathcal{K}_1,\partial\mathcal{K}_2)]^{-N+(\vartheta-1)q}$.

To prove (4.7), once fixed $\mathcal{K}_1$, find $\mathcal{K}_2$ such that $\mathrm{dist}(\mathcal{K}_1,\partial\mathcal{K}_2)=\mathrm{dist}(\mathcal{K}_1,\partial\Omega_T)/2$, apply (4.6) and then estimate the integrals on the right-hand side using Lemma 4.1. $\square$

The following lemma is essentially contained, as particular case, in [4, Theorem 1], see also [3]. We show how to adapt this result to the case $p=2$ to obtain a perfect analogue of [28, Theorem 1.8 & Remark 9.1].

**Lemma 4.6.** *Let $u$ be a solution to (4.2) and suppose $f \in L^{1,\vartheta}(\Omega_T)$, $\vartheta \in [2,N]$. Then $Du \in \mathcal{M}^{\frac{\vartheta}{\vartheta-1},\vartheta}_{\rm loc}(\Omega_T)$ and moreover the following local estimate holds: for every cylinder*

$$
(4.8)\quad R^{\vartheta-N}\big\||Du|+s\big\|^{\frac{\vartheta}{\vartheta-1}}_{\mathcal{M}^{\frac{\vartheta}{\vartheta-1}}(Q_R)} \le c\,\|f\|^{\frac{\vartheta}{\vartheta-1}}_{L^{1,\vartheta}(Q_{2R})} + c\,R^{\vartheta}\Bigg(\fint_{Q_{2R}}(|Du|+s)^q\,dz\Bigg)^{\frac{1}{q}\cdot\frac{\vartheta}{\vartheta-1}}
$$

*for a constant depending on $n,\nu,L,q$, and therefore for every compact set $\mathcal{K}\Subset_{\mathcal{P}}\Omega_T$*

$$
(4.9)\qquad \big\||Du|+s\big\|_{\mathcal{M}^{\frac{\vartheta}{\vartheta-1},\vartheta}(\mathcal{K})} \le c\Big[s+\|f\|_{L^{1,\vartheta}(\Omega_T)}\Big],
$$

*with $q$ as above and $c\equiv c(n,\nu,L,q,\mathrm{dist}_{\mathcal{P}}(\mathcal{K},\partial_{\mathcal{P}}\Omega_T),|\Omega|,T)$.*

*Proof.* Once given $u$ and $Q_{2R}$ as in the statement of the lemma, rescale the problem as in Remark 4.2 so that $\|f\|_{L^{1,\vartheta}(Q_{2R})}=1$. That is, avoiding the trivial case $\|f\|_{L^{1,\vartheta}(Q_{2R})}=0$, set $A=\|f\|_{L^{1,\vartheta}(Q_{2R})}$; clearly $\|\tilde f\|_{L^{1,\vartheta}(Q_{2R})}=1$. Theorem 1 of [4] now gives

$$
R^{-N}\big\||D\tilde u|+\tilde s\big\|^{\frac{\vartheta}{\vartheta-1}}_{\mathcal{M}^{\frac{\vartheta}{\vartheta-1}}(Q_R)} \le c\,\frac{1}{|Q_{2R}|}\int_{Q_{2R}}|\tilde f|\,dz + c\Bigg(\fint_{Q_{2R}}(|D\tilde u|+\tilde s)\,dz\Bigg)^{\frac{\vartheta}{\vartheta-1}}
$$

and the constant now depends on $n,\nu,L,\vartheta$ but not on $\|f\|_{L^{1,\vartheta}(Q_{2R})}$. Rescaling back to $u$ and $f$, taking into account (1.2) and using Hölder's inequality yields

$$
R^{-N}\big\||Du|+s\big\|^{\frac{\vartheta}{\vartheta-1}}_{\mathcal{M}^{\frac{\vartheta}{\vartheta-1}}(Q_R)} \le c\,R^{-\vartheta}\|f\|^{\frac{\vartheta}{\vartheta-1}}_{L^{1,\vartheta}(Q_{2R})} + c\Bigg(\fint_{Q_{2R}}(|Du|+s)^q\,dz\Bigg)^{\frac{1}{q}\cdot\frac{\vartheta}{\vartheta-1}},
$$

which is (4.8). To infer (4.9) we simply take into account (4.7) and the definition of the norm in (3.8). Finally note that the dependence of the constant on $\vartheta$ can be avoided, since this parameter varies in a compact set and the dependence of the constant upon $\vartheta$ is continuous. □

## 5. Proof of the main results

Fix a cylinder $Q_{4R}(z_0)\Subset\Omega_T$ and rescale as in Remark 4.2 to a function $\tilde u$ solution to $\partial\tilde u - \mathrm{div}\,\tilde a(x,t,D\tilde u)=\tilde f$ in $Q_4$. For this function we know from Theorem 2.11 that

$$
Du \in W^{1-\varepsilon,\frac{1-\varepsilon}{2};1}(Q_2)
$$

for every $\varepsilon\in(0,1)$; once fixed $\beta\in(0,1)$ we can apply Proposition 8.2 from [2] with $\kappa$ appropriately close to one, depending on $n,\beta,\Omega_{T,1}=Q_2,\Omega_{T,2}=Q_4$ to infer that

$$
\begin{aligned}
\|\tau_{h^2}D\tilde u\|_{L^1(Q_2)}+\sum_{i=1}^n\|\tau_{i,h}D\tilde u\|_{L^1(Q_2)} &\le c\,|h|^\beta\Big[\big\||D\tilde u|+s\big\|_{L^1(Q_4)}+\|\tilde f\|_{L^1(Q_4)}\Big]\\
&\le c\,|h|^\beta\Big[\big\||D\tilde u|+s\big\|_{L^{1,\vartheta-1}(Q_4)}+\|\tilde f\|_{L^{1,\vartheta}(Q_4)}\Big]
\end{aligned}
$$

for every $h \in (0, \mathcal{D})$, with $\mathcal{D} < 1/4$ a small constant depending on the dimension $n$, and the constant $c$ depending on $n, \nu, L, \beta$, so that $\tau_{h^2} D\tilde{u}(x,t)$ and $\tau_{i,h} D\tilde{u}(x,t)$ belong to $Q_4$ for $(x,t) \in Q_2$. Scaling back to $Q_{4R}(z_0)$, using Lemma 3.3

$$\|\tau_{(Rh)^2} Du\|_{L^1(Q_{2R})} + \sum_{i=1}^{n} \|\tau_{i,Rh} D\tilde{u}\|_{L^1(Q_{2R})} \leq c\, R^{N+1-\vartheta}\, |h|^{\beta} \Big[ \big\| |Du| + s \big\|_{L^{1,\vartheta-1}(Q_{4R})} + \|f\|_{L^{1,\vartheta}(Q_{4R})} \Big],$$

and replacing $h$ with $h/R$

$$\|\tau_{h^2} Du\|_{L^1(Q_{2R})} + \sum_{i=1}^{n} \|\tau_{i,h} Du\|_{L^1(Q_{2R})} \leq c\, R^{N+1-\vartheta-\beta}\, |h|^{\beta} \Big[ \big\| |Du| + s \big\|_{L^{1,\vartheta-1}(Q_{4R})} + \|f\|_{L^{1,\vartheta}(Q_{4R})} \Big], \tag{5.1}$$

for $0 < |h| < \mathcal{D}R$. Now fix $q \in (1, \vartheta/(\vartheta-1))$ and define $\bar{\gamma} \in (0,1)$ from

$$q = \frac{\vartheta - \bar{\gamma}}{\vartheta - 1} \qquad \Longleftrightarrow \qquad \bar{\gamma} = \vartheta - q(\vartheta - 1); \tag{5.2}$$

if $\gamma \in (0, \bar{\gamma})$ then

$$q < m_0 := \frac{\vartheta - \gamma}{\vartheta - 1} < \frac{\vartheta}{\vartheta - 1} =: m$$

and we can write, for $t \in (0,1)$,

$$q = (1-t) + t m_0 = \frac{\bar{\gamma} - \gamma}{1 - \gamma} + \frac{1 - \bar{\gamma}}{1 - \gamma} m_0. \tag{5.3}$$

We also have

$$\frac{m - m_0}{m} = \frac{\gamma}{\vartheta}, \qquad \frac{m_0}{m} = \frac{\vartheta - \gamma}{\vartheta}. \tag{5.4}$$

Now estimate (4.8) with $q = 1$, taking $m$-th root, yields

$$\big\| |Du| + s \big\|_{\mathcal{M}^m(Q_R)} \leq c\, R^{\frac{N-\vartheta}{m}} \Big[ \big\| |Du| + s \big\|_{L^{1,\vartheta-1}(Q_{2R})} + \|f\|_{L^{1,\vartheta}(Q_{2R})} \Big];$$

at this point, using Hölder's inequality for Marcinkiewicz spaces, see [28, Lemma 2.8], we infer

$$\begin{aligned}
\|Du\|^{m_0}_{L^{m_0}(Q_R)} &\leq \frac{m}{m - m_0} R^{N(1-\frac{m_0}{m})} \|Du\|^{m_0}_{\mathcal{M}^m(Q_R)} \\
&\leq \frac{c}{\gamma}\, R^{N(1-\frac{m_0}{m}) + m_0 \frac{N-\vartheta}{m}} \Big[ \big\| |Du| + s \big\|^{m_0}_{L^{1,\vartheta-1}(Q_{2R})} + \|f\|^{m_0}_{L^{1,\vartheta}(Q_{2R})} \Big] \\
&= \frac{c}{\gamma}\, R^{N-\vartheta+\gamma} \Big[ \big\| |Du| + s \big\|^{m_0}_{L^{1,\vartheta-1}(Q_{2R})} + \|f\|^{m_0}_{L^{1,\vartheta}(Q_{2R})} \Big]
\end{aligned} \tag{5.5}$$

by a direct computation, taking into account (4.9) and $(5.4)_2$. Here $c \equiv c(n, \nu, L, \vartheta)$. Using again Hölder's inequality and (5.3) we get, for $1 < q < \vartheta/(\vartheta - 1)$ and $0 < |h| <$

$\mathcal{D}R \leq R/4$

$$\begin{aligned}\|\tau_{i,h}Du\|^q_{L^q(Q_{R/2})} &\leq \|\tau_{i,h}Du\|^{1-t}_{L^1(Q_{R/2})}\|\tau_{i,h}Du\|^{m_0 t}_{L^{m_0}(Q_{R/2})}\\ &\leq \|\tau_{i,h}Du\|^{1-t}_{L^1(Q_{R/2})}\|Du\|^{m_0 t}_{L^{m_0}(Q_R)}.\end{aligned}$$

In turn, exploiting (5.1) and (5.5) we have

$$\begin{aligned}&\|\tau_{i,h}Du\|^q_{L^q(Q_{R/2})}\\ &\quad\leq c\,R^{N-\vartheta+(1-\beta)(1-t)+\gamma t}\,|h|^{\beta(1-t)}\Big[\big\||Du|+s\big\|_{L^{1,\vartheta-1}(Q_{2R})}+\|f\|_{L^{1,\vartheta}(Q_{2R})}\Big]^q\end{aligned}$$

for any $i \in \{1,\dots,n\}$, and similarly

$$\begin{aligned}&\|\tau_{h^2}Du\|^q_{L^q(Q_{R/2})}\\ &\quad\leq c\,R^{N-\vartheta+(1-\beta)(1-t)+\gamma t}\,|h|^{\beta(1-t)}\Big[\big\||Du|+s\big\|_{L^{1,\vartheta-1}(Q_{2R})}+\|f\|_{L^{1,\vartheta}(Q_{2R})}\Big]^q.\end{aligned}$$

Now a direct computation shows that $(1-\beta)(1-t)+\gamma t > \bar{\gamma}(1-\beta)$ since $\beta,\bar{\gamma}\in(0,1)$; therefore, enlarging the constant by a factor depending on the diameter of $\Omega_T$ with respect to the parabolic distance and summing up the last two inequalities we get

$$\begin{aligned}\text{(5.6)}\quad &\|\tau_{h^2}Du\|^q_{L^q(Q_{R/2})}+\sum_{i=1}^n\|\tau_{i,h}Du\|^q_{L^q(Q_{R/2})}\\ &\qquad\leq c\,R^{N-\vartheta+\bar{\gamma}(1-\beta)}\,|h|^{\beta(1-t)}\Big[\big\||Du|+s\big\|_{L^{1,\vartheta-1}(Q_{2R})}+\|f\|_{L^{1,\vartheta}(Q_{2R})}\Big]^q\end{aligned}$$

for $1<q<\vartheta/(\vartheta-1)$; the reader might recall that $\bar{\gamma}:=\vartheta-q(\vartheta-1)$. Note that (5.6) holds also for $q=1$, and this is just (5.1), and *in this case* even for $\gamma=0$. Note that the previous estimate extend to increases in any spatial direction $e\in\partial B_1$, from an averaging argument (i.e. in place of $\tau_{i,h}Du$ we can write, up to a change in the constant, $[\tau_{e,h}Du](x,t)=Du(x+he,t)-Du(x,t)$, for any $e$ as before).

We conclude the proof. Take $\tilde{\gamma}\in(0,\beta(1-t))$; calling $I_{R/4}$ the interval $(t_0-(R/4)^2,t_0+(R/4)^2)$ and denoting $B_{R/2}\equiv B_{R/2}(x_0)$, changing variables, using coarea formula, for $q$ as in (3.3) and finally taking into account (5.6)

$$\begin{aligned}\text{(5.7)}\quad &\int_{I_{R/4}}\int_{B_{R/2}}\int_{B_{R/2}}\frac{|Du(x,t)-Du(y,t)|^q}{|x-y|^{n+\beta(1-t)-\tilde{\gamma}}}\,dx\,dy\,dt\\ &\leq c\int_0^R\frac{1}{|h|^{1-\tilde{\gamma}}}\int_{I_{R/4}}\int_{\partial B_1}\int_{B_{R/2}}\frac{|Du(y+he,t)-Du(y,t)|^q}{|h|^{\beta(1-t)}}\,dy\,d\mathcal{H}^{n-1}(e)\,dt\,dh\\ &\leq c\int_0^R\frac{dh}{|h|^{1-\tilde{\gamma}}}\int_{\partial B_1}\sup_{h\in(0,R/2)}\int_{I_{R/4}}\int_{B_{R/2}}\frac{|Du(y+he,t)-Du(y,t)|^q}{|h|^{\beta(1-t)}}\,dy\,dt\,d\mathcal{H}^{n-1}(e).\end{aligned}$$

Now, since

$$\sup_{h\in(0,R/2)}\int_{I_{R/4}}\int_{B_{R/2}}\frac{|Du(y+he,t)-Du(y,t)|^q}{|h|^{\beta(1-t)}}\,dy\,dt$$
$$\leq c\,R^{N-\vartheta+\bar{\gamma}(1-\beta)}\Big[\big\||Du|+s\big\|_{L^{1,\vartheta-1}(Q_{2R})}+\|f\|_{L^{1,\vartheta}(Q_{2R})}\Big]^q$$
$$+\,c\,\big[\mathcal{D}R\big]^{-\beta(1-t)}R^{N-(\vartheta-1)}\big\||Du|+s\big\|^q_{L^{1,\vartheta-1}(Q_{2R})}$$

by (5.6) and Lemma 4.5; we then have, since both $N-\vartheta+1-\beta(1-t)$ and $N-\vartheta+\bar{\gamma}(1-\beta)+\tilde{\gamma}$ are greater than $N-\vartheta$,

$$\sup_{h\in(0,R/2)}\int_{I_{R/4}}\int_{B_{R/2}}\frac{|Du(y+he,t)-Du(y,t)|^q}{|h|^{\beta(1-t)}}\,dy\,dt$$
$$\leq c\,R^{N-\vartheta}\Big[\big\||Du|+s\big\|_{L^{1,\vartheta-1}(Q_{2R})}+\|f\|_{L^{1,\vartheta}(Q_{2R})}\Big]^q,$$

where $c$ also depends on $\mathrm{diam}(\Omega_T)$. Similarly, for $\tilde{\tilde{\gamma}}\in(0,\beta(1-t)/2)$

$$\int_{B_{R/2}}\int_{I_{R/4}}\int_{I_{R/4}}\frac{|Du(x,t)-Du(x,s)|^q}{|t-s|^{1+\beta(1-t)/2-\tilde{\tilde{\gamma}}}}\,dt\,ds\,dx \tag{5.8}$$
$$\leq c\,R^{N-\vartheta}\Big[\big\||Du|+s\big\|_{L^{1,\vartheta-1}(Q_{2R})}+\|f\|_{L^{1,\vartheta}(Q_{2R})}\Big]^q.$$

This, together with the facts that $\beta,\tilde{\gamma},\tilde{\tilde{\gamma}},\gamma$ are arbitrary (recall (5.3) and (5.2)), and also straightforward changes in the radii involved in the various estimates, shows (3.4).

Estimate (5.8), together with (4.7), yields also (3.5). □

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