QUASISTATIC EVOLUTION IN PERFECT PLASTICITY AS LIMIT OF DYNAMIC PROCESSES

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Abstract. We introduce a model of dynamic visco-elasto-plastic evolution in the linearly elastic regime and we prove an existence and uniqueness result. Then we study the limit of (a rescaled version of) the solutions when the data vary slowly. We prove that they converge, up to a subsequence, to a quasistatic evolution in perfect plasticity.

Keywords: visco-elasto-plasticity, perfect plasticity, dynamic evolution, quasistatic evolution, discrete time approximation, implicit Euler scheme, incremental minimum problems.

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1. Introduction

The quasistatic evolution of rate independent systems has been often obtained as the limit case of a viscosity driven evolution (see [27], [20], [9], [6], [30], [22], [13], [14], [16], [17], [21], [23]). In this paper we present a case study on the approximation of a quasistatic evolution by dynamic evolutions, in a mechanical problem governed by partial differential equations. For a similar problem in finite dimension we refer to [1].

More precisely we approximate the solutions of the quasistatic evolution in linearly elastic perfect plasticity (see [27] and [5]) by the solutions of suitable dynamic visco-elasto-plastic problems, when a parameter connected with the speed of the process tends to 0.

In the first part of the paper we consider a model of dynamic visco-elasto-plastic evolution in the linearly elastic regime. The reference configuration is a bounded open set $\Omega \subset \mathbb{R}^n$ with sufficiently smooth boundary. The linearized strain Eu, defined as the symmetric part of the gradient of the displacement u, is decomposed as $Eu = e + p$, where e is the elastic part and p is the plastic part. The stress $\sigma = A^0 e + A^1 \dot{e}$ is the sum of an elastic part $A^0 e$ and a viscous part $A^1 \dot{e}$, where A^0 is the elasticity tensor, A^1 is the viscosity tensor, and $\dot{e}$ is the derivative of e with respect to time. The balance of momentum gives the equation

$$\ddot{u} - \mathrm{div}\sigma = f,$$

where f is the volume force, and we have supposed, for simplicity, that the mass density is identically equal to 1. The evolution of the plastic part is governed by the flow rule

$$\dot{p} = \sigma_D - \pi_K \sigma_D,$$

where σ_D is the deviatoric part of σ and π_K is the projection onto a prescribed convex set K in the space of deviatoric symmetric matrices, which can be interpreted as the domain of visco-elasticity. Indeed, if σ_D belongs to K during the evolution,

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then there is no production of plastic strain, so that, if $p = 0$ at the initial time, then $p = 0$ for every time and the solution is purely visco-elastic.

The complete system of equations is then

$$Eu = e + p, \tag{1.1a}$$

$$\sigma = A^0 e + A^1 \dot{e}, \tag{1.1b}$$

$$\ddot{u} - \mathrm{div}\sigma = f, \tag{1.1c}$$

$$\dot{p} = \sigma_D - \pi_K \sigma_D, \tag{1.1d}$$

supplemented by initial and boundary conditions.

Under natural assumptions on A^0, A^1, f, and K we prove existence and uniqueness of a solution to (1.1) with initial and boundary conditions (Theorem 3.1). In analogy with the energy method for rate independent processes developed by Mielke (see [20] and the references therein), we first prove that system (1.1) has a weak formulation expressed in terms of an energy balance together with a stability condition (Theorem 3.3). The proof of the existence of a solution to this weak formulation is obtained by time discretization. In the discrete formulation we solve suitable incremental minimum problems and then we pass to the limit as the time step tends to 0.

In the second part of the work we analyze the behavior of the solution to system (1.1) as the data of the problem become slower and slower. After a standard change of variables described at the beginning of Section 6, we are led to study the behavior of the solutions to the system

$$Eu^\epsilon = e^\epsilon + p^\epsilon, \tag{1.2a}$$

$$\sigma^\epsilon = A^0 e^\epsilon + \epsilon A^1 \dot{e}^\epsilon, \tag{1.2b}$$

$$\epsilon^2 \ddot{u}^\epsilon - \mathrm{div}\sigma^\epsilon = f, \tag{1.2c}$$

$$\epsilon \dot{p}^\epsilon = \sigma^\epsilon_D - \pi_K \sigma^\epsilon_D, \tag{1.2d}$$

as ϵ tends to 0.

Under suitable assumptions we show (Theorem 6.2) that these solutions converge, up to a subsequence, to a weak solution of the quasistatic evolution problem in perfect plasticity (see [27] and [5]), whose strong formulation is given by

$$Eu = e + p, \tag{1.3a}$$

$$\sigma = A^0 e, \tag{1.3b}$$

$$-\mathrm{div}\sigma = f, \tag{1.3c}$$

$$\sigma_D \in K \text{ and } \dot{p} \in N_K \sigma_D, \tag{1.3d}$$

where $N_K \sigma_D$ denotes the normal cone to K at σ_D.

The proof of this convergence result is obtained using the weak formulation of (1.1) expressed by energy balance and stability condition. We show that we can pass to the limit in this formulation obtaining the energy formulation of (1.3) developed in [5]. A remarkable difficulty in this proof is due to the fact that problems (1.1) and (1.3) are formulated in completely different function spaces (see Theorem 3.1 and Definition 5.1).

2. Preliminaries

2.1. **Notation. Vectors and Matrices.** If $a, b \in \mathbb{R}^n$, their scalar product is defined by $a \cdot b := \sum_i a_i b_i$, and $|a| := (a \cdot a)^{1/2}$ is the norm of a. If $\eta = (\eta_{ij})$ and $\xi = (\xi_{ij})$ belong to the space $\mathbb{M}^{n\times n}$ of $n \times n$ matrices with real entries, their scalar product is defined by $\eta \cdot \xi := \sum_{ij} \eta_{ij}\xi_{ij}$. Similary $|\eta| := (\eta \cdot \eta)^{1/2}$ is the norm of η. $\mathbb{M}^{n\times n}_{\mathrm{sym}}$ is the subspace of $\mathbb{M}^{n\times n}$ composed of symmetric matrices. Moreover $\mathbb{M}^{n\times n}_D$

denotes the subspace of symmetric matrices with null trace, i.e., $\eta \in \mathbb{M}^{n\times n}_D$ if η is symmetric and $\mathrm{tr}\eta = \sum_i \eta_{ii} = 0$. The space $\mathbb{M}^{n\times n}_{\mathrm{sym}}$ can be split as

$$\mathbb{M}^{n\times n}_{\mathrm{sym}} = \mathbb{M}^{n\times n}_D \oplus \mathbb{R}I,$$

where I is the identity matrix, so that every $\eta \in \mathbb{M}^{n\times n}_{\mathrm{sym}}$ can be written as $\eta = \eta_D + \mathrm{tr}\eta I$, where η_D, called the deviatoric part of η, is the projection of η into $\mathbb{M}^{n\times n}_D$.

Duality and Norms. If X is a Banach space and $u \in X$, we usually denote the norm of u by $\|u\|_X$. If X is $L^p(\Omega)$ or $L^p(\Omega;\mathbb{R}^n)$ the norm is denoted by $\|u\|_{L^p}$. If u, v are functions in $L^2(\Omega;\mathbb{R}^n)$ the scalar product of u and v is denoted by $\langle u, v\rangle_\Omega$. In general, if X is a Banach space, X' is its dual space and $\langle u, v\rangle_X$ denotes the duality product between $u \in X'$ and $v \in X$. The subscript X is sometimes omitted, if it is clear from the context.

If Γ is an oriented hypersurface in $\mathbb{R}^n$ and v, w are two $\mathbb{R}^n$-valued maps defined on Γ, we write

$$\langle v, w\rangle_\Gamma := \int_\Gamma v \cdot w \, d\mathcal{H}^{n-1},$$

where $\mathcal{H}^{n-1}$ denotes the $n-1$ dimensional Hausdorff measure.

3. Visco-Elasto-Plastic Evolution

3.1. **Kinematical Setting. The Reference Configuration.** The reference configuration is a bounded connected open set Ω in $\mathbb{R}^n$, $n \geq 2$, with Lipschitz boundary. We suppose that $\partial\Omega = \Gamma_0 \cup \Gamma_1 \cup \partial\Gamma$, where Γ_0, Γ_1, and $\partial\Gamma$ are pairwise disjoint, Γ_0 and Γ_1 are relatively open in $\partial\Omega$, and $\partial\Gamma$ is the relative boundary in $\partial\Omega$ both of Γ_0 and Γ_1. We assume that $\Gamma_0 \neq \emptyset$ and that $\mathcal{H}^{n-1}(\partial\Gamma) = 0$. On Γ_0 we will prescribe a Dirichlet condition on the displacement u, while on Γ_1 we will impose a Neumann condition on the stress σ.

Elastic and Plastic Strain. If u is the displacement, the linearized strain Eu is its symmetrized gradient, defined as the $\mathbb{M}^{n\times n}_{\mathrm{sym}}$-valued distribution with components $E_{ij}u = \frac{1}{2}(D_i u_j + D_j u_i)$. The linearized strain is decomposed as the sum of the elastic strain e and the plastic strain p. Given $w \in H^1(\Omega,\mathbb{R}^n)$, we say that a triple (u, e, p) is kinematically admissible for the visco-elasto-plastic problem with boundary datum w if $u \in H^1(\Omega;\mathbb{R}^n)$, $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$, $p \in L^2(\Omega;\mathbb{M}^{n\times n}_D)$, and

$$Eu = e + p \quad \text{on } \Omega, \tag{3.1a}$$

$$u|_{\Gamma_0} = w \quad \text{on } \Gamma_0. \tag{3.1b}$$

We denote the set of these triples by $A(w)$. It is convenient to introduce the subspace of $H^1(\Omega;\mathbb{R}^n)$ defined by

$$H^1_{\Gamma_0}(\Omega;\mathbb{R}^n) := \{u \in H^1(\Omega;\mathbb{R}^n) : u|_{\Gamma_0} = 0\}$$

and its dual space, denoted by $H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n)$. It is clear that $(u, e, p) \in A(w)$ if and only if $u - w \in H^1_{\Gamma_0}(\Omega;\mathbb{R}^n)$ and $Eu = e + p$, with $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$ and $p \in L^2(\Omega;\mathbb{M}^{n\times n}_D)$.

Stress and External Forces. In the visco-elasto-plastic model the stress σ depends linearly on the elastic part e of the strain Eu and on its time derivative $\dot{e}$. To express this dependence we introduce the elastic tensor A^0 and the visco-elastic tensor A^1. These are positive definite symmetric linear operators of $\mathbb{M}^{n\times n}_{\mathrm{sym}}$ into itself, therefore there exist positive constants α_0, α_1 and β_0, β_1 such that

$$|A^i\xi| \leq \beta_i|\xi|, \tag{3.2a}$$

$$A^i\xi \cdot \xi \geq \alpha_i|\xi|^2, \tag{3.2b}$$

for every $\xi \in \mathbb{M}^{n\times n}_{\rm sym}$ and $i = 0, 1$. The stress satisfies the constitutive relation

$$\sigma = A^0 e + A^1 \dot{e}. \tag{3.3}$$

The term $A^1\dot{e}$ in the equation above is the component of the stress due to internal frictions. To express the energy balance it is useful to introduce the quadratic forms

$$Q_0(\xi) = \frac{1}{2} A^0 \xi \cdot \xi \quad \text{and} \quad Q_1(\xi) = A^1 \xi \cdot \xi.$$

For every $e \in L^2(\Omega; \mathbb{M}^{n\times n}_{\rm sym})$ we define

$$\mathcal{Q}_0(e) = \int_\Omega Q_0(e)dx \quad \text{and} \quad \mathcal{Q}_1(e) = \int_\Omega Q_1(e)dx.$$

These function turn out to be lower semicontinuous with respect to the weak topology of $L^2(\Omega; \mathbb{M}^{n\times n}_{\rm sym})$. $\mathcal{Q}_0(e)$ represents the *stored elastic energy* associated to $e \in L^2(\Omega; \mathbb{M}^{n\times n}_{\rm sym})$ while $\mathcal{Q}_1(\dot{e})$ represents the rate of viscoelastic dissipation.

We assume that the time dependent body force $f(t)$ belongs to $L^2(\Omega; \mathbb{R}^n)$ and that the time dependent surface force $g(t)$ belongs to $L^2(\Gamma_1; \mathbb{R}^n)$. It is convenient to introduce the total load $\mathcal{L}(t) \in H^{-1}_{\Gamma_0}(\Omega; \mathbb{R}^n)$ of external forces acting on the body, defined by

$$\langle \mathcal{L}(t), u\rangle := \langle f(t), u\rangle_\Omega + \langle g(t), u\rangle_{\Gamma_1}, \tag{3.4}$$

where $\langle \cdot, \cdot\rangle$ denotes the duality pairing between $H^{-1}_{\Gamma_0}(\Omega; \mathbb{R}^n)$ and $H^1_{\Gamma_0}(\Omega; \mathbb{R}^n)$. When dealing with the visco-elasto-plastic problem we will only suppose that the total load $\mathcal{L}(t)$ belongs to $H^{-1}_{\Gamma_0}(\Omega; \mathbb{R}^n)$, without assuming the particular form (3.4). The hypotheses on the functions $t \mapsto \mathcal{L}(t)$ and $t \mapsto w(t)$ and the regularity of $t \mapsto (u(t), e(t), p(t))$ will be made precise in the statement of Theorems 3.1 and 3.3 below.

The law which expresses the second principle of dynamic is

$$\ddot{u}(t) - \operatorname{div}\sigma(t) = f(t) \quad \text{in } \Omega, \tag{3.5}$$

where we assume that the mass density of the elasto-plastic body is 1. Equation (3.5) is supplemented with the boundary conditions

$$u(t) = w(t) \quad \text{on } \Gamma_0, \tag{3.6a}$$

$$\sigma(t)\nu = g(t) \quad \text{on } \Gamma_1. \tag{3.6b}$$

To deal with (3.5) and (3.6), it is convenient to introduce the continuous linear operator $\operatorname{div}_{\Gamma_0} : L^2(\Omega; \mathbb{M}^{n\times n}_{\rm sym}) \to H^{-1}_{\Gamma_0}(\Omega; \mathbb{R}^n)$ defined by

$$\langle \operatorname{div}_{\Gamma_0}\sigma, \varphi\rangle := -\langle \sigma, E\varphi\rangle \tag{3.7}$$

for every $\sigma \in L^2(\Omega; \mathbb{M}^{n\times n}_{\rm sym})$ and every $\varphi \in H^1_{\Gamma_0}(\Omega; \mathbb{R}^n)$.

If $f(t)$, $g(t)$, $\sigma(t)$, $u(t)$, Γ_0, and Γ_1 are sufficiently regular and $\mathcal{L}(t)$ is the total external load defined by (3.4), then we can prove, using integration by parts, that (3.5) and (3.6b) are equivalent to

$$\ddot{u}(t) - \operatorname{div}_{\Gamma_0}\sigma(t) = \mathcal{L}(t), \tag{3.8}$$

interpreted as equality between elements of $H^{-1}_{\Gamma_0}(\Omega; \mathbb{R}^n)$. In other words (3.8) is satisfied if and only if

$$\langle \ddot{u}(t), \varphi\rangle + \langle \sigma(t), E\varphi\rangle = \langle \mathcal{L}(t), \varphi\rangle \tag{3.9}$$

for every $\varphi \in H^1_{\Gamma_0}(\Omega; \mathbb{R}^n)$. In the irregular case, equation (3.9) represents the weak formulation of problem (3.5) with boundary condition (3.6b).

Plastic Dissipation. The elastic domain K is a convex and compact set in $\mathbb{M}^{n\times n}_D$. We will suppose that there exist two positive real numbers $r_1 < R_1$ such that

$$B(0, r_1) \subseteq K \subseteq B(0, R_1). \tag{3.10}$$

It is convenient to introduce the set

$$\mathcal{K}(\Omega) := \{\xi \in L^2(\Omega; \mathbb{M}^{n\times n}_D) : \xi(x) \in K \text{ for a.e. } x \in \Omega\}. \tag{3.11}$$

If π_K denotes the minimal distance projection of $\mathbb{M}^{n\times n}_D$ into K, and $\pi_{\mathcal{K}(\Omega)}$ denotes the projection of $L^2(\Omega; \mathbb{M}^{n\times n}_D)$ into $\mathcal{K}(\Omega)$, then it is easy to check that

$$(\pi_{\mathcal{K}(\Omega)}\xi)(x) = \pi_K \xi(x) \quad \text{for a.e. } x \in \Omega, \tag{3.12}$$

for every $\xi \in L^2(\Omega; \mathbb{M}^{n\times n}_D)$.

The evolution of the plastic strain $p(t,x)$ will be expressed by the Maximum Dissipation Principle, or Principle of Maximum Work of Hill (see, e.g., [11], [18], [27]): if σ is the stress, then p will satisfy the following

$$\begin{aligned} &(\sigma_D(t,x) - \xi)\cdot \dot p(t,x) \ge 0 \quad \text{for every } \xi \in K \text{ and a.e. } x \text{ in } \Omega \\ &\sigma_D(t,x) - \dot p(t,x) \in K, \quad \text{for a.e. } x \text{ in } \Omega, \end{aligned}$$

where we assume for simplicity that the viscosity coefficient is 1. Thanks to the characterization of the projection onto convex sets (see, e.g., [12]), this condition is satisfied if and only if $\sigma_D(t,x) - \dot p(t,x)$ coincides with $\pi_K \sigma_D(t,x)$, for a.e. $x \in \Omega$. By (3.12), this can be written as

$$\dot p(t) = \sigma_D(t) - \pi_{\mathcal{K}(\Omega)}\sigma_D(t). \tag{3.13}$$

We define the support function $H : \mathbb{M}^{n\times n}_D \to [0, +\infty[$ of K by

$$H(\xi) = \sup_{\zeta\in K} \zeta\cdot\xi. \tag{3.14}$$

It turns out that H is convex and positively homogeneous of degree one. In particular it satisfies the triangle inequality

$$H(\xi+\zeta) \le H(\xi) + H(\zeta)$$

and the following inequality, due to (3.10):

$$r_1|\xi| \le H(\xi) \le R_1|\xi|. \tag{3.15}$$

We define $\mathcal{H} : L^2(\Omega; \mathbb{M}^{n\times n}_D) \to \mathbb{R}$ by

$$\mathcal{H}(p) = \int_\Omega H(p(x))dx. \tag{3.16}$$

If $p \in H^1([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_D))$ and $\dot p(t)$ is its time derivative, then $\mathcal{H}(\dot p)$ represents the rate of plastic dissipation, so that,

$$\int_0^T \mathcal{H}(\dot p)dt \tag{3.17}$$

is the total plastic dissipation in the time interval $[0,T]$.

We notice that, by the definition of H, the subdifferential of H satisfies (see e.g. [25, Theorem 13.1])

$$\partial H(0) = K. \tag{3.18}$$

From (3.18), it easily follows

$$\partial\mathcal{H}(0) = \mathcal{K}(\Omega), \tag{3.19}$$

where $\partial\mathcal{H}(\xi)$ denotes the subdifferential of $\mathcal{H}$ at ξ.

3.2. **Existence Results for Elasto-Visco-Plastic Evolutions.** Given an elasto-visco-plastic body Ω in $\mathbb{R}^n$ satisfying all the properties described in the previous section, we fix an external load $\mathcal{L}$ and a Dirichlet boundary datum w, and look for a solution of the dynamic equation (3.8) and of the flow rule (3.13), with stress σ defined by (3.3) and strain satisfying equation (3.1). The main existence result for an elasto-visco-plastic evolution is the following theorem.

Theorem 3.1. *Let $T>0$, let $\mathcal{L}\in L^2([0,T];H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n))$, and let w be a function such that*

$$w\in L^\infty([0,T];H^1(\Omega;\mathbb{R}^n)), \tag{3.20a}$$
$$\dot w\in C^0([0,T];L^2(\Omega;\mathbb{R}^n))\cap L^2([0,T];H^1(\Omega;\mathbb{R}^n)), \tag{3.20b}$$
$$\ddot w\in L^2([0,T];H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n)). \tag{3.20c}$$

Then for every $(u_0,e_0,p_0)\in A(w(0))$ and $v_0\in L^2(\Omega;\mathbb{R}^n)$ there exists a unique quadruple (u,e,p,σ) of functions, with

$$u\in L^\infty([0,T];H^1(\Omega;\mathbb{R}^n)), \tag{3.21a}$$
$$\dot u\in L^\infty([0,T];L^2(\Omega;\mathbb{R}^n))\cap L^2([0,T];H^1(\Omega;\mathbb{R}^n)), \tag{3.21b}$$
$$\ddot u\in L^2([0,T];H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n)), \tag{3.21c}$$
$$e\in L^\infty([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{sym})), \tag{3.21d}$$
$$p\in L^\infty([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{D})), \tag{3.21e}$$
$$\dot e\in L^2([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{sym})), \tag{3.21f}$$
$$\dot p\in L^2([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{D})), \tag{3.21g}$$
$$\sigma\in L^2([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{sym})), \tag{3.21h}$$

such that for a.e. $t\in[0,T]$ we have

$$Eu(t)=e(t)+p(t), \tag{3.22a}$$
$$\sigma(t)=A^0e(t)+A^1\dot e(t), \tag{3.22b}$$
$$\ddot u(t)-\mathrm{div}_{\Gamma_0}\sigma(t)=\mathcal{L}(t), \tag{3.22c}$$
$$\dot p(t)=\sigma_D(t)-\pi_{\mathcal{K}(\Omega)}\sigma_D(t), \tag{3.22d}$$

and

$$u(t)=w(t)\quad on\ \Gamma_0, \tag{3.23}$$
$$u(0)=u_0,\ \ e(0)=e_0,\ \ p(0)=p_0,\ \ \dot u(0)=v_0. \tag{3.24}$$

Moreover (u,e,p,σ) satisfies the equilibrium condition

$$\begin{aligned}-\mathcal{H}(q)&\le\langle A^0e(t),\eta\rangle+\langle A^1\dot e(t),\eta\rangle+\langle\dot p(t),q\rangle\\&\quad+\langle\ddot u(t),\varphi\rangle-\langle\mathcal{L}(t),\varphi\rangle\le\mathcal{H}(-q),\end{aligned} \tag{3.25}$$

for a.e. $t\in[0,T]$ and for every $(\varphi,\eta,q)\in A(0)$, where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n)$ and $H^1_{\Gamma_0}(\Omega;\mathbb{R}^n)$ in the terms containing $\ddot u$ and $\mathcal{L}$, while it denotes the scalar product in L^2 in all other terms.

Remark 3.2. In view of (3.20) and (3.21) we see that u, w, $\dot u$, $\dot w$, e and p are absolutely continuous, i.e.,

$$u,w\in AC([0,T];H^1(\Omega;\mathbb{R}^n)), \tag{3.26a}$$
$$\dot u,\dot w\in AC([0,T];H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n)), \tag{3.26b}$$
$$e\in AC([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})), \tag{3.26c}$$
$$p\in AC([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{D})) \tag{3.26d}$$

(see, e.g., [4], Proposition A.3 and following Corollary). Moreover [31, Proposition 23.23] implies that

$$\dot u - \dot w \in C^0([0,T];L^2(\Omega;\mathbb{R}^n)), \tag{3.27a}$$

$$\|\dot u - \dot w\|^2_{L^2} \in AC([0,T]), \tag{3.27b}$$

$$\tfrac{d}{dt}\|\dot u - \dot w\|^2_{L^2} = 2\langle \ddot u - \ddot w, \dot u - \dot w\rangle \text{ a.e. } t\in[0,T], \tag{3.27c}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n)$ and $H^1_{\Gamma_0}(\Omega;\mathbb{R}^n)$. Since $\dot w \in C^0([0,T];L^2(\Omega;\mathbb{R}^n))$, from (3.27a) we obtain

$$\dot u \in C^0([0,T];L^2(\Omega;\mathbb{R}^n)). \tag{3.28}$$

This property gives a precise meaning to the initial conditions (3.24).

Before proving Theorem 3.1 we will first state the following result, which characterizes the solutions of equations (3.22c) and (3.22d).

Theorem 3.3. *Under the hypotheses of Theorem 3.1, we assume that* (u,e,p,σ) *satisfies* (3.21)*,* (3.22a)*,* (3.22b)*,* (3.23)*, and* (3.24)*. Then* (u,e,p,σ) *satisfies* (3.22c) *and* (3.22d) *for a.e.* $t\in[0,T]$ *if and only if both the following conditions hold:*

(a) *Energy balance: for every* $t\in[0,T]$ *we have*

$$\mathcal{Q}_0(e(t)) + \frac12\|\dot u(t)-\dot w(t)\|^2_{L^2} + \int_0^t \mathcal{Q}_1(\dot e)ds + \int_0^t \|\dot p\|^2_{L^2}ds + \int_0^t \mathcal{H}(\dot p)ds =$$
$$= \int_0^t \langle \sigma, E\dot w\rangle ds + \int_0^t \langle \mathcal{L}-\ddot w, \dot u - \dot w\rangle ds + \mathcal{Q}_0(e_0) + \frac12\|v_0 - \dot w(0)\|^2_{L^2}, \tag{3.29}$$

(b) *For a.e.* $t\in[0,T]$ *the equilibrium condition* (3.25) *holds for every* $(\varphi,\eta,q)\in A(0)$.

Moreover, if the two previous conditions are satisfied, then

$$\langle \sigma_D(t) - \dot p(t), \dot p(t)\rangle = \mathcal{H}(\dot p(t)), \tag{3.30}$$

for a.e. $t\in[0,T]$.

Remark 3.4. If the data w and $\mathcal{L}$ are sufficiently regular and $\mathcal{L}$ has the form (3.4) with $f\in L^2([0,T];L^2(\Omega;\mathbb{R}^n))$ and $g\in L^\infty([0,T];L^2(\Gamma_1;\mathbb{R}^n))$, then we can integrate by parts the terms $\int_0^t\langle \ddot w, \dot u\rangle ds$ and $\int_0^t\langle \ddot w, \dot w\rangle ds$ obtaining that we can rewrite the energy balance formula as follows:

$$\mathcal{Q}_0(e(t)) + \frac12\|\dot u(t)\|^2_{L^2} + \int_0^t \mathcal{Q}_1(\dot e)ds + \int_0^t \|\dot p\|^2_{L^2}ds + \int_0^t \mathcal{H}(\dot p)ds =$$
$$= \int_0^t \langle \sigma, E\dot w\rangle ds + \int_0^t \langle f, \dot u - \dot w\rangle ds + \int_0^t \langle g, \dot u - \dot w\rangle_{\Gamma_1} ds$$
$$+ \int_0^t \langle \ddot u, \dot w\rangle ds + \mathcal{Q}_0(e_0) + \frac12\|v_0\|^2_{L^2},$$

which becomes, using $\ddot u = \mathrm{div}_{\Gamma_0}\sigma + \mathcal{L}$:

$$\mathcal{Q}_0(e(t)) + \frac12\|\dot u(t)\|^2_{L^2} + \int_0^t \mathcal{Q}_1(\dot e)ds + \int_0^t \|\dot p\|^2_{L^2}ds + \int_0^t \mathcal{H}(\dot p)ds =$$
$$= \int_0^t \langle \sigma\nu, \dot u\rangle_{\Gamma_0} ds + \int_0^t \langle f, \dot u\rangle ds + \int_0^t \langle g, \dot u\rangle_{\Gamma_1} ds + \mathcal{Q}_0(e_0) + \frac12\|v_0\|^2_{L^2},$$

where we have used $\dot u = \dot w$ on Γ_0. This is the usual formulation of the energy conservation law. Indeed $\mathcal{Q}_0(e(t))$ is the stored elastic energy, $\frac12\|\dot u(t)\|^2_{L^2}$ is the kinetic energy, $\int_0^t \mathcal{Q}_1(\dot e(t))ds$ is the visco-elastic dissipation, $\int_0^t \|\dot p\|^2_{L^2}ds$ is the visco-plastic dissipation, and $\int_0^t \mathcal{H}(\dot p)ds$ is the plastic dissipation. On the right-hand side the terms $\int_0^t \langle \sigma\nu, \dot u\rangle_{\Gamma_0} ds$, $\int_0^t \langle g, \dot u\rangle_{\Gamma_1} ds$, and $\int_0^t \langle f, \dot u\rangle ds$ represent the work done by the external forces on the Dirichlet boundary, on the Neumann boundary, and on the body itself, while the two terms $\mathcal{Q}_0(e_0)$ and $\frac{1}{2}\|v_0\|^2_{L^2}$ are the stored elastic energy and the kinetic energy at the initial time.

Proof of Theorem 3.3. Let us suppose that the quadruple (u, e, p, σ) satisfies (3.25) and (3.29); let us prove (3.22c). Let $\varphi \in H^1_{\Gamma_0}(\Omega;\mathbb{R}^n)$; since $(\varphi, E\varphi, 0) \in A(0)$, we choose $\eta = E(\varphi)$ and $q = 0$ in (3.25) and for a.e. $t \in [0,T]$ we get

$$\langle A^0 e(t) + A^1 \dot e(t), E\varphi\rangle + \langle \ddot u(t), \varphi\rangle - \langle \mathcal{L}(t), \varphi\rangle = 0,$$

which is equivalent to (3.22c), thanks to (3.9) and (3.22b).

It remains to prove (3.22d). Choosing $(0, q, -q) \in A(0)$ in (3.25) for some $q \in L^2(\Omega, \mathbb{M}^{n\times n}_D)$, for a.e. $t \in [0,T]$ we get

$$-\mathcal{H}(-q) \leq \langle A^0 e(t) + A^1 \dot e(t), q\rangle - \langle \dot p(t), q\rangle \leq \mathcal{H}(q),$$

which, by (3.22b), says that

$$\sigma_D(t) - \dot p(t) \in \partial\mathcal{H}(0) = \mathcal{K}(\Omega) \tag{3.31}$$

thanks to the arbitraryness of q (see (3.19)).

By (3.20), (3.21), and (3.23) for a.e. $t \in [0,T]$ the function $\varphi := \dot u(t) - \dot w(t)$ belongs to $H^1_{\Gamma_0}(\Omega;\mathbb{R}^n)$. Then we use this function in (3.9) and integrate with respect to time, taking into account (3.22a), (3.22b), (3.26c), (3.27b), and (3.27c). We finally get

$$\begin{aligned} &\mathcal{Q}_0(e(t)) - \mathcal{Q}_0(e_0) + \int_0^t \mathcal{Q}_1(\dot e)ds - \int_0^t \langle \sigma, E\dot w\rangle ds + \int_0^t \langle \sigma_D, \dot p\rangle ds \\ &+ \frac{1}{2}\|\dot u(t) - \dot w(t)\|^2_{L^2} - \frac{1}{2}\|v_0 - \dot w(0)\|^2_{L^2} - \int_0^t \langle \mathcal{L} - \ddot w, \dot u - \dot w\rangle ds = 0. \end{aligned} \tag{3.32}$$

for every $t \in [0,T]$. This equality, together with the energy balance (3.29), implies that (3.30) holds for a.e. $t \in [0,T]$. As a consequence, by the definition of $\mathcal{H}$, we deduce that for a.e. $t \in [0,T]$ and for every $\xi \in \mathcal{K}(\Omega)$ we have

$$\langle \sigma_D(t) - \dot p(t), \dot p(t)\rangle \geq \langle \xi, \dot p(t)\rangle,$$

which is equivalent to

$$\langle \sigma_D(t) - (\sigma_D(t) - \dot p(t)), \xi - (\sigma_D(t) - \dot p(t))\rangle \leq 0.$$

Thanks to (3.31), $\sigma_D(t) - \dot p(t)$ belongs to $\mathcal{K}(\Omega)$; therefore the arbitrariness of ξ and the well-known characterization of the projection onto convex sets (see, e.g., [12], Chapter 1.2) give that $\sigma_D(t) - \dot p(t) = \pi_{\mathcal{K}(\Omega)}\sigma_D(t)$ for a.e. $t \in [0,T]$.

Conversely suppose (u, e, p, σ) to be a solution of the system of equations (3.22). Formula (3.25) is proved in Theorem 3.1. In order to get the energy balance we first prove that, if a function (u, e, p, σ) satisfies (3.22), then (3.30) holds. Indeed, if $\xi \in \mathcal{K}(\Omega)$, then from the properties of convex sets it follows that for a.e. $t \in [0,T]$

$$\begin{aligned} &(\sigma_D - \dot p)\cdot \dot p = \pi_K\sigma_D \cdot (\sigma_D - \pi_K\sigma_D) \geq \\ &\geq \pi_K\sigma_D \cdot (\sigma_D - \pi_K\sigma_D) + (\xi - \pi_K\sigma_D)\cdot(\sigma_D - \pi_K\sigma_D) = \xi \cdot (\sigma_D - \pi_K\sigma_D) \end{aligned}$$

almost everywhere in Ω, that is $(\sigma_D - \dot p)\cdot \dot p \geq H(\sigma_D - \pi_K\sigma_D) = H(\dot p)$ thanks to the definition of H. Since $\sigma_D - \dot p \in K$ a.e. in Ω and for a.e. $t \in [0,T]$ by (3.22d), the definition of H gives also the opposite inequality. So integrating on Ω we get (3.30).

Choosing again $\varphi = \dot u(t) - \dot w(t)$ in (3.9) and integrating with respect to time, we get (3.32), which together with (3.30) gives the energy balance (3.29).

□

Proof of Theorem 3.1. We will obtain the solution by time discretization, considering the limit of approximate solutions constructed by solving incremental minimum problems. Given an integer $N > 0$ we define $\tau = T/N$ and subdivide the interval $[0, T)$ into N subintervals $[t_i, t_{t+1})$, $i = 0, \dots, N-1$ of length τ, with $t_i = i\tau$. Let us set

$$u_{-1} = u_0 - \tau v_0, \quad w_{-1} = w_0 - \tau \dot{w}(0),$$
$$w_i = w(t_i), \qquad \mathcal{L}_i = \frac{1}{\tau}\int_{t_i}^{t_{i+1}} \mathcal{L}(s)ds.$$

We construct a sequence (u_i, e_i, p_i) with $i = 0, 1, \dots, N$ by induction. First (u_0, e_0, p_0) coincides with the initial data (3.24). Let us fix i and let us suppose $(u_j, e_j, p_j) \in A(w_j)$ to have been defined for $j = 0, \dots, i$. Then $(u_{i+1}, e_{i+1}, p_{i+1})$ is defined as the unique minimizer on $A(w_{i+1})$ of the functional

$$\begin{aligned} V_i(u, e, p) =& \frac{1}{2}\langle A^0 e, e\rangle + \frac{1}{2\tau}\langle A^1(e - e_i), e - e_i\rangle + \frac{1}{2\tau}\|p - p_i\|^2_{L^2} \\ &+ \mathcal{H}(p - p_i) + \frac{1}{2}\|\frac{u - u_i}{\tau} - \frac{u_i - u_{i-1}}{\tau}\|^2_{L^2} - \langle \mathcal{L}_i, u\rangle, \end{aligned} \tag{3.33}$$

which turns out to be coercive and strictly convex on $A(w_{i+1})$.

To obtain the Euler conditions we observe that $(u_{i+1}, e_{i+1}, p_{i+1}) + \lambda(\varphi, \eta, q) \in A(w_{i+1})$ for every $(\varphi, \eta, q) \in A(0)$, and for every $\lambda \in \mathbb{R}$. Evaluating V_i in this point and differentiating with respect to λ at $0^{\pm}$ we get

$$\begin{aligned} -\mathcal{H}(q) \leq& \langle A^0 e_{i+1}, \eta\rangle + \frac{1}{\tau}\langle A^1(e_{i+1} - e_i), \eta\rangle + \frac{1}{\tau}\langle p_{i+1} - p_i, q\rangle \\ &+ \frac{1}{\tau}\langle v_{i+1} - v_i, \varphi\rangle - \langle \mathcal{L}_i, \varphi\rangle \leq \mathcal{H}(-q), \end{aligned} \tag{3.34}$$

where we have set

$$v_j = \frac{1}{\tau}(u_j - u_{j-1}). \tag{3.35}$$

We now define the piecewise affine interpolation $u_\tau, e_\tau, p_\tau, w_\tau$ on $[0, T]$ by

$$u_\tau(t) = u_i + \frac{u_{i+1} - u_i}{\tau}(t - t_i) \qquad \text{if } t \in [t_i, t_{i+1}) \tag{3.36a}$$
$$e_\tau(t) = e_i + \frac{e_{i+1} - e_i}{\tau}(t - t_i) \qquad \text{if } t \in [t_i, t_{i+1}) \tag{3.36b}$$
$$p_\tau(t) = p_i + \frac{p_{i+1} - p_i}{\tau}(t - t_i) \qquad \text{if } t \in [t_i, t_{i+1}) \tag{3.36c}$$
$$w_\tau(t) = w_i + \frac{w_{i+1} - w_i}{\tau}(t - t_i) \qquad \text{if } t \in [t_i, t_{i+1}) \tag{3.36d}$$

The proof now is divided into four steps: in the first one we obtain that a subsequence of (u_τ, e_τ, p_τ) has a limit (u, e, p) as $\tau \to 0$, and we show that such a limit satisfies the regularity conditions (3.21). In the second step we pass to the limit in (3.34), obtaining the equilibrium condition (3.25). In the third step we obtain the energy balance (3.29) for (u, e, p). From this and Theorem 3.3 it will follow that (u, e, p) satisfies the required equations (3.22). In the last step we prove the uniqueness.

Step 1. To simplify the notation we set $\omega_i = \frac{1}{\tau}(w_i - w_{i-1})$ and define, for $t \in [0, T]$,

$$\omega_\tau(t) = \omega_i + (\omega_{i+1} - \omega_i)\frac{t - t_i}{\tau} \qquad \text{if } t \in [t_i, t_{i+1}), \tag{3.37a}$$
$$v_\tau(t) = v_i + (v_{i+1} - v_i)\frac{t - t_i}{\tau} \qquad \text{if } t \in [t_i, t_{i+1}). \tag{3.37b}$$

We shall use the three following identities:

$$\langle A^0 e_{i+1}, e_{i+1} - e_i\rangle = \int_{t_i}^{t_{i+1}} \langle A^0 e_\tau, \dot e_\tau\rangle ds + \frac{\tau}{2}\int_{t_i}^{t_{i+1}} \langle A^0 \dot e_\tau, \dot e_\tau\rangle ds, \tag{3.38}$$

$$\begin{aligned}&\langle A^0 e_{i+1}, Ew_{i+1} - Ew_i\rangle = \\ &\quad = \int_{t_i}^{t_{i+1}} \langle A^0 e_\tau, E\dot w_\tau\rangle ds + \frac{\tau}{2}\int_{t_i}^{t_{i+1}} \langle A^0 \dot e_\tau, E\dot w_\tau\rangle ds,\end{aligned} \tag{3.39}$$

$$\begin{aligned}&\langle (v_{i+1} - v_i) - (\omega_{i+1} - \omega_i), v_{i+1} - \omega_{i+1}\rangle = \\ &\quad = \frac{1}{2}\|v_{i+1} - \omega_{i+1}\|^2_{L^2} - \frac{1}{2}\|v_i - \omega_i\|^2_{L^2} + \frac{\tau}{2}\int_{t_i}^{t_{i+1}} \|\dot v_\tau - \dot\omega_\tau\|^2_{L^2} ds.\end{aligned} \tag{3.40}$$

We put

$$\begin{aligned}\varphi &= u_{i+1} - u_i - (w_{i+1} - w_i),\\ \eta &= e_{i+1} - e_i - (Ew_{i+1} - Ew_i),\\ q &= p_{i+1} - p_i,\end{aligned} \tag{3.41}$$

into (3.34) and take the sum over $i = 0, \dots, j-1$. Using (3.38)-(3.40) we get

$$\begin{aligned}&\int_0^{t_j} \langle A^0 e_\tau, \dot e_\tau\rangle ds + \frac{\tau}{2}\int_0^{t_j} \langle A^0 \dot e_\tau, \dot e_\tau\rangle ds + \int_0^{t_j} \langle A^1 \dot e_\tau, \dot e_\tau\rangle ds \\ &+ \int_0^{t_j} \|\dot p_\tau\|^2_{L^2} ds + \frac{\tau}{2}\int_0^{t_j} \|\dot v_\tau - \dot\omega_\tau\|^2 ds + \frac{1}{2}\|v_\tau(t_{j+1}) - \omega_\tau(t_{j+1}))\|^2 \le \\ &\le \int_0^{t_j} \mathcal{H}(-\dot p_\tau) ds + \int_0^{t_j} \langle \mathcal{L} - \dot\omega_\tau, \dot u_\tau - \dot w_\tau\rangle ds \\ &+ \frac{1}{2}\|v_0 - \omega_0\|^2 + \int_0^t \langle A^0 e_\tau + A^1 \dot e_\tau + \frac{\tau}{2} A^0 \dot e_\tau, E\dot w_\tau\rangle ds,\end{aligned} \tag{3.42}$$

By (3.15) there exists a constant C such that $\mathcal{H}(q) \le C\|q\|_{L^2}$ for every $q \in L^2(\Omega; \mathbb{M}^{n\times n}_D)$. Therefore we obtain

$$\begin{aligned}&\frac{1}{2}\langle A^0 e_\tau(t_j), e_\tau(t_j)\rangle + \int_0^{t_j} \langle A^1 \dot e_\tau, \dot e_\tau\rangle ds \\ &+ \int_0^{t_j} \|\dot p_\tau\|^2_{L^2} ds + \frac{1}{2}\|v_\tau(t_{j+1}) - \omega_\tau(t_{j+1})\|^2_{L^2} \le \\ &\le C\int_0^{t_j} \|\dot p_\tau\|_{L^2} dt + \frac{1}{2\lambda}\int_0^{t_j} \|\mathcal{L} - \dot\omega_\tau\|^2_{H^{-1}_{\Gamma_0}} ds + \frac{\lambda}{2}\int_0^{t_j} \|\dot u_\tau - \dot w_\tau\|^2_{H^1_{\Gamma_0}} ds \\ &+ \frac{3}{2\lambda}\int_0^{t_j} \|E\dot w_\tau\|^2_{L^2} ds + \frac{\lambda}{2}\int_0^{t_j} \|A^0 e_\tau\|^2_{L^2} ds \\ &+ \frac{\lambda}{2}\int_0^{t_j} \|A^1 \dot e_\tau\|^2_{L^2} ds + \frac{\tau\lambda}{4}\int_0^{t_j} \|A^0 \dot e_\tau\|^2_{L^2} ds + D,\end{aligned} \tag{3.43}$$

where λ is an arbitrary positive number, that we will choose later, and C and D are positive constants independent of λ.

Since $\ddot w \in L^2([0,T]; H^{-1}_{\Gamma_0}(\Omega; \mathbb{R}^n))$ and $\dot w \in L^2([0,T]; H^1(\Omega; \mathbb{R}^n))$, we see that

$$\dot w_\tau \to \dot w \text{ strongly in } L^2([0,T]; H^1(\Omega; \mathbb{R}^n)), \tag{3.44a}$$

$$\omega_\tau \to \dot w \text{ strongly in } L^2([0,T]; H^1(\Omega; \mathbb{R}^n)), \tag{3.44b}$$

$$\dot\omega_\tau \to \ddot w \text{ strongly in } L^2([0,T]; H^{-1}_{\Gamma_0}(\Omega; \mathbb{R}^n)). \tag{3.44c}$$

The proof of the first two properties is straightforward. To prove (3.44c) we first put $\tilde w_\tau(t) := \frac{1}{\tau}\int_{t_i}^{t_{i+1}} \ddot w(s) ds \in H^{-1}_{\Gamma_0}(\Omega; \mathbb{R}^n))$ for $t \in [t_i, t_{i+1})$. Since $\tilde w_\tau$ tends to $\ddot w$, it suffices to show that $\tilde{w}_\tau - \dot{\omega}_\tau$ tends to 0 strongly in $L^2([0,T]; H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n))$. So we write

$$\|\dot{\omega}_\tau - \tilde{w}_\tau\|^2_{L^2(H^{-1}_{\Gamma_0})} = \sum_{i=0}^{N-1} \tau \Big\| \frac{1}{\tau}\int_{t_i}^{t_{i+1}} \Big(\frac{1}{\tau}\int_{s-\tau}^{s} \ddot{w}(r)dr - \ddot{w}(s)\Big)ds\Big\|^2_{H^{-1}_{\Gamma_0}} \leq$$
$$\leq \frac{1}{\tau}\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}}\int_{s-\tau}^{s} \|\ddot{w}(r) - \ddot{w}(s)\|^2_{H^{-1}_{\Gamma_0}} drds \leq$$
$$\leq \frac{1}{\tau}\sum_{i=0}^{N-1} \int_{t_{i-1}}^{t_{i+1}}\int_{t_{i-1}}^{t_{i+1}} \|\ddot{w}(r) - \ddot{w}(s)\|^2_{H^{-1}_{\Gamma_0}} drds,$$

where we set $\ddot{w}(s) = 0$ for $s < 0$. Defining $W(r,s) = \|\ddot{w}(r) - \ddot{w}(s)\|^2_{H^{-1}_{\Gamma_0}}$, we see that the integral in the last line is bounded by

$$\frac{2}{\tau}\int_{2\tau}^{2\tau} dh \int_0^T W(r, r+h)dr,$$

that turns out to go to 0 as $\tau \to 0$, because $h \mapsto \int_0^T W(r, r+h)dr$ is continuous and vanishes at $h = 0$.

Therefore from (3.44) we see that the term

$$\int_0^{t_j} \|\mathcal{L} - \dot{\omega}_\tau\|^2_{H^{-1}_{\Gamma_0}} ds + \int_0^{t_j} \|E\dot{w}_\tau\|^2_{L^2} ds$$

is bounded from above. By Poincaré and Korn inequalities there exists a constant γ such that

$$\|\dot{u}_\tau - \dot{w}_\tau\|^2_{H^1_{\Gamma_0}} \leq \gamma(\|\dot{e}_\tau\|^2_{L^2} + \|\dot{p}_\tau\|^2_{L^2} + \|E\dot{w}_\tau\|^2_{L^2}),$$

and since for some constant $C_1 > 0$

$$C\int_0^{t_j} \|\dot{p}_\tau\|_{L^2} ds \leq C_1 + \frac{1}{2}\int_0^{t_j} \|\dot{p}_\tau\|^2_{L^2} ds,$$

using formula (3.2) we get from (3.43)

$$\frac{\alpha_0}{2}\|e_\tau(t_j)\|^2_{L^2} + \alpha_1\int_0^{t_j} \|\dot{e}_\tau\|^2_{L^2} ds + \frac{1}{2}\int_0^{t_j} \|\dot{p}_\tau\|^2_{L^2} ds$$
$$+ \frac{1}{2}\|v_\tau(t_{j+1}) - \omega_\tau(t_{j+1})\|^2_{L^2} \leq$$
$$\leq \frac{\lambda\gamma}{2}\int_0^{t_j} \|\dot{e}_\tau\|^2_{L^2} ds + \frac{\lambda\gamma}{2}\int_0^{t_j} \|\dot{p}_\tau\|^2_{L^2} ds + \frac{\lambda\beta_0^2}{2}\int_0^{t_j} \|e_\tau\|^2_{L^2} ds$$
$$+ \frac{\lambda\beta_1^2}{2}\int_0^{t_j} \|\dot{e}_\tau\|^2_{L^2} ds + T\frac{\lambda\beta_0^2}{4}\int_0^{t_j} \|\dot{e}_\tau\|^2_{L^2} ds + M_\lambda,$$

where M_λ is a constant depending on λ. Choosing now λ in such a way that $\lambda\beta_0^2 < \alpha_0$, $2\lambda\gamma < 1$, and $2\lambda\gamma + 2\lambda\beta_1^2 + T\lambda\beta_0^2 < 2\alpha_1$ we obtain

$$\frac{\alpha_0}{2}\|e_\tau(t_j)\|^2_{L^2} + \frac{\alpha_1}{2}\int_0^{t_j} \|\dot{e}_\tau\|^2_{L^2} ds + \frac{1}{4}\int_0^{t_j} \|\dot{p}_\tau\|^2_{L^2} ds$$
$$+ \frac{1}{2}\|v_\tau(t_{j+1}) - \omega_\tau(t_{j+1})\|^2_{L^2} \leq \frac{\alpha_0}{2}\int_0^{t_j} \|e_\tau\|^2_{L^2} ds + M_\lambda. \tag{3.45}$$

Now neglecting some non-negative terms in the left-hand side we get

$$\|e_\tau(t)\|^2_{L^2} \leq K + \int_0^t \|e_\tau\|^2_{L^2} ds \qquad \forall t \in [0,T] \tag{3.46}$$

K being a positive constant independent of τ. So we can use Gronwall lemma to obtain that e_τ is bounded in $L^\infty([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym}))$ uniformly with respect to τ. Going back to (3.45) we also obtain:

$$\dot u_\tau \in L^\infty([0,T];L^2(\Omega;\mathbb{R}^n)), \tag{3.47a}$$
$$e_\tau \in L^\infty([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym})), \tag{3.47b}$$
$$\dot e_\tau \in L^2([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym})), \tag{3.47c}$$
$$\dot p_\tau \in L^2([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_D)), \tag{3.47d}$$

and $\dot u_\tau, e_\tau, \dot e_\tau, \dot p_\tau$ are bounded in these spaces uniformly with respect to τ. For the first condition above we have used that, as a consequence of (3.20) and (3.37), ω_τ is uniformly bounded in $L^\infty([0,T];L^2(\Omega;\mathbb{R}^n))$. We can then pass to the limit as τ tends to 0 in a subsequence, and find functions v, e, h and q such that

$$\dot u_\tau \rightharpoonup v \quad \text{weakly* in } L^\infty([0,T];L^2(\Omega;\mathbb{R}^n)), \tag{3.48a}$$
$$e_\tau \rightharpoonup e \quad \text{weakly* in } L^\infty([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym})), \tag{3.48b}$$
$$\dot e_\tau \rightharpoonup h \quad \text{weakly in } L^2([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym})), \tag{3.48c}$$
$$\dot p_\tau \rightharpoonup q \quad \text{weakly in } L^2([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_D)). \tag{3.48d}$$

Integrating by parts we get $h = \dot e$ almost everywhere in $[0,T]$; thanks to the properties of the distributional derivatives of functions on a real interval into a Banach space (see [4], Appendix), we obtain that e is absolutely continuous and its strong derivative coincides with h a.e. in $[0,T]$.

From the estimates (3.48) and from the equalities $u_\tau(t) = \int_0^t \dot u_\tau ds + u_0$ and $p_\tau(t) = \int_0^t \dot p_\tau ds + p_0$ it follows that

$$u_\tau \text{ is bounded in } L^\infty([0,T];L^2(\Omega;\mathbb{R}^n)), \tag{3.49a}$$
$$p_\tau \text{ is bounded in } L^\infty([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_D)), \tag{3.49b}$$
$$u_\tau(t) \rightharpoonup u(t) := \int_0^t v(s)ds + u_0 \quad \text{weakly in } L^2(\Omega;\mathbb{R}^n), \tag{3.49c}$$
$$p_\tau(t) \rightharpoonup p(t) := \int_0^t q(s)ds + p_0 \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{n\times n}_D) \tag{3.49d}$$

for every $t \in [0,T]$. Note that, by (3.48a) and (3.48d) we deduce that

$$u \in L^\infty([0,T];L^2(\Omega;\mathbb{R}^n)), \tag{3.50a}$$
$$p \in L^\infty([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_D)). \tag{3.50b}$$

Arguing in a similar way for e and h we see

$$e_\tau(t) \rightharpoonup e(t) := \int_0^t \dot e(s)ds + e_0 \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym}) \tag{3.51}$$

for every $t \in [0,T]$.

In view of (3.49) we see that u and p are absolutely continuous and that their derivatives with respect to t coincide with v and q almost everywhere in $[0,T]$. Moreover from (3.47a) and by definition of v_τ we see that also v_τ is uniformly bounded in $L^\infty([0,T];L^2(\Omega;\mathbb{R}^n))$ and we claim that it tends weakly* to $v = \dot u$ in $L^\infty([0,T];L^2(\Omega;\mathbb{R}^n))$. Indeed let v^* be a weak* limit of a subsequence of v_τ and let $\varphi \in H^1_{\Gamma_0}(\Omega;\mathbb{R}^n)$. Putting $\eta = E\varphi$ and $q = 0$ in (3.34) we get

$$-\mathrm{div}_{\Gamma_0}(A^0 e_{i+1}) - \mathrm{div}_{\Gamma_0}(A^1 \frac{e_{i+1}-e_i}{\tau}) + \frac{v_{i+1}-v_i}{\tau} = \mathcal{L}_i,$$

which allows us to deduce from (3.47b) and (3.47c) that $\dot v_\tau = \frac{v_{i+1}-v_i}{\tau}$ is bounded in $L^2([0,T];H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n))$ uniformly with respect to τ, thanks to the continuity of the operator div_{Γ_0}.

So, using the Hölder inequality, we estimate

$$\|v_\tau(t)-v_\tau(t_{i+1})\|_{H^{-1}_{\Gamma_0}} \le \tau^{1/2}M \quad \text{for } t\in[t_i,t_{i+1}),$$

for some positive constant M independent of τ, t, and i. Since $\dot u_\tau(t) = v_\tau(t_{i+1})$ for $t\in[t_i,t_{i+1})$ we have

$$\|v_\tau(t)-\dot u_\tau(t)\|_{H^{-1}_{\Gamma_0}} \le \tau^{1/2}M,$$

so that $v_\tau - \dot u_\tau$ tends to 0 strongly in $L^\infty([0,T],H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n))$. From this it easily follows that the two sequences v_τ and $\dot u_\tau$ must have the same weak* limit in $L^\infty([0,T];H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n))$, so $v=v^*$, proving our claim.

The boundness condition proved above implies that $\dot v_\tau$ tends, up to a subsequence, to a function ζ weakly in $L^2([0,T];H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n))$, and it easily follows that $\zeta=\dot v=\ddot u$. Therefore

$$\dot v_\tau \rightharpoonup \ddot u \quad \text{weakly in } L^2([0,T];H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n)). \tag{3.52}$$

Now, the identity

$$Eu_\tau(t) = e_\tau(t)+p_\tau(t), \tag{3.53}$$

together with conditions (3.47b) and (3.49b), implies that $Eu_\tau(t)$ is bounded in $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$ uniformly with respect to τ and t. Then the Korn inequality implies that $Du_\tau(t)$ is actually uniformly bounded in $L^2(\Omega;\mathbb{M}^{n\times n})$, so since $u_\tau(t)\rightharpoonup u(t)$ weakly in $L^2(\Omega;\mathbb{R}^n)$), we get $u(t)\in H^1(\Omega;\mathbb{R}^n)$ and

$$u_\tau(t)\rightharpoonup u(t) \text{ weakly in } H^1(\Omega;\mathbb{R}^n) \text{ and strongly in } L^2(\Omega;\mathbb{R}^n) \tag{3.54}$$

for all $t\in[0,T]$. Hence (3.53) passes to the limit giving

$$Eu(t) = e(t)+p(t) \tag{3.55}$$

for all $t\in[0,T]$. By (3.48b) and (3.50b) we deduce that $Eu(t)$ is bounded in $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$ uniformly with respect to t. By (3.50a) and Korn inequality this implies that $u\in L^\infty([0,T];H^1(\Omega;\mathbb{R}^n))$. On the other hand since (3.55) gives

$$E\dot u(t) = \dot e(t)+\dot p(t), \tag{3.56}$$

from (3.48c), (3.48d), and (3.49d) we deduce that $E\dot u\in L^2([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}))$. Together with (3.48a) and (3.49) this implies that $\dot u\in L^2([0,T];H^1(\Omega;\mathbb{R}^n))$. Moreover (3.53), (3.48c), (3.48d), and (3.56) give

$$E\dot u_\tau \rightharpoonup E\dot u \text{ weakly in } L^2([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})), \tag{3.57}$$

so that (3.48a) implies

$$\dot u_\tau \rightharpoonup \dot u \text{ weakly in } L^2([0,T];H^1(\Omega;\mathbb{R}^n)). \tag{3.58}$$

We now define $\sigma(t) := A^0 e(t) + A^1\dot e(t)$. The results proved so far imply that (u,e,p,σ) satisfies (3.21).

Step 2. In order to show that the functions above satisfy (3.22) we need to pass to the limit in (3.34). We consider the piecewise constant interpolation $\tilde e_\tau$ defined by

$$\tilde e_\tau(t) = e_{i+1} \quad \text{if } t\in[t_i,t_{i+1}).$$

Using (3.47c), it is easily seen that $\tilde e_\tau - e_\tau \to 0$ strongly in $L^\infty([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}))$. Together with (3.48b), this gives

$$\tilde e_\tau \rightharpoonup e \quad \text{weakly* in } L^\infty([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})). \tag{3.59}$$

We also define the piecewise constant interpolation $\mathcal{L}_\tau$ by

$$\mathcal{L}_\tau(t) = \mathcal{L}_i = \frac{1}{\tau}\int_{t_i}^{t_{i+1}} \mathcal{L}(s)ds \quad \text{if } \ t \in [t_i, t_{i+1}).$$

By standard properties of L^2 functions and of their approximation by averaging on subintervals, we have that

$$\mathcal{L}_\tau \to \mathcal{L} \quad \text{strongly in } L^2([0,T]; H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n)). \tag{3.60}$$

For fixed τ (3.34) says that for a.e. $t \in [0,T]$ we have

$$\begin{aligned} -\mathcal{H}(q) \le \langle A^0\tilde{e}_\tau, \eta\rangle + \langle A^1\dot{e}_\tau, \eta\rangle + \langle \dot{p}_\tau, q\rangle \\ + \langle \dot{v}_\tau, \varphi\rangle - \langle \mathcal{L}_\tau, \varphi\rangle \le \mathcal{H}(-q) \end{aligned}$$

for every $(\varphi, \eta, q) \in A(0)$. All the terms in the formula above converge weakly in $L^1([0,T])$ as $\tau \to 0$, thanks to (3.48), (3.52), and (3.59). So for every $(\varphi, \eta, q) \in A(0)$ we can pass to the limit obtaining

$$\begin{aligned} -\mathcal{H}(q) \le \langle A^0 e, \eta\rangle + \langle A^1\dot{e}, \eta\rangle + \langle \dot{p}, q\rangle \\ + \langle \ddot{u}, \varphi\rangle - \langle \mathcal{L}, \varphi\rangle \le \mathcal{H}(-q) \end{aligned} \tag{3.61}$$

for a.e. $t \in [0,T]$. Since the space $A(0)$ is separable, we can construct a set of full measure in $[0,T]$ such that (3.61) holds in this set for every $(\varphi, \eta, q) \in A(0)$, which gives (3.25).

Step 3. We will now prove the energy balance (3.29): let $\lambda \in (0,1)$ and put

$$\begin{aligned} \varphi &= u_{i+1} - \lambda(u_{i+1} - u_i) + \lambda(w_{i+1} - w_i) \\ \eta &= e_{i+1} - \lambda(e_{i+1} - e_i) + \lambda(Ew_{i+1} - Ew_i) \\ q &= p_{i+1} - \lambda(p_{i+1} - p_i), \end{aligned} \tag{3.62}$$

by the minimality of $(u_{i+1}, e_{i+1}, p_{i+1})$ for the functional V_i defined by (3.33) we have $V_i(u_{i+1}, e_{i+1}, p_{i+1}) \le V_i(\varphi, \eta, q)$. This implies

$$\begin{aligned}
&\frac{1}{2}\langle A^0 e_{i+1}, e_{i+1}\rangle + \frac{1}{2\tau}\langle A^1(e_{i+1} - e_i), e_{i+1} - e_i\rangle + \frac{1}{2\tau}\|p_{i+1} - p_i\|^2_{L^2} \\
&+ \mathcal{H}(p_{i+1} - p_i) + \frac{1}{2}\|v_{i+1} - v_i\|^2_{L^2} - \langle \mathcal{L}_i, u_{i+1}\rangle \le \\
&\le \frac{(1-\lambda)^2}{2}\langle A^0 e_{i+1}, e_{i+1}\rangle + \lambda(1-\lambda)\langle A^0 e_{i+1}, e_i\rangle + \frac{\lambda^2}{2}\langle A^0 e_i, e_i\rangle \\
&+ \frac{\lambda^2}{2}\langle A^0(Ew_{i+1} - Ew_i), Ew_{i+1} - Ew_i\rangle + \lambda\langle A^0 e_{i+1}, Ew_{i+1} - Ew_i\rangle \\
&- \lambda^2\langle A^0(e_{i+1} - e_i), Ew_{i+1} - Ew_i\rangle + \frac{(1-\lambda)^2}{2\tau}\langle A^1(e_{i+1} - e_i), e_{i+1} - e_i\rangle \\
&+ \frac{\lambda^2}{2\tau}\langle A^1(Ew_{i+1} - Ew_i), Ew_{i+1} - Ew_i\rangle \\
&+ \frac{\lambda(1-\lambda)}{\tau}\langle A^1(e_{i+1} - e_i), Ew_{i+1} - Ew_i\rangle \\
&+ \frac{(1-\lambda)^2}{2\tau}\|p_{i+1} - p_i\|^2_{L^2} + (1-\lambda)\mathcal{H}(p_{i+1} - p_i) + \frac{1}{2}\|v_{i+1} - v_i\|^2_{L^2} \\
&+ \frac{\lambda^2}{2}\|v_{i+1} - \omega_{i+1}\|^2_{L^2} - \lambda\langle v_{i+1} - v_i - (\omega_{i+1} - \omega_i), v_{i+1} - \omega_{i+1}\rangle \\
&- \langle \mathcal{L}_i, u_{i+1}\rangle + \lambda\tau\langle \mathcal{L}_i - \frac{\omega_{i+1} - \omega_i}{\tau}, v_{i+1} - \omega_{i+1}\rangle.
\end{aligned}$$

Dividing by λ we get

$$\begin{aligned}
&\frac{2-\lambda}{2}(A^0 e_{i+1}, e_{i+1}) - (1-\lambda)\langle A^0 e_{i+1}, e_i\rangle \\
&- \langle A^0 e_{i+1}, Ew_{i+1} - Ew_i\rangle + \lambda\langle A^0(e_{i+1} - e_i), Ew_{i+1} - Ew_i\rangle \\
&- \frac{\lambda}{2}\langle A^0(Ew_{i+1} - Ew_i), Ew_{i+1} - Ew_i\rangle + \frac{2-\lambda}{2\tau}\langle A^1(e_{i+1} - e_i), e_{i+1} - e_i\rangle \\
&+ \frac{2-\lambda}{2\tau}\|p_{i+1} - p_i\|^2_{L^2} + \mathcal{H}(p_{i+1} - p_i) - \frac{\lambda}{2\tau}\langle A^1(Ew_{i+1} - Ew_i), Ew_{i+1} - Ew_i\rangle \\
&- \frac{1-\lambda}{\tau}\langle A^1(e_{i+1} - e_i), Ew_{i+1} - Ew_i\rangle + \langle v_{i+1} - v_i - (\omega_{i+1} - \omega_i), v_{i+1} - \omega_{i+1}\rangle \\
&- \tau\langle \mathcal{L}_i - \frac{\omega_{i+1} - \omega_i}{\tau}, v_{i+1} - \omega_{i+1}\rangle \leq \frac{\lambda}{2}\langle A^0 e_i, e_i\rangle + \frac{\lambda}{2}\|v_{i+1} - \omega_{i+1}\|^2_{L^2}.
\end{aligned}$$

Since $\langle A^0 e_{i+1}, e_{i+1}\rangle \geq 0$ and $\lambda \in (0,1)$ it follows that

$$\begin{aligned}
&(1-\lambda)\langle A^0 e_{i+1}, e_{i+1} - e_i\rangle + \frac{2-\lambda}{2}\tau\langle A^1 \frac{e_{i+1} - e_i}{\tau}, \frac{e_{i+1} - e_i}{\tau}\rangle \\
&- \langle A^0 e_{i+1}, Ew_{i+1} - Ew_i\rangle + \lambda\tau^2\langle A^0 \frac{e_{i+1} - e_i}{\tau}, \frac{Ew_{i+1} - Ew_i}{\tau}\rangle \\
&- \tau^2\frac{\lambda}{2}\langle A^0 \frac{Ew_{i+1} - Ew_i}{\tau}, \frac{Ew_{i+1} - Ew_i}{\tau}\rangle \\
&- (1-\lambda)\tau\langle A^1 \frac{e_{i+1} - e_i}{\tau}, \frac{Ew_{i+1} - Ew_i}{\tau}\rangle + \frac{2-\lambda}{2}\tau\|\frac{p_{i+1} - p_i}{\tau}\|^2_{L^2} \\
&+ \tau\mathcal{H}(\frac{p_{i+1} - p_i}{\tau}) + \langle (v_{i+1} - v_i) - (\omega_{i+1} - \omega_i), v_{i+1} - \omega_{i+1}\rangle \leq \\
&\leq \tau\langle \mathcal{L}_i - \frac{\omega_{i+1} - \omega_i}{\tau}, v_{i+1} - \omega_{i+1}\rangle \\
&+ \frac{\lambda}{2}\langle A^0 e_i, e_i\rangle + \frac{\lambda}{2}\|v_{i+1} - \omega_{i+1}\|^2_{L^2} + \frac{\lambda\tau}{2}\langle A^1 \frac{Ew_{i+1} - Ew_i}{\tau}, \frac{Ew_{i+1} - Ew_i}{\tau}\rangle.
\end{aligned}$$

Now, thanks to (3.38)-(3.40), from the last inequality we get

$$\begin{aligned}
&(1-\lambda)\int_{t_i}^{t_{i+1}} \langle A^0 e_\tau, \dot e_\tau\rangle ds + \frac{2-\lambda}{2}\int_{t_i}^{t_{i+1}} \langle A^1 \dot e_\tau, \dot e_\tau\rangle ds \\
&+ \frac{2-\lambda}{2}\int_{t_i}^{t_{i+1}} \|\dot p_\tau\|^2_{L^2} ds + \int_{t_i}^{t_{i+1}} \mathcal{H}(\dot p_\tau) ds \\
&+ \frac{\tau}{2}\int_{t_i}^{t_{i+1}} \|\dot v_\tau - \dot\omega_\tau\|^2_{L^2} ds + \frac{1}{2}\|v_{i+1} - \omega_{i+1}\|^2_{L^2} - \frac{1}{2}\|v_i - \omega_i\|^2_{L^2} \leq \\
&\leq \int_{t_i}^{t_{i+1}} \langle \mathcal{L} - \dot\omega_\tau, \dot u_\tau - \dot w_\tau\rangle ds - \frac{6-7\lambda}{12}\tau\int_{t_i}^{t_{i+1}} \langle A^0 \dot e_\tau, \dot e_\tau\rangle ds \\
&+ \frac{\lambda}{2\tau}\int_{t_i}^{t_{i+1}} \|\dot u_\tau - \dot w_\tau\|^2_{L^2} ds + \frac{\lambda}{2}\int_{t_i}^{t_{i+1}} \langle \tau A^0 E\dot w_\tau + A^1 E\dot w_\tau, E\dot w_\tau\rangle ds \\
&+ \int_{t_i}^{t_{i+1}} \langle A^0 e_\tau + A^1 \dot e_\tau, E\dot w_\tau\rangle ds + \int_{t_i}^{t_{i+1}} \langle (\frac{\tau}{2} - \lambda\tau)A^0 \dot e_\tau - \lambda A^1 \dot e_\tau, E\dot w_\tau\rangle ds \\
&+ \frac{\lambda}{2\tau}\int_{t_i}^{t_{i+1}} \langle A^0 e_\tau, e_\tau\rangle ds - \frac{\lambda}{2}\int_{t_i}^{t_{i+1}} \langle A^0 e_\tau, \dot e_\tau\rangle ds,
\end{aligned}$$

where we have used that

$$\frac{\lambda}{2}\langle A^0 e_i, e_i\rangle = \frac{\lambda}{2\tau}\int_{t_i}^{t_{i+1}} \langle A^0 e_\tau, e_\tau\rangle ds$$

$$-\frac{\lambda}{2}\int_{t_i}^{t_{i+1}}\langle A^0 e_\tau, \dot e_\tau\rangle ds + \frac{\lambda\tau}{12}\int_{t_i}^{t_{i+1}}\langle A^0 \dot e_\tau, \dot e_\tau\rangle ds.$$

We now sum over $i = 0, \dots, j$ and we obtain

$$\begin{aligned}
&\frac{1-\lambda}{2}\langle A^0 e_\tau(t_{j+1}), e_\tau(t_{j+1})\rangle - \frac{1-\lambda}{2}\langle A^0 e_0, e_0\rangle \\
&+ \frac{2-\lambda}{2}\int_0^{t_{j+1}}\langle A^1 \dot e_\tau, \dot e_\tau\rangle ds + \frac{2-\lambda}{2}\int_0^{t_{j+1}}\|\dot p_\tau\|^2_{L^2} ds + \int_0^{t_{j+1}}\mathcal{H}(\dot p_\tau) ds \\
&+ \frac{\tau}{2}\int_0^{t_{j+1}}\|\dot v_\tau - \dot\omega_\tau\|^2_{L^2} ds + \frac{1}{2}\|v_{j+1} - \omega_{j+1}\|^2_{L^2} - \frac{1}{2}\|v_0 - \omega_0\|^2_{L^2} \le \\
&\le \int_0^{t_{j+1}}\langle \mathcal{L} - \dot\omega_\tau, \dot u_\tau - \dot w_\tau\rangle ds + \int_0^{t_{j+1}}\langle A^0 e_\tau + A^1 \dot e_\tau, E\dot w_\tau\rangle ds \\
&+ \frac{\lambda}{2\tau}\int_0^{t_{j+1}}\langle A^0 e_\tau, e_\tau\rangle ds + \frac{\lambda}{2\tau}\int_0^{t_{j+1}}\|\dot u_\tau - \dot w_\tau\|^2_{L^2} ds \\
&- \frac{6-7\lambda}{12}\tau\int_0^{t_{j+1}}\langle A^0 \dot e_\tau, \dot e_\tau\rangle ds + \frac{\lambda}{2}\int_0^{t_{j+1}}\langle \tau A^0 E\dot w_\tau + A^1 E\dot w_\tau, E\dot w_\tau\rangle ds \\
&+ \int_0^{t_{j+1}}\langle (\frac{\tau}{2} - \lambda\tau) A^0 \dot e_\tau - \lambda A^1 \dot e_\tau, E\dot w_\tau\rangle ds - \frac{\lambda}{2}\int_0^{t_{j+1}}\langle A^0 e_\tau, \dot e_\tau\rangle ds.
\end{aligned}$$

We now take $\lambda = o(\tau)$ and then pass to the limit as $\tau \to 0$. To this aim we fix $t \in [0, T]$ and, for every $\tau > 0$, we define $\hat t_\tau = t_{j+1}$, where j is the unique index such that $t_j \le t < t_{j+1}$. For the third, fourth, and fifth term in the left-hand side of the previous inequality we just use the lower semicontinuity with respect to the convergences in (3.48); for the sixth term we use (3.44a) and (3.44b); to deal with the first and the seventh term we apply Lemma 3.5 below taking into account (3.44b), (3.48b), (3.48c), and (3.52), obtaining

$$\begin{aligned}
&e_\tau(t_{j+1}) = e_\tau(\hat t_\tau) \rightharpoonup e(t) \text{ weakly in } L^2(\Omega; \mathbb{M}^{n\times n}_{\rm sym}), \\
&v_{j+1} - \omega_{j+1} = v_\tau(\hat t_\tau) - \omega(\hat t_\tau) \rightharpoonup \dot u(t) - \dot w(t) \text{ weakly in } H^{-1}_{\Gamma_0}(\Omega; \mathbb{R}^n).
\end{aligned}$$

Since the L^2 norm is lower semicontinuous with respect to weak convergence in $H^{-1}_{\Gamma_0}(\Omega; \mathbb{R}^n)$ (this can be proved by a duality argument as in the classical case of $H^{-1}(\Omega; \mathbb{R}^n)$), we obtain a lower semicontinuity inequality also for these terms.

As for the right-hand side of the previous inequality, we can pass to the limit in the first term thanks to (3.44) and (3.58), and in the second term thanks to (3.44a), (3.48b), and (3.48c). All other terms tend to 0 by (3.44) and (3.47). Thus we obtain

$$\begin{aligned}
&\mathcal{Q}_0(e(t)) - \mathcal{Q}_0(e(0)) + \int_0^t \mathcal{Q}_1(\dot e) ds + \int_0^t \|\dot p\|^2_{L^2} ds + \int_0^t \mathcal{H}(\dot p) ds + \frac{1}{2}\|\dot u(t) - \dot w(t)\|^2_{L^2} \\
&- \int_0^t \langle A^0 e + A^1 \dot e, E\dot w\rangle ds - \frac{1}{2}\|v_0 - \dot w(0)\|^2_{L^2} - \int_0^t \langle \mathcal{L} - \ddot w, \dot u - \dot w\rangle ds \le 0.
\end{aligned}$$

To prove the energy balance (3.29) we need to show that also the opposite inequality holds. To this aim, for a.e. $t \in [0, T]$, we use the first inequality of (3.25) with $\varphi = \dot u(t) - \dot w(t)$, $\eta = \dot e(t) - E\dot w(t)$, and $q = \dot p(t)$. This gives

$$\begin{aligned}
&\langle A^0 e(t), \dot e(t)\rangle + \langle A^1 \dot e(t), \dot e(t)\rangle - \langle A^0 e(t) + A^1 \dot e(t), E\dot w(t)\rangle + \|\dot p(t)\|^2_{L^2} \\
&+ \langle \ddot u(t) - \ddot w(t), \dot u(t) - \dot w(t)\rangle + \mathcal{H}(\dot p(t)) - \langle \mathcal{L}(t) - \ddot w(t), \dot u(t) - \dot w(t)\rangle \ge 0;
\end{aligned}$$

integrating from 0 to t and using (3.26c), (3.27b), and (3.27c) we get the thesis.

Now thanks to Theorem 3.3 the quadruple (u, e, p, σ) satisfies the system of equations (3.22), since the two conditions (3.25) and (3.29) hold.

Step 4. It only remains to prove that the solution is unique. Let (u_1, e_1, p_1) and (u_2, e_2, p_2) be two solutions with the same initial data. Setting $u = u_1 - u_2$, $e = e_1 - e_2$, and $p = p_1 - p_2$, we will prove that (u, e, p) must be costantly zero. In order to show this, we prove that

$$\int_0^t \langle A^0 e, \dot{e}\rangle ds + \int_0^t \langle A^1 \dot{e}, \dot{e}\rangle ds + \int_0^t \|\dot{p}\|^2_{L^2} ds + \int_0^t \langle \ddot{u}, \dot{u}\rangle ds \leq 0 \tag{3.63}$$

for every $t \in [0, T]$. Since the initial data for u, e, p, and $\dot{u}$ are zero, from this inequality, from (3.26), and from [31, Proposition 23.23] it follows that

$$\mathcal{Q}_0(e(t)) + \int_0^t \mathcal{Q}_1(\dot{e}) ds + \int_0^t \|\dot{p}\|^2_{L^2} ds + \frac{1}{2}\|\dot{u}(t)\|^2_{L^2} \leq 0.$$

We then deduce that all the terms are zero for every $t \in [0, T]$. Together with the initial conditions this implies that (u, e, p) is constantly zero.

Inequality (3.63) is equivalent to

$$\begin{aligned}
&\int_0^t \langle A^0 e_1, \dot{e}_1\rangle ds + \int_0^t \langle A^1 \dot{e}_1, \dot{e}_1\rangle ds + \int_0^t \|\dot{p}_1\|^2_{L^2} ds \\
&+ \int_0^t \langle A^0 e_2, \dot{e}_2\rangle ds + \int_0^t \langle A^1 \dot{e}_2, \dot{e}_2\rangle ds + \int_0^t \|\dot{p}_2\|^2_{L^2} ds \\
&+ \int_0^t \langle \ddot{u}_1 - \ddot{u}_2, \dot{u}_1 - \dot{u}_2\rangle ds - \int_0^t \langle A^0 e_1, \dot{e}_2\rangle ds - \int_0^t \langle A^0 e_2, \dot{e}_1\rangle ds \\
&- \int_0^t \langle A^1 \dot{e}_1, \dot{e}_2\rangle ds - \int_0^t \langle A^1 \dot{e}_2, \dot{e}_1\rangle ds - 2\int_0^t \langle \dot{p}_1, \dot{p}_2\rangle ds \leq 0.
\end{aligned} \tag{3.64}$$

From (3.26c), (3.27b), and (3.27c), and from the energy balance (3.29) we get, for $i = 1, 2$

$$\begin{aligned}
&\int_0^t \langle A^0 e_i, \dot{e}_i\rangle ds + \int_0^t \langle A^1 \dot{e}_i, \dot{e}_i\rangle ds + \int_0^t \|\dot{p}_i\|^2_{L^2} ds = \\
&= \int_0^t \langle \sigma_i, E\dot{w}\rangle ds - \int_0^t \mathcal{H}(\dot{p}_i) ds \\
&- \int_0^t \langle \ddot{u}_i - \ddot{w}, \dot{u}_i - \dot{w}\rangle ds + \int_0^t \langle \mathcal{L} - \ddot{w}, \dot{u}_i - \dot{w}\rangle ds,
\end{aligned}$$

where $\sigma_i = A^0 e_i + A^1 \dot{e}_i$. Substituting in (3.64) we obtain

$$\begin{aligned}
&\int_0^t \langle \mathcal{L} - \ddot{w}, \dot{u}_1 - \dot{w}\rangle ds + \int_0^t \langle \mathcal{L} - \ddot{w}, \dot{u}_2 - \dot{w}\rangle ds \\
&- \int_0^t \langle A^0 e_1, \dot{e}_2\rangle ds - \int_0^t \langle A^0 e_2, \dot{e}_1\rangle ds - \int_0^t \langle A^1 \dot{e}_1, \dot{e}_2\rangle ds \\
&- \int_0^t \langle A^1 \dot{e}_2, \dot{e}_1\rangle ds - 2\int_0^t \langle \dot{p}_1, \dot{p}_2\rangle ds + \int_0^t \langle \sigma_1, E\dot{w}\rangle ds \\
&+ \int_0^t \langle \sigma_2, E\dot{w}\rangle ds - \int_0^t \langle \ddot{u}_1 - \ddot{w}, \dot{u}_2 - \dot{w}\rangle ds - \int_0^t \langle \ddot{u}_2 - \ddot{w}, \dot{u}_1 - \dot{w}\rangle ds \leq \\
&\leq \int_0^t \mathcal{H}(\dot{p}_1) ds + \int_0^t \mathcal{H}(\dot{p}_2) ds
\end{aligned} \tag{3.65}$$

Since $\varphi := \dot{u}_1(t) - \dot{w}(t) \in H^1_{\Gamma_0}(\Omega; \mathbb{R}^n)$ for a.e. $t \in [0, T]$, by (3.22c) we can use it in (3.9) for u_2, σ_2, and obtain

$$\langle \ddot{u}_2 - \ddot{w}, \dot{u}_1 - \dot{w}\rangle = \langle \mathcal{L} - \ddot{w}, \dot{u}_1 - \dot{w}\rangle - \langle A^0 e_2 + A^1 \dot{e}_2, \dot{e}_1\rangle - \langle \sigma_2, \dot{p}_1\rangle + \langle \sigma_2, E\dot{w}\rangle.$$

Similarly we have

$$\langle \ddot{u}_1 - \ddot{w}, \dot{u}_2 - \dot{w}\rangle = \langle \mathcal{L} - \ddot{w}, \dot{u}_2 - \dot{w}\rangle - \langle A^0 e_1 + A^1 \dot{e}_1, \dot{e}_2\rangle - \langle \sigma_1, \dot{p}_2\rangle + \langle \sigma_1, E\dot{w}\rangle,$$

and substituting in (3.65) we find that (3.63) is equivalent to

$$\int_0^t \langle \sigma_1, \dot{p}_2\rangle ds + \int_0^t \langle \sigma_2, \dot{p}_1\rangle ds - 2\int_0^t \langle \dot{p}_1, \dot{p}_2\rangle ds \le \le \int_0^t \mathcal{H}(\dot{p}_1) ds + \int_0^t \mathcal{H}(\dot{p}_2) ds.$$

This follows easily from the inequalities

$$\langle \sigma_1(t) - \dot{p}_1(t), \dot{p}_2(t)\rangle \le \mathcal{H}(\dot{p}_2(t)), \quad \langle \sigma_2(t) - \dot{p}_2(t), \dot{p}_1(t)\rangle \le \mathcal{H}(\dot{p}_1(t)),$$

which are direct consequences of the the definition of $\mathcal{H}$ and of the inclusion $(\sigma_i)_D(t) - \dot{p}_i(t) \in \mathcal{K}(\Omega)$, due to (3.22d).

□

Here we prove the lemma we have used in the previous proof.

Lemma 3.5. *Let X be a Banach space. Assume that q_τ tends to q_0 weakly in $H^1([0,T];X)$ as τ tends to zero. Then it holds*

$$q_\tau(t_\tau) \rightharpoonup q_0(t_0) \quad \textit{weakly in } X \tag{3.66}$$

for every $t_\tau, t_0 \in [0,T]$ with $t_\tau \to t_0$ as $\tau \to 0$.

Proof. Since $H^1([0,T];X)$ is continuously embedded in $C^{0,1/2}([0,T];X)$, we have $q_\tau \rightharpoonup q_0$ weakly in $C^{0,1/2}([0,T];X)$. This implies in particular that

$$q_\tau(t) \rightharpoonup q_0(t) \ \text{ weakly in } X \tag{3.67}$$

for all $t \in [0,T]$. If $t_\tau \to t_0$ we have

$$\|q_\tau(t_\tau) - q_\tau(t_0)\| \le \int_{t_0}^{t_\tau} \|\dot{q}_\tau\| dt \le M(t_\tau - t_0)^{1/2},$$

where $\|\cdot\|$ is the norm in X and M is an upper bound for the norm of q_τ in $H^1([0,T];X)$. Now (3.66) follows from the previous inequality and (3.67). □

4. Perfect Plasticity

In this and in the next sections we study the behavior of the solutions of (3.22) when the data of the problem, i.e., the external load and the boundary conditions, vary very slowly. We are going to prove that the inertial and viscosity terms become negligible in the limit, and that the solutions of the dynamic problems actually approach the quasistatic evolution for perfect plasticity. To this aim we provide in this section the mathematical setting and tools to formulate and solve the perfect plasticity problem.

4.1. **Preliminary Tools. Space BD.** In perfect plasticity the displacement u belongs to the space of functions with bounded deformation on Ω, defined as

$$BD(\Omega) = \{u \in L^1(\Omega;\mathbb{R}^n) : Eu \in \mathcal{M}_b(\Omega;\mathbb{M}^{n\times n}_{\text{sym}})\}.$$

Here and henceforth, if V is a finite dimensional vector space and A is a locally compact subset of $\mathbb{R}^n$, the symbol $\mathcal{M}_b(A;V)$ denotes the space of V-valued bounded Radon measures on A, endowed with the norm $\|\lambda\|_{\mathcal{M}_b} := |\lambda|(A)$, where $|\lambda|$ is the variation of λ.

The space $BD(\Omega)$ is endowed with the norm

$$\|u\|_{BD} = \|u\|_{L^1} + \|Eu\|_{\mathcal{M}_b}.$$

Besides the strong convergence, we shall also consider a notion of weak* convergence in $BD(\Omega)$. We say that a sequence u_k converges to u weakly* in $BD(\Omega)$ if and only if u_k converges to u weakly in $L^1(\Omega;\mathbb{R}^n)$ and Eu_k converges to Eu weakly* in $\mathcal{M}_b(\Omega;\mathbb{M}^{n\times n}_{\rm sym})$. Every function u in $BD(\Omega)$ has a trace in $L^1(\partial\Omega;\mathbb{R}^n)$, that we will still denote by u, or sometimes by $u|_{\partial\Omega}$. By [28, Proposition 2.4 and Remark 2.5] there exists a constant C depending only on Ω such that

$$\|u\|_{L^1(\Omega)} \le C(\|u\|_{L^1(\Gamma_0)} + \|Eu\|_{\mathcal{M}_b(\Omega)}). \tag{4.1}$$

For technical reasons related to the stress-strain duality, in addition to the assumption already introduced in Section 2.1, we now suppose that

$$\partial\Omega \text{ and } \partial\Gamma \text{ are of class } C^2. \tag{4.2}$$

Elastic and Plastic Strain. In perfect plasticity the plastic strain p belongs to $\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$. The singular part of this measure describes plastic slips. Given $w \in H^1(\Omega;\mathbb{R}^n)$, we say that a triple (u,e,p) is kinematically admissible for the perfectly plastic problem with boundary datum w if $u \in BD(\Omega;\mathbb{R}^n)$, $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym})$, $p \in \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$, and

$$Eu = e + p \quad \text{ on } \Omega, \tag{4.3a}$$

$$p = (w-u)\odot\nu\mathcal{H}^{n-1} \quad \text{ on } \Gamma_0, \tag{4.3b}$$

where ν denotes the outer unit normal to $\partial\Omega$ and $\odot$ denotes the symmetrized tensor product.

The set of these triples will be denoted by $A_{BD}(w)$. Note that in this definition of kinematical admissibility, the Dirichlet boundary condition (3.1b) is replaced by the relaxed condition (4.3b), which represents a plastic slip occurring at Γ_0. It is also easily seen that the inclusion $A(w) \subset A_{BD}(w)$ holds, so that every admissible triple for the visco-elasto-plastic problem is also admissible for the perfectly plastic problem.

The following closure property is proved in [5, Lemma 2.1].

Lemma 4.1. *Let w_k be a sequence in $H^1(\Omega;\mathbb{R}^n)$ and $(u_k,e_k,p_k) \in A_{BD}(w_k)$. Let us suppose that $w_k \rightharpoonup w_\infty$ weakly in $H^1(\Omega;\mathbb{R}^n)$, $u_k \rightharpoonup u_\infty$ weakly* in $BD(\Omega)$, $e_k \rightharpoonup e_\infty$ weakly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, and $p_k \rightharpoonup p_\infty$ weakly* in $M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$. Then $(u_\infty,e_\infty,p_\infty) \in A_{BD}(w_\infty)$.*

Stress. In addition to the assumptions of Section 2.1, we now suppose that the elastic tensor A^0 maps the orthogonal spaces $\mathbb{M}^{n\times n}_D$ and $\mathbb{R}I$ into themselves. This is equivalent to require that there exist a positive definite symmetric operator $A^0_D : \mathbb{M}^{n\times n}_D \to \mathbb{M}^{n\times n}_D$ and a positive constant κ^0 such that

$$A^0\xi = A^0_D\xi_D + \kappa^0(\mathrm{tr}\xi)I. \tag{4.4}$$

In the perfectly plastic model the stress σ is related to the strain by the equation

$$\sigma = A^0 e \tag{4.5}$$

where e is the elastic component of the strain Eu. Therefore if (u,e,p) is kinematically admissible, then σ belongs to $L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym})$.

In perfect plasticity the stress satisfies the constraint

$$\sigma_D \in \mathcal{K}(\Omega), \tag{4.6}$$

where $\mathcal{K}(\Omega)$ is defined in (3.11). In particular

$$\sigma_D \in L^\infty(\Omega;\mathbb{M}^{n\times n}_D). \tag{4.7}$$

Convex Functions of Measures. In perfect plasticity we need to define the functional (3.16) for $p \in \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$. This is done by using the theory of convex functions of measures (see [10] and [28]): for every $p \in \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ we consider the non-negative Radon measure $H(p)$ on $\Omega \cup \Gamma_0$ defined by

$$H(p)(B) := \int_B H(p/|p|)d|p| \tag{4.8}$$

for every Borel set $B \subset \Omega \cup \Gamma_0$, where $p/|p|$ is the Radon-Nikodym derivative of p with respect to its variation $|p|$. We also define

$$\mathcal{H}(p) := H(p)(\Omega \cup \Gamma_0) = \int_{\Omega\cup\Gamma_0} H(p/|p|)d|p|.$$

The function $p \mapsto \mathcal{H}(p)$ turns out to be lower semicontinuous with respect to the weak* topology of $\mathcal{M}_b(\Omega\cup\Gamma_0; \mathbb{M}^{n\times n}_D)$, and satisfies the triangle inequality. Moreover if $p_k \rightharpoonup p$ weakly* and $|p_k|(\Omega \cup \Gamma_0) \to |p|(\Omega \cup \Gamma_0)$, then $\mathcal{H}(p_k) \to \mathcal{H}(p)$.

Stress-Strain Duality. If $\sigma \in L^2(\Omega; \mathbb{M}^{n\times n}_{\rm sym})$, with $\mathrm{div}\sigma \in L^2(\Omega; \mathbb{R}^n)$, we define the distribution $[\sigma\nu]$ on $\partial\Omega$ by setting

$$\langle [\sigma\nu], \varphi\rangle_{\partial\Omega} := \langle \mathrm{div}\sigma, \varphi\rangle + \langle \sigma, E\varphi\rangle, \tag{4.9}$$

for each $\varphi \in H^1(\Omega; \mathbb{R}^n)$. It turns out that $[\sigma\nu] \in H^{-\frac{1}{2}}(\partial\Omega; \mathbb{R}^n)$ (see e.g. [28, Theorem 1.2, Chapter I]). We define the normal and tangential part of $[\sigma\nu]$ by

$$[\sigma\nu]_\nu := ([\sigma\nu] \cdot \nu)\nu, \qquad\qquad [\sigma\nu]^\perp_\nu := [\sigma\nu] - [\sigma\nu]_\nu, \tag{4.10}$$

and we have that $[\sigma\nu]_\nu$ and $[\sigma\nu]^\perp_\nu$ belong to $H^{-\frac{1}{2}}(\partial\Omega; \mathbb{R}^n)$ thanks to the regularity assumption (4.2) on $\partial\Omega$. If $\sigma_D \in L^\infty(\Omega; \mathbb{M}^{n\times n}_D)$, by [15, Lemma 2.4] we also have that $[\sigma\nu]^\perp_\nu \in L^\infty(\partial\Omega; \mathbb{R}^n)$ and

$$\|[\sigma\nu]^\perp_\nu\|_{\infty,\partial\Omega} \le \frac{1}{\sqrt{2}}\|\sigma_D\|_{L^\infty}. \tag{4.11}$$

The set of admissible stresses for the perfectly plastic problem is defined by

$$\Sigma(\Omega) := \{\sigma \in L^2(\Omega; \mathbb{M}^{n\times n}_{\rm sym}) : \mathrm{div}\sigma \in L^n(\Omega; \mathbb{R}^n) \text{ and } \sigma_D \in L^\infty(\Omega; \mathbb{M}^{n\times n}_D)\}.$$

The set of admissible plastic strains $\Pi_{\Gamma_0}(\Omega)$ is the set of all $p \in \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ such that there exist $u \in BD(\Omega)$, $e \in L^2(\Omega; \mathbb{M}^{n\times n}_{\rm sym})$ and $w \in H^1(\Omega; \mathbb{R}^n)$ satisfying $(u, e, p) \in A_{BD}(w)$.

If $\sigma \in \Sigma(\Omega)$ it turns out that $\sigma \in L^r(\Omega; \mathbb{M}^{n\times n}_{\rm sym})$ for all $r < +\infty$ (see [29, Proposition 2.5]). For every $u \in BD(\Omega)$ with $\mathrm{div}u \in L^2(\Omega)$ we define the distribution $[\sigma_D \cdot E_D u]$ by

$$\langle [\sigma_D \cdot E_D u], \varphi\rangle = -\langle \mathrm{div}\sigma, \varphi u\rangle - \frac{1}{n}\langle \mathrm{tr}\sigma, \varphi \mathrm{div}u\rangle - \langle \sigma, u \odot \nabla\varphi\rangle \tag{4.12}$$

for every $\varphi \in C^\infty_c(\Omega)$. As proved in [29, Theorem 3.2] the distribution $[\sigma_D \cdot E_D u]$ is a bounded Radon measure in Ω.

As in [5], if $\sigma \in \Sigma(\Omega)$ and $p \in \Pi_{\Gamma_0}(\Omega)$, we define the bounded Radon measure $[\sigma_D \cdot p]$ on $\Omega \cup \Gamma_0$ by setting

$$[\sigma_D \cdot p] := [\sigma_D \cdot E_D u] - \sigma_D \cdot e_D \qquad \text{on } \Omega,$$
$$[\sigma_D \cdot p] := [\sigma\nu]^\perp_\nu \cdot (w - u)\mathcal{H}^{n-1} \qquad \text{on } \Gamma_0,$$

where $u \in BD(\Omega)$, $e \in L^2(\Omega; \mathbb{M}^{n\times n}_{\rm sym})$ and $w \in H^1(\Omega; \mathbb{R}^n)$ satisfy $(u, e, p) \in A_{BD}(w)$, and we notice that this definition does not depend on the particular choice of u, e, w (see [5, page 250]). We also define the duality pairing between $\sigma \in \Sigma(\Omega)$ and $p \in \Pi_{\Gamma_0}(\Omega)$ by

$$\langle \sigma_D, p\rangle := [\sigma_D \cdot p](\Omega \cup \Gamma_0). \tag{4.13}$$

The following inequalities between measures hold (see [5, (2.33) and Proposition 2.4]):

$$|[\sigma_D \cdot p]| \leq \|\sigma_D\|_{L^\infty} |p| \quad \text{on } \Omega \cup \Gamma_0, \tag{4.14}$$

$$[\sigma_D \cdot p] \leq H(p) \quad \text{on } \Omega \cup \Gamma_0, \tag{4.15}$$

where $H(p)$ is the measure introduced in (4.8). The following integration by parts formula is proved in [5, Proposition 2.2] when $\varphi \in C^1(\bar{\Omega})$. The extension to Lipschitz functions is straightforward.

Proposition 4.2. *Let $\sigma \in \Sigma(\Omega)$, $f \in L^n(\Omega;\mathbb{R}^n)$, $g \in L^\infty(\Gamma_1;\mathbb{R}^n)$ and suppose $(u,e,p) \in A_{BD}(w)$ with $w \in H^1(\Omega;\mathbb{R}^n)$. If $-\mathrm{div}\sigma = f$ on Ω and $[\sigma\nu] = g$ on Γ_1, then it holds*

$$\langle \sigma_D, p\rangle + \langle \sigma, e - Ew\rangle = \langle f, u - w\rangle + \langle g, u - w\rangle_{\Gamma_1}. \tag{4.16}$$

Moreover

$$\begin{aligned} &\langle [\sigma_D \cdot p], \varphi\rangle + \langle \sigma \cdot (e - Ew), \varphi\rangle + \langle \sigma, \nabla\varphi \odot (u - w)\rangle = \\ &= \langle f, \varphi(u - w)\rangle + \langle g, \varphi(u - w)\rangle_{\Gamma_1}, \end{aligned} \tag{4.17}$$

for every $\varphi \in C^{0,1}(\bar{\Omega})$.

As a consequence of the formula above we obtain the following lemma.

Lemma 4.3. *Let $\sigma_k, \sigma \in \Sigma(\Omega)$, $w_k, w \in H^1(\Omega;\mathbb{R}^n)$, $(u_k, e_k, p_k) \in A_{BD}(w_k)$, and $(u,e,p) \in A_{BD}(w)$ be such that*

$$\begin{aligned} &\sigma_k \to \sigma \text{ strongly in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym}),\\ &\mathrm{div}\sigma_k \to \mathrm{div}\sigma \text{ strongly in } L^n(\Omega;\mathbb{R}^n),\\ &(\sigma_k)_D \text{ are uniformly bounded in } L^\infty(\Omega;\mathbb{M}^{n\times n}_D),\\ &u_k \rightharpoonup u \text{ weakly in } L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^n),\\ &w_k \rightharpoonup w \text{ weakly in } H^1(\Omega;\mathbb{R}^n),\\ &e_k \rightharpoonup e \text{ weakly in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym}), \end{aligned}$$

then $\langle [(\sigma_k)_D \cdot p_k], \varphi\rangle \to \langle [\sigma \cdot p], \varphi\rangle$ for every $\varphi \in C^{0,1}_c(\Omega \cup \Gamma_0)$.

Proof. Our hypotheses imply that $\sigma_k \to \sigma$ strongly in $L^n(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$ by [29, Proposition 2.5]. The conclusion follows now from (4.17). □

4.2. **Hypotheses on the Data.** We discuss here the hypotheses on the data for the quasistatic evolution problem in perfect plasticity.

External Load. In contrast to the dynamic case, in perfect plasticity it is not enough to assume that the total load $\mathcal{L}(t)$ belongs to $H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n)$. Instead, we assume that $\mathcal{L}(t)$ takes the form (3.4), with $f(t) \in L^n(\Omega;\mathbb{R}^n)$ and $g(t) \in L^\infty(\Gamma_1;\mathbb{R}^n)$, so that now the duality $\langle \mathcal{L}(t), u\rangle$ is well defined by (3.4) for every $u \in BD(\Omega)$.

The balance equations for the forces are

$$-\,\mathrm{div}\sigma(t) = f(t) \quad \text{in } \Omega, \tag{4.19}$$

$$[\sigma(t)\nu] = g(t) \quad \text{on } \Gamma_1, \tag{4.20}$$

where $[\sigma(t)\nu]$ denotes the normal component of $\sigma(t)$, which can be defined as a distribution according to (4.9), since $\mathrm{div}\sigma(t) \in L^2(\Omega;\mathbb{R}^n)$ by (4.19). As for the time dependence, we assume that

$$f \in AC([0,T]; L^n(\Omega;\mathbb{R}^n)), \tag{4.21a}$$

$$g \in AC([0,T]; L^\infty(\Gamma_1;\mathbb{R}^n)). \tag{4.21b}$$

This implies that for a.e. $t \in [0,T]$ there exists an element of the dual of $BD(\Omega)$, denoted by $\dot{\mathcal{L}}(t)$, such that

$$\langle \dot{\mathcal{L}}(t), u\rangle = \lim_{s\to t} \langle \frac{\mathcal{L}(s)-\mathcal{L}(t)}{s-t}, u\rangle \tag{4.22}$$

for every $u \in BD(\Omega)$ (see [5, Remark 4.1]).

As usual in perfect plasticity problems, we assume a uniform safe-load condition: there exist a function $\varrho : [0,T] \to L^2(\Omega, \mathbb{M}^{n\times n}_{\mathrm{sym}})$ and a positive constant δ such that for every $t \in [0,T]$ we have

$$-\operatorname{div}\varrho(t) = f(t) \ \text{ on } \Omega, \tag{4.23a}$$

$$[\varrho(t)\nu] = g(t) \ \text{ on } \Gamma_1, \tag{4.23b}$$

and

$$\varrho_D(t) + \xi \in \mathcal{K}(\Omega) \ \text{ for every } \xi \in \mathbb{M}^{n\times n}_D \text{ with } |\xi| \le \delta. \tag{4.24}$$

Moreover we require that

$$t \mapsto \varrho(t) \ \text{ and } \ t \mapsto \varrho_D(t) \quad \text{are absolutely continuous} \tag{4.25}$$

from $[0,T]$ to $L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}})$ and $L^\infty(\Omega; \mathbb{M}^{n\times n}_D)$ respectively, so that the function $t \mapsto \dot\varrho(t)$ belongs to $L^1([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_{\mathrm{sym}}))$ and

$$\frac{\varrho_D(t)-\varrho_D(s)}{t-s} \to \dot\varrho_D(s) \ \text{ weakly* in } L^\infty(\Omega; \mathbb{M}^{n\times n}_D) \quad \text{as } t\to s, \tag{4.26}$$

for a.e. $s \in [0,T]$, and

$$t \mapsto \|\dot\varrho(t)\|_{L^\infty} \text{ belongs to } L^1([0,T]) \tag{4.27}$$

(see [5, Theorem 7.1]).

Using (4.14) and (4.25) we see that for every $p \in \Pi_{\Gamma_0}(\Omega)$ the function

$$t \mapsto \langle \varrho_D(t), p\rangle \quad \text{belongs to } \ AC([0,T]). \tag{4.28}$$

Moreover, by (4.21a), (4.23a), (4.24), and (4.25), we obtain

$$\tfrac{d}{dt}\langle \varrho_D(t), p\rangle = \langle \dot\varrho_D(t), p\rangle \quad \text{for a.e. } t \in [0,T], \tag{4.29}$$

thanks to [5, formula (2.38)].

Boundary Conditions. The boundary condition on Γ_0 is given in the relaxed form considered in (4.3b) with a time dependent function $t \to w(t)$. We assume that

$$w \in AC([0,T]; H^1(\Omega; \mathbb{R}^n)). \tag{4.30}$$

Plastic Dissipation. In the energy formulation for the quasistatic evolution problem for perfect plasticity, it is not convenient to use formulas like (3.17), because they require the existence of the time derivative of $p(t)$. Instead, for an arbitrary function $p : [0,T] \to \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ we define the plastic dissipation in $[a,b] \subset [0,T]$ as

$$\mathcal{D}_H(a,b;p) := \sup \sum_{i=0}^{N-1} \mathcal{H}(p(t_{i+1}) - p(t_i)), \tag{4.31}$$

where the supremum is taken over all the possible choices of the integer $N > 0$ and of the real numbers $a = t_0 < t_1 < ... < t_{N-1} < t_N = b$. One can prove (see [5, Chapter 7]) that, if $p : [0,T] \to \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$ is absolutely continuous, then

$$\mathcal{D}_H(a,b;p) = \int_a^b \mathcal{H}(\dot p(t))dt, \tag{4.32}$$

where $\dot p$ is the derivative of p defined by

$$\dot p(t) := w^*\text{-}\lim_{s\to t} \frac{p(s)-p(t)}{s-t}. \tag{4.33}$$

As a consequence of the safe-load condition (4.24) we can easily prove that for every $t \in [0,T]$

$$\mathcal{H}(q) - \langle \varrho(t), q\rangle \geq \gamma \|q\|_{\mathcal{M}_b}, \tag{4.34}$$

for every $q \in L^1(\Omega, \mathbb{M}^{n\times n}_D)$, where the positive constant γ is independent of q and t (see [5, Lemma 3.2]). Moreover we have that

$$H(q) - \varrho(t)\cdot q \geq 0 \text{ a.e. in } \Omega, \tag{4.35}$$

for every $q \in L^1(\Omega, \mathbb{M}^{n\times n}_D)$.

5. Quasistatic Evolution in Perfect Plasticity

We recall here the energy formulation of a perfectly plastic quasistatic evolution.

Definition 5.1. Let $u_0 \in BD(\Omega)$, $e_0 \in L^2(\Omega; \mathbb{M}^{n\times n}_{\rm sym})$, and $p_0 \in M_b(\Omega\cup\Gamma_0; \mathbb{M}^{n\times n}_D)$. Suppose that f, g, $\mathcal{L}$, ϱ, and w satisfy (3.4), (4.21), (4.23), (4.24), (4.25), and (4.30). A *quasistatic evolution in perfect plasticity* with initial conditions u_0, e_0, p_0, and boundary condition w on Γ_0 is a function (u,e,p,σ) from $[0,T]$ into $BD(\Omega,\mathbb{R}^n)\times L^2(\Omega, \mathbb{M}^{n\times n}_{\rm sym}) \times \mathcal{M}_b(\Omega\cup\Gamma_0, \mathbb{M}^{n\times n}_D)\times L^2(\Omega, \mathbb{M}^{n\times n}_{\rm sym})$, with

$$u(0) = u_0, \;\; e(0) = e_0, \;\; p(0) = p_0, \tag{5.1}$$

$$\sigma(t) = A^0 e(t) \quad \text{for every } t\in[0,T], \tag{5.2}$$

such that $t \mapsto p(t)$ has bounded variation and the following two conditions are satisfied for every $t \in [0,T]$:

(a) $(u(t), e(t), p(t)) \in A_{BD}(w(t))$ and

$$\mathcal{Q}_0(e(t)) - \langle \mathcal{L}(t), u(t)\rangle \leq \mathcal{Q}_0(\eta) - \langle \mathcal{L}(t), \varphi\rangle + \mathcal{H}(q - p(t)) \tag{5.3}$$

for every $(\varphi, \eta, q) \in A_{BD}(w(t))$;

(b) $$\mathcal{Q}_0(e(t)) - \mathcal{Q}_0(e_0) + \mathcal{D}_H(p; 0, t) = \int_0^t \langle \sigma, E\dot{w}\rangle ds - \int_0^t \langle \mathcal{L}, \dot{w}\rangle ds + \langle \mathcal{L}(t), u(t)\rangle - \langle \mathcal{L}(0), u_0\rangle - \int_0^t \langle \dot{\mathcal{L}}, u\rangle ds, \tag{5.4}$$

where $\mathcal{D}_H(p; 0, t)$ is defined by (4.31).

The integrals in the right-hand side of (5.4) are well defined thanks to [5, Theorem 3.8 and Remark 4.3].

If $(u_0, e_0, p_0) \in A_{BD}(w(0))$ satisfies the following stability condition

$$\mathcal{Q}_0(e_0) - \langle \mathcal{L}(0), u_0\rangle \leq \mathcal{Q}_0(\eta) - \langle \mathcal{L}(0), \varphi\rangle + \mathcal{H}(q - p_0) \tag{5.5}$$

for every $(\varphi, \eta, q) \in A_{BD}(w(0))$, then there exists a quasistatic evolution in perfect plasticity with initial conditions u_0, e_0, p_0, and boundary condition w on Γ_0 (see [5, Theorem 4.5]). Moreover the function $t \mapsto (u(t), e(t), p(t))$ is absolutely continuous from $[0,T]$ into $BD(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym})\times\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$ ([5, Theorem 5.1]).

In our analysis of the behavior of the solutions $(u^\epsilon, e^\epsilon, p^\epsilon, \sigma^\epsilon)$ of (1.2) as $\epsilon \to 0$ we find that $(u^\epsilon, e^\epsilon, p^\epsilon, \sigma^\epsilon)$ converges to a function (u,e,p,σ) which satisfies conditions (5.3) and (5.4) only for a.e. $t \in [0,T]$. The following theorem shows that this is enough to guarantee that (u,e,p,σ) is a quasistatic evolution, according to Definition 5.1.

Theorem 5.2. *Let u_0, e_0, p_0, f, g, $\mathcal{L}$, w, and ϱ be as in Definition 5.1. Let S be a subset of $[0,T]$ of full $\mathcal{L}^1$ measure containing 0 and let $(u,e,\sigma) : S \to BD(\Omega)\times L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\times L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ be a bounded and measurable function satisfying (5.1) and (5.2) for all $t\in S$. Suppose that $p : [0,T] \to \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$ has bounded variation and that conditions (a) and (b) of Definition 5.1 are satisfied for every $t \in S$. Then there exists an absolutely continuous function $(u,e,\sigma) : [0,T] \to$*

$BD(\Omega)\times L^2(\Omega;\mathbb{M}^{n\times n}_{sym})\times L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ which extends (u,e,σ). Moreover p is absolutely continuous and (u,e,p,σ) is a quasistatic evolution in perfect plasticity with initial conditions u_0, e_0, p_0, and boundary condition w on Γ_0.

Remark 5.3. Let $t\in S$, $(u(t),e(t),p(t))\in A_{BD}(w(t))$ and $\sigma(t):=A^0e(t)$. As shown in [5, Theorem 3.6] the following conditions are equivalent:

(a) Inequality (5.3) is satisfied for every $(\varphi,\eta,q)\in A_{BD}(w(t))$;
(b) $-\mathcal{H}(q)\le\langle A^0e(t),\eta\rangle-\langle\mathcal{L}(t),v\rangle\le\mathcal{H}(-q)$ for every $(v,\eta,q)\in A_{BD}(0)$;
(c) $\sigma(t)\in\Sigma(\Omega)$, $\sigma_D(t)\in\mathcal{K}(\Omega)$, $-\mathrm{div}\sigma(t)=f(t)$ in Ω, and $[\sigma(t)\nu]=g(t)$ on Γ_1.

The following lemma gives an elementary but useful tool for the proof of Theorem 5.2.

Lemma 5.4. *Let $p:[0,T]\to\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$ be a function with bounded variation and let $\psi(t):=\mathcal{D}_H(p;0,t)$ for $t\in[0,T]$. Assume that there exists a set $S\subseteq[0,T]$ of full $\mathcal{L}^1$ measure such that $p|_S$ and $\psi|_S$ are absolutely continuous on S. Then p is absolutely continuous on $[0,T]$.*

Proof. The absolute continuity on S implies that

$$\lim_{\substack{s\to t^-\\ s\in S}}\psi(s)=\lim_{\substack{s\to t^+\\ s\in S}}\psi(s)$$

for every $t\in[0,T]$. Since ψ is non-decreasing, we deduce that the common value of the limit coincides with $\psi(t)$. This shows that ψ is continuous on $[0,T]$. Since

$$\|p(t_1)-p(t_2)\|_{\mathcal{M}_b}\le\mathcal{D}_H(p;t_1,t_2)=\psi(t_2)-\psi(t_1)$$

for every $0\le t_1\le t_2\le T$, we conclude that also p is continuous on $[0,T]$. Moreover the fact that the restriction of p to S is absolutely continuous implies that it is absolutely continuous on $[0,T]$ as well. □

Proof of Theorem 5.2. We first prove that the functions e, p and u are absolutely continuous on S. We argue as in the proof of [5, Theorem 5.2] using only times t_1, t_2 and s in the set S, and we obtain that for any t_1, $t_2\in S$ with $t_1<t_2$ we have that

$$\|e(t_2)-e(t_1))\|^2_{L^2}\le\int_{t_1}^{t_2}\|e(s)-e(t_1)\|_{L^2}\phi(s)ds+(\int_{t_1}^{t_2}\phi(s)ds)^2,$$

where ϕ is a suitable non-negative integrable function. As a consequence of [5, Lemma 5.3] we get that $\|e(t_2)-e(t_1))\|_{L^2}\le\frac{3}{2}\int_{t_1}^{t_2}\phi(s)ds$ so that $t\mapsto e(t)$ is absolutely continuous from S into $L^2(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}})$. Continuing as in the proof of [5, Theorem 5.2] we obtain also that p and u are absolutely continuous on S. From equation (5.4) it follows that $t\mapsto\mathcal{D}_H(p;0,t)$ is absolutely continuous on S, so that, applying Lemma 5.4, we get that p is absolutely continuous on $[0,T]$. Now (u,e) admits an absolutely continuous extension to $[0,T]$ that we still denote by (u,e). By continuity this extension satisfies (5.3) and (5.4) for every $t\in[0,T]$. This completes the proof. □

Remark 5.5. Under the hypotheses of Definition 5.1, for every $t\in[0,T]$ condition (b) of Definition 5.1 is equivalent to the following condition:

(b') The function $p:[0,T]\to\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$ has bounded variation and

$$\begin{aligned}&\mathcal{Q}_0(e(t))+\mathcal{D}_H(p;0,t)-\langle\varrho(t),e(t)-Ew(t)\rangle-\langle\varrho_D(t),p(t)\rangle=\\&=\mathcal{Q}_0(e_0)-\langle\varrho(0),e(0)-Ew(0)\rangle-\langle\varrho_D(0),p(0)\rangle+\int_0^t\langle\sigma,E\dot w\rangle ds\\&-\int_0^t\langle\dot\varrho,e-Ew\rangle ds-\int_0^t\langle\dot\varrho_D,p\rangle ds.\end{aligned}\tag{5.6}$$

This is proved in [5, Theorem 4.4] using the integration by parts formula (4.16). Note that the duality product $\langle \dot{\varrho}_D(t), p(t)\rangle$ is well defined for a.e. $t \in [0,T]$ by (4.21a), (4.23a), (4.25), and (4.26).

6. Limit of Dynamic Solutions

Here we formulate in a precise way the asymptotic analysis of the dynamic problem as the data become slower and slower. This will be done by a suitable change of variables. We start from an external load $\mathcal{L}(t)$, a boundary datum $w(t)$ defined on the interval $[0,T]$, and initial conditions u_0, e_0, p_0, and v_0. We then consider the rescaled problem with external load $\mathcal{L}_\epsilon(t) = \mathcal{L}(\epsilon t)$, boundary condition $w_\epsilon(t) = w(\epsilon t)$ on the interval $[0, T/\epsilon]$, and initial conditions $u_\epsilon(0) = u_0$, $e_\epsilon(0) = e_0$, $p_\epsilon(0) = p_0$, and $\dot u_\epsilon(0) = \epsilon v_0$. The dynamic solutions of the corresponding systems (3.22) are denoted by $(u_\epsilon(t), e_\epsilon(t), p_\epsilon(t), \sigma_\epsilon(t))$.

To study the limit behavior of $(u_\epsilon(t), e_\epsilon(t), p_\epsilon(t), \sigma_\epsilon(t))$ on the whole interval $[0, T/\epsilon]$ it is convenient to consider the rescaled functions $(u^\epsilon(t), e^\epsilon(t), p^\epsilon(t), \sigma^\epsilon(t))$ $:= (u_\epsilon(t/\epsilon), e_\epsilon(t/\epsilon), p_\epsilon(t/\epsilon), \sigma_\epsilon(t/\epsilon))$, defined on $[0,T]$, and to study their limit as $\epsilon \downarrow 0$. A straightforward change of variables shows that $(u^\epsilon, e^\epsilon, p^\epsilon, \sigma^\epsilon)$ will satisfy the following system of equations on $[0,T]$

$$Eu^\epsilon = e^\epsilon + p^\epsilon, \tag{6.1a}$$

$$\sigma^\epsilon = A^0 e^\epsilon + \epsilon A^1 \dot e^\epsilon, \tag{6.1b}$$

$$\epsilon^2 \ddot u^\epsilon - \mathrm{div}_{\Gamma_0}(\sigma^\epsilon) = \mathcal{L}, \tag{6.1c}$$

$$\epsilon \dot p^\epsilon = \sigma^\epsilon - \pi_K \sigma^\epsilon, \tag{6.1d}$$

with boundary and initial conditions

$$u^\epsilon(t) = w(t) \quad \text{on } \Gamma_0 \quad \text{for every } t \in [0,T], \tag{6.2}$$

$$u^\epsilon(0) = u_0, \quad e^\epsilon(0) = e_0, \quad p^\epsilon(0) = p_0, \quad \dot u^\epsilon(0) = v_0. \tag{6.3}$$

We shall prove (Theorem 6.2) that, under suitable assumptions, the solutions $(u^\epsilon, e^\epsilon, p^\epsilon, \sigma^\epsilon)$ of (6.1) tend to a solution of the quasi-static evolution problem in perfect plasticity, according to Definition 5.1.

Hypotheses on the Data. The regularity assumptions on the data considered in the dynamical problem are not sufficient to study the limit of the solutions of (6.1). Therefore we introduce a new set of hypotheses, which includes also the case of data depending on ϵ and converging in a suitable way as ϵ tends to 0.

Let $M > 0$ be a constant. For $\epsilon \in (0,1)$ we consider the following assumptions.

(i) Hypotheses on w^ϵ and w:

$$w^\epsilon \in L^\infty([0,T]; H^1(\Omega;\mathbb{R}^n)), \tag{6.4a}$$

$$\dot w^\epsilon \in C^0([0,T]; L^2(\Omega;\mathbb{R}^n)) \cap L^2([0,T]; H^1(\Omega;\mathbb{R}^n)), \tag{6.4b}$$

$$\ddot w^\epsilon \in L^2([0,T]; H^{-1}_{\Gamma_0}(\Omega;\mathbb{R}^n)), \tag{6.4c}$$

$$w \in AC([0,T]; H^1(\Omega;\mathbb{R}^n)), \tag{6.4d}$$

$$w^\epsilon \to w \text{ strongly in } W^{1,1}([0,T]; H^1(\Omega;\mathbb{R}^n)), \tag{6.4e}$$

$$\epsilon \|\dot w^\epsilon(0)\|_{L^2} \to 0, \tag{6.4f}$$

$$\epsilon \|\dot w^\epsilon(t)\|_{L^2} \le M \text{ for all } t \in [0,T], \tag{6.4g}$$

$$\epsilon \int_0^T \|\dot w^\epsilon\|^2_{H^1} dt \to 0, \tag{6.4h}$$

$$\epsilon^2 \int_0^T \|\ddot w^\epsilon\|^2_{H^{-1}_{\Gamma_0}} dt \le M. \tag{6.4i}$$

(ii) Hypotheses on f^ϵ, g^ϵ, f, and g: we assume that there exist ϱ^ϵ and ϱ satisfying (4.23) and (4.24) with f^ϵ, g^ϵ and f, g respectively, and with δ independent of ϵ. We also suppose that

$$f^\epsilon \in AC([0,T]; L^n(\Omega;\mathbb{R}^n)), \tag{6.5a}$$

$$\varrho^\epsilon \in AC([0,T]; L^n(\Omega;\mathbb{M}^{n\times n}_{\rm sym})), \tag{6.5b}$$

$$f \in AC([0,T]; L^n(\Omega;\mathbb{R}^n)), \tag{6.5c}$$

$$g \in AC([0,T]; L^\infty(\Gamma_1;\mathbb{R}^n)), \tag{6.5d}$$

$$\varrho \in AC([0,T]; L^n(\Omega;\mathbb{M}^{n\times n}_{\rm sym})), \tag{6.5e}$$

$$\varrho_D \in AC([0,T]; L^\infty(\Omega;\mathbb{M}^{n\times n}_D)), \tag{6.5f}$$

$$f^\epsilon \to f \text{ strongly in } W^{1,1}([0,T]; L^n(\Omega;\mathbb{R}^n)), \tag{6.5g}$$

$$\varrho^\epsilon \to \varrho \text{ strongly in } W^{1,1}([0,T]; L^n(\Omega;\mathbb{M}^{n\times n}_{\rm sym})). \tag{6.5h}$$

(iii) Hypotheses on the initial data $(u_0^\epsilon, e_0^\epsilon, p_0^\epsilon)$, (u_0, e_0, p_0), and v_0^ϵ.

$$(u_0^\epsilon, e_0^\epsilon, p_0^\epsilon) \in A(w^\epsilon(0)), \tag{6.6a}$$

$$(u_0, e_0, p_0) \in A_{BD}(w(0)), \tag{6.6b}$$

$$(u_0, e_0, p_0) \text{ satisfies the stability condition (5.5)}, \tag{6.6c}$$

$$u_0^\epsilon \to u_0 \text{ strongly in } L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^n), \tag{6.6d}$$

$$e_0^\epsilon \to e_0 \text{ strongly in } L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym}), \tag{6.6e}$$

$$p_0^\epsilon \rightharpoonup p_0 \text{ weakly* in } \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D), \tag{6.6f}$$

$$v_0^\epsilon \in L^2(\Omega;\mathbb{R}^n) \text{ and } \epsilon\|v_0^\epsilon\|_{L^2} \to 0. \tag{6.6g}$$

Remark 6.1. If we assume that

$$\varrho_D^\epsilon \in AC([0,T]; L^\infty(\Omega;\mathbb{M}^{n\times n}_D)), \tag{6.7a}$$

$$\int_0^T \|\dot\varrho_D^\epsilon - \dot\varrho_D\|_{L^\infty}\,dt \to 0, \tag{6.7b}$$

then we can replace (6.5b), (6.5e), and (6.5h) by the weaker conditions

$$\varrho_\epsilon,\ \varrho \in AC([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym})), \tag{6.7c}$$

$$\varrho^\epsilon \to \varrho \text{ strongly in } W^{1,1}([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym})). \tag{6.7d}$$

Indeed using [29, Proposition 2.5] (see also [28, Chapter 2, Proposition 7.1]) from (4.27), (6.5g), and (6.7) we deduce that ϱ^ϵ, $\varrho \in AC([0,T]; L^n(\Omega;\mathbb{M}^{n\times n}_{\rm sym}))$ and that (6.5h) holds.

We now state the main result.

Theorem 6.2. *Assume hypotheses (i)-(iii) above. Let $(u^\epsilon, e^\epsilon, p^\epsilon, \sigma^\epsilon)$ be the solution of* (6.1) *satisfying the boundary condition w^ϵ on Γ_0 for every $t\in[0,T]$, and the initial data*

$$u^\epsilon(0) = u_0^\epsilon,\ e^\epsilon(0) = e_0^\epsilon,\ p^\epsilon(0) = p_0^\epsilon \quad \dot u^\epsilon(0) = v_0^\epsilon.$$

Then there exist a quasistatic evolution in perfect plasticity (u,e,p,σ), with initial conditions (u_0,e_0,p_0) and boundary condition w on Γ_0, and a subsequence of $(u^\epsilon, e^\epsilon, p^\epsilon, \sigma^\epsilon)$, not relabeled, such that

$$u^\epsilon(t) \rightharpoonup u(t) \quad \textit{weakly* in } BD(\Omega), \tag{6.8}$$

$$e^\epsilon(t) \to e(t) \quad \textit{strongly in } L^2(\Omega;\mathbb{M}^{n\times n}_{sym}), \tag{6.9}$$

for a.e. $t\in[0,T]$, and

$$p^\epsilon(t) \rightharpoonup p(t) \ \textit{weakly* in } \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D), \tag{6.10}$$

for all $t \in [0,T]$. Moreover there exists $M > 0$ such that

$$\|u^\epsilon(t)\|_{L^1} + \|e^\epsilon(t)\|_{L^2} + \|p^\epsilon(t)\|_{\mathcal{M}_b} \leq M \tag{6.11}$$

for every $\epsilon \in (0,1)$ and every $t \in [0,T]$.

Proof. From Theorem 3.3 we get the energy balance formula

$$\begin{aligned}
&\mathcal{Q}_0(e^\epsilon(t)) + \frac{\epsilon^2}{2}\|\dot u^\epsilon(t) - \dot w^\epsilon(t)\|^2_{L^2} + \epsilon\int_0^t \mathcal{Q}_1(\dot e^\epsilon)ds + \epsilon\int_0^t \|\dot p^\epsilon\|^2_{L^2}ds + \int_0^t \mathcal{H}(\dot p^\epsilon)ds = \\
&= \int_0^t \langle \sigma^\epsilon, E\dot w^\epsilon\rangle ds + \int_0^t \langle f^\epsilon, \dot u^\epsilon - \dot w^\epsilon\rangle ds + \int_0^t \langle g^\epsilon, \dot u^\epsilon - \dot w^\epsilon\rangle_{\Gamma_1} ds \\
&- \epsilon^2\int_0^t \langle \ddot w^\epsilon, \dot u^\epsilon - \dot w^\epsilon\rangle ds + \mathcal{Q}_0(e_0^\epsilon) + \frac{\epsilon^2}{2}\|v_0^\epsilon - \dot w^\epsilon(0)\|^2_{L^2},
\end{aligned} \tag{6.12}$$

where $\sigma^\epsilon = A^0 e^\epsilon + \epsilon A^1 \dot e^\epsilon$. Using the safe-load condition (4.23) and (4.24) and integrating by parts in space, we get

$$\begin{aligned}
&\mathcal{Q}_0(e^\epsilon(t)) + \frac{\epsilon^2}{2}\|\dot u^\epsilon(t) - \dot w^\epsilon(t)\|^2_{L^2} + \epsilon\int_0^t \mathcal{Q}_1(\dot e^\epsilon)ds + \epsilon\int_0^t \|\dot p^\epsilon\|^2_{L^2}ds + \int_0^t \mathcal{H}(\dot p^\epsilon)ds = \\
&= \int_0^t \langle \sigma^\epsilon, E\dot w^\epsilon\rangle ds + \int_0^t \langle \varrho^\epsilon, E\dot u^\epsilon - E\dot w^\epsilon\rangle ds - \epsilon^2\int_0^t \langle \ddot w^\epsilon, \dot u^\epsilon - \dot w^\epsilon\rangle ds \\
&+ \mathcal{Q}_0(e_0^\epsilon) + \frac{\epsilon^2}{2}\|v_0^\epsilon - \dot w^\epsilon(0)\|^2_{L^2}.
\end{aligned} \tag{6.13}$$

By (3.2), (6.4e), (6.4g),(6.4i), (6.5h), (6.6e), and (6.6g), using the Cauchy inequality, we get a positive constant D_0 such that

$$\begin{aligned}
&\frac{\alpha_0}{2}\|e^\epsilon(t)\|^2_{L^2} + \epsilon\alpha_1\int_0^t \|\dot e^\epsilon\|^2_{L^2}ds + \epsilon\int_0^t \|\dot p^\epsilon\|^2_{L^2}ds + \int_0^t \mathcal{H}(\dot p^\epsilon)ds \leq \\
&\leq \beta_0\int_0^t \|e^\epsilon\|_{L^2}\|E\dot w^\epsilon\|_{L^2}ds + \epsilon\beta_1\int_0^t \|\dot e^\epsilon\|_{L^2}\|E\dot w^\epsilon\|_{L^2}ds + \int_0^t \langle \varrho^\epsilon, \dot e^\epsilon\rangle ds \\
&+ \int_0^t \langle \varrho_D^\epsilon, \dot p^\epsilon\rangle ds + \frac{\epsilon^2}{2}\int_0^t \|\dot u^\epsilon - \dot w^\epsilon\|^2_{H^1_{\Gamma_0}} ds + D_0,
\end{aligned} \tag{6.14}$$

for every $\epsilon \in (0,1)$. By Poincaré and Korn inequalities there exists a constant c such that

$$\|\dot u^\epsilon - \dot w^\epsilon\|^2_{H^1_{\Gamma_0}} \leq 2c\|\dot e^\epsilon\|^2_{L^2} + 2c\|\dot p^\epsilon\|^2_{L^2} + 2c\|E\dot w^\epsilon\|^2_{L^2}.$$

Integrating by parts in time the term $\langle \varrho^\epsilon, \dot e^\epsilon\rangle$ and using again the Cauchy inequality and the inequality $\|e^\epsilon\|_{L^2} \leq 1 + \|e^\epsilon\|^2_{L^2}$, we obtain that for every $\lambda > 0$ the right-hand side of (6.14) can be estimated from above by

$$\begin{aligned}
&\beta_0\int_0^t \|e^\epsilon\|^2_{L^2}\|E\dot w^\epsilon\|_{L^2}ds + \epsilon\lambda\beta_1\int_0^t \|\dot e^\epsilon\|^2_{L^2}ds + \lambda\|e^\epsilon(t)\|^2_{L^2} + \int_0^t \|\dot\varrho^\epsilon\|_{L^2}\|e^\epsilon\|^2_{L^2}ds \\
&+ \int_0^t \langle \varrho_D^\epsilon, \dot p^\epsilon\rangle ds + c\epsilon^2\int_0^t \|\dot e^\epsilon\|^2_{L^2}ds + c\epsilon^2\int_0^t \|\dot p^\epsilon\|^2_{L^2}ds + D_\lambda,
\end{aligned} \tag{6.15}$$

for a suitable constant D_λ independent of ϵ that can be obtained using (6.4e), (6.4h), (6.5h), and (6.6e). Taking $\lambda = \min\{\frac{\alpha_0}{4}, \frac{\alpha_1}{2\beta_1}\}$, from (4.34), (6.14), and (6.15), we get

$$\begin{aligned}
&\frac{\alpha_0}{4}\|e^\epsilon(t)\|^2_{L^2} + (\frac{\alpha_1}{2}\epsilon - c\epsilon^2)\int_0^t \|\dot e^\epsilon\|^2_{L^2}ds + (\epsilon - c\epsilon^2)\int_0^t \|\dot p^\epsilon\|^2_{L^2}dt + \gamma\int_0^t \|\dot p^\epsilon\|_{L^1}ds \leq \\
&\leq \int_0^t \psi^\epsilon\|e^\epsilon\|^2_{L^2}ds + D_\lambda,
\end{aligned} \tag{6.16}$$

where $\psi^\epsilon = \beta_0 \|E\dot w^\epsilon\|_{L^2} + \|\dot\varrho^\epsilon\|_{L^2}$. Since ψ^ϵ is bounded in $L^1([0,T])$ by (6.4e) and (6.5h), using the Gronwall Lemma we obtain that $\|e^\epsilon(t)\|_{L^2}$ is bounded by some constant independent of t and ϵ. Together with (6.16), this gives

$$\|e^\epsilon(t)\|_{L^2} \leq M \quad \text{for all } t \in [0,T], \tag{6.17a}$$

$$\int_0^T \|\dot p^\epsilon\|_{L^1} ds \leq M, \tag{6.17b}$$

$$\epsilon \int_0^T \|\dot e^\epsilon\|^2_{L^2} ds \leq M, \tag{6.17c}$$

$$\epsilon \int_0^T \|\dot p^\epsilon\|^2_{L^2} ds \leq M. \tag{6.17d}$$

for all $\epsilon \in (0,1)$ and some constant $M > 0$ independent of t and ϵ. Using the Korn inequality, from (6.4e), (6.4h), (6.17c), and (6.17d), we get

$$\epsilon \int_0^T \|\dot u^\epsilon\|^2_{H^1} ds \leq M. \tag{6.17e}$$

Since $L^1(\Omega;\mathbb{M}^{n\times n}_D)$ is naturally embedded into $\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$, the functions p^ϵ are actually continuous functions from $[0,T]$ into $\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$, and inequality (6.17b) says that the total variation of p^ϵ is bounded uniformly with respect to ϵ. Taking into account (6.6f), we can employ a generalization of Helly Theorem (see [5, Lemma 7.2] and [3, Theorem 3.5, Chapter 1]), which implies that there exist a subsequence, still denoted by p^ϵ, and a function $p : [0,T] \to \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$, with bounded variation, such that, as $\epsilon \to 0$,

$$p^\epsilon(t) \rightharpoonup p(t) \ \text{ weakly* in } \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D) \ \text{ for every } t \in [0,T]. \tag{6.18}$$

It then follows that $p(t)$ is bounded in $M_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$ uniformly with respect to t.

From (6.17a) we also get, possibly passing to another subsequence, that there exists $e \in L^\infty([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym}))$ such that

$$e^\epsilon \rightharpoonup e \text{ weakly* in } L^\infty([0,T];L^2(\Omega;\mathbb{M}^{n\times n}_{\rm sym})), \tag{6.19}$$

as $\epsilon \to 0$.

Writing $E(u^\epsilon - w^\epsilon) = e^\epsilon + p^\epsilon - Ew^\epsilon$, by (6.4e), (6.6f), (6.17a), and (6.17b), we see that $E(u^\epsilon - w^\epsilon)$ is bounded in $L^\infty([0,T];L^1(\Omega;\mathbb{M}^{n\times n}_{\rm sym}))$ uniformly with respect to ϵ, so that, thanks to (4.1), $u^\epsilon - w^\epsilon$ is bounded in $L^\infty([0,T];BD(\Omega,\mathbb{R}^n))$ uniformly with respect to ϵ. Then, as a consequence of the embedding $BD(\Omega) \hookrightarrow L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^n)$, there exists $u \in L^\infty([0,T];L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^n))$ such that

$$u^\epsilon \rightharpoonup u \text{ weakly* in } L^\infty([0,T];L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^n)), \tag{6.20}$$

again for a suitable subsequence, as $\epsilon \to 0$. Using the equality $Eu^\epsilon = e^\epsilon + p^\epsilon$, from (6.18) and (6.19) we obtain that $u \in L^\infty([0,T];BD(\Omega))$ and $Eu = e + p$.

By (3.25) we see that the function $(u^\epsilon, e^\epsilon, p^\epsilon)$ satisfies the equilibrium condition

$$\begin{aligned} -\mathcal{H}(q) \leq \langle A^0 e^\epsilon(t), \eta\rangle + \langle \epsilon A^1 \dot e^\epsilon(t), \eta\rangle + \langle \epsilon \dot p^\epsilon(t), q\rangle \\ + \langle \epsilon^2 \ddot u^\epsilon(t), \varphi\rangle - \langle f^\epsilon(t), \varphi\rangle - \langle g^\epsilon(t), \varphi\rangle_{\Gamma_1} \leq \mathcal{H}(-q), \end{aligned} \tag{6.21}$$

for every $(\varphi, \eta, q) \in A(0)$ and a.e. $t \in [0,T]$.

Let us fix a smooth and non-negative real function ψ on $[0,T]$. Multipling the previous formula by ψ and integrating on $[0,T]$ we get

$$\begin{aligned}-\int_0^T \mathcal{H}(q)\psi(s)ds &\leq \int_0^T \langle A^0 e^\epsilon(s),\eta\rangle\psi(s)ds + \int_0^T \langle \epsilon A^1 \dot{e}^\epsilon(s),\eta\rangle\psi(s)ds \\ &+ \int_0^T \langle \epsilon\dot{p}^\epsilon(s),q\rangle\psi(s)ds + \int_0^T \langle \epsilon^2\ddot{u}^\epsilon(s),\varphi\rangle\psi(s)ds - \int_0^T \langle f^\epsilon(s),\varphi\rangle\psi(s)ds \\ &- \int_0^T \langle g^\epsilon(s),\varphi\rangle_{\Gamma_1}\psi(s)ds \leq \int_0^T \mathcal{H}(-q)\psi(s)ds, \end{aligned} \tag{6.22}$$

for every $(\varphi,\eta,q)\in A(0)$. It is easily seen that, if ψ has compact support, thanks to (6.17e) the term

$$\int_0^T \langle \epsilon^2\ddot{u}^\epsilon(s),\varphi\rangle\psi(s)ds = -\epsilon^2\int_0^T \langle \dot{u}^\epsilon(s),\varphi\rangle\dot{\psi}(s)ds$$

vanishes as $\epsilon\to 0$, and the same is true for the terms

$$\int_0^T \langle \epsilon A^1\dot{e}^\epsilon(s),\eta\rangle\psi(s)ds + \int_0^T \langle \epsilon\dot{p}^\epsilon(s),q\rangle\psi(s)ds$$

thanks to (3.2a), (6.17c) and (6.17d). Moreover, by (4.23) we can write

$$\int_0^T (\langle f^\epsilon(s),\varphi\rangle + \langle g^\epsilon(s),\varphi\rangle_{\Gamma_1})\psi(s)ds = \int_0^T \langle \varrho^\epsilon(s),\eta+q\rangle\psi(s)ds,$$

and, thanks to (6.5h), we obtain that the last expression tends to

$$\int_0^T \langle \varrho(s),\eta+q\rangle\psi(s)ds = \int_0^T (\langle f(s),\varphi\rangle + \langle g(s),\varphi\rangle_{\Gamma_1})\psi(s)ds.$$

So from (6.19) and (6.22) we get

$$\begin{aligned}-\int_0^T \mathcal{H}(q)\psi(s)ds &\leq \int_0^T \langle A^0 e(s),\eta\rangle\psi(s)ds - \int_0^T \langle f(s),\varphi\rangle\psi(s)ds \\ &- \int_0^T \langle g(s),\varphi\rangle_{\Gamma_1}\psi(s)ds \leq \int_0^T \mathcal{H}(-q)\psi(s)ds,\end{aligned}$$

and thanks to the arbitrariness of ψ:

$$-\mathcal{H}(q) \leq \langle A^0 e(t),\eta\rangle - \langle f(t),\varphi\rangle - \langle g(t),\varphi\rangle_{\Gamma_1} \leq \mathcal{H}(-q), \tag{6.23}$$

for a fixed $(\varphi,\eta,q)\in A(0)$ and for a.e. $t\in[0,T]$. The fact that $A(0)$ is separable allows us to prove that for a.e. $t\in[0,T]$ inequalities (6.23) hold for every $(\varphi,\eta,q)\in A(0)$.

Let us define $\sigma(t) := A^0 e(t)$. For each $q\in L^2(\Omega;\mathbb{M}^{n\times n}_D)$, since $(0,q,-q)\in A(0)$, we see that

$$-\mathcal{H}(-q) \leq \langle \sigma(t),q\rangle \leq \mathcal{H}(q), \tag{6.24}$$

which says that $\sigma_D(t)\in\partial\mathcal{H}(0)=\mathcal{K}(\Omega)$ (see (3.19)). Moreover, since for each $\varphi\in H^1_{\Gamma_0}(\Omega;\mathbb{R}^n)$ we have $(\varphi,E\varphi,0)\in A(0)$, from (6.23) we obtain

$$\langle \sigma(t),E\varphi\rangle - \langle f(t),\varphi\rangle = \langle g(t),\varphi\rangle_{\Gamma_1} \quad \text{for all } \varphi\in H^1_{\Gamma_0}(\Omega;\mathbb{R}^n). \tag{6.25}$$

From this we get $\mathrm{div}\sigma(t) = f(t)$ a.e. in Ω, and $[\sigma(t)\nu] = g(t)$ on Γ_1. Therefore, $(u(t),e(t),p(t))$ satisfies condition (c) of Remark 5.3. This implies that for a.e. $t\in[0,T]$, $(u(t),e(t),p(t))$ satisfies the minimality condition (5.3) for all $(\varphi,\eta,q)\in A_{BD}(w(t))$. We now set $S := \{0\}\cup\{t\in(0,T] :\ (5.3) \text{ is satisfied}\}$ and we define $u(0) := u_0$ and $e(0) := e_0$. Since $p(0) = p_0$ by (6.6f) and (6.18), we deduce from (6.6c) that condition (5.3) is also satisfied for $t=0$.

Since $t \mapsto p(t)$ has bounded variation from $[0,T]$ into $\mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D)$, it is globally bounded and there exists a countable set $N \subset [0,T]$ such that for every $t \in [0,T] \setminus N$

$$p(s) \to p(t) \quad \text{strongly in } \mathcal{M}_b(\Omega \cup \Gamma_0; \mathbb{M}^{n\times n}_D) \ \text{ as } s \to t. \tag{6.26a}$$

By the minimality property of $(u(s), e(s), p(s))$ for $s \in S$ we can apply [5, Theorem 3.8] and for every $t \in S \setminus N$ we obtain

$$e(s) \to e(t) \quad \text{strongly in } L^2(\Omega; \mathbb{M}^{n\times n}_{\rm sym}) \ \text{ as } s \to t, \tag{6.26b}$$

$$u(s) \to u(t) \quad \text{strongly in } BD(\Omega) \ \text{ as } s \to t. \tag{6.26c}$$

By the continuity of the embedding $BD(\Omega) \hookrightarrow L^{\frac{n}{n-1}}(\Omega; \mathbb{R}^n)$ we also get

$$u(s) \to u(t) \quad \text{strongly in } L^{\frac{n}{n-1}}(\Omega; \mathbb{R}^n) \ \text{ as } s \to t. \tag{6.26d}$$

In order to prove the energy balance (5.4) we fix $t \in S \setminus (N \cup \{0\})$. For every k let $0 = t^k_0 < t^k_1 < ... < t^k_k = t$ be elements of $(S \setminus N) \cup \{0\}$ such that $\max_i (t^k_i - t^k_{i-1}) \to 0$ as $k \to \infty$. Then, since $(u(t^k_i) - (w(t^k_i) - w(t^k_{i-1})), e(t^k_i) - (Ew(t^k_i) - Ew(t^k_{i-1})), p(t^k_i)) \in A_{BD}(w(t^k_{i-1}))$ by (5.3), we have

$$\begin{aligned}
&\mathcal{Q}_0(e(t^k_{i-1})) - \langle f(t^k_{i-1}), u(t^k_{i-1})\rangle - \langle g(t^k_{i-1}), u(t^k_{i-1})\rangle_{\Gamma_1} \le \mathcal{Q}_0(e(t^k_i)) \\
&- \langle A^0 e(t^k_i), Ew(t^k_i) - Ew(t^k_{i-1})\rangle + \mathcal{Q}_0(Ew(t^k_i)) - Ew(t^k_{i-1})) \\
&- \langle f(t^k_{i-1}), u(t^k_i) - (w(t^k_i) - w(t^k_{i-1}))\rangle \\
&- \langle g(t^k_{i-1}), u(t^k_i) - (w(t^k_i) - w(t^k_{i-1}))\rangle_{\Gamma_1} + \mathcal{H}(p(t^k_i) - p(t^k_{i-1})).
\end{aligned}$$

Employing the integration by parts formula (4.16) and then summing up over $i = 1, \dots, k$, we obtain

$$\begin{aligned}
&\mathcal{Q}_0(e(t)) - \mathcal{Q}_0(e_0) + \sum_{i=1}^k \mathcal{H}(p(t^k_i) - p(t^k_{i-1})) + \sum_{i=1}^k \mathcal{Q}_0(Ew(t^k_i) - Ew(t^k_{i-1})) \ge \\
&\ge \sum_{i=1}^k \langle A^0 e(t^k_i), Ew(t^k_i) - Ew(t^k_{i-1})\rangle + \langle \varrho(t), e(t) - Ew(t)\rangle - \langle \varrho(0), e(0) - Ew(0)\rangle \\
&+ \langle \varrho_D(t), p(t)\rangle - \langle \varrho_D(0), p(0)\rangle - \sum_{i=1}^k \langle \varrho(t^k_i) - \varrho(t^k_{i-1}), e(t^k_i)\rangle \\
&+ \sum_{i=1}^k \langle \varrho(t^k_i) - \varrho(t^k_{i-1}), Ew(t^k_i)\rangle - \sum_{i=1}^k \langle \varrho_D(t^k_i) - \varrho_D(t^k_{i-1}), p(t^k_i)\rangle.
\end{aligned} \tag{6.27}$$

By (4.28), (4.29), (6.4d), (6.5e), (6.5f), and (6.26) we can apply Lemmas 7.1 and 7.2, with S replaced by $S \setminus (N \cup \{0\})$, and we obtain that the four Riemann sums in the right-hand side of (6.27) converge to

$$\int_0^t \langle \sigma, Ew\rangle ds, \quad \int_0^t \langle \dot\varrho, e\rangle ds, \quad \int_0^t \langle \dot\varrho, Ew\rangle ds, \quad \int_0^t \langle \dot\varrho_D, p\rangle ds.$$

Moreover we see that $\sum_{i=1}^k \mathcal{Q}_0(Ew(t^k_i) - Ew(t^k_{i-1}))$ tends to 0 as $k \to \infty$, thanks to the absolute continuity of $t \mapsto Ew(t)$. Therefore, passing to the limit in (6.27) we obtain

$$\begin{aligned}
&\mathcal{Q}_0(e(t)) + \mathcal{D}_H(p; 0, t) - \langle \varrho(t), e(t) - Ew(t)\rangle - \langle \varrho_D(t), p(t)\rangle \ge \\
&\ge \mathcal{Q}_0(e_0) - \langle \varrho(0), e(0) - Ew(0)\rangle - \langle \varrho_D(0), p(0)\rangle + \int_0^t \langle \sigma, E\dot w\rangle ds \\
&- \int_0^t \langle \dot\varrho, e - Ew\rangle ds - \int_0^t \langle \dot\varrho_D, p\rangle ds,
\end{aligned} \tag{6.28}$$

for a.e. $t \in [0,T]$, where $\sigma = A^0 e$.

We want to show that actually equality holds. In order to prove the opposite inequality we consider equation (6.13).

Thanks to the semicontinuity of $\mathcal{Q}_0(\cdot)$, by (6.19) we have

$$\int_a^b \mathcal{Q}_0(e(t))dt \le \liminf_{\epsilon\to 0} \int_a^b \mathcal{Q}_0(e^\epsilon(t))dt \tag{6.29}$$

for all $0 < a < b < T$. We claim that

$$\begin{aligned}&\int_a^b \Big(\mathcal{D}_H(p;0,t) - \langle \varrho_D(t), p(t)\rangle + \langle \varrho_D(0), p_0\rangle + \int_0^t \langle \dot\varrho_D, p\rangle ds\Big)dt \le \\ &\le \liminf_{\epsilon\to 0} \int_a^b \Big(\int_0^t \mathcal{H}(\dot p^\epsilon)ds - \int_0^t \langle \varrho_D^\epsilon, \dot p^\epsilon\rangle ds\Big)dt,\end{aligned} \tag{6.30}$$

for all $0 < a < b < T$. This, together with (6.29), implies

$$\begin{aligned}&\int_a^b \Big(\mathcal{Q}_0(e(t)) + \mathcal{D}_H(p;0,t) - \langle \varrho_D(t), p(t)\rangle + \langle \varrho_D(0), p_0\rangle + \int_0^t \langle \dot\varrho_D, p\rangle ds\Big)dt \le \\ &\le \liminf_{\epsilon\to 0} \int_a^b \Big(\mathcal{Q}_0(e^\epsilon(t)) + \frac{\epsilon^2}{2}\|\dot u^\epsilon(t) - \dot w^\epsilon(t)\|_{L^2}^2 + \epsilon \int_0^t \mathcal{Q}_1(\dot e^\epsilon)ds \\ &+ \epsilon \int_0^t \|\dot p^\epsilon\|_{L^2}^2 ds + \int_0^t \mathcal{H}(\dot p^\epsilon)ds - \int_0^t \langle \varrho_D^\epsilon, \dot p^\epsilon\rangle ds\Big)dt = \\ &= \liminf_{\epsilon\to 0} \int_a^b \Big(\int_0^t \langle \sigma^\epsilon, E\dot w^\epsilon\rangle ds + \langle \varrho^\epsilon(t), e^\epsilon(t) - Ew^\epsilon(t)\rangle \\ &- \langle \varrho^\epsilon(0), e^\epsilon(0) - Ew^\epsilon(0)\rangle - \int_0^t \langle \dot\varrho^\epsilon, e^\epsilon - Ew^\epsilon\rangle ds \\ &- \epsilon^2 \int_0^t \langle \ddot w^\epsilon, \dot u^\epsilon - \dot w^\epsilon\rangle ds + \mathcal{Q}_0(e_0^\epsilon) + \frac{\epsilon^2}{2}\|v_0^\epsilon - \dot w^\epsilon(0)\|_{L^2}^2\Big)dt,\end{aligned} \tag{6.31}$$

where the equality follows from (6.13) after an integration by parts in time.

Using (6.4f), (6.4h), (6.4i), (6.6g), and (6.17e) it is easily seen that

$$\epsilon^2 \int_a^b \Big(\int_0^t \langle \ddot w^\epsilon, \dot u^\epsilon - \dot w^\epsilon\rangle ds\Big)dt \to 0, \tag{6.32a}$$

$$\epsilon^2 \|v_0^\epsilon - \dot w^\epsilon(0)\|_{L^2}^2 \to 0, \tag{6.32b}$$

while

$$\int_a^b \Big(\int_0^t \langle \sigma^\epsilon, E\dot w^\epsilon\rangle ds\Big)dt \to \int_a^b \Big(\int_0^t \langle \sigma, E\dot w\rangle ds\Big)dt, \tag{6.32c}$$

$$\mathcal{Q}_0(e_0^\epsilon) \to \mathcal{Q}_0(e_0), \tag{6.32d}$$

$$\int_a^b \langle \varrho^\epsilon(t), e^\epsilon(t) - Ew^\epsilon(t)\rangle dt \to \int_a^b \langle \varrho(t), e(t) - Ew(t)\rangle dt, \tag{6.32e}$$

$$\langle \varrho^\epsilon(0), e^\epsilon(0) - Ew^\epsilon(0)\rangle \to \langle \varrho(0), e(0) - Ew(0)\rangle, \tag{6.32f}$$

$$\int_a^b \Big(\int_0^t \langle \dot\varrho^\epsilon, e^\epsilon - Ew^\epsilon\rangle ds\Big)dt \to \int_a^b \Big(\int_0^t \langle \dot\varrho, e - Ew\rangle ds\Big)dt, \tag{6.32g}$$

thanks to (6.4e), (6.4h), (6.5h), (6.6e), (6.17c), and (6.19) . This implies that

$$\int_a^b \Big(\mathcal{Q}_0(e(t)) + \mathcal{D}_H(p;0,t) - \langle \varrho_D(t), p(t)\rangle + \langle \varrho_D(0), p_0\rangle + \int_0^t \langle \dot\varrho_D, p\rangle ds\Big) dt \leq$$
$$\leq \int_a^b \Big(\int_0^t \langle \sigma, E\dot w\rangle ds + \mathcal{Q}_0(e_0) + \langle \varrho(t), e(t) - Ew(t)\rangle$$
$$- \langle \varrho(0), e(0) - Ew(0)\rangle - \int_0^t \langle \dot\varrho, e - Ew\rangle ds\Big) dt. \tag{6.33}$$

From the arbitrariness of a and b and from (6.28) for a.e. $t \in [0,T]$ we obtain (5.6), which is equivalent to (5.4).

It remains to prove claim (6.30). This will be done by adapting the proof of [5, Theorem 4.5]. Let $\varphi : [0,+\infty) \to \mathbb{R}$ be a non-negative C^∞ function such that $\phi(s) = 0$ for $s \leq 1$ and $\phi(s) = 1$ for $s \geq 2$. For $\delta > 0$ we define $\psi_\delta(x) := \phi(\frac{1}{\delta}\mathrm{dist}(x,\Gamma_1))$ for $x \in \bar\Omega$.

Since H is positively 1-homogeneous and satisfies (4.35) we have that

$$\int_0^t \mathcal{H}(\psi_\delta \dot p^\epsilon) ds - \int_0^t \langle \varrho^\epsilon_D, \dot p^\epsilon \psi_\delta\rangle ds \leq \int_0^t \mathcal{H}(\dot p^\epsilon) ds - \int_0^t \langle \varrho^\epsilon_D, \dot p^\epsilon\rangle ds. \tag{6.34}$$

Integrating by parts with respect to time and using then (4.17), this is equivalent to

$$\int_0^t \mathcal{H}(\psi_\delta \dot p^\epsilon) ds - \int_0^t \langle \dot\varrho^\epsilon, (e^\epsilon - Ew^\epsilon)\psi_\delta\rangle ds + \int_0^t \langle \dot f^\epsilon, \psi_\delta(u^\epsilon - w^\epsilon)\rangle ds$$
$$- \int_0^t \langle \dot\varrho^\epsilon, (u^\epsilon - w^\epsilon) \odot \nabla\psi_\delta\rangle ds - \langle [\varrho^\epsilon_D(t) \cdot p^\epsilon(t)], \psi_\delta\rangle + \langle [\varrho^\epsilon_D(0) \cdot p^\epsilon(0)], \psi_\delta\rangle \leq$$
$$\leq \int_0^t \mathcal{H}(\dot p^\epsilon) ds - \int_0^t \langle \varrho^\epsilon_D, \dot p^\epsilon\rangle ds. \tag{6.35}$$

The lower semicontinuity of the variation, together with (4.32) and (6.18), implies

$$\mathcal{D}_H(\psi_\delta p; 0,t) \leq \liminf_{\epsilon\to 0} \int_0^t \mathcal{H}(\psi_\delta \dot p^\epsilon(s)) ds. \tag{6.36}$$

By (4.24), (6.4e), (6.5g), (6.5h),(6.6d), and (6.6e), using Lemma 4.3 we obtain

$$\langle [\varrho^\epsilon_D(0) \cdot p^\epsilon(0)], \psi_\delta\rangle \to \langle [\varrho_D(0) \cdot p(0)], \psi_\delta\rangle. \tag{6.37}$$

For what concerns the term $\langle [\varrho^\epsilon_D(t) \cdot p^\epsilon(t)], \psi_\delta\rangle$, we fix $0 \leq a < b \leq T$ and integrate on $[a,b]$ with respect to time. Using (4.17) we write

$$\int_a^b \langle [\varrho^\epsilon_D \cdot p^\epsilon], \psi_\delta\rangle ds = - \int_a^b \langle \varrho^\epsilon \cdot (e^\epsilon - Ew^\epsilon), \psi_\delta\rangle ds$$
$$+ \int_a^b \langle f^\epsilon, \psi_\delta(u^\epsilon - w^\epsilon)\rangle ds - \int_a^b \langle \varrho^\epsilon, (u^\epsilon - w^\epsilon) \odot \nabla\psi_\delta\rangle ds,$$

where we have used the fact that ψ_δ is zero in a neighborhood of Γ_1. The last three terms pass to the limit thanks to (6.4e), (6.5g), (6.5h), (6.19), and (6.20). Therefore, using again (4.17) we obtain

$$\int_a^b \langle [\varrho^\epsilon_D \cdot p^\epsilon], \psi_\delta\rangle ds \to \int_a^b \langle [\varrho_D \cdot p], \psi_\delta\rangle ds. \tag{6.38}$$

We now integrate in (6.35) with respect to time. By (6.4e), (6.5g), (6.5h), (6.19), (6.20), and (6.36)-(6.38) we get

$$\int_a^b \Big(\mathcal{D}_H(\psi_\delta p; 0,t) - \int_0^t \langle \dot\varrho \cdot (e - Ew), \psi_\delta\rangle ds + \int_0^t \langle \dot f, \psi_\delta(u-w)\rangle ds$$
$$- \int_0^t \langle \dot\varrho, (u-w)\odot\nabla\psi_\delta\rangle ds - \langle[\varrho_D(t)\cdot p(t)],\psi_\delta\rangle + \langle[\varrho_D(0)\cdot p(0)],\psi_\delta\rangle\Big) dt \leq$$
$$\leq \liminf_{\epsilon\to 0}\int_a^b \Big(\int_0^t \mathcal{H}(\dot p^\epsilon) ds - \int_0^t \langle \varrho_D^\epsilon, \dot p^\epsilon\rangle ds\Big) dt. \tag{6.39}$$

Using (4.17) we get

$$\int_a^b \Big(\mathcal{D}_H(\psi_\delta p; 0,t) - \langle[\varrho_D(t)\cdot p(t)],\psi_\delta\rangle + \langle[\varrho_D(0)\cdot p(0)],\psi_\delta\rangle + \int_0^t \langle[\dot\varrho_D\cdot p],\psi_\delta\rangle ds\Big) dt \leq$$
$$\leq \liminf_{\epsilon\to 0}\int_a^b \Big(\int_0^t \mathcal{H}(\dot p^\epsilon) ds - \int_0^t \langle \varrho_D^\epsilon, \dot p^\epsilon\rangle ds\Big) dt.$$

Letting $\delta \to 0$ and using the semicontinuity of $\mathcal{D}_H$ we then obtain (6.30). This concludes the proof of (5.4) for a.e. $t \in [0,T]$.

Since (5.3) and (5.4) are satisfied for a.e. $t \in [0,T]$, and in particular for $t = 0$, we can apply Theorem 5.2. We obtain that $p : [0,T] \to \mathcal{M}_b(\Omega\cup\Gamma_0; \mathbb{M}^{n\times n}_D)$ is absolutely continuous and we can redefine $u(t)$ and $e(t)$ on a set of times with measure zero so that $u : [0,T] \to BD(\Omega)$ and $e : [0,T] \to L^2(\Omega, \mathbb{M}^{n\times n}_{\rm sym})$ are absolutely continuous and the function (u,e,p,σ), with $\sigma(t) = A^0 e(t)$, is a quasistatic evolution in perfect plasticity with initial conditions u_0, e_0, p_0, and boundary condition w on Γ_0.

From (6.32) and from the energy balance (5.4) it follows that the inequality in (6.31) is actually an equality and that the liminf is a limit. So, since

$$\int_a^b \Big(\frac{\epsilon^2}{2}\|\dot u^\epsilon(t) - \dot w^\epsilon(t)\|^2_{L^2} + \epsilon\int_0^t \mathcal{Q}_1(\dot e^\epsilon) ds + \epsilon\int_0^t \|\dot p^\epsilon\|^2_{L^2} ds\Big) dt \geq 0,$$

it follows that equality holds also in (6.29) and (6.30), and that the liminf is a limit also in this formulae. In particular

$$\int_0^T \mathcal{Q}_0(e^\epsilon(t)) dt \to \int_0^T \mathcal{Q}_0(e(t)) dt, \tag{6.40}$$

Since $e^\epsilon \rightharpoonup e$ weakly by (6.19), from (6.40) it follows that

$$e^\epsilon \to e \quad \text{strongly in } L^2([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_{\rm sym})), \tag{6.41}$$

which gives (6.9) for a suitable subsequence. From this and (6.18) we conclude that

$$Eu^\epsilon(t) \rightharpoonup Eu(t) \quad \text{weakly* in } \mathcal{M}_b(\Omega\cup\Gamma_0; \mathbb{M}^{n\times n}_{\rm sym}), \tag{6.42}$$

for a.e. $t \in [0,T]$.

Let us fix t for which (6.9) and (6.42) hold. Since $u^\epsilon(t) \in A(w^\epsilon(t))$, it follows from (4.1) that $u^\epsilon(t)$ is bounded in $BD(\Omega)$ uniformly with respect to ϵ. Up to a subsequence we may assume that $u^\epsilon(t)$ converges weakly* in $BD(\Omega)$ to a function v. By Lemma 4.1 it follows that $(v, e(t), p(t)) \in A_{BD}(w(t))$. Since we have also $(u(t), e(t), p(t)) \in A_{BD}(w(t))$, we deduce that $Ev = Eu(t)$ in Ω and $(w(t)-v)\odot\nu = (w(t)-u(t))\odot\nu$ $\mathcal{H}^{n-1}$-almost everywhere on Γ_0. This implies that $v = u(t)$ $\mathcal{H}^{n-1}$ almost everywhere on Γ_0, and applying inequality (4.1) to $v - u(t)$ we obtain that $v = u(t)$ almost everywhere in Ω. This concludes the proof of (6.8).

□

7. Appendix

This section contains the proof of two technical results concerning the convergence of suitable Riemann sums for functions with values in Banach spaces.

Lemma 7.1. *Let X be a Banach space, let $\phi \in W^{1,1}([0,T];X)$, let $S \subset (0,T]$ be a set of full measure containing T and let $\psi : S \to X'$ be a bounded weakly* continuous function. For every $k > 0$ let $\{t_i^k\}_{0\le i\le k}$ be a subset of $S\cup\{0\}$ such that $0 = t_0^k < t_1^k < \cdots < t_k^k = T$ and $\max_{i=1}^k |t_i^k - t_{i-1}^k| \to 0$ as $k \to +\infty$. Then*

$$\lim_{k\to\infty}\sum_{i=1}^k \langle \psi(t_i^k), \phi(t_i^k) - \phi(t_{i-1}^k)\rangle = \int_0^T \langle \psi(t), \dot\phi(t)\rangle dt,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality product between X' and X.

Proof. Let $\psi_k : [0,T] \to X'$ be the piecewise constant function defined by $\psi_k(t) = \psi(t_i^k)$ for $t_{i-1}^k < t \le t_i^k$. Then

$$\sum_{i=1}^k \langle \psi(t_i^k), \phi(t_i^k) - \phi(t_{i-1}^k)\rangle = \int_0^T \langle \psi_k(t), \dot\phi(t)\rangle dt.$$

Since $\psi_k(t) \rightharpoonup \psi(t)$ weakly* for every $t \in S$ we have $\langle \psi_k(t), \dot\phi(t)\rangle \to \langle \psi(t), \dot\phi(t)\rangle$ for a.e. $t \in [0,T]$. The conclusion follows from the Dominated Convergence Theorem. □

The next lemma extends the previous result to the case of the duality product introduced in (4.13).

Lemma 7.2. *Let ϱ be the function introduced in the safe-load condition* (4.23)-(4.25) *and let $p : [0,T] \to \mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$ be a bounded function. Assume that there exists a set $S \subset (0,T]$ of full measure containing T such that for every $t \in S$ the function p is continuous at t with respect to the strong topology of $\mathcal{M}_b(\Omega\cup\Gamma_0;\mathbb{M}^{n\times n}_D)$ and $p(t) \in \Pi_{\Gamma_0}(\Omega)$. For every $k > 0$ let $\{t_i^k\}_{0\le i\le k}$ be a subset of $S\cup\{0\}$ such that $0 = t_0^k < t_1^k < \cdots < t_k^k = T$ and $\max_{i=1}^k |t_i^k - t_{i-1}^k| \to 0$ as $k \to +\infty$. Then*

$$\lim_{k\to\infty}\sum_{i=1}^k \langle \varrho_D(t_i^k) - \varrho_D(t_{i-1}^k), p(t_i^k)\rangle = \int_0^T \langle \dot\varrho_D(t), p(t)\rangle dt,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality product introduced in (4.13).

Proof. Let $p_k : [0,T] \to \Pi_{\Gamma_0}(\Omega)$ be the piecewise constant function defined by $p_k(t) = p(t_i^k)$ for $t_{i-1}^k < t \le t_i^k$. Using (4.28) and (4.29) we obtain that

$$\sum_{i=1}^k \langle \varrho_D(t_i^k) - \varrho_D(t_{i-1}^k), p(t_i^k)\rangle = \int_0^T \langle \dot\varrho_D(t), p_k(t)\rangle dt =$$
$$= \int_0^T \langle \dot\varrho_D(t), p_k(t) - p(t)\rangle dt + \int_0^T \langle \dot\varrho_D(t), p(t)\rangle dt. \tag{7.1}$$

By (4.14) we have

$$\int_0^T |\langle \dot\varrho_D(t), p_k(t) - p(t)\rangle| dt \le \int_0^T \|\dot\varrho_D(t)\|_{L^\infty} \|p_k(t) - p(t)\|_{\mathcal{M}_b} dt$$

Since $\|p_k(t) - p(t)\|_{\mathcal{M}_b} \to 0$ for a.e. $t \in S$ by our continuity assumption and $t \mapsto \|\dot\varrho(t)\|_{L^\infty}$ belongs to $L^1([0,T])$ (see [5, Theorem 7.1]), we obtain

$$\lim_{k\to\infty}\int_0^T |\langle \dot\varrho_D(t), p_k(t) - p(t)\rangle| dt = 0 \tag{7.2}$$

by the Dominated Convergence Theorem. The conclusion follows from (7.1) and (7.2). □

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