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Ill-posedness for Bounded Admissible Solutions of the 2-dimensional p-system

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Abstract. Consider the p–system of isentropic gas dynamics in n space dimensions, where $n \geq 2$. In a recent joint work with László Székelyhidi we showed bounded initial data for which this system has infinitely many admissible solutions. Moreover, the solutions and the initial data are bounded away from the void. Our result builds on an earlier work where we introduced a new tool to generate wild solutions to the Euler equations for incompressible fluids.

1. Introduction

The p–system of isentropic gas dynamics in Eulerian coordinates is perhaps the oldest hyperbolic system of conservation laws. The unknowns of the system, which consists of $n+1$ equations, are the density ρ and the velocity v of the gas:

$$\left\{\begin{array}{l} \partial_t \rho + \mathrm{div}_x(\rho v) \;=\; 0 \\ \partial_t(\rho v) + \mathrm{div}_x(\rho v \otimes v) + \nabla[p(\rho)] \;=\; 0 \\ \rho(0,\cdot) \;=\; \rho^0 \\ v(0,\cdot) \;=\; v^0 \end{array}\right. \tag{1.1}$$

(cf. (3.3.17) in [**2**] and Section 1.1 of [**9**] p7). The pressure p is a function of ρ, which is determined from the constitutive thermodynamic relations of the gas in question and satisfies the assumption $p' > 0$. A typical example is $p(\rho) = k\rho^\gamma$, with constants $k > 0$ and $\gamma > 1$, which gives the constitutive relation for a polytropic gas (cf. (3.3.19) and (3.3.20) of [**2**]). Weak solutions of (1.1) are bounded functions in $\mathbb{R}^n$, which solve it in the sense of distributions. Thus, weak solutions satisfy the following identities for every test function $\psi, \varphi \in C_c^\infty(\mathbb{R}^n \times [0,\infty[)$:

$$\int_0^\infty \int_{\mathbb{R}^n} \Big[\rho \partial_t \psi + \rho v \cdot \nabla_x \psi\Big]\, dx\, dt \;+\; \int_{\mathbb{R}^n} \rho^0(x)\, \psi(x,0)\, dx = 0, \tag{1.2}$$

$$\int_0^\infty \int_{\mathbb{R}^n} \Big[\rho v \cdot \partial_t \varphi + \rho \langle v \otimes v, \nabla \varphi\rangle\Big]\, dx\, dt \;+\; \int_{\mathbb{R}^n} \rho^0(x) v^0(x) \cdot \varphi(x,0)\, dx \;=0. \tag{1.3}$$

Admissible solutions have to satisfy an additional inequality, coming from the conservation law for the energy of the system. More precisely, consider the internal energy $\varepsilon : \mathbb{R}^+ \to \mathbb{R}$ given through the law $p(r) = r^2 \varepsilon'(r)$.

Definition 1.1. A weak solution of (1.1) is admissible if the following inequality holds for every nonnegative $\psi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R})$:

$$\int_0^\infty \int_{\mathbb{R}^n} \left[\left(\rho\varepsilon(\rho) + \frac{\rho|v|^2}{2}\right)\partial_t\psi + \left(\rho\varepsilon(\rho) + \frac{\rho|v|^2}{2} + p(\rho)\right) v\cdot\nabla_x\psi\right]$$
$$+ \int_{\mathbb{R}^n}\left(\rho^0\varepsilon(\rho^0) + \frac{\rho^0|v^0|^2}{2}\right)\psi(\cdot,0) \;\geq\; 0\,. \tag{1.4}$$

In the paper [**4**] we have given a proof of the following result.

Theorem 1.2. *Let $n \geq 2$. Then, for any given function p, there exist bounded initial data (ρ^0, v^0) with $\rho^0 \geq c > 0$ for which there are infinitely many bounded admissible solutions (ρ, v) of (1.1) with $\rho \geq c > 0$.*

Remark 1.3. In fact, all the solutions constructed in our proof of Theorem 1.2 satisfy the energy *equality*, that is, the equality sign holds in (1.4). They are therefore also entropy solutions of the full compressible Euler system (see for instance example (d) of Section 3.3 of [**2**]) and they show nonuniqueness in this case as well.

2. Ill–posedness for incompressible Euler

Theorem 1.2 is a byproduct of some ideas which we have originally introduced to study certain celabrated examples of wild solutions to the Euler equations of incompressible fluid dynamics:

$$\begin{cases} \partial_t v + \mathrm{div}\,(v\otimes v) + \nabla p \;=\; 0,\\ \mathrm{div}\, v \;=\; 0,\\ v(x,0) \;=\; v^0(x), \end{cases} \tag{2.1}$$

where the initial data v^0 satisfies the compatibility condition

$$\mathrm{div}\, v^0 \;=\; 0\,. \tag{2.2}$$

In his pioneering work [**8**] V. Scheffer showed that weak solutions to the 2–dimensional Euler equations are not unique. In particular Scheffer constructed a nontrivial weak solution which is compactly supported in space and time, thus disproving uniqueness for (2.1) even when $v^0 = 0$. A simpler construction was later proposed by A. Shnirelman in [**10**].

In a recent paper [**3**], we have shown how the general framework of convex integration [**1, 7, 5**] combined with Tartar's programme on oscillation phenomena in conservation laws [**12**] (see also [**6**] for an overview) can be applied to (2.1). In this way, one can easily recover Scheffer's and Shnirelman's counterexamples in all dimensions and with bounded velocity and pressure. Moreover, the construction yields as a simple corollary the existence of energy–decreasing solutions, thus recovering another groundbreaking result of Shnirelman [**11**], again with the additional features that our examples have bounded velocity and pressures and can be shown to exist in any dimension.

These results left open the question of whether one might achieve the uniqueness of weak solutions by imposing a form of the energy inequality. In the work [**4**] we answered this question on the negative for several known criteria. Though the motivation for (1.4) comes from the theory of shock waves, which are obviously absent in incompressible Euler, these admissibility criteria are formally very similar to that of Definition 1.1, Therefore, the ideas introduced in [**4**] can be successfully exported to admissible solutions of the p–system, yielding Theorem 1.2 as a corollary.

3. Plane wave analysis of Euler's equations

We start by briefly explaining Tartar's framework [**12**]. One considers nonlinear PDEs that can be expressed as a system of linear PDEs (conservation laws)

$$\sum_{i=1}^{m} A_i \partial_i z = 0 \tag{3.1}$$

coupled with a pointwise nonlinear constraint (constitutive relations)

$$z(x) \in K \subset \mathbb{R}^d \text{ a.e.,} \tag{3.2}$$

where $z : \Omega \subset \mathbb{R}^m \to \mathbb{R}^d$ is the unknown state variable. The idea is then to consider *plane wave* solutions to (3.1), that is, solutions of the form

$$z(x) = ah(x \cdot \xi), \tag{3.3}$$

where $h : \mathbb{R} \to \mathbb{R}$. The *wave cone* Λ is given by the states $a \in \mathbb{R}^d$ such that for any choice of the profile h the function (3.3) solves (3.1), that is,

$$\Lambda := \left\{ a \in \mathbb{R}^d : \exists \xi \in \mathbb{R}^m \setminus \{0\} \quad \text{with} \quad \sum_{i=1}^{m} \xi_i A_i a = 0 \right\}. \tag{3.4}$$

The oscillatory behavior of solutions to the nonlinear problem is then determined by the compatibility of the set K with the cone Λ.

The Euler equations can be naturally rewritten in this framework. The domain is $\mathbb{R}^m = \mathbb{R}^{n+1}$, and the state variable z is defined as $z = (v, u, q)$, where

$$q = p + \frac{1}{n}|v|^2, \text{ and } u = v \otimes v - \frac{1}{n}|v|^2 I_n,$$

so that u is a symmetric $n \times n$ matrix with vanishing trace and I_n denotes the $n \times n$ identity matrix. From now on the linear space of symmetric $n \times n$ matrices will be denoted by $\mathcal{S}^n$ and the subspace of trace–free symmetric matrices by $\mathcal{S}^n_0$. The following lemma is straightforward.

Lemma 3.1. *Suppose $v \in L^\infty(\mathbb{R}^n_x \times \mathbb{R}_t; \mathbb{R}^n)$, $u \in L^\infty(\mathbb{R}^n_x \times \mathbb{R}_t; \mathcal{S}^n_0)$, and $q \in L^\infty(\mathbb{R}^n_x \times \mathbb{R}_t)$ solve*

$$\begin{aligned} \partial_t v + \mathit{div}\, u + \nabla q &= 0, \\ \mathit{div}\, v &= 0, \end{aligned} \tag{3.5}$$

in the sense of distributions. If in addition

$$u = v \otimes v - \frac{1}{n}|v|^2 I_n \qquad \text{a.e. in } \mathbb{R}^n_x \times \mathbb{R}_t, \tag{3.6}$$

then v and $p := q - \frac{1}{n}|v|^2$ are a solution to (2.1) with $f \equiv 0$. Conversely, if v and p solve (2.1) distributionally, then v, $u := v \otimes v - \frac{1}{n}|v|^2 I_n$ and $q := p + \frac{1}{n}|v|^2$ solve (3.5) and (3.6).

Consider the $(n+1)\times(n+1)$ symmetric matrix in block form

$$U = \begin{pmatrix} u+qI_n & v \\ v & 0 \end{pmatrix}, \tag{3.7}$$

where I_n is the $n\times n$ identity matrix. Notice that by introducing new coordinates $y=(x,t)\in\mathbb{R}^{n+1}$ the equation (3.5) becomes simply

$$\mathrm{div}_y U = 0.$$

Here, as usual, a divergence–free matrix field is a matrix of functions with rows that are divergence–free vectors. Therefore the wave cone corresponding to (3.5) is given by

$$\Lambda = \left\{(v,u,q)\in\mathbb{R}^n\times\mathcal{S}^n_0\times\mathbb{R} : \det\begin{pmatrix} u+qI_n & v \\ v & 0 \end{pmatrix} = 0\right\}.$$

Remark 3.2. A simple linear algebra computation shows that for every $v\in\mathbb{R}^n$ and $u\in\mathcal{S}^n_0$ there exists $q\in\mathbb{R}$ such that $(v,u,q)\in\Lambda$, revealing that the wave cone is very large. Indeed, let $V^\perp\subset\mathbb{R}^n$ be the linear space orthogonal to v and consider on $V^\perp$ the quadratic form $\xi\mapsto\xi\cdot u\xi$. Then, $\det U = 0$ if and only if $-q$ is an eigenvalue of this quadratic form.

In order to exploit this fact for constructing irregular solutions to the nonlinear system, one needs plane wave–like solutions to (3.5) which are localized in space. Clearly an exact plane–wave as in (3.3) has compact support only if it is identically zero. Therefore this can only be done by introducing an error in the range of the wave, deviating from the line spanned by the wave state $a\in\mathbb{R}^d$. A crucial point is, therefore, to control this error.

4. The generalized energy and subsolutions

Next, for every $r\geq 0$, we consider the set of *Euler states of speed r*

$$K_r := \left\{(v,u)\in\mathbb{R}^n\times\mathcal{S}^n_0 : u = v\otimes v - \frac{r^2}{n}I_n,\ |v| = r\right\} \tag{4.1}$$

(c.f. Section of [**3**], in particular (25) therein). Lemma 3.1 says simply that solutions to the Euler equations can be viewed as evolutions on the manifold of Euler states subject to the linear conservation laws (3.5).

Next, we denote by K^{co}_r the convex hull in $\mathbb{R}^n\times\mathcal{S}^n_0$ of K_r. This convex set has been computed in [**4**].

Lemma 4.1. *For any $w\in\mathcal{S}^n$ let $\lambda_{max}(w)$ denote the largest eigenvalue of w. For $(v,u)\in\mathbb{R}^n\times\mathcal{S}^n_0$ let*

$$e(v,u) := \frac{n}{2}\lambda_{max}(v\otimes v - u). \tag{4.2}$$

Then

(i) *$e:\mathbb{R}^n\times\mathcal{S}^n_0\to\mathbb{R}$ is convex;*
(ii) *$\frac{1}{2}|v|^2\leq e(v,u)$, with equality if and only if $u = v\otimes v - \frac{|v|^2}{n}I_n$;*
(iii) *$|u|_\infty\leq 2\frac{n-1}{n}e(v,u)$, where $|u|_\infty$ denotes the operator norm of the matrix;*
(iv) *The $\frac{1}{2}r^2$–sublevel set of e is the convex hull of K_r, i.e.*

$$K^{co}_r = \left\{(v,u)\in\mathbb{R}^n\times\mathcal{S}^n_0 : e(v,u)\leq\frac{r^2}{2}\right\}. \tag{4.3}$$

(v) *If* $(u,v) \in \mathbb{R}^n \times \mathcal{S}_0^n$, *then* $\sqrt{2e(v,u)}$ *gives the smallest* ρ *for which* $(u,v) \in K_\rho^{co}$.

In view of (ii) if a triple (v,u,q) solving (3.5) corresponds a solution of the Euler equations via the correspondence in Lemma 3.1, then $e(v,u)$ is simply the energy density of the solution. In view of this remark, if (v,u,q) is a solution of (3.5), $e(v,u)$ will be called the *generalized energy density*, and $E(t) = \int_{\mathbb{R}^n} e(v(x,t),u(x,t))dx$ will be called the *generalized energy*.

The key proposition of [**4**] states, roughly speaking, that, given an energy profile $\bar{e}$ satisfying certain technical assumptions, the existence of some suitable "subsolution" for the Cauchy problem (2.1)–(2.2) implies the existence of weak solutions having energy density $\bar{e}$.

PROPOSITION 4.2. *Let* $\Omega \subset \mathbb{R}^n$ *be an open set (not necessarily bounded) and let*

$$\bar{e} \in C\big(\overline{\Omega}\times]0,T[\big) \cap C\big([0,T];L^1(\Omega)\big).$$

Assume there exists (v_0,u_0,q_0) *smooth solution of* (3.5) *on* $\mathbb{R}^n\times]0,T[$ *with the following properties:*

$$v_0 \in C\big([0,T];L^2_w\big), \tag{4.4}$$

$$\operatorname{supp}(v_0(\cdot,t),u_0(\cdot,t)) \subset\subset \Omega \text{ for all } t\in]0,T[, \tag{4.5}$$

$$e\big(v_0(x,t),u_0(x,t)\big) < \bar{e}(x,t) \text{ for all } (x,t)\in\Omega\times\,]0,T[\,. \tag{4.6}$$

Then there exist infinitely many weak solutions v *of the Euler equations* (2.1) *with pressure*

$$p = q_0 - \frac{1}{n}|v|^2 \tag{4.7}$$

such that

$$v \in C\big([0,T];L^2_w\big), \tag{4.8}$$

$$v(\cdot,t) = v_0(\cdot,t) \qquad \text{for } t = 0,T, \tag{4.9}$$

$$\frac{1}{2}|v(\cdot,t)|^2 = \bar{e}(\cdot,t)\,\mathbf{1}_\Omega \qquad \text{for every } t\in]0,T[. \tag{4.10}$$

Proposition 4.2 is proved in [**4**] combined Tartar's plane wave analysis with the so called Baire category argument. A different approach is instead given by the Lipschitz convex integration.

In order to give an idea of this second mechanism, consider the particular case $\bar{e} \equiv 1$ and $v_0 \equiv 0$. Moreover, let us neglect the technical condition (4.8). Our goal would then be to construct a weak bounded solution of Euler which

- is supported in $\Omega \times [0,T]$;
- takes the values 0 at the times 0 and T;
- has energy identically equal to 1 on $\Omega\times]0,T[$.

Note that such solution can be extended to 0 for times $t \notin [0,T]$, thus achieving the celebrated example of Scheffer. In fact, this is the solution constructed in [**3**].

The idea of the Lipschitz convex integration would be to construct (v,u) as an infinite sum

$$(v,u) = \sum_{i=1}^{\infty}(v_i,u_i)$$

with the properties that

(a) the partial sums $S_k = \sum_{i=0}^{k}(v_i, u_i)$ are smooth and compactly supported in Ω
(b) S_k takes value in the interior of the set K_1^{co}.
(c) (v, u) takes values in the extremal points K_1 a.e. in Ω,
(d) (v, u) solves the linear partial differential equations (3.5).

Now, (d) is achieved because in fact each summand (v_i, u_i) is a smooth solution of (3.5). As for (c) the key is that:

- S_k converges strongly in L^1_{loc};
- (v_i, u_i) are chosen inductively so to let $\varepsilon_k := \|\text{dist}\,(S_k, K_1)$ tend to 0.

Each (v_{k+1}, u_{k+1}) is in fact the sum of finitely many localized waves with disjoint supports, which "move" S_k closer to the K_1 in an average sense. The strong convergence is triggered by the choice of the frequencies λ_k of the localized waves, which grow very fast.

The Baire category method, instead, achieves the sequence S_k using a "stability argument". As a byproduct we obtain that, if one looks at the weak* closure of smooth maps S_k's satisfying (a) and (b), a "typical" element of this set takes its values in K_1. One advantage of the Baire category method is therefore that it produces automatically infinitely many solutions.

5. Construction of suitable initial data

Another main discovery of [**4**] is the existence of "interesting" subsolutions. Consider for instance the example discussed in the previous section: $\bar{e} \equiv 1$ and $v_0 \equiv 0$. It is then obvious that, for any v exhibited by Proposition 4.2, we have

- $v(0, \cdot) \equiv 0$;
- $\int |v(x, t)|^2\, dx = |\Omega|$ for $t \in]0, T[$.

It is therefore obvious that the initial data is achieved in a "weak sense", that is, for $t \downarrow 0$, $v(\cdot, t)$ converges only weakly to 0, but not strongly. Thus such a v obviously violate any form of energy inequality.

In order to achieve solutions v which fulfill any energy inequality we then need a subsolution v_0 which satisfies $v_0(0, x) \in K_{\bar{e}(0,x)}$ for a.e $x \in \Omega$. On the other hand, the existence of such subsolutions is by no mean obvious. For instance, the typical "weak–strong uniqueness" of admissible solutions (valid for Euler and for hyperbolic systems of conservation laws, see ?? and ??) implies that such v_0 is necessarily nonsmooth at $t = 0$.

In [**4**] we showed the existence of many "interesting subsolutions". Having fixed a bounded open set $\Omega \subset \mathbb{R}^n$ we indeed have

PROPOSITION 5.1. *There exist triples* $(\bar{v}, \bar{u}, \bar{q})$ *solving* (3.5) *in* $\mathbb{R}^n \times \mathbb{R}$ *and enjoying the following properties:*

$$\bar{q} = 0,\ (\bar{v}, \bar{u}) \text{ is smooth in } \mathbb{R}^n \times (\mathbb{R} \setminus \{0\}) \text{ and } \bar{v} \in C\big(\mathbb{R}; L^2_w\big)\,, \tag{5.1}$$

$$\text{supp}\,(\bar{v}, \bar{u}) \subset \Omega \times]-T, T[\,, \tag{5.2}$$

$$\text{supp}\,(\bar{v}(\cdot, t), \bar{u}(\cdot, t)) \subset\subset \Omega \text{ for all } t \neq 0\,, \tag{5.3}$$

$$e\big(\bar{v}(x, t), \bar{u}(x, t)\big) < 1 \text{ for all } (x, t) \in \mathbb{R}^n \times (\mathbb{R} \setminus \{0\})\,. \tag{5.4}$$

Moreover

$$\frac{1}{2}|\bar{v}(x,0)|^2 = 1 \ \textit{a.e. in } \Omega.$$

Indeed it turns out that the proof of this Proposition is just an adaptation of the same ideas of the proof of Proposition 4.2.

6. Proof of Theorem 1.2

Let $\alpha := p(1)$, $\beta := p(2)$ and $\gamma = \beta - \alpha$. Recall that $p' > 0$, we conclude $\gamma > 0$.

We let Ω be the unit ball, $T = 1/2$ and $(\bar{v}, \bar{u})$ be as in Proposition 5.1. Define $\bar{e} := n\gamma/2$, $q_0 := 0$,

$$v_0(x,t) \ := \ \frac{n\gamma}{2} \begin{cases} \bar{v}(x,t) & \text{for } t \in [0,1/2] \\ \bar{v}(x,t-1/2) & \text{for } t \in [1/2,1], \end{cases} \tag{6.1}$$

$$u_0(x,t) \ := \ \frac{n\gamma}{2} \begin{cases} \bar{u}(x,t) & \text{for } t \in [0,1/2] \\ \bar{u}(x,t-1/2) & \text{for } t \in [1/2,1]. \end{cases} \tag{6.2}$$

It is easy to see that the triple (v_0, u_0, q_0) satisfies the assumptions of Proposition 4.2 with $\bar{e} \equiv \frac{n\gamma}{2}$. Therefore, there exists infinitely many solutions $v \in C([0,1], L^2_w)$ of (2.1) in $\mathbb{R}^n \times [0,1]$ with

$$v(x,0) = \bar{v}(x,0) = v(x,1) \text{ for a.e. } x \in \Omega,$$

and such that

$$\frac{1}{2}|v(\cdot,t)|^2 \ = \ \frac{n\gamma}{2}\mathbf{1}_\Omega \qquad \text{for every } t \in]0,1[\,. \tag{6.3}$$

Since $\frac{1}{2}|v_0(\cdot,0)|^2 = \frac{n\gamma}{2}\mathbf{1}_\Omega$ as well, it turns out that the map $t \mapsto v(\cdot,t)$ is continuous in the strong topology of L^2.

Each such v can be extended to $\mathbb{R}^n \times [0,\infty[$ 1-periodically in time, by setting $v(x,t) = v(x,t-k)$ for $t \in [k,k+1]$. Summarizing, we have found infinitely many solutions (v,p) of (2.1) with the following properties:

- $v \in C([0,\infty[, L^2)$ and $|v|^2 = n\gamma\,\mathbf{1}_{\Omega\times[0,\infty[}$;
- $p = -|v|^2/n = -\gamma\,\mathbf{1}_{\Omega\times[0,\infty[}$.

Therefore, we conclude that

$$\partial_t v + \operatorname{div} v \otimes v + \nabla\big(\alpha\,\mathbf{1}_{\Omega\times[0,\infty[} + \beta\,\mathbf{1}_{\mathbb{R}^n\setminus\Omega\times[0,\infty[}\big) \ = \ 0\,.$$

Hence, if we set

$$\rho \ = \ \mathbf{1}_{\Omega\times[0,\infty[} + 2\,\mathbf{1}_{\mathbb{R}^n\setminus\Omega\times[0,\infty[}$$

for any such v, the pair (ρ, v) is a weak solution of (1.1) with initial data (ρ^0, v^0), where $\rho_0 = \mathbf{1}_\Omega + 2\,\mathbf{1}_{\mathbb{R}^n\setminus\Omega}$.

Each such solution is admissible. Indeed

$$\begin{aligned} & \partial_t \left[\rho\varepsilon(\rho) + \frac{\rho|v|^2}{2}\right] + \operatorname{div}_x \left[\left(\rho\varepsilon(\rho) + \frac{\rho|v|^2}{2} + p(\rho)\right) v\right] \\ = \ & \partial_t \left[\left(\varepsilon(1) + \frac{n\gamma}{2}\right) \mathbf{1}_{\Omega\times[0,\infty[} + 2\varepsilon(2)\,\mathbf{1}_{\mathbb{R}^n\setminus\Omega\times[0,\infty[}\right] \\ & + \left(\varepsilon(1) + p(1) + \frac{n\gamma}{2}\right) \operatorname{div} v \ = \ 0\,. \end{aligned} \tag{6.4}$$

In order to conclude requirement (1.4) of Definition 1.1, it suffices to notice that $(\rho(\cdot,t), v(\cdot,t)) \to (\rho^0, v^0)$ strongly in L^2_{loc}. In fact, this shows that (1.4) holds with the equality sign.

7. Final comments

Clearly, the solutions constructed in the previous section are discontinuous along the interface $\partial\Omega \times [0, \infty[$. However this discontinuity is not at all a classical shock wave because v does not have a "strong trace from the interior" at $\partial\Omega\times[0, \infty[$. In particulat the interface is not that of a jump discontinuity, not even in a weak sense. More precisely, though the normal trace of v at $\partial\Omega \times [0, \infty[$ is zero (for divergence–free fields, left and right normal traces coincide), the normal trace of $v \otimes v$ is:

- 0 from the "outside";
- nonzero from the "inside".

This is not at all surprising. The construction outlined in Section 4 shows that v is build by adding highly oscillatory solutions: it is quite obvious that v does not have a "strong trace from the interior" at $\partial\Omega \times [0, \infty[$. Moreover, if such a strong trace existed, the solution (v, ρ) would violate the Rankine–Hugoniot condition.

This remark brings yet an important point. The ill–posedness proved in our works do not seem to bear any relation to the formation of shock waves. One might instead conjecture that it is an effect of accumulation of vorticity. However, the reader should be extremely cautious in interpreting our solutions in terms of classical concepts of fluid dynamics. As an example, let us come back to the solutions v produced by Proposition 4.2 when $\bar{e} \equiv 1$ and $v_0 \equiv 0$. If any such solution described the motion of a physical fluid, we would have a number of paradoxes:

- The fluid would be totally at rest at time 0, move at positive time and go back at rest at time T, in spite of the total absence of external forces;
- the fluid would in fact remain at rest outside Ω and move *any* particle in Ω at speed 1, but the interface would not resemble at all that of a shear flow;
- the pressure would be constant inside Ω, displaying a total absence of interaction for the particles staying in Ω.

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