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Almgren's Q-Valued Functions Revisited

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Abstract

In a pioneering work written 30 years ago, Almgren developed a far-reaching regularity theory for area-minimizing currents in codimension higher than 1. Building upon Almgren's work, Chang proved later the optimal regularity statement for 2-dimensional currents. In some recent papers the author, in collaboration with Emanuele Spadaro, has simplified and extended some results of Almgren's theory, most notably the ones concerning Dir-minimizing multiple valued functions and the approximation of area-minimizing currents with small cylindrical excess. In this talk I will give an overview of our contributions and illustrate some possible future directions.

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1. Introduction

1.1. The regularity theory for area-minimizing currents.

In this note we will describe some recent contributions to the regularity theory for integer rectifiable area-minimizing currents. For the sake of simplicity we will restrict ourselves to currents in the Euclidean space. For all the relevant definitions concerning currents we refer the reader to the classical textbooks [16] and [39].

As it is well known there is a dramatic difference in the theory depending on the codimension of the current. In codimension 1 currents without boundary

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are boundaries of sets of finite perimeter. This allows several important simplifications in the theory (see for instance [23]) and it also implies that area-minimizing currents of codimension 1 enjoy much better regularity properties. Let us briefly review the main results in the interior regularity theory.

Codimension 1. Let T be an area-minimizing current of dimension n in $\mathbb{R}^{n+1}$.

(a1) For $n \leq 6$, T is an analytic submanifold in $\mathbb{R}^{n+1} \setminus \mathrm{supp}\,(\partial T)$ (see for instance [16, Theorem 5.4.15]);

(a2) for $n = 7$, T is an analytic submanifold in $\mathbb{R}^{n+1} \setminus \mathrm{supp}\,(\partial T)$ with the exception of a discrete set $\mathrm{Sing}(T)$ of singular points (see for instance [16, Section 5.4.16]);

(a3) for $n = 7$, in a neighborhood of each $x \in \mathrm{Sing}(T)$ the current is a perturbation of an area-minimizing cone (see [40]);

(a4) for $n > 7$, T is an analytic submanifold in $\mathbb{R}^{n+1} \setminus \mathrm{supp}\,(\partial T)$ with the exception of a closed set $\mathrm{Sing}(T)$ of (Hausdorff) dimension at most $n-7$ (see for instance [39, Theorem 37.7]);

(a5) if $n > 7$, the singular set $\mathrm{Sing}(T)$ is rectifiable and has locally finite $\mathcal{H}^{n-7}$-measure (see [42, Lecture 4, Theorem 4] and [41]; here $\mathcal{H}^\alpha$ denotes, as usual, the α-dimensional Hausdorff measure).

The results in (a2) and (a5) give the optimal estimates of the size of $\mathrm{Sing}(T)$. The optimality of (a2) is shown by the Simons cone. The minimizing property of this cone was first proved in the celebrated paper of Bombieri, De Giorgi, and Giusti [8]. In order to prove the optimality of (a5) it suffices to take the product of the Simons cone with a linear space of dimension $n-7$ (cp. with [16, Theorem 5.4.9]).

Codimension $k > 1$. Let T be an integer rectifiable area-minimizing current of dimension n in $\mathbb{R}^{n+k}$.

(b1) If $n = 1$, T is the union of nonintersecting straight lines;

(b2) if $n = 2$, T is an analytic submanifold in $\mathbb{R}^{n+k} \setminus \mathrm{supp}\,(\partial T)$ with the exception of a discrete set $\mathrm{Sing}(T)$ (see [7]);

(b3) if $n = 2$, in a neighborhood of each $x \in \mathrm{Sing}(T)$ the current is a perturbation of a suitable "branched holomorphic curve" (see [32]);

(b4) for $n > 2$, T is an analytic submanifold in $\mathbb{R}^{n+k} \setminus \mathrm{supp}\,(\partial T)$ with the exception of a closed set $\mathrm{Sing}(T)$ of dimension at most $n-2$ (see [4]).

The size estimate of (b2) is optimal, as shown by taking any holomorphic curve in $\mathbb{R}^4 = \mathbb{C}^2$ with branch points. This example plays a crucial role in the rest of our discussion and will be examined in further detail later on.

One first striking difference between these series of results is that in the latter singularities appear quite naturally as soon as we depart from the trivial case $n = 1$. Moreover, this appearance, linked to the well-known phenomenon of branching of holomorphic curves, is far much easier to understand than the minimizing property of the Simons cone, which is the simplest example of a singular area-minimizing current with codimension 1.

The second striking difference is in the length, the intricacy and the technical complications presented by Almgren's and Chang's results ((b2) and (b4)) in comparison with Federer's size estimates of $\mathrm{Sing}(T)$ ((a2) and (a4)). Assuming indeed a certain amount of prerequisites in geometric measure theory, (a1), (a2) and (a4) are essentially the combination of three ingredients: the pioneering work of De Giorgi on the excess-decay [9], the classical work of Simons on stable minimal cones [43] and Federer's reduction argument, see [17]. Moreover, only a relatively small portion of the theorems in [43] are needed to prove (a4). Let me also mention that, before the work of [43] completed the proof of (a1), lower-dimensional versions were achieved in the works of Fleming, De Giorgi and Almgren [19, 10, 3]. The interested reader might find a complete and quite readable account in the beautiful book of Giusti [23].

Assuming the same amount of prerequisites, the theorem in (b4) is instead a monograph of about 950 pages, see [4]. This monograph contains, among many other things, far-reaching generalizations of both De Giorgi's and Federer's arguments. The proof of (b2) is contained in the paper [7], where the author builds upon (essentially all) the techniques developed in Almgren's monograph and on the important papers [33] and [47]. Indeed, some of the constructions needed in [7] are claimed to be suitable modifications of the ones in [4], but the detailed proofs of these statements have never appeared.

1.2. Branching.

Let us examine in more details the first obstruction to the full regularity in the case of higher codimension. The key observation relies on a classical computation of Wirtinger [49], used by Federer in his elegant proof of the following statement (cp. to [16, Section 5.4.19]).

Theorem 1.1. *If M is a Kähler manifold of real dimension $2m$ and Γ a complex submanifold of M of real dimension $2j$, then Γ represents an integer rectifiable area minimizing current. More precisely, if U is a bounded open set with $U \cap \mathrm{supp}\,(\partial\Gamma) = \emptyset$ and Σ is an integer rectifiable current of dimension $2j$ such that*

- $\partial(\Gamma - \Sigma) = 0$,
- $\mathrm{supp}\,(\Gamma - \Sigma) \subset U$,

then the mass of Σ in U is larger than the mass of Γ in U. Moreover, the inequality is strict unless $\Gamma = \Sigma$.

In a more modern language, the Wirtinger-Federer result can be rephrased in the following way: the k-th exterior power of the Kähler form is a calibration for holomorphic submanifolds of complex dimension k. For a beautiful account of calibrating forms we refer the reader to the paper [27].

The presence of branching phenomena in area-minimizing currents of codimension larger than 1 is also the principal reason for the difficulty of Almgren's monumental result. Much of Section 2 will be devoted to give an intuitive explanation of this.

1.3. Looking for a manageable proof. The intricacy of Almgren's big regularity paper [4] has essentially stopped the research in the area till few years ago, in spite of the abundance of interesting geometric objects which are naturally minimal submanifolds of "large" codimension (see again the paper [27]). Recently, in view of some applications to geometry and topology, alternative proofs of Chang's result have been found for J-holomorphic curves. The first of these proofs has been given by Taubes in [45] for J-holomorphic curves in symplectic 4-manifolds. The generalization of Taubes' approach to $1-1$ currents in (even-dimensional) manifolds carrying a certain complex structure has been given by Rivière and Tian (see [36], [35] and [37]). This proof contains several beautiful ideas and faces some of the same problems which are solved in Almgren's monograph. However, its applicability seems limited to 2-dimensional currents which are calibrated by some complex structure. At present, the general theorem of Chang (not to speak of the result of Almgren) does not seem reachable with similar approaches.

The remarkable papers [35] and [37] and several discussions of the author with Tristan Rivière have been the starting point of the line of research which will be presented here. The results which will be described in this note have appeared in the papers [13], [14], [12], [44] and [15]. A substantial part of these papers is dedicated to give self-contained and much simpler proofs of a considerable portion of Almgren's monograph. In the remaining part we take advantage of some new ideas to expand Almgren's theories in other directions. Though some fundamental ideas behind these papers are still the ones of Almgren, our approaches highlight some rather new aspects. In some cases we have taken advantage of modern techniques of metric analysis, in some other we have discovered new phenomena. The overall result is that we can handle the complexity of the subject in a much more efficient way. Our obvious final goal is to give a less complex, yet complete account of Almgren's and Chang's regularity results and possibly go beyond them in a not so far future.

In the next sections we will describe roughly the contents of the papers [13], [14] and [15]. In the final section we collect several interesting related open problems.

2. Why Multiple Valued Functions?

2.1. De Giorgi's excess decay. The first breakthrough of the regularity theory for area-minimizing currents is due to De Giorgi. In order to state

De Giorgi's main theorem, we have to introduce the so-called (spherical) excess $\mathrm{Ex}(T, B_r(p))$ of the current T in the ball $B_r(p)$. For every simple unitary n-vector $\vec{\pi}$, we set

$$\mathrm{Ex}(T, B_r(p), \pi) := \frac{1}{2} \fint_{B_r(p)} |\vec{T} - \vec{\pi}|^2 d\|T\| \,. \tag{2.1}$$

The measure $\|T\|$ is the localized mass of the current: for every open set U, $\|T\|(U)$ is the total mass of the current in U. $\vec{T}$ is the simple unitary n-vector field orienting T.

The spherical excess is then defined as

$$\mathrm{Ex}(T, B_r(p)) \;:=\; \min_{\pi} \mathrm{Ex}(T, B_r(p), \pi) \,.$$

This definition is valid in any codimension. For the reader who is not very familiar with the notation of geometric measure theory, the formulas can be considerably simplified in codimension 1. First of all, the minimum can be taken over all oriented n-dimensional planes π ($\vec{\pi}$ is then just the unitary n-vector orienting π). Moreover $|\vec{T} - \vec{\pi}|$ can be substituted by $|\nu_T - \nu|$, where:

- ν_T is the unit vector field normal to the current, compatible with the orientation of the tangent n-vector $\vec{T}$;
- ν is the unit vector normal to π compatible with the orientation $\vec{\pi}$.

A third important object that we need to introduce is the density of the current at a point, which is defined as

$$\theta(T, p) \;:=\; \lim_{r \downarrow 0} \frac{\|T\|(B_r(p))}{\omega_n r^n} \,, \tag{2.2}$$

where ω_n denotes, as usual, the n-dimensional measure of the n-dimensional ball. The existence of the limit in (2.2) is guaranteed by the monotonicity formula (cp. with [39, Section 4.17]).

Theorem 2.1. *Let Q be a positive integer. There exist constants $\varepsilon, \beta > 0$ depending only on Q and n such that the following holds. Let T be an area-minimizing integral current of dimension n in $\mathbb{R}^{n+1}$. Assume that, for $r > 0$ and $p \in \mathrm{supp}\,(T)$, the following hypotheses are satisfied:*

(i) $\theta(T, p) = Q$;

(ii) $\mathrm{supp}\,(\partial T) \cap B_r(p) = \emptyset$;

(iii) $\|T\|(B_r(p)) \leq (Q + \varepsilon)\omega_n r^n$;

(iv) the spherical excess of T in $B_r(p)$ is smaller than ε.

Then $\mathrm{supp}\,(T) \cap B_{r/2}(p)$ *is the graph of a $C^{1,\beta}$ function f.*

To be more precise, De Giorgi in [9] proved the case $Q = 1$ of this theorem. However the general case $Q > 1$ can be easily recovered from De Giorgi's statement using the decomposition of T in boundaries of sets of finite perimeter as in [16, Section 4.5.17].

To get some intuitive idea about the theorem above, consider the extreme case where the spherical excess in $B_r(p)$ is 0. Using assumption (ii) we then conclude that T in $B_r(p)$ consists of (possibly countably many) parallel disks. Exploiting (i), (iii) and the minimality of T, from the monotonicity formula we easily conclude that, in a slightly smaller ball $B_{r-C\varepsilon}(p)$, T consists of a single disk containing the origin and counted with multiplicity Q. Thus, the assumptions (i)–(iv) tell us that the current T is close, in an "average" sense, to Q copies of a single disk. Theorem 2.1 could be therefore classified as an "ε-regularity theorem".

2.2. Again branching.

As already mentioned, De Giorgi's original proof covers the case $Q = 1$ and the extension to $Q > 1$ uses heavily the features of codimension 1 currents. In higher codimension the statement is still correct for $Q = 1$ (see for instance [16, Theorem 5.4.7]; in fact much more is true, see [2]), but fails dramatically if $Q > 1$. Once again, the main reason for this breakdown is the existence of branching points.

Remark 2.2. Consider in $\mathbb{R}^4 = \mathbb{C}^2$ the holomorphic curve $\Gamma = \{(z, w) : z^2 = w^3\}$. Theorem 1.1 implies that Γ is an area–minimizing current of real dimension 2 in any bounded open subset of $\mathbb{R}^4$. Moreover, set $p = 0$. Then

$$\theta(T, 0) = \lim_{r \downarrow 0} \frac{\|T\|(B_r(0))}{\omega_2 r^2} \; = \; 2\,.$$

Obviously, given any positive $\varepsilon > 0$ there is a δ such that (i)–(iv) are satisfied for every $r < \delta$. On the other hand, no matter how small r is, $B_r(0) \cap \Gamma$ is never the graph of a smooth function.

We proceed our discussion by giving an oversimplified description of De Giorgi's proof of Theorem 2.1 in the case $Q = 1$. In a first step, the hypotheses (i)–(iv) are used to approximate the current T with the graph G of a Lipschitz (real valued) function f with small Lipschitz constant. In particular, the approximation algorithm ensures that the area of T and the area of G are close. On the other hand, recall that the area of the graph of a function over a domain Ω is given by the formula

$$\int_\Omega \sqrt{1 + |\nabla f|^2}\,. \tag{2.3}$$

If $|\nabla f|$ is small, this integral is close to

$$\int_\Omega \left(1 + \frac{|\nabla f|^2}{2}\right) \tag{2.4}$$

(in higher codimension, i.e. when f is vector-valued, the formula for (2.3) is more complicated, but the second order expansion is nonetheless given by (2.4)).

Thus, the minimality of the current T implies that f is close, in a suitable integral sense, to a minimum of the Dirichlet energy, i.e. to an harmonic function. Using the decay properties of harmonic functions, one can infer that the excess $\mathrm{Ex}(T, B_\rho(p))$ is decaying like $\rho^{2\beta}$ for some $\beta > 0$. This decay leads then to the $C^{1,\beta}$ regularity via a "Morrey-type" argument.

2.3. Dealing with branching. As already noticed, in codimension 1 the higher multiplicity case can be reduced to the case of multiplicity 1. Obviously, Remark 2.2 shows that this reduction is impossible in codimension larger than 1. In that example the very beginning of De Giorgi's strategy fails, since it is simply not possible to approximate efficiently Γ with the graph of a (single valued) function. This discussion motivates the starting idea of Almgren's monograph. In order to tackle the regularity question in codimension larger than 1 we need to approximate currents with "multiple valued functions".

It is interesting to notice that, if we turn our attention to stationary currents (or, more generally, stationary integral varifolds), the reduction to multiplicity 1 becomes false even in the codimension 1 case. In this setting, the best result available at present is Allard's Theorem [2], which ensures regularity in a dense open set. Nothing better is known, even assuming stability, in spite of the fact that all available examples have singularities of dimension at most $n-1$. If we assume stability and an a-priori knowledge that the singular set has zero $\mathcal{H}^{n-2}$-measure, then the classical curvature estimates of Schoen and Simon imply that the singular set has in fact dimension at most $n-7$ (see [38]). In a very recent paper [48], Wickramasekera has extended this result to the optimal assumption that the $\mathcal{H}^{n-1}$-measure of the singular set is 0. Related questions are open for "stationary multiple valued functions" as well (see Section 8 below).

3. The Dirichlet Energy for Multiple Valued Functions

3.1. The metric space of unordered *Q*-tuples. Roughly the first fifth of Almgren's monograph is devoted to develop the theory of multiple valued functions. The obvious model case to keep in mind is the following. Given two integers k, Q with $\mathrm{MCD}(k, Q) = 1$, look at the function which maps each point $z \in \mathbb{C}$ into the set $M(z) := \{w^k : w^Q = z\} \subset \mathbb{C}$. Obviously for each z we can order the elements of the set $M(z)$ as $\{u_1, \ldots, u_Q\}$. However, it is not possible to do it *globally* in such a way that the maps $z \mapsto u_i(z)$ are continuous.

This motivates the following definition. Given an integer Q we define a Q-valued map from a set $E \subset \mathbb{R}^m$ into $\mathbb{R}^n$ as a function which to each point $x \in E$ associates an unordered Q-tuple of vectors in $\mathbb{R}^n$. There is a fairly efficient formulation of this definition which will play a pivotal role in our discussion.

Following Almgren, we consider the group $\mathscr{P}_Q$ of permutations of Q elements and we let $\mathcal{A}_Q(\mathbb{R}^n)$ be the set $(\mathbb{R}^n)^Q$ modulo the equivalence relation

$$(v_1, \ldots, v_Q) \equiv (v_{\pi(1)}, \ldots, v_{\pi(Q)}) \qquad \forall \pi \in \mathscr{P}_Q\,.$$

The set $\mathcal{A}_Q(\mathbb{R}^n)$ can be naturally identified with a subset of the set of measures (cp. with [4] and [13, Definition 0.1]).

Definition 3.1 (Unordered Q-tuples). Denote by $[\![P_i]\!]$ the Dirac mass in $P_i \in \mathbb{R}^n$. Then,

$$\mathcal{A}_Q(\mathbb{R}^n) := \left\{\sum_{i=1}^{Q} [\![P_i]\!] \,:\, P_i \in \mathbb{R}^n \text{ for every } i = 1, \ldots, Q\right\}.$$

This set has a natural metric structure; cp. with [4] and [13, Definition 0.2] (the experts will recognize the well-known Wasserstein 2-distance, cp. with [46]).

Definition 3.2. For every $T_1, T_2 \in \mathcal{A}_Q(\mathbb{R}^n)$, with $T_1 = \sum_i [\![P_i]\!]$ and $T_2 = \sum_i [\![S_i]\!]$, we set

$$\mathcal{G}(T_1, T_2) \;:=\; \min_{\sigma \in \mathscr{P}_Q} \sqrt{\sum_i \left|P_i - S_{\sigma(i)}\right|^2}\,. \tag{3.1}$$

3.2. Almgren's extrinsic maps.

The metric $\mathcal{G}$ is "locally euclidean" at most of the points. Consider for instance the model case $Q = 2$ and a point $P = [\![P_1]\!] + [\![P_2]\!]$ with $P_1 \neq P_2$. Then, obviously, in a sufficiently small neighborhood of P, the metric space $\mathcal{A}_2(\mathbb{R}^n)$ is isomorphic to the Euclidean space $\mathbb{R}^{2n}$. This fails instead in any neighborhood of a point of type $P = 2\,[\![P_1]\!]$. On the other hand, if we restrict our attention to the closed subset $\{2\,[\![X]\!] : X \in \mathbb{R}^n\}$, we obtain the metric structure of $\mathbb{R}^n$. A remarkable observation of Almgren is that $\mathcal{A}_Q(\mathbb{R}^n)$ is biLipschitz equivalent to a deformation retract of the Euclidean space (cp. with [4, Section 1.3]). For a simple presentation of this fact we refer the reader to [13, Section 2.1].

Theorem 3.3. *There exists $N = N(Q, n)$ and an injective $\boldsymbol{\xi} : \mathcal{A}_Q(\mathbb{R}^n) \to \mathbb{R}^N$ such that:*

(*i*) $\mathrm{Lip}(\boldsymbol{\xi}) \leq 1$;

(*ii*) *if $\mathcal{Q} = \boldsymbol{\xi}(\mathcal{A}_Q)$, then* $\mathrm{Lip}(\boldsymbol{\xi}^{-1}|_{\mathcal{Q}}) \leq C(n, Q)$.

Moreover there exists a Lipschitz map $\boldsymbol{\rho} : \mathbb{R}^N \to \mathcal{Q}$ which is the identity on $\mathcal{Q}$.

In fact much more can be said: the set $\mathcal{Q}$ is a cone and a polytope. On each separate face of the polytope the metric structure induced by $\mathcal{G}$ is euclidean, essentially for the reasons outlined a few paragraphs above (cp. again with [4, Section 1.3] or with [14, Section 6.1]).

3.3. The generalized Dirichlet energy. Using the metric structure on $\mathcal{A}_Q(\mathbb{R}^n)$ one defines obviously measurable, Lipschitz and Hölder maps from subsets of $\mathbb{R}^m$ into $\mathcal{A}_Q(\mathbb{R}^n)$. However, if we want to approximate area-minimizing currents with multiple valued functions and "linearize" the area functional in the spirit of De Giorgi, we need to define a suitable concept of Dirichlet energy. We will now show how this can be done naturally. However, the approach outlined below is not the one of Almgren.

Consider again the model case of $Q = 2$ and assume $u : \Omega \to \mathcal{A}_2(\mathbb{R}^n)$ is a Lipschitz map. If, at some point x, $u(x) = [\![P_1]\!] + [\![P_2]\!]$ is "genuinely 2-valued", i.e. $P_1 \neq P_2$, then there exist obviously a ball $B_r(x) \subset \Omega$ and two Lipschitz functions $u_1, u_2 : B_r(x) \to \mathbb{R}^n$ such that $u(y) = [\![u_1(y)]\!] + [\![u_2(y)]\!]$ for every $y \in B_r(x)$ (in this and similar situations, we will then say that there is a *regular selection* for u in $B_r(x)$, cp. with [13, Definition 1.1]). For each separate function u_i, the classical Theorem of Rademacher ensures the differentiability almost everywhere.

Recall that our ultimate goal is to define the Dirichlet energy so that it is a suitable approximation of the area of the graph of u. The "graph of u over $B_r(x)$" is simply to union of the graphs of the two functions u_i. When the gradients ∇u_i are close to 0, the area of each graph is close to

$$\int_{B_r(x)} \left(1 + \frac{1}{2}|\nabla u_i|^2\right) .$$

Thus, the only suitable definition of Dirichlet energy of u on the domain $B_r(x)$ is given by

$$\int_{B_r(x)} |Du|^2 \;:=\; \int_{B_r(x)} (|Du_1|^2 + |Du_2|^2)\,.$$

By an obvious localization procedure, this definition can be extended to the (open!) set $\Omega_2 \subset \Omega$ where u is genuinely 2-valued.

For each element z in the complement set $\Omega_1 := \Omega \setminus \Omega_2$, $u(z)$ is a single point counted with multiplicity 2. Then there is a Lipschitz map $v : \Omega_1 \to \mathbb{R}^n$ such that $u(z) = 2\,[\![v(z)]\!]$ for every $z \in \Omega_1$. Again in view of our goal, the only suitable definition of the Dirichlet energy of u over Ω_1 is twice the Dirichlet energy of v. We thus are left with only one possibility for the Dirichlet energy on the global set Ω:

$$\mathrm{Dir}(u,\Omega) \;:=\; \int_{\Omega_2} (|Du_1|^2 + |Du_2|^2) + 2\int_{\Omega_1} |Dv|^2\,.$$

This analysis can be obviously generalized to any positive integer Q, leading to a general definition of Dirichlet energy for Lipschitz multiple valued functions. The graphs of Lipschitz multiple valued functions carry naturally a structure of integer rectifiable currents (see [4, Section 1.6] or [14, Appendix C]). It is not difficult to see that, when the Lipschitz constant is small, the Dirichlet energy defined in this section is the second order approximation of the area of the corresponding graph (we refer the reader to [14, Section 2.3]).

Almgren's definition of Dir goes instead through a suitable concept of differentiability for multiple valued functions and a corresponding Rademacher's theorem (in [4] the derivation of this result is quite involved and a much simpler proof has been published in [24]). The arguments in [13, Section 1] easily show that the two points of view are equivalent. In fact the "stratification" strategy outlined above yields a fairly straightforward proof of Almgren's generalized Rademacher's Theorem (see [13, Section 1.3.2]).

Having established the correct notion of Dirichlet energy for Lipschitz functions, one could define the Sobolev space $W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n))$ through a "completion strategy": a measurable map $v : \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$ is in $W^{1,2}$ if and only if there is a sequence of Lipschitz maps u_k converging to v a.e. and enjoying a uniform bound $\mathrm{Dir}(\Omega, u_k) \leq C$. The Dirichlet energy of v is then defined via a "relaxation procedure": $\mathrm{Dir}(\Omega, v)$ is the infimum of all constants C for which there is a sequence with the properties above.

Almgren's approach is again rather different. $W^{1,2}$ maps are defined as those maps u for which $\boldsymbol{\xi} \circ u$ is $W^{1,2}$. The Dirichlet energy is again defined via a suitable notion of approximate differentiability. In our paper [13] we start from a third definition of Dirichlet energy and Sobolev space. However, all these points of view are completely equivalent, as one can easily conclude from the arguments in [13, Section 4] (cp. in particular with the Lipschitz approximation technique of [13, Proposition 4.4]).

3.4. The cornerstones of the theory of Dir-minimizers. We are now ready to state the three main theorems of Almgren concerning Dir-minimizers. Their proofs occupy essentially Chapters 1 and 2, i.e. the first fifth of Almgren's monograph. In what follows, Ω is always assumed to be a bounded open set with a sufficiently regular boundary (in fact, in order to give a complete account, we should have defined the trace at $\partial\Omega$ of $W^{1,2}$ multiple valued functions; we have avoided to enter in the details to keep our presentation short: the interested reader can consult, for instance, [13, Definition 0.7]).

Theorem 3.4 (Existence for the Dirichlet Problem). *Let $g \in W^{1,2}(\Omega; \mathcal{A}_Q)$. Then there exists a* Dir*-minimizing $f \in W^{1,2}(\Omega; \mathcal{A}_Q)$ such that $f|_{\partial\Omega} = g|_{\partial\Omega}$.*

Theorem 3.5 (Hölder regularity). *There is a constant $\alpha = \alpha(m, Q) > 0$ with the following property. If $f \in W^{1,2}(\Omega; \mathcal{A}_Q)$ is* Dir*-minimizing, then $f \in C^{0,\alpha}(\Omega')$ for every $\Omega' \subset\subset \Omega \subset \mathbb{R}^m$. For two-dimensional domains, we have the explicit constant $\alpha(2, Q) = 1/Q$.*

For the second regularity theorem we need the definition of the singular set of f.

Definition 3.6 (Regular and singular points). A Dir-minimizing f is regular at a point $x \in \Omega$ if there exists a neighborhood B of x and Q analytic functions $f_i : B \to \mathbb{R}^n$ such that

$$f(y) = \sum_i [\![f_i(y)]\!] \qquad \text{for almost every } y \in B \tag{3.2}$$

and either $f_i(x) \neq f_j(x)$ for every $x \in B$, or $f_i \equiv f_j$. The singular set Σ_f of f is the complement of the set of regular points.

Theorem 3.7 (Estimate of the singular set). *Let f be* Dir*-minimizing. Then, the singular set Σ_f of f is relatively closed in Ω. Moreover, if $m = 2$, then Σ_f is at most countable, and if $m \geq 3$, then the Hausdorff dimension of Σ_f is at most $m - 2$.*

Note in particular the striking similarity between the estimate of the size of the singular set in the case of multiple valued Dir-minimizers and in that of area-minimizing currents. It will be discussed later that, even in the case of Dir-minimizers, there are singular solutions (which are no better than Hölder continuous).

Complete and self-contained proofs of these theorems can be found in [13]. The key idea beyond the estimate for the singular set is the celebrated frequency function (cp. with [13, Section 3.4]), which has been indeed used in a variety of different contexts in the theory of unique continuation of partial differential equations (see for instance the papers [20], [21]). This is the central tool of our proofs as well. However, our arguments manage much more efficiently the technical intricacies of the problem and some aspects of the theory are developed in further details. For instance, we present in [13, Section 3.1] the Euler-Lagrange conditions derived from first variations in a rather general form. This is to our knowledge the first time that these conditions appear somewhere in this generality.

Largely following ideas of [7] and of White, we improve the second regularity theorem to the following optimal statement for planar maps.

Theorem 3.8 (Improved estimate of the singular set). *Let f be* Dir*-minimizing and $m = 2$. Then, the singular set Σ of f consists of isolated points.*

This result was announced in [7]. However, to our knowledge the proof has never appeared so far. For a discussion of the optimality of these regularity results, we refer the reader to Section 5 below.

4. Metric Analysis

4.1. An intrinsic approach. One of the less satisfactory points of Almgren's theory is the heavy use of the Lipschitz maps $\boldsymbol{\xi}$ and $\boldsymbol{\rho}$. First of all, this makes the arguments often counterintuitive. Second, there is the obvious disturbing fact that, while several choices of $\boldsymbol{\xi}$ and $\boldsymbol{\rho}$ are possible, the objects of the study and the ultimate conclusions of the theory are totally independent of this choice. This fact has been pointed out for the first time in [24]. In the papers [24] and [25] the author made some progress in the program of making Almgren's theory "intrinsic", i.e. independent of the euclidean embedding.

As far as the theory of Dir-minimizers is concerned, this program has been completed in our paper [13]. This work also makes a clear link between Almgren's theory and the vast existing literature about metric analysis, metric geometry and general harmonic maps, which started with the pioneering papers [22], [30] and [5] (we refer the interested reader to [13, Section 4.1]).

The metric approach has several features:

- One first advantage is that it allows to separate "hard" and "soft" parts in Almgren's theory. Several conclusions can indeed be reached in a straightforward way by "abstract nonsense". Only few key points need deeply the structure of $\mathcal{A}_Q(\mathbb{R}^n)$ and some "hard" computations. By quickly discarding the minor points, the metric theory is a powerful tool to recognize plausible statements and crucial issues.

- A second advantage is the natural link to the metric theory of currents developed by Ambrosio and Kirchheim in [6]. This theory recovers many of the central theorems of Federer and Fleming's work [18] in a clean way and offers some new powerful tools (like the Jerrard-Soner BV estimates for the slicing theory). The reason why this connection is useful will be explored in detail in Section 6.

4.2. Intrinsic definition of the Dirichlet energy. The metric point of view relies upon the following alternative definitions of Dirichlet energy and Sobolev functions (cp. with the general theory developed in [5] and [34]; the careful reader will notice, however, that there is a crucial difference between the definition of Dirichlet energy in [34] and the one given below).

Definition 4.1 (Sobolev Q-valued functions). A measurable $f : \Omega \to \mathcal{A}_Q$ is in the Sobolev class $W^{1,p}$ $(1 \leq p \leq \infty)$ if there exist m functions $\varphi_j \in L^p(\Omega; \mathbb{R}^+)$ such that

(i) $x \mapsto \mathcal{G}(f(x), T) \in W^{1,p}(\Omega)$ for all $T \in \mathcal{A}_Q$;

(ii) $|\partial_j \,\mathcal{G}(f, T)| \leq \varphi_j$ a.e. in Ω for all $T \in \mathcal{A}_Q$ and for all $j \in \{1, \ldots, m\}$.

It is not difficult to show the existence of minimal functions $\tilde{\varphi}_j$ fulfilling (ii), i.e. such that, for any other φ_j satisfying (ii), $\tilde{\varphi}_j \leq \varphi_j$ a.e. (cp. with [13, Proposition 4.2]). Such "minimal bounds" will be denoted by $|\partial_j f|$ and we note that they are characterized by the following property (see again [13, Proposition 4.2]): for every countable dense subset $\{T_i\}_{i \in \mathbb{N}}$ of $\mathcal{A}_Q$ and for every $j = 1, \ldots, m$,

$$|\partial_j f| \;=\; \sup_{i \in \mathbb{N}} |\partial_j \,\mathcal{G}(f, T_i)| \quad \text{almost everywhere in } \Omega. \tag{4.1}$$

We are now ready to define the Dirichlet energy.

Definition 4.2. The function $|Df|^2$ is defined to be the sum of $|\partial_j f|^2$. The Dirichlet energy of $f \in W^{1,2}(U; \mathcal{A}_Q)$ is then defined by $\mathrm{Dir}(f, U) := \int_U |Df|^2$.

As already mentioned, this definition is equivalent to the one proposed in the previous section.

The paper [13] gives therefore two different approaches to the theorems stated in the previous section. One can follow a (considerably simpler) version of Almgren's "extrinsic" approach, exploiting the maps $\boldsymbol{\xi}$ and $\boldsymbol{\rho}$. Or one can use the intrinsic approach starting from the definitions above, without using the maps $\boldsymbol{\xi}$ and $\boldsymbol{\rho}$. However, proceeding further in Almgren's program for the regularity of area-minimizing currents, there is a point at which we have not been able to avoid these extrinsic maps (see Sections 6.4 and 8).

5. Higher Integrability of Dir Minimizers and Other Results

5.1. Multiple valued functions beyond Almgren. Many results of Almgren have been extended in several directions. In particular

- The papers [11], [25], [52], [53] extend some of Almgren's results to ambient spaces which are more general than the euclidean one;
- The papers [51], [54], [26] and [24] consider some other objects in the multiple valued setting (such as differential inclusions, geometric flows and quasiminima);
- The papers [31] and [12] extend some of Almgren's theorems to more general energy functionals.

5.2. Higher integrability. In this section we focus on a recent new contribution to the theory, which plays an important role in our derivation of the second main step in Almgren's program. Dir-minimizing functions enjoy higher integrability of the gradient. We believe that several intricate arguments and complicated constructions in Almgren's third chapter can be reinterpreted as rather particular cases of this key observation (see for instance [4, Section 3.20]). Surprisingly, this higher integrability can be proved in a very simple way by deriving a suitable reverse Hölder inequality and using a (nowadays) very standard version of the classical Gehring's Lemma.

Theorem 5.1 (Higher integrability of Dir-minimizers). *Let $\Omega' \subset\subset \Omega \subset\subset \mathbb{R}^m$ be open domains. Then, there exist $p > 2$ and $C > 0$ such that*

$$\|Du\|_{L^p(\Omega')} \leq C \, \|Du\|_{L^2(\Omega)} \quad \text{for every Dir-minimizing } u \in W^{1,2}(\Omega, \mathcal{A}_Q(\mathbb{R}^n)). \tag{5.1}$$

This theorem has been stated and proved for the first time in [14]. The relevant reverse Hölder inequality has been derived using a comparison argument and hence relying heavily on the minimality of the Dir-minimizers. A second proof, exploiting the Euler-Lagrange conditions to give a Caccioppoli-type inequality, has been given in [44]. This last proof still uses the regularity theory for Dir-minimizers. However, this occurs only at one step: one could hope to remove this restriction and generalize the higher integrability to "critical" points of the Dirichlet energy (cp. with Section 8).

5.3. Optimality. In [44] a yet different proof for the planar case is proposed, yielding the optimal range of exponents p for which (5.1) holds. The optimality of this result, as well as the optimality of Theorems 3.5 and 3.8, is shown by another remarkable observation of Almgren. Besides giving area-minimizing currents, holomorphic varieties are locally graphs of Dir minimizing Q-valued functions. In [4, Section 2.20] Almgren proves this statement appealing to his powerful approximation results for area-minimizing currents (see Section 6 below). However this is unnecessary and a rather elementary proof can be found in [44].

6. Approximation of Area-Minimizing Currents

After developing the theory of multiple valued functions, Almgren devotes the third chapter of his monograph to a suitable approximation theorem for area-minimizing currents, which is the multiple valued counterpart of the classical approximation theorem of De Giorgi in his proof of the excess-decay property.

6.1. Almgren's main approximation theorem. We start by giving the exact statement of Almgren's approximation result in the euclidean setting. Compared to the rest of the note, this part is rather technical. On the other hand, in order to get an understanding of Almgren's approximation theorem, a certain familiarity with the theory of currents can hardly be avoided.

Consider integer rectifiable m-dimensional currents T supported in some open cylinder $\mathcal{C}_r(y) = B_r(y) \times \mathbb{R}^n \subset \mathbb{R}^m \times \mathbb{R}^n$ and satisfying the following assumption:

$$\pi_\# T = Q\,[\![B_r(y)]\!] \qquad \text{and} \qquad \partial T = 0, \tag{6.1}$$

where $\pi : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ is the orthogonal projection and m, n, Q are fixed positive integers. In an informal language, the hypothesis (6.1) means that the current "covers" Q times the base of the cylinder.

We denote by $\mathfrak{e}_T$ the non-negative *excess measure* and by $\mathrm{Ex}(T, \mathcal{C}_r(y))$ the *cylindrical excess*, respectively defined by

$$\mathfrak{e}_T(A) := \mathbf{M}\big(T \llcorner (A \times \mathbb{R}^n)\big) - Q\,|A| \quad \text{for every Borel } \; A \subset B_r(y), \tag{6.2}$$

$$\mathrm{Ex}(T, \mathcal{C}_r(y)) := \frac{\mathfrak{e}_T(B_r(x))}{|B_r(x)|} = \frac{\mathfrak{e}_T(B_r(x))}{\omega_m r^m}. \tag{6.3}$$

Though it is not apparent from the definition given above, the cylindrical excess bears some similarities with the spherical excess.

Theorem 6.1. *There exist constants $C, \delta, \varepsilon_0 > 0$ with the following property. Let T be an area-minimizing, integer rectifiable m-dimensional current in the cylinder $\mathcal{C}_4$ which satisfies* (6.1)*. If $E = \mathrm{Ex}(T, \mathcal{C}_4) < \varepsilon_0$, then there exist a Q-valued function $f \in \mathrm{Lip}(B_1, \mathcal{A}_Q(\mathbb{R}^n))$ and a closed set $K \subset B_1$ such that*

$$\mathrm{Lip}(f) \leq C E^{\delta}, \tag{6.4}$$

$$\mathrm{graph}(f|_K) = T \llcorner (K \times \mathbb{R}^n) \quad \textit{and} \quad |B_1 \setminus K| \leq C E^{1+\delta}, \tag{6.5}$$

$$\left| \mathbf{M}(T \llcorner \mathcal{C}_1) - Q\,\omega_m - \int_{B_1} \frac{|Df|^2}{2} \right| \leq C\, E^{1+\delta}. \tag{6.6}$$

An interesting aspect which makes the proof of Theorem 6.1 quite hard is the gain of a small power E^{δ} in the three estimates (6.4), (6.5) and (6.6). Observe that the usual approximation theorems stated commonly in the literature, which cover the case $Q = 1$ and "stationary currents" (in fact, stationary integral varifolds), are stated with $\delta = 0$. On the other hand, the gain of Theorem 6.1 plays a crucial role in some of the estimates needed for the third main step of Almgren's program, i.e. the "construction of the center manifold" (cp. with Section 7).

6.2. Higher integrability for area-minimizing currents.

The note [14] provides a different, much simpler proof of Almgren's theorem. A key point is a higher integrability estimate for the Lebesgue density $\mathfrak{d}_T$ of the measure $\mathfrak{e}_T$, called the *excess density*,

$$\mathfrak{d}_T(x) := \limsup_{s \to 0} \frac{\mathfrak{e}_T(B_s(x))}{\omega_m\, s^m}.$$

Theorem 6.2. *There exist constants $p > 1$ and $C, \varepsilon > 0$ with the following property. Assume T is an area-minimizing, integer rectifiable current of dimension m. If T satisfies* (6.1) *and $E = \mathrm{Ex}(T, \mathcal{C}_4) < \varepsilon$, then*

$$\int_{\{\mathfrak{d} \leq 1\} \cap B_2} \mathfrak{d}^p \leq C\, E^p. \tag{6.7}$$

This estimate, which can be thought as the "current counterpart" of Theorem 5.1, is not explicitly stated in [4], but it can be deduced from some of the arguments therein. These arguments, which include quite elaborate constructions and use several intricate covering algorithms, are the most involved part of Almgren's proof.

One comment is in order. In the case $Q = 1$ we know a posteriori that T coincides with the graph of a $C^{1,\alpha}$ function over B_2 (cp. with Theorem 2.1). However, the branching phenomenon makes Theorem 6.2 much more interesting in the higher codimension, since essentially it cannot be improved (except in the sense of optimizing the exponent p and the constant C). Consider in particular the following example. Let η be a rather small constant and T be the current associated to the holomorphic variety $\{z^2 = \eta w\} \subset \mathbb{C}^2 = \mathbb{R}^4$. Set $\mathcal{C}_4 := \{|w| < 4\}$ and $Q = 2$. If η is chosen very small compared to ε, then T satisfies all the assumptions of Theorem 6.1. On the other hand, the corresponding function $\mathfrak{d}_T$ does not belong to L^2 and one can easily check that estimate (6.7) does not hold if $p \geq 2$.

6.3. Some new techniques coming from metric analysis.

The main contribution of [14] is to give a much shorter and conceptually clearer derivation of (6.7) (in fact, since Theorem 6.2 is not stated by Almgren, the real point is to establish Theorem 6.6 below, which however is trivially equivalent). Moreover, in [14] we introduce several new ideas. In particular:

(i) we introduce a powerful maximal function truncation technique to approximate general integer rectifiable currents with multiple valued functions;

(ii) we give a simple compactness argument to conclude directly a first harmonic approximation of T;

(iii) we give a new proof of the existence of Almgren's "almost projections" $\boldsymbol{\rho}^\star$.

In the rest of this section we look more closely at these ideas.

Given a normal m-current T, following [6] we can view the slice map $x \mapsto \langle T, \pi, x\rangle$ as a BV function taking values in the space of 0-dimensional currents (endowed with the flat metric). Indeed, by a key estimate of Jerrard and Soner (see [6] and [29]), the total variation of the slice map is controlled by the mass of T and ∂T. In the same vein, following [13], Q-valued functions can be viewed as Sobolev maps into the space of 0-dimensional currents. These two points of view can be combined with standard maximal function truncation arguments to develop a powerful and simple Lipschitz approximation technique, which gives a systematic tool to find graphical approximations of integer rectifiable currents.

To give a more precise idea of this method, we introduce the maximal function of the excess measure of a current T (satisfying (6.1)):

$$M_T(x) := \sup_{B_s(x)\subset B_r(y)} \frac{\mathbf{e}_T(B_s(x))}{\omega_m\, s^m} = \sup_{B_s(x)\subset B_r(y)} \mathrm{Ex}(T, \mathcal{C}_s(x)).$$

Our main approximation result is the following and relies on an improvement of the usual Jerrard–Soner estimate.

Proposition 6.3 (Lipschitz approximation). *There exist constants $c, C > 0$ with the following property. Let T be an integer rectifiable m-current in $\mathcal{C}_{4s}(x)$ satisfying* (6.1) *and let $\eta \in (0, c)$ be given. Set $K := \big\{M_T < \eta\big\} \cap B_{3s}(x)$. Then,* *there exists* $u \in \mathrm{Lip}(B_{3s}(x), \mathcal{A}_Q(\mathbb{R}^n))$ *such that* $\mathrm{graph}(u|_K) = T\llcorner(K \times \mathbb{R}^n)$, $\mathrm{Lip}(u) \leq C\,\eta^{\frac{1}{2}}$ *and*

$$|B_{3s}(x) \setminus K| \leq \frac{C}{\eta}\,\mathbf{e}_T\big(\{M_T > \eta/2\}\big). \tag{6.8}$$

In the rest of this section, we will often choose $\eta = E^{2\alpha}$ $(= \mathrm{Ex}(T, \mathcal{C}_{4s}(x))^{2\alpha})$, for some $\alpha \in (0, (2m)^{-1})$. The map u given by Proposition 6.3 will then be called *the* E^α*-Lipschitz* (or briefly *the Lipschitz*) approximation of T in $\mathcal{C}_{3s}(x)$. We therefore conclude the following estimates:

$$\mathrm{Lip}(u) \leq C\,E^\alpha, \tag{6.9}$$

$$|B_{3s}(x) \setminus K| \leq C\,E^{-2\,\alpha}\,\mathbf{e}_T\big(\{M_T > E^{2\,\alpha}/2\}\big), \tag{6.10}$$

$$\int_{B_{3s}(x)\setminus K} |Du|^2 \leq \mathbf{e}_T\big(\{M_T > E^{2\,\alpha}/2\}\big). \tag{6.11}$$

In particular, the function f in Theorem 6.1 is given by the E^α-Lipschitz approximation of T in $\mathcal{C}_1$, for a suitable choice of α.

The second step in the proof of Theorem 6.2 is a compactness argument which shows that, when T is area-minimizing, the approximation f is close to a Dir-minimizing function w, with an $o(E)$ error.

Theorem 6.4 ($o(E)$-improvement). *Let* $\alpha \in (0, (2m)^{-1})$*. For every* $\eta > 0$*, there exists* $\varepsilon_1 = \varepsilon_1(\eta) > 0$ *with the following property. Let* T *be a rectifiable, area-minimizing* m*-current in* $\mathcal{C}_{4s}(x)$ *satisfying* (6.1)*. If* $E \leq \varepsilon_1$ *and* f *is the* E^α*-Lipschitz approximation of* T *in* $\mathcal{C}_{3s}(x)$*, then*

$$\int_{B_{2s}(x)\setminus K} |Df|^2 \leq \eta\,\mathbf{e}_T(B_{4s}(x)), \tag{6.12}$$

and there exists a Dir*-minimizing* $w \in W^{1,2}(B_{2s}(x), \mathcal{A}_Q(\mathbb{R}^n))$ *such that*

$$\int_{B_{2s}(x)} \mathcal{G}(f,w)^2 + \int_{B_{2s}(x)} \big(|Df| - |Dw|\big)^2 \leq \eta\,\mathbf{e}_T(B_{4s}(x)). \tag{6.13}$$

This theorem is the multi-valued analog of De Giorgi's harmonic approximation, which is ultimately the heart of all the regularity theories for minimal surfaces. Our compactness argument is, to our knowledge, new (even for $n = 1$) and particularly robust. Indeed, we expect it to be useful in more general situations.

Next, Theorems 6.4 and 5.1 imply the following key estimate, which leads to Theorem 6.2 via an elementary "covering and stopping radius" argument.

Proposition 6.5. *For every* $\kappa > 0$*, there is* $\varepsilon_2 > 0$ *with the following property. Let* T *be an integer rectifiable, area-minimizing current in* $\mathcal{C}_{4s}(x)$ *satisfying* (6.1)*. If* $E \leq \varepsilon_2$*, then*

$$\mathbf{e}_T(A) \leq \kappa\,E s^m \quad \text{for every Borel } A \subset B_s(x) \text{ with } |A| \leq \varepsilon_2 |B_s(x)|. \tag{6.14}$$

Using now Theorem 6.2, we can prove the most important estimate contained in Chapter 3 of [4].

Theorem 6.6. *There exist constants $\sigma, C > 0$ with the following property. Let T be an area-minimizing, integer rectifiable T of dimension m in $\mathcal{C}_4$. If T satisfies* (6.1) *and $E = \mathrm{Ex}(T, \mathcal{C}_4) < \varepsilon_0$, then*

$$\mathfrak{e}_T(A) \leq C\,E\left(E^\sigma + |A|^\sigma\right) \quad \textit{for every Borel } A \subset B_{4/3}. \tag{6.15}$$

6.4. Almgren's "almost projection" $\boldsymbol{\rho}^\star$. The proof of Theorem 6.6 is then the only part where we follow essentially Almgren's strategy. The main point is to estimate the size of the set over which the graph of the Lipschitz approximation f differs from T. As in many standard references, in the case $Q = 1$ this is achieved comparing the mass of T with the mass of the graph of $f * \rho_{E^\omega}$, where ρ is a smooth convolution kernel and $\omega > 0$ a suitably chosen constant (this idea is, essentially, already contained in De Giorgi's original proof).

However, for $Q > 1$, the space $\mathcal{A}_Q(\mathbb{R}^n)$ is not linear and we cannot regularize f by convolution. To bypass this problem, we follow Almgren and view $\mathcal{A}_Q$ as a subset of a large Euclidean space (via the biLipschitz embedding $\boldsymbol{\xi}$). We can then take the convolution of the map $\boldsymbol{\xi} \circ f$ and project it back on the set $\boldsymbol{\xi}(\mathcal{A}_Q)$. However, in order to do this efficiently in terms of the energy, we need an "almost" projection, denoted by $\boldsymbol{\rho}^\star_\mu$, which is almost 1-Lipschitz in the μ-neighborhood of $\boldsymbol{\xi}(\mathcal{A}_Q(\mathbb{R}^n))$ (μ is a parameter which must be tuned accordingly). At this point Theorem 6.2 enters in a crucial way in estimating the size of the set where the regularization of $\boldsymbol{\xi} \circ f$ is far from $\boldsymbol{\xi}(\mathcal{A}_Q(\mathbb{R}^n))$.

The maps $\boldsymbol{\rho}^\star_\mu$ are slightly different from Almgren's almost projections, but similar in spirit. In [14] we propose on original argument for the construction of $\boldsymbol{\rho}^\star_\mu$. One advantage of this argument is that it yields more explicit estimates in terms of the crucial parameter μ. As mentioned earlier, this is so far the only stage where we cannot avoid Almgren's extrinsic maps. It would be of interest to develop a more intrinsic approximation procedure, bypassing this "convolution and projection" technique (cp. Section 8 below).

7. Center Manifold: A Case Study

The fourth chapter of the big regularity paper (and roughly half of this monograph) is devoted to the construction of the so called "center manifold". In that chapter Almgren succeeds in constructing a $C^{3,\alpha}$ regular surface, which he calls center manifold and, roughly speaking, approximates the "average of the sheets of the current" (we refer to [4] for further details) in a neighborhood of a branching point. In the model example of Remark 2.2, the "ideal center manifold" would be the plane $\{z = 0\}$.

Essentially, the center manifold plays the same role of the barycenters of the measures $u(x)$ when u is a Q-valued map. In the latter example, it is rather straightforward to prove that the resulting "average function" is a classical harmonic function (see for example [13, Lemma 3.23]). Unfortunately for the case of area-minimizing current, due to the "nonlinear nature" of the problem, there is no obvious PDE allowing for a similar conclusion.

7.1. Higher regularity "without PDEs".

In the introduction of [4] Almgren observes that, in the case $Q = 1$, the center manifold coincides necessarily with the current itself, thus implying directly its $C^{3,\alpha}$ regularity. Compared to the usual proofs, this is rather striking. In fact, after proving Theorem 2.1, the "usual" regularity theory proceeds further by deriving the well-known Euler–Lagrange equations for the function f. It then turns out that f solves a system of elliptic partial differential equations and the Schauder theory implies that f is smooth (in fact analytic, using the classical result by Hopf [28]).

The corollary of Almgren's construction is that the $C^{3,\alpha}$ regularity can be concluded without appealing to "nonparametric techniques". In the note [15] we give a simple direct proof of this remark, essentially following Almgren's strategy for the construction of the center manifold in the case $Q = 1$. Though in a very simplified situation, this model case retains several key estimates of Almgren's construction. For instance it makes transparent the fundamental role played by the E^δ-gain in the estimates of the Approximation Theorem 6.1.

Our hope is that this will be a first step in the full understanding of Almgren's result. It is worthwhile to notice that, compared to the extremely long construction of the center manifold, the last portion of [4], containing the concluding arguments of Almgren's regularity theorem for area-minimizing integral currents, is much shorter. The construction of the center manifold seems the last big obstacle which needs to be overcome in order to understand the full regularity results of Almgren and Chang.

It is of a certain interest to notice that this "higher regularity" result stops a little after three derivatives. It does not seem possible, for instance, to get an estimate for the C^4 norm. In the proof presented in [15], this is quite transparent. In some sense, one can think of Almgren's strategy as an extremely careful approximation of the current obtained by pasting together (suitably rotated) graphs of harmonic functions.

One reason for the $C^{3,\alpha}$ estimate might be the fact that the Dirichlet energy is a quite accurate approximation of the area functional. Loosely speaking, one can think of De Giorgi's theorem as a consequence of the fact that the harmonic functions are first order expansions of solutions to the minimal surfaces. One gains almost 2 derivatives in this way (a careful look at the proof of Theorem 2.1 would show that it works for every $\beta < 1$, cp. with the Appendix of [15]). Taking the Taylor expansion to the next level, it turns out that harmonic functions approximate solutions of the minimal surface equations even "to the next order". To illustrate this phenomenon, consider the simpler situation of a surface of codimension 1, given by the graph of a Lipschitz function f. The key ingredient in De Giorgi's argument for the excess-decay is the following observation on the integrand of the area functional:

$$I(\nabla f) \;:=\; \sqrt{1+|\nabla f|^2} \;=\; 1+\frac{1}{2}|\nabla f|^2 + o(|\nabla f|^2)\,.$$

However, the Taylor expansion yields a much more precise information:

$$I(\nabla f) \;=\; 1+\frac{1}{2}|\nabla f|^2 + O(|\nabla f|^4)\,. \tag{7.1}$$

The identity (7.1) is correct also in higher codimension.

8. Open Problems

In this section we collect a list of open problems on Q-valued functions. As already mentioned, there are several directions in which Almgren's theory could be extended, in particular in generalizing it to non-euclidean ambient spaces. However, in this list we have decided to focus on the euclidean setting and on problems which would deliver new information rather than generalizing existing theorems to different contexts. Many of these problems have been proposed by Almgren and the reader might find them in the collection [1].

(1) In the proof of Theorem 3.7 a pivotal role is played by the so called "tangent functions". The key idea (which ultimately might be regarded as the most important discovery of Almgren) is that, when suitably rescaling a Dir-minimizer in a neighborhood of a singularity, the resulting maps converge, up to subsequences, to Dir-minimizers which are radially homogeneous. This theorem is achieved through the monotonicity of the celebrated frequency function, which in this context plays the same role of the monotonicity formula for area-minimizing currents.

    The uniqueness of the "blow-up" at a singularity is not known, except for the planar maps (see [13, Theorem 5.3], where it is proved before Theorem 3.8 exploiting some ideas of [7]; assuming Theorem 3.8, this uniqueness is an obvious consequence of the considerations in [32]). Almgren suggests that a relevant role in this problem might be played by the techniques developed in [40] (cp. with [1, Problem 5.6]).

(2) A tentative conjecture is that the singular set of a Dir-minimizer map on an m–dimensional domain should have (locally) finite $\mathcal{H}^{m-2}$ measure and be rectifiable. This is only known to hold in the case $m = 2$ (cp. with Theorem 3.8).

(3) In [1, Problem 5.5] Almgren asks whether the graph of a Dir-minimizer is always a real analytic set. To our knowledge this is unknown even in the case of planar maps, where rather detailed information is available (after combining Theorem 3.8 with the results of [32]).

(4) It would be interesting to get other examples of Dir-minimizers. To our knowledge, no other systematic class of examples is known apart from

that of holomorphic varieties (cp. with the discussion in Section 5.3). Is there any other similar class that one could derive from other calibrated geometries?

(5) Essentially nothing is known if we replace the minimizing property with stationarity. Different notions of stationary maps are possible, due to the difference between inner and outer variations (cp. with [13, Section 3.1]) and to the possibility of introducing more general type of deformations. Does the singular set have measure zero? It is easy to see that there are maps which are stationary with respect to both inner and outer variations and have a singular set of dimension $m-1$. Does the singular set have dimension $m-2$ if the map is stationary with respect to any one-parameter family of deformations (cp. with [4, Problem 5.5])?

(6) Very little is known if we change the Dirichlet energy. The paper [12] shows the existence of a large class of semicontinuous functionals. If we restrict to planar maps and quadratic (semicontinuous) functionals, the only information available for minimizers is the Hölder continuity (proved in [31]).

(7) Are Dir-minimizers continuous, or ever Hölder, up to the boundary, if the boundary data are sufficiently regular? The only known result is the continuity for 2–dimensional domains (proved in [50]).

(8) Can one avoid the map $\boldsymbol{\rho}^\star_\mu$ in the proof of Theorem 6.1? Another way to phrase this question is the following. Is there an "intrinsic" efficient smoothing procedure for Q-valued functions? So far the following are the only two available techniques:

  – The (intrinsic) maximal function truncation argument which allows to approximate general Q-valued functions in $W^{1,p}$ with Lipschitz maps.
  – Almgren's extrinsic smoothing: the given map u is transformed into a Euclidean map $\boldsymbol{\xi} \circ u$; this map is than regularized (for instance with a convolution) and, to produce again a Q-valued map, the regularization is projected on the set $\boldsymbol{\xi}(\mathcal{A}_Q)$.

The latter yields efficient estimates when dealing with the Dirichlet energies of the corresponding maps. We do not know of any intrinsic method to achieve regularizations with the same estimates.

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