Continuous dissipative Euler flows and a conjecture of Onsager

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Abstract. It is known since the pioneering works of Scheffer and Shnirelman that there are nontrivial distributional solutions to the Euler equations which are compactly supported in space and time. Obviously these solutions do not respect the classical conservation law for the total kinetic energy and they are therefore very irregular. In recent joint works we have proved the existence of *continuous* and even Hölder continuous solutions which dissipate the kinetic energy. Our theorem might be regarded as a first step towards a conjecture of Lars Onsager, which in 1949 asserted the existence of dissipative Hölder solutions for any Hölder exponent smaller than $\frac{1}{3}$.

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1. The Euler equations

The incompressible Euler equations are a system of partial differential equations which were derived more than 250 years ago by Euler to describe the motion of an inviscid fluid. If we assume that the density of the fluid is a constant ρ_0, the unknowns of the system are the velocity v, a vector field, and the pressure p, a scalar field. For convenience we will assume that these fields are defined on $\mathbb{T}^n \times I$ or in $\mathbb{R}^n \times I$, where $\mathbb{T}^n = \mathbb{S}^1 \times \ldots \times \mathbb{S}^1$ is the n-dimensional torus and I is either an open interval $]0, T[$, or the open halfline $]0, \infty[$ or the entire real line $\mathbb{R}$. In general we assume $n \geq 2$, but the case of interests here are obviously $n = 2, 3$. The equations take then the following form

$$\begin{cases} \partial_t v + \mathrm{div}_x (v \otimes v) + \nabla p = 0 \\ \mathrm{div}_x v = 0 \,, \end{cases} \tag{1.1}$$

where the density ρ_0 of the fluid is normalized to 1.

The velocity $v(x, t)$ represents the speed of the fluid particle which at times t occupies the point x. If Ω is a smooth (bounded) open domain, then

$$\int_{\partial\Omega} p(x,t) \nu(x) \, dA(x)$$

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is the total force exerted at time t by the fluid outside Ω upon the portion of fluid inside Ω. ν denotes the exterior unit normal to Ω. Note that p is then well-defined up to an arbitrary function of time, since

$$\int_{\partial\Omega} \nu = 0$$

for every smooth bounded open set Ω. This arbitrariness in the definition of p can be seen directly from (1.1) and it is natural to mod it out by normalizing p so that $\int_{\mathbb{T}^n} p(x,t)\,dx = 0$, which from now on will always be assumed to hold.

The two equations in (1.1) express simply the conservation of mass and momentum. Indeed, if (v,p) is a pair of C^1 functions satisfying (1.1) and Ω an arbitrary domain, the divergence theorem implies

$$\int_{\partial\Omega} v\cdot\nu = 0 \tag{1.2}$$

$$\frac{d}{dt}\int_{\Omega} v = \int_{\partial\Omega} v(v\cdot\nu) + \int_{\partial\Omega} p\nu\,. \tag{1.3}$$

The identity (1.2) expresses the conservation of mass: the total amount of fluid particles "getting out" of Ω is balanced by the total amount "getting in". The identity (1.3) is the counterpart of the conservation of momentum: the rate of change of the momentum of the fluid contained in Ω is given by the sum of the flux of momentum through Ω and the total force exerted on Ω by the portion of fluid lying outside.

In continuum mechanics it is often the case that balance laws as in (1.2) and (1.3) (valid for any "fluid element" Ω) are derived, under suitable assumptions, from first principles, whereas the differential equations (as (1.1)) are deduced as consequences when the functions are sufficiently smooth. In the case at hand (1.1) can be easily derived from (1.2)-(1.3) if the pair (v,p) is C^1. However we can make sense of (1.2) and (1.3) even if (v,p) are much less smooth: the continuity of the pair is, for instance, enough to make sense of all the integrals in (1.2) and (1.3) whenever Ω has C^1 (or even Lipschitz) boundary. Though this looks quite natural, we will see that there are pairs of continuous functions satisfying (1.2) and (1.3) which display a quite counterintuitive behavior.

2. Anomalous dissipation

If (v,p) is a C^1 solution of (1.1), we can scalar multiply the first equation by v and use the chain rule to derive the identity

$$\partial_t \frac{|v|^2}{2} + \mathrm{div}_x\left(\left(\frac{|v|^2}{2}+p\right)v\right) = 0\,.$$

Assume that the domain of definition is $\mathbb{T}^n\times I$ and integrate this last equality in space. We then derive the conservation of the total kinetic energy

$$\frac{d}{dt}\int_{\mathbb{T}^n} |v|^2(x,t)\,dx = 0\,. \tag{2.1}$$

Thus, classical solutions of the incompressible Euler equations are energy conservative.

Nonetheless, in [42] Onsager suggested the existence of solutions to the 3-dimensional incompressible Euler equations which dissipate the energy. Such solutions cannot be interpreted in classical terms and it is remarkable that indeed Onsager himself suggests a concept of solution which coincides with our modern notion of weak (distributional) solutions.

Before coming to this, let us briefly describe the considerations of Onsager on the energy spectrum for 3-dimensional isotropic turbulence. We start by introducing the Navier Stokes equations, namely the system

$$\begin{cases} \partial_t v + \mathrm{div}_x (v \otimes v) + \nabla p = \nu \Delta v \\ \mathrm{div}_x v = 0\,, \end{cases} \tag{2.2}$$

where the viscosity ν is considered to be fairly small (or, in the language of fluid dynamics, the Reynolds number of the flow is high). For a smooth solution of (2.2) the balance law for the energy (2.1) would then take the form

$$\frac{d}{dt} \int_{\mathbb{T}^n} |v|^2(x,t)\, dx = -2\nu \int_{\mathbb{T}^n} |\nabla \times v|^2(x,t)\, dx\,. \tag{2.3}$$

It is well known that, in 2 dimensions, the right hand side of (2.3), called the *enstrophy*, is a conserved quantity and hence there is no mechanism of "inflation" for the dissipation term. However, this conservation does not hold for 3-dimensional solutions, where the energy is the only constant of motion and there are several experimental reasons to believe that typically the enstrophy becomes quite large.

If we were considering a family of solutions u_ν with $\nu \to 0$ and if these solutions were to converge to a classical solution of (1.1), then the right hand side of (2.3) would behave as $O(\nu)$. However, in the theory of hydrodynamic turbulence it is expected that, in 3-dimensions and for "typical" turbulent solutions of (2.2), the right hand side of (2.3) is independent of the viscosity. Thus, one may advance the hypothesis that the dissipation of the energy is not primarily driven by the viscous term $\nu\Delta u$ and that the main responsible for this dissipation is indeed the nonlinear term of the equations, which appear as well in (1.1).

This hypothesis and a corresponding "energy spectrum" law has been first put forward by Kolmogorov in [34] (nowadays often cited as K41 theory) and, as pointed out by Onsager in [42], rediscovered independently at least twice (in [41] and [56]; see also [30], which refers to [45]). We briefly explain here the motivations given by [42] for the Kolmogorov's law (and refer to [28] for a nice and much more detailed analysis of Onsager's discoveries).

Denote by $E(t)$ the average of the total kinetic energy (divided by the density of the fluid) and by $Q = -\frac{dE}{dt}$ its rate of dissipation. Moreover, we let L be the "macroscale" of the flow (in our case we can suppose this is the side length of the torus, i.e. 2π). If we assume that Q depends only on L and E a simple dimensional analysis suggests the law

$$Q = -\frac{dE}{dt} = cE^{\frac{3}{2}}L^{-1} \tag{2.4}$$

where c is a dimensionless constant. Indeed, if σ denotes the unit of space and τ the unit of time, then E is measured in σ^2/τ^2, Q in σ^2/τ^3 and L in σ: it can be readily checked that the law (2.4) is the only possible one of the form $cE^\alpha L^\beta$ for which c is a dimensionless constant. The law (2.4) has been verified extensively in experiments and it turns out to be valid as long as the viscosity is very small compared to $E(t)$.

In order to get into Onsager's explanation of how this might be possible, we expand the velocity v in Fourier series:

$$v(x,t) = \sum_{k\in\mathbb{Z}^3} a_k(t)e^{ik\cdot x}\,.$$

Obviously $a_{-k} = \overline{a_k}$, because v is real-valued. Moreover the divergence-free constraint translates into the identity $k \cdot a_k = 0$. We then rewrite the remaining equations of (2.2) as an infinite-dimensional system of ODEs for the a_k:

$$\frac{da_k}{dt} = i\sum_\ell a_{k-\ell}\cdot\ell\left[-a_\ell + \frac{(a_\ell\cdot k)k}{|k|^2}\right] - \nu|k|^2 a_k \tag{2.5}$$

Clearly the total kinetic energy is (up to constant factors) $\sum_k |a_k|^2$. Observe, moreover, that L is essentially the smallest λ such that $\sum_{|k|=\lambda}|a_k|^2$ is comparable to E.

We next derive the rate of change of the energy carried by a given wave number:

$$\frac{d}{dt}|a_k|^2 = -2|k|^2\nu|a_k|^2 + \sum_\ell Q(k,\ell)\,, \tag{2.6}$$

where the term $Q(k,\ell)$ is given by

$$Q(k,\ell) = -2\mathrm{Im}\left((a_{k+\ell}\cdot\ell)(\overline{a_k}\cdot\overline{a_\ell}) + (a_{\ell-k}\cdot k)(a_k\cdot\overline{a_\ell})\right)\,.$$

Note that $Q(k,\ell) = -Q(\ell,k)$: this term accounts for the "energy exchange" between different Fourier modes. As long as $-\nu|k|^2$ is small (i.e. for sufficiently small k), we can assume that the term $Q(k,\ell)$ is the dominating one in (2.6).

The picture proposed by Onsager for a "typical" chaotic flow is the following: in the infinite sum at the right hand side of (2.6) only the terms where a_k, a_ℓ, $a_{k+\ell}$ have a comparable size are dominating. So, the energy gets redistributed from wave lengths of a certain size to wave length of, say, double that size. As λ grows the redistribution process happens faster and faster, so that after a short time (i.e. before E becomes too smal for the validity of (2.4)) the energy is redistributed at all scales. If this transfer is a chaotic process, after few steps the information about the low wave numbers (i.e. the macroscopic features of the flow). It is therefore plausible that the energy flux of the energy distribution depends only on the total dissipation rate $Q = -\frac{dE}{dt}$ and on the modulus of the wave number $|k|$.

If we set $f(\lambda) := \sum_{|k|\leq\lambda} |a_k|^2$ the energy distribution $E(\lambda)$ is "formally" $\frac{df}{d\lambda}$, so that $E = \int E(\lambda)d\lambda$. Since the frequency is measured in σ^{-1}, $E(\lambda)$ is measured in $\sigma^3\tau^{-2}$. The same dimensional analysis leading to (2.4) gives then

$$E(\lambda) = \beta Q^{\frac{2}{3}}\lambda^{-\frac{5}{3}} \tag{2.7}$$

where β is a dimensional constant. The last identity is the famous Kolmogorov's law.

3. Weak solutions

At the end of his note Onsager remarks that *... in principle, turbulent dissipation as described could take place just as readily without the final assistance of viscosity. In the absence of viscosity the standard proof of conservation of energy does not apply, because the velocity field does not remain differentiable! In fact it is possible to show that the velocity field in such "ideal turbulence" cannot obey any Lipschitz condition of the form* $|v(x) - v(y)| \leq C|x|^\alpha$ *for any* α *greater than* $\frac{1}{3}$*; otherwise the energy is conserved.*

First of all, translated in modern PDE terminology, Onsager is simply proposing to look at weak solutions. Again, quoting him directly *...Of course, under the circumstances, the ordinary formulation of the laws of motion in terms of differential equations becomes inadequate and must be replaced by a more general description; for example, the formulation* (2.5) *in terms of Fourier series will do.* Thus he is simply looking at functions with Fourier coefficients satisfying (2.5), where ν is set equal to 0.

However, he is not assuming any differentiability of these solutions: the only assumption is that the right hand side of (2.5) makes sense, or in other words that the series:

$$\sum_\ell a_{k-\ell} \cdot \ell \left[-a_\ell + \frac{(a_\ell \cdot k)k}{|k|^2}\right]$$

converges. Recall that $a_k \cdot k = 0$. Thus the series above can be rewritten as

$$\sum_\ell a_{k-\ell} \cdot k \left[-a_\ell + \frac{(a_\ell \cdot k)k}{|k|^2}\right] .$$

For the converge it is then sufficient to assume that $\sum |a_k|^2 < \infty$, i.e. that the velocity field is square summable. Summarizing, "Onsager's solutions" are those divergence free real valued fields

$$v(x,t) = \sum a_k(t)e^{ik\cdot x}$$

with Fourier coefficients satisfying

$$\frac{da_k}{dt} = i \sum_{\ell} a_{k-\ell} \cdot k \left[-a_\ell + \frac{(a_\ell \cdot k)k}{|k|^2} \right] \tag{3.1}$$

and such that

$$\sum |a_k|^2(t) < \infty \qquad \text{for every } t.$$

Consider now the (time-dependent) vector

$$b(t) := -i \sum_{\ell} a_{k-\ell} \cdot k a_\ell \,.$$

Observe that the right hand side of (3.1) is the vector $b - \frac{(k \cdot b)}{|k|^2} k$, i.e. the projection of $b(t)$ on the vector space V_k orthogonal to k. Since $t \mapsto a_k(t)$ is a curve in V_k, (3.1) is satisfied if and only if

$$w \cdot \frac{da_k}{dt} - w \cdot b(t) = 0 \qquad \text{for all } w \in V_k.$$

In turn the last identity can be rewritten as

$$w \cdot \frac{da_k}{dt} + w \otimes k : \sum_{\ell} a_{k-\ell} \otimes a_\ell = 0\,. \tag{3.2}$$

Introduce next the vector field $\varphi(x) := w e^{ik \cdot x}$ and observe that, for $w \in V_k$, φ is divergence-free. The identity (3.2) is simply

$$\int_{\mathbb{T}^n} \varphi(x) \cdot \frac{\partial v}{\partial t}(x,t)\, dx - \int_{\mathbb{T}^n} \nabla \varphi(x) : v \otimes v(x,t)\, dx = 0 \tag{3.3}$$

A simple density argument shows then that (3.1) holds if and only if v is a weak solution in the sense of distributions. We recall the latter notion for the reader's convenience.

Definition 3.1. A vector field $v \in L^2(\mathbb{T}^n \times I)$ is a weak solution of the incompressible Euler equations if

$$\int \partial_t \varphi \cdot v + \nabla \varphi : (v \otimes v)\, dxdt = 0 \tag{3.4}$$

for all $\varphi \in C_c^\infty(\mathbb{T}^n \times I; \mathbb{R}^n)$ with $\operatorname{div} \varphi = 0$ and

$$\int v \cdot \nabla \psi \, dxdt = 0 \qquad \text{for all } \psi \in C_c^\infty(\mathbb{T}^n \times I). \tag{3.5}$$

4. The Onsager's conjecture

The final sentence of Onsager's note is then *...The detailed conservation of energy* (2.6) *does not imply conservation of the total energy if the number of steps in the cascade is infinite, as expected, and the double sum of $Q(\ell, k)$ converges only conditionally.* Here, as implicit in the discussion, we are setting $\nu = 0$. Thus Onsager claims that a closer inspection of the identity (2.6) shows that the total conservation of the energy can be inferred from the weak formulation of the equation (3.1) only when the solution is Hölder continuous with exponent larger than $\frac{1}{3}$, whereas this might fail for smaller exponents.

Following this suggestion, the claim about the energy conservation has been shown by Eyink in [27] under the assumption that $\sum_k |k|^\alpha |a_k| < \infty$ (which does imply the α-Hölder regularity, in space, of the function v, but it is obviously a stronger condition). Onsager's exact claim has then been sown by Constantin, E and Titi with an elegant and fairly short argument (we refer also to [49] for more precise results). However, much less is known on the other side of the conjecture, namely on the existence of solutions with lower regularity which do not preserve energy. This will be the main focus of the rest of the note, where we will explore what has been proved up to now.

The exponent $\frac{1}{3}$ has a direct significance in isotropic turbulence, since it is related to another famous law of the Kolmogorov's theory, namely the fact that, in isotropic turbulent flows, the spatial variance of velocities is comparable to the distance to the power $\frac{2}{3}$ (see the discussion in the paper [28]). These laws are always derived by scaling arguments and thus a proof of the Onsager's conjecture would give a first justification purely based on rigorous mathematical considerations pertaining to the equations of motions.

5. Weak solutions with compact support in time

The first proof that weak solutions of the Euler's equations might not be energy conservative is due to Scheffer in his groundbreaking paper [46]. The main theorem of [46] states the existence of a non-trivial weak solution in $L^2(\mathbb{R}^2 \times \mathbb{R})$ with compact support in space and time. Later on Shnirelman in [47] gave a different proof of the existence of a non-trivial weak solution in $L^2(\mathbb{T}^2 \times \mathbb{R})$ with compact support in time. In these constructions it is not clear if the solution belongs to the energy space, i.e. whether *each* time-slice belongs to L^2. In the note [48] Shnirelman gave the first existence proof of a solution of the 3-dimensional Euler equations which dissipates the energy: obviously this solution does belong to the energy space and hence satisfies the requirement that the kinetic energy be finite at each time.

In the paper [20] we provided a relatively simple proof of the following stronger statement.

Theorem 5.1. *There exist infinitely many compactly supported bounded weak solutions of the incompressible Euler equations in any space dimension.*

The proof in [20] is based on a suitable notion of subsolution and it embeds the examples of Theorem 5.1 in a long tradition of (rather counterintuitive) constructions in the theory of differential inclusions. As pointed out in the important paper [38] by Müller and Šverak, these results (see, for instance, [10, 11, 19, 32, 33]) have a close relation to Gromov's h-principle. In particular the method of convex integration, introduced by Gromov and extended by Müller and Šverak to Lipschitz mappings, provides a very powerful tool to construct such examples. In the paper [20] these tools were suitably modified and used for the first time to explain Scheffer's non-uniqueness theorem. It was also noticed immediately that this approach allows to go way beyond the result of Scheffer. Indeed it has lead to new developments for several equations in fluid dynamics (see [12, 18, 50, 52, 53, 54, 57]), for which we refer to the survey article [22].

We now motivate the definition of subsolution following [22]. Let us first recall the concept of Reynolds stress. It is generally accepted that the appearance of high-frequency oscillations in the velocity field is the main reason responsible for turbulent phenomena in incompressible flows. One related major problem is therefore to understand the dynamics of the coarse-grained, in other words macroscopically averaged, velocity field. If $\overline{v}$ denotes the macroscopically averaged velocity field, then it satisfies

$$\begin{aligned} &\partial_t \overline{v} + \operatorname{div}(\overline{v} \otimes \overline{v} + R) + \nabla \overline{p} = 0 \\ &\operatorname{div} \overline{v} = 0, \end{aligned} \tag{5.1}$$

where

$$R = \overline{v \otimes v} - \overline{v} \otimes \overline{v}\,.$$

The latter quantity is called Reynolds stress and arises because the averaging does not commute with the nonlinearity $v \otimes v$. On this formal level the precise definition of averaging plays no role, be it long-time averages, ensemble-averages or local space-time averages. The latter can be interpreted as taking weak limits. Indeed, weak limits of Leray solutions of the Navier-Stokes equations with vanishing viscosity have been proposed in the literature as a deterministic approach to turbulence (see [1], [2], [13], [37]). We are now ready to introduce our notion of subsolution. In what follows we will use $\mathcal{S}^{n\times n}$ for the space of $n \times n$ symmetric matrices.

Definition 5.2 (Subsolutions). Let $\overline{e} \in L^1_{loc}(\mathbb{R}^n \times (0,T))$ with $\overline{e} \geq 0$. A subsolution to the incompressible Euler equations with given kinetic energy density $\overline{e}$ is a triple

$$(v, R, p) : \mathbb{R}^n \times (0,T) \to \mathbb{R}^n \times \mathcal{S}^{n\times n} \times \mathbb{R}$$

with the following properties:

(i) $v \in L^2_{loc}, \quad u \in L^1_{loc}, \quad p$ is a distribution;

(ii) (5.1) is satisfied in the sense of distributions

(iii) $R \geq \frac{1}{n}(2\overline{e} - |\overline{v}|^2)\mathrm{Id} \geq 0$ a.e..

Remark 5.3. Though in the various reference [20, 21, 22] the notion of subsolution is seemingly different from the one given above, the two concepts are easily shown to be equivalent. Consider, for instance the triple (v, u, q) of [22, Definition 2.3] and impose the relations

$$\operatorname{tr} R = 2\bar{e} - |v|^2 , \qquad q = p + \frac{2}{n}\bar{e} \qquad \text{and} \qquad u = (R + v \otimes v) - \frac{2}{n}\bar{e}\,\mathrm{Id}\,.$$

It is then obvious that (v, R, p) is a subsolution in the sense of Definition 5.2 if and only if (v, u, q) is a subsolution in the sense of [22, Definition 2.3].

Observe that if $R = 0$, then the v component of the subsolution is in fact a weak solution of the Euler equations. As mentioned above, in passing to weak limits (or when considering any other averaging process), the high-frequency oscillations in the velocity are responsible for the appearance of a non-trivial Reynolds stress. Equivalently stated, this phenomenon is responsible for the inequality sign in (iii).

The key point in our approach to prove Theorem 5.1 is that, starting from a subsolution, an appropriate iteration process reintroduces the high-frequency oscillations. In the limit of this process one obtains weak solutions. However, since the oscillations are reintroduced in a very non-unique way, in fact this generates *many* solutions from the same subsolution. In the next theorem we give a precise formulation of the previous discussion.

Theorem 5.4 (Subsolution criterion). *Let $\bar{e} \in C(\mathbb{R}^n \times (0, T))$ and $(\bar{v}, \overline{R}, \bar{p})$ be a smooth subsolution such that $2\bar{e} - |\bar{v}|^2 > 0$. Then there exist infinitely many weak solutions $v \in L^\infty_{loc}(\mathbb{R}^n \times (0, T))$ of the Euler equations such that*

$$\tfrac{1}{2}|v|^2 = \bar{e}$$

almost everywhere. Infinitely many among these belong to $C((0, T), L^2)$.

This theorem corresponds to Proposition 2 of [21] (cp. with Theorem 2.4 of [22]). From it we derived quite severe counterexamples to the uniqueness of solutions to the Euler equations, even when imposing quite restrictive additional constraints.

6. The Nash-Kuiper Theorem and Gromov's *h*-principle

The origin of convex integration lies in the famous Nash-Kuiper theorem. In this section we briefly recall some landmark results from the theory of isometric embeddings.

Let M^n be a smooth compact manifold of dimension $n \geq 2$, equipped with a Riemannian metric g. An isometric immersion of (M^n, g) into $\mathbb{R}^m$ is a map $u \in C^1(M^n; \mathbb{R}^m)$ such that the induced metric $u^\sharp e$ agrees with g. In local coordinates this amounts to the system

$$\partial_i u \cdot \partial_j u = g_{ij} \tag{6.1}$$

consisting of $\frac{n}{2}(n+1)$ equations in m unknowns. If in addition u is injective, it is an isometric embedding. Assume for the moment that $g \in C^\infty$. The two classical theorems concerning the solvability of this system are:

(A) if $m \geq \frac{1}{2}(n+2)(n+3)$, then any short embedding can be uniformly approximated by isometric embeddings of class C^∞ (Nash [40], Gromov [29]);

(B) if $m \geq n+1$, then any short embedding can be uniformly approximated by isometric embeddings of class C^1 (Nash [39], Kuiper [36]).

Recall that a short embedding is an injective map $u : M^n \to \mathbb{R}^m$ such that the metric induced on M by u is shorter than g. In coordinates this means that

$$(\partial_i u \cdot \partial_j u) \leq (g_{ij}) \tag{6.2}$$

in the sense of quadratic forms. Thus, (A) and (B) are not merely existence theorems, they show that there exists a huge (essentially C^0-dense) set of solutions. This type of abundance of solutions is a central aspect of Gromov's h-principle, for which the isometric embedding problem is a primary example (see [26, 29]).

There is a clear formal analogy between (6.1)-(6.2) and (1.1)-(5.1). First of all, note that the Reynolds stress measures the defect to being a solution of the Euler equations and it is in general a nonnegative symmetric tensor, whereas $g_{ij} - \partial_i u \cdot \partial_j u$ measures the defect to being isometric and, for a short map, is also a nonnegative symmetric tensor. More precisely (6.1) can be formulated for the deformation gradient $A := Du$ as the coupling of the linear constraint

$$\operatorname{curl} A = 0$$

with the nonlinear relation

$$A^t A = g.$$

In this sense short maps are "subsolutions" to the isometric embedding problem in the spirit of Definition 5.2. Along this line of thought, Theorem 5.4 is then the analogue for the Euler equations of the Nash-Kuiper result (B). However note that, strictly speaking, the formal analog of statement (B) would be replacing L^∞ by C^0 in Theorem 5.4.

Statement (B) is rather surprising for two reasons. First of all, for $n \geq 3$ and $m = n+1$, the system (6.1) is overdetermined. Moreover, for $n = 2$ we can compare (B) to the classical rigidity result concerning the Weyl problem: if $(\mathbb{S}^2, g)$ is a compact Riemannian surface with positive Gauss curvature and $u \in C^2$ is an isometric immersion into $\mathbb{R}^3$, then u is uniquely determined up to a rigid motion ([14, 31], see also [51] Chapter 12 for a thorough discussion). Thus it is clear that isometric immersions have a completely different qualitative behavior at low and high regularity (i.e. below and above C^2).

A strikingly similar phenomenon holds for the Euler equations since, when coupled with the energy constraint $|v|^2 = 2\bar{e}$, they are also formally overdetermined. Moreover C^1 solutions of the Cauchy problem are unique. There are further analogies when we look at embeddings with Hölder regularity, as we will see in Section 8 below.

7. Continuous and Hölder dissipative solutions

In the paper [23] we have succeeded in constructing the first example of a dissipative continuous solutions of the Euler equations. More precisely, we can prove the following statement.

Theorem 7.1. *Assume $e : [0,1] \to \mathbb{R}$ is a positive smooth function. Then there is a continuous vector field $v : \mathbb{T}^3 \times [0,1] \to \mathbb{R}^3$ and a continuous scalar field $p : \mathbb{T}^3 \times [0,1] \to \mathbb{R}$ which solve* (1.1) *in the sense of distributions and such that*

$$e(t) = \int |v|^2(x,t)\,dx \qquad \forall t \in [0,1]\,. \tag{7.1}$$

Moreover, in the more recent note [24] we have achieved a version of Theorem 7.1 which allows for a small Hölder exponent.

Theorem 7.2. *There is $\theta \in]0, \frac{1}{3}[$ with the following property. For every smooth positive function $e : \mathbb{S}^1 \to \mathbb{R}$ there is a vector field $v \in C^\theta(\mathbb{T}^3 \times \mathbb{S}^1, \mathbb{R}^3)$ and a scalar field $p \in C^\theta(\mathbb{T}^3 \times \mathbb{S}^1)$ which solve the incompressible Euler equations in the sense of distributions and such that*

$$e(t) = \int |v|^2(x,t)\,dx \qquad \forall t \in \mathbb{S}^1\,. \tag{7.2}$$

This represents obviously the first instance that Onsager's suggestion might indeed be correct. The construction in [23] is much more complicated and more surprising than the one in [20]. Note indeed that by a simple approximation argument continuous weak solutions of (1.1) satisfy the much stronger balance laws (1.2) and (1.3) for *any* C^1 open domain Ω.

Clearly, Theorem 7.1 is not the C^0 counterpart of Theorem 5.4. The way the theorem is derived share, however, several similarities with the Nash-Kuiper approach to the approximation of short maps with C^1 isometric embeddings. Indeed, Theorem 7.1 is achieved through an iteration procedure: the final product of this scheme can be seen as a superposition of infinitely many (perturbed) and weakly interacting Beltrami flows. Curiously, the idea that turbulent flows can be understood as a superposition of Beltrami flows has already been proposed almost 30 years ago in the fluid dynamics literature: see the work of Constantin and Majda [16].

Along the iteration the maps will be subsolutions of the Euler equations in the sense of Definition 5.2. In what follows $\mathcal{S}^{3\times 3}_0$ denotes the vector space of symmetric trace-free 3×3 matrices.

Definition 7.3. Assume $v, p, \mathring{R}$ are smooth functions on $\mathbb{T}^3 \times [0,1]$ taking values, respectively, in $\mathbb{R}^3, \mathbb{R}, \mathcal{S}^{3\times 3}_0$. We say that they solve the Euler-Reynolds system if

$$\left\{\begin{array}{l} \partial_t v + \mathrm{div}\,(v \otimes v) + \nabla p = \mathrm{div}\,\mathring{R} \\ \\ \mathrm{div}\, v = 0\,. \end{array}\right. \tag{7.3}$$

Clearly, the tensor $-\mathring{R}$ is just the traceless part of the Reynolds stress R introduced in (5.1). We are now ready to state the main proposition of [23], of which Theorem 7.1 is a simple corollary.

Proposition 7.4. *Let e be as in Theorem 7.1. Then there are positive constants η and M with the following property.*

Let $\delta \leq 1$ be any positive number and $(v, p, \mathring{R})$ a solution of the Euler-Reynolds system (7.3) *such that*

$$\tfrac{3\delta}{4} e(t) \leq e(t) - \int |v|^2(x,t)\,dx \leq \tfrac{5\delta}{4} e(t) \qquad \forall t \in [0,1] \tag{7.4}$$

and

$$\sup_{x,t} |\mathring{R}(x,t)| \leq \eta\delta\,. \tag{7.5}$$

Then there is a second triple $(v_1, p_1, \mathring{R}_1)$ which solves as well the Euler-Reynolds system and satisfies the following estimates:

$$\tfrac{3\delta}{8} e(t) \leq e(t) - \int |v_1|^2(x,t)\,dx \leq \tfrac{5\delta}{8} e(t) \qquad \forall t \in [0,1]\,, \tag{7.6}$$

$$\sup_{x,t} |\mathring{R}_1(x,t)| \leq \tfrac{1}{2}\eta\delta\,, \tag{7.7}$$

$$\sup_{x,t} |v_1(x,t) - v(x,t)| \leq M\sqrt{\delta} \tag{7.8}$$

and

$$\sup_{x,t} |p_1(x,t) - p(x,t)| \leq M\delta\,. \tag{7.9}$$

Proof of Theorem 7.1. We start by setting $v_0 = 0$, $p_0 = 0$, $\mathring{R}_0 = 0$ and $\delta := 1$. We then apply Proposition 7.4 iteratively to reach a sequence $(v_n, p_n, \mathring{R}_n)$ which solves (7.3) and such that

$$\frac{3}{4}\frac{e(t)}{2^n} \leq e(t) - \int |v_n|^2(x,t)\,dx \;\leq\; \frac{5}{4}\frac{e(t)}{2^n} \qquad \text{for all } t \in [0,1] \tag{7.10}$$

$$\sup_{x,t} |\mathring{R}_n(x,t)| \;\leq\; \frac{\eta}{2^n} \tag{7.11}$$

$$\sup_{x,t} |v_{n+1}(x,t) - v_n(x,t)| \;\leq\; M\sqrt{\frac{1}{2^n}} \tag{7.12}$$

$$\sup_{x,t} |p_{n+1}(x,t) - p_n(x,t)| \;\leq\; \frac{M}{2^n}\,. \tag{7.13}$$

Then $\{v_n\}$ and $\{p_n\}$ are both Cauchy sequences in $C(\mathbb{T}^3 \times [0,1])$ and converge uniformly to two continuous functions v and p. Similarly $\mathring{R}_n$ converges uniformly to 0. Moreover, by (7.10)

$$\int_{\mathbb{T}^3} |v|^2(x,t)\,dx = e(t) \qquad \forall t \in [0,1]\,.$$

Passing into the limit in (7.3) we therefore conclude that (v, p) solves (1.1). □

The proof of Proposition 7.4 shares several similarities with Nash's scheme. The most important one, common to all instances of the h principle, is that the map v_1 consists of adding two perturbations to v:

$$v_1 = v + w_o + w_c =: v + w. \tag{7.14}$$

where the leading term of the perturbation has the form

$$w_o(x,t) = W(x,t,\lambda x,\lambda t)$$

with W smooth and λ very large. Thus v_1 is derived from v by adding very fast oscillations.

On the other hand there are several points where our method departs dramatically from Nash's, due to some issues which are typical of the Euler equations and are not present for the isometric embeddings. We just highlight the two ones which are, in our opinion, the most relevant.

First of all, our scheme has to deal with a "transport term" which arises, roughly speaking, as the linearization of the first equation in (1.1). This term is typical of an evolution equation, whereas, instead, the equations for isometric embeddings are "static". At a first glance this transport term makes it impossible to use a scheme like the one of Nash to prove Theorem 7.1. To overcome this obstruction we need to introduce a phase-function that acts as a kind of discrete Galilean transformation of the (stationary) Beltrami flows, and to introduce an "intermediate" scale along each iteration step on which this transformation acts.

Secondly, convex integration heavily relies on one-dimensional oscillations - the simple reason being that these can be "integrated", hence the name convex integration. As already mentioned, the main building blocks of our iteration scheme are Beltrami flows, which are truly three-dimensional oscillations. The issue of going beyond one-dimensional oscillations has been raised by Gromov (p219 of [29]) as well as Kirchheim-Müller-Šverák (p52 of [33]), but as far as we know, there have been no such examples in the literature so far. In fact, it seems that with one-dimensional oscillations alone one cannot reach a proof of Proposition 7.4.

8. $C^{1,\alpha}$ isometric embeddings

The question of a sharp regularity threshold has been the object of investigation for the isometric embedding of surfaces as well (see for instance [29], [58]). Consider a smooth Riemannian 2-dimensional manifold $M = (\mathbb{S}^2, g)$ with positive curvature. As already mentioned, the isometric embeddings of M into $\mathbb{R}^3$ are rigid in the class C^2, whereas the h-principle holds for C^1. Borisov investigated embeddings of class $C^{1,\alpha}$ and proved the rigidity for $\alpha > \frac{2}{3}$ (as a culminating result of the investigations in [3, 4, 5, 6, 7]) and the local h-principle for $\alpha < \frac{1}{13}$ (although the latter was announced in 1965, see [8], a partial proof only appeared in 2004 [9]). In [17] we returned to this problem, and gave a more modern PDE proof of the loca h-principle for $\alpha < \frac{1}{7}$, together with more general statements in all dimensions (here by locality we mean that the h-principle holds in this form for Riemannian 2-dimensional manifolds diffeomorphic to $\mathbb{R}^2$: for purely technical reasons, when the topology is more complicated the proofs yield a lower treshold, cp. with [17, Corollaries 1 and 2]).

The arguments of [3, 4, 5, 6, 7] for the rigidity when $\alpha > \frac{2}{3}$ are geometric but quite involved. A short proof of Borisov's rigidity result was provided in [17]. Note that if $u \in C^3$ one can compute the area distorsion of the Gauss map from the Riemann-curvature tensor, which in turn depends only on the metric. When the curvature is positive, the image $u(M)$ is therefore locally convex. Even if the metric g is smooth, this is nonetheless false in general when the isometry is not regular enough, as shown precisely by the Nash-Kuiper theorem. However, by a result of Pogorelov (see [43] and [44]), the convexity of $u(M)$ holds even for C^1 maps u, provided one could show that the area distortion is always "positive" (cp. with [17] for the exact definition). The theory developed by Borisov in [3, 4, 5, 6, 7] shows that this positivity holds when the isometric immersion u is of class $C^{1,\frac{2}{3}+\varepsilon}$. In [17] we recover Borisov's statement expressing the equality between the Riemann-curvature tensor and the area distortion of the Gauss map with a suitable integrable formula. The latter resenbles, in structure, the integral identity leading to the energy conservation for the Euler equations. Indeed our computations in [17] bear striking similarities with those used in [15] for proving the energy conservation of $C^{1,\frac{1}{3}+\varepsilon}$ solutions of Euler.

In the case of isometric embeddings there does not seem to be a universally accepted critical exponent (see Problem 27 in [58]), even though $\frac{1}{2}$ and $\frac{1}{3}$ seem both relevant (compare with the discussion in [9] and with that in [22]).

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