Dissipative continuous Euler flows in two and three dimensions

A. Choffrut, C. De Lellis, L. Székelyhidi Jr.

April 2, 2012

Contents

In [1] the authors construct dissipative continuous (weak) solutions to the incompressible Euler equations on the three-dimensional torus $\mathbb{T}^3$. The building blocks in their proof are Beltrami flows, which are inherently three-dimensional. The purpose of this note is to show that the techniques can nevertheless be adapted to the two-dimensional case. (For motivation for the search of such flows with "anomalous energy dissipation", in particular for more on the Onsager conjecture, the interested reader is referred to the Introduction of [1].)

1 Main results

In this paper we will take

$$d = 2 \quad \text{or} \quad 3.$$

Theorem 1 *Let $T > 0$. Assume $e\colon [0,T] \to \mathbb{R}$ is a smooth positive function. Then, there is a continuous vector field $v\colon \mathbb{T}^d \times [0,T] \to \mathbb{R}^d$ and a continuous scalar field $p\colon \mathbb{T}^d \times [0,T] \to \mathbb{R}$ solving the incompressible Euler equations*

$$\partial_t v + \operatorname{div}(v \otimes v) + \nabla p = 0, \qquad \operatorname{div} v = 0 \tag{1}$$

in the sense of distributions and such that

$$e(t) = \int_{\mathbb{T}^d} |v|^2(x,t)\, dx, \qquad t \in [0,T]. \tag{2}$$

The case $d = 3$ was proved in [1]. Here we show that the proof can be adpated to the case $d = 2$. It is based on an iteration procedure in each step of which one solves a system of equations closely related to the Euler equations. We introduce some terminology. We let $\mathcal{S}^{d\times d}$ denote the space of symmetric $d \times d$ matrices and $\mathcal{S}_0^{d\times d}$ its subspace of trace-free matrices. Assume $v, p, \mathring{R}$ are smooth functions on $\mathbb{T}^d \times [0,T]$ taking values in $\mathbb{R}^d, \mathbb{R}$, and $\mathcal{S}_0^{d\times d}$ respectively. We say that $(v, p, \mathring{R})$ solves the Euler-Reynolds system if they satisfy

$$\partial_t v + \operatorname{div}(v \otimes v) + \nabla p = \operatorname{div} \mathring{R}, \qquad \operatorname{div} v = 0. \tag{3}$$

Theorem 1 is a consequence of the following

Proposition 2 *Let $e\colon [0,T] \to \mathbb{R}$ be a smooth positive function. Then, there exist constants η and M, depending on e, with the following property.*

Let $\delta \leq 1$ and $(v, p, \mathring{R})$ a solution of the Euler-Reynolds system

$$\partial_t v + \operatorname{div}(v \otimes v) + \nabla p = \operatorname{div} \mathring{R}$$

such that

$$\frac{3}{4}\delta e(t) \leq e(t) - \int_{\mathbb{T}^d} |v|^2(x,t)\, dx \leq \frac{5}{4}\delta e(t), \qquad t \in [0,T] \tag{4}$$

and

$$\sup_{x,t} |\mathring{R}(x,t)| \leq \eta\delta. \tag{5}$$

Then there exists $(v_1, p_1, \mathring{R}_1)$ *solving the Euler-Reynolds system and satisfying the following estimates:*

$$\frac{3}{8}\delta e(t) \le e(t) - \int |v_1|^2(x,t)\,dx \le \frac{5}{8}\delta e(t), \qquad t \in [0,T] \tag{6}$$

$$\sup_{x,t} |\mathring{R}_1(x,t) \le |\frac{1}{2}\eta\delta, \tag{7}$$

$$\sup_{x,t} |v_1(x,t) - v(x,t)| \le M\sqrt{\delta}, \tag{8}$$

and

$$\sup_{x,t} |p_1(x,t) - p(x,t)| \le M\delta. \tag{9}$$

Proof of Theorem 1: Start with $v_0 = 0$, $p_0 = 0$, $\mathring{R}_0 = 0$, and $\delta = 1$. Apply Proposition 2 iteratively to obtain sequences v_n, p_n, and $\mathring{R}_n$ solving the Euler-Reynolds system (3) and satisfying

$$\begin{aligned}
\frac{3}{4}\frac{e(t)}{2^n} \le e(t) - \int_{\mathbb{T}^d} |v_n|^2(x,t)\,dx &\le \frac{5}{4}\frac{e(t)}{2^n}, \qquad t \in [0,T], \\
\sup_{x,t} |\mathring{R}_n(x,t)| &\le \frac{\eta}{2^n}, \\
\sup_{x,t} |v_{n+1}(x,t) - v_n(x,t)| &\le M\sqrt{\frac{1}{2^n}}, \\
\sup_{x,t} |p_{n+1}(x,t) - p_n(x,t)| &\le \frac{M}{2^n}.
\end{aligned}$$

The sequences v_n and p_n are Cauchy in $C(\mathbb{T}^d \times [0,T])$ and hence converge (uniformly) to continuous functions v and p respectively. Likewise, $\mathring{R}_n$ converges (uniformly) to 0. Moreover, taking limits in the estimates on the energy,

$$\int_{\mathbb{T}^d} |v|^2(x,t)\,dx = e(t), \qquad t \in [0,T].$$

Also, we may pass to the limit in the (weak formulation of the) Euler-Reynolds system (3) and this shows that v, p satisfy the (weak formulation of the) Euler equations (1). ■

We also show that the 3D flows constructed in Theorem 1 are genuinely three-dimensional.

Theorem 3 *Let* $d = 3$. *Then, a typical solution constructed in Theorem 1 satsifies*

$$\text{rank}\left(\int_{\mathbb{T}^3} v(x,t) \otimes v(x,t)\,dx\right) = 3, \qquad t \in [0,T].$$

Furthermore, v *can be made arbitrarily small in the* H^{-1}*-topology: for any prescribed* $\epsilon > 0$*, we have*

$$\|v\|_{H^{-1}} < \epsilon. \tag{10}$$

This is done by a careful analysis of the proof of [1].

Remark A similar analysis in the two-dimensional case leads to the conclusion that the two-dimensional flows constructed in Theorem 1 are typically not parallel flows. However, it is classical that such flows are necessarily stationary, and this is not possible if $e(t)$ is not constant.

2 General considerations on the construction

In this Section we collect the main ingredients in the proofs of Theorems 1 and 3.

2.1 Construction of $(v_1, p_1, \mathring{R}_1)$

Assume $(v, p, \mathring{R})$ as in Proposition 2. Write $v_1 = v + w$ and $p_1 = p + q$ where w and q are to be determined. Then,

$$\partial_t v_1 + \operatorname{div}(v_1 \otimes v_1) + \nabla p_1 = \operatorname{div}(w \otimes w + q\mathrm{Id} + \mathring{R}) + \partial_t w + \operatorname{div}(v \otimes w + w \otimes v).$$

The perturbation w should be so chosen as to eliminate the first term in the right-hand side, viewing the remainder as a small error. Note how this first term is reminiscent of the stationary Euler equations. Roughly speaking, the perturbation w will be chosen as a high oscillation modulation of a stationary solution. More specifically, it will be taken of the form

$$w = w_o + w_c$$

where w_o is a highly oscillator term given quite explicitly and the corrector term w_c enforce the divergence-free condition $\operatorname{div} w = 0$. The oscillation term w_o will depend on two parameters λ and μ such that

$$\lambda, \mu, \frac{\lambda}{\mu} \in \mathbb{N}.$$

In fact, λ and μ will have to be chosen sufficiently large, depending on v, $\mathring{R}$, and e, in order that the desired estimates (6), (7), (8), and (9) be satisfied.

2.2 A complete set of stationary flows

The stationary Euler equation is nonlinear. The following Proposition says that there exists a *linear* space of stationary solutions. We first introduce some notation.

- **Case** $d = 2$ For $k \in \mathbb{Z}^2$, we let

$$b_k(\xi) := i\frac{k^\perp}{|k|}e^{ik\cdot\xi}, \qquad \psi_k(\xi) = \frac{e^{ik\cdot\xi}}{|k|}. \tag{11}$$

so that

$$b_k(\xi) = \nabla_\xi^\perp \psi_k(\xi), \quad \text{where} \quad \nabla_\xi^\perp = (-\partial_{\xi^2}, \partial_{\xi^1}).$$

hence $\mathrm{div}_\xi\ b_k = 0$. Observe also that $\overline{b_k} = b_{-k}$ and $\overline{\psi_k} = \psi_{-k}$.

- **Case** $d = 3$ For $k \in \mathbb{Z}^3$, we let

$$b_k(\xi) := B_k e^{ik\cdot\xi} \tag{12}$$

where B_k will be chosen appropriately, see Lemma 5. In particular, one should have $\mathrm{div}_\xi\ b_k = 0$ and $\overline{b_k} = b_{-k}$.

Lemma 4 (A complete set of stationary flows in 2D) *Let $\nu \geq 1$. For $k \in \mathbb{Z}^2$ such that $|k|^2 = \nu$, let $a_k \in \mathbb{C}$ such that $\overline{a_k} = a_{-k}$. Then,*

$$W(\xi) = \sum_{|k|^2=\nu} a_k b_k(\xi), \qquad \Psi(\xi) = \sum_{|k|^2=\nu} a_k \psi_k(\xi)$$

are $\mathbb{R}$-valued and satisfy

$$\mathrm{div}_\xi(W \otimes W) = \nabla_\xi \left(\frac{|W|^2}{2} - \nu \frac{\Psi^2}{2} \right). \tag{13}$$

Furthermore,

$$\langle W \otimes W \rangle_\xi := \fint_{\mathbb{T}^2} W \otimes W \, d\xi = \sum_{|k|^2=\nu} |a_k|^2 \left(\mathrm{Id} - \frac{k}{|k|} \otimes \frac{k}{|k|} \right) \tag{14}$$

In other words, W and Ψ are the velocity field and stream function, respectively, of a stationary flow with pressure $P = -\frac{|W|^2}{2} + \nu \frac{\Psi^2}{2}$.

Proof By direct computation one finds $\Delta_\xi \psi_k = -|k|^2 \psi_k$, and hence that $\Delta_\xi \Psi = -\nu \Psi$. Recall the identities

$$\mathrm{div}_\xi(W \otimes W) = \frac{1}{2} \nabla_\xi |W|^2 + (\mathrm{curl}_\xi W) W^\perp$$

where $\mathrm{curl}_\xi W = \partial_{\xi^1} W^2 - \partial_{\xi^2} W^1 = \Delta_\xi \Psi$ and $W^\perp = (-W^2, W^1)$. Then,

$$\mathrm{div}_\xi(W \otimes W) = \nabla_\xi \frac{|W|^2}{2} - \nu \Psi \nabla_\xi \Psi$$

as desired.

As for the average, write

$$\begin{aligned} \fint_{\mathbb{T}^2} W \otimes W(\xi) \, d\xi &= \sum_{j,k} a_j a_k b_j \otimes b_k \\ &= -\sum_{j,k} a_j a_k e^{i(k+j)\cdot\xi} \frac{j^\perp}{|j|} \otimes \frac{k^\perp}{|k|} \\ &= \sum_{j,k} a_k \overline{a_j} e^{i(k-j)\cdot\xi} \frac{j^\perp}{|j|} \otimes \frac{k^\perp}{|k|}. \end{aligned} \tag{15}$$

If $j \neq k$, $\fint_{\mathbb{T}^2} e^{i(k-j)\cdot\xi}\, d\xi = 0$ and it is 1 if $j = k$. Thus,

$$\langle W \otimes W \rangle_\xi = \sum_{|k|^2=\nu} |a_k|^2 \frac{k^\perp}{|k|} \otimes \frac{k^\perp}{|k|} = \sum_{|k|^2=\nu} |a_k|^2 \left(\mathrm{Id} - \frac{k}{|k|} \otimes \frac{k}{|k|} \right)$$

where the last identity follows from $\frac{k^\perp}{|k|} \otimes \frac{k^\perp}{|k|} = \left(\mathrm{Id} - \frac{k}{|k|} \otimes \frac{k}{|k|} \right)$ by direct calculation. ■

The following three-dimensional version was proved in [1].

Lemma 5 (Beltrami flows) *Let $\nu \geq 1$. There exist $B_k \in \mathbb{C}^3$ for $|k| = \nu$ such that, for any choice of $a_k \in \mathbb{C}$ such that $\overline{a_k} = a_{-k}$, the vector field*

$$W(\xi) = \sum_{|k|=\nu} a_k b_k(\xi) \tag{16}$$

is divergence-free and satisfies

$$\mathrm{div}\, (W \otimes W) = \nabla \frac{|W|^2}{2}. \tag{17}$$

Furthermore,

$$\fint_{\mathbb{T}^3} W \otimes W \, d\xi = \sum_{|k|=\nu} |a_k|^2 \left(\mathrm{Id} - \frac{k}{|k|} \otimes \frac{k}{|k|} \right). \tag{18}$$

Note that in the three-dimensional case the pressure is given by $P = -\frac{|W|^2}{2}$.
Proof This is Proposition 3.1 in [1]. ■

2.3 The geometric lemma

Lemma 6 (Geometric Lemma) *For every $N \in \mathbb{N}$ we can choose $r_0 > 0$ and $\nu \geq 1$ with the following property. There exist pairwise disjoint subsets*

$$\Lambda_j \subset \begin{cases} \{k \in \mathbb{Z}^2 \ : \ |k|^2 = \nu\} & \text{if} \quad d = 2 \\ \{k \in \mathbb{Z}^3 \ : \ |k| = \nu\} & \text{if} \quad d = 3 \end{cases}, \qquad j \in \{1, \ldots, N\}$$

and smooth positive functions

$$\gamma_k^{(j)} \in C^\infty(B_{r_0}(\mathrm{Id})), \qquad j \in \{1, \ldots, N\}, \quad k \in \Lambda_j$$

such that

1. *$k \in \Lambda_j$ implies $-k \in \Lambda_j$ and $\gamma_k^{(j)} = \gamma_{-k}^{(j)}$;*

2. for each $R \in B_{r_0}(\mathrm{Id})$ we have the identity

$$R = \sum_{k\in\Lambda_j} \left(\gamma_k^{(j)}(R)\right)^2 \left(\mathrm{Id} - \frac{k}{|k|} \otimes \frac{k}{|k|}\right) \qquad R \in B_{r_0}(\mathrm{Id}). \tag{19}$$

Proof This was proved for the case $d = 3$ in Lemma 3.2 of [1] (up to a normalization constant). Here we treat the case $d = 2$ only. For each $v \in \mathbb{R}^2 \setminus \{0\}$ we define

$$M_v = \mathrm{Id} - \frac{v}{|v|} \otimes \frac{v}{|v|}.$$

Step 1 Fix $\nu \geq 1$ and for each set $F \subset \{k \in \mathbb{Z}^2 \mid |k|^2 = \nu\}$ denote $c(F)$ the interior of the convex hull in $\mathcal{S}^{2\times 2}$ (the space of symmetric 2×2 matrices) of the set $M_F := \{M_k \,:\, k \in F\}$. We claim in this step that it sufices to find ν and N disjoint subsets $F_j \subset \{k \in \mathbb{Z}^2 \,:\, |k|^2 = \nu\}$ such that

- $-F_j = F_j$;
- $c(F_j)$ contains a positive multiple of the identity.

Indeed, we will show below that, if F_j satisfies these two conditions, then we can find $r_0 > 0$, a subset $\Gamma_j \subset F_j$, and smooth positive functions $\lambda_k^{(j)} \in C^\infty(B_{2r_0}(\mathrm{Id}))$ for $k \in \Gamma_j$ such that

$$R = \sum_{k\in\Gamma_j} \lambda_k^{(j)}(R) M_k.$$

We then define the sets Λ_j and the functions $\gamma_k^{(j)}$ for $k \in \Lambda_j$ according to

- $\Lambda_j := \Gamma_j \cup -\Gamma_j$;
- $\lambda_k^{(j)} = 0$ if $k \in \Lambda_j \setminus \Gamma_j$;
- $\gamma_k^{(j)} := \sqrt{\lambda_k^{(j)} + \lambda_k^{(j)}}$ for $k \in \Lambda_j$.

Note that the sets Λ_j are then symmetric and that the functions $\gamma_k^{(j)}$ satisfy (19). Moreover, since at least one of $\lambda_{\pm k}^{(j)}$ is positive on $B_{2r_0}(\mathrm{Id})$, $\gamma_k^{(j)}$ is smooth (and positive) in $B_{r_0}(\mathrm{Id})$.

We now come to the existence of Γ_j. The open set $c(F_j)$ contains an element $\alpha\mathrm{Id}$ for some $\alpha > 0$. Observe that the space $\mathcal{S}^{2\times 2}$ of symmetric 2×2 matrices has dimension 3. Since $\alpha\mathrm{Id}$ sits inside the open set $c(F_j)$, there exists a 4-simplex S with vertices $A_1, \ldots, A_4 \in c(F_j)$ such that $\alpha\mathrm{Id}$ belongs to the interior of S. Let then ϑ sothat the ball $\tilde{U}$ centered at $\alpha\mathrm{Id}$ and radius ϑ si contained in S. Then, each $R \in \tilde{U}$ can be written in a unique way as a convex combination of A_i's:

$$R = \sum_{i=1}^{4} \beta_i(R) A_i$$

where the functions β_i are positive and smooth in $\tilde{U}$.

Using now Carathéodory's Theorem, each A_i is the convex combination $\sum_n \lambda_{i,n} M_{v_{i,n}}$ of at most 4 matrices $M_{v_{i,n}}$ where $v_{i,n} \in F_j$, where we require that each $\lambda_{i,n}$ be positive. (Carathéodory's Theorem guarantees the existence of 4 elements $M_{v_{i,n}}$ such that A_i belongs to their *closed* convex hull. If we insist that all coefficients should be positive, then some of these elements should be thrown away.)

Set now $r_0 := \frac{\vartheta}{2\alpha}$. Then

$$R = \sum_{i,n} \frac{1}{\alpha} \beta_i(\alpha R) \lambda_{i,n} M_{v_{i,n}}, \qquad R \in B_{2r_0}(\mathrm{Id})$$

and each coefficient $\frac{1}{\alpha}\beta_i(\alpha R)\lambda_{i,n}$ is positive for $R \in B_{2r_0}(\mathrm{Id})$. The set Γ_j is then taken as $\{v_{i,n}\}$. Since one might have $v_{i,n} = v_{l,m}$ for distinct (i,n) and (l,m), the function λ_k will be defined by

$$\lambda_k(R) = \sum_{(i,n): v_{i,n}=k} \frac{1}{\alpha} \beta_i(\alpha R) \lambda_{i,n}$$

and this completes Step 1.

Step 2 By Step 1, in order to prove the lemma, it suffices to find ν and N disjoint families $F_1, \cdots, F_N \subset \sqrt{\nu}\mathbb{S}^1 \cap \mathbb{Z}^2$ such that each set $c(F_j)$ contains a positive multiple of the identity. Note that $\mathbb{S}^1 \cap \mathbb{Q}^2$ is dense in $\mathbb{S}^1$. Indeed, let

$$u \in \mathbb{R} \mapsto s(u) := \left(\frac{2u}{u^2+1}, \frac{u^2-1}{u^2+1} \right) \in \mathbb{S}^1 \subset \mathbb{R}^2.$$

Clearly $s(\mathbb{Q}) \subset \mathbb{Q}^2$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ and s is a diffeomorphism onto $\mathbb{S}^1 \setminus \{(0,1)\}$, the claim is proved.

In turn, there exists a sequence $\nu_k \to +\infty$ such that the sets $\mathbb{S}^1 \cap \frac{1}{\sqrt{\nu_k}}\mathbb{Z}^2$ converge, in the Hausdorff sense, to the entire circle $\mathbb{S}^1$. Given the sequence ν_k, one can easily partition each $\sqrt{\nu_k}\mathbb{S}^1 \cap \mathbb{Z}^2$ into N disjoint symmetric families $\{F_j^k\}_{j=1,\dots,N}$ in such a way that, for each fixed j, the corresponding sequence of sets $\left\{\frac{1}{\sqrt{\nu_k}} F_j^k\right\}_k$ converges in the Hausdorff sense to $\mathbb{S}^1$. Hence, any point of $c(\mathbb{S}^1)$ is contained in $c(\frac{1}{\sqrt{\nu_k}} F_j^k)$ for k sufficiently large. On the other hand, it is easy to see that $c(\mathbb{S}^1)$ contains a multiple $\alpha\mathrm{Id}$ of the identity in its interior. (One can adapt for instance the argument of Lemma 4.2 in [2].)

By Step 1, this concludes the proof. ■

3 The maps $v_1, p_1, \mathring{R}_1$

Let $e(t)$ be as in Theorem 1 and suppose $(v, p, \mathring{R})$ satisfies the hypothesis of Proposition 2. In this Section we define the next iterates $v_1 = v + w$, $p_1 = p + q$, and $\mathring{R}_1$. The perturbation

w will be defined as $w = w_o + w_c$ where w_o is given by an explicit formula, see (27), and the corrector w_c guarantees that $\operatorname{div} w = 0$.

3.1 The perturbation w_o

We apply Lemma 6 with

$$N = 2^d$$

(where $d \in \{2, 3\}$ denotes the number of space dimensions) to obtain $\nu \geq 1$ and $r_0 > 0$, and pairwise disjoint families Λ_j with corresponding functions $\gamma_k^{(j)} \in C^\infty(B_{r_0}(\mathrm{Id}))$.

The following is proved in [1] and works for any number of space dimensions. Denote $\mathcal{C}_1, \ldots, \mathcal{C}_N$ the equivalence classes of $\mathbb{Z}^d / \sim$ where $k \sim l$ if $k - l \in (2\mathbb{Z})^d$. The parameter $\mu \in \mathbb{N}$ will be fixed later.

Proposition 7 (Partition of the space of velocities) *There exists a partition of the space of velocities, namely $\mathbb{R}$-valued functions $\alpha_l(v)$ for $l \in \mathbb{Z}^d$ satisfying*

$$\sum_{l \in \mathbb{Z}^d} (\alpha_l(v))^2 \equiv 1 \tag{20}$$

such that, setting

$$\phi_k^{(j)}(v, \tau) = \sum_{l \in \mathcal{C}_j} \alpha_l(\mu v) e^{-i(k \cdot \frac{l}{\mu})\tau}, \qquad j = 1, \ldots, N, \quad k \in \mathbb{Z}^d, \tag{21}$$

we have the following estimates:

$$\sup_{v,\tau} |D_v^m \phi_k^{(j)}(v, \tau)| \leq C(m, d)\mu^m, \tag{22}$$

$$\sup_{v,\tau} |D_v^m (\partial_\tau \phi_k^{(j)} + i(k \cdot v)\phi_k^{(j)})(v, \tau)| \leq C(m, |k|, d)\mu^{m-1}. \tag{23}$$

Furthermore, $\overline{\phi_k^{(j)}} = \phi_{-k}^{(j)}$ and

$$|\phi_k^{(j)}(v, \tau)|^2 = \sum_{l \in \mathcal{C}_j} \alpha_l^2(v). \tag{24}$$

Set now

$$\rho(s) := \frac{1}{d(2\pi)^d} \left(e(t) \left(1 - \frac{\delta}{2}\right) - \int_{\mathbb{T}^d} |v|^2(x, t)\, dx \right) \tag{25}$$

and

$$R(y, s) := \rho(s)\mathrm{Id} - \mathring{R}(y, s). \tag{26}$$

Define

$$w_o(x, t) := W(x, t, \lambda t, \lambda x) \tag{27}$$

where

$$\begin{aligned} W(y,s,\tau,\xi) &= \sum_{|k|^2=\nu} a_k(y,s,\tau) b_k(\xi) \\ &= \sqrt{\rho(s)} \sum_{j=1}^{N} \sum_{k\in\Lambda_j} \gamma_k^{(j)} \left(\frac{R(y,s)}{\rho(s)}\right) \phi_k^{(j)}(v(y,s),\tau) b_k(\xi). \end{aligned} \tag{28}$$

(The velocity fields b_k were defined in (11).) We also introduce the corresponding stream function

$$\psi_o(x,t) := \Psi(x,t,\lambda t,\lambda x) \tag{29}$$

where

$$\begin{aligned} \Psi(y,s,\tau,\xi) &= \sum_{|k|^2=\nu} a_k(y,s,\tau) \psi_k(\xi) \\ &= \sqrt{\rho(s)} \sum_{j=1}^{N} \sum_{k\in\Lambda_j} \gamma_k^{(j)} \left(\frac{R(y,s)}{\rho(s)}\right) \phi_k^{(j)}(v(y,s),\tau) \psi_k(\xi). \end{aligned}$$

(The stream functions ψ_k were defined in (11).)

3.2 The constants η and M

In this Section we fix the values of the constants η and M from Proposition 2.

The perturbation w_o is well defined provided $\frac{R}{\rho} \in B_{r_0}(\mathrm{Id})$ where r_0 is as in Lemma 6. By definition (25) of ρ and assumption (4),

$$\rho(t) \geq \frac{1}{d(2\pi)^d} \frac{\delta}{4} e(t) \geq c\delta \min_{t\in[0,T]} e(t)$$

where $c = \frac{1}{4d(2\pi)^d}$. Then,

$$\left| \frac{R}{\rho(t)} - \mathrm{Id} \right| \leq \frac{1}{c\delta \min_{t\in[0,T]} e(t)} \left| \mathring{R} \right| \leq \frac{\eta}{c \min_{t\in[0,T]} e(t)}.$$

Thus, we choose η satisfying

$$\eta \leq \frac{1}{2} c \min_{t\in[0,T]} e(t) r_0. \tag{30}$$

Observe that this restriction is independent of δ.

We choose first a constant $M' > 1$ such that

$$M' > 2 \left(\sum_{j=1}^{N} \sum_{k\in\Lambda_j} \sup_R |\gamma_k^{(j)}(R)| \sup_{v,\tau} |\phi_k^{(j)}(v,\tau)| \sup_\xi |b_k(\xi)| \right)^2 \tag{31}$$

and, in case $d = 2$, we additionally impose that

$$M' > \frac{1}{\sqrt{\nu}} \sum_{j=1}^{N} \sum_{k \in \Lambda_j} \sup_R |\gamma_k^{(j)}(R)| \sup_{v,\tau} |\phi_k^{(j)}(v,\tau)| \sup_\xi |\psi_k(\xi)| \tag{32}$$

and then choose $M > 1$ such that

$$M \geq M' \sqrt{\frac{\sup_{0 \leq t \leq T} e(t)}{4d(2\pi)^d}}.$$

Observe that with these choices we have

$$\sqrt{\nu}\|\psi_o\|_0 \quad \text{and} \quad \|w_o\|_0 \quad \leq \quad \frac{\sqrt{M}}{2}\sqrt{\delta}. \tag{33}$$

since $\rho(t) \leq \frac{1}{d(2\pi)^d}\delta e(t)$ by (25) and (4).

3.3 The correction w_c

To obtain w from w_o we need to introduce the Leray projection onto divergence-free vector fields with zero average.

Definition 8 (The Leray projector) *Let $v \in C^\infty(\mathbb{T}^d, \mathbb{R}^d)$ be a smooth vector field. Let*

$$\mathcal{Q}v := \nabla\phi + \fint_{\mathbb{T}^d} v$$

where $\phi \in C^\infty(\mathbb{T}^d)$ is the solution to

$$\Delta\phi = \operatorname{div} v \quad \text{in } \mathbb{T}^d \qquad \text{subject to} \quad \fint_{\mathbb{T}^d} \phi = 0.$$

We denote by

$$\mathcal{P} := I - \mathcal{Q}$$

the Leray projector onto divergence-free vector fields with zero average.

The iterate v_1 is then expressed as

$$v_1 = v + \mathcal{P}w_o = v + w_o + w_c, \qquad w_c = -\mathcal{Q}w_o = w - w_o. \tag{34}$$

3.4 The pressure term p_1

We set

$$p_1 := \begin{cases} p - \left(\frac{|w_o|^2}{2} - \nu\frac{\psi_o^2}{2}\right) & \text{if} \quad d = 2, \\ p - \frac{|w_o|^2}{2} & \text{if} \quad d = 3. \end{cases} \tag{35}$$

This choice for p_1 will become clearer at the end of the proof, see Lemma 18.

3.5 The Reynolds stress tensor $\mathring{R}_1$

To construct $\mathring{R}_1$, we introduce another operator.

Definition 9 *Let $v \in C^\infty(\mathbb{T}^d, \mathbb{R}^d)$ be a smooth vector field. We define $\mathcal{R}v$ to be the matrix-valued periodic function*

$$\mathcal{R}v = \begin{cases} \nabla u + (\nabla u)^\top - (\text{div } u)\text{Id} & \text{if} \quad d = 2 \\ \frac{1}{4}\left(\nabla \mathcal{P}u + (\nabla \mathcal{P}u)^\perp\right) + \frac{3}{4}\left(\nabla u + (\nabla u)^\perp\right) - \frac{1}{2}(\text{div}\, u)\text{Id} & \text{if} \quad d = 3 \end{cases} \tag{36}$$

where $u \in C^\infty(\mathbb{T}^d, \mathbb{R}^d)$ is the solution to

$$\Delta u = v - \fint_{\mathbb{T}^d} v \quad \text{in } \mathbb{T}^d, \qquad \text{subject to} \quad \fint_{\mathbb{T}^d} u = 0.$$

The operator $\mathcal{R}$ satisfies the following properties.

Lemma 10 **($\mathcal{R} = \text{div}^{-1}$)** *For any $v \in C^\infty(\mathbb{T}^d, \mathbb{R}^d)$ we have*

1. *$\mathcal{R}v(x)$ is a symmetric trace-free matrix for each $x \in \mathbb{T}^d$;*
2. *$\text{div}\, \mathcal{R}v = v - \fint_{\mathbb{T}^d} v$.*

Proof The case $d = 3$ is treated in [1] and thus we assume $d = 2$. Clearly, $\mathcal{R}v$ is symmetric. Next,

$$\text{tr}\, \mathcal{R}v = 2\text{div}\, u - 2\text{div}\, u = 0$$

and

$$\begin{aligned} \text{div}\, \mathcal{R}v &= \Delta u + \text{div}\, (\nabla u)^\top - \text{div}\, ((\text{div}\, u)\text{Id}) \\ &= \Delta u + \text{div}\, (\nabla u)^\top - \nabla \text{div}\, u \\ &= \Delta u \\ &= v - \fint_{\mathbb{T}^2} v \end{aligned}$$

by definition of u. ■

Then we set

$$\mathring{R}_1 := \mathcal{R}\left(\partial_t v_1 + \text{div}\, (v_1 \otimes v_1) + \nabla p_1\right). \tag{37}$$

One verifies, as in [1] (after Lemma 4.3, p. 15), that the argument in the right-hand side has zero average: $\text{div}\, (v_1 \otimes v_1 + p_1 \text{Id})$ clearly has average zero, and so does $\partial_t v_1 = \partial_t v + \partial_t w$ since $\partial_t v = -\text{div}\, (v \otimes v + p\text{Id})$ has average zero as well as w by definition of $\mathcal{P}$. In turn, Lemma 10 yields

$$\partial_t v_1 + \text{div}\, (v_1 \otimes v_1) + \nabla p_1 = \text{div}\, \mathring{R}_1.$$

4 Notation and assumptions

The letter m will denote a natural number (in $\mathbb{N}$), and α a real number in the interval $(0,1)$. The letter C will always denote a generic constant which may depend on $e, v, \mathring{R}, \nu, \alpha$, and δ, but not on λ nor μ. We will further impose that

$$1 \leq \mu \leq \lambda. \tag{38}$$

The sup-norm is denoted $\|f\|_0 = \sup_{\mathbb{T}^d} |f|$. The Hölder seminorms are given by

$$\begin{aligned} [f]_m &:= \max_{|\gamma|=m} \|D^\gamma f\|_0, \\ [f]_{m+\alpha} &:= \max_{|\gamma|=m} \sup_{x\neq y} \frac{|D^\gamma f(x) - D^\gamma f(y)|}{|x-y|^\alpha} \end{aligned}$$

and the Hölder norms are given by

$$\begin{aligned} \|f\|_m &:= \sum_{j=0}^m [f]_j\,, \\ \|f\|_{m+\alpha} &:= \|f\|_m + [f]_{m+\alpha}\,. \end{aligned}$$

We also recall the following elementariy identity: for $0 \leq r \leq 1$

$$[fg]_r \leq C([f]_r\,\|g\|_0 + \|f\|_0\,[g]_r).$$

5 Schauder estimates

In this Section we record estimates in Hölder spaces ("Schauder estimates") established in [1].

The next Proposition collects estimates in Hölder spaces ("Schauder estimates") for varous operators used in the remainder.

Proposition 11 *For any $\alpha \in (0,1)$ and $m \in \mathbb{N}$ there exists a constant $C = C(m,\alpha,d)$ satisfying the following properties. If $\phi, \psi \colon \mathbb{T}^d \to \mathbb{R}$ are the unique solutions to*

$$\left\{ \begin{array}{l} \Delta\phi = f \\ \fint_{\mathbb{T}^d} \phi = 0 \end{array} \right. , \qquad \left\{ \begin{array}{l} \Delta\phi = \operatorname{div} F \\ \fint_{\mathbb{T}^d} \psi = 0 \end{array} \right. ,$$

then

$$\|\phi\|_{m+2,\alpha} \leq C\|f\|_{m,\alpha}, \quad \text{and} \quad \|\psi\|_{m+1,\alpha} \leq C\|F\|_{m,\alpha}.$$

Moreover, we have the following estimates:

$$\begin{aligned}
\|\mathcal{Q}v\|_{m+\alpha} &\leq C\|v\|_{m+\alpha}, &(39)\\
\|\mathcal{P}v\|_{m+\alpha} &\leq C\|v\|_{m+\alpha}, &(40)\\
\|\mathcal{R}v\|_{m+1,\alpha} &\leq C\|v\|_{m+\alpha}, &(41)\\
\|\mathcal{R}\mathrm{div}\, A\|_{m+\alpha} &\leq C\|A\|_{m+\alpha}, &(42)\\
\|\mathcal{R}\mathcal{Q}\mathrm{div}\, A\|_{m+\alpha} &\leq C\|A\|_{m+\alpha}. &(43)
\end{aligned}$$

These estimates are proved in [1] in three dimensions, and it is easy to see that they should also hold in two dimensions as well. (The difference in the expressions for the operator $\mathcal{R}$ is only superficial.) Suffice it to say that these estimates are the expected ones: $\mathcal{P}$ and $\mathcal{Q}$ are differential operators of degree 0; $\mathcal{R}$ is a differential operator of degree -1; and div is a differential operator of degree 1.

The effect of the oscillation parameter λ is described in the following

Proposition 12 *Let $k \in \mathbb{Z}^d \setminus 0$ and $\lambda \geq 1$ be fixed.*

1. *For any $a \in C^\infty(\mathbb{T}^d)$ and $m \in \mathbb{N}$ we have*

$$\left| \int_{\mathbb{T}^d} a(x) e^{i\lambda k\cdot x}\, dx \right| \leq \frac{[a]_m}{\lambda^m}.$$

2. *Let $\phi_\lambda \in C^\infty(\mathbb{T}^d)$ be the solution to*

$$\Delta\phi_\lambda = f_\lambda \quad \text{in} \quad \mathbb{T}^d$$

subject to $\fint_{\mathbb{T}^d} \phi_\lambda = 0$ where $f_\lambda(x) = a(x)e^{i\lambda k\cdot x} - \fint_{\mathbb{T}^d} a(y)e^{i\lambda k\cdot y}dy$. Then, for any $\alpha \in (0,1)$ we have the estimate

$$\|\nabla\phi_\lambda\|_\alpha \leq \frac{C}{\lambda^{1-\alpha}}\|a\|_0 + \frac{C}{\lambda^{m-\alpha}}[a]_m + \frac{C}{\lambda^m}[a]_{m+\alpha}$$

where $C = C(m,\alpha,d)$.

This was established in [1] and is in fact valid in any dimension.

The following is a consequence of the definition (36) of $\mathcal{R}$, the Schauder estimate (40) for $\mathcal{P}$, and Proposition 12.

Corollary 13 (Estimates for the operator $\mathcal{R}$) *Let $k \in \mathbb{Z}^d \setminus 0$ be fixed. For a smooth vector field $a \in C^\infty(\mathbb{T}^d, \mathbb{R}^d)$, let $F(x) := a(x)e^{i\lambda k\cdot x}$. Then, we have*

$$\|\mathcal{R}(F)\|_\alpha \leq \frac{C}{\lambda^{1-\alpha}}\|a\|_0 + \frac{C}{\lambda^{m-\alpha}}[a]_m + \frac{C}{\lambda^m}[a]_{m+\alpha}$$

for $C = C(m,\alpha,d)$.

6 Estimates on the corrector and the energy

We recall that $w_o(x,t)$ is defined in (27).

Lemma 14 *1. Let $a_k \in C^\infty(\mathbb{T}^d \times [0,T] \times \mathbb{R})$ be as in (27). Then, for any $r \geq 0$,*

$$\begin{aligned} \|a_k(\cdot, s, \tau)\|_r &\leq C\mu^r, \\ \|\partial_s a_k(\cdot, s, \tau)\|_r &\leq C\mu^{r+1}, \\ \|\partial_\tau a_k(\cdot, s, \tau)\|_r &\leq C\mu^r, \\ \|(\partial_\tau a_k + i(k\cdot v)a_k)(\cdot, s, \tau)\|_r &\leq C\mu^{r-1}. \end{aligned}$$

2. The matrix-valued function $W \otimes W$ is given by

$$(W \otimes W)(y, s, \tau, \xi) = R(y, s) + \sum_{1 \leq |k| \leq 2\nu} U_k(y, s, \tau) e^{ik\cdot\xi} \tag{44}$$

where the coefficients $U_k \in C^\infty(\mathbb{T}^d \times [0,T] \times \mathcal{S}^{d\times d})$ satisfy, for any $r \geq 0$,

$$\begin{aligned} \|U_k(\cdot, s, \tau)\|_r &\leq C\mu^r, \\ \|\partial_s U_k(\cdot, s, \tau)\|_r &\leq C\mu^{r+1}, \\ \|\partial_\tau U_k(\cdot, s, \tau)\|_r &\leq C\mu^r, \\ \|(\partial_\tau + i(k\cdot v)U_k)(\cdot, s, \tau)\|_r &\leq C\mu^{r-1}. \end{aligned}$$

3. For $d = 2$, we have

$$\frac{|W|^2}{2} - \nu\frac{\Psi^2}{2} = \sum_{1 \leq |k| \leq 2\nu} \tilde{a}_k e^{ik\cdot\xi}$$

where the functions $\tilde{a}_k$ satisfy for any $r \geq 0$

$$\|\tilde{a}_k(\cdot, s, \tau)\|_r \leq C\mu^r.$$

In all these estimates the constant C depends on r, e, v, and $\mathring{R}$, but is independent of (s, τ) and μ.

Proof The first two items are proved in [1] for $d = 3$ and we only briefly recall their proof since it is identical for $d = 2$. The estimates on the coefficients a_k follow from the estimates (22) and (23) on $\phi_k^{(j)}$. Next, write $W \otimes W$ as a Fourier series in ξ:

$$W \otimes W(y, s, \tau, \xi) = U_0(y, s, \tau) + \sum_{1 \leq |k| \leq 2\nu} U_k(y, s, \tau) e^{ik\cdot\xi}$$

where the entries in the U_k's are quadratic in the a_k's and $\|a_k\|_0 \leq C$. Thus, the U_k's satisfy the claimed the estimates. Furthermore,

$$\begin{aligned}
U_0(y,s,\tau) &= \fint_{\mathbb{T}^d} W \otimes W \, d\xi \\
&\overset{(14)}{=} \rho \sum_{j=1}^{N} \sum_{k\in\Lambda_j} \left(\gamma_k^{(j)}\left(\frac{R}{\rho}\right)\right)^2 |\phi_k^{(j)}(v,\tau)|^2 \left(\mathrm{Id} - \frac{k}{|k|}\otimes\frac{k}{|k|}\right) \\
&\overset{(24)}{=} \rho \sum_{j=1}^{N} \sum_{k\in\Lambda_j} \sum_{l\in\mathcal{C}_j} \left(\gamma_k^{(j)}\left(\frac{R}{\rho}\right)\right)^2 \alpha_l^2(v) \left(\mathrm{Id} - \frac{k}{|k|}\otimes\frac{k}{|k|}\right) \\
&\overset{(19)}{=} R \sum_{j=1}^{N} \sum_{l\in\mathcal{C}_j} \alpha_l^2(v) \\
&\overset{(20)}{=} R. \qquad (45)
\end{aligned}$$

As for the third item of the Proposition, concerning $\frac{|W|^2}{2} - \nu\frac{\Psi^2}{2}$ when $d=2$, we compute, omitting variables and remembering that the sums are over j and k such that $|j|^2 = |k|^2 = \nu$,

$$\begin{aligned}
\Psi^2 &= \sum_{j,k} a_j a_k \psi_j \psi_k \\
&= \frac{1}{\nu} \sum_{j,k} a_j a_k e^{i(k+j)\cdot\xi} \\
&= \frac{1}{\nu} \sum_{j,k} a_k \overline{a_j} e^{i(k-j)\cdot\xi} \\
&= \frac{1}{\nu} \sum_{|k|^2=\nu} |a_k|^2 + \frac{1}{\nu} \sum_{j\neq k} a_k \overline{a_j} e^{i(k-j)\cdot\xi} \\
&= \frac{1}{\nu} \sum_{|k|^2=\nu} |a_k|^2 + \sum_{1\leq|k|\leq 2\nu} \underline{a}_k(y) e^{i(k\cdot\xi)}
\end{aligned}$$

where the coefficients $\underline{a}_k$ are quadratic in the a_k's. But from the expression (14) for $\langle W\otimes W\rangle_\xi$ we deduce

$$\begin{aligned}
|W|^2 = \mathrm{tr}\,(W\otimes W) &= \mathrm{tr}\, R + \sum_{1\leq|k|\leq 2\nu} \mathrm{tr}\, U_k e^{ik\cdot\xi} \\
&= \sum_{|k|^2=\nu} |a_k|^2 + \sum_{1\leq|k|\leq 2\nu} \mathrm{tr}\, U_k e^{ik\cdot\xi}.
\end{aligned}$$

Subtracting the above expression for $\nu\Psi^2$ from that of $|W|^2$ we obtain the desired expression with suitable coefficients $\tilde{a}_k$ which satisfy the same estimates as the a_k's. ■

Lemma 15 (Estimate on the corrector)

$$\|w_c\|_\alpha \le C\frac{\mu}{\lambda^{1-\alpha}}. \tag{46}$$

Proof This is proved in Lemma 6.2, p. 21, in [1] for $d=3$. For clarity we reprove it for $d=2$ with the appropriate adjustments. Recall that

$$w_o(x,t) = \sum_{|k|^2=\nu} a_k(x,t,\lambda t)\nabla^\perp\psi_k(\lambda x) = \nabla_\xi^\perp \Psi(x,t,\lambda t,\lambda x)$$

where $\Psi(y,s,\tau,\xi) = \sum_{|k|^2=\nu} a_k(y,s,\tau)\psi_k(\xi)$. But

$$\nabla^\perp\left(a_k(x,t,\lambda t)\psi_k(\lambda x)\right) = \lambda a_k(x,t,\lambda t)(\nabla^\perp\psi_k)(\lambda x) + \psi_k(\lambda x)\nabla^\perp a_k(x,t,\lambda t)$$

or

$$a_k(x,t,\lambda t)(\nabla^\perp\psi_k)(\lambda x) = \frac{1}{\lambda}\left\{\nabla^\perp\left(a_k(x,t,\lambda t)\psi_k(\lambda x)\right) - \psi_k(\lambda x)\nabla^\perp a_k(x,t,\lambda)\right\}$$

and thus

$$w_o(x,t) \;=\; \frac{1}{\lambda}\nabla^\perp\left(\sum_{|k|^2=\nu} a_k(x,t,\lambda t)\psi_k(\lambda x)\right) - \frac{1}{\lambda}\sum_{|k|^2=\nu}\psi_k(\lambda x)\nabla^\perp a_k(x,t,\lambda t).$$

Since $\mathcal{Q}$ eliminates the divergence-free part and div $\circ\,\nabla^\perp = 0$, we have by (34)

$$w_c(x,t) = \mathcal{Q}w_o(x,t) = \frac{1}{\lambda}\mathcal{Q}\left(\sum_{|k|^2=\nu}\psi_k(\lambda x)\nabla^\perp a_k(x,t,\lambda t)\right) =: \frac{1}{\lambda}\mathcal{Q}u_c(x,t). \tag{47}$$

Thus, by Schauder estimate (39) for $\mathcal{Q}$ and the estimates on the coefficients a_k from Lemma 14, we find

$$\|w_c\|_\alpha \le \frac{C}{\lambda}\|u_c\|_\alpha \le C\frac{\mu}{\lambda^{1-\alpha}}.$$

■

Lemma 16 (Estimate on the energy)

$$\left| e(t)(1-\frac{\delta}{2}) - \int_{\mathbb{T}^d}|v_1|^2\,dx\right| \le C\frac{\mu}{\lambda^{1-\alpha}}. \tag{48}$$

Proof This is proved in Lemma 6.3, p. 21 of [1] for $d=3$. For clarity we briefly recall it in the case $d=2$. Taking the trace in the expression (44) for $W\otimes W$ gives

$$|W|^2 = \operatorname{tr} R + \sum_{1\le|k|\le 2\nu}\operatorname{tr} U_k e^{ik\cdot\xi}.$$

From part (1) of Proposition 12 with $m = 1$, and using estimates on U_k from Lemma 14, we find

$$\left| \int_{\mathbb{T}^d} |w_o|^2 - \mathrm{tr}\ R \, dx \right| \quad \text{and} \quad \left| \int_{\mathbb{T}^d} v \cdot w_o \, dx \right| \quad \leq \quad C \frac{\mu}{\lambda}.$$

This, along with the estimate (46) on w_c and $\|w_o\|_0 \leq C$, implies

$$\left| \int_{\mathbb{T}^d} |v_1|^2 - |v|^2 - |w_o|^2 \, dx \right| \leq C \frac{\mu}{\lambda^{1-\alpha}}.$$

Now by definition (25) of ρ we have $\mathrm{tr}\ R = d\rho = \frac{1}{(2\pi)^2} \left(e(t) \left(1 - \frac{\delta}{2}\right) - \int_{\mathbb{T}^d} |v|^2 \, dx \right)$. Putting the above together finishes the proof. ■

7 Estimates on the Reynolds stress

In order to clarify the choice for p_1 as in (35), we will temporarily write $p_1 = p + q$. With this, we have

$$\begin{aligned} \mathrm{div}\, \mathring{R}_1 &= \partial_t v_1 + \mathrm{div}\,(v_1 \otimes v_1) + \nabla p_1 \\ &= \quad \partial_t w_o + v \cdot \nabla w_o \\ &\quad + \mathrm{div}\,(w_o \otimes w_o + q\mathrm{Id} + \mathring{R}) \\ &\quad + \partial_t w_c + \mathrm{div}\,(v_1 \otimes w_c + w_c \otimes v_1 - w_c \otimes w_c + v \otimes w_o). \end{aligned}$$

We split the Reynolds stress tensor into the *transport part*, the *oscillation part*, and the *error* as shown on the right-hand side of the above identity. In the remainder of this Section we estimate these terms separately.

Lemma 17 (The transport part)

$$\|\mathcal{R}(\partial_t w_o + v \cdot \nabla w_o)\|_\alpha \leq C \left(\frac{\lambda^\alpha}{\mu} + \frac{\mu^2}{\lambda^{1-\alpha}} \right). \tag{49}$$

Proof This is proved in Lemma 7.1, p. 22 of [1] for $d = 3$ and is valid for $d = 2$ as well. ■

Lemma 18 (The oscillation term)

$$\left\| \mathcal{R} \left(\mathrm{div}\,(w_o \otimes w_o + q\mathrm{Id} + \mathring{R}) \right) \right\|_\alpha \leq C \frac{\mu^2}{\lambda^{1-\alpha}}. \tag{50}$$

Proof This is proved in Lemma 7.2, p. 23 of [1] for $d = 3$. The main difference in the case $d = 2$ is in the role of q.

Recalling that $R(y, s) = \rho(s)\mathrm{Id} - \mathring{R}(y, s)$, see (26), and noting that $\rho = \rho(t)$ is a function of t only,

$$\begin{aligned}
& \operatorname{div}(w_o \otimes w_o + \mathring{R} + q\mathrm{Id}) \\
= \quad & \operatorname{div}(w_o \otimes w_o - R + q\mathrm{Id}) \\
= \quad & \operatorname{div}\left(w_o \otimes w_o - R - \left(\frac{|w_o|^2}{2} - \nu\frac{\psi_o^2}{2}\right)\mathrm{Id}\right) + \nabla\left(q + \frac{|w_o|^2}{2} - \nu\frac{\psi_o^2}{2}\right) \\
= \quad & \operatorname{div}_y\left(W \otimes W - R - \left(\frac{|W|^2}{2} - \nu\frac{\Psi^2}{2}\right)\mathrm{Id}\right) \\
+ \quad & \lambda\operatorname{div}_\xi\left(W \otimes W - \left(\frac{|W|^2}{2} - \nu\frac{\Psi^2}{2}\right)\mathrm{Id}\right) \\
+ \quad & \nabla\left(q + \frac{|w_o|^2}{2} - \nu\frac{\psi_o^2}{2}\right) \\
= \quad & \operatorname{div}_y\left(W \otimes W - R - \left(\frac{|W|^2}{2} - \nu\frac{\Psi^2}{2}\right)\mathrm{Id}\right) \\
= \quad & \operatorname{div}_y\left(\sum_{|k|^2=\nu} U_k e^{i\lambda k\cdot x} - \sum_{|k|^2=\nu} \tilde{a}_k e^{i\lambda k\cdot x}\mathrm{Id}\right)
\end{aligned}$$

where two cancelations occur by construction of w_o, see (13), and by definition of q, see (35). The estimate follows from Corollary 13 with $m = 1$. ■

Lemma 19 (The error - I)

$$\|\mathcal{R}\partial_t w_c\|_\alpha \le C\frac{\mu^2}{\lambda^{1-\alpha}}. \tag{51}$$

Proof This is proved in Lemma 7.3, p. 23 of [1] for $d = 3$. For clarity we briefly recall it for $d = 2$ with the appropriate adjustments.

Recall that $u_c = \sum_{|k|^2=\nu} \psi_k(\lambda x)\nabla^\perp a_k(x, t, \lambda t)$ was defined in (47) and thus

$$\begin{aligned}
\partial_t u_c(x, t) = \quad & \lambda \sum_{|k|^2=\nu} \psi_k(\lambda x)\nabla^\perp \partial_\tau a_k(x, t, \lambda t) \\
+ \quad & \sum_{|k|^2=\nu} \psi_k(\lambda x)\nabla^\perp \partial_s a_k(x, t, \lambda t)
\end{aligned}$$

But for any vector-valued function $A(y, s, \tau)$, we have

$$\operatorname{div}\left(A(x, t, \lambda t) \otimes \frac{k}{|k|^3}e^{i\lambda k\cdot x}\right) = i\lambda A(x, t, \lambda t)\frac{e^{i\lambda k\cdot x}}{|k|} + e^{i\lambda k\cdot x}\left(\frac{k}{|k|^3}\cdot\nabla\right)A(x, t, \lambda t)$$

hence

$$\psi_k(\lambda x)A(x,t,\lambda t)e^{i\lambda k\cdot x} = \frac{1}{i\lambda}\mathrm{div}\,\left(A(x,t,\lambda t)\otimes\frac{k}{|k|^3}e^{i\lambda k\cdot x}\right) - \frac{1}{i\lambda}e^{i\lambda k\cdot x}\left(\frac{k}{|k|^3}\cdot\nabla\right)A(x,t,\lambda t).$$

Therefore $\partial_t u_c$ is of the form

$$\partial_t u_c = \mathrm{div}\, U_c + \tilde{u}_c$$

where

$$\|U_c\|_\alpha \le C\mu\lambda^\alpha, \qquad \|\tilde{u}_c\|_\alpha \le C\mu^2\lambda^\alpha$$

owing to the estimates from Lemma 14. In turn,

$$\begin{aligned}
\|\mathcal{R}\partial_t w_c\|_\alpha &\le \frac{1}{\lambda}\left(\|\mathcal{R}\mathcal{Q}\mathrm{div}\, U_c\|_\alpha + \|\mathcal{R}\mathcal{Q}\tilde{u}_c\|_\alpha\right) \\
&\le \frac{C}{\lambda}\left(\|U_c\|_\alpha + \|\tilde{u}_c\|_\alpha\right) \\
&\le C\frac{\mu^2}{\lambda^{1-\alpha}}
\end{aligned}$$

owing to the estimates (39), (41), and (43). ■

Lemma 20 (The error - II)

$$\|\mathcal{R}(\mathrm{div}\,(v_1\otimes w_c + w_c\otimes v_1 - w_c\otimes w_c))\|_\alpha \le C\frac{\mu}{\lambda^{1-2\alpha}}. \tag{52}$$

Proof This is proved in Lemma 7.4, p. 24 of [1] for $d = 3$ and the proof is valid in the case $d = 2$. ■

Lemma 21 (The error - III)

$$\|\mathcal{R}(\mathrm{div}\,(v\otimes w_o))\|_\alpha \le C\frac{\mu^2}{\lambda^{1-\alpha}}. \tag{53}$$

Proof This is proved in Lemma 7.4, p. 24 of [1] for $d = 3$ and we briefly indicate the proof in the case $d = 2$. Using that $\mathrm{div}\,_\xi b_k = 0$, we find

$$\begin{aligned}
\mathrm{div}\,(v\otimes w_o) &= w_o\cdot\nabla v + (\mathrm{div}\, w_o)v \\
&= \sum_{|k|^2=\nu} a_k(b_k\cdot\nabla)v + (\nabla a_k\cdot b_k)v \\
&= \sum_{|k|^2=\nu}\left[a_k\left(\frac{ik^\perp}{|k|}\cdot\nabla\right)v + \left(\nabla a_k\cdot\frac{ik^\perp}{|k|}\right)v\right]e^{i\lambda k\cdot x}.
\end{aligned}$$

The estimate follows from Corollary 13 with $m = 1$. ■

8 Conclusion: proof of Proposition 2

Recall that $e(t)$ is given as in Proposition 2 and that $(v, p, \mathring{R})$ is assumed to solve the Euler-Reynolds system (3) and to satisfy the bounds (4) and (5).

We have now all estimates available in order to fix the parameters μ, λ, and α so that the estimates (6), (7), (8), and (9) may hold. For simplicity we will take

$$\mu = \lambda^{\beta} \tag{54}$$

for some β to be determined (although strictly speaking this can only hold up to some constant depending only on β since it is required that $\mu \in \mathbb{N}$). Recall that C denotes a generic constant (possibly) depending on e, $v, \mathring{R}$, ν, α, and δ, but not on λ nor μ.

Recall that the constant M of Proposition 2 has already been fixed in (33) so that

$$\sqrt{\nu}\|\psi_o\|_0 \quad \text{and} \quad \|w_o\|_0 \leq \frac{\sqrt{M}}{2}\sqrt{\delta}. \tag{55}$$

Since $v_1 - v = w_o + w_c$, the bound (8) on $v_1 - v$ follows provided

$$\|w_c\|_\alpha \leq C\frac{\mu}{\lambda^{1-\alpha}} = C\lambda^{\alpha+\beta-1} \leq \sqrt{\delta}.$$

The bound (9) on $p_1 - p = -\left(\frac{|w_o|^2}{2} - \nu\frac{\psi_o^2}{2}\right)$ also follows from (55).

The bound (6) on the energy follows from (48) provided

$$C\frac{\mu}{\lambda^{1-\alpha}} = C\lambda^{\alpha+\beta-1} \leq \frac{\delta}{8}\min_{t\in[0,T]} e(t).$$

Finally, the estimates (17), (50), (51), (52), and (53), imply that

$$\|\mathring{R}_1\|_\alpha \leq C(\lambda^{\alpha-\beta} + \lambda^{\alpha+2\beta-1} + \lambda^{2\alpha+\beta-1}).$$

In conclusion, imposing

$$\alpha < \beta \qquad \text{and} \qquad \alpha + 2\beta < 1 \tag{56}$$

ensures that the bounds (6), (7), (8), and (9) hold provided λ is chosen sufficiently large.

This concludes the proof of Proposition 2.

9 Proof of Theorem 3

In this Section we fix $d = 3$. The proof of Theorem 3 follows from a closer analysis of the proof of Theorem 1.

Lemma 22 *Let $e, v, p, \mathring{R}$ be given. Let $R, w_o, w_c, v_1, \mathring{R}_1$ be defined as in (26), (27), (34), and (37). Then, there exist constants $C = C(e, v, p, \mathring{R})$ and $\gamma > 0$ such that*

$$\|w_c\|_\alpha \leq C\lambda^{-\gamma} \tag{57}$$

$$\left|\int_{\mathbb{T}^3} v \cdot w_o \, dx\right| \leq C\lambda^{-\gamma} \tag{58}$$

$$\left|\int_{\mathbb{T}^3} w_o \otimes w_o \, dx - R \, dx\right| \leq C\lambda^{-\gamma} \tag{59}$$

$$\left|\int_{\mathbb{T}^3} v_1 \otimes v_1 - v \otimes v - w_o \otimes w_o \, dx\right| \leq C\lambda^{-\gamma} \tag{60}$$

$$\|\mathring{R}_1\|_\alpha \leq C\lambda^{-\gamma}. \tag{61}$$

Proof Take $0 < \alpha < \beta < 1$ satisfying $\alpha + 2\beta < 1$, see (56), and set

$$\gamma = \min\left(1 - \alpha - 2\beta, \beta - \alpha, 1 - 2\alpha - \beta\right).$$

Estimate (57) is Lemma 15. Estimates (58), (59) and (60) follow from the proof of Lemma 16. Specifically, (58) follows since v is fixed and w_o is oscillatory, and (59) follows since $w_o \otimes w_o - R$ is the sum of oscillatory terms. For (60) we write

$$\int_{\mathbb{T}^3} v_1 \otimes v_1 \, dx = \int_{\mathbb{T}^3} v \otimes v \, dx + \int_{\mathbb{T}^3} w_o \otimes w_o \, dx + \int_{\mathbb{T}^3} \left(v \otimes w + w \otimes v + w_o \otimes w_c + w_c \otimes w_o\right) dx.$$

The estimate follows since w_o is oscillatory and w_c is small (v is fixed). Estimate (61) follows from Lemmas 7.1-7.4 of [1]. ∎

The following is a refined version of Proposition 2.

Proposition 23 *Let e be as above and let $v, p, \mathring{R}$ solve the Euler-Reynolds system. Suppose that there exist $0 < \delta \leq 1$ satisfying*

$$\left|e(t)(1-\delta) - \int_{\mathbb{T}^3} |v(t)|^2 \, dx\right| \leq \frac{\delta}{4} e(t)$$

$$\|\mathring{R}\|_0 \leq \eta\delta.$$

Let $R, v_1 = v + w = v + w_o = w_c, p_1 = p + q, \mathring{R}_1$ be defined as in (26), (27), (34), (35), and (37), so that they satisfy the Euler-Reynolds system:

$$\partial_t v_1 + \operatorname{div}(v_1 \otimes v_1) + \nabla p = \operatorname{div} \mathring{R}_1.$$

For any ϵ', the following inequalities are satisfied provided λ is chosen sufficiently large *depending on* $(v, p, \mathring{R})$.

$$\left| e(t)\left(1-\frac{\delta}{2}\right) - \int_{\mathbb{T}^3} |v_1(t)|^2\,dx \right| \leq \frac{\delta}{8} e(t) \tag{62}$$

$$\|\mathring{R}_1\|_0 \leq \min\left(\frac{1}{2}\eta\delta, \epsilon'\right) \tag{63}$$

$$\|v_1 - v\|_0 \leq M\sqrt{\delta}, \tag{64}$$

$$\|v_1 - v\|_{H^{-1}} \leq \epsilon', \tag{65}$$

$$\|p_1 - p\|_0 \leq M\delta \tag{66}$$

$$\left| \int_{\mathbb{T}^3} v_1 \otimes v_1\,dx - \int_{\mathbb{T}^3} v \otimes v\,dx - \int_{\mathbb{T}^3} w_o \otimes w_o\,dx \right| \leq \epsilon' \tag{67}$$

$$\left| \int_{\mathbb{T}^3} w_o \otimes w_o - R\,dx \right| \leq \epsilon'. \tag{68}$$

Proof With

$$\rho(t) = \frac{1}{3(2\pi)^3}\left[e(t)\left(1-\frac{\delta}{2}\right) - \int_{\mathbb{T}^3} |v(x,t)|^2\,dx \right] \tag{69}$$

we have

$$\frac{\min_{0\leq t\leq T} e(t)\delta}{12(2\pi)^3} < \rho(t) \leq \frac{\max_{0\leq t\leq T} e(t)\delta}{4(2\pi)^3}. \tag{70}$$

Owing to the definition (30) of η we obtain

$$\left\| \frac{R}{\rho} - \mathrm{Id} \right\|_0 = \left\| \frac{\mathring{R}}{\rho} \right\|_0 < r_0.$$

Thus, v_1, p_1, R_1 etc. can be defined and estimated as in Lemma 22. We will now choose λ in Lemma 22 sufficiently large (depending on $e, v, p, \tilde{R}$) so that the desired estimates hold.

Recalling the definition (26) of R in terms of ρ, we have

$$\begin{aligned} & e(t)\left(1-\frac{\delta}{2}\right) - \int_{\mathbb{T}^3} |v_1(t)|^2\,dx \\ = \; & e(t)\left(1-\frac{\delta}{2}\right) - \int_{\mathbb{T}^3} |v(t)|^2\,dx - \int_{\mathbb{T}^3} |w_o|^2\,dx + \int_{\mathbb{T}^3} \left(|v_1|^2 - |v|^2 - |w_o|^2\right)\,dx \\ = \; & - \int_{\mathbb{T}^3} \left(|w_o|^2 - \operatorname{tr} R\right)\,dx + \int_{\mathbb{T}^3} \left(|v_1|^2 - |v|^2 - |w_o|^2\right)\,dx. \end{aligned}$$

Invoking (59) and (60), we obtain (62) with large λ.

From (61) it is clear that the bound (63) on $\mathring{R}_1$ holds with λ large.

From (57), w_c can be made arbitrarily small with λ large, while from (27) and (32) we have $|w_o| \leq \frac{1}{2}M'\sqrt{\tilde{\rho}} \leq \frac{1}{2}M\sqrt{\delta}$. Thus, the bounds (64) on $v_1 - v = w_o + w_c$ and (66) on $p_1 - p = -\frac{|w_o|^2}{2}$ follow with λ large. Furthermore, w_o is a fast oscillating function and w_c is small in the sup-norm, so that we achieve the H^{-1}-bound (65) on $v_1 - v$ as well with λ large.

Finally, (59) and (60) trivially imply (67) and (68) with large λ. ■

9.1 Proof of Theorem 3

We are now ready to prove that the solution v, p constructed in Theorem 1 in the case $d = 3$ satisfies the estimates of Theorem 3 provided λ is chosen sufficiently large at each iteration. Set

$$\epsilon_0 := \frac{1}{2} \min_{0 \leq t \leq T} e(t).$$

We recall that the solution (v, p) is obtained as a limit

$$v := \lim_n v^{(n)}, \qquad p := \lim_n p^{(n)}$$

where the sequences $v^{(n)}$, $p^{(n)}$ are as follows. We let

$$v^{(0)} = 0, \qquad p^{(0)} = 0, \qquad \mathring{R}^{(0)} = 0.$$

For $n \geq 0$, we construct

$$v^{(n+1)} = v^{(n)} + w^{(n+1)}, \qquad p^{(n+1)} = p^{(n)} + q^{(n+1)}, \qquad \mathring{R}^{(n+1)}$$

using Proposition 23 with $v^{(n)} = v$, $p^{(n)} = p$, $\mathring{R}^{(n)} = \mathring{R}_1$, $v^{(n+1)} = v_1$, $w_o^{(n+1)} = w_o$, $w_c^{(n+1)} = w_c$, $R^{(n+1)} = R$, and

$$\delta = 2^{-n-1}, \quad \epsilon' = \min\left(\frac{1}{6}\epsilon_0 2^{-n-1}, \epsilon 2^{-n-1}\right). \tag{71}$$

Observe in particular that

$$\rho^{(1)}(t) \geq \frac{\epsilon_0}{3(2\pi)^3}, \qquad t \in [0, T]$$

according to (69) with $v = 0$ and $\delta = 1$ (step $n = 0$).

Now

$$\|v\|_{H^{-1}} \leq \sum_{n=0}^{\infty} \|w^{(n+1)}\|_{H^{-1}} \leq \epsilon$$

easily follows from (65). This establishes (10).

Next, for each n,

$$\begin{aligned}
\int_{\mathbb{T}^3} v^{(n+1)} \otimes v^{(n+1)}\,dx &= \int_{\mathbb{T}^3} v^{(n)} \otimes v^{(n)} + \int_{\mathbb{T}^3} w_o^{(n+1)} \otimes w_o^{(n+1)}\,dx \\
&\quad + \int_{\mathbb{T}^3} \left(v^{(n+1)} \otimes v^{(n+1)} - v^{(n)} \otimes v^{(n)} - w_o^{(n+1)} \otimes w_o^{(n+1)} \right) dx \\
&= \int_{\mathbb{T}^3} v^{(n)} \otimes v^{(n)} + \int_{\mathbb{T}^3} R^{(n+1)}\,dx \\
&\quad + \int_{\mathbb{T}^3} \left(w_o^{(n+1)} \otimes w_o^{(n+1)} - R^{(n+1)} \right) dx \\
&\quad + \int_{\mathbb{T}^3} \left(v^{(n+1)} \otimes v^{(n+1)} - v^{(n)} \otimes v^{(n)} - w_o^{(n+1)} \otimes w_o^{(n+1)} \right) dx \\
&= \int_{\mathbb{T}^3} v^{(n)} \otimes v^{(n)} + 3(2\pi)^3 \rho^{(n+1)} \mathrm{Id} - \int_{\mathbb{T}^3} \mathring{R}^{(n+1)}\,dx \\
&\quad + \int_{\mathbb{T}^3} \left(w_o^{(n+1)} \otimes w_o^{(n+1)} - R^{(n+1)} \right) dx \\
&\quad + \int_{\mathbb{T}^3} \left(v^{(n+1)} \otimes v^{(n+1)} - v^{(n)} \otimes v^{(n)} - w_o^{(n+1)} \otimes w_o^{(n+1)} \right) dx.
\end{aligned}$$

But the sum of the last two integrals is bounded by $\frac{1}{3}\epsilon_0 2^{-n-1}$ using (67), (68), and (71). Then,

$$\int_{\mathbb{T}^3} v \otimes v\,dx \;=\; -\sum_{n=0}^{\infty} \int_{\mathbb{T}^3} \mathring{R}^{(n+1)}\,dx + 3(2\pi)^3 \sum_{n=0}^{\infty} \rho^{(n+1)} \mathrm{Id} + \text{error}$$

where "error" denotes a term which is bounded by $\frac{1}{3}\epsilon_0$. Also, by (63) and (71) we have

$$\left| \sum_{n=0}^{\infty} \int_{\mathbb{T}^3} \mathring{R}^{(n+1)}\,dx \right| \le \frac{1}{6}\epsilon_0$$

and so

$$\int_{\mathbb{T}^3} v \otimes v\,dx = 3(2\pi)^3 \sum_{n=0}^{\infty} \rho^{(n+1)} \mathrm{Id} + \text{error}'$$

where the new error term is bounded by $\frac{1}{2}\epsilon_0$. Since $\rho^{(n+1)} \ge 0$ for $n \ge 1$ and $3(2\pi)^3\rho^{(1)} \ge \epsilon_0$, and setting $i = j$ we have

$$\left| \int_{\mathbb{T}^3} v \otimes v\,dx - \epsilon_0 \mathrm{Id} \right| \ge \frac{1}{2} \left| \epsilon_0 \mathrm{Id} \right|.$$

This implies that $\int_{\mathbb{T}^3} v \otimes v\,dx$ has full rank. ■

References

[1] C. De Lellis, L. Székelyhidi Jr., *Dissipative continuous Euler flows*, preprint

[2] C. De Lellis, L. Székelyhidi Jr., *The Euler equations as a differential inclusion*, Ann. of Math. (2) 170, 3 (2009), 1417-1436